Extending Homomorphism Theorem to Multi-Systems

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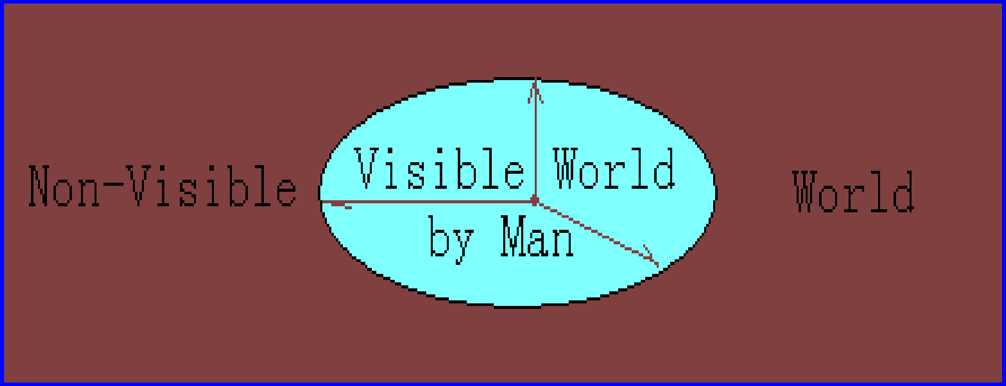
1. What is the Essence of Smarandache's Notion?

- **What can be acknowledged by mankind?**

TAO TEH KING**（道德经）** said**:**

All things that we can acknowledge is determined by our eyes, or ears, or nose, or tongue, or body or passions, i.e., these six organs.

- What is this sentence meaning? The non-visible world can be only known by the other five organs, particularly, the passion.



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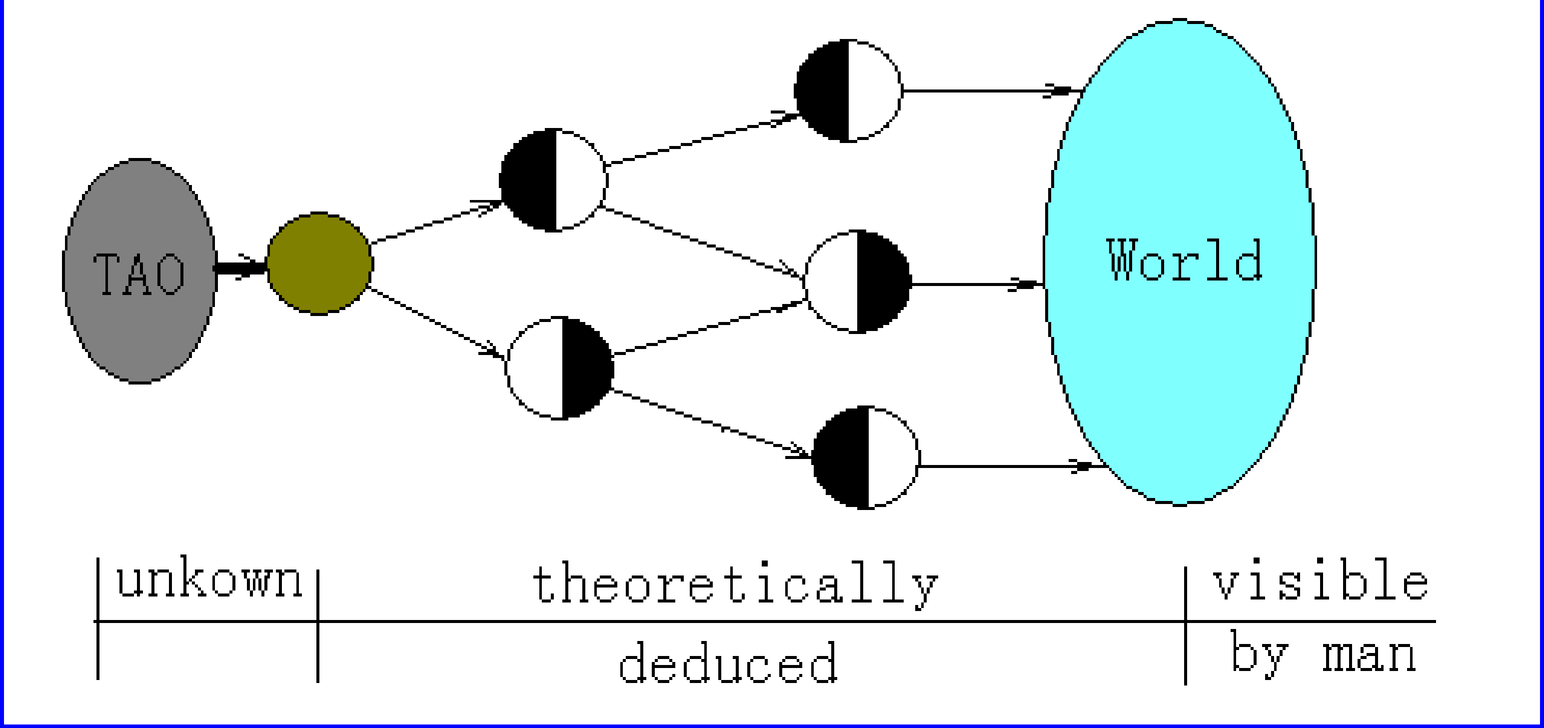
道德经

道生一，一生二，二生三，三生万物。万物负阴而抱阳，冲气以为和。

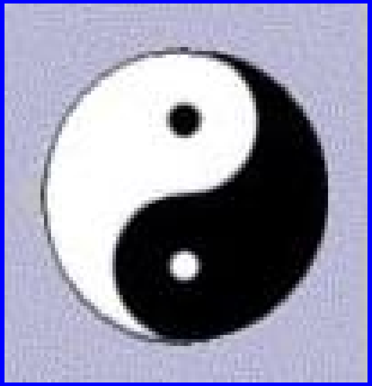
人法地，地法天，天法道，道法自然。

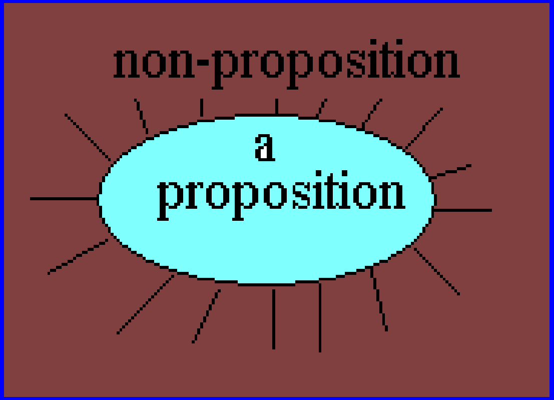
- What are these words meaning?



Here, the *theoretically deduced* is done by logic, particularly, Mathematical deduction.

The **combined positive and negative notion** in *TAO TEH KING* comes into being the idea of S-denied in the following, i.e., a proposition with its non-proposition.





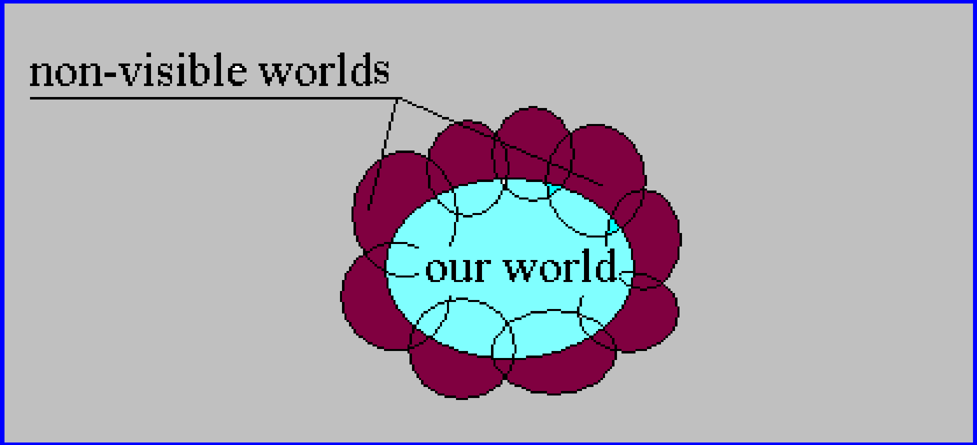
Smarandachely denied axioms:

An axiom is said smarandachely denied (S-denied) if in the same space the axiom behaves differently, i.e., validated and Invalided, or only invalidated but in at least two distinct ways.

- How can we know the non-visible world? We can only know it by mathematical deduction. Then HOW TO?

Smarandache multi-space:

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geqq 2$.



Applying it to mathematics, what we can obtain?

Combinatorial Conjecture(Mao,2005):

Every mathematical science can be reconstructed from or made by combinatorization.

- Why is this conjecture important?

It means that:

(i) One can selects finite combinatorial rulers to reconstruct or make generalization for classical mathematics and

(ii) One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences. Whence, it produces infinite creativity for math..

- How is it working? See the following sections.

2. A Review of Homomorphism Theorem on Groups

A set G with a binary operation "$\circ$", denoted by $(G;\circ)$, is called a *group* if $x \circ y \in G$ for $\forall x, y \in G$ such that the following conditions hold.

(*i*) $(x \circ y) \circ z = x \circ (y \circ z)$ for $\forall x, y, z \in G$;

(*ii*) There is an element $1_G, 1_G \in G$ such that $x \circ 1_G = x$;

(*iii*) For $\forall x \in G$, there is an element $y, y \in G$, such that $x \circ y = 1_G$.

For two groups G, G', let σ be a mapping from G to G'. If

$$\sigma(x \circ y) = \sigma(x) \circ \sigma(y),$$

for $\forall x, y \in G$, then call σ a *homomorphism* from G to G'. The *image* $Im\sigma$ and the *kernel* $Ker\sigma$ of a homomorphism $\sigma : G \to G'$ are defined as follows:

$$Im\sigma = G^{\sigma} = \{\sigma(x) | \ \forall x \in G\ \}, \quad Ker\sigma = \{x | \ \forall x \in G, \ \sigma(x) = 1_{G'}\ \}.$$

Homomorphism Theorem. *Let $\sigma : G \to G'$ be a homomorphism* from G to G'. *Then*

$$(G;\circ)/Ker\sigma \cong Im\sigma.$$

3. Definition of Multi-systems

3.1. Algebraic Systems. Let $\mathscr{A}$ be a set and $\circ$ an operation on $\mathscr{A}$. If $\circ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, i.e., closed then we call $\mathscr{A}$ an *algebraic system under the operation* $\circ$, denoted by $(\mathscr{A}; \circ)$. For example, let $\mathscr{A} = \{1, 2, 3\}$. Define operations $\times_1, \times_2$ on $\mathscr{A}$ by following tables.

$\times_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$\times_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Then we get two algebraic systems $(\mathscr{A}; \times_1)$ and $(\mathscr{A}; \times_2)$.

3.2. Multi-Operation Systems. A *multi-operation system* is a pair $(\mathscr{H};\widetilde{O})$ with a set $\mathscr{H}$ and an operation set $\widetilde{O}=\{\circ_i \mid 1\leq i\leq l\}$ on $\mathscr{H}$ such that each pair $(\mathscr{H};\circ_i)$ is an algebraic system. A multi-operation system $(\mathscr{H};\widetilde{O})$ is *associative* if for $\forall a,b,c\in\mathscr{H}$, $\forall\circ_1,\circ_2\in\widetilde{O}$, there is

$$(a\circ_1 b)\circ_2 c=a\circ_1(b\circ_2 c).$$

Two multi-operation systems $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ are called *homomorphic* if there is a mapping $\omega:\mathscr{H}_1\rightarrow\mathscr{H}_2$ with $\omega:\widetilde{O}_1\rightarrow\widetilde{O}_2$ such that for $a_1,b_1\in\mathscr{H}_1$ and $\circ_1\in\widetilde{O}_1$, there exists an operation $\circ_2=\omega(\circ_1)\in\widetilde{O}_2$ enables that

$$\omega(a_1\circ_1 b_1)=\omega(a_1)\circ_2\omega(b_1).$$

4. Extending to Algebraic Systems

Let $(\mathscr{A};\circ)$ be an algebraic system and $\mathscr{B} \prec \mathscr{A}$. For $\forall a \in \mathscr{A}$, define a coset $a \circ \mathscr{B}$ of $\mathscr{B}$ in $\mathscr{A}$ by $a \circ \mathscr{B} = \{a \circ b | \forall b \in \mathscr{B}\}$. Define a *quotient set* $\mathfrak{S} = \mathscr{A}/\mathscr{B}$ consists of all cosets of $\mathscr{B}$ in $\mathscr{A}$ and let R be a minimal set with $\mathfrak{S} = \{r \circ \mathscr{B} | r \in R\}$.

Theorem 4.1. *If $(\mathscr{B};\circ)$ is a subgroup of an associative system $(\mathscr{A};\circ)$, then*

(i) for $\forall a, b \in \mathscr{A}$, $(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) = \emptyset$ or $a \circ \mathscr{B} = b \circ \mathscr{B}$, i.e., $\mathfrak{S}$ is a partition of $\mathscr{A}$;

(ii) define an operation $\bullet$ on $\mathfrak{S}$ by

$$(a \circ \mathscr{B}) \bullet (b \circ \mathscr{B}) = (a \circ b) \circ \mathscr{B},$$

then $(\mathfrak{S};\bullet)$ is an associative algebraic system, called a quotient system of $\mathscr{A}$ to $\mathscr{B}$. Particularly, if there is a representation R whose each element has an inverse in $(\mathscr{A};\circ)$ with unit $1_{\mathscr{A}}$, then $(\mathfrak{S};\bullet)$ is a group, called a quotient group of $\mathscr{A}$ to $\mathscr{B}$.

Proof For (i), notice that if $(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) \neq \emptyset$ for $a, b \in \mathscr{A}$, then there are elements $c_1, c_2 \in \mathscr{B}$ such that $a \circ c_1 = b \circ c_2$. By assumption, $(\mathscr{B}; \circ)$ is a group of $(\mathscr{A}; \circ)$, we know that there exists an inverse element $c_1^{-1} \in \mathscr{B}$, i.e., $a = b \circ c_2 \circ c_1^{-1}$. Therefore, we get that

$$\begin{aligned} a \circ \mathscr{B} &= (b \circ c_2 \circ c_1^{-1}) \circ \mathscr{B} = \{(b \circ c_2 \circ c_1^{-1}) \circ c | \forall c \in \mathscr{B}\} \\ &= \{b \circ c | \forall c \in \mathscr{B}\} = b \circ \mathscr{B} \end{aligned}$$

By definition of $\bullet$ on $\mathfrak{S}$ and (i), we know that $(\mathfrak{S}; \bullet)$ is an algebraic system. For $\forall a, b, c \in \mathscr{A}$, by the associative laws in $(\mathscr{A}; \circ)$, we find that

$$\begin{aligned} ((a \circ \mathscr{B}) \bullet (b \circ \mathscr{B})) \bullet (c \circ \mathscr{B}) &= ((a \circ b) \circ \mathscr{B}) \bullet (c \circ \mathscr{B}) \\ &= ((a \circ b) \circ c) \circ \mathscr{B} = (a \circ (b \circ c)) \circ \mathscr{B} \\ &= (a \circ \mathscr{B}) \bullet ((b \circ \mathscr{B}) \bullet (c \circ \mathscr{B})). \end{aligned}$$

Now if there is a representation R whose each element has an inverse in $(\mathscr{A}; \circ)$ with unit $1_{\mathscr{A}}$, then it is easy to know that $1_{\mathscr{A}} \circ \mathscr{B}$ is the unit and $a^{-1} \circ \mathscr{B}$ the inverse element of $a \circ \mathscr{B}$ in $\mathfrak{S}$. Whence, $(\mathfrak{S}; \bullet)$ is a group. $\square$

Now let $\varpi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ be a homomorphism from an algebraic system $(\mathscr{A}_1; \circ_1)$ with unit $1_{\mathscr{A}_1}$ to $(\mathscr{A}_2; \circ_2)$ with unit $1_{\mathscr{A}_2}$. Define the *inverse set* $\varpi^{-1}(a_2)$ *for an element* $a_2 \in \mathscr{A}_2$ by $\varpi^{-1}(a_2) = \{a_1 \in \mathscr{A}_1 | \varpi(a_1) = a_2\}$. Particularly, if $a_2 = 1_{\mathscr{A}_2}$, the inverse set $\varpi^{-1}(1_{\mathscr{A}_2})$ is important in algebra and called the *kernel of* ϖ and denoted by $\mathrm{Ker}(\varpi)$, a normal subgroup of $(\mathscr{A}_1; \circ_1)$ if it is associative and each element in $\mathrm{Ker}(\varpi)$ has inverse element in $(\mathscr{A}_1; \circ_1)$.

Theorem 4.2. *Let $\varpi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ be an onto homomorphism from associative systems $(\mathscr{A}_1; \circ_1)$ to $(\mathscr{A}_2; \circ_2)$ with units $1_{\mathscr{A}_1}$, $1_{\mathscr{A}_2}$. Then*

$$\mathscr{A}_1/\mathrm{Ker}(\varpi) \cong (\mathscr{A}_2; \circ_2)$$

if each element of $\mathrm{Ker}(\varpi)$ *has an inverse in* $(\mathscr{A}_1; \circ_1)$.

Proof We have known that $\mathrm{Ker}(\varpi)$ is a subgroup of $(\mathscr{A}_1; \circ_1)$. Whence $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ is a quotient system. Define a mapping $\varsigma : \mathscr{A}_1/\mathrm{Ker}(\varpi) \rightarrow \mathscr{A}_2$ by

$$\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) = \varpi(a).$$

We prove this mapping is an isomorphism. Notice that ς is onto by that ϖ is an onto homomorphism. Now if $a \circ_1 \mathrm{Ker}(\varpi) \neq b \circ_1 \mathrm{Ker}(\varpi)$, then $\varpi(a) \neq \varpi(b)$. Otherwise, we find that $a \circ_1 \mathrm{Ker}(\varpi) = b \circ_1 \mathrm{Ker}(\varpi)$, a contradiction. Whence, $\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \neq \varsigma(b \circ_1 \mathrm{Ker}(\varpi))$, i.e., ς is a bijection from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $\mathscr{A}_2$.

Since ϖ is a homomorphism, we get that

$$\begin{aligned}
&\varsigma((a \circ_1 \mathrm{Ker}(\varpi)) \circ_1 (b \circ_1 \mathrm{Ker}(\varpi))) \\
&= \varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \circ_2 \varsigma(b \circ_1 \mathrm{Ker}(\varpi)) \\
&= \varpi(a) \circ_2 \varpi(b),
\end{aligned}$$

i.e., ς is an isomorphism from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $(\mathscr{A}_2; \circ_2)$. □

5. Extending to Multi-Systems

Assume $(\mathscr{G};\widetilde{O}) \prec (\mathscr{H},\widetilde{O})$. For $\forall a \in \mathscr{H}$ and $\circ_i \in \widetilde{O}$, where $1 \leq i \leq l$, define a coset $a \circ_i \mathscr{G}$ by $a \circ_i \mathscr{G} = \{a \circ_i b | \text{ for } \forall b \in \mathscr{G}\}$, and let

$$\mathscr{H} = \bigcup_{a \in R, \circ \in \widetilde{P} \subset \widetilde{O}} a \circ \mathscr{G}.$$

Then the set

$$\mathscr{Q} = \{a \circ \mathscr{G} | a \in R, \circ \in \widetilde{P} \subset \widetilde{O}\}$$

is called a *quotient set of* $\mathscr{G}$ *in* $\mathscr{H}$ *with a representation pair* $(R, \widetilde{P})$, denoted by $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$. Similar to Theoren 4.1 , we get the following result.

Theorem 5.1. *Let $(\mathscr{H}, \widetilde{O})$ be an associative multi-operation system with a unit $1_\circ$ for $\forall \circ \in \widetilde{O}$ and $\mathscr{G} \subset \mathscr{H}$.*

(i) If $\mathscr{G}$ is closed for operations in $\widetilde{O}$ and for $\forall a \in \mathscr{G}, \circ \in \widetilde{O}$, there exists an inverse element $a_\circ^{-1}$ in $(\mathscr{G}; \circ)$, then there is a representation pair $(R, \widetilde{P})$ such that the quotient set $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is a partition of $\mathscr{H}$, i.e., for $a, b \in \mathscr{H}, \forall \circ_1, \circ_2 \in \widetilde{O}$, $(a \circ_1 \mathscr{G}) \cap (b \circ_2 \mathscr{G}) = \emptyset$ or $a \circ_1 \mathscr{G} = b \circ_2 \mathscr{G}$.

(ii) For $\forall \circ \in \widetilde{O}$, define an operation $\circ$ on $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ by

$$(a \circ_1 \mathscr{G}) \circ (b \circ_2 \mathscr{G}) = (a \circ b) \circ_1 \mathscr{G}.$$

Then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}; \widetilde{O})$ is an associative multi-operation system. Particularly, if there is a representation pair $(R, \widetilde{P})$ such that for $\circ' \in \widetilde{P}$, any element in R has an inverse in $(\mathscr{H}; \circ')$, then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}, \circ')$ is a group.

Let $\mathcal{I}(\widetilde{O})$ be the set of all units $1_\circ, \circ \in \widetilde{O}$ in a multi-operation system $(\mathscr{H}; \widetilde{O})$. Define a *multi-kernel* $\widetilde{\mathrm{Ker}}\omega$ of a homomorphism $\omega : (\mathscr{H}_1; \widetilde{O}_1) \to (\mathscr{H}_2; \widetilde{O}_2)$ by

$$\widetilde{\mathrm{Ker}}\omega = \{\ a \in \mathscr{H}_1 \mid \omega(a) = 1_\circ \in \mathcal{I}(\widetilde{O}_2)\ \}.$$

Theorem 5.2. *Let ω be an onto homomorphism from associative systems $(\mathscr{H}_1; \widetilde{O}_1)$ to $(\mathscr{H}_2; \widetilde{O}_2)$ with $(\mathcal{I}(\widetilde{O}_2); \widetilde{O}_2)$ an algebraic system with unit $1_{\circ^-}$ for $\forall \circ^- \in \widetilde{O}_2$ and inverse x^{-1} for $\forall x \in (\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2); \circ^-)$. Then there are representation pairs $(R_1, \widetilde{P}_1)$ and $(R_2, \widetilde{P}_2)$, where $\widetilde{P}_1 \subset \widetilde{O}, \widetilde{P}_2 \subset \widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1; \widetilde{O}_1)}{(\widetilde{\mathrm{Ker}}\omega; \widetilde{O}_1)}\Big|_{(R_1, \widetilde{P}_1)} \cong \frac{(\mathscr{H}_2; \widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2); \widetilde{O}_2)}\Big|_{(R_2, \widetilde{P}_2)}$$

if each element of $\widetilde{\mathrm{Ker}}\omega$ has an inverse in $(\mathscr{H}_1; \circ)$ for $\circ \in \widetilde{O}_1$.

Proof Notice that $\widetilde{\mathrm{Ker}\omega}$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. In fact, for $\forall k_1,k_2\in\widetilde{\mathrm{Ker}\omega}$ and $\forall\circ\in\widetilde{O}_1$, there is an operation $\circ^-\in\widetilde{O}_2$ such that

$$\omega(k_1\circ k_2)=\omega(k_1)\circ^-\omega(k_2)\in\mathcal{I}(\widetilde{O}_2)$$

since $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system. Whence, $\widetilde{\mathrm{Ker}\omega}$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. By assumption, for any operation $\circ\in\widetilde{O}_1$ each element $a\in\widetilde{\mathrm{Ker}\omega}$ has an inverse a^{-1} in $(\mathscr{H}_1;\circ)$. Let $\omega:(\mathscr{H}_1;\circ)\to(\mathscr{H}_2;\circ^-)$. We know that

$$\omega(a\circ a^{-1})=\omega(a)\circ^-\omega(a^{-1})=1_{\circ^-},$$

i.e., $\omega(a^{-1})=\omega(a)^{-1}$ in $(\mathscr{H}_2;\circ^-)$. Because $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system with an inverse x^{-1} for $\forall x\in\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^-)$, we find that $\omega(a^{-1})\in\mathcal{I}(\widetilde{O}_2)$, namely, $a^{-1}\in\widetilde{\mathrm{Ker}\omega}$.

Define a mapping $\sigma : \frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)} \to \frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ by

$$\sigma(a \circ \mathrm{Ker}\omega) = \sigma(a) \circ^{-} \mathcal{I}(\widetilde{O}_2)$$

for $\forall a \in R_1, \circ \in \widetilde{P}_1$, where $\omega : (\mathscr{H}_1; \circ) \to (\mathscr{H}_2; \circ^{-})$. We prove σ is an isomorphism. Notice that σ is onto by that ω is an onto homomorphism. Now if $a \circ_1 \widetilde{\mathrm{Ker}\omega} \neq b \circ_2 \mathrm{Ker}(\varpi)$ for $a, b \in R_1$ and $\circ_1, \circ_2 \in \widetilde{P}_1$, then $\omega(a) \circ_1^{-} \mathcal{I}(\widetilde{O}_2) \neq \omega(b) \circ_2^{-} \mathcal{I}(\widetilde{O}_2)$. Otherwise, we find that $a \circ_1 \widetilde{\mathrm{Ker}\omega} = b \circ_2 \widetilde{\mathrm{Ker}\omega}$, a contradiction. Whence, $\sigma(a \circ_1 \widetilde{\mathrm{Ker}\omega}) \neq \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega})$, i.e., σ is a bijection from $\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$.

Since ω is a homomorphism, we get that

$$\begin{aligned}
&\sigma((a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ (b \circ_2 \widetilde{\mathrm{Ker}\omega})) \\
&= \sigma(a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ^- \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega}) \\
&= (\omega(a) \circ_1^- \mathcal{I}(\tilde{O}_2)) \circ^- (\omega(b) \circ_2^- \mathcal{I}(\tilde{O}_2)) \\
&= \sigma((a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ^- \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega}),
\end{aligned}$$

i.e., σ is an isomorphism from $\frac{(\mathscr{H}_1;\tilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\tilde{O}_1)}\Big|_{(R_1,\tilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\tilde{O}_2)}{(\mathcal{I}(\tilde{O}_2);\tilde{O}_2)}\Big|_{(R_2,\tilde{P}_2)}$. □

Corollary 5.1. *Let* $(\mathscr{H}_1;\widetilde{O}_1)$, $(\mathscr{H}_2;\widetilde{O}_2)$ *be multi-operation systems with groups* $(\mathscr{H}_2;\circ_1)$, $(\mathscr{H}_2;\circ_2)$ *for* $\forall\circ_1\in\widetilde{O}_1$, $\forall\circ_2\in\widetilde{O}_2$ *and* $\omega:(\mathscr{H}_1;\widetilde{O}_1)\rightarrow(\mathscr{H}_2;\widetilde{O}_2)$ *a homomorphism. Then there are representation pairs* $(R_1,\widetilde{P}_1)$ *and* $(R_2,\widetilde{P}_2)$, *where* $\widetilde{P}_1\subset\widetilde{O}_1,\widetilde{P}_2\subset\widetilde{O}_2$ *such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{\widetilde{(\mathrm{Ker}\omega;\widetilde{O}_1)}}\Big|_{(R_1,\widetilde{P}_1)}\cong\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}\Big|_{(R_2,\widetilde{P}_2)}.$$

Corollary 5.2. *Let* $(\mathscr{H};\widetilde{O})$ *be a multi-operation system and* $\omega:(\mathscr{H};\widetilde{O})\rightarrow(\mathscr{A};\circ)$ *a onto homomorphism from* $(\mathscr{H};\widetilde{O})$ *to a group* $(\mathscr{A};\circ)$. *Then there are representation pairs* $(R,\widetilde{P})$, $\widetilde{P}\subset\widetilde{O}$ *such that*

$$\frac{(\mathscr{H};\widetilde{O})}{\widetilde{(\mathrm{Ker}\omega;\widetilde{O})}}\Big|_{(R,\widetilde{P})}\cong(\mathscr{A};\circ).$$

An *algebraic multi-system* is a pair $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ with

$$\widetilde{\mathscr{A}}=\bigcup_{i=1}^{m}\mathscr{H}_i \text{ and } \widetilde{\mathscr{O}}=\bigcup_{i=1}^{m}\mathcal{O}_i$$

such that for any integer $i, 1\leq i\leq m$, $(\mathscr{H}_i;\mathcal{O}_i)$ is a multi-operation system.

Theorem 5.3. *Let $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ be algebraic multi-systems, where $\widetilde{\mathscr{A}_k}=\bigcup_{i=1}^{m}\mathscr{H}_i^k$, $\widetilde{\mathscr{O}_k}=\bigcup_{i=1}^{m}\mathcal{O}_i^k$ for $k=1,2$ and $o:(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)\to(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ a onto homomorphism with a multi-group $(\mathcal{I}_i^2;\mathcal{O}_i^2)$ for any integer $i,1\leq i\leq m$. Then there are representation pairs $(\widetilde{R}_1,\widetilde{P}_1)$ and $(\widetilde{R}_2,\widetilde{P}_2)$ such that*

$$\frac{(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)}{(\widetilde{\mathrm{Ker}(o)};\mathcal{O}_1)}\Big|_{(\widetilde{R}_1,\widetilde{P}_1)}\cong\frac{(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)}{(\widetilde{\mathcal{I}}(\mathcal{O}_2);\mathcal{O}_2)}\Big|_{(\widetilde{R}_2,\widetilde{P}_2)}$$

where $(\widetilde{\mathcal{I}}(\mathcal{O}_2);\mathcal{O}_2)=\bigcup_{i=1}^{m}(\mathcal{I}_i^2;\mathcal{O}_i^2)$.