BOUNDING THE SMARANDACHE FUNCTION

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Let $S(n)$, for $n \in \mathbb{N}^+$ denote the Smarandache function, then $S(n)$ is defined as the smallest $m \in \mathbb{N}^+$, with $n|m!$. From the definition one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the canonical prime factorization of n, then $S(n) = \max\{S(p_i^{\alpha_i})\}$, where the maximum is taken over the i's from 1 to k. This observation illustrates the importance of being able to calculate the Smarandache function for prime powers. This paper will be considering that process. We will give an upper and lower bound for $S(p^\alpha)$ in Theorem 1.4. A recursive procedure of calculating $S(p^\alpha)$ is then given in Proposition 1.8. Before preceeding we offer these trivial observations:

Observation 1. *If p is prime, then $S(p) = p$.*

Observation 2. *If p is prime, then $S(p^k) \leq kp$.*

Observation 3. *p divides $S(p^k)$*

Observation 4. *If p is prime and $k < p$, then $S(p^k) = kp$.*

To see that observation 4 holds, one need only consider the sequence

$$2, 3, 4 \dots, p-1, p, p+1, \dots, 2p, 2p+1, \dots, 3p, \dots, kp$$

and count the elements which have a factor of p.

Define $T_p(n) = \sum_{k=1}^{\infty} [\frac{n}{p^k}]$, where $[\cdot]$ represents the greatest integer function. The function T_p counts the number of powers of p in $n!$. To relate $T_p(n)$ and $S(n)$ note that $S(p^\alpha)$ is the smallest n such that $T_p(n) \geq \alpha$. In other words $S(p^\alpha)$ is characterized by

$$(*) \qquad T_p(S(p^\alpha)) \geq \alpha \quad \text{and} \quad T_p(S(p^\alpha) - 1) \leq \alpha - 1.$$

Lemma 1.0. *For* $n \geq 1$, $T_p(n) < \frac{n}{p-1}$

Proof. $T_p(n) = \sum_{k=1}^{\infty}[\frac{n}{p^k}] < \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1}$ □

Corollary 1.1. $(p-1)\alpha < S(p^\alpha) \leq p\alpha$

Recall this basic fact about the p-adic representation of a number n. Given $n, p \in \mathbb{Z}$ and $p \geq 2$, $n \geq 0$, we can uniquely represent $n = \sum_{j=0}^{\infty} a_j(n)p^j$, where each $a_j \in \{0, 1, 2, \ldots, p-1\}$.

Lemma 1.2. $T_p(n) = \frac{1}{p-1}(n - \sum_{j=0}^{\infty} a_j(n))$

Proof.

$$\begin{aligned}
T_p(n) &= \sum_{k=1}^{\infty}[\frac{n}{p^k}] = \sum_{k=1}^{\infty}\Big[\frac{\sum_{j=0}^{\infty} a_j(n)p^j}{p^k}\Big] \\
&= \sum_{k=1}^{\infty} \frac{\sum_{j=k}^{\infty} a_j(n)p^j}{p^k} = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} a_j(n)p^{j-k} \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{j} a_j(n)p^{j-k} = \sum_{j=1}^{\infty} a_j(n)\sum_{k=1}^{j} p^{j-k} \\
&= \sum_{k=1}^{\infty} a_k(n)\sum_{j=1}^{k} p^{k-j} = \frac{1}{p-1}\sum_{k=1}^{\infty} a_k(n)(p^k - 1) \\
&= \frac{1}{p-1}\sum_{k=1}^{\infty}(a_k(n)p^k - a_k(n)) \\
&= \frac{1}{p-1}(n - \sum_{k=0}^{\infty} a_k(n)) \quad \square
\end{aligned}$$

Lemma 1.3. *If* $n \geq 1$ *then*

$$1 \leq \sum_{j=0}^{\infty} a_j(n) \leq (p-1)[[\log_p(n)] + 1].$$

Proof. For each a_j we have $a_j \leq p-1$. Note that in the p-adic expansion of n, $a_j(n) = 0$ for all $j > [\log_p(n)]$. Thus we have $1 \leq \sum_{j=0}^{\infty} a_j(n) \leq (p-1)([\log_p(n)] + 1)$.

Now using the characterization $*$ and Lemma 1.2, we get the following

$$S(p^\alpha) - \sum_{j=0}^{\infty} a_j(S(p^\alpha)) \geq (p-1)\alpha \quad \text{and}$$

$$(**) \qquad S(p^\alpha) - 1 - \sum_{j=0}^{\infty} a_j(S(p^\alpha) - 1) \leq (\alpha - 1)(p-1).$$

Applying Lemma 1.3 to the first inequality for $S(p^\alpha)$, yields a lower bound of

$$S(p^\alpha) \geq (p-1)\alpha + 1.$$

This lower bound cannot be improved since we obtain equality when $\alpha = p + 1$, in fact we achieve equality whenever $\alpha = p^t + p^{t-1} + \cdots + p + 1$ for $t \geq 1$. Now $S(p^\alpha)$ is clearly integer valued, so one may choose to write the lower bound as $S(p^\alpha) > (p-1)\alpha$.

From the latter inequality (**), we get the following.

$$\begin{aligned}
S(p^\alpha) &\leq (p-1)(\alpha-1) + 1 + \sum_{j=0}^{\infty} a_j(S(p^\alpha) - 1) \\
&\leq (p-1)(\alpha-1) + 1 + (p-1)([\log_p(S(p^\alpha) - 1)] + 1) \\
&= (p-1)(\alpha-1) + 1 + (p-1)[\log_p(S(p^\alpha) - 1)] + (p-1) \\
&= \alpha(p-1) + (p-1)[\log_p(S(p^\alpha) - 1)] + 1 \\
&\leq \alpha(p-1) + (p-1)[\log_p(p\alpha - 1)] + 1 \\
&\leq \alpha(p-1) + (p-1)[\log_p(p\alpha)] + 1 \\
&= \alpha(p-1) + (p-1)[\log_p(\alpha) + 1] + 1 \\
&= \alpha(p-1) + (p-1)[\log_p(\alpha)] + (p-1) + 1 \\
&= (p-1)[\alpha + 1 + \log_p(\alpha)] + 1
\end{aligned}$$

Theorem 1.4. *For any prime p and any integer α, we have*

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p(\alpha)] + 1.$$

We now consider the sharpness of this upper bound. Note that when $\alpha = p^k - k$ the upper bound yields the value $(p-1)p^k + 1$. As it turns out $S(p^{p^k - k})$ is one less than this yield.

Lemma 1.5. *$S(p^{p^k-k}) = (p-1)p^k$, for $k \geq 1$.*

Proof. Consider

$$\begin{aligned} T_p(p^{k+1}-p^k) &= \sum_{l=1}^{\infty}[\frac{p^{k+1}-p^k}{p^l}] \\ &= (p^k - p^{k-1}) + (p^{k-1} - p^{k-2}) + \cdots + (p^2 - p) + (p-1) = p^k - 1 \end{aligned}$$

and

$$\begin{aligned} T_p(p^{k+1}-p^k-1) &= \sum_{l=1}^{\infty}[\frac{p^{k+1}-p^k-1}{p^l}] \\ &= \left[p^k - p^{k-1} - \frac{1}{p}\right] + \left[p^{k-1} - p^{k-2} - \frac{1}{p^2}\right] + \cdots + \left[1 - \frac{1}{p} - \frac{1}{p^{k+1}}\right] \\ &= (p^k - p^{k-1} - 1) + (p^{k-1} - p^{k-2} - 1) + \cdots + (p - 1 - 1) + 0 \\ &= p^k - (k+1). \end{aligned}$$

Since $T_p(p^{k+1}-p^k-1) < p^k - k \leq T_p(p^{k+1}-p^k)$, we have $S(p^{p^k-k}) = (p-1)p^k$. □

Thus we have produced infinitely many values that are within one of the upper bound. If we recall Observation 3, the upper bound should be congruent to 0 mod p. So one could subtract the remainder of the upper bound when dividing by p from the upper bound and make it sharp. We shall omit that task in this paper.

We now turn our attention to answering the question when is $S(p^\alpha) = p^\beta$. Consider the following calculations, verification is left for the reader.

$$\begin{aligned} T_p(p^{\beta+1}) &= p^\beta + p^{\beta-1} + \cdots + p + 1 \\ T_p(p^{\beta+1}-1) &= p^\beta + p^{\beta-1} + \cdots + p - \beta \\ T_p(p^\beta) &= p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \\ T_p(p^\beta - 1) &= p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 - \beta \end{aligned}$$

Thus we have $S(p^\alpha) = p^{\beta+1}$ if $p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta \leq \alpha \leq p^\beta + p^{\beta-1} + \cdots + p + 1$. If $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \leq \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta$, then we have $p^\beta \leq S(p^\alpha) < p^{\beta+1}$.

We now offer a recursive procedure for calculating $S(p^\alpha)$. The following is a technical lemma that will be used in proving the recursion formula.

Lemma 1.6. *Suppose we have* $p^\beta \le r < p^{\beta+1}$, *for some* $\beta \ge 0$, *then* $T_p(r) = T_p(p^\beta) + T_p(r - p^\beta)$.

Proof.

$$T_p(r) = \sum_{k=1}^{\infty}[\frac{r}{p^k}] = \sum_{k=1}^{\beta}[\frac{p^\beta + (r-p^\beta)}{p^k}]$$
$$= \sum_{k=1}^{\beta}(\frac{p^\beta}{p^k}) + \sum_{k=1}^{\beta}[\frac{r-p^\beta}{p^k}]$$
$$= T_p(p^\beta) + T_p(r - p^\beta) \quad \square$$

Lemma 1.7. *If* $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1$, *then* $S(p^\alpha) = p^\beta + S(p^{\alpha-(p^{\beta-1}+p^{\beta-2}+\cdots+p+1)})$.

Proof. Case 1: Assume that $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta$.

$$S(p^\alpha) = \min\{r | T_p(r) \ge \alpha\}$$
$$= \min\{r | T_p(r) \ge \alpha \quad \text{and} \quad p^\beta \le r < p^{\beta+1}\}$$
$$= \min\{r | T_p(p^\beta) + T_p(r - p^\beta) \ge \alpha \quad \text{and} \quad p^\beta \le r < p^{\beta+1}\}$$
$$= p^\beta + \min\{r - p^\beta | T_p(r - p^\beta) \ge \alpha - T_p(p^\beta) \quad \text{and} \quad 0 \le r - p^\beta < p^{\beta+1} - p^\beta\}$$
$$= p^\beta + \min\{r | T_p(r) \ge \alpha - T_p(p^\beta) \quad \text{and} \quad 0 \le r < p^{\beta+1} - p^\beta = p^\beta(p-1)\}$$
$$= p^\beta + S(p^{\alpha - T_p(p^\alpha)})$$
$$= p^\beta + S(p^{\alpha-(p^{\beta-1}+p^{\beta-2}+\cdots+p+1)})$$

Case 2: Assume that $p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1$. From the prior calculations of $T_p(p^{\beta+1})$ and $T_p(p^{\beta+1} - 1)$ we have the $S(p^\alpha) = p^{\beta+1}$ for any α in this range. Now consider the right hand side of the equation, $p^\beta + S(p^{\alpha-(p^{\beta-1}+p^{\beta-2}+\cdots+p+1)})$. We can restate this expression as $p^\beta + S(p^t)$, where $p^\beta - \beta \le t < p^\beta$. From the proof of Lemma 1.4 we see that $T_p(p^{\beta+1} - p^\beta) = p^\beta - 1$ and $T_p(p^{\beta+1} - p^\beta - 1) = p^\beta - \beta - 1$, thus it must be that $S(p^t) = p^{\beta+1} - p^\beta$. Therefore the right hand side is $p^{\beta+1}$. $\square$

Clearly this lemma can be repeated as long as $\alpha - (p^{\beta-1} + \ldots 1) \geq p^{\beta-1} + \ldots 1$, so we can strengthen Lemma 1.6.

Proposition 1.8. *If* $d = p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \leq \alpha < p^{\beta} + p^{\beta-1} + \cdots + p + 1$, *write* $\alpha = qd + r$ *with* $0 \leq r < d$, *then* $S(p^{\alpha}) = qp^{\beta} + S(p^r)$.

Now $p^{\beta} + p^{\beta-1} + \cdots + p + 1 = p^{\beta}(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\beta}}) \leq \frac{p^{\beta+1}}{p-1}$. Therefore we get $\log_p \alpha < \log_p(p^{\beta} + \cdots + 1) = \beta + 1 - \log_p(p-1) < \beta + 1$, and similarly $\beta - 1 < \beta - \log_p(p-1) < \log_p(\alpha) < \beta + 1$, or $\log_p \alpha - 1 < \beta < \log_p \alpha + 1$. Hence the exact value of $S(p^{\alpha})$ can be obtained by applying the proposition on the order of $\log_p \alpha$ times.

References

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