Sigmoidal Model for Belief Function-Based Electre Tri Method

Jean Dezert
Jean-Marc Tacnet

Originally published as Dezert J., Tacnet J.-M., *Sigmoidal Model for Belief Function-based Electre Tri method*, in Belief 2012, Compiègne, May 2012, and reprinted with permission.

Abstract. Main decision-making problems can be described into choice, ranking or sorting of a set of alternatives or solutions. The principle of Electre TRI (ET) method is to sort alternatives a_i according to criteria g_j into categories C_h whose lower and upper limits are respectively b_h and b_{h+1}. The sorting procedure is based on the evaluations of outranking relations based firstly on calculation of partial concordance and discordance indexes and secondly on global concordance and credibility indexes. In this paper, we propose to replace the calculation of the original concordance and discordance indexes of ET method by a more effective sigmoidal model. Such model is part of a new Belief Function ET (BF-ET) method under development and allows a comprehensive, elegant and continuous mathematical representation of degree of concordance, discordance and the uncertainty level which is not directly taken into account explicitly in the classical Electre Tri.

1 Introduction

The Electre Tri (ET) method, developed by Yu [13], remains one of the most successful and applied methods for multiple criteria decision aiding (MCDA) sorting problems [5]. ET method assigns a set of given alternatives $a_i \in \mathbf{A}$, $i = 1,2,\dots,n$ according to criteria g_j, $j = 1,2,\dots,m$ to a pre-defined (and ordered) set of categories $C_h \in \mathbf{C}$, $h = 1,2,\dots,p+1$ whose lower and upper limits are respectively b_h and b_{h+1} for all $h = 1,\dots,p$), with $b_0 \leq b_1 \leq b_2 \leq \dots \leq b_{h-1} \leq b_h \leq \dots \leq b_p$. The assignment of an alternative a_i to a category C_h (limited by profiles b_h and $b_h + 1$) consists in four steps involving at first the computation of global concordance $c(a_i, b_h)$ and discordance $d(a_i, b_h)$ indexes[1] (steps 1 & 2), secondly their fusion into a credibility

---

[1] Themselves computed from partial concordance and discordance indexes based on a given set criteria $g_j(.)$, $j \in \mathbf{J}$.

index $\rho(a_i,b_h)$ (step 3), and finally the decision and choice of the category based on the evaluations of outranking relations [13, 6] (step 4). The partial concordance index $c_j(a_i,b_h)$ measures the concordance of a_i and b_h in the assertion *"a_i is at least as good as b_h"*. The partial discordance index $d_j(a_i,b_h)$ measures the opposition of a_i and b_h in the assertion *"a_i is at least as good as b_h"*. The global concordance index $c(a_i,b_h)$ measures the concordance of a_i and b_h on all criteria in the assertion *"a_i outranks b_h"*. The degree of credibility of the outranking relation denoted as $\rho(a_i,b_h)$ expresses to which extent *"a_i outranks b_h"* according to $c(a_i,b_h)$ and $d_j(a_i,b_h)$ for all criteria. The main steps of ET method are described below:

1. **Concordance Index:** The concordance index $c(a_i,b_h) \in [0,1]$ between the alternative a_i and the category C_h is computed as the weighted average of partial concordance indexes $c_j(a_i,b_h)$, that is

$$c(a_i,b_h) = \sum_{j\in\mathbf{J}} w_j c_j(a_i,b_h) \tag{1}$$

   where the weights $w_i \in [0,1]$ represent the relative importance of each criterion $g_j(.)$ in the evaluation of the global concordance index. They must satisfy $\sum_{j\in\mathbf{J}} w_j = 1$. The partial concordance index $c_j(a_i,b_h) \in [0,1]$ based on a given criterion $g_j(.)$ is computed from the difference of the criteria evaluated for the profil b_h, and the criterion evaluated for the alternative a_i. If the difference $g_j(b_h) - g_j(a_i)$ is less (or equal) to a given preference threshold $q_j(g_j(b_h))$ then a_i and C_h are considered as different based on the criterion $g_j(.)$ so that a preference of a_i with respect to C_h can be clearly done. If the difference $g_j(b_h) - g_j(a_i)$ is strictly greater to another given threshold $p_j(g_j(b_h))$ then a_i and C_h are considered as indifferent (similar) based on $g_j(.)$). When $g_j(b_h) - g_j(a_i) \in [q_j(g_j(b_h)), p_j(g_j(b_h))]$, the partial concordance index $c_j(a_i,b_h)$ is computed from a linear interpolation. Mathematically, the partial concordance index is obtained by:

$$c_j(a_i,b_h) \triangleq \begin{cases} 1 \quad \text{if} \quad g_j(b_h) - g_j(a_i) \leq q_j(g_j(b_h)) \\ 0 \quad \text{if} \quad g_j(b_h) - g_j(a_i) > p_j(g_j(b_h)) \\ \frac{g_j(a_i)+p_j(g_j(b_h))-g_j(b_h)}{p_j(g_j(b_h))-q_j(g_j(b_h))} \quad \text{otherwise} \end{cases} \tag{2}$$

2. **Discordance Index:** The discordance index between the alternative a_i and the category C_h depends on a possible veto condition expressed by the choice of a veto threshold $v_j(g_j(b_h))$ imposed on some criterion $g_j(.)$. The (global) discordance index $d(a_i,b_h)$ is computed from the partial discordance indexes:

$$d_j(a_i,b_h) \triangleq \begin{cases} 1 \quad \text{if} \quad g_j(b_h) - g_j(a_i) > v_j(g_j(b_h)) \\ 0 \quad \text{if} \quad g_j(b_h) - g_j(a_i) \leq p_j(g_j(b_h)) \\ \frac{g_j(b_h)-g_j(a_i)-p_j(g_j(b_h))}{v_j(g_j(b_h))-p_j(g_j(b_h))} \quad \text{otherwise} \end{cases} \tag{3}$$

One defines by $\mathbf{V}$ the set of indexes $j \in \mathbf{J}$ where the veto applies (where the partial discordance index is greater than the global concordance index), that is

$$\mathbf{V} \triangleq \{j \in \mathbf{J} | d_j(a_i, b_h) > c(a_i, b_h)\} \tag{4}$$

Then a global discordance index can be defined [12] as

$$d(a_i, b_h) \triangleq \begin{cases} 1 & \text{if} \quad \mathbf{V} = \emptyset \\ \prod_{j \in \mathbf{V}} \frac{1 - d_j(a_i, b_h)}{1 - c_j(a_i, b_h)} & \text{if} \quad \mathbf{V} \neq \emptyset \end{cases} \tag{5}$$

3. **Global Credibility Index:** In ET method, the (global) credibility index $\rho(a_i, b_h)$ is computed by the simple discounting of the concordance index $c(a_i, b_h)$ given by (1) by the discordance index (discounting factor) $d(a_i, b_h)$ given in (5). Mathematically, this is given by

$$\rho(a_i, b_h) = c(a_i, b_h) d(a_i, b_h) \tag{6}$$

4. **Assignment Procedure:** The assignment of a given action a_i to a certain category C_h results from the comparison of a_i to the profile defining the lower and upper limits of the categories. For a given category limit b_h, this comparison relies on the credibility of the assertions a_i outranks b_h. Once all credibility indexes $\rho(a_i, b_h)$ for $i = 1, 2, \dots, m$ and $h = 1, 2, \dots, k$ have been computed, the assignment matrix $\mathbf{M} \triangleq [\rho(a_i, b_h)]$ is available for helping in the final decision-making process. In ELECTRE TRI method, a simple λ-cutting level strategy (for a given choice of $\lambda \in [0.5, 1]$) is used in order to transform the fuzzy outranking relation into a crisp one to determine if each alternative outranks (or not) each category. This is done by testing if $\rho(a_i, b_h) \geq \lambda$. If the inequality is satisfied, it means that indeed a_i outranks the category C_h. Based on outranking relations between all pairs of alternatives and profiles of categories, two approches are proposed in ELECTRE TRI to finally assign the alternatives into categories, see [5] for details:

   - Pessimistic (conjunctive) approach: a_i is compared with b_k, b_{k-1}, b_{k-2}, $\dots$, until a_i outranks b_h where $h \leq k$. The alternative a_i is then assigned to the highest category C_h if $\rho(a_i, b_h) \geq \lambda$ for a given threshold λ.
   - Optimistic (disjunctive) approach: a_i is compared with b_1, b_2, $\dots b_h$, $\dots$ until b_h outranks a_i. The alternative a_i is assigned to the lowest category C_h for which the upper profile b_h is preferred to a_i.

The objective and motivation of this paper is to develop a new Belief Function based ET method taking into account the potential of BF to model uncertainties. The whole BF-ET method is under development and will be presented and evaluated on a detailed practical example in a forthcoming publication. Due to space limitation constraints, we just present here what we propose to compute the new concordance and discordance indexes useful in our BF-ET.

2 Limitations of the Classical Electre Tri

ET method remains rather based on heuristic approach than on a theoretical one for each of its steps. Belief functions can improve ET method because of their ability to model and manage conflicting as well as uncertainty information in a theoretical framework. We only focus here on steps 1 and 2 and we propose a solution to overcome their limitations in the next section.

Example 1: Let's consider $g_j(a_i) \in [0,100]$, and let's take $g_j(b_h) = 50$ and the following thresholds: $q_j(g_j(b_h)) = 20$ (indifference threshold), $p_j(g_j(b_h)) = 25$ (preference threshold) and $v_j(g_j(b_h) = 40$ (veto threshold). Then the local concordance and discordances indexes obtained in steps 1 and 2 of ET are shown on the Fig. 1.

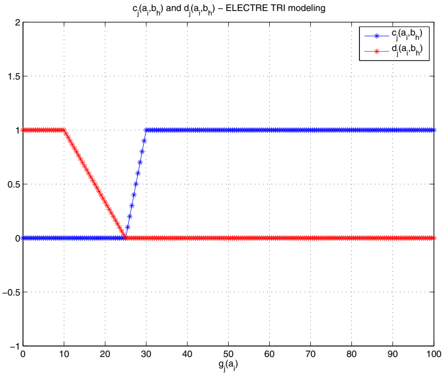

Fig. 1 Example of partial concordance and discordance indexes.

From this very simple example, one sees that ET modeling of partial concordance and discordance indexes is not very satisfactory since there is no clear (explicit and consistent) modeling of the uncertainty area where the action a_i is not totally discordant, nor totally concordant with the profile b_h. In such simplistic modeling, there exist points $g_j(a_i)$ (lying on the slope of the blue or red curves) that can be not totally concordant while being totally not discordant (and vice-versa), which is counter-intuitive and rather abnormal. This drawback will be solved using our new sigmoidal basic belief assignment (bba) modeling presented in the next section.

3 Sigmoidal Model for Concordance and Discordance Indexes

In fact, there are several ways to compute partial concordances and discordances indexes and to combine them in order to provide the global credibility indexes $\rho(a_i,b_h)$. Electre Tri proposes a simple and basic approach based on hard thresholding techniques for doing this. It can fail to work efficiently in practice in some cases, or may require a lot of experience to calibrate/tune all setting parameters in order to apply it to get pertinent results for decision-making support. Usually, a sensitivity analysis must be done very carefully before applying ET in real applications. Here, we propose a more flexible approach based on sigmoidal modeling where no hard thresholding technique is required.

In ET approach, we are mainly concerned in the evaluation of the credibility indexes $\rho(a_i,b_h) \in [0,1]$ for $i = 1,2,\ldots,m$ and $h = 1,2,\ldots,k$ (step 3) from which the final decision (assignment) will be drawn in step 4. Step 3 is conditioned by the results of steps 1 and 2 which can be improved using belief functions. For such purpose, we consider, a binary frame of discernment[2] $\Theta \triangleq \{c,\bar{c}\}$ where c means that the alternative a_i is concordant with the assertion "a_i is at least as good as profile b_h", and $\bar{c}$ means that the alternative a_i is opposed (discordant) to this assertion. This must obviously be done with all the assertions to check in the ET framework. The basic idea is for each pair (a_i,b_h) to evaluate its bba $m_{ih}(.)$ defined on the power-set of Θ, denoted 2^Θ. Such bba's have of course to be defined from the combination (fusion) of the local bba's $m^j_{ih}(.)$ evaluated from each possible criteria $g_j(.)$ (as in steps 1 and 2). The main issue is to derive the local bba's $m^j_{ih}(.)$ defined in 2^Θ from the knowledge of the criteria $g_j(.)$ and preference, indifference and veto thresholds $p_j(g_j(b_h))$, $q_j(g_j(b_h))$ and $v_j(g_j(b_h))$ respectively. It turns out that this can be easily obtained from the new method of construction of bba presented in [4] and adapted here in the ET context as follows:

- Let $g_j(a_i)$ be the evaluation of the criterion $g_j(.)$ for the alternative a_i, following ET approach when $g_j(a_i) \geq g_j(b_h) - q_j(g_j(b_h))$ then the belief in concordance c must be high (close to one), whereas it must be low (close to zero) as soon as $g_j(a_i) < g_j(b_h) - p_j(g_j(b_h))$. Similarly, the belief in discordance $\bar{c}$ must be high (close to one) if $g_j(a_i) < g_j(b_h) - v_j(g_j(b_h))$, and it must be low (close to zero) when $g_j(a_i) \geq g_j(b_h) - p_j(g_j(b_h))$. Such behavior can be modeled directly from the sigmoid functions defined by $f_{s,t}(g) \triangleq 1/(1+e^{-s(g-t)})$ where g is the criterion magnitude of the alternative under consideration; t is the abscissa of the inflection point of the sigmoid. $s/4$ is the slope[3] of the tangent at the inflection point. It can be easily verified that the bba $m^j_{ih}(.)$ satisfying the expected behavior can be obtained by the fusion[4] of the two following simple bba's defined by: where the abscisses of inflection points are given by $t_c = g_j(b_h) - \frac{1}{2}(p_j(g_j(b_h)) + q_j(g_j(b_h)))$ and $t_{\bar{c}} = g_j(b_h) - \frac{1}{2}(p_j(g_j(b_h)) + v_j(g_j(b_h)))$ and the parameters s_c and $s_{\bar{c}}$ are given by[5] $s_c = 4/(p_j(g_j(b_h)) - q_j(g_j(b_h)))$ and $s_{\bar{c}} = 4/(v_j(g_j(b_h)) - p_j(g_j(b_h)))$.

Table 1 Construction of $m_1(.)$ and $m_2(.)$.

focal element	$m_1(.)$	$m_2(.)$
c	$f_{s_c,t_c}(g)$	0
$\bar{c}$	0	$f_{-s_{\bar{c}},t_{\bar{c}}}(g)$
$c \cup \bar{c}$	$1 - f_{s_c,t_c}(g)$	$1 - f_{-s_{\bar{c}},t_{\bar{c}}}(g)$

---

[2] Here we assume that Shafer's model holds, that is $c \cap \bar{c} = \emptyset$.

[3] i.e. the ratio of the vertical and horizontal distances between two points on a line; zero if the line is horizontal, undefined if it is vertical.

[4] With averaging rule, PCR5 rule, or Dempster-Shafer rule [8].

[5] The coefficient 4 appearing in s_c and $s_{\bar{c}}$ expressions comes from the fact that for a sigmoid of parameter s, the tangent at its inflection point is $s/4$.

• From the setting of threshold parameters $p_j(g_j(b_h))$, $q_j(g_j(b_h))$ and $v_j(g_j(b_h))$, it is easy to compute the parameters of the sigmoids (t_c, s_c) and $(t_{\bar{c}}, t_{\bar{c}})$, and thus to get the values of bba's $m_1(.)$ and $m_2(.)$. Once this has been done the local bba $m^j_{ih}(.)$ is computed by the fusion (denoted $\oplus$) of bba's $m_1(.)$ and $m_2(.)$, that is $m^j_{ih}(.) = [m_1 \oplus m_2](.)$. As shown in [4], the choice of a particular rule of combination (Dempster, PCR5, or hybrid rule) has only a little impact on the result of the combined bba $m^j_{ih}(.)$. But since PCR5 proposes a better management of conflicting bba's yielding to more specific results than with other rules [1], we use it to combine $m_1(.)$ with $m_2(.)$ to compute $m^j_{ih}(.)$ associated with the criterion $g_j(.)$ and the pair (a_i, b_h). In adopting such sigmoidal modeling, we get now from $m^j_{ih}(.)$ a fully consistent and elegant representation of local concordance $c_j(a_i, b_h)$ (step 1 of ET), local discordance $d_j(a_i, b_h)$ (step 2 of ET), as well as of the local uncertainty $u_j(a_i, b_h)$ by considering: $c_j(a_i, b_h) \triangleq m^j_{ih}(c) \in [0, 1]$, $d_j(a_i, b_h) \triangleq m^j_{ih}(\bar{c}) \in [0, 1]$ and $u_j(a_i, b_h) \triangleq m^j_{ih}(c \cup \bar{c}) \in [0, 1]$. Of course, one has also $c_j(a_i, b_h) + d_j(a_i, b_h) + u_j(a_i, b_h) = 1$.

4 Example of a Sigmoidal Model

If one takes back the example 1, the inflection points of the sigmoids $f_1(g) \triangleq f_{s_c, t_c}(g)$ and $f_2(g) \triangleq f_{-s_{\bar{c}}, t_{\bar{c}}}(g)$ have the following abscisses $t_c = 50 - (25 + 20)/2 = 27.5$ and $t_{\bar{c}} = 50 - (25 + 40)/2 = 17.5$ and parameters $s_c = 4/(25 - 20) = 4/5 = 0.8$ and $s_{\bar{c}} = 4/(40 - 25) = 4/15 \approx 0.2666$. The two sigmoids $f_1(g_j(a_i))$ and $f_2(g_j(a_i))$ are shown on the Fig. 2.

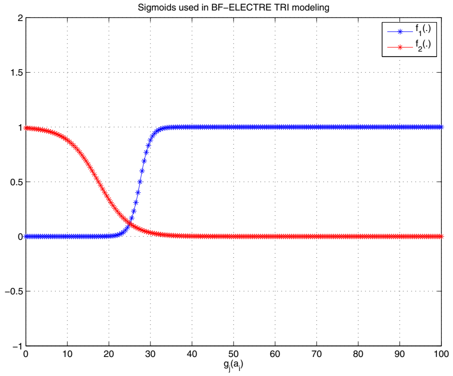

Fig. 2 $f_1(g_j(a_i))$ and $f_2(g_j(a_i))$ sigmoids.

It is interesting to note the resemblance of Fig. 2 with Fig. 1. From these sigmoids, the bba's $m_1(.)$ and $m_2(.)$ are computed according to Table 1 and shown on the Figure 3.

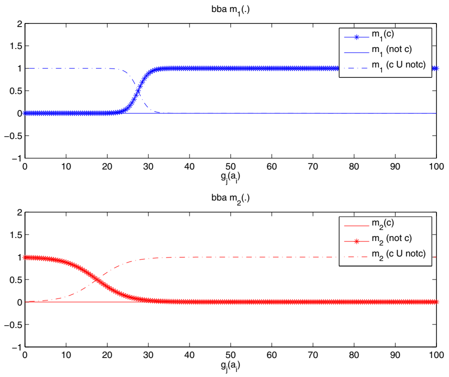

Fig. 3 Bba's $m_1(.)$ and $m_2(.)$ to combine.

The construction of the consistent bba $m_{ih}^{j}(.)$ is obtained by the PCR5 fusion of the bba's $m_1(.)$ and $m_2(.)$. The result is shown on Fig. 4.

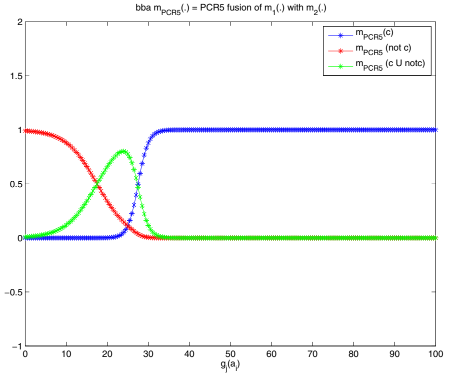

Fig. 4 $m_{ih}^{j}(.)$ obtained from the PCR5 fusion of $m_1(.)$ with $m_2(.)$.

From this new sigmoidal modeling, we can compute the local bba's $m_{ih}^{j}(.)$ derived from the knowledge of criterion $g_j(.)$ and setting parameters. This is a smooth appealing and elegant technique to build all the local bba's: no hard thresholding is necessary because of the continuity of sigmoid functions.

One can then compute the global concordance and discordance indexes of steps 1 and 2 from the computation of the combined bba $m_{ih}(.)$ resulting of the fusion of local bba's $m_{ih}^{j}(.)$ taking eventually into account their importance and reliability[6] (if one wants). This can be done using the recent fusion techniques proposed in [9], or by a simple weighted averaging. From $m_{ih}(.)$ we can use the same credibility index as in step 3 of ET, or just skip this third step and define a decision-making based directly on the bba $m_{ih}(.)$ using classical approaches used in belief function framework (say the max of belief, plausibility, or pignistic probability, etc).

---

[6] In classical ET, the reliability of criteria is not taken into account.

5 Conclusions

After a brief presentation of the classical ET method, we have proposed a new approach to model and compute the concordance and discordance indexes based on belief functions in order to overcome the limitations of steps 1 and 2 of the ET approach. The advantages of our modeling is to provide an elegant and simple way not only to compute the concordance and discordance indexes, but also the uncertainty level that may occur when information appears partially concordant and discordant. The Improvements of other steps of ET method are under development. In future reaserch works, we will evaluate and compare on real MCDA problem our BF-ET with the original ET method and with other belief functions based methods already available in MCDA frameworks [10, 11].

References

1. Dezert, J., Smarandache, F.: Proportional Conflict Redistribution Rules for Information Fusion. In: [8], vol. 2, pp. 3–68 (2006)
2. Dezert, J., Smarandache, F.: An introduction to DSmT. In: [8], vol. 3, pp. 3–73 (2009)
3. Dezert, J., Smarandache, F., Tacnet, J.-M., Batton-Hubert, M.: Multi-criteria decision making based on DSmT/AHP. In: International Workshop on Belief Functions, Brest, France (April 2010)
4. Dezert, J., Liu, Z., Mercier, G.: Edge Detection in Color Images Based on DSmT. In: Proceedings of Fusion 2011, Chicago, USA (July 2011)
5. Figueira, J., Mousseau, V., Roy, B.: ELECTRE methods. In: Multiple Criteria Decision Analysis: State of Art Surveys, ch. 4. Springer Science+Business Media Inc. (2005)
6. Mousseau, V., Slowinski, R., Zielniewicz, P.: Electre tri 2.0a - methological guide and user's manual - document no 111. In: Cahier et Documents du Lamsade, Lamsade, Université Paris-Dauphine, Paris (1999)
7. Shafer, G.: A mathematical theory of evidence. Princeton University Press (1976)
8. Smarandache, F., Dezert, J.: Advances and applications of DSmT for information fusion (Collected works), vol. 1-3. American Research Press (2004-2009), `http://fs.gallup.unm.edu/DSmT.htm`
9. Smarandache, F., Dezert, J., Tacnet, J.-M.: Fusion of sources of evidence with different importances and reliabilities. In: Proc. of Fusion 2010 Conf., Edinburgh, UK (July 2010)
10. Tacnet, J.-M., Dezert, J.: Cautious OWA and Evidential Reasoning for Decision Making under Uncertainty. In: Proceedings of Fusion 2011, Chicago, USA (July 2011)
11. Tacnet, J.-M., Batton-Hubert, M., Dezert, J.: A two-step fusion process for multi-criteria decision applied to natural hazards in mountains. In: Proceedings of International Workshop on the Theory of Belief Functions, Belief 2010, Brest (France), April 1-2 (2010)
12. Tervonen, T., Figueira, J.R., Lahdelma, R., Dias, J.A., Salminen, P.: A stochastic method for robustness analysis in sorting problems. European Journal of Operational Research 192, 236–242 (2009)
13. Yu, W.: Aide multicritère à la décision dans le cadre de la problématique du tri: Concepts, méthodes et applications. Ph.D Thesis, University Paris-Dauphine, France (1992)