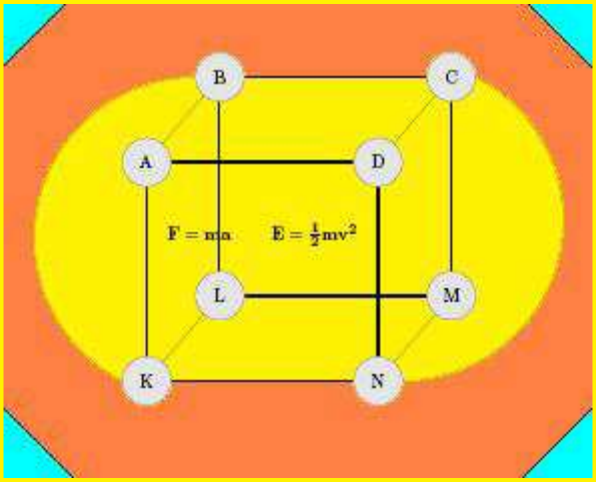
Proceedings of the First International Conference on Smarandache Multispace & Multistructure

Edited by Linfan Mao



Beijing University of Civil Engineering and Architecture
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Proceedings of the First International Conference On Smarandache Multispace & Multistructures (28-30 June 2013, Beijing, China)

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The First International Conference on Smarandache Multispace and Multistructure was held in China

In recent decades, Smarandache's notions of multispace and multistructure were widely spread and have shown much importance in sciences around the world. Organized by Prof.Linfan Mao, a professional conference on multispaces and multistructures, named the *First International Conference on Smarandache Multispace and Multistructure* was held in Beijing University of Civil Engineering and Architecture of P. R. China on June 28-30, 2013, which was announced by American Mathematical Society in advance.

The Smarandache multispace and multistructure are qualitative notions, but both can be applied to metric and non-metric systems. There were 46 researchers haven taken part in this conference with 14 papers on Smarandache multispaces and geometry, birings, neutrosophy, neutrosophic groups, regular maps and topological graphs with applications to non-solvable equation systems.

Prof.Yanpei Liu reports on topological graphs

[1] http://fs.gallup.unm.edu/Multispace.htm

Prof.Linfan Mao reports on non-solvable systems of differential equations with graphs in $\mathbb{R}^n$

Prof.Shaofei Du reports on regular maps with developments

Applications of Smarandache multispaces and multistructures underline a combinatorial mathematical structure and interchangeability with other sciences, including gravitational fields, weak and strong interactions, traffic network, etc.

All participants have showed a genuine interest on topics discussed in this conference and would like to carry these notions forward in their scientific works.

首届 Smarandache 重空间与重结构学术交流会召开

近十年来，重空间与重结构的泛科学思想在全世界范围内得到了广泛传播，并在科学研究中日益显现出了重要作用，形成了一种泛组合科学。为使这种泛组合科学得到进一步应用，推进人类认识自然的能力，由我校兼职教授毛林繁博士（教授级高级工程师）主持，北京建筑大学经济与管理工程学院和理学院共同承办的“首届 Smarandache 重空间与重结构国际学术研讨会”，于 2013 年 6 月 28 日在北京建筑大学西城校区成功举办，我校理学院院长崔景安教授、副院长梁昔明教授、经管学院公共管理系主任张俊副教授等相关教师和研究生，以及北京交通大学、首都师范大学和廊坊师范学院等学校师生参加了本次会议。该会议此前曾在美国数学会网站注册公告，得到了国内外学者响应。

Smarandache 重空间与重结构是一种定性思想，但其应用于数学、物理等科学研究中业已取得了丰硕成果，特别是与组合学的有机融合，可应用于度量或无度量的数学系统，例如，代数、几何、拓扑、分析等数学学科，以及理论物理、天体物理、宇宙学等。本次会议有四十多位学者出席，先后收到来自美国、伊朗、尼日利亚、印度和中国学者提交的 14 篇大会交流论文，内容包括 Smarandache 重空间与 Smarandache 几何，双环，中智学，中智群及其子群，Smarandache 有向 n- 标号图，对称地图和拓扑图及其在不可解微分方程组拓扑结构刻画中的应用等。

刘彦佩教授（北京交通大学教授、博导）作
“我所认识的拓扑图论”大会报告

[1] http://www.bucea.edu.cn/xzdt/45363.htm

毛林繁教授（北京建筑大学经管学院兼职教授）作
“不可解微分方程组拓扑结构研究”大会报告

杜少飞教授（首都师范大学教授、博导）作
“对称地图及近年研究进展”大会报告

给定组合结构的 Smarandache 重空间，即数学组合对其他科学的应用，如引力场、强、弱作用场、传染病控制以及交通流分析等论题在本次会上也得到了交流。与会者对本次会议论题表现出了极大兴趣，同时有意致力于将重空间和重结构思想应用于其科学研究，以进一步提高人类认识自然的能力。

S-Denying a Theory

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Abstract: In this paper we introduce the operators of validation and invalidation of a proposition, and we extend the operator of S-denying a proposition, or an axiomatic system, from the geometric space to respectively any theory in any domain of knowledge, and show six examples in geometry, in mathematical analysis, and in topology.

Key Words: operator of S-denying, axiomatic system

AMS(2010): 51M15, 53B15, 53B40, 57N16

§1. Introduction

Let T be a theory in any domain of knowledge, endowed with an ensemble of sentences E, on a given space M.

E can be for example an axiomatic system of this theory, or a set of primary propositions of this theory, or all valid logical formulas of this theory, etc. E should be closed under the logical implications, i.e. given any subset of propositions $P_1, P_2, \cdots$ in this theory, if Q is a logical consequence of them then Q must also belong to this theory.

A sentence is a logic formula whose each variable is quantified i.e. inside the scope of a quantifier such as: $\exists$ (exist), $\forall$ (for all), modal logic quantifiers, and other various modern logics' quantifiers. With respect to this theory, let P be a proposition, or a sentence, or an axiom, or a theorem, or a lemma, or a logical formula, or a statement, etc. of E. It is said that P is S-denied on the space M if P is valid for some elements of M and invalid for other elements of M, or P is only invalid on M but in at least two different ways.

An ensemble of sentences E is considered S-denied if at least one of its propositions is S-denied. And a theory T is S-denied if its ensemble of sentences is S-denied, which is equivalent to at least one of its propositions being S-denied.

The proposition P is partially or totally denied/negated on M. The proposition P can be simultaneously validated in one way and invalidated in (finitely or infinitely) many different ways on the same space M, or only invalidated in (finitely or infinitely) many different ways.

The invalidation can be done in many different ways. For example the statement $A =: x \neq 5$ can be invalidated as $x = 5$ (total negation), but $x \in \{5, 6\}$ (partial negation). (Use a notation for S-denying, for invalidating in a way, for invalidating in another way a different

[1]The multispace operator S-denied (Smarandachely-denied) has been inherited from the previously published scientific literature (see for example Ref. [1] and [2]).

notation; consider it as an operator: neutrosophic operator? A notation for invalidation as well.)

But the statement $B =$: $x > 3$ can be invalidated in many ways, such as $x \leq 3$, or $x = 3$, or $x < 3$, or $x = -7$, or $x = 2$, etc. A negation is an invalidation, but not reciprocally - since an invalidation signifies a (partial or total) degree of negation, so invalidation may not necessarily be a complete negation. The negation of B is $B =$: $x \leq 3$, while $x = -7$ is a partial negation (therefore an invalidation) of B.

Also, the statement $C =$: *John's car is blue and Steve's car is red* can be invalidated in many ways, as: *John's car is yellow and Steve's car is red*, or *John's car is blue and Steve's car is black*, or *John's car is white and Steve's car is orange*, or *John's car is not blue and Steve's car is not red*, or *John's car is not blue and Steve's car is red*, etc.

Therefore, we can S-deny a theory in finitely or infinitely many ways, giving birth to many partially or totally denied versions/deviations/alternatives theories: $T_1, T_2, \cdots$. These new theories represent degrees of negations of the original theory T.

Some of them could be useful in future development of sciences.

Why do we study such S-denying operator? Because our reality is heterogeneous, composed of a multitude of spaces, each space with different structures. Therefore, in one space a statement may be valid, in another space it may be invalid, and invalidation can be done in various ways. Or a proposition may be false in one space and true in another space or we may have a degree of truth and a degree of falsehood and a degree of indeterminacy. Yet, we live in this mosaic of distinct (even opposite structured) spaces put together.

S-denying involved the creation of the *multi-space* in geometry and of the *S-geometries* (1969). It was spelt *multi-space*, or *multispace*, of *S-multispace*, or *mu-space*, and similarly for its: *multi-structure*, or *multistructure*, or *S-multistructure*, or *mu-structure*.

§2. **Notations**

Let $< A >$ be a statement (or proposition, axiom, theorem, etc.).

a) For the classical Boolean *logic negation* we use the same notation. The negation of $< A >$ is noted by $\neg A$ and $\neg A =< nonA >$. An invalidation of $< A >$ is noted by $i(A)$, while a validation of $< A >$ is noted by $v(A)$:

$$i(A) \subset 2^{<nonA>} \backslash \{\emptyset\} \text{ and } v(A) \subset 2^{<A>} \backslash \{\emptyset\},$$

where 2^X means the power-set of X, or all subsets of X.

All possible invalidations of $< A >$ form a set of invalidations, notated by $I(A)$. Similarly for all possible validations of $< A >$ that form a set of validations, and noted by $V(A)$.

b) S-denying of $< A >$ is noted by $S_\neg(A)$. S-denying of $< A >$ means some validations of $< A >$ together with some invalidations of $< A >$ in the same space, or only invalidations of $< A >$ in the same space but in many ways. Therefore, $S_\neg(A) \subset V(A) \bigcup I(A)$ or $S_\neg(A) \subset I(A)^k$ for $k \geq 2$.

§3. **Examples**

Let's see some models of S-denying, three in a geometrical space, and other three in mathematical analysis (calculus) and topology.

3.1 The first S-denying model was constructed in 1969. This section is a compilation of ideas from paper [1]:

An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space (i.e., validated and invalided, or only invalidated but in multiple distinct ways). A Smarandache Geometry [SG] is a geometry which has at least one Smarandachely denied axiom.

Let's note any point, line, plane, space, triangle, etc. in such geometry by s-point, s-line, s-plane, s-space, s-triangle respectively in order to distinguish them from other geometries. *Why these hybrid geometries*? Because in reality there does not exist isolated homogeneous spaces, but a mixture of them, interconnected, and each having a different structure. These geometries are becoming very important now since they combine many spaces into one, because our world is not formed by perfect homogeneous spaces as in pure mathematics, but by non-homogeneous spaces. Also, SG introduce the degree of negation in geometry for the first time (for example an axiom is denied 40% and accepted 60% of the space) that's why they can become revolutionary in science and it thanks to the idea of partial denying/accepting of axioms/propositions in a space (making multi-spaces, i.e. a space formed by combination of many different other spaces), as in fuzzy logic the degree of truth (40% false and 60% true). They are starting to have applications in physics and engineering because of dealing with non-homogeneous spaces.

The first model of S-denying and of SG was the following:

The axiom that through a point exterior to a given line there is only one parallel passing through it (Euclid's Fifth Postulate), was S-denied by having in the same space: no parallel, one parallel only, and many parallels.

In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated. In the Lobachevsky-Bolyai-Gauss geometry, called hyperbolic geometry, this fifth Euclidean postulate is invalidated in the following way: there are infinitely many lines parallels to a given line passing through an exterior point.

While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point. Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some SG's. These last geometries can be partially Euclidean and partially Non-Euclidean simultaneously.

3.2 Geometric Model

Suppose we have a rectangle ABCD. See Fig.1 below.

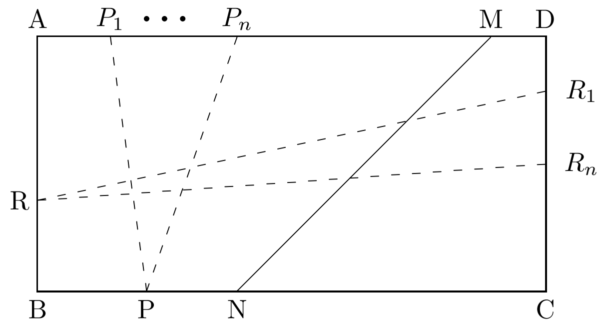

Fig.1

In this model we define as:

Point = any point inside or on the sides of this rectangle;

Line = a segment of line that connects two points of opposite sides of the rectangle;

Parallel lines = lines that do not have any common point (do not intersect);

Concurrent lines = lines that have a common point.

Let's take the line MN, where M lies on side AD and N on side BC as in the above Fig. 1. Let P be a point on side BC, and R a point on side AB.

Through P there are passing infinitely many parallels ($PP_1, \cdots, PP_n, \cdots$) to the line MN, but through R there is no parallel to the line MN (the lines $RR_1, \cdots, RR_n$ cut line MN). Therefore, the Fifth Postulate of Euclid (that though a point exterior to a line, in a given plane, there is only one parallel to that line) in S-denied on the space of the rectangle ABCD since it is invalidated in two distinct ways.

3.3 Another Geometric Model

We change a little the Geometric Model 1 such that:

The rectangle ABCD is such that side AB is smaller than side BC. And we define as line the arc of circle inside (and on the borders) of ABCD, centered in the rectangle's vertices A, B, C, or D.

The axiom that: through two distinct points there exist only one line that passes through is S-denied (in three different ways):

a) Through the points A and B there is no passing line in this model, since there is no arc of circle centered in A, B, C, or D that passes through both points. See Fig.2.

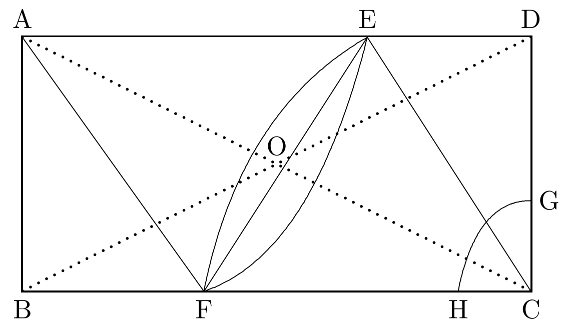

Fig.2

b) We construct the perpendicular EF⊥AC that passes through the point of intersection of the diagonals AC and BD. Through the points E and F there are two distinct lines the dark green (left side) arc of circle centered in C since CE≡FC, and the light green (right side) arc of circle centered in A since AE≡AF. And because the right triangles ⨆COE, ⨆COF, ⨆AOE, and ⨆AOF are all four congruent, we get CE≡FC≡AE≡AF.

c) Through the points G and H such that CG≡CH (their lengths are equal) there is only one passing line (the dark green arc of circle GH, centered in C) since AG≠AH (their lengths are different), and similarly BG≠BH and DG≠DH.

3.4 Example for the Axiom of Separation

The Axiom of Separation of Hausdorff is the following:

$$\forall x, y \in M, \exists N(x), N(y) \Rightarrow N(x) \bigcap N(y) = \emptyset,$$

where $N(x)$ is a neighborhood of x, and respectively $N(y)$ is a neighborhood of y.

We can S-deny this axiom on a space M in the following way:

a) $\exists x_1, y_1 \in M$ and $\exists N_1(x_1), N_1(y_1) \Rightarrow N_1(x_1) \bigcap N_1(y_1) = \emptyset$, where $N_1(x_1)$ is a neighborhood of x_1, and respectively $N_1(y_1)$ is a neighborhood of y_1. [validated]

b) $\exists x_2, y_2 \in M \Rightarrow \forall N_2(x_2), N_2(y_2), N_2(x_2) \bigcap N_2(y_2) = \emptyset$, where $N_2(x_2)$ is a neighborhood of x_2, and respectively $N_2(y_2)$ is a neighborhood of y_2. [invalidated]

Therefore we have two categories of points in M: some points that verify The Axiom of Separation of Hausdorff and other points that do not verify it. So M becomes a partially separable and partially inseparable space, or we can see that M has some degrees of separation.

3.5 Example for the Norm

If we remove one or more axioms (or properties) from the definition of a notion $< A >$ we get a pseudo-notion $< pseudoA >$. For example, if we remove the third axiom (inequality of the triangle) from the definition of the $< norm >$ we get a $< pseudonorm >$. The axioms of a norm on a real or complex vectorial space V over a field F, $x \to \| \cdot \|$, are the following:

a) $\|x\| = 0 \Leftrightarrow x = 0$;

b) $\forall x \in V, \forall \alpha \in F$, $\|\alpha x\| = |\alpha|\|x\|$;

c) $\forall x, y \in V$, $\|x + y\| \leq \|x\| \cdot \|y\|$ (inequality of the triangle).

For example, a pseudo-norm on a real or complex vectorial space V over a field F, $x \rightarrow {}_p\|\cdot\|$, may verify only the first two above axioms of the norm.

A pseudo-norm is a particular case of an S-denied norm since we may have vectorial spaces over some given scalar fields where there are some vectors and scalars that satisfy the third axiom [validation], but others that do not satisfy [invalidation]; or for all vectors and scalars we may have either $\|x + y\| = 5\|x\| \cdot \|y\|$ or $\|x + y| = 6\|x\| \cdot \|y\|$, so invalidation (since we get $\|x + y\| > \|x\| \cdot \|y\|$) in two different ways.

Let's consider the complex vectorial space $\mathscr{C} = \{a + bi,\ \text{where } a, b \in R, i = \sqrt{-1}\}$ over the field of real numbers R. If $z = a + bi \in \mathscr{C}$ then its pseudo-norm is $\|z\| = \sqrt{a^2 + b^2}$. This verifies the first two axioms of the norm, but do not satisfy the third axiom of the norm since:

For $x = 0 + bi$ and $y = a + 0i$ we get $\|x + y\| = \|a + bi\| = \sqrt{a^2 + b^2} \leq \|x\| \cdot \|y\| = \|0 + bi\| \cdot \|a + 0i\| = |ab|$, or $a^2 + b^2 \leq a^2 b^2$. But this is true for example when $a = b \geq \sqrt{2}$ (validation), and false if one of a or b is zero and the other is strictly positive (invalidation).

Pseudo-norms are already in use in today's scientific research, because for some applications the norms are considered too restrictive. Similarly one can define a pseudo-manifold (relaxing some properties of the manifold), etc.

3.6 Example in Topology

A topology $\mathscr{O}$ on a given set E is the ensemble of all parts of E verifying the following properties:

a) E and the empty set $\emptyset$ belong to $\mathscr{O}$;

b) Intersection of any two elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too;

c) Union of any family of elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too.

Let's go backwards. Suppose we have a topology $\mathscr{O}_1$ on a given set E_1, and the second or third (or both) previous axioms have been S-denied, resulting an S-denied topology $S\neg(\mathscr{O}_1)$ on the given set E_1.

In general, we can go back and *recover* (*reconstruct*) the original topology $\mathscr{O}_1$ from $S\neg(\mathscr{O}_1)$ by recurrence: if two elements belong to $S\neg(\mathscr{O}_1)$ then we set these elements and their intersection to belong to $\mathscr{O}_1$, and if a family of elements belong to $S\neg(\mathscr{O}_1)$ then we set these family elements and their union to belong to $\mathscr{O}_1$; and so on: we continue this recurrent process until it does not bring any new element to $\mathscr{O}_1$.

§4. **Conclusion**

Decidability changes in an S-denied theory, i.e. a defined sentence in an S-denied theory can be partially deducible and partially undeducible (we talk about degrees of deducibility of a sentence in an S-denied theory).

Since in classical deducible research, a theory T of language L is said complete if any sentence of L is decidable in T, we can say that an S-denied theory is partially complete (or has some degrees of completeness and degrees of incompleteness).

References

[1] Kuciuk L. and Antholy M., An Introduction to Smarandache Geometries, *Mathematics Magazine*, Aurora, Canada, Vol.12, 2003.

[2] Mao Linfan, An introduction to Smarandache geometries on maps, *2005 International Conference on Graph Theory and Combinatorics*, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005. also appear in *Smarandache geometries & map theory with applications*(I), Chinese Branch Xiquan House, 2007.

[3] Dmitri Rabounski, Smarandache spaces as a new extension of the basic space-time of general relativity, *Progress in Physics*, Vol.2, L1-L2, 2010.

[4] F.Smarandache, *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania, 1969, and in *Paradoxist Mathematics*, Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[5] F.Smarandache, Multi-space and multi-structure, in *Neutrosophy. Neutrosophic Logic, Set, Probability and Statistics*, American Research Press, 1998.

[6] Matthew Szudzik, *Sentence*, From MathWorld – A Wolfram Web Resource, created by Eric W. Weisstein, http://mathworld.wolfram.com/Sentence.html.

Smarandache Geometry

— A geometry with philosophical notion

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Abstract: A *Smarandache geometry* is such a geometry with an axiom validated and invalided, or only invalided but in multiple distinct ways, which can be used to globally characterize behavior of in the world. This paper introduces such geometry and discusses Smarandache n-manifolds, particularly, pseudo-Euclidean spaces and combinatorial manifolds.

Key Words: Smarandachely axiom, map geometry, Smarandachely manifold, pseudo-Euclidean space, combinatorial manifold.

AMS(2010): 51M15, 53B15, 53B40, 57N16

§1. Introduction

A famous words claims that *every thing is constituted both by the positive and the negative.* In fact, it is philosophical proposition for human beings knowing the world. We are used to harmonious systems both in life and scientific research. But such a system can be only existent in one's assumption, can not appearing in the world really because a thing in the world constituted by its contradict parts. Thus for a global, not a unilateral knowing a thing in the world, we are needed to research such system with contradiction, i.e., Smarandache systems [5-6].

Definition 1.1([2-4]) *A rule $\mathcal{R}$ in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule $\mathcal{R}$. Particularly, if such a system $(\Sigma;\mathcal{R})$ is a geometry, then it is a Smarandache geometry (1969).

A Smarandache n-manifold is an n-dimensional manifold that support a Smarandache geometry.

For example, let us consider a Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C, such as those shown in Fig.1. Define s-points to be all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C.

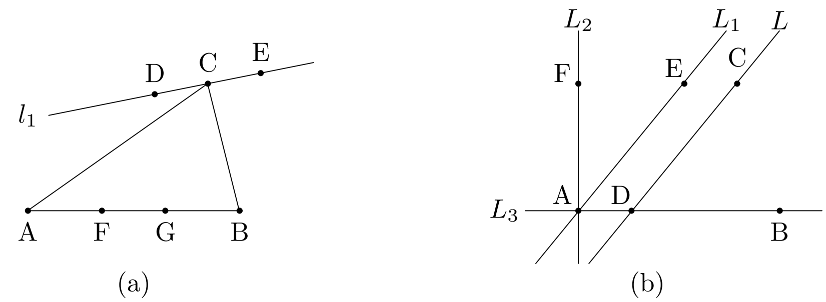


Fig.1

Then such a system is a Smarandache geometry by observations following.

Observation 1. The axiom (E1) in Euclidean geometry that through any two distinct points there exist one line passing through them is now replaced by: *one s-line* and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.1(a).

Observation 2. The axiom (E5) in Euclidean geometry that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*. Let L be an s-line passes through points C and D on AB, L_1 parallel to L passing through A and E a point on L_1. Then there is one line L_1 passing through E parallel L but no lines passing through A and a point F not on L_1, such as those shown in Fig.1(b).

For $\forall R \in \mathcal{R}$ Let $\sum_R$ be the maximal system such that $(\sum_R; R)$ without Smarandachely denied axiom, i.e., validated or invalided in one way. Then we are easily know that $(\sum; \mathcal{R}) = \bigcup\limits_{R\in\mathcal{R}} (\sum_R; R)$. Thus we can naturally get a Smarandache multi-space defined following.

Definition 1.2([2-4]) *Let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, i.e., the rule $\mathcal{R}_i$ on Σ_i for integers $1 \leq i \leq m$, denoted by $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$. Clearly, a Smarandache multi-space is a Smarandache geometry.*

A Smarandache multi-space $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ inherits a vertex-edge labeled graph defined following:

$$V\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{S}, \widetilde{R}\right]\right) \to l^E(\Sigma_i, \Sigma_j) = \varpi\left(\Sigma_i \bigcap \Sigma_j\right),$$

where ϖ is a characteristic on $\Sigma_i \bigcap \Sigma_j$ such that $\Sigma_i \bigcap \Sigma_j$ is isomorphic to $\Sigma_k \bigcap \Sigma_l$ if and only if $\varpi(\Sigma_i \bigcap \Sigma_j) = \varpi\left(\Sigma_k \bigcap \Sigma_l\right)$ for integers $1 \leq i, j, k, l \leq m$.

§2. Smarandache 2-Manifolds

A point P on $\mathbf{R}^2$ is in fact associated with a real number π. Generally, we consider a function $\mu : \mathbf{R}^2 \to [0, 2\pi)$ and classify points on $\mathbf{R}^2$ into three classes following:

Elliptic Type. All points $P \in \mathbf{R}^2$ with $\mu(P) < \pi$.
Euclidean Type. All points $Q \in \mathbf{R}^2$ with $\mu(Q) = \pi$.
Hyperbolic Type. All points $U \in \mathbf{R}^2$ with $\mu(U) > \pi$.

Such a Euclidean plane $\mathbf{R}^2$ with elliptic or hyperbolic points is called a *Smarandache plane*, denoted by $(\mathbf{R}^2, \mu)$ and these elliptic or hyperbolic points are called *non-Euclidean points*. A *Smarandache 2-manifold* is a 2-manifold M that there are points $p \in M$ with neighborhood $U(p)$ homeomorphic to a Smarandache plane and it is called *finite* if there are only finitely non-Euclidean points on the 2-manifold.

Definition 2.1 *A map geometry without boundary is such a combinatorial map M on a 2-manifold associates a real number $\mu(u)$ to each vertex u, $u \in V(M)$ with $\rho_M(u) \geq 3$. A map geometry with boundary $f_1, f_2, \cdots, f_l$ is a map geometry (M, μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$ such that $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected.*

For example, Fig.2 shows a map geometry on wheel $W_{1.4}$ without boundary.

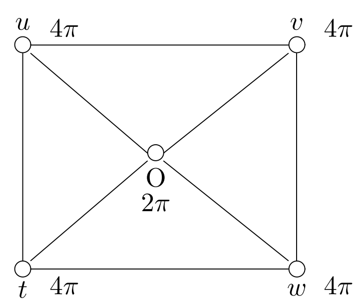


Fig.2

Then we know the following results.

Theorem 2.2([3]) *Let M be a map with $\rho_M(v) \geq 3$ for $\forall v \in V(M)$. Then for $\forall u \in V(M)$, there is a map geometry with or without boundary such that u is elliptic, Euclidean or hyperbolic.*

Theorem 2.3([3]) *Let M be a map of order≥ 3 and $\rho_M(v) \geq 3$ for $\forall v \in V(M)$,. Then there exists a map geometry with or without boundary in which elliptic, Euclidean and hyperbolic points appear simultaneously.*

A nice model for Smarandache geometry called *s-manifolds* was constructed by Iseri in [1]:

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \leq i \leq n$ satisfying the following conditions:

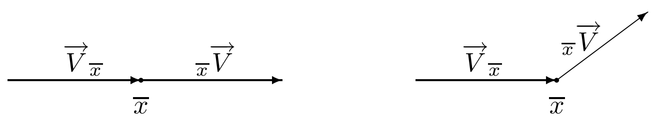
(i) each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i, j \leq n$ and $i \neq j$;

(ii) each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.

It should be noted that all these s-manifolds are in fact planar map geometries with vertex valencies 5, 6 or 7.

§3. Pseudo-Euclidean Spaces

A *pseudo-Euclidean space* $(\mathbf{R}^n, \mu)$ is such a Euclidean space $\mathbf{R}^n$ associated with a mapping $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ for $\overline{x} \in \mathbf{R}^n$, such as those shown in Fig.3, where $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$ is the normal basis of $\mathbf{R}^n$ and $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ two vectors with end or initial point at $\overline{x}$, respectively.



(a) (b)

Fig.3

where $\overrightarrow{V}_{\overline{x}}$ and ${}_{\overline{x}}\overrightarrow{V}$ are in the same orientation in case (a), but not in case (b). Such points in case (a) are called *Euclidean* and in case (b) *non-Euclidean*. A pseudo-Euclidean $(\mathbf{R}^n, \mu)$ is *finite* if it has finite non-Euclidean points, otherwise, *infinite*.

Denote these elliptic, Euclidean and hyperbolic point sets by

$\overrightarrow{V}_{eu} = \{\overline{u} \in \mathbf{R}^n \mid \overline{\mathrm{u}} \text{ an Euclidean point}\}$,
$\overrightarrow{V}_{el} = \{\overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{v}} \text{ an elliptic point}\}$.
$\overrightarrow{V}_{hy} = \{\overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{w}} \text{ a hyperbolic point}\}$.

Then we get a partition $\mathbf{R}^n = \overrightarrow{V}_{eu} \bigcup \overrightarrow{V}_{el} \bigcup \overrightarrow{V}_{hy}$ on points in $\mathbf{R}^n$. The points in $\overrightarrow{V}_{el} \cup \overrightarrow{V}_{hy}$ are called *non-Euclidean points*. It should be noted that a straight line L is determined by an initial point $(x_1^0, x_2^0, \cdots, x_n^0)$ and an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$. We get results following.

Theorem 3.1([3]) *A pseudo-Euclidean space* $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ *is a Smarandache geometry if* $\overrightarrow{V}_{eu}$, $\overrightarrow{V}_{el} \neq \emptyset$, *or* $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, *or* $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ *for an orientation* $\overrightarrow{O}$ *in* $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$.

Denoted by $\mathcal{O}_{\overline{x}}$ the set of all normal bases at point $\overline{x}$. Then a *pseudo-Euclidean space* $(\mathbf{R}, \mu)$ is nothing but a Euclidean space $\mathbf{R}^n$ associated with a linear mapping $\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \to \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\} \in \mathcal{O}_{\overline{x}}$ such that $\mu(\overline{\epsilon}_1) = \overline{\epsilon}'_1$, $\mu(\overline{\epsilon}_2) = \overline{\epsilon}'_2$, $\cdots$, $\mu(\overline{\epsilon}_n) = \overline{\epsilon}'_n$ at point $\overline{x} \in \mathbf{R}^n$. Thus if $\overrightarrow{V}_{\overline{x}} = c_1\overline{\epsilon}_1 + c_2\overline{\epsilon}_2 + \cdots + c_n\overline{\epsilon}_n$ and

$$\mu(\overline{\epsilon}_i) = x_{i1}\overline{\epsilon}_1 + x_{i2}\overline{\epsilon}_2 + \cdots + x_{in}\overline{\epsilon}_n, \quad 1 \leq i \leq n.$$

Then

$$\begin{aligned} \mu\left({}_{\overline{x}}\overrightarrow{V}\right) &= (c_1, c_2, \cdots, c_n)(\mu(\overline{\epsilon}_1), \mu(\overline{\epsilon}_2), \cdots, \mu(\overline{\epsilon}_n))^t \\ &= (c_1, c_2, \cdots, c_n)[x_{ij}]_{n\times n}(\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t. \end{aligned}$$

Thus $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ is determined by $\mu(\overline{x}) = [x_{ij}]_{n\times n}$ for $\overline{x} \in \mathbf{R}^n$.

Theorem 3.2([2-3]) *If* $(\mathbf{R}^n, \mu)$ *is a pseudo-Euclidean space, then* $\mu(\overline{x}) = [x_{ij}]_{n\times n}$ *is an* $n \times n$ *orthogonal matrix for* $\forall\ \overline{x} \in \mathbf{R}^n$*, i.e.,* $[\overline{x}]\,[\overline{x}]^t = I_{n\times n}$.

The *curvature* $R(L)$ of an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$ for $m \geq 1$ in $(\mathbf{R}^n, \mu)$ is defined by

$$R(L) = \prod_{i=1}^{m} \mu(\overline{x}_i).$$

It is *Euclidean* if $R(L) = I_{n\times n}$, otherwise, *non-Euclidean*. Obviously, a point in a Euclidean space $\mathbf{R}^n$ is indeed Euclidean by this definition.

Theorem 3.3([2-3]) *Let* $(\mathbf{R}^n, \mu)$ *be a pseudo-Euclidean space and* L *an s-line in* $(\mathbf{R}^n, \mu)$ *passing through non-Euclidean points* $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$*. Then* L *is closed if and only if* L *is Euclidean.*

For example, we consider the pseudo-Euclidean space $(\mathbf{R}^2, \mu)$ and find the rotation matrix $\mu(\overline{x})$ for points $\overline{x} \in \mathbf{R}^2$. Let $\theta_{\overline{x}}$ be the angle form $\overline{\epsilon}_1$ to $\mu\overline{\epsilon}_1$. Then it is easily to know that

$$\mu(\overline{x}) = \begin{pmatrix} \cos\theta_{\overline{x}} & \sin\theta_{\overline{x}} \\ \sin\theta_{\overline{x}} & -\cos\theta_{\overline{x}} \end{pmatrix}.$$

Now if an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^2$, then Theorem 3.3 implies that

$$\begin{pmatrix} \cos\theta_{\overline{x}_1} & \sin\theta_{\overline{x}_1} \\ \sin\theta_{\overline{x}_1} & -\cos\theta_{\overline{x}_1} \end{pmatrix} \begin{pmatrix} \cos\theta_{\overline{x}_2} & \sin\theta_{\overline{x}_2} \\ \sin\theta_{\overline{x}_2} & -\cos\theta_{\overline{x}_2} \end{pmatrix} \cdots \begin{pmatrix} \cos\theta_{\overline{x}_m} & \sin\theta_{\overline{x}_m} \\ \sin\theta_{\overline{x}_m} & -\cos\theta_{\overline{x}_m} \end{pmatrix} = I_{n\times n}.$$

Thus

$$\mu(\overline{x}) = \begin{pmatrix} \cos(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) & \sin(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) \\ \sin(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) & \cos(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) \end{pmatrix} = I_{n\times n}.$$

Whence, $\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m} = 2k\pi$ for an integer k. This fact is in agreement with that of Theorem 3.3.

An embedded graph $G_{\mathbf{R}^n}$ is called *Smarandachely* if there exists a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ with a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ such that all of its vertices are non-Euclidean points in $(\mathbf{R}^n, \mu)$.

Theorem 3.4([2-3]) *An embedded* 2*-connected graph* $G_{\mathbf{R}^n}$ *is Smarandachely if and only if there is a mapping* $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ *and a directed circuit-decomposition*

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \qquad 1 \leq i \leq s$$

are solvable.

Furthermore, we know the conditions for a curve C exists in a pseudo-Euclidean space $(\mathbf{R}^n, \mu|_{\overrightarrow{O}})$.

Theorem 3.5([3]) *A curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \mu|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$ if and only if*

$$\left.\frac{df_i(t)}{dt}\right|_{\overline{u}} = \sqrt{(\frac{1}{\mu_i(\overline{u})})^2 - 1}, \quad 1 \leq i \leq n$$

for $\forall \overline{u} \in C$, where $\mu|_{\overrightarrow{O}} = (\mu_1, \mu_2, \cdots, \mu_n)$.

Corollary 3.6 *A straight line L exists in $(\mathbf{R}^n, \mu|_{\overrightarrow{O}})$ if and only if $\mu|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ for $\forall \overline{u} \in L$ and $\forall \overrightarrow{O} \in \mathscr{O}$.*

§4. Combinatorial Manifolds

A *combinatorial manifold* is a combination of manifolds. By definition it is a Smarandache manifold if the dimensions are different. A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. For the Euler-Poincaré characteristics $\chi\left(\widetilde{M}\right)$ of combinatorial manifolds $\widetilde{M}$, we know the following results.

Theorem 4.1([4]) *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then*

$$\chi\left(\widetilde{M}\right) = \sum_{K^k \in Cl(k), k \geq 2} \sum_{M_{i_j} \in V(K^k), 1 \leq j \leq s \leq k} (-1)^{s+1} \chi\left(M_{i_1} \bigcap \cdots \bigcap M_{i_s}\right)$$

Corollary 4.2 *Let $\widetilde{M}$ be a finitely combinatorial manifold. If $G^L\left[\widetilde{M}\right]$ is K^3-free, then*

$$\chi\left(\widetilde{M}\right) = \sum_{M \in V(G^L[\widetilde{M}])} \chi^2(M) - \sum_{(M_1, M_2) \in E(G^L[\widetilde{M}])} \chi\left(M_1 \bigcap M_2\right).$$

Particularly, if $\dim(M_1 \bigcap M_2)$ is a constant for any $(M_1, M_2) \in E\left(G^L\left[\widetilde{M}\right]\right)$, then

$$\chi\left(\widetilde{M}\right) = \sum_{M \in V(G^L[\widetilde{M}])} \chi^2(M) - \chi\left(M_1 \bigcap M_2\right)\left|E\left(G^L\left[\widetilde{M}\right]\right)\right|.$$

§5. Discussions

The Smarandache geometry presents a mathematical approach for characterizing globally behavior of things in the world, which is equivalent to Smarandache multi-space. As its a special case, combinatorial manifolds enable one to establish mathematical model for things in multi-spaces, particularly, the multilateral of a thing, such as those of mechanical fields, gravitational fields, electromagnetic fields and gauge fields,..., etc.. All these notions are included in references [2-4].

In physics, all manifolds are assumed to be smooth, which needs differential theory on combinatorial manifolds for calculation. The reader is refereed to [4] of mine for further reading on combinatorial differential geometry.

References

[1] H.Iseri, *Smarandache Manifolds*, American Research Press, Rehoboth, NM,2002.

[2] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Second edition), The Education Publisher Inc. 2011.

[3] Linfan Mao, *Smarandache Multi-Space Theory* (Second edition), The Education Publisher Inc. 2011.

[4] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory* (Second edition), The Education Publisher Inc. 2011.

[5] F.Smarandache, Mixed noneuclidean geometries, *Eprint arXiv: math/0010119*, 10/2000.

[6] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

Neutrosophic Transdisciplinarity — Multi-Space & Multi-Structure

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§1. Definitions

Neutrosophic Transdisciplinarity means to find common features to uncommon entities, i.e., for vague, imprecise, not-clear-boundary entity < A > one has:

$< A > \bigcap < nonA > \neq \emptyset$, or even more $< A > \bigcap < antiA > \neq \emptyset$. Similarly, $< A > \bigcap < neutA > = \emptyset$ and $< antiA > \bigcap < neutA > = \emptyset$, up to $< A > \bigcap < neutA > \bigcap < antiA > = \emptyset$, where $< nonA >$ means what is not A, and $< antiA >$ means the opposite of $< A >$.

There exists a *principle of attraction* not only between the opposites $< A >$ and $< antiA >$ (as in dialectics), but also between them and their neutralities $< neutA >$ related to them, since $< neutA >$ contributes to the Completeness of knowledge. $< neutA >$ means neither $< A >$ nor $< antiA >$, but in between; $< neutA >$ is included in $< nonA >$.

As part of Neutrosophic Transdisciplinarity we have the following important conceptions.

§2. Multi-Structure and Multi-Space

2.1 Multi-Concentric-Structure

Let S_1 and S_2 be two distinct structures, induced by the ensemble of laws L, which verify the ensembles of axioms A_1 and A_2 respectively, such that A_1 is strictly included in A_2. One says that the set M, endowed with the properties:

a) M has an S_1-structure;

b) there is a proper subset P (different from the empty set $\emptyset$, from the unitary element, from the idempotent element if any with respect to S_2, and from the whole set M) of the initial set M, which has an S_2-structure;

c) M doesn't have an S_2-structure,

is called a 2-concentric-structure. We can generalize it to an n-concentric-structure, for $n \geq 2$ (even infinite-concentric-structure).

(By default, 1-concentric structure on a set M means only one structure on M and on its proper subsets.)

An n-concentric-structure on a set S means a weak structure $\{w(0)\}$ on S such that there exists a chain of proper subsets

$$P(n-1) < P(n-2) < \cdots < P(2) < P(1) < S,$$

where $<$ means *included in*, whose corresponding structures verify the inverse chain

$$\{w(n-1)\} > \{w(n-2)\} > \cdots > \{w(2)\} > \{w(1)\} > \{w(0)\},$$

where $>$ signifies *strictly stronger* (i.e., structure satisfying more axioms).

For example, say a groupoid D, which contains a proper subset S which is a semigroup, which in its turn contains a proper subset M which is a monoid, which contains a proper subset NG which is a non-commutative group, which contains a proper subset CG which is a commutative group, where D includes S, which includes M, which includes NG, which includes CG. In fact, this is a 5-concentric-structure.

2.2 **Multi-Space**

Let $S_1, S_2, \cdots, S_n$ be n structures on respectively the sets $M_1, M_2, \cdots, M_n$, where $n \geq 2$ (n may even be infinite). The structures $S_i,\ i = 1, 2, \cdots, n$, may not necessarily be distinct two by two; each structure S_i may be or not n_i-concentric, for $n_i \geq 1$. And the sets $M_i,\ i = 1, 2, \cdots, n$, may not necessarily be disjoint, also some sets M_i may be equal to or included in other sets $M_j,\ j = 1, 2, \cdots, n$. We define the *multi-space* M as a union of the previous sets:

$$M = M_1 \bigcup M_2 \bigcup \cdots \bigcup M_n,$$

hence we have n (different or not, overlapping or not) structures on M. A multi-space is a space with many structures that may overlap, or some structures may include others or may be equal, or the structures may interact and influence each other as in our everyday life.

Therefore, a region (in particular a point) which belong to the intersection of $1 \leq k \leq n$ sets M_i may have k different structures in the same time. And here it is the difficulty and beauty of the a multi-space and its overlapping multi-structures.

(We thus may have $< R > \neq < R >$, i.e. a region R different from itself, since R could be endowed with different structures simultaneously.)

For example, we can construct a geometric multi-space formed by the union of three distinct subspaces: an Euclidean subspace, a hyperbolic subspace and an elliptic subspace.

As particular cases when all M_i sets have the same type of structure, we can define the Multi-Group (or n-group; for example; bigroup, tri-group, etc., when all sets M_i are groups), Multi- Ring (or n-ring, for example biring, tri-ring, etc. when all sets M_i are rings), Multi-Field (n-field), Multi-Lattice (n-lattice), Multi-Algebra (n-algebra), Multi-Module (n-module), and so on - which may be generalized to infinite-structure-space (when all sets have the same type of structure), etc.

§3. **Conclusion**

The multi-space comes from reality, it is not artificial, because our reality is not homogeneous, but has many spaces with different structures. A multi-space means a combination of any spaces (may be all of the same dimensions, or of different dimensions - it doesn' t matter). For example, a Smarandache geometry (SG) is a combination of geometrical (manifold or pseudo-manifold, etc.) spaces, while the multi-space is a combination of any (algebraic, geometric, analytical, physics, chemistry, etc.) space. So, the multi-space can be interdisciplinary, i.e. math and physics spaces, or math and biology and chemistry spaces, etc. Therefore, an SG is a particular case of a multi-space. Similarly, a Smarandache algebraic structure is also a particular case of a multi-space.

This multi-space is a combination of spaces on the horizontal way, but also on the vertical way (if needed for certain applications). On the horizontal way means a simple union of spaces (that may overlap or not, may have the same dimension or not, may have metrics or not, the metrics if any may be the same or different, etc.). On the vertical way means more spaces overlapping in the same time, every one different or not. The multi-space is really very general because it tries to model our reality. The parallel universes are particular cases of the multi-space too. So, they are multi-dimensional (they can have some dimensions on the horizontal way, and other dimensions on the vertical way, etc.).

The multi-space with its multi-structure is a *Theory of Everything*. It can be used, for example, in the *Unified Field Theory* that tries to unite the gravitational, electromagnetic, weak, and strong interactions (in physics).

Reference

[1] F.Smarandache, *Mixed Non-Euclidean geometries*, Arhivele Statului, Filiala Valcea, Rm. Valcea, 1969.

Non-Solvable Equation Systems with Graphs Embedded in $\mathbb{R}^n$

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Abstract: Different from the homogenous systems, a Smarandache system is a contradictory system in which an axiom behaves in at least two different ways within the same system, i.e., validated and invalided, or only invalided but in multiple distinct ways. Such systems widely exist in the world. In this report, we discuss such a kind of Smarandache system, i.e., non-solvable equation systems, such as those of non-solvable algebraic equations, non-solvable ordinary differential equations and non-solvable partial differential equations by topological graphs, classify these systems and characterize their global behaviors, particularly, the sum-stability and prod-stability of such equations. Applications of such systems to other sciences, such as those of controlling of infectious diseases, interaction fields and flows in network are also included in this report.

Key Words: Non-solvable equation, Smarandache system, topological graphs, vertex-edge labeled graph, G-solution, sum-stability, prod-stability.

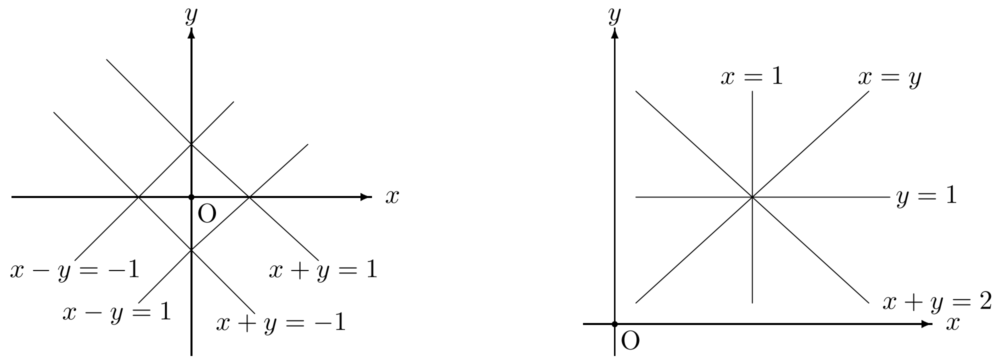
§1. Introduction

Consider two systems of linear equations following:

$$(LES_4^N)\ \left\{\begin{array}{rcr} x+y & = & 1 \\ x+y & = & -1 \\ x-y & = & -1 \\ x-y & = & 1 \end{array}\right. \qquad\qquad (LES_4^S)\ \left\{\begin{array}{l} x=y \\ x+y=2 \\ x=1 \\ y=1 \end{array}\right.$$

Clearly, (LES_4^N) is non-solvable because $x+y=-1$ is contradictious to $x+y=1$, and so that for equations $x-y=-1$ and $x-y=1$. Thus there are no solutions x_0, y_0 hold with all equations in this system. But (LES_4^S) is solvable clearly with a solution $x=1$ and $y=1$.

It should be noted that each equation in systems (LES_4^N) and (LES_4^S) is a straight line in Euclidean space $\mathbb{R}^2$, such as those shown in Fig.1.



(LES_4^N) (LES_4^S)

Fig.1

What is the geometrical essence of a non-solvable or solvable system of linear equations? It is clear that each linear equation $ax+by=c$ with $ab \neq 0$ is in fact a point set $L_{ax+by=c} = \{(x,y)|ax+by=c\}$ in $\mathbb{R}^2$. Then, the system (LES_4^n) is non-solvable but (LES_4^S) solvable in sense because of

$$L_{x+y=1}\bigcap L_{x+y=-1}\bigcap L_{x-y=1}\bigcap L_{x-y=-1}=\emptyset$$

and

$$L_{x=y}\bigcap L_{x=1}\bigcap L_{y=1}\bigcap L_{x+y=2}=\{(1,1)\}$$

in Euclidean plane $\mathbb{R}^2$. Generally, let

$$(ES_m)\quad \begin{cases} f_1(x_1,x_2,\cdots,x_n)=0 \\ f_2(x_1,x_2,\cdots,x_n)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1,x_2,\cdots,x_n)=0 \end{cases}$$

be a system of algebraic equations in Euclidean space $\mathbb{R}^n$ for an integer $n \geq 1$ with point set S_{f_i} such that $f_i(x_1,x_2,\cdots,x_n)=0$ for any point $(x_1,x_2,\cdots,x_n) \in S_{f_i}$, $1 \leq i \leq m$. Then, it is clear that the system (ES_m) is solvable or not dependent on $\bigcap\limits_{i=1}^{m} S_{f_i}=\emptyset$ or $\neq \emptyset$. This fact implies the following interesting result.

Proposition 1.1 *Any system (ES_m) of algebraic equations with each equation solvable posses a geometrical figure in $\mathbb{R}^n$, no matter it is solvable or not.*

Conversely, for a geometrical figure $\mathscr{G}$ in $\mathbb{R}^n, n \geq 2$, *how can we get an algebraic representation for geometrical figure $\mathscr{G}$*? As a special case, let G be a graph embedded in Euclidean space $\mathbb{R}^n$ and

$$(ES_e)\quad \begin{cases} f_1^e(x_1, x_2, \cdots, x_n) = 0 \\ f_2^e(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^e(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of equations for determining an edge $e \in E(G)$ in $\mathbb{R}^n$. Then the system

$$\left.\begin{array}{r} f_1^e(x_1, x_2, \cdots, x_n) = 0 \\ f_2^e(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^e(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} \forall e \in E(G)$$

is a non-solvable system of equations. Generally, let $\mathscr{G}$ be a geometrical figure consisting of m parts $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_m$, where $\mathscr{G}_i$ is determined by a system of algebraic equations

$$\begin{cases} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

Similarly, we get a non-solvable system

$$\left.\begin{array}{l} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} 1 \leq i \leq m.$$

Thus we obtain the following result.

Proposition 1.2 *Any geometrical figure $\mathscr{G}$ consisting of m parts, each of which is determined by a system of algebraic equations in $\mathbb{R}^n, n \geq 2$ posses an algebraic representation by system of equations, solvable or not in $\mathbb{R}^n$.*

For example, let G be a planar graph with vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_1$, shown in Fig.2.

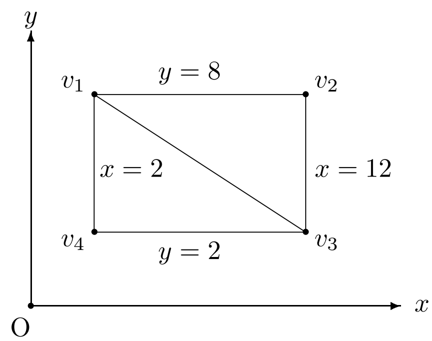


Fig.2

Then we get a non-solvable system of linear equations

$$\begin{cases} x = 2 \\ y = 8 \\ x = 12 \\ y = 2 \\ 3x + 5y = 46. \end{cases}$$

More results on non-solvable linear systems of equations can be found in [9]. Terminologies and notations in this paper are standard. For those not mentioned in this paper, we follow [12] and [15] for partial or ordinary differential equations. [5-7], [13-14] for algebra, topology and Smarandache systems, and [1] for mechanics.

§2. Smarandache Systems with Labeled Topological Graphs

A non-solvable system of algebraic equations is in fact a contradictory system in classical meaning of mathematics. As we have shown, such systems extensively exist in mathematics and possess real meaning even if in classical mathematics. This fact enables one to introduce the conception of Smarandache system following.

Definition 2.1([5-7]) *A rule $\mathcal{R}$ in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule $\mathcal{R}$.

Without loss of generality, let $(\Sigma_1; \mathcal{R}_1)$ $(\Sigma_2; \mathcal{R}_2), \cdots, (\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1 \leq i \leq m$. If for two integers i, j, $1 \leq i, j \leq m$, $\Sigma_i \neq \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \neq \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*. If we can list all systems of a Smarandache system $(\Sigma; \mathcal{R})$, then we get a *Smarandache multi-space* defined following.

Definition 2.2([5-7],[11]) *Let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$.*

The conception of Smarandache multi-space reflects the notion of the whole $\widetilde{\Sigma}$ is consisting of its parts $(\Sigma_i; \mathcal{R}_i), i \geq 1$ for a thing in philosophy. The laterality of human beings implies that one can only determines lateral feature of a thing in general. Such a typical example is the proverb of blind men with an elephant.

Fig. 3

In this proverb, there are 6 blind men were be asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They then entered into an endless argument and each of them insisted his view right. *All of you are right*! A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*, i.e., a Smarandache multi-space consisting of these 6 parts.

Usually, a man is blind for an unknowing thing and takes himself side as the dominant factor. That makes him knowing only the lateral features of a thing, not the whole. That is also the reason why one used to harmonious, not contradictory systems in classical mathematics. But the world is filled with contradictions. Being a wise man knowing the world, we need to find the whole, not just the parts. Thus the Smarandache multi-space is important for sciences.

Notice that a Smarandache multi-space $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ naturally inherits a combinatorial structure, i.e., a vertex-edge labeled topological graph defined following.

Definition 2.3(([5-7])) *Let $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ be a Smarandache multi-space with $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{m} \Sigma_i$ and $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$. Then a inherited graph $G\left[\widetilde{\Sigma}, \widetilde{R}\right]$ of $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ is a labeled topological graph defined by*

$$V\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}$$

with an edge labeling

$$l^E: (\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{S}, \widetilde{R}\right]\right) \to l^E(\Sigma_i, \Sigma_j) = \varpi\left(\Sigma_i \bigcap \Sigma_j\right),$$

where ϖ *is a characteristic on* $\Sigma_i \bigcap \Sigma_j$ *such that* $\Sigma_i \bigcap \Sigma_j$ *is isomorphic to* $\Sigma_k \bigcap \Sigma_l$ *if and only if* $\varpi(\Sigma_i \bigcap \Sigma_j) = \varpi(\Sigma_k \bigcap \Sigma_l)$ *for integers* $1 \le i, j, k, l \le m$.

For example, let $S_1 = \{a, b, c\}$, $S_2 = \{c, d, e\}$, $S_3 = \{a, c, e\}$ and $S_4 = \{d, e, f\}$. Then the multi-space $\widetilde{S} = \bigcup_{i=1}^{4} S_i = \{a, b, c, d, e, f\}$ with its labeled topological graph $G[\widetilde{S}]$ is shown in Fig.4.

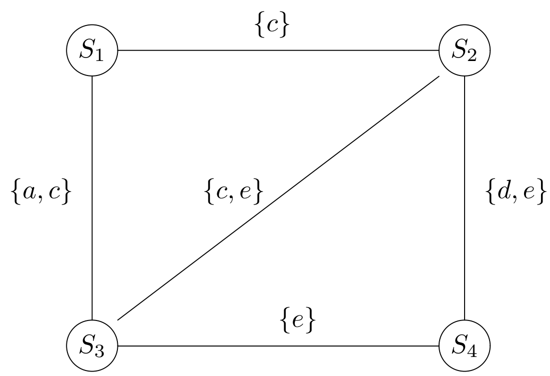


Fig.4

The labeled topological graph $G\left[\widetilde{\Sigma}, \widetilde{R}\right]$ reflects the notion that there exists linkage between things in philosophy. In fact, each edge $(\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right)$ is such a linkage with coupling $\varpi(\Sigma_i \bigcap \Sigma_j)$. For example, let $a = \{\text{tusk}\}$, $b = \{\text{nose}\}$, $c_1, c_2 = \{\text{ear}\}$, $d = \{\text{head}\}$, $e = \{\text{neck}\}$, $f = \{\text{belly}\}$, $g_1, g_2, g_3, g_4 = \{\text{leg}\}$, $h = \{\text{tail}\}$ for an elephant $\mathscr{C}$. Then its labeled topological graph is shown in Fig.5,

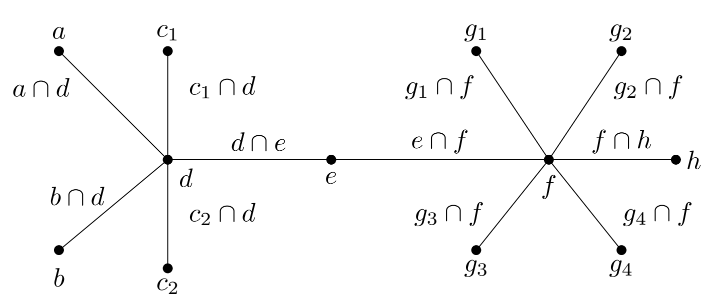


Fig.5

which implies that one can characterizes the geometrical behavior of an elephant combinatorially.

§3. Non-Solvable Systems of Ordinary Differential Equations

3.1 Linear Ordinary Differential Equations

For integers $m,\ n \geq 1$, let

$$\dot{X} = F_1(X),\ \dot{X} = F_2(X), \cdots, \dot{X} = F_m(X) \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i : \mathbf{R}^n \to \mathbf{R}^n$ such that $F_i(\overline{0}) = \overline{0}$, particularly, let

$$\dot{X} = A_1 X, \cdots, \dot{X} = A_k X, \cdots, \dot{X} = A_m X \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

Definition 3.1 *An ordinary differential equation system (DES_m^1) or $(LDES_m^1)$ are called non-solvable if there are no function $X(t)$ hold with (DES_m^1) or $(LDES_m^1)$ unless the constants.*

As we known, the general solution of the ith differential equation in $(LDES_m^1)$ is a linear space spanned by the elements in the solution basis

$$\mathscr{B}_i = \{\ \overline{\beta}_k(t)e^{\alpha_k t} \mid 1 \leq k \leq n\ \}$$

for integers $1 \leq i \leq m$, where

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n, \end{cases}$$

λ_i is the k_i-fold zero of the characteristic equation

$$\det(A - \lambda I_{n\times n}) = |A - \lambda I_{n\times n}| = 0$$

with $k_1 + k_2 + \cdots + k_s = n$ and $\overline{\beta}_i(t)$ is an n-dimensional vector consisting of polynomials in t with degree$\leq k_i - 1$.

In this case, we can simplify the labeled topological graph $G\left[\widetilde{\sum}, \widetilde{R}\right]$ replaced each $\sum_i$ by the solution basis $\mathscr{B}_i$ and $\sum_i \bigcap \sum_j$ by $\mathscr{B}_i \bigcap \mathscr{B}_j$ if $\mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset$ for integers $1 \leq i, j \leq m$, called the *basis graph* of $(LDES_m^1)$, denoted by $G[LDES_m^1]$. For example, let $m = 4$ and $\mathscr{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$, $\mathscr{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}$, $\mathscr{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}$ and $\mathscr{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}$, where $\lambda_i,\ 1 \leq i \leq 6$ are real numbers different two by two. Then $G[LDES_m^1]$ is shown in Fig.6.

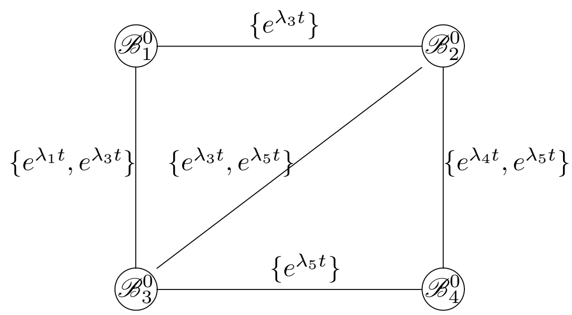


Fig.6

We get the following results.

Theorem 3.2([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *uniquely determines a basis graph* $G[LDES_m^1]$ *inherited in* $(LDES_m^1)$*. Conversely, every basis graph* G *uniquely determines a homogeneous differential equation system* $(LDES_m^1)$ *such that* $G[LDES_m^1] \simeq G$*.*

Such a basis graph $G[LDES_m^1]$ is called the G-solution of $(LDES_m^1)$. Theorem 3.2 implies that

Theorem 3.3([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *has a unique* G*-solution, and for every basis graph* H*, there is a unique linear homogeneous differential equation system* $(LDES_m^1)$ *with* G*-solution* H*.*

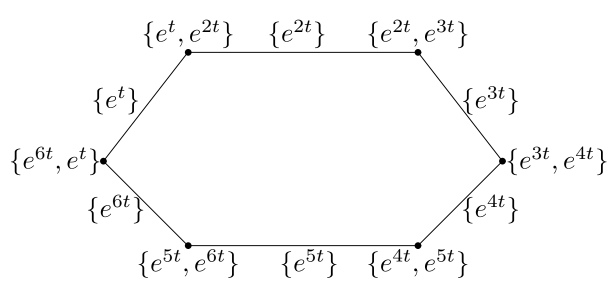


Fig.7 **A basis graph**

Example 3.4 Let (LDE_m^n) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1)-(6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ and its basis graph is shown in Fig.7.

3.2 Combinatorial Characteristics of Linear Differential Equations

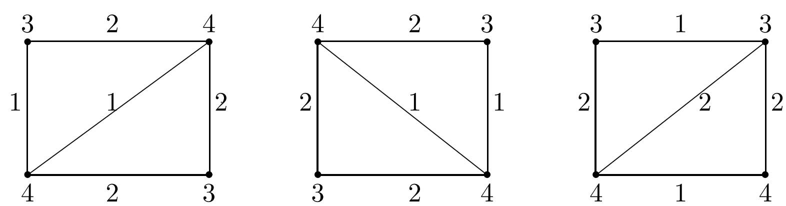
Definition 3.5 *Let* $(LDES^1_m)$, $(LDES^1_m)'$ *be two linear homogeneous differential equation systems with G-solutions* H, H'. *They are called combinatorially equivalent if there is an isomorphism* $\varphi : H \rightarrow H'$, *thus there is an isomorphism* $\varphi :\ H \rightarrow H'$ *of graph and labelings* θ, τ *on* H *and* H' *respectively such that* $\varphi\theta(x) = \tau\varphi(x)$ *for* $\forall x \in V(H)\bigcup E(H)$, *denoted by* $(LDES^1_m) \stackrel{\varphi}{\simeq} (LDES^1_m)'$.

We introduce the conception of *integral graph* for $(LDES^1_m)$ following.

Definition 3.6 *Let* G *be a simple graph. A vertex-edge labeled graph* $\theta :\ G \rightarrow \mathbb{Z}^+$ *is called integral if* $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ *for* $\forall uv \in E(G)$, *denoted by* G^{I_θ}.

Let $G_1^{I_\theta}$ *and* $G_2^{I_\tau}$ *be two integral labeled graphs. They are called identical if* $G_1 \stackrel{\varphi}{\simeq} G_2$ *and* $\theta(x) = \tau(\varphi(x))$ *for any graph isomorphism* φ *and* $\forall x \in V(G_1)\bigcup E(G_1)$, *denoted by* $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.8 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.



$G_1^{I_\theta}$ $\qquad$ $G_2^{I_\tau}$ $\qquad$ $G_3^{I_\sigma}$

Fig.8

Then we get a combinatorial characteristic for combinatorially equivalent $(LDES^1_m)$ following.

Theorem 3.5([10]) *Let* $(LDES^1_m)$, $(LDES^1_m)'$ *be two linear homogeneous differential equation systems with integral labeled graphs* H, H'. *Then* $(LDES^1_m) \stackrel{\varphi}{\simeq} (LDES^1_m)'$ *if and only if* $H = H'$.

3.3 Non-Linear Ordinary Differential Equations

If some functions $F_i(X)$, $1 \leq i \leq m$ are non-linear in (DES^1_m), we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\overline{0}$, i.e., if

$$F_i(X) = F_i'(\overline{0})X + R_i(X),$$

where $F_i'(\overline{0})$ is an $n \times n$ matrix, we replace the ith equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F_i'(\overline{0})X$$

in (DES_m^1). Whence, we get a uniquely linear differential equation system $(LDES_m^1)$ from (DES_m^1) and its basis graph $G[LDES_m^1]$. Such a basis graph $G[LDES_m^1]$ of linearized differential equation system (DES_m^1) is defined to be the *linearized basis graph* of (DES_m^1) and denoted by $G[DES_m^1]$. We can also apply G-solutions $G[DES_m^1]$ for characterizing the behavior of (DES_m^1).

§4. Cauchy Problem on Non-Solvable Partial Differential Equations

Let $(PDES_m)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \end{cases}$$

on a function $u(x_1, \cdots, x_n, t)$. Then its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0, \end{cases}$$

i.e., substitute $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ into $(PDES_m)$ for the term $u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}$, where $\alpha_i \geq 0$ for integers $1 \leq i \leq n$.

Definition 4.1 *A non-solvable $(PDES_m)$ is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

The following result characterizes the non-solvable partial differential equations of first order by applying the method of characteristic curves.

Theorem 4.2([11]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \leq k \leq m$$

is algebraically contradictory, in this case, there must be an integer k_0, $1 \leq k_0 \leq m$ *such that*

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer j_0, $1 \leq j_0 \leq n-1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

Particularly, we get conclusions following by Theorem 4.2.

Corollary 4.3 *Let*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

be an algebraically contradictory system of partial differential equations of first order. Then there are no values x_i^0, u_0, p_i^0, $1 \leq i \leq n$ *such that*

$$\begin{cases} F_1(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0, \\ F_2(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0. \end{cases}$$

Corollary 4.4 *A Cauchy problem* $(LPDES_m^C)$ *of quasilinear partial differential equations with initial values* $u|_{x_n=x_n^0} = u_0$ *is non-solvable if and only if the system* $(LPDES_m)$ *of partial differential equations is algebraically contradictory.*

Denoted by $\widehat{G}[PDES_m^C]$ such a graph $G[PDES_m^C]$ eradicated all labels. Particularly, replacing each label $S^{[i]}$ by $S_0^{[i]} = \{u_0^{[i]}\}$ and $S^{[i]} \bigcap S^{[j]}$ by $S_0^{[i]} \bigcap S_0^{[j]}$ for integers $1 \leq i, j \leq m$, we get a new labeled topological graph, denoted by $G_0[PDES_m^C]$. Clearly, $\widehat{G}[PDES_m^C] \simeq \widehat{G}_0[PDES_m^C]$.

Theorem 4.5([11]) *For any system* $(PDES_m^C)$ *of partial differential equations of first order,* $\widehat{G}[PDES_m^C]$ *is simple. Conversely, for any simple graph* G*, there is a system* $(PDES_m^C)$ *of partial differential equations of first order such that* $\widehat{G}[PDES_m^C] \simeq G$.

Particularly, if $(PDES_m^C)$ is linear, we can immediately find its underlying graph following.

Corollary 4.6 *Let* $(LPDES_m)$ *be a system of linear partial differential equations of first order with maximal contradictory classes* $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$ *on equations in* $(LPDES)$*. Then* $\widehat{G}[LPDES_m^C] \simeq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s)$*, i.e., an s-partite complete graph.*

Definition 4.7 *Let ($PDES_m^C$) be the Cauchy problem of a partial differential equation system of first order. Then the labeled topological graph $G[PDES_m^C]$ is called its topological graph solution, abbreviated to G-solution.*

Combining this definition with that of Theorems 4.5, the following conclusion is holden immediately.

Theorem 4.8([11]) *A Cauchy problem on system ($PDES_m$) of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ for the kth equation in ($PDES_m$), $1 \leq k \leq m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G-solvable, i.e., $G[PDES_m^C]$ is uniquely determined.

§5. Global Stability of Non-Solvable Differential Equations

Definition 5.1 *Let H be a spanning subgraph of $G[LDES_m^1]$ of systems $(LDES_m^1)$ with initial value $X_v(0)$. Then $G[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,*

$$\left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| = 0.$$

Similarly, a system ($PDES_m^C$) is sum-stable if for any number $\varepsilon > 0$ there exists $\delta_v > 0$, $v \in V(\widehat{G}[0])$ such that each $G(t)$-solution with $\left| {u'}_0^{[v]} - u_0^{[v]} \right| < \delta_v, \forall v \in V(\widehat{G}[0])$ exists for all $t \geq 0$ and with the inequality

$$\left| \sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if there exists a number $\beta_v > 0$, $v \in V(\widehat{G}[0])$ such that every $G'[t]$-solution with $\left| {u'}_0^{[v]} - u_0^{[v]} \right| < \beta_v, \forall v \in V(\widehat{G}[0])$ satisfies

$$\lim_{t \to \infty} \left| \sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| = 0,$$

then the $G[t]$-solution is called asymptotically stable, denoted by $G[t] \xrightarrow{\Sigma} G[0]$.

We get results on the global stability for G-solutions of $(LDES_m^1)$ and $(PDES_m^C)$.

Theorem 5.2([10]) *A zero G-solution of linear homogenous differential equation systems $(LDES_m^1)$ is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *hold for* $\forall v \in V(H)$.

Example 5.3 Let a G-solution of $(LDES_m^1)$ or (LDE_m^n) be the basis graph shown in Fig.4.1, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$. In fact, it is prod-stable on the triangle $v_1v_2v_3$.

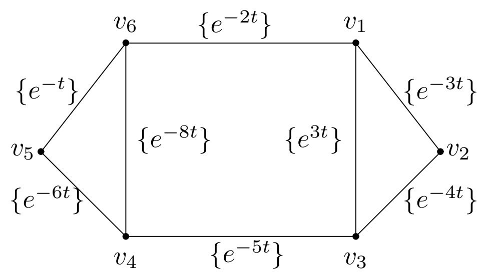


Fig.9

For partial differential equations, let the system $(PDES_m^C)$ be

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ &u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1}) \end{aligned}\right\} \quad 1 \le i \le m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for $1 \le i \le m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. Then we know that

Theorem 5.4([11]) *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in ($APDES_m$) for each integer $1 \le i \le m$. If $\sum\limits_{i=1}^{m} H_i(X) > 0$ and $\sum\limits_{i=1}^{m} \frac{\partial H_i}{\partial t} \le 0$ for $X \ne \sum\limits_{i=1}^{m} X_0^{[i]}$, then the system ($APDES_m$) is sum-stability, i.e., $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if $\sum\limits_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0$ for $X \ne \sum\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Sigma}{\to} G[0]$.*

§6. Applications

6.1 Application to Geometry

First, it is easily to shown that the G-solution of $(PDES_m^C)$ is nothing but a differentiable manifold.

Theorem 6.1([11]) *Let the Cauchy problem be ($PDES_m^C$). Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable n-manifold with atlas $\mathscr{A} = \{(U_v, \phi_v) | v \in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n-dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^n$ and ϕ_v the projection ϕ_v : $((x_1, x_2, \cdots, x_n), u(x_1, x_2, \cdots, x_n))) \to (x_1, x_2, \cdots, x_n)$ for $\forall (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.*

Theorems 4.8 and 6.1 enables one to find the following result for vector fields on differentiable manifolds by non-solvable system $(PDES_m^C)$.

Theorem 6.2([11]) *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^n$ underlying a connected graph defined by*

$$V(G) = \{U_i | 1 \leq i \leq m\}, \quad E(G) = \{(U_i, U_j) | U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathscr{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

More results on geometrical structure of manifold can be found in references [2-3] and [8].

6.2 Global Control of Infectious Diseases

Consider two cases of virus for infectious diseases:

Case 1 *There are m known virus $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$ and an person infected a virus $\mathscr{V}_i$ will never infects other viruses $\mathscr{V}_j$ for $j \neq i$.*

Case 2 *There are m varying $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ from a virus $\mathscr{V}$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$.*

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

$$\left\{\begin{array}{l} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{array}\right. \quad \left\{\begin{array}{l} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{array}\right. \quad \cdots \quad \left\{\begin{array}{l} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{array}\right. \qquad (DES_m^1)$$

and know the following result by Theorem 5.2 that

Conclusion 6.3([10]) *For m infectious viruses $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ in an area with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$, then they decline to 0 finally if $0 < S < \sum\limits_{i=1}^{m} h_i \Big/ \sum\limits_{i=1}^{m} k_i$, i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.*

6.3 Flows in Network

Let O be a node in N incident with m in-flows and 1 out-flow shown in Fig.10.

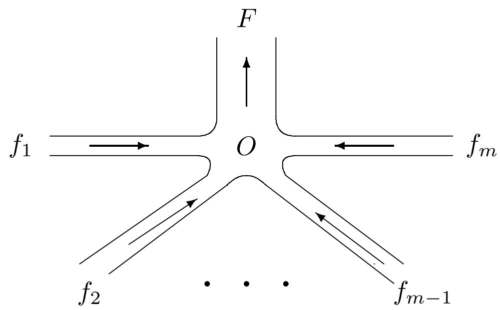


Fig.10

How can we characterize the behavior of flow F? Denote the rate, density of flow f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$ and that of F by $\rho^{[F]}$, respectively. Then we know that

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0, \ 1 \leq i \leq m.$$

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x, t_0)$ at time t_0. Replacing each $\rho^{[i]}$ by ρ in these flow equations of f_i, $1 \leq i \leq m$ enables one getting a non-solvable system $(PDES_m^C)$ of partial differential equations following.

$$\left.\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \phi_i(\rho)\dfrac{\partial \rho}{\partial x} = 0 \\ \rho\,|_{t=t_0} \ = \rho^{[i]}(x, t_0) \end{array}\right\} \ 1 \leq i \leq m.$$

Let $\rho_0^{[i]}$ be an equilibrium point of the ith equation, i.e., $\phi_i(\rho_0^{[i]})\dfrac{\partial \rho_0^{[i]}}{\partial x} = 0$. Applying Theorem 5.4, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \geq 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then we know that the flow F is stable and furthermore, if

$$\sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then it is also asymptotically stable.

References

[1] R.Abraham and Marsden, *Foundation of Mechanics*(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[3] Linfan Mao, Combinatorial fields - an introduction, *International J. Math. Combin.*, Vol.3, 2009, 01-22.

[4] Linfan Mao, Relativity in combinatorial gravitational fields, *Progress in Physics*, Vol.3(2010), 39-50.

[5] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[6] Linfan Mao, *Smarandache Multi-Space Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[7] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[8] Linfan Mao, Graph structure of manifolds with listing, *International J.Contemp. Math. Science*, Vol.5, 2011, No.2, 71-85.

[9] Linfan Mao, Non-solvable spaces of linear equation systems, *International J. Math. Combin.*, Vol.2 (2012), 9-23.

[10] Linfan Mao, Global stability of non-solvable ordinary differential equations with applications, *International J.Math. Combin.*, Vol.1 (2013), 1-37.

[11] Linfan Mao, Non-solvable partial differential equations with applications, Submitted, 2013.

[12] M.Renardy, R.C.Rogers, *An Introduction to Partial Differential Equations*(2th Edition), Springer-Verlag, Berlin Heidelberg, 2004.

[13] F.Smarandache, Mixed noneuclidean geometries, *Eprint arXiv: math/0010119*, 10/2000.

[14] F.Smarandache, *A Unifying Field in Logics–Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[15] Wolfgang Walter, *Ordinary Differential Equations*, Springer-Verlag New York, Inc., 1998.

Some Properties of Birings

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Abstract: Let R be any ring and let $S = R_1 \cup R_2$ be the union of any two subrings of R. Since in general S is not a subring of R but R_1 and R_2 are algebraic structures on their own under the binary operations inherited from the parent ring R, S is recognized as a bialgebraic structure and it is called a biring. The purpose of this paper is to present some properties of such bialgebraic structures.

Key Words: Biring, bi-subring, bi-ideal, bi-field and bidomain

AMS(2010): 08A05, 13A05, 13A15

§1. Introduction

Generally speaking, the unions of any two subgroups of a group, subgroupoids of a groupoid, subsemigroups of a semigroup, submonoids of a monoid, subloops of a loop, subsemirings of a semiring, subfields of a field and subspaces of a vector space do not form any nice algebraic structures other than ordinary sets. Similarly, if S_1 and S_2 are any two subrings of a ring R, I_1 and I_2 any two ideals of R, the unions $S = S_1 \cup S_2$ and $I = I_1 \cup I_2$ generally are not subrings and ideals of R, respectively [2]. However, the concept of bialgebraic structures recently introduced by Vasantha Kandasamy [9] recognises the union $S = S_1 \cup S_2$ as a biring and $I = I_1 \cup I_2$ as a bi-ideal. One of the major advantages of bialgebraic structures is the exhibition of distinct algebraic properties totally different from those inherited from the parent structures. The concept of birings was introduced and studied in [9]. Other related bialgebraic structures introduced in [9] included binear-rings, bisemi-rings, biseminear-rings and group birings. Agboola and Akinola in [1] studied bicoset of a bivector space. Also, we refer the readers to [3-7]. In this paper, we will present and study some properties of birings.

§2. Definitions and Elementary Properties of Birings

Definition 2.1 *Let R_1 and R_2 be any two proper subsets of a non-empty set R. Then, $R = R_1 \cup R_2$ is said to be a biring if the following conditions hold:*

(1) R_1 *is a ring;*

(2) R_2 *is a ring.*

Definition 2.2 *A biring $R = R_1 \cup R_2$ is said to be commutative if R_1 and R_2 are commutative rings. $R = R_1 \cup R_2$ is said to be a non-commutative biring if R_1 is non-commutative or R_2 is non-commutative.*

Definition 2.3 *A biring $R = R_1 \cup R_2$ is said to have a zero element if R_1 and R_2 have different zero elements. The zero element 0 is written $0_1 \cup 0_2$ (notation is not set theoretic union) where $0_i, i = 1, 2$ are the zero elements of R_i. If R_1 and R_2 have the same zero element, we say that the biring $R = R_1 \cup R_2$ has a mono-zero element.*

Definition 2.4 *A biring $R = R_1 \cup R_2$ is said to have a unit if R_1 and R_2 have different units. The unit element u is written $u_1 \cup u_2$, where $u_i, i = 1, 2$ are the units of R_i. If R_1 and R_2 have the same unit, we say that the biring $R = R_1 \cup R_2$ has a mono-unit.*

Definition 2.5 *A biring $R = R_1 \cup R_2$ is said to be finite if it has a finite number of elements. Otherwise, R is said to be an infinite biring. If R is finite, the order of R is denoted by $o(R)$.*

Example 1 Let $R = \{0, 2, 4, 6, 7, 8, 10, 12\}$ be a subset of $\mathcal{Z}_{14}$. It is clear that $(R, +, \cdot)$ is not a ring but then, $R_1 = \{0, 7\}$ and $R_2 = \{0, 2, 4, 6, 8, 10, 12\}$ are rings so that $R = R_1 \cup R_2$ is a finite commutative biring.

Definition 2.6 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset S of R is said to be a sub-biring of R if $S = S_1 \cup S_2$ and S itself is a biring and $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.*

Theorem 2.7 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset $S = S_1 \cup S_2$ of R is a sub-biring of R if and only if $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$ are subrings of R_1 and R_2, respectively.*

Definition 2.8 *Let $R = R_1 \cup R_2$ be a biring and let x be a non-zero element of R. Then,*

(1) *x is a zero-divisor in R if there exists a non-zero element y in R such that $xy = 0$;*

(2) *x is an idempotent in R if $x^2 = x$;*

(3) *x is nilpotent in R if $x^n = 0$ for some $n > 0$.*

Example 2 Consider the biring $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \{0, 2, 4, 6\}$ a subset of $\mathcal{Z}_8$.

(1) If $S_1 = 4\mathcal{Z}$ and $S_2 = \{0, 4\}$, then S_1 is a subring of R_1 and S_2 is a subring of R_2. Thus, $S = S_1 \cup S_2$ is a bi-subring of R since $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.

(2) If $S_1 = 3\mathcal{Z}$ and $S_2 = \{0, 4\}$, then $S = S_1 \cup S_2$ is a biring but not a bi-subring of R because $S_1 \neq S \cap R_1$ and $S_2 \neq S \cap R_2$. This can only happen in a biring structure.

Theorem 2.9 *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be sub-birings of R and S, respectively. Then,*

(1) *$R \times S = (R_1 \times S_1) \cup (R_2 \times S_2)$ is a biring;*

(2) *$I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ is a sub-biring of $R \times S$.*

Definition 2.10 *Let $R = R_1 \cup R_2$ be a biring and let I be a non-empty subset of R.*

(1) *I is a right bi-ideal of R if $I = I_1 \cup I_2$, where I_1 is a right ideal of R_1 and I_2 is a right ideal of R_2;*

(2) *I is a left bi-ideal of R if $I = I_1 \cup I_2$, where I_1 is a left ideal of R_1 and I_2 is a left ideal of R_2;*

(3) *$I = I_1 \cup I_2$ is a bi-ideal of R if I_1 is an ideal of R_1 and I_2 is an ideal of R_2.*

Definition 2.11 *Let $R = R_1 \cup R_2$ be a biring and let I be a non-empty subset of R. Then, $I = I_1 \cup I_2$ is a mixed bi-ideal of R if I_1 is a right (left) ideal of R_1 and I_2 is a left (right) ideal of R_2.*

Theorem 2.12 *Let $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ and $K = K_1 \cup K_2$ be left (right) bi-ideals of a biring $R = R_1 \cup R_2$. Then,*

(1) *$IJ = (I_1J_1) \cup (I_2J_2)$ is a left(right) bi-ideal of R;*

(2) *$I \cap J = (I_1 \cap J_1) \cup (I_2 \cap J_2)$ is a left(right) bi-ideal of R;*

(3) *$I + J = (I_1 + J_1) \cup (I_2 + J_2)$ is a left(right) bi-ideal of R;*

(4) *$I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ is a left(right) bi-ideal of R;*

(5) *$(IJ)K = \Big((I_1J_1)K_1\Big) \cup \Big((I_2J_2)K_2\Big) = I(JK) = \Big(I_1(J_1K_1)\Big) \cup \Big(I_2(J_2K_2)\Big)$;*

(6) *$I(J+K) = \Big(I_1(J_1+K_1)\Big) \cup \Big(I_2(J_2+K_2)\Big) = IJ+IK = (I_1J_1+I_1K_1) \cup (I_2J_2+I_2K_2)$;*

(7) *$(J+K)I = \Big((J_1+K_1)I_1\Big) \cup \Big((J_2+K_2)I_2\Big) = JI+KI = (J_1I_1+K_1I_1) \cup (J_2I_2+K_2I_2)$.*

Example 3 Let R be the collection of all 2×2 upper triangular and lower triangular matrices over a field F and let

$$R_1 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in F \right\},$$

$$R_2 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\},$$

$$I_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in F \right\},$$

$$I_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in F \right\}.$$

Then, $R = R_1 \cup R_2$ is a non-commutative biring with a mono-unit $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I = I_1 \cup I_2$ is a right bi-ideal of $R = R_1 \cup R_2$.

Definition 2.13 *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings. The mapping $\phi : R \to S$ is called a biring homomorphism if $\phi = \phi_1 \cup \phi_2$ and $\phi_1 : R_1 \to S_1$ and $\phi_2 : R_2 \to S_2$ are ring homomorphisms. If $\phi_1 : R_1 \to S_1$ and $\phi_2 : R_2 \to S_2$ are ring isomorphisms, then $\phi = \phi_1 \cup \phi_2$ is a biring isomorphism and we write $R = R_1 \cup R_2 \cong S = S_1 \cup S_2$. The image of ϕ denoted by $Im\phi = Im\phi_1 \cup Im\phi_2 = \{y_1 \in S_1, y_2 \in S_2 : y_1 = \phi_1(x_1), y_2 = \phi_2(x_2)$ for some $x_1 \in R_1, x_2 \in R_2\}$. The* *kernel of ϕ denoted by*

$$Ker\phi = Ker\phi_1 \cup Ker\phi_2 = \{a_1 \in R_1, a_2 \in R_2 : \phi_1(a_1) = 0 \text{ and } \phi_2(a_2) = 0\}.$$

Theorem 2.14 *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings and let $\phi = \phi_1 \cup \phi_2 : R \to S$ be a biring homomorphism. Then,*

(1) *$Im\phi$ is a sub-biring of the biring S;*

(2) *$Ker\phi$ is a bi-ideal of the biring R;*

(3) *$Ker\phi = \{0\}$ if and only if $\phi_i, i = 1, 2$ are injective.*

Proof (1) It is clear that $Im\phi = Im\phi_1 \cup Im\phi_2$, where $\phi_1 : R_1 \to S_1$ and $\phi_2 : R_2 \to S_2$ are ring homomorphisms, is not an empty set. Since $Im\phi_1$ is a subring of S_1 and $Im\phi_2$ is a subring of S_2, it follows that $Im\phi = Im\phi_1 \cup Im\phi_2$ is a biring. Lastly, it can easily be shown that $Im\phi \cap S_1 = Im\phi_1$,$Im\phi \cap S_2 = Im\phi_2$ and consequently, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a sub-biring of the biring $S = S_1 \cup S_2$.

(2) The proof is similar to (1).

(3) It is clear. □

Let $I = I_1 \cup I_2$ be a left bi-ideal of a biring $R = R_1 \cup R_2$. We know that R_1/I_1 and R_2/I_2 are factor rings and therefore $(R_1/I_1) \cup (R_2/I_2)$ is a biring called *factor-biring*. Since $\phi_1 : R_1 \to R_1/I_1$ and $\phi_2 : R_2 \to R_2/I_2$ are natural homomorphisms with kernels I_1 and I_2, respectively, it follows that $\phi_1 \cup \phi_2 = \phi : R \to R/I$ is a natural biring homomorphism whose kernel is $Ker\phi = I_1 \cup I_2$.

Theorem 2.15(First Isomorphism Theorem) *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings and let $\phi_1 \cup \phi_2 = \phi : R \to S$ be a biring homomorphism with kernel $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. Then, $R/K \cong Im\phi$.*

Proof Suppose that $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ are birings and suppose that $\phi_1 \cup \phi_2 = \phi : R \to S$ is a biring homomorphism with kernel $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. Then, K is a bi-ideal of R, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a bi-subring of S and $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2)$ is a biring. From the classical rings (first isomorphism theorem), we have $R_i/Ker\phi_i \cong Im\phi_i, i = 1, 2$ and therefore, $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2$. □

Theorem 2.16(Second Isomorphism Theorem) *Let $R = R_1 \cup R_2$ be a biring. If $S = S_1 \cup S_2$ is a sub-biring of R and $I = I_1 \cup I_2$ is a bi-ideal of R, then*

(1) *$S + I$ is a sub-biring of R;*

(2) *I is a bi-ideal of $S + I$;*

(3) *$S \cap I$ is a bi-ideal of S;*

(4) *$(S + I)/I \cong S/(S \cap I)$.*

Proof Suppose that $R = R_1 \cup R_2$ is a biring, $S = S_1 \cup S_2$ a sub-biring and $I = I_1 \cup I_2$ a bi-ideal of R.

(1) $S + I = (S_1 + I_1) \cup (S_2 + I_2)$ is a biring since $S_i + I_i$ are subrings of R_i, where $i = 1, 2$.

Now, $R_1 \cap (S+I) = \Big(R_1 \cap (S_1 + I_1)\Big) \cup \Big(R_1 \cap (S_2 + I_2)\Big) = S_1 + I_1$. Similarly, we have $R_2 \cap (S+I) = S_2 + I_2$. Thus, $S+I$ is a sub-biring of R.

(2) and (3) are clear.

(4) It is clear that $(S+I)/I = \Big((S_1+I_1)/I_1\Big) \cup \Big((S_2+I_2)/I_2\Big)$ is a biring since $(S_1+I_1)/I_1$ and $(S_2+I_2)/I_2$ are rings. Similarly, $S/(S\cap I) = \Big(S_1/(S_1\cap I_1)\Big) \cup \Big(S_2/(S_2\cap I_2)\Big)$ is a biring. Consider the mapping $\phi = \phi_1 \cup \phi_2 : S_1 \cup S_2 \to \Big((S_1+I_1)/I_1\Big) \cup \Big((S_2+I_2)/I_2\Big)$. It is clear that ϕ is a biring homomorphism since $\phi_i : S_i \to (S_i + I_i)/I_i, i = 1,2$ are ring homomorphisms. Also, since $Ker\phi_i = S_i \cap I_i, i = 1,2$, it follows that $Ker\phi = (S_1\cap I_1)\cup(S_2\cap I_2)$. The required result follows from the first isomorphism theorem. □

Theorem 2.17(Third Isomorphism Theorem) *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be two bi-ideals of R such that $J_i \subseteq I_i, i = 1,2$. Then,*

(1) *I/J is a bi-ideal of R/J;*

(2) *$R/I \cong (R/J)/(I/J)$.*

Proof Suppose that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ are two bi-ideals of the biring $R = R_1 \cup R_2$ such that $J_i \subseteq I_i, i = 1,2$.

(1) It is clear that $R/J = (R_1/J_1)\cup(R_2/J_2)$ and $I/J = (I_1/J_1)\cup(I_2/J_2)$ are birings. Now, $(R_1/J_1) \cap \Big((I_1/J_1)\cup(I_2/J_2)\Big) = \Big((R_1/J_1)\cap(I_1/J_1)\Big) \cup \Big((R_1/J_1)\cap(I_2/J_2)\Big) = I_1/J_1$ (since $J_i \subseteq I_i \subseteq R_i, i = 1,2$). Similarly, $(R_2/J_2) \cap \Big((I_1/J_1)\cup(I_2/J_2)\Big) = I_2/J_2$. Consequently, I/J is a sub-biring of R/J and in fact a bi-ideal.

(2) Let us consider the mapping $\phi = \phi_1 \cup \phi_2 : (R_1/J_1)\cup(R_2/J_2) \to (R_1/I_1)\cup(R_2/I_2)$. Since $\phi_i : R_i/J_i \to R_i/I_i, i = 1,2$ are ring homomorphisms with $Ker\phi_i = I_i/J_i$, it follows that $\phi = \phi_1 \cup \phi_2$ is a biring homomorphism and $Ker\phi = Ker\phi_1 \cup Ker\phi_2 = (I_1/J_1)\cup(I_2/J_2)$. Applying the first isomorphism theorem, we have $\Big((R_1/J_1)/(I_1/J_1)\Big) \cup \Big((R_2/J_2)/(I_2/J_2)\Big) \cong (R_1/I_1) \cup /(R_2/I_2)$. □

Definition 2.18 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then,*

(1) *I is said to be a principal bi-ideal of R if I_1 is a principal ideal of R_1 and I_2 is a principal ideal of R_2;*

(2) *I is said to be a maximal (minimal) bi-ideal of R if I_1 is a maximal (minimal) ideal of R_1 and I_2 is a maximal (minimal) ideal of R_2;*

(3) *I is said to be a primary bi-ideal of R if I_1 is a primary ideal of R_1 and I_2 is a primary ideal of R_2;*

(4) *I is said to be a prime bi-ideal of R if I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2.*

Example 4 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = \mathcal{R}[x]$, the ring of polynomials over $\mathcal{R}$. Let $I_1 = (2)$ and $I_2 = (x^2+1)$. Then, $I = I_1 \cup I_2$ is a principal bi-ideal of R.

Definition 2.19 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then, I is said to be a quasi maximal (minimal) bi-ideal of R if I_1 or I_2 is a maximal (minimal) ideal.*

Definition 2.20 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a simple biring if R has no non-trivial bi-ideals.*

Theorem 2.21 *Let $\phi = \phi_1 \cup \phi_2 : R \to S$ be a biring homomorphism. If $J = J_1 \cup J_2$ is a prime bi-ideal of S, then $\phi^{-1}(J)$ is a prime bi-ideal of R.*

Proof Suppose that $J = J_1 \cup J_2$ is a prime bi-ideal of S. Then, $J_i, i = 1, 2$ are prime ideals of S_i. Since $\phi^{-1}(J_i), i = 1, 2$ are prime ideals of R_i, we have $I = \phi^{-1}(J_1) \cup \phi^{-1}(J_2)$ to be a prime bi-ideal of R. □

Definition 2.22 *Let $R = R_1 \cup R_2$ be a commutative biring. Then,*

(1) *R is said to be a bidomain if R_1 and R_2 are integral domains;*

(2) *R is said to be a pseudo bidomain if R_1 and R_2 are integral domains but R has zero divisors;*

(3) *R is said to be a bifield if R_1 and R_2 are fields. If R is finite, we call R a finite bifield. R is said to be a bifield of finite characteristic if the characteristic of both R_1 and R_2 are finite. We call R a bifield of characteristic zero if the characteristic of both R_1 and R_2 is zero. No characteristic is associated with R if R_1 or R_2 is a field of zero characteristic and one of R_1 or R_2 is of some finite characteristic.*

Definition 2.23 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a bidivision ring if R is non-commutative and has no zero-divisors that is R_1 and R_2 are division rings.*

Example 5 (1) Let $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \mathcal{R}[x]$ the ring of integers and the ring of polynomials over $\mathcal{R}$, respectively. Since R_1 and R_2 are integral domains, it follows that R is a bidomain.

(2) The biring $R = R_1 \cup R_2$ of Example **??** is a pseudo bidomain.

(3) Let $F = F_1 \cup F_2$ where $F_1 = \mathcal{Q}(\sqrt{p_1})$, $F_2 = \mathcal{Q}(\sqrt{p_2})$ where $p_i, i = 1, 2$ are different primes. Since F_1 and F_2 are fields of zero characteristics, it follows that F is a bi-field of zero characteristic.

Theorem 2.24 *Let $R = R_1 \cup R_2$ be a biring. Then, R is a bidomain if and only if the zero bi-ideal $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.*

Proof Suppose that R is a bidomain. Then, $R_i, i = 1, 2$ are integral domains. Since the zero ideals (0_i) in R_i are prime, it follows that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.

Conversely, suppose that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal. Then, $(0_i), i = 1, 2$ are prime ideals in R_i and hence $R_i, i = 1, 2$ are integral domains. Therefore, $R = R_1 \cup R_2$ is a bidomain. □

Theorem 2.25 *Let $F = F_1 \cup F_2$ be a bi-field. Then, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain.*

Proof Since F_1 and F_2 are fields which are integeral domains, it follows that $F_1[x]$ and $F_2[x]$ are integral domains and consequently, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain. □

§3. Further Properties of Birings

Except otherwise stated in this section, all birings are assumed to be commutative with zero and unit elements.

Theorem 3.1 *Let R be any ring and let S_1 and S_2 be any two distinct subrings of R. Then, $S = S_1 \cup S_2$ is a biring.*

Proof Suppose that S_1 and S_2 are two distinct subrings of R. Then, $S_1 \nsubseteq S_2$ or $S_2 \nsubseteq S_1$ but $S_1 \cap S_2 \neq \emptyset$. Since S_1 and S_2 are rings under the same operations inherited from R, it follows that $S = S_1 \cup S_2$ is a biring. □

Corollary 3.2 *Let R_1 and R_2 be any two unrelated rings that is $R_1 \nsubseteq R_2$ or $R_2 \nsubseteq R_1$ but $R_1 \cap R_2 \neq \emptyset$. Then, $R = R_1 \cup R_2$ is a biring.*

Example 6 (1) Let $R = \mathcal{Z}$ and let $S_1 = 2\mathcal{Z}$, $S_2 = 3\mathcal{Z}$. Then, $S = S_1 \cup S_2$ is a biring.

(2) Let $R_1 = \mathcal{Z}_2$ and $R_2 = \mathcal{Z}_3$ be rings of integers modulo 2 and 3, respectively. Then, $R = R_1 \cup R_2$ is a biring.

Example 7 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = C[0,1]$, the ring of all real-valued continuous functions on $[0,1]$. Let $I_1 = (p)$, where p is a prime number and let $I_2 = \{f(x) \in R_2 : f(x) = 0\}$. It is clear that I_1 and I_2 are maximal ideals of R_1 and R_2, respectively. Hence, $I = I_1 \cup I_2$ is a maximal bi-ideal of R.

Theorem 3.3 *Let $R = \{0, a, b\}$ be a set under addition and multiplication modulo 2. Then, R is a biring if and only if a and b $(a \neq b)$ are idempotent (nilpotent) in R.*

Proof Suppose that $R = \{0, a, b\}$ is a set under addition and multiplication modulo 2 and suppose that a and b are idempotent (nilpotent) in R. Let $R_1 = \{0, a\}$ and $R_2 = \{0, b\}$, where $a \neq b$. Then, R_1 and R_2 are rings and hence $R = R_1 \cup R_2$ is a biring. The proof of the converse is clear. □

Corollary 3.4 *There exists a biring of order 3.*

Theorem 3.5 *Let $R = R_1 \cup R_2$ be a finite bidomain. Then, R is a bi-field.*

Proof Suppose that $R = R_1 \cup R_2$ is a finite bidomain. Then, each $R_i, i = 1, 2$ is a finite integral domain which is a field. Therefore, R is a bifield. □

Theorem 3.6 *Let $R = R_1 \cup R_2$ be a bi-field. Then, R is a bidomain.*

Proof Suppose that $R = R_1 \cup R_2$ is a bi-field. Then, each $R_i, i = 1, 2$ is a field which is an integral domain. The required result follows from the definition of a bidomain. □

Remark 1 Every finite bidivision ring is a bi-field.

Indeed, suppose that $R = R_1 \cup R_2$ is a finite bidivision ring. Then, each $R_i, i = 1, 2$ is a finite division ring which is a field. Consequently, R is a bi-field.

Theorem 3.7 *Every biring in general need not have bi-ideals.*

Proof Suppose that $R = R_1 \cup R_2$ is a biring and suppose that $I_i, i = 1, 2$ are ideals of R_i. If $I = I_1 \cup I_2$ is such that $I_i \neq I \cap R_i$, where $i = 1, 2$, then I cannot be a bi-ideal of R. $\square$

Corollary 3.8 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$, where $I_i, i = 1, 2$ are ideals of R_i. Then, I is a bi-ideal of R if and only if $I_i = I \cap R_i$, where $i = 1, 2$.*

Corollary 3.9 *A biring $R = R_1 \cup R_2$ may not have a maximal bi-ideal.*

Theorem 3.10 *Let $R = R_1 \cup R_2$ be a biring and let $M = M_1 \cup M_2$ be a bi-ideal of R. Then, R/M is a bi-field if and only if M is a maximal bi-ideal.*

Proof Suppose that M is a maximal bi-ideal of R. Then, each M_i, $i = 1, 2$ is a maximal ideal in R_i, $i = 1, 2$ and consequently, each R_i/I_i is a field and therefore R/M is a bi-field.

Conversely, suppose that R/M is a bi-field. Then, each R_i/M_i, $i = 1, 2$ is a field so that each M_i, $i = 1, 2$ is a maximal ideal in R_i. Hence, $M = I_1 \cup I_2$ is a maximal bi-ideal. $\square$

Theorem 3.11 *Let $R = R_1 \cup R_2$ be a biring and let $P = P_1 \cup P_2$ be a bi-ideal of R. Then, R/P is a bidomain if and only if P is a prime bi-ideal.*

Proof Suppose that P is a prime bi-ideal of R. Then, each P_i, $i = 1, 2$ is a prime ideal in R_i, $i = 1, 2$ and so, each R_i/P_i is an integral domain and therefore R/P is a bidomain.

Conversely, suppose that R/P is a bidomain. Then, each R_i/P_i, $i = 1, 2$ is an integral domain and therefore each P_i, $i = 1, 2$ is a prime ideal in R_i. Hence, $P = P_1 \cup P_2$ is a prime bi-ideal. $\square$

Theorem 3.12 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. If I is maximal, then I is prime.*

Proof Suppose that I is maximal. Then, $I_i, i = 1, 2$ are maximal ideals of R_i so that R_i/I_i are fields which are integral domains. Thus, $R/I = (R_1/I_1) \cup (R_2/I_2)$ is a bidomain and by Theorem 3.11, $I = I_1 \cup I_2$ is a prime bi-ideal. $\square$

Theorem 3.13 *Let $\phi : R \to S$ be a biring homomorphism from a biring $R = R_1 \cup R_2$ onto a biring $S = S_1 \cup S_2$ and let $K = Ker\phi_1 \cup Ker\phi_2$ be the kernel of ϕ.*

(1) *If S is a bi-field, then K is a maximal bi-ideal of R;*

(2) *If S is a bidomain, then K is a prime bi-ideal of R.*

Proof By Theorem 2.7, we have $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2 = S_1 \cup S_2 = S$. The required results follow by applying Theorems 3.10 and 3.11. $\square$

Definition 3.14 *Let $R = R_1 \cup R_2$ be a biring and let $N(R)$ be the set of nilpotent elements of R. Then, $N(R)$ is called the bi-nilradical of R if $N(R) = N(R_1) \cup N(R_2)$, where $N(R_i)$,*

$i = 1, 2$ are the nilradicals of R_i.

Theorem 3.15 *Let $R = R_1 \cup R_2$ be a biring. Then, $N(R)$ is a bi-ideal of R.*

Proof $N(R)$ is non-empty since $0_1 \in N(R_1)$ and $0_2 \in N(R_2)$. Now, if $x = x_1 \cup x_2, y_1 \cup y_2 \in N(R)$ and $r = r_1 \cup r_2 \in R$ where $x_i, y_i \in N(R_i), r_i \in R_i, i = 1, 2$, then it follows that $x - y, xr \in N(R)$. Lastly, $R_1 \cap \Big(N(R_1) \cup N(R_2)\Big) = \Big(R_1 \cap N(R_1)\Big) \cup \Big(R_1 \cap N(R_2)\Big) = N(R_1)$. Similarly, we have $R_2 \cap \Big(N(R_1) \cup N(R_2)\Big) = N(R_2)$. Hence, $N(R)$ is a bi-ideal. □

Definition 3.16 *Let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be any two bi-ideals of a biring $R = R_1 \cup R_2$. The set $(I : J)$ is called a bi-ideal quotient of I and J if $(I : J) = (I_1 : J_1) \cup (I_2 : J_2)$, where $(I_i : J_i), i = 1, 2$ are ideal quotients of I_i and J_i. If $I = (0) = (0_1) \cup (0_2)$, a zero bi-ideal, then $\Big((0) : J\Big) = \Big((0_1) : J_1\Big) \cup \Big((0_2) : J_2\Big)$ which is called a bi-annihilator of J denoted by $Ann(J)$. If $0 \neq x \in R_1$ and $0 \neq y \in R_2$, then $Z(R_1) = \bigcup_x Ann(x)$ and $Z(R_2) = \bigcup_y Ann(y)$, where $Z(R_i), i = 1, 2$ are the sets of zero-divisors of R_i.*

Theorem 3.17 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be any two bi-ideals of R. Then, $(I : J)$ is a bi-ideal of R.*

Proof For $0 = 0_1 \cup 0_2 \in R$, we have $0_1 \in (I_1 : J_1)$ and $0_2 \in (I_2 : J_2)$ so that $(I : J) \neq \emptyset$. If $x = x_1 \cup x_2, y = y_1 \cup y_2 \in (I : J)$ and $r = r_1 \cup r_2 \in R$, then $x - y, xr \in (I : J)$ since $(I_i : J_i), i = 1, 2$ are ideals of R_i. It can be shown that $R_1 \cap \Big((I_1 : J_1) \cup (I_2 : J_2)\Big) = (I_1 : J_1)$ and $R_2 \cap \Big((I_1 : J_1) \cup (I_2 : J_2)\Big) = (I_2 : J_2)$. Accordingly, $(I : J)$ is a bi-ideal of R. □

Example 8 Under addition and multplication modulo 6, consider the biring $R = \{0, 2, 3, 4\}$, where $R_1 = \{0, 3\}$ and $R_2 = \{0, 2, 4\}$. It is clear that $Z(R) \neq Z(R_1) \cup Z(R_2)$. Hence, for $0 \neq z = x \cup y \in R$, $0 \neq x \in R_1$ and $0 \neq y \in R_2$, we have

$$\bigcup_{z = x \cup y} Ann(z) \neq \left(\bigcup_x Ann(x)\right) \cup \left(\bigcup_y Ann(y)\right).$$

Definition 3.18 *Let $I = I_1 \cup I_2$ be any bi-ideal of a biring $R = R_1 \cup R_2$. The set r(I) is called a bi-radical of I if $r(I) = r(I_1) \cup r(I_2)$, where $r(I_i), i = 1, 2$ are radicals of I_i. If $I = (0) = (0_1) \cup (0_2)$, then $r(I) = N(R)$.*

Definition 3.19 *If $R = R_1 \cup R_2$ is a biring and $I = I_1 \cup I_2$ is a bi-ideal of R, then $r(I)$ is a bi-ideal.*

References

[1] Agboola A.A.A. and Akinola L.S., On the Bicoset of a Bivector Space, Int. J. Math. Comb., 4(2009), 1-8.

[2] Davvaz B., *Fundamentals of Ring Theory*, Yazd University, Yazd , 2006.

[3] Jun Young Bae, Biideals in BCK/BCI-bialgebras, *Kyungpook Math. J.*, 48(2008), 577-584.

[4] Jun Young Bae and Lee Kyoung Ja., Ideal theory in BCK/BCI-bialgebras, *Adv. Stud. Contemp. Math.* (Kyungshang) 20(2010), 481-488.

[5] Jun Young Bae, Öztürk, Mehmet Ali and Roh Eun Hwan, BCK/BCI-bialgebras, *Sci. Math. Jpn.*, 64(2006), 595-600.

[6] Maggu P.L., On Introduction of Bigroup Concept with its Applications in Industry, *Pure and App. Math Sci.*, 39(1994), 171-173.

[7] Maggu P.L. and Rajeev K., On Sub-bigroup and its Applications in Industry, *Pure and App. Math Sci.*, 43(1996), 85-88.

[8] Smarandache F., Special Algebraic Structures, in Collected Papers, Abaddaba, Oradea, 3(2000), 78-81.

[9] Vasantha Kandasamy W.B., *Bialgebraic Structures and Smarandache Bialgebraic Structures*, American Research Press, Rehoboth, 2003. http://www.gallup.unm.edu/~ smarandache /eBooks-otherformats.htm.

[10] Vasantha Kandasamy W.B., Bivector Spaces, *U. Sci. Phy. Sci.*, 11(1999), 186-190.

Surface Embeddability of Graphs via Tree-travels

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Abstract: This paper provides a characterization for surface embeddability of a graph with any given orientable and nonorientable genus not zero via a method discovered by the author thirty years ago.

Key Words: Surface, graph,Smarandache λ^S-drawing, embeddability, tree-travel.

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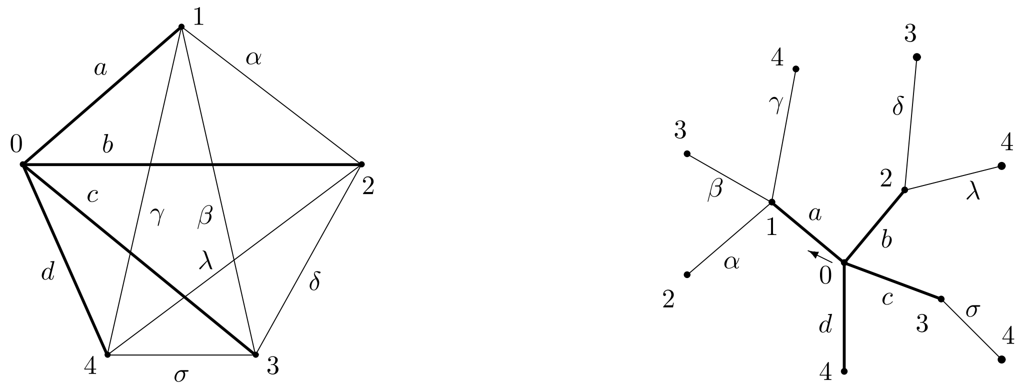
§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S. Along the Kurotowski research line for determining the embeddability of a graph on a surface of genus not zero, the number of forbidden minors is greater than a hundred even for the projective plane, a nonorientable surface of genus 1 in [1].

However, this paper extends the results in [3] which is on the basis of the method established in [3-4] by the author himself for dealing with the problem on the maximum genus of a graph in 1979. Although the principle idea looks like from the joint trees, a main difference of a tree used here is not corresponding to an embedding of the graph considered.

Given a graph $G = (V, E)$, let T be a spanning tree of G. If each cotree edge is added to T as an articulate edge, what obtained is called a *protracted tree* of G, denoted by $\breve{T}$. An protracted tree $\breve{T}$ is oriented via an orientation of T or its fundamental circuits. In order to guarantee the well-definedness of the orientation for given rotation at all vertices on G and a selected vertex of T, the direction of a cotree edge is always chosen in coincidence with its direction firstly appeared along the the face boundary of $\breve{T}$. For convenience, vertices on the boundary are marked by the ordinary natural numbers as the root vertex, the starting vertex, by 0. Of course, the boundary is a travel on G, called a *tree-travel*.

In Fig.1, (a) A spanning tree T of K_5(i.e., the complete graph of order 5), as shown by bold lines; (b) the protracted tree $\breve{T}$ of T.



(a) T (b) $\breve{T}$

Fig.1

§2. Tree-Travels

Let $C = C(V;e)$ be the tree travel obtained from the boundary of $\breve{T}$ with 0 as the starting vertex. Apparently, the travel as a edge sequence $C = C(e)$ provides a double covering of $G = (V, E)$, denoted by

$$C(V;e) = 0P_{0,i_1}v_{i_1}P_{i_1,i_2}v_{i_2}P_{i_2,i'_1}v_{i'_1}P_{i'_1,i'_2}v_{i'_2}P_{i'_2,2\epsilon}0 \tag{1}$$

where $\epsilon = |E|$.

For a vertex-edge sequence Q as a tree-travel, denote by $[Q]_{\mathrm{eg}}$ the edge sequence induced from Q missing vertices, then $C_{\mathrm{eg}} = [C(V;e)]_{\mathrm{eg}}$ is a polyhegon(*i.e.*, a polyhedron with only one face).

Example 1 From $\breve{T}$ in Fig.1(b), obtain the thee-travel

$$C(V;e) = 0P_{0,8}0P_{8,14}0P_{14,18}0P_{18,20}0$$

where $v_0 = v_8 = v_{14} = v_{18} = v_{20} = 0$ and

$$\begin{aligned} P_{0,8} &= a1\alpha2\alpha^{-1}1\beta3\beta^{-1}1\gamma4\gamma^{-1}1a^{-1}; \\ P_{8,14} &= b2\delta3\delta^{-1}2\lambda4\lambda^{-1}2b^{-1}; \\ P_{14,18} &= c3\sigma4\sigma^{-1}3c^{-1}; \\ P_{18,20} &= d4d^{-1}. \end{aligned}$$

For natural number i, if av_ia^{-1} is a segment in C, then a is called a *reflective edge* and then v_i, the *reflective vertex* of a.

Because of nothing important for articulate vertices(1-valent vertices) and 2-valent vertices in an embedding, we are allowed to restrict ourselves only discussing graphs with neither 1-valent nor 2-valent vertices without loss of generality. From vertices of all greater than or equal to 3, we are allowed only to consider all reflective edges as on the cotree.

If v_{i_1} and v_{i_2} are both reflective vertices in (1), their reflective edges are adjacent in G and $v_{i'_1} = v_{i_1}$ and $v_{i'_2} = v_{i_2}$, $[P_{v_{i_1,i_2}}]_{\rm eg} \cap [P_{v_{i'_1,i'_2}}]_{\rm eg} = \emptyset$, but neither $v_{i'_1}$ nor $v_{i'_2}$ is a reflective vertex, then the transformation from C to

$$\triangle_{v_{i_1},v_{i_2}} C(V;e) = 0P_{0,i_1} v_{i_1} P_{i'_1,i'_2} v_{i_2} P_{i_2,i'_1} v_{i'_1} P_{i_1,i_2} v_{i'_2} P_{i'_2,0} 0. \quad (2)$$

is called an operation of *interchange segments* for $\{v_{i_1}, v_{i_2}\}$.

Example 2 In $C = C(V;e))$ of Example 1, $v_2 = 2$ and $v_4 = 3$ are two reflective vertices, their reflective edges α and β, $v_9 = 2$ and $v_15 = 3$. For interchange segments once on C, we have

$$\triangle_{2,3} C = 0P_{0,2} 2P_{9,15} 3P_{4,9} 2P_{2,4} 3P_{15,20} 0 \ (= C_1).$$

where

$$\begin{aligned}
&P_{0,2} = a1\alpha \ (= P_{1;0,2});\\
&P_{9,15} = \delta 3\delta^{-1} 2\lambda 4\lambda^{-1} 2b^{-1} 0c3 \ (= P_{1;2,8});\\
&P_{4,9} = \beta^{-1} 1\gamma 4\gamma^{-1} 1a^{-1} 0b2 \ (= P_{1;8,13});\\
&P_{2,4} = \alpha^{-1} 1\beta \ (= P_{1;13,15});\\
&P_{15,20} = \sigma 4\sigma^{-1} 3c^{-1} 0d4d^{-1} \ (= P_{1;15,20}).
\end{aligned}$$

Lemma 1 Polyhegon $\triangle_{v_i,v_j} C_{\rm eg}$ is orientable if, and only if, $C_{\rm eg}$ is orientable and the genus of $\triangle_{v_{i_1},v_{i_2}} C_{\rm eg}$ is exactly 1 greater than that of $C_{\rm eg}$.

Proof Because of the invariant of orientability for $\triangle$-operation on a polyhegon, the first statement is true.

In order to prove the second statement, assume cotree edges α and β are reflective edges at vertices, respectively,v_{i_1} and v_{i_2}. Because of

$$C_{\rm eg} = A\alpha\alpha^{-1} B\beta\beta^{-1} CDE$$

where

$$\begin{aligned}
&A\alpha = [P_{0,i_1}]_{\rm eg};\ \alpha^{-1} B\beta = [P_{i_1,i_2}]_{\rm eg};\\
&\beta^{-1} C = [P_{i_2,i'_1}]_{\rm eg};\ D = [P_{i'_1,i'_2}]_{\rm eg};\\
&E = [P_{i'_2,i_\epsilon}]_{\rm eg},
\end{aligned}$$

we have

$$\begin{aligned}
\triangle_{v_{i_1},v_{i_2}} C_{\rm eg} &= A\alpha D\beta^{-1} C\alpha^{-1} B\beta E\\
&\sim_{\rm top} ABCDE\alpha\beta\alpha^{-1}\beta^{-1},\ \text{(Theorem 3.3.3 in [5])}\\
&= C_{\rm eg}\alpha\beta\alpha^{-1}\beta^{-1}\ \text{(Transform 1, in §3.1 of [5])}.
\end{aligned}$$

Therefore, the second statement is true. □

If interchange segments can be done on C successively for k times, then C is called a *k-tree travel*. Since one reflective edge is reduced for each interchange of segments on C and C has at most $m = \lfloor \beta/2 \rfloor$ reflective edges, we have $0 \leqslant k \leqslant m$ where $\beta = \beta(G)$ is the Betti number(or corank) of G. When $k = m$, C is also called *normal*.

For a k-tree travel $C_k(V; e, e^{-1})$ of G, graph G_k is defined as

$$G_k = T \bigcup [E_{\rm ref} \bigcap E_{\bar{T}} - \sum_{j=1}^{k} \{e_j, e'_j\}] \tag{3}$$

where T is a spanning tree, $[X]$ represents the edge induced subgraph by edge subset X, and $e \in E_{\rm ref}$, $e \in E_{\bar{T}}$, $\{e_j, e'_j\}$ are, respectively, reflective edge, cotree edge, pair of reflective edges for interchange segments.

Example 3 On C_1 in Example 2, $v_{1;3} = 3$ and $v_{1;5} = 4$ are two reflective vertices, $v_{1;8} = 3$ and $v_{1;10} = 4$. By doing interchange segments on C_1, obtain

$$\triangle_{3,4} C_1 = 0P_{1;0,10}3P_{1;17,19}4P_{1;12,15}3P_{1;10.12}4P_{1;19,20}0\ (= C_2)$$

where

$$\begin{aligned}
&P_{1;0,10} = a1\alpha 2b^{-1}0c3\beta^{-1}1\gamma 4\gamma^{-1}1a^{-1}0b2\delta(= P_{2;0,10});\\
&P_{1;17,19} = c^{-1}0d(= P_{2;10,12});\\
&P_{1;12.17} = \alpha^{-1}2\alpha^{-1}1\beta 3\sigma 4\sigma^{-1}(= P_{2;12,17});\\
&P_{1;10,12} = \delta^{-1}2\lambda(= P_{2;17,19});\\
&P_{1;19,20} = d^{-1}(= P_{2;19,20}).
\end{aligned}$$

Because of $[P_{2;6,16}]_{\rm eg} \cap [P_{2;12,19}]_{\rm eg} \neq \emptyset$ for $v_{2;12} = 4$ and $v_{2;19} = 4$, only $v_{2;6} = 4$ and $v_{2;16} = 4$ with their reflective edges γ and σ are allowed for doing interchange segments on C_2. The protracted tree $\breve{T}$ in Fig.1(b) provides a 2-tree travel C, and then a 1-tree travel as well.

However, if interchange segments are done for pairs of cotree edges as $\{\beta, \gamma\}$, $\{\delta, \lambda\}$ and $\{\alpha, \sigma\}$ in this order, it is known that C is also a 3-tree travel.

On C of Example 1, the reflective vertices of cotree edges β and γ are, respectively, $v_4 = 3$ and $v_6 = 4$, choose $4' = 15$ and $6' = 19$, we have

$$\triangle_{4,6} C = 0P_{1;0,4}3P_{1;4,8}4P_{1;8,17}3P_{1;17,19}4P_{1;19,20}0(= C_1)$$

where

$$\begin{aligned}
&P_{1;0,4} = P_{0,4};\ P_{1;4,8} = P_{15,19};\ P_{1;8,17} = P_{6,15};\\
&P_{1;17,19} = P_{4,6};\ P_{1;19,20} = P_{19,20}.
\end{aligned}$$

On C_1, subindices of the reflective vertices for reflective edges δ and λ are 5 and 8, choose $5' = 17$ and $8' = 19$, find

$$\triangle_{5,8} C_1 = 0P_{2;0,5}3P_{2;5,7}4P_{2;7,16}3P_{2;16,19}4P_{2;19,20}0(= C_2)$$

where

$$P_{2;0,12} = P_{1;0,12};\ P_{2;12,14} = P_{1;17,19};\ P_{2;14,17} = P_{1;14,17};$$
$$P_{2;17,19} = P_{1;12,14};\ P_{2;19,20} = P_{1;19,20}.$$

On C_2, subindices of the reflective vertices for reflective edges α and σ are 2 and 5, choose $2' = 18$ and $5' = 19$, find

$$\triangle_{5,8}C_2 = 0P_{3;0,2}3P_{3;2,3}4P_{3;3,16}3P_{3;16,19}4P_{3;19,20}0(= C_3)$$

where

$$P_{3;0,2} = P_{2;0,2};\ P_{2;2,3} = P_{2;18,19};\ P_{3;3,16} = P_{2;5,18};$$
$$P_{3;16,19} = P_{2;2,5};\ P_{3;19,20} = P_{2;19,20}.$$

Because of $\beta(K_5) = 6$, $m = 3 = \lfloor \beta/2 \rfloor$. Thus, the tree-travel C is normal.

This example tells us the problem of determining the maximum orientable genus of a graph can be transformed into that of determining a k-tree travel of a graph with k maximum as shown in [4].

Lemma 2 Among all k-tree travel of a graph G, the maximum of k is the maximum orientable genus $\gamma_{\max}(G)$ of G.

Proof In order to prove this lemma, the following two facts have to be known(both of them can be done via the finite recursion principle in §1.3 of [5]!).

Fact 1 In a connected graph G considered, there exists a spanning tree such that any pair of cotree edges whose fundamental circuits with vertex in common are adjacent in G.

Fact 2 For a spanning tree T with Fact 1, there exists an orientation such that on the protracted tree $\breve{T}$, no two articulate subvertices(articulate vertices of T) with odd out-degree of cotree have a path in the cotree.

Because of that if two cotree edges for a tree are with their fundamental circuits without vertex in common then they for any other tree are with their fundamental circuits without vertex in common as well, Fact 1 enables us to find a spanning tree with number of pairs of adjacent cotree edges as much as possible and Fact 2 enables us to find an orientation such that the number of times for dong interchange segments successively as much as possible. From Lemma 1, the lemma can be done. □

§3. Tree-Travel Theorems

The purpose of what follows is for characterizing the embeddability of a graph on a surface of genus not necessary to be zero via k-tree travels.

Theorem 1 *A graph G can be embedded into an orientable surface of genus k if, and only if, there exists a k-tree travel $C_k(V;e)$ such that G_k is planar.*

Proof Necessity. Let $\mu(G)$ be an embedding of G on an orientable surface of genus k. From Lemma 2, $\mu(G)$ has a spanning tree T with its edge subsets E_0, $|E_0| = \beta(G) - 2k$, such that $\hat{G} = G - E_0$ is with exactly one face. By successively doing the inverse of interchange segments for k times, a k-tree travel is obtained on $\hat{G}$. Let K be consisted of the k pairs of cotree edge subsets. Thus, from Operation 2 in §3.3 of [5], $G_k = G - K = \hat{G} - K + E_0$ is planar.

Sufficiency. Because of G with a k-tree travel $C_k(V;e)$, Let K be consisted of the k pairs of cotree edge subsets in successively doing interchange segments for k times. Since $G_k = G - K$ is planar, By successively doing the inverse of interchange segments for k times on $C_k(V;e)$ in its planar embedding, an embedding of G on an orientable surface of genus k is obtained. □

Example 4 In Example 1, for $G = K_5$, C is a 1-tree travel for the pair of cotree edges α and β. And, $G_1 = K_5 - \{\alpha, \beta\}$ is planar. Its planar embedding is

$$[4\sigma^{-1}3c^{-1}0d4]_{\text{eg}} = (\sigma^{-1}c^{-1}d);$$
$$[4d^{-1}0a1\gamma4]_{\text{eg}} = (d^{-1}a\gamma);$$
$$[3\sigma4\lambda^{-1}2\delta3]_{\text{eg}} = (\sigma\lambda^{-1}\delta);\ [0c3\delta^{-1}2b^{-1}0]_{\text{eg}} = (c\delta^{-1}b^{-1});$$
$$[2\lambda4\gamma^{-1}1a^{-1}0b2]_{\text{eg}} = (\lambda\gamma^{-1}a^{-1}b).$$

By recovering $\{\alpha, \beta\}$ to G and then doing interchange segments once on C, obtain C_1. From C_1 on the basis of a planar embedding of G_1, an embedding of G on an orientable surface of genus 1(the torus) is produced as

$$[4\sigma^{-1}3c^{-1}0d4]_{\text{eg}} = (\sigma^{-1}c^{-1}d);\ [4d^{-1}0a1\gamma4]_{\text{eg}} = (d^{-1}a\gamma);$$
$$[3\sigma4\lambda^{-1}2\delta3\beta^{-1}1a^{-1}0b2\alpha^{-1}1\beta3]_{\text{eg}} = (\sigma\lambda^{-1}\delta\beta^{-1}a^{-1}b2\alpha^{-1}\beta);$$
$$[0c3\delta^{-1}2b^{-1}0]_{\text{eg}} = (c\delta^{-1}b^{-1});\ [2\lambda4\gamma^{-1}1\alpha2]_{\text{eg}} = (\lambda\gamma^{-1}\alpha).$$

Similarly, we further discuss on nonorientable case. Let $G = (V, E)$, T a spanning tree, and

$$C(V;e) = 0P_{0,i}v_iP_{i,j}v_jP_{j,2\epsilon}0 \tag{4}$$

is the travel obtained from 0 along the boundary of protracted tree $\breve{T}$. If v_i is a reflective vertex and $v_j = v_i$, then

$$\widetilde{\triangle}_\xi C(V;e) = 0P_{0,i}v_iP_{i,j}^{-1}v_jP_{j,2\epsilon}0 \tag{5}$$

is called what is obtained by doing a *reverse segment* for the reflective vertex v_i on $C(V;e)$.

If reverse segment can be done for successively k times on C, then C is called a $\widetilde{k}$*-tree travel*. Because of one reflective edge reduced for each reverse segment and at most β reflective edges on C, we have $0 \leqslant k \leqslant \beta$ where $\beta = \beta(G)$ is the Betti number of G(or corank). When $k = \beta$, C(or G) is called *twist normal*.

Lemma 3 A connected graph is twist normal if, and only if, the graph is not a tree.

Proof Because of trees no cotree edge themselves, the reverse segment can not be done, this leads to the necessity. Conversely, because of a graph not a tree, the graph has to be with a circuit, a tree-travel has at least one reflective edge. Because of no effect to other reflective edges after doing reverse segment once for a reflective edge, reverse segment can always be done for successively $\beta = \beta(G)$ times, and hence this tree-travel is twist normal. Therefore, sufficiency holds. □

Lemma 4 Let C be obtained by doing reverse segment at least once on a tree-travel of a graph. Then the polyhegon $[\triangle_i C]_{\rm eg}$ is nonorientable and its genus

$$\widetilde{g}([\triangle_\xi C]_{\rm eg}) = \begin{cases} 2g(C)+1, & \text{when } C \text{ orientable;} \\ \widetilde{g}(C)+1, & \text{when } C \text{ nonorientable.} \end{cases} \tag{6}$$

Proof Although a tree-travel is orientable with genus 0 itself, after the first time of doing the reverse segment on what are obtained the nonorientability is always kept unchanged. This leads to the first conclusion. Assume $C_{\rm eg}$ is orientable with genus $g(C)$(in fact, only $g(C)=0$ will be used!). Because of

$$[\triangle_i C]_{\rm eg} = A\xi B^{-1}\xi C$$

where $[P_{0,i}]_{\rm eg} = A\xi$, $[P_{i,j}]_{\rm eg} = \xi^{-1}B$ and $[P_{j,\epsilon}]_{\rm eg} = C$, From (3.1.2) in [5]

$$[\triangle_i C]_{\rm eg} \sim_{\rm top} ABC\xi\xi.$$

Noticing that from Operation 0 in §3.3 of [5], $C_{rseg} \sim_{\rm top} ABC$, Lemma 3.1.1 in [5] leads to

$$\widetilde{g}([\triangle_\xi C]_{\rm eg}) = 2g([C]_{\rm eg}) + 1 = 2g(C) + 1.$$

Assume $C_{\rm eg}$ is nonorientable with genus $g(C)$. Because of

$$C_{\rm eg} = A\xi\xi^{-1}BC \sim_{\rm top} ABC,$$

$\widetilde{g}([\triangle_\xi C]_{\rm eg}) = \widetilde{g}(C)+1$. Thus, this implies the second conclusion. □

As a matter of fact, only reverse segment is enough on a tree-travel for determining the nonorientable maximum genus of a graph.

Lemma 5 Any connected graph, except only for trees, has its Betti number as the nonorientable maximum genus.

Proof From Lemmas 3-4, the conclusion can soon be done. □

For a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ on G, the graph $G_{\widetilde{k}}$ is defined as

$$G_{\widetilde{k}} = T\bigcup[E_{\rm ref} - \sum_{j=1}^{k}\{e_j\}] \tag{7}$$

where T is a spanning tree, $[X]$ the induced graph of edge subset X, and $e \in E_{\rm ref}$ and $\{e_j, e'_j\}$, respectively, a reflective edge and that used for reverse segment.

Theorem 2 *A graph G can be embedded into a nonorientable surface of genus k if, and only if, G has a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ such that $G_{\widetilde{k}}$ is planar.*

Proof From Lemma 3, for k, $1 \leqslant k \leqslant \beta(G)$, any connected graph G but tree has a $\widetilde{k}$-tree travel.

Necessity. Because of G embeddable on a nonorientable surface $S_{\widetilde{k}}$ of genus k, let $\widetilde{\mu}(G)$ be an embedding of G on $S_{\widetilde{k}}$. From Lemma 5, $\widetilde{\mu}(G)$ has a spanning tree T with cotree edge set E_0, $|E_0| = \beta(G) - k$, such that $\widetilde{G} = G - E_0$ has exactly one face. By doing the inverse of reverse segment for k times, a $\widetilde{k}$-tree travel of $\widetilde{G}$ is obtained. Let K be a set consisted of the k cotree edges. From Operation 2 in §3.3 of [5], $G_{\widetilde{k}} = G - K = \widetilde{G} - K + E_0$ is planar.

Sufficiency. Because of G with a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V; e)$, let K be the set of k cotree edges used for successively dong reverse segment. Since $G_{\widetilde{k}} = G - K$ is planar, by successively doing reverse segment for k times on $C_{\widetilde{k}}(V; e)$ in a planar embedding of $G_{\widetilde{k}}$, an embedding of G on a nonorientable surface $S_{\widetilde{k}}$ of genus k is then extracted. □

Example 5 On $K_{3,3}$, take a spanning tree T, as shown in Fig.2(a) by bold lines. In (b), given a protracted tree $\breve{T}$ of T. From $\breve{T}$, get a tree-travel

$$C = 0P_{0,11}2P_{11,15}2P_{15,0}0 \ (= C_0)$$

where $v_0 = v_{18}$ and

$$\begin{aligned} &P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1}; \\ &P_{11,15} = d^{-1}0a1b5\alpha; \\ &P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}. \end{aligned}$$

Because of $v_{15} = 2$ as the reflective vertex of cotree edge α and $v_{11} = v_{15}$,

$$\triangle_3 C_0 = 0P_{1;0,11}2P_{1;11,15}2P_{1;15,0}0 \ (= C_1)$$

where

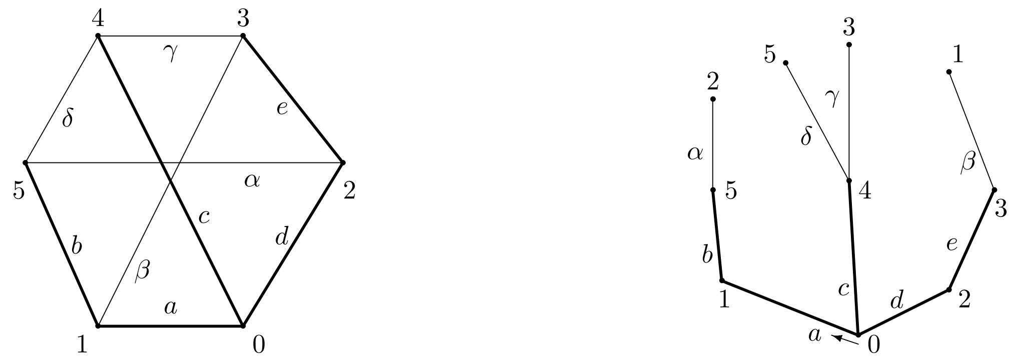
$$\begin{aligned} &P_{1;0,11} = P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1}; \\ &P_{1;11,15} = P^{-1}_{11,15} = \alpha^{-1}5b^{-1}1a^{-1}0d; \\ &P_{1;15,0} = P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}. \end{aligned}$$

Since $G_{\widetilde{1}} = K_{3,3} - \alpha$ is planar, from C_0 we have its planar embedding

$$\begin{aligned} &f_1 = [5P_{16,0}0P_{0,2}0]_{\rm eg} = (b^{-1}a^{-1}c\delta); \\ &f_2 = [3P_{4,8}3]_{\rm eg} = (\gamma^{-1}c^{-1}de); \\ &f_3 = [1P_{13,14}5P_{2,4}3P_{8,9}1]_{\rm eg} = (\delta^{-1}\gamma\beta b); \\ &f_4 = [1P_{9,13}1]_{\rm eg} = (\beta^{-1}e^{-1}d^{-1}a). \end{aligned}$$

By doing reverse segment on C_0, get C_1. On this basis, an embedding of $K_{3,3}$ on the projective plane(*i.e.*, nonorientable surface $S_{\widetilde{1}}$ of genus 1) is obtained as

$$\begin{cases} \widetilde{f_1} = [5P_{1;16,0}0P_{1;0,2}0]_{\rm eg} = f_1 = (b^{-1}a^{-1}c\delta); \\ \widetilde{f_2} = [3P_{1;4,8}3]_{\rm eg} = f_2 = (\gamma^{-1}c^{-1}de); \\ \widetilde{f_3} = [1P_{1;9,11}2P_{1;11,13}1]_{\rm eg} = be^{-1}e^{-1}\alpha^{-1}b^{-1}); \\ \widetilde{f_4} = [0P_{1;14,15}2P_{1;15,16}5P_{1;2,4}3P_{1;8,9}1P_{1;13,14}0]_{\rm eg} \\ \quad = (d\alpha^{-1}\delta^{-1}\gamma\beta a^{-1}). \end{cases}$$



(a) T (b) $\check{T}$

Fig.2

§4. **Research Notes**

A. For the embeddability of a graph on the torus, double torus *etc* or in general orientable surfaces of genus small, more efficient characterizations are still necessary to be further contemplated on the basis of Theorem 1.

B. For the embeddability of a graph on the projective plane(1-crosscap), Klein bottle(2-crosscap), 3-crosscap *etc* or in general nonorientable surfaces of genus small, more efficient characterizations are also necessary to be further contemplated on the basis of Theorem 2.

C. Tree-travels can be extended to deal with all problems related to embedings of a graph on surfaces as joint trees in a constructive way.

References

[1] Archdeacon D., A Kuratowski theorem for the projective plane, *J. Comb. Theory*, 5 (1981), 243–246.

[2] Liu Yi. and Liu, Y.P., A characterization of the embeddability of graphs on the surface of given genus, *Chin. Ann. Math.*, 17B(1996), 457–462.

[3] Liu Y.P., The nonorientable maximum genus of a graph(in Chinese with English abstract), *Scintia Sinica*, Special Issue on Math. I(1979), 191–201.

[4] Liu Y.P., The maximum orientable genus of a graph, *Scientia Sinica*, Special Issue on Math. II(1979), 41–55.

[5] Liu Y.P., *Topological Theory on Graphs*, USTC Press, 2008.

Surface Embeddability of Graphs via Joint Trees

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Abstract: This paper provides a way to observe embedings of a graph on surfaces based on join trees and then characterizations of orientable and nonorientable embeddabilities of a graph with given genus.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, joint tree.

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§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The term *joint three* looks firstly appeared in [1] and then in [2] in a certain detail and [3] firstly in English. However, the theoretical idea was initiated in early articles of the author [4–5] in which maximum genus of a graph in both orientable and nonorientable cases were investigated.

The central idea is to transform a problem related to embeddings of a graph on surfaces i.e., compact 2-manifolds without boundary in topology into that on polyhegons (or polygons of even size with binary boundaries). The following two principles can be seen in [3].

Principle A *Joint trees of a graph have a 1–to–1 correspondence to embeddings of the graph with the same orientability and genus i.e., on the same surfaces.*

Principle B *Associate polyhegons (as surfaces) of a graph have a 1–to–1 correspondence to joint trees of the graph with the same orientability and genus, i.e., on the same surfaces.*

The two principle above are employed in this paper as the theoretical foundation. These enable us to discuss in any way among associate polyhegons, joint trees and embeddings of a graph considered.

§2. Layers and Exchangers

Given a surface $S = (A)$. it is divided into segments layer by layer as in the following.

The 0th layer contains only one segment, *i.e.*, $A(= A_0)$;

The 1st layer is obtained by dividing the segment A_0 into l_1 segments, *i.e.*, $S = (A_1, A_2, \cdots, A_{l_1})$, where $A_1, A_2, \cdots, A_{l_1}$ are called the 1*st layer segments*;

Suppose that on $k-1$st layer, the $k-1$st layer segments are $A_{\underline{n}_{(k-1)}}$ where $\underline{n}_{(k-1)}$ is an integral $k-1$-vector satisfied by

$$\underline{1}_{(k-1)} \leqslant (n_1, n_2, \cdots, n_{k-1}) \leqslant \underline{N}_{(k-1)}$$

with $\underline{1}_{(k-1)} = (1, 1, \cdots, 1)$, $\underline{N}_{(k-1)} = (N_1, N_2, \cdots, N_{k-1})$, $N_1 = l_1 = N_{(1)}$, $N_2 = l_{A_{N_{(1)}}}$, $N_3 = l_{A_{\underline{N}_{(2)}}}$, $\cdots$, $N_{k-1} = l_{A_{\underline{N}_{(k-2)}}}$, then the kth layer segments are obtained by dividing each $k-1$st layer segment as

$$A_{\underline{n}_{(k-1)},1}, A_{\underline{n}_{(k-1)},2}, \cdots, A_{\underline{n}_{(k-1)},l_{A_{\underline{n}_{(k-1)}}}} \tag{1}$$

where $\underline{1}_{(k)} = (\underline{n}_{(k-1)}, 1) \leqslant (\underline{n}_{(k-1)}, i) \leqslant \underline{N}_{(k)} = (\underline{N}_{(k-1)}, N_k)$ and $N_k = l_{A_{\underline{N}_{(k-1)}}}$. Segments in (1) are called *successors* of $A_{\underline{n}_{(k-1)}}$. Conversely, $A_{\underline{n}_{(k-1)}}$ is the *predecessor* of any one in (1).

A layer segment which has only one element is called an *end segment* and others, *principle segments*. For an example, let

$$S = (1, -7, 2, -5, 3, -1, 4, -6, 5, -2, 6, 7, -3, -4).$$

Fig.2.1 shows a layer division of S and Tab.2.1, the principle segments in each layer.

For a layer division of a surface, if principle segments are dealt with vertices and edges are with the relationship between predecessor and successor, then what is obtained is a tree denoted by T. On T, by adding cotree edges as end segments, a graph $G = (V, E)$ is induced. For example, the graph induced from the layer division shown in Fig.1 is as

$$V = \{A, B, C, D, E, F, G, H, I\} \tag{2}$$

and

$$E = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4, 5, 6, 7\}, \tag{3}$$

where

$$a = (A, B), b = (A, C), c = (A, D), d = (B, E),$$
$$e = (C, F), f = (C, G), g = (D, H), h = (D, I),$$

and

$$1 = (B, F), 2 = (E, H), 3 = (F, I), 4 = (G, I),$$
$$5 = (B, C), 6 = (G, H), 7 = (D, E).$$

By considering $E_T = \{a, b, c, d, e, f, g, h\}$, $\overline{E}_T = \{1, 2, 3, 4, 5, 6, 7\}$, $\delta_i = 0, i = 1, 2, \cdots, 7$, and the rotation σ implied in the layer division, a joint tree $\widehat{T}^{\delta}_{\sigma}$ is produced.

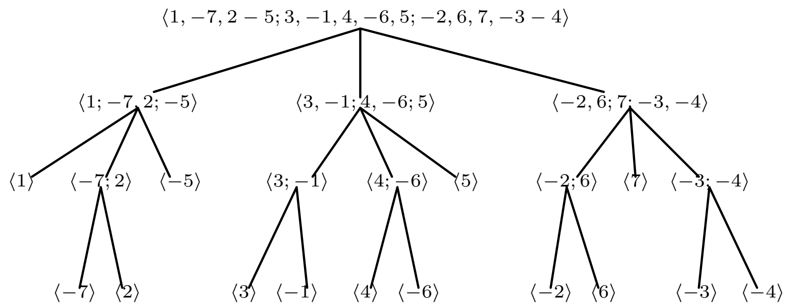

Fig.1 Layer division of surface S

Layers	Principle segments
0th layer	$A = \langle 1, -7, 2-5; 3, -1, 4, -6, 5; -2, 6, 7, -3-4\rangle$
1st layer	$B = \langle 1; -7, 2; -5\rangle, C = \langle 3, -1; 4, -6; 5\rangle,$
	$D = \langle -2, 6; 7; -3, -4\rangle$
2nd layer	$E = \langle -7; 2\rangle, F = \langle 3; -1\rangle, G = \langle 4; -6\rangle,$
	$H = \langle -2; 6\rangle, I = \langle -3; -4\rangle$

Tab.1 Layers and principle segments

Theorem 1 *A layer division of a polyhegon determines a joint tree. Conversely, a joint tree determines a layer division of its associate polyhegon.*

Proof For a layer division of a polyhegon as a polyhegon, all segments are treated as vertices and two vertices have an edge if, and only if, they are in successive layers with one as a subsegment of the other. This graph can be shown as a tree. Because of each non-end vertex with a rotation and end vertices pairwise with binary indices, this tree itself is a joint tree.

Conversely, for a joint tree, it is also seen as a layer division of the surface determined by the boundary polyhegon of the tree. □

Then, an operation on a layer division is discussed for transforming an associate polyhegon into another in order to visit all associate polyhegon without repetition.

A layer segment with all its successors is called a *branch* in the layer division. The operation of interchanging the positions of two layer segments with the same predecessor in a layer division is called an *exchanger*.

Lemma 1 *A layer division of an associate polyhegon of a graph under an exchanger is still a layer division of another associate polyhegon. Conversely, the later under the same exchanger becomes the former.*

Proof On the basis of Theorem 1, only necessary to see what happens by exchanger on a joint tree once. Because of only changing the rotation at a vertex for doing exchanger once, exchanger transforms a joint tree into another joint tree of the same graph. This is the first conclusion. Because of exchanger inversible, the second conclusion holds. □

On the basis of this lemma, an exchanger can be seen as an operation on the set of all associate surfaces of a graph.

Lemma 2 *The exchanger is closed in the set of all associate polyhegons of a graph.*

Proof From Theorem 1, the lemma is a direct conclusion of Lemma 1. □

Lemma 3 *Let $\mathcal{A}(G)$ be the set of all associate polyhegons of a graph G, then for any S_1, $S_2 \in \mathcal{A}(G)$, there exist a sequence of exchangers on the set such that S_1 can be transformed into S_2.*

Proof Because of exchanger corresponding to transposition of two elements in a rotation at a vertex, in virtue of permutation principle that any two rotation can be transformed from one into another by transpositions, from Theorem 1 and Lemma 1, the conclusion is done. □

If $\mathcal{A}(G)$ is dealt as the vertex set and an edge as an exchanger, then what is obtained in this way is called the *associate polyhegon graph* of G, and denoted by $\mathcal{H}(G)$. From Principle A, it is also called the *surface embedding graph* of G.

Theorem 2 *In $\mathcal{H}(G)$, there is a Hamilton path. Further, for any two vertices, $\mathcal{H}(G)$ has a Hamilton path with the two vertices as ends.*

Proof Since a rotation at each vertex is a cyclic permutation(or in short a cycle) on the set of semi-edges with the vertex, an exchanger of layer segments is corresponding to a transposition on the set at a vertex.

Since any two cycles at a vertex v can be transformed from one into another by $\rho(v)$ transpositions where $\rho(v)$ is the valency of v, *i.e.*, the order of cycle(rotation), This enables us to do exchangers from the 1st layer on according to the order from left to right at one vertex to the other. Because of the finiteness, an associate polyhegon can always transformed into another by $|\mathcal{A}(G)|$ exchangers. From Theorem 1 with Principles 1–2, the conclusion is done. □

First, starting from a surface in $\mathcal{A}(G)$, by doing exchangers at each principle segments in one layer to another, a Hamilton path can always be found in considering Theorem 2 and Theorem 1. Then, a Hamilton path can be found on $\mathcal{H}(G)$.

Further, for chosen $S_1, S_2 \in \mathcal{A}(G) = V(\mathcal{H}(G))$ adjective, starting from S_1, by doing exchangers avoid S_2 except the final step, on the basis of the strongly finite recursion principle, a Hamilton path between S_1 and S_2 can be obtained. In consequence, a Hamilton circuit can be found on $\mathcal{H}(G)$.

Corollary 1 *In $\mathcal{H}(G)$, there exists a Hamilton circuit.*

Theorem 2 tells us that the problem of determining the minimum, or maximum genus of graph G has an algorithm in time linear on $\mathcal{H}(G)$.

§3. Main Theorems

For a graph G, let $\mathcal{S}(G)$ be the the associate polehegons (or surfaces) of G, and $\mathbf{S}_p$ and $\mathbf{S}_{\widetilde{q}}$, the subsets of, respectively, orientable and nonorientable polyhegons of genus $p \geqslant 0$ and $q \geqslant 1$.

Then, we have

$$\mathcal{S}(G) = \sum_{p \geqslant 0} \mathbf{S}_p + \sum_{q \geqslant 1} \mathbf{S}_{\widetilde{q}}.$$

Theorem 3 *A graph G can be embedded on an orientable surface of genus p if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. Moreover, for an embedding of G, there exist a sequence of exchangers by which the corresponding polyhegon of the embedding can be transformed into one in $\mathbf{S}_p$.*

Proof For an embedding of G on an orientable surface of genus p, from Theorem 1 there is an associate polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. This is the necessity of the first statement.

Conversely, given an associate polyhegen in $\mathbf{S}_p$, $p \geqslant 0$, from Theorems 1–2 with Principles A and B, an embedding of G on an orientable surface of genus p can be done. This is the sufficiency of the first statement.

The last statement of the theorem is directly seen from the proof of Theorem 2. $\square$

For an orientable embedding $\mu(G)$ of G, denote by $\widetilde{\mathbf{S}}_\mu$ the set of all nonorientable associate polyhegons induced from $\mu(G)$.

Theorem 4 *A graph G can be embedded on a nonorientable surface of genus $q(\geqslant 1)$ if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. Moreover, if G has an embedding $\widetilde{\mu}$ on a nonorientable surface of genus q, then it can always be done from an orientable embedding μ arbitrarily given to another orientable embedding μ' by a sequence of exchangers such that the associate polyhegon of $\widetilde{\mu}$ is in $\widetilde{\mathbf{S}}_{\mu'}$.*

Proof For an embedding of G on a nonorientable surface of genus q, Theorem 1 and Principle B lead to that its associate polyhegon is in $\mathbf{S}_q$, $q \geqslant 1$. This is the necessity of the first statement.

Conversely, let $S_{\widetilde{q}}$ be an associate polyhegon of G in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. From Principles A and B, an embedding of G on a nonorietable surface of genus q can be found from $S_{\widetilde{q}}$. This is the sufficiency of the first statement.

Since a nonorientable embedding of G has exactly one under orientable embedding of G by Principle A, Theorem 2 directly leads to the second statement. $\square$

§4. Research Notes

A. Theorems 1 and 2 enable us to establish a procedure for finding all embeddings of a graph G in linear space of the size of G and in linear time of size of $\mathcal{H}(G)$. The implementation of this procedure on computers can be seen in [6].

B. In Theorems 3 and 4, it is necessary to investigate a procedure to extract a sequence of transpositions considered for the corresponding purpose efficiently.

C. On the basis of the associate polyhegons, the recognition of operations from a polyhegon of genus p to that of genus $p+k$ for given $k \geqslant 0$ have not yet be investigated. However, for the case $k = 0$ the operations are just Operetions 0–2 all topological that are shown in [1–3].

D. It looks worthful to investigate the associate polyhegon graph of a graph further for accessing the determination of the maximum(orientable) and minimum(orientable or nonorientable) genus of a graph.

References

[1] Y.P.Liu, *Advances in Combinatorial Maps*(in Chinese), Northern Jiaotong University Press, Beijing, 2003.

[2] Y.P.Liu, *Algebraic Principles of Maps*(in Chinese), Higher Education Press, Beijing, 2006.

[3] Y.P.Liu, *Theory of Polyhedra*, Science Press, Beijing, 2008.

[4] Y.P.Liu, The nonorientable maximum genus of a graph(in Chinese with English abstract), *Scintia Sinica*, Special Issue on Math. I(1979), 191–201.

[5] Y.P.Liu, The maximum orientable genus of a graph, *Scientia Sinica*, Special Issue on Math. II(1979), 41–55.

[6] Wang Tao and Y.P. Liu, Implements of some new algorithms for combinatorial maps, *OR Trans.*, 12(2008), 2: 58–66.

Surface Embeddability of Graphs via Reductions

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Abstract: On the basis of reductions, polyhedral forms of Jordan axiom on closed curve in the plane are extended to establish characterizations for the surface embeddability of a graph.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, Jordan closed cure axiom, forbidden minor.

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§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The classical version of Jordan curve theorem in topology states that a single closed curve C separates the sphere into two connected components of which C is their common boundary. In this section, we investigate the polyhedral statements and proofs of the Jordan curve theorem.

Let $\Sigma = \Sigma(G; F)$ be a polyhedron whose underlying graph $G = (V, E)$ with F as the set of faces. If any circuit C of G not a face boundary of Σ has the property that there exist two proper subgraphs In and Ou of G such that

$$In \bigcup Ou = G;\ \ In \bigcap Ou = C, \tag{A}$$

then Σ is said to have the *first Jordan curve property*, or simply write as 1-JCP. For a graph G, if there is a polyhedron $\Sigma = \Sigma(G; F)$ which has the 1-JCP, then G is said to have the 1-JCP as well.

Of course, in order to make sense for the problems discussed in this section, we always suppose that all the members of F in the polyhedron $\Sigma = \Sigma(G; F)$ are circuits of G.

Theorem A(First Jordan curve theorem) *G has the 1-JCP If, and only if, G is planar.*

Proof Because of $\mathcal{H}_1(\Sigma) = 0, \Sigma = \Sigma(G; F)$, from Theorem 4.2.5 in [1], we know that Im ∂_2 = Ker $\partial_1 = \mathcal{C}$, the cycle space of G and hence Im $\partial_2 \supseteq F$ which contains a basis of $\mathcal{C}$.

Thus, for any circuit $C \notin F$, there exists a subset D of F such that

$$C = \sum_{f \in D} \partial_2 f;\ C = \sum_{f \in F \setminus D} \partial_2 f. \tag{B}$$

Moreover, if we write

$$Ou = G[\bigcup_{f \in D} f];\ In = G[\bigcup_{f \in F \setminus D} f],$$

then Ou and In satisfy the relations in (A) since any edge of G appears exactly twice in the members of F. This is the sufficiency.

Conversely, if G is not planar, then G only have embedding on surfaces of genus not 0. Because of the existence of non contractible circuit, such a circuit does not satisfy the 1-JCP and hence G is without 1-JCP. This is the necessity. □

Let $\Sigma^* = \Sigma(G^*; F^*)$ be a dual polyhedron of $\Sigma = \Sigma(G; F)$. For a circuit C in G, let $C^* = \{e^* | \ \forall e \in C\}$, or say the corresponding vector in $\mathcal{G}_1^*$, of $C \in \mathcal{G}_1$.

Lemma 1 *Let C be a circuit in Σ. Then, $G^* \backslash C^*$ has at most two connected components.*

Proof Suppose H^* be a connected component of $G^* \backslash C^*$ but not the only one. Let D be the subset of F corresponding to $V(H^*)$. Then,

$$C' = \sum_{f \in D} \partial_2 f \subseteq C.$$

However, if $\emptyset \neq C' \subset C$, then C itself is not a circuit. This is a contradiction to the condition of the lemma. From that any edge appears twice in the members of F, there is only one possibility that

$$C = \sum_{f \in F \setminus D} \partial_2 f.$$

Hence, $F \backslash D$ determines the other connected component of $G^* \backslash C^*$ when $C' = C$. □

Any circuit C in G which is the underlying graph of a polyhedron $\Sigma = \Sigma(G; F)$ is said to have the *second Jordan curve property,* or simply write 2-JCP for Σ with its dual $\Sigma^* = \Sigma(G^*; F^*)$ if $G^* \backslash C^*$ has exactly two connected components. A graph G is said to have the 2-JCP if all the circuits in G have the property.

Theorem B(Second Jordan curve theorem) *A graph G has the 2-JCP if, and only if, G is planar.*

Proof To prove the necessity. Because for any circuit C in G, $G^* \backslash C^*$ has exactly two connected components, any C^* which corresponds to a circuit C in G is a cocircuit. Since any edge in G^* appears exactly twice in the elements of V^*, which are all cocircuits, from Lemma 1, V^* contains a basis of Ker δ_1^*. Moreover, V^* is a subset of Im δ_0^*. Hence, Ker $\delta_1 \subseteq$ Im δ_0. From Lemma 4.3.2 in [1], Im $\delta_0^* \subseteq$ Ker δ_1^*. Then, we have Ker δ_1^* =Im δ_0^*, *i.e.*, $\widetilde{\mathcal{H}}_1(\Sigma^*) = 0$. From the dual case of Theorem 4.3.2 in [1], G^* is planar and hence so is G. Conversely, to prove the sufficiency. From the planar duality, for any circuit C in G, C^* is a cocircuit in G^*. Then, $G^* \backslash C^*$ has two connected components and hence C has the 2- JCP. □

For a graph G, of course connected without loop, associated with a polyhedron $\Sigma = \Sigma(G; F)$, let C be a circuit and E_C, the set of edges incident to, but not on C. We may define an equivalence on E_C, denoted by $\sim_C$ as the transitive closure of that $\forall a, b \in E_C$,

$$\begin{aligned} a \sim_C b \Leftrightarrow \quad & \exists f \in F,\ (a^\alpha C(a,b) b^\beta \subset f) \\ & \vee (b^{-\beta} C(b,a) a^{-\alpha} \subset f), \end{aligned} \tag{C}$$

where $C(a, b)$, or $C(b, a)$ is the common path from a to b, or from b to a in $C \cap f$ respectively. It can be seen that $|E_C/\sim_C| \leqslant 2$ and the equality holds for any C not in F only if Σ is orientable.

In this case, the two equivalent classes are denoted by $E_{\mathcal{L}} = E_{\mathcal{L}}(C)$ and $E_{\mathcal{R}} = E_{\mathcal{R}}(C)$. Further, let $V_{\mathcal{L}}$ and $V_{\mathcal{R}}$ be the subsets of vertices by which a path between the two ends of two edges in $E_{\mathcal{L}}$ and $E_{\mathcal{R}}$ without common vertex with C passes respectively.

From the connectedness of G, it is clear that $V_{\mathcal{L}} \cup V_{\mathcal{R}} = V \setminus V(C)$. If $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$, then C is said to have the *third Jordan curve property,* or simply write 3-JCP. In particular, if C has the 3-JCP, then every path from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$ (or vice versa) crosses C and hence C has the 1-JCP. If every circuit which is not the boundary of a face f of $\Sigma(G)$, one of the underlain polyhedra of G has the 3-JCP, then G is said to have the 3-JCP as well.

Lemma 2 Let C be a circuit of G which is associated with an orientable polyhedron $\Sigma = \Sigma(G; F)$. If C has the 2-JCP, then C has the 3-JCP. Conversely, if $V_{\mathcal{L}}(C) \neq \emptyset$, $V_{\mathcal{R}}(C) \neq \emptyset$ and C has the 3-JCP, then C has the 2-JCP.

Proof For a vertex $v^* \in V^* = V(G^*)$, let $f(v^*) \in F$ be the corresponding face of Σ. Suppose In^* and Ou^* are the two connected components of $G^* \setminus C^*$ by the 2-JCP of C. Then,

$$In = \bigcup_{v^* \in In^*} f(v^*) \text{ and } Ou = \bigcup_{v^* \in Ou^*} f(v^*)$$

are subgraphs of G such that $In \cup Ou = G$ and $In \cap Ou = C$. Also, $E_{\mathcal{L}} \subset In$ and $E_{\mathcal{R}} \subset Ou$ (or vice versa). The only thing remained is to show $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$. By contradiction, if $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$, then In and Ou have a vertex which is not on C in common and hence have an edge incident with the vertex, which is not on C, in common. This is a contradiction to $In \cap Ou = C$.

Conversely, from Lemma 1, we may assume that $G^* \setminus C^*$ is connected by contradiction. Then there exists a path P^* from v_1^* to v_2^* in $G^* \setminus C^*$ such that $V(f(v_1^*)) \cap V_{\mathcal{L}} \neq \emptyset$ and $V(f(v_2^*)) \cap V_{\mathcal{R}} \neq \emptyset$. Consider

$$H = \bigcup_{v^* \in P^*} f(v^*) \subseteq G.$$

Suppose $P = v_1 v_2 \cdots v_l$ is the shortest path in H from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$.

To show that P does not cross C. By contradiction, assume that v_{i+1} is the first vertex of P crosses C. From the shortestness, v_i is not in $V_{\mathcal{R}}$. Suppose that subpath $v_{i+1} \cdots v_{j-1}$, $i+2 \leqslant j < l$, lies on C and that v_j does not lie on C. By the definition of $E_{\mathcal{L}}$, $(v_{j-1}, v_j) \in E_{\mathcal{L}}$ and hence $v_j \in V_{\mathcal{L}}$. This is a contradiction to the shortestness. However, from that P does not cross C, $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$. This is a contradiction to the 3-JCP. $\square$

Theorem C(Third Jordan curve theorem) *Let $G = (V, E)$ be with an orientable polyhedron $\Sigma = \Sigma(G; F)$. Then, G has the 3-JCP if, and only if, G is planar.*

Proof From Theorem B and Lemma 2, the sufficiency is obvious. Conversely, assume that G is not planar. By Lemma 4.2.6 in [1], $\mathrm{Im}\partial_2 \subseteq \mathrm{Ker}\partial_1 = \mathcal{C}$, the cycle space of G. By Theorem 4.2.5 in [1], $\mathrm{Im}\partial_2 \subset \mathrm{Ker}\partial_1$. Then, from Theorem B, there exists a circuit $C \in \mathcal{C} \setminus \mathrm{Im}\partial_2$ without the 2-JCP. Moreover, we also have that $V_{\mathcal{L}} \neq \emptyset$ and $V_{\mathcal{R}} \neq \emptyset$. If otherwise $V_{\mathcal{L}} = \emptyset$, let

$$D = \{f | \exists e \in E_{\mathcal{L}}, e \in f\} \subset F.$$

Because $V_{\mathcal{L}} = \emptyset$, any $f \in D$ contains only edges and chords of C, we have

$$C = \sum_{f \in D} \partial_2 f$$

that contradicts to $C \notin \mathrm{Im}\partial_2$. Therefore, from Lemma 2, C does not have the 3-JCP. The necessity holds. □

§2 Reducibilities

For S_g as a surface(orientable, or nonorientable) of genus g, If a graph H is not embedded on a surface S_g but what obtained by deleting an edge from H is embeddable on S_g, then H is said to be *reducible* for S_g. In a graph G, the subgraphs of G homeomorphic to H are called a type of *reducible configuration* of G, or shortly a *reduction*. Robertson and Seymour in [2] has been shown that graphs have their types of reductions for a surface of genus given finite. However, even for projective plane the simplest nonorientable surface, the types of reductions are more than 100 [3,7].

For a surface S_g, $g \geqslant 1$, let $\mathcal{H}_{g-1}$ be the set of all reductions of surface S_{g-1}. For $H \in \mathcal{H}_{g-1}$, assume the embeddings of H on S_g have ϕ faces. If a graph G has a decomposition of ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, such that

$$\bigcup_{i=1}^{\phi} H_i = G; \ \bigcup_{i \neq j}^{\phi} (H_i \bigcap H_j) = H; \tag{1}$$

all H_i, $1 \leqslant i \leqslant \phi$, are planar and the common vertices of each H_i with H in the boundary of a face, then G is said to be with the *reducibility* 1 for the surface S_g.

Let $\Sigma^* = (G^*; F^*)$ be a polyhedron which is the dual of the embedding $\Sigma = (G; F)$ of G on surface S_g. For surface S_{g-1}, a reduction $H \subseteq G$ is given. Denote $H^* = [e^* | \forall e \in E(H)]$. Naturally, $G^* - E(H^*)$ has at least $\phi = |F|$ connected components. If exact ϕ components and each component planar with all boundary vertices are successively on the boundary of a face, then Σ is said to be with the *reducibility* 2.

A graph G which has an embedding with reducibility 2 , then G is said to be with *reducibility* 2 as well.

Given $\Sigma = (G; F)$ as a polyhedron with under graph $G = (V, E)$ and face set F. Let H be a reduction of surface S_{p-1} and, $H \subseteq G$. Denote by C the set of edges on the boundary of H

in G and E_C, the set of all edges of G incident to but not in H. Let us extend the relation $\sim_C$: $\forall a, b \in E_C$,

$$a \sim_C b \Leftrightarrow \exists f \in F_H,\ a, b \in \partial_2 f \tag{2}$$

by transitive law as a equivalence. Naturally, $|E_C/\sim_C| \leqslant \phi_H$. Denote by $\{E_i | 1 \leqslant i \leqslant \phi_C\}$ the set of equivalent classes on E_C. Notice that $E_i = \emptyset$ can be missed without loss of generality. Let V_i, $1 \leqslant i \leqslant \phi_C$, be the set of vertices on a path between two edges of E_i in G avoiding boundary vertices. When $E_i = \emptyset$, $V_i = \emptyset$ is missed as well. By the connectedness of G , it is seen that

$$\bigcup_{i=1}^{\phi_C} V_i = V - V_H. \tag{3}$$

If for any $1 \leqslant i < j \leqslant \phi_C$, $V_i \cap V_j = \emptyset$, and all $[V_i]$ planar with all vertices incident to E_i on the boundary of a face, then H, G as well, is said to be with *reducibility* 3.

§3. Reducibility Theorems

Theorem 1 *A graph G can be embedded on a surface $S_g(g \geqslant 1)$ if, and only if, G is with the reducibility* 1.

Proof Necessity. Let $\mu(G)$ be an embedding of G on surface $S_g(g \geqslant 1)$. If $H \in \mathcal{H}_{g-1}$, then $\mu(H)$ is an embedding on $S_g(g \geqslant 1)$ as well. Assume $\{f_i | 1 \leqslant i \leqslant \phi\}$ is the face set of $\mu(H)$, then $G_i = [\partial f_i + E([f_i]_{in})]$, $1 \leqslant i \leqslant \phi$, provide a decomposition satisfied by (1). Easy to show that all G_i, $1 \leqslant i \leqslant \phi$, are planar. And, all the common edges of G_i and H are successively in a face boundary. Thus, G is with reducibility 1.

Sufficiency. Because of G with reducibility 1, let $H \in \mathcal{H}_{g-1}$, assume the embedding $\mu(H)$ of H on surface S_g has ϕ faces. Let G have ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, satisfied by (1), and all H_i planar with all common edges of H_i and H in a face boundary. Denote by $\mu_i(H_i)$ a planar embedding of H_i with one face whose boundary is in a face boundary of $\mu(H)$, $1 \leqslant i \leqslant \phi$. Put each $\mu_i(H_i)$ in the corresponding face of $\mu(H)$, an embedding of G on surface $S_g(g \geqslant 1)$ is then obtained. $\square$

Theorem 2 *A graph G can be embedded on a surface $S_g(g \geqslant 1)$ if, and only if, G is with the reducibility* 2.

Proof Necessity. Let $\mu(G) = \Sigma = (G; F)$ be an embedding of G on surface $S_g(g \geqslant 1)$ and $\mu^*(G) = \mu(G^*) = (G^*, F^*)(= \Sigma^*)$, its dual. Given $H \subseteq G$ as a reduction. From the duality between the two polyhedra $\mu(H)$ and $\mu^*(H)$, the interior domain of a face in $\mu(H)$ has at least a vertex of G^*, $G^* - E(H^*)$ has exactly $\phi = |F_{\mu(H)}|$ connected components. Because of each component on a planar disc with all boundary vertices successively on the boundary of the disc, H is with the reducibility 2. Hence, G has the reducibility 2.

Sufficiency. By employing the embedding $\mu(H)$ of reduction H of G on surface $S_g(g \geqslant 1)$ with reducibility 2, put the planar embedding of the dual of each component of $G^* - E(H^*)$ in the corresponding face of $\mu(H)$ in agreement with common boundary, an embedding of $\mu(G)$ on surface $S_g(g \geqslant 1)$ is soon done. $\square$

Theorem 3 *A 3-connected graph G can be embedded on a surface $S_g(g \geqslant 1)$ if, and only if, G is with reducibility 3.*

Proof Necessity. Assume $\mu(G) = (G, F)$ is an embedding of G on surface $S_g(g \geqslant 1)$. Given $H \subseteq G$ as a reduction of surface S_{p-1}. Because of $H \subseteq G$, the restriction $\mu(H)$ of $\mu(G)$ on H is also an embedding of H on surface $S_g(g \geqslant 1)$. From the 3-connectedness of G, edges incident to a face of $\mu(H)$ are as an equivalent class in E_C. Moreover, the subgraph determined by a class is planar with boundary in coincidence, *i.e.*, H has the reducibility 3. Hence, G has the reducibility 3.

Sufficiency. By employing the embedding $\mu(H)$ of the reduction H in G on surface $S_g(g \geqslant 1)$ with the reducibility 3, put each planar embedding of $[V_i]$ in the interior domain of the corresponding face of $\mu(H)$ in agreement with the boundary condition, an embedding $\mu(G)$ of G on $S_g(g \geqslant 1)$ is extended from $\mu(H)$. □

§4. Research Notes

A. On the basis of Theorems 1–3, the surface embeddability of a graph on a surface(orientabl or nonorientable) of genus smaller can be easily found with better efficiency.

For an example, the sphere S_0 has its reductions in two class described as $K_{3,3}$ and K_5. Based on these, the characterizations for the embeddability of a graph on the torus and the projective plane has been established in [4]. Because of the number of distinct embeddings of K_5 and $K_{3,3}$ on torus and projective plane much smaller as shown in the Appendix of [5], the characterizations can be realized by computers with an algorithm much efficiency compared with the existences, *e.g.*, in [7].

B. The three polyhedral forms of Jordan closed planar curve axiom as shown in section 2 initiated from Chapter 4 of [6] are firstly used for surface embeddings of a graph in [4]. However, characterizations in that paper are with a mistake of missing the boundary conditions as shown in this paper.

C. The condition of 3-connectedness in Theorem 3 is not essential. It is only for the simplicity in description.

D. In all of Theorem 1–3, the conditions on planarity can be replaced by the corresponding Jordan curve property as shown in section 2 as in [4] with the attention of the boundary conditions.

References

[1] Liu Y.P., *Topological Theory on Graphs*, USTC Press, Hefei, 2008.

[2] Robertson N. and P. Seymour, Generalizing Kuratowski's theorem, *Cong. Numer.*, **45** (1984), 129–138.

[3] Archdeacon, D., A Kuratowski theorem for the projective plane, *J. Graph Theory*, **5** (1981), 243– 246.

[4] Liu Yi. and Y.P. Liu, A characterization of the embeddability of graphs on the surface of given genus, *Chin. Ann. Math.*, 17B(1996), 457–462.

[5] Liu Y.P., *General Theory of Map Census*, Science Press, Beijing, 2009.

[6] Liu Y.P., *Embeddability in Graphs*, Kluwer, Dordrecht/Boston/London, 1995.

[7] Glover H., J. Huneke and C.S. Wang, 103 graphs that are irreduclble for the projective plane, *J. Combin. Theory*, **B27**(1979), 232–370.

Recent Developments in Regular Maps

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Abstract: A 2-cell embedding of a graph into an orientable or nonorientable closed surface is called regular if its automorphism group acts regularly on its arcs and flags respectively. One of central problems in topological graph theory is to classify regular maps by given underlying graphs or automorphism groups. In this talk, we shall present some recent results in regular maps.

Key Words: Surface, regular map, classification, automorphism group of graph.

AMS(2010): 05C15, 05C25

§1. **Surfaces and Embeddings**

2-manifold M: a topological space M which is Hausdorf and is covered by countably many open sets isomorphic to either 2-dim open ball or 2-dim half-ball;

Closed 2-manifold M: compact, boundary is empty;

Surface S: closed, connected 2-manifold;

Classification of Surfaces:

(i) Orientable Surfaces: S_g, $g = 0, 1, 2, \cdots$,
$v + f - e = 2 - 2g$

(ii) Nonorientable Surfaces: N_k, $k = 0, 1, 2, \cdots$,
$v + f - e = 2 - k$

Embeddings of a graph X in the surface is a continuous one-to-one function $i : X \to S$.

2-cell Embeddings: each region is homemorphic to an open disk.

The primitive objective of topological graph theory is to draw a graph on a surface so that no two edges cross.

Topological Map $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph X is called the *underlying graph* of the map.

Automorphism of a map $\mathcal{M}$: an automorphism of the underlying graph X which can be extended to self-homeomorphism of the surface.

Automorphism group Aut $(\mathcal{M})$: all the automorphisms of the map $\mathcal{M}$.

Remark Aut $(\mathcal{M})$ acts semi-regularly on the arcs of X.

Regular Map: Aut $(\mathcal{M})$ acts regularly on the arcs of X.

Three main research directions:

1. Classifying regular maps by groups;
2. Classifying regular maps by underlying graphs
3. Classifying regular maps by genus

§2. Combinatorial and Algebraic Map

Combinatorial Orientable Map: connected simple graph $\mathcal{G} = \mathcal{G}(V, D)$, with vertex set $V = V(\mathcal{G})$, dart (arc) set $D = D(\mathcal{G})$.

Arc-reversing involution L: interchanging the two arcs underlying every given edge.

Rotation R: cyclically permutes the arcs initiated at v for each vertex $v \in V(\mathcal{G})$.

Map $\mathcal{M}$ with underlying graph $\mathcal{G}$: the triple $\mathcal{M} = \mathcal{M}(\mathcal{G};\, R,\, L)$.

Remarks *Monodromy group* $\mathrm{Mon}(\mathcal{M}) := \langle R, L\rangle$ acts transitively on D.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(\mathcal{G}_1;\, R_1,\, L_1),\ \mathcal{M}_2 = \mathcal{M}(\mathcal{G}_2;\, R_2\, L_2),$$

Map isomorphism: bijection $\phi : D(\mathcal{G}_1) \to D(\mathcal{G}_2)$ such that

$$L_1\phi = \phi L_2, R_1\phi = \phi R_2$$

Automorphism ϕ of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: Aut $(\mathcal{M})$

Remarks Aut $(\mathcal{M}) = C_{S_D}(\mathrm{Mon}(\mathcal{M}))$ and Aut $(\mathcal{M})$ acts semi-regularly on D,

Regular Map: Aut $(\mathcal{M})$ acts regularly on D.

Remarks For regular map, we have

(i) Aut $(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;

(ii) Aut $(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ on D can be viewed as the right and the left regular representations of an abstract group $G = \mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$

Algebraic Orientable Maps:

Coset graph: Group G, $H \leqslant G$ free-core, $B = HgH$ with $B = B^{-1}$. Define *coset graph* $\mathcal{G} = \mathcal{G}(G; H, B)$ by $V(\mathcal{G}) = \{Hg \mid g \in G\}$ and $D(\mathcal{G}) = \{(Hg, Hbg) \mid b \in B, g \in G\}$.

Definition 2.1 *Let $G = \langle r, \ell\rangle$ be a finite two-generator group with $\ell^2 = 1$ and $\langle r\rangle \cap \langle r\rangle^\ell = 1$. By an algebraic map $\mathcal{M}(G;\, r, \ell) = (\mathcal{G};\, R)$, we mean the map whose underlying graph is the coset graph $\mathcal{G} = \mathcal{G}(G; \langle r\rangle, \langle r\rangle\ell\langle r\rangle)$ and rotation R is determined by*

$$(\langle r\rangle g, \langle r\rangle l r^i g)^R = (\langle r\rangle g, \langle r\rangle l r^{i+1} g),$$

for any $g \in G$.

See:

S.F. Du, J.H. Kwak and R. Nedela, A Classification of regular embeddings of graphs of order a product of two primes, *J. Algeb. Combin.* **19**(2004), 123–141.

(i) any algebraic map $\mathcal{M}$ is regular with $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M}) \cong G$.

(ii) Each regular map can be represented an algebraic map.

(iii) $\mathcal{M}(G; r_1, \ell_1) \cong \mathcal{M}(G;\, r_2, \ell_2)$ if and only if there exists an element $\sigma \in \mathrm{Aut}\,(G)$ such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

Classify all regular maps of a given underlying arc-transitive graph $\mathcal{G}$ with valency s in the following two steps:

(1) Find the representatives G (as abstract groups) of the isomorphic classes of arc-regular subgroups of $\mathrm{Aut}\,(\mathcal{G})$ with cyclic vertex-stabilizers.

(2) For each group G given in (1), determine all the algebraic regular maps $\mathcal{M}(G; r, \ell)$ with underlying graphs isomorphic to $\mathcal{G}$, or equivalently, determine the representatives of the orbits of $\mathrm{Aut}\,(G)$ on the set of generating pairs (r, ℓ) of G such that $|r| = n$, $|\ell| = 2$ and $\mathcal{G}(G; \langle r\rangle, \langle r\rangle\ell\langle r\rangle) \cong \mathcal{G}$.

Combinatorial Nonorientable Map:

Definition 2.2 *For a given finite set F and three fixed-point free involutory permutations t, r, ℓ on F, a quadruple $\mathcal{M} = \mathcal{M}(F; t, r, \ell)$ is called a combinatorial map if they satisfy two conditions: (1) $t\ell = \ell t$; (2) the group $\langle t, r, \ell\rangle$ acts transitively on F.*

F: *flag* set;

t, r, ℓ are called *longitudinal, rotary, and transverse involution,* respectively.

$\mathrm{Mon}(\mathcal{M}) = \langle t, r, \ell\rangle$: *Monodromy group* of $\mathcal{M}$,

Vertices, edges and *face-boundaries* of $\mathcal{M}$ to be orbits of the subgroups $\langle t, r\rangle$, $\langle t, \ell\rangle$ and $\langle r, \ell\rangle$, respectively.

The incidence in $\mathcal{M}$ can be represented by nontrivial intersection.

The map $\mathcal{M}$ is called *unoriented*.

The even-word subgroup $\langle tr, r\ell \rangle$ of Mon($\mathcal{M}$) has the index at most 2.

Orientable: if the index is 2,

Nonorientable: if the index is 1

Given two maps $\mathcal{M}_1 = \mathcal{M}(F_1; t_1, r_1, \ell_1)$ and $\mathcal{M}_2 = \mathcal{M}_2(F_2; t_2, r_2, \ell_2)$,

Map isomorphism: bijection $\phi : F_1 \to F_2$ such that

$$\phi t_1 = t_2\phi, \quad \phi r_1 = r_2\phi, \quad \phi \ell_1 = \ell_2\phi.$$

Automorphism of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: Aut ($\mathcal{M}$)

Remarks Aut $(\mathcal{M}) = C_{S_F}(\mathrm{Mon}(\mathcal{M}))$ and Aut ($\mathcal{M}$) acts semi-regularly on F,

Regular Map: Aut ($\mathcal{M}$) acts regularly on F.

Remarks For regular map, we have

(i) Aut $(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;

(ii) Aut ($\mathcal{M}$) and Mon($\mathcal{M}$) on F can be viewed as the right and the left regular representations of an abstract group G.

(iii) $\mathcal{M}(G; t_1, r_1, \ell_1) \cong \mathcal{M}(G; t_2, r_2, \ell_2)$ if and only if there exists $\sigma \in \mathrm{Aut}\,(G)$ such that $t_1^\sigma = t_2$, $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

§3. Classify Regular Maps by Given Graphs

3.1 Complete Graphs

Orientrable:

N.L. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat.* (6) **4** (1971), 132-138.

L.D. James and G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

The proof uses the characterization of sharply doubly transitive permutation groups, $n = p^k$ and $G = \mathrm{AGL}(1, p^k)$.

Nonorientrable:

S. E. Wilson, Cantankerous maps and rotary embeddings of K_n, *J. Combin. Theory Ser. B* **47** (1989), 262–273.

n must be of order 3, 4 or 6.

3.2 Complete Multipartite Graphs $K_n[\bar{K}_p]$, p Prime

Orientable maps.

S.F. Du, J.H. Kwak and R. Nedela, Regular embeddings of complete multipartite graphs, *European J. Combin.* **26** (2005), 505–519.

Independent of CFSG.

3.3 Graphs of Order pq

Orienable maps:

S.F. Du, J.H. Kwak and R. Nedela, A Classification of regular embeddings of graphs of order a product of two primes, *J. Algeb. Combin.* **19**(2004), 123–141.

Independent of CFSG.

Nonorienable maps:

S.F. Du and F.R.Wang, Nonorientable regular embeddings of graphs of order a product of two distinct primes, submitted to *J. Graph Theory*.

depends on CFSG.

3.4 Complete Bipartite Graphs $K_{n,n}$

Nonorientable Maps:

J.H.Kwak and Y.S.Kwon, Classification of nonorientable regular embeddings complete bipartite graphs, Submitted.

Orientable Maps:

Regular embeddings of $K_{n,n}$ are very important, which have been studied in connection with various branches of mathematics including Riemann surfaces and algebraic curves, Galois groups, see survey paper:

G.A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca* **47** (1997), 1-33.

Classification processes was begun by Nedela, Škovuera and Zlatoš:

R. Nedela, M. Škoviera and A. Zlatoš, Regular embeddings of complete bipartite graphs, *Discrete Math.*, **258** (1-3), 2002, p. 379–381.

n is a product of two primes:

J.H. Kwak and Y.S. Kwon, Regular orientable embeddings of complete bipartite graphs, *J. Graph Theory* **50** (2005), 105–122.

Reflexible maps:

J. H. Kwak and Y. S. Kwon, Classification of reflexible regular embeddings and self-Petrie dual regular embeddings of complete bipartite graphs, Submitted.

$(n, \phi(n)) = 1$:

G.A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with unique regular embeddings, submitted.

$n = p^k$, p is odd prime:

G.A. Jones, R. Nedela and M. Škoviera, Regular Embeddings of $K_{n,n}$ where n is an odd prime power, *European J. Combin.* **28** (2007), 1863-1875.

$n = 2^k$:

S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I: Metacyclic case, *European J. Combin.* **28** (2007), 1595-1608.

S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. II: Nonmetacyclic case, submitted

Any n:

G.A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, (preparation, 2007.)

The key point for this work is to determine the structure of group
$G = \langle a\rangle\langle b\rangle$, where $|a| = |b| = n$, $\langle a\rangle \cap \langle b\rangle = 1$, and $a^\alpha = b$ for some involution α in Aut (G).

If $n = p^k$ and p is odd, then a result of Huppert implies that such a group G must be metacyclic.
If $n = 2^k$, we need to classify non-metacyclic case.

Theorem 3.1 (Du, Jones, Kwak, Nedela, Skoviera)

Suppose that $G = \langle a\rangle\langle b\rangle$, where $|a| = |b| = 2^e$, $e \geqslant 2$, $\langle a\rangle \cap \langle b\rangle = 1$, and $a^\alpha = b$ for some involution α in Aut (G). Then one of the following cases hold:

(1) G is metacyclic and G has presentation

$$G_1(e, f) = \langle h, g \mid h^{2^e} = g^{2^e} = 1, h^g = h^{1+2^f}\rangle$$

where $f = 2, \ldots, e$, and we may set $a = g^m$ and $b = g^m h$, where m is odd, $1 \leq m \leq 2^{e-f}$;

(2) G is not metacyclic, $G' \cong C_2$, and G has presentation

$$G_2 = \langle a, b \mid a^4 = b^4 = [a^2, b] = [b^2, a] = 1, [b, a] = a^2 b^2\rangle;$$

(3) G' is generated by two elements, and G has presentation

$$\begin{aligned} G_3(e, k, l) = \langle a, b \mid\ & a^{2^e} = b^{2^e} = [b^2, a^2] = 1, \\ & [b, a] = a^{2+k2^{e-1}} b^{-2+k2^{e-1}}, \\ & (b^2)^a = a^{l2^{e-1}} b^{-2+l2^{e-1}}, (a^2)^b = a^{-2+l2^{e-1}} b^{l2^{e-1}}\rangle, \end{aligned}$$

where $e \geqslant 3$, and $k, l \in \{0, 1\}$. Moreover, $G_3(e, 0, 1) \cong G_3(e, 1, 1)$.

Generally, for the structure of groups which is a product of two abelian groups, here I recommend the following papers:

B. Huppert, Über das Produkt von paarweise vertauschbaren zyklischen Gruppen, *Math. Z.* **58** (1953), 243–264.

N. Itô, Über das Produkt von zwei abelschen Gruppen, *Math. Z.* **62** (1955), 400–401.

N. Itô, Über das Produkt von zwei zyklischen 2-Gruppen, *Publ. Math. Debrecen* **4** (1956), 517–520.

M. D. E. Conder and I. M. Isaacs, Derived subgroups of product an abelian and a cyclic subgroup, *J. London Math. Soc.* **69** (2004), 333–348.

(4) *n-dimensional hypercubes Q_n:*

Graph Q_n : vertex set $V = V(n, 2)$, while two vertices $\mathbf{x_1}$ and $\mathbf{x_2}$ are adjacent if and only if $\mathbf{x_1}+\mathbf{x_2}$ is an unit vector. This graph has valency n and automorphism group $\mathrm{Aut}\,(Q_n) = Z_2^n : S_n$

Nonorientable maps:

Y.S. Kwon and R. Nedela, Non-existence of nonorientable regular embedings of $n-$dimensional cubes, to appear in *Discrete Math.*.

Only Q_2, which an embedding in projective plane.

Orientable maps: n is odd:

S.F. Du, J.H. Kwak and R. Nedela, Classification of regular embeddings of hypercubes of odd dimension, *Discrete Math.* **307**(1) (2007), 119-124.

$n = 2m$, m is odd: Jing Xu, Classification of regular embeddings of hypercubes of dimension $2m$, when m is odd, *Science in China, 2007*

Problem 3.2 *Classify regular embeddings of hypercubes dimension n for $n = 2^k m$,$k \geqslant 2$ and $m \geqslant 3$ is odd.*

Key point is to determine the arc regular subgroups $\langle r, \ell\rangle$ of $\mathrm{Aut}\,(Q_n) = Z_2^n : S_n$ s.t. $|r| = n$ and $|\ell| = 2$.

§4. Classify Regular Maps by Given Groups

Question 4.1 *Study a finite group G, as quotients of triangle groups, realize these groups as automorphism groups of compact Riemann surfaces.*

G.A. Jones and D. Singerman, *Complex function: an algebraic and geometric viewpoint,* Cambridge Univ. Press, 1987.

Question 4.2 *Given a group G, classify all the regular maps with the automorphism groups isomorphic to G.*

1. Orientable cases

(i) $G = \mathrm{PSL}(2, q)$:

Macbeath described all triples x, y, z with $xyz = 1$ generating PSL$(2, q)$ (in fact $SL(2, q)$) in terms of their orders.

A.M.Macbeath, Generators of the linear fractional groups. 1969 Number Theory Proc. Sympos. Pure Math., Vol. XII, Houston, Tex., 1967, 14–32 Amer. Math. Soc., Providence, R.I. 20.75 (30.00)

Other results:

A.A. Albert and J. Thompson, Two-element generation of the projective unimodular group, *Illinois J. Math.* **3**(1959), 421–439.

M. Downs, Some enumerations of regular hypermaps with automorphism group isomorphic to $\mathrm{PSL}_2(q)$, *Quart. J. Math. Oxford Ser.* (2)**48**(1997), 39–58.

D. Garbe, Über eine Klasse von arithmetisch definierbaren Normalteilern der Modulgruppe. (German) *Math. Ann.* (3)**235**(1978), 195–215.

H. Glover and D. Sjerve, The genus of $\mathrm{PSL}_2(q)$, *J. Reine Angew. Math.* **380**(1987), 59–86.

H. Glover and D. Sjerve, Representing $\mathrm{PSL}_2(p)$ on a Riemann surface of least genus, *Enseign. Math.* (2)**31** (1985), 305–325.

U. Langer and R. Rosenberger, Erzeugende endlicher projektiver linearer Gruppen (German), *Results Math.* **15**(1989), 119–148.

C.H. Sah, Groups realted to compact Riemann surfaces, *Acta Math.* **1969**, 13–42.

D.B. Surowski, Vertex-transitive triangulations of compact orientable 2-manifolds, *J. Combin. Theory Ser. B* (3)**39** (1985), 371–375.

(ii) Hurwitz groups

$(2, 3, 7)$ triangle group, where many cases of finite (usually simple) groups have been recently shown to be quotients or non-quotients, Hurwitz groups or non-Hurwitz groups

A_n and S_n:

M.D.E. Conder, The summetric genus of alternating and symmetric groups, *J. Combin. Theroy Ser. B* **39**(1985), 179-186.

Suzuki groups:

G.A. Jones and S.A. Silver, Suzuki groups and surfaces, *J. London Math. Soc.* (2)**48**(1993), 117–125.

Ree groups:

G.A. Jones, Ree groups and Riemann surfaces, *J. Algebra*, (1)**165**(1994), 41–62.

C.H. Sah, Groups realted to compact Riemann surfaces, *Acta Math.* **1969**, 13–42.

Survey paper:

M.D.E. Conder, Hurwitz groups: A brief survey, *Bull. Amer. Math. Soc.* **23**(1990), 359-370.

2. Nonorientalbe cases

very few results.

Singerman showed that $\mathrm{PSL}(2,q)$ is a homomorphic image of the extended modular group for all q except for $q=7$, 11 and 3^n, where $n=2$ or n is odd and he gave some applications to group actions on surfaces.

D. Singerman, $\mathrm{PSL}(2,q)$ as an image of the extended modular group with applications to group actions on surfaces, Groups—St. Andrews 1985. *Proc. Edinburgh Math. Soc.* (2)**30**(1987), no. 1, 143–151.

S.F. Du and J.H.Kwak, Groups $\mathrm{PSL}(3,p)$ and Nonorientable Regular Maps, *Journal of Algebra.* in reversion, 2007, 23 pages.

Theorem 4.3 *For a prime p, set $G=\mathrm{SL}(3,p)$ and $\overline{G}=\mathrm{PSL}(3,p)$. Let $\mathcal{M}$ be a nonorientable regular map with the automorphism group isomorphic to $\overline{G}$. Then $\mathcal{M}$ is isomorphic to one of the maps $\mathcal{M}(\alpha,\beta)=\mathcal{M}(\overline{G};\bar{t},\bar{r},\bar{l}_{(\alpha,\beta)})$, where*

$$t=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad r=\begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\ell_{(\alpha,\beta)}=\begin{pmatrix} \alpha & \beta & 0 \\ \beta^{-1}(1-\alpha^2) & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $p\geqslant 5$, $\alpha,\beta\in F_p^*$ and $\alpha\neq\pm1$. Moreover, $\mathcal{M}(\alpha_1,\beta_1)\cong\mathcal{M}(\alpha_2,\beta_2)$ if and only if $(\alpha_1,\beta_1)=(\alpha_2,\beta_2)$. In particular, there are p^2-4p+3 maps, each of which has the simple underlying graph of valency p.

Problem 4.4 *Given a finite simple group G, classify all the regular maps with the automorphism groups isomorphic to G.*

Smarandache Directionally n-Signed Graphs — A Survey

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Abstract: Let $G = (V, E)$ be a graph. By *directional labeling (or d-labeling)* of an edge $x = uv$ of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$, we mean a labeling of the edge x such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ in the direction from u to v, and the label on x as $(a_n, a_{n-1}, \cdots, a_1)$ in the direction from v to u. In this survey, we study graphs, called *(n, d)-sigraphs*, in which every edge is d-labeled by an n-tuple $(a_1, a_2, \cdots, a_n)$, where $a_k \in \{+, -\}$, for $1 \leq k \leq n$. Several variations and characterizations of directionally n-signed graphs have been proposed and studied. These include the various notions of balance and others.

Key Words: Signed graphs, directional labeling, complementation, balance.

AMS(2010): 05C 22

§1. Introduction

For graph theory terminology and notation in this paper we follow the book [3]. All graphs considered here are finite and simple.

There are two ways of labeling the edges of a graph by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ (See [10]).

1. *Undirected labeling* or *labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ irrespective of the direction from u to v or v to u.

2. *Directional labeling* or *d-labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ in the direction from u to v, and $(a_n, a_{n-1}, \cdots, a_1)$ in the direction from v to u.

Note that the d-labeling of edges of G by ordered n-tuples is equivalent to labeling the symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where uv is a symmetric arc in $\overrightarrow{G}$ if, and only if, uv is an edge in G, so that if $(a_1, a_2, \cdots, a_n)$ is the d-label on uv in G, then the labels on the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are $(a_1, a_2, \cdots, a_n)$ and $(a_n, a_{n-1}, \cdots, a_1)$ respectively.

Let H_n be the *n-fold sign group*, $H_n = \{+, -\}^n = \{(a_1, a_2, \cdots, a_n) : a_1, a_2, \cdots, a_n \in \{+, -\}\}$ with co-ordinate-wise multiplication. Thus, writing $a = (a_1, a_2, \cdots, a_n)$ and $t = (t_1, t_2, \cdots, t_n)$ then $at := (a_1t_1, a_2t_2, \cdots, a_nt_n)$. For any $t \in H_n$, the *action* of t on H_n is $a^t = at$, the co-ordinate-wise product.

Let $n \geq 1$ be a positive integer. An *n-signed graph (n-signed digraph)* is a graph $G = (V, E)$

in which each edge (arc) is labeled by an ordered n-tuple of signs, i.e., an element of H_n. A *signed graph* $G = (V, E)$ is a graph in which each edge is labeled by $+$ or $-$. Thus a 1-signed graph is a signed graph. Signed graphs are well studied in literature (See for example [1, 4-7, 13-21, 23, 24].

In this survey, we study graphs in which each edge is labeled by an ordered n-tuple $a = (a_1, a_2, \cdots, a_n)$ of signs (i.e, an element of H_n) in one direction but in the other direction its label is the reverse: $a^r = (a_n, a_{n-1}, \cdots, a_1)$, called *directionally labeled n-signed graphs* (or *(n, d)-signed graphs*).

Note that an n-signed graph $G = (V, E)$ can be considered as a symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where both $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are arcs if, and only if, uv is an edge in G. Further, if an edge uv in G is labeled by the n-tuple $(a_1, a_2, \cdots, a_n)$, then in $\overrightarrow{G}$ both the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are labeled by the n-tuple $(a_1, a_2, \cdots, a_n)$.

In [1], the authors study voltage graph defined as follows: A *voltage graph* is an ordered triple $\overrightarrow{G} = (V, \overrightarrow{E}, M)$, where V and $\overrightarrow{E}$ are the vertex set and arc set respectively and M is a group. Further, each arc is labeled by an element of the group M so that if an arc $\overrightarrow{uv}$ is labeled by an element $a \in M$, then the arc $\overrightarrow{vu}$ is labeled by its inverse, a^{-1}.

Since each n-tuple $(a_1, a_2, \cdots, a_n)$ is its own inverse in the group H_n, we can regard an n-signed graph $G = (V, E)$ as a voltage graph $\overrightarrow{G} = (V, \overrightarrow{E}, H_n)$ as defined above. Note that the d-labeling of edges in an (n, d)-signed graph considering the edges as symmetric directed arcs is different from the above labeling. For example, consider a $(4, d)$-signed graph in **F**igure 1. As mentioned above, this can also be represented by a symmetric 4-signed digraph. Note that this is not a voltage graph as defined in [1], since for example; the label on $\overrightarrow{v_2v_1}$ is not the (group) inverse of the label on $\overrightarrow{v_1v_2}$.

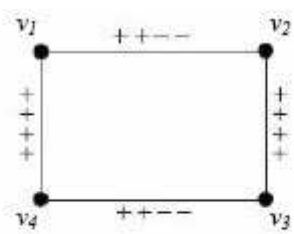

Fig.1

In [8-9], the authors initiated a study of $(3, d)$ and $(4, d)$-Signed graphs. Also, discussed some applications of $(3, d)$ and $(4, d)$-Signed graphs in real life situations.

In [10], the authors introduced the notion of complementation and generalize the notion of balance in signed graphs to the directionally n-signed graphs. In this context, the authors look upon two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge. Also given some motivation to study (n, d)-signed graphs in connection with relations among human beings in society.

In [10], the authors defined complementation and isomorphism for (n, d)-signed graphs as follows: For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^r = (a_n, a_{n-1}, \cdots, a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an (n,d)-signed graph $G = (V,E)$, written G^t, is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^t. The *reversal* G^r is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^r.

Let $G = (V,E)$ and $G' = (V', E')$ be two (n,d)-signed graphs. Then G is said to be *isomorphic* to G' and we write $G \cong G'$, if there exists a bijection $\phi : V \to V'$ such that if uv is an edge in G which is d-labeled by $a = (a_1, a_2, \cdots, a_n)$, then $\phi(u)\phi(v)$ is an edge in G' which is d-labeled by a, and conversely.

For each $t \in H_n$, an (n,d)-signed graph $G = (V,E)$ is *t-self complementary*, if $G \cong G^t$. Further, G is *self reverse*, if $G \cong G^r$.

Proposition 1.1(E. Sampathkumar et al. [10]) *For all $t \in H_n$, an (n,d)-signed graph $G = (V,E)$ is t-self complementary if, and only if, G^a is t-self complementary, for any $a \in H_n$.*

For any cycle C in G, let $\mathcal{P}(\overrightarrow{C})$ [10] denotes the product of the n-tuples on C given by $(a_{11}, a_{12}, \cdots, a_{1n})(a_{21}, a_{22}, \cdots, a_{2n}) \cdots (a_{m1}, a_{m2}, \cdots, a_{mn})$ and

$$\mathcal{P}(\overleftarrow{C}) = (a_{mn}, a_{m(n-1)}, \cdots, a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, \cdots, a_{(m-1)1}) \cdots (a_{1n}, a_{1(n-1)}, \cdots, a_{11}).$$

Similarly, for any path P in G, $\mathcal{P}(\overrightarrow{P})$ denotes the product of the n-tuples on P given by $(a_{11}, a_{12}, \cdots, a_{1n})(a_{21}, a_{22}, \cdots, a_{2n}) \cdots (a_{m-1,1}, a_{m-1,2}, \cdots, a_{m-1,n})$ and

$$\mathcal{P}(\overleftarrow{P}) = (a_{(m-1)n}, a_{(m-1)(n-1)}, \cdots, a_{(m-1)1}) \cdots (a_{1n}, a_{1(n-1)}, \cdots, a_{11}).$$

An n-tuple $(a_1, a_2, \cdots, a_n)$ is *identity n-tuple*, if each $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. Further an n-tuple $a = (a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a^r = a$, otherwise it is a *non-symmetric n-tuple*. In (n,d)-signed graph $G = (V,E)$ an edge labeled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Note that the above products $\mathcal{P}(\overrightarrow{C})$ $(\mathcal{P}(\overrightarrow{P}))$ as well as $\mathcal{P}(\overleftarrow{C})$ $(\mathcal{P}(\overleftarrow{P}))$ are n-tuples. In general, these two products need not be equal.

§2. Balance in an (n,d)-Signed Graph

In [10], the authors defined two notions of balance in an (n,d)-signed graph $G = (V,E)$ as follows:

Definition 2.1 *Let $G = (V,E)$ be an (n,d)-sigraph. Then,*

(i) *G is* identity balanced *(or* i-balanced*), if $P(\overrightarrow{C})$ on each cycle of G is the identity n-tuple, and*

(ii) *G is* balanced*, if every cycle contains an even number of non-identity edges.*

Note: An i-balanced (n,d)-sigraph need not be balanced and conversely. For example, consider the $(4,d)$-sigraphs in Figure.2. In Figure.2(a) G is an i-balanced but not balanced, and in Figure.2(b) G is balanced but not i-balanced.

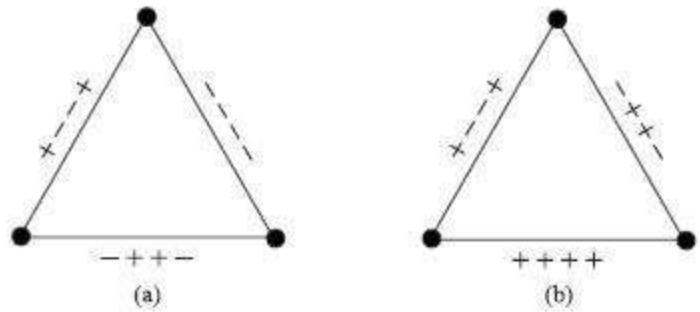

Fig.2

2.1 Criteria for balance

An (n, d)-signed graph $G = (V, E)$ is i-balanced if each non-identity n-tuple appears an even number of times in $P(\overrightarrow{C})$ on any cycle of G.

However, the converse is not true. For example see Figure.3(a). In Figure.3(b), the number of non-identity 4-tuples is even and hence it is balanced. But it is not i-balanced, since the 4-tuple $(+ + --)$ (as well as $(- - ++)$) does not appear an even number of times in $P(\overrightarrow{C})$ of 4-tuples.

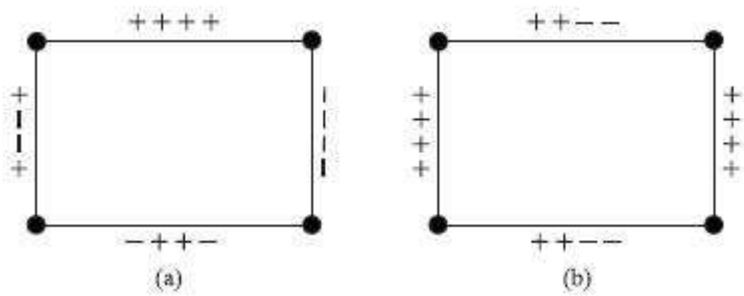

Fig.3

In [10], the authors obtained following characterizations of balanced and i-balanced (n, d)-sigraphs:

Proposition 2.2(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is balanced if, and only if, there exists a partition $V_1 \cup V_2$ of V such that each identity edge joins two vertices in V_1 or V_2, and each non-identity edge joins a vertex of V_1 and a vertex of V_2.*

As earlier discussed, let $P(C)$ denote the product of the n-tuples in $P(\overrightarrow{C})$ on any cycle C in an (n, d)-sigraph $G = (V, E)$.

Theorem 2.3(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in $P(C)$ whose k^{th} co-ordinate is $-$ is even.*

In H_n, let S_1 denote the set of non-identity symmetric n-tuples and S_2 denote the set of non-symmetric n-tuples. The product of all n-tuples in each $S_k, 1 \leq k \leq 2$ is the identity n-tuple.

Theorem 2.4(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced, if both of the following hold:*

(i) In $P(C)$, each n-tuple in S_1 occurs an even number of times, or each n-tuple in S_1 occurs odd number of times (the same parity, or equal mod 2).

(ii) In $P(C)$, each n-tuple in S_2 occurs an even number of times, or each n-tuple in S_2 occurs an odd number of times.

In [11], the authors obtained another characterization of i-balanced (n, d)-signed graphs as follows:

Theorem 2.5(E.Sampathkumar et al. [11]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced if, and only if, any two vertices u and v have the property that for any two edge distinct $u - v$ paths $\overrightarrow{P_1} = (u = u_0, u_1, \cdots, u_m = v$ and $\overrightarrow{P_2} = (u = v_0, v_1, \cdots, v_n = v)$ in G, $\mathcal{P}(\overrightarrow{P_1}) = (\mathcal{P}(\overrightarrow{P_2}))^r$ and $\mathcal{P}(\overrightarrow{P_2}) = (\mathcal{P}(\overrightarrow{P_1}))^r$.*

From the above result, the following are the easy consequences:

Corollary 2.6 *In an i-balanced (n, d)-signed graph G if two vertices are joined by at least 3 paths then the product of n tuples on any paths joining them must be symmetric.*

A graph $G = (V, E)$ is said to be k-connected for some positive integer k, if between any two vertices there exists at least k disjoint paths joining them.

Corollary 2.7 *If the underlying graph of an i-balanced (n, d)-signed graph is 3-connected, then all the edges in G must be labeled by a symmetric n-tuple.*

Corollary 2.8 *A complete (n, d)-signed graph on $p \geq 4$ is i-balanced then all the edges must be labeled by symmetric n-tuple.*

2.2 Complete (n, d)-Signed Graphs

In [11], the authors defined: an (n, d)-sigraph is *complete*, if its underlying graph is complete. Based on the complete (n, d)-signed graphs, the authors proved the following results: An (n, d)-signed graph is *complete*, if its underlying graph is complete.

Proposition 2.9(E.Sampathkumar et al. [11]) *The four triangles constructed on four vertices $\{a, b, c, d\}$ can be directed so that given any pair of vertices say (a, b) the product of the edges of these 4 directed triangles is the product of the n-tuples on the arcs $\overrightarrow{ab}$ and $\overrightarrow{ba}$.*

Corollary 2.10 *The product of the n-tuples of the four triangles constructed on four vertices $\{a, b, c, d\}$ is identity if at least one edge is labeled by a symmetric n-tuple.*

The *i-balance base* with axis a of a complete (n, d)-signed graph $G = (V, E)$ consists list of the product of the n-tuples on the triangles containing a [11].

Theorem 2.11(E.Sampathkumar et al. [11]) *If the i-balance base with axis a and n-tuple of an edge adjacent to a is known, the product of the n-tuples on all the triangles of G can be deduced from it.*

In the statement of above result, it is not necessary to know the n-tuple of an edge incident at a. But it is sufficient that an edge incident at a is a symmetric n-tuple.

Theorem 2.12(E.Sampathkumar et al. [11]) *A complete (n,d)-sigraph $G=(V,E)$ is i-balanced if, and only if, all the triangles of a base are identity.*

Theorem 2.13(E.Sampathkumar et al. [11]) *The number of i-balanced complete (n,d)-sigraphs of m vertices is p^{m-1}, where $p=2^{\lceil n/2\rceil}$.*

§3. Path Balance in (n,d)-Signed Graphs

In [11], E.Sampathkumar et al. defined the path balance in an (n,d)-signed graphs as follows:

Let $G=(V,E)$ be an (n,d)-sigraph. Then G is

1. *Path i-balanced*, if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$.

2. *Path balanced* if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v have same number of non identity n-tuples.

Clearly, the notion of path balance and balance coincides. That is an (n,d)-signed graph is balanced if, and only if, G is path balanced.

If an (n,d) signed graph G is i-balanced then G need not be path i-balanced and conversely.

In [11], the authors obtained the characterization path i-balanced (n,d)-signed graphs as follows:

Theorem 3.1(Characterization of path i-balanced $(n;d)$ signed graphs) *An (n,d)-signed graph is path i-balanced if, and only if, any two vertices u and v satisfy the property that for any two vertex disjoint $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$.*

§4. Local Balance in (n,d)-Signed Graphs

The notion of local balance in signed graph was introduced by F. Harary [5]. A signed graph $S=(G,\sigma)$ is locally at a vertex v, or S is *balanced at* v, if all cycles containing v are balanced. A cut point in a connected graph G is a vertex whose removal results in a disconnected graph. The following result due to Harary [5] gives interdependence of local balance and cut vertex of a signed graph.

Theorem 4.1(F.Harary [5]) *If a connected signed graph $S = (G, \sigma)$ is balanced at a vertex u. Let v be a vertex on a cycle C passing through u which is not a cut point, then S is balanced at v.*

In [11], the authors extend the notion of local balance in signed graph to (n, d)-signed graphs as follows: Let $G = (V, E)$ be a (n, d)-signed graph. Then for any vertices $v \in V(G)$, G is *locally i-balanced at v* (*locally balanced at v*) if all cycles in G containing v is i-balanced (balanced).

Analogous to the above result, in [11] we have the following for an (n, d) signed graphs:

Theorem 4.2 *If a connected (n, d)-signed graph $G = (V, E)$ is locally i-balanced (locally balanced) at a vertex u and v be a vertex on a cycle C passing through u which is not a cut point, then S is locally i-balanced(locally balanced) at v.*

§5. Symmetric Balance in (n, d)-Signed Graphs

In [22], P.S.K.Reddy and U.K.Misra defined a new notion of balance called *symmetric balance* or *s-balanced* in (n, d)-signed graphs as follows:

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil n/2 \rceil$. Let $G = (V, E)$ be an (n, d)-signed graph. Then G is *symmetric balanced* or *s-balanced* if $P(\overrightarrow{C})$ on each cycle C of G is symmetric n-tuple.

Note: If an (n, d)-signed graph $G = (V, E)$ is i-balanced then clearly G is s-balanced. But a s-balanced (n, d)-signed graph need not be i-balanced. For example, the $(4, d)$-signed graphs in Figure 4. G is an s-balanced but not i-balanced.

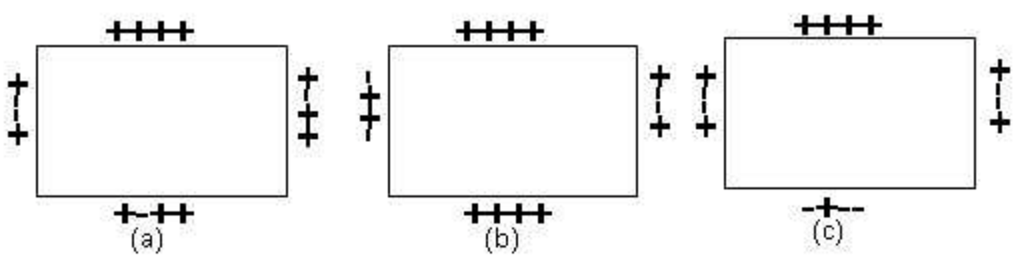

Fig.4

In [22], the authors obtained the following results based on symmetric balance or s-balanced in (n, d)-signed graphs.

Theorem 5.1(P.S.K.Reddy and U.K.Mishra [22]) *A (n, d)-signed graph is s-balanced if and only if every cycle of G contains an even number of non-symmetric n-tuples.*

The following result gives a necessary and sufficient condition for a balanced (n, d)-signed graph to be s-balanced.

Theorem 5.2(P.S.K.Reddy and U.K.Mishra [22]) *A balanced (n, d) signed graph $G = (V, E)$ is s-balanced if and only if every cycle of G contains even number of non identity symmetric n tuples.*

In [22], the authors obtained another characterization of s-balanced (n, d)-signed graphs, which is analogous to the partition criteria for balance in signed graphs due to Harary [4].

Theorem 5.3(Characterization of s-balanced (n, d)-sigraph) *An (n, d)-signed graph $G = (V, E)$ is s balanced if and only if the vertex set $V(G)$ of G can be partitioned into two sets V_1 and V_2 such that each symmetric edge joins the vertices in the same set and each non-symmetric edge joins a vertex of V_1 and a vertex of V_2.*

An n-marking $\mu : V(G) \to H_n$ of an (n, d)-signed graph $G = (V, E)$ is an assignment n-tuples to the vertices of G. In [22], the authors given another characterization of s-balanced (n, d)-signed graphs which gives a relationship between the n-marking and s-balanced (n, d)-signed graphs.

Theorem 5.4(P.S.K.Reddy and U.K.Mishra [22]) *An (n, d)-signed graph $G = (V, E)$ is s-balanced if and only if there exists an n-marking μ of vertices of G such that if the n-tuple on any arc $\overrightarrow{uv}$ is symmetric or nonsymmetric according as the n-tuple $\mu(u)\mu(v)$ is.*

§6. Directionally 2-Signed Graphs

In [12], E.Sampathkumar et al. proved that the directionally 2-signed graphs are equivalent to bidirected graphs, where each end of an edge has a sign. A bidirected graph implies a signed graph, where each edge has a sign. Signed graphs are the special case $n = 1$, where directionality is trivial. Directionally 2-signed graphs (or $(2, d)$-signed graphs) are also special, in a less obvious way. A bidirected graph B $= (G, \beta)$ is a graph $G = (V, E)$ in which each end (e, u) of an edge $e = uv$ has a sign $\beta(e, u) \in \{+, -\}$. G is the underlying graph and β is the bidirection. (The $+$ sign denotes an arrow on the u-end of e pointed into the vertex u; a $-$ sign denotes an arrow directed out of u. Thus, in a bidirected graph each end of an edge has an independent direction. Bidirected graphs were defined by Edmonds [2].) In view of this, E.Sampathkumar et al. [12] proved the following result:

Theorem 6.1(E.Sampathkumar et al. [12]) *Directionally 2-signed graphs are equivalent to bidirected graphs.*

§7. Conclusion

In this brief survey, we have described directionally n-signed graphs (or (n, d)-signed graphs) and their characterizations. Many of the characterizations are more recent. This in an active area of research. We have included a set of references which have been cited in our description. These references are just a small part of the literature, but they should provide a good start for readers interested in this area.

References

[1] B.D.Acharya and M. Acharya, New algebraic models of a Social System, *Indian J. of Pure and Appl. Math.*, 17(2) 1986, 152-168.

[2] J.Edmonds and E.L.Johnson, Matching: a well-solved class of integral linear programs, in: Richard Guy et al., eds., *Combinatorial Structures and Their Applications* (Proc. Calgary Int. Conf., Calgary, 1969), pp. 89-92. Gordon and Breach, New York, 1970.

[3] F.Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[4] F.Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

[5] F.Harary, On local balance and N-balance in signed graphs, *Michigan Math. J.*, 3(1955), 37-41.

[6] F.Harary, R. Norman and D. Cartwright, *Structural models: An introduction to the theory of directed graphs*, J. Wiley, New York, 1965.

[7] R.Rangarajan, M. S.Subramanya and P.Siva Kota Reddy, The H-Line Signed Graph of a Signed Graph, *International J. Math. Combin.*, 2 (2010), 37-43.

[8] E.Sampathkumar, P.Siva Kota Reddy and M. S.Subramanya, $(3, d)$-sigraph and its applications, *Advn. Stud. Contemp. Math.*, 17(1) (2008), 57-67.

[9] E.Sampathkumar, P.Siva Kota Reddy and M. S.Subramanya, $(4, d)$-sigraph and its applications, *Advn. Stud. Contemp. Math.*, 20(1) (2010), 115-124.

[10] E.Sampathkumar, P.Siva Kota Reddy, and M.S.Subramanya, Directionally n-signed graphs, in: B.D. Acharya et al., eds., *Advances in Discrete Mathematics and Applications: Mysore, 2008* (Proc. Int. Conf. Discrete Math., ICDM-2008), pp. 153-160, Ramanujan Math. Soc. Lect. Notes Ser., 13, Ramanujan Mathematical Society, Mysore, India, 2010.

[11] E.Sampathkumar, P.Siva Kota Reddy, and M.S.Subramanya, Directionally n-signed graphs-II, *Int. J. Math. Combin.*, 4 (2009), 89-98.

[12] E.Sampathkumar, M.A.Sriraj and T.Zaslavsky, Directionally 2-signed and bidirected graphs.

[13] P.Siva Kota Reddy and M.S.Subramanya, Signed graph equation $L^k(S) \sim \overline{S}$, *International J. Math. Combin.*, 4 (2009), 84-88.

[14] P.Siva Kota Reddy, S.Vijay and V.Lokesha, n^{th} Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, 12(3) (2009), 307-313.

[15] P.Siva Kota Reddy, S.Vijay and H.C.Savithri, A Note on Path Sidigraphs, *International J. Math. Combin.*, 1 (2010), 42-46.

[16] P.Siva Kota Reddy, S.Vijay and V.Lokesha, n^{th} Power signed graphs-II, *International J. Math. Combin.*, 1 (2010), 74-79.

[17] P.Siva Kota Reddy and S.Vijay, Total minimal dominating signed graph, *International J. Math. Combin.*, 3 (2010), 11-16.

[18] P.Siva Kota Reddy and K.V.Madhusudhan, Negation switching equivalence in signed graphs, *International J. Math. Combin.*, 3 (2010), 85-90.

[19] P.Siva Kota Reddy, t-Path Sigraphs, *Tamsui Oxford J. of Math. Sciences*, 26(4) (2010), 433-441.

[20] P.Siva Kota Reddy, B.Prashanth and Kavita S. Permi, A Note on Antipodal Signed Graphs, *International J. Math. Combin.*, 1 (2011), 107-112.

[21] P.Siva Kota Reddy, B.Prashanth, and T.R.Vasanth Kumar, Antipodal Signed Directed Graphs, *Advn. Stud. Contemp. Math.*, 21(4) (2011), 355-360.

[22] P.Siva Kota Reddy and U.K.Misra, Directionally n-signed graphs-III: The notion of symmetric balance, Submitted.

[23] T.Zaslavsky, Signed Graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.

[24] T.Zaslavsky, A mathematical bibliography of Signed and Gain graphs and its allied areas, *Electron. J. Combin.*, Dynamic Surveys in Combinatorics (1998), No. DS8. Eighth ed. (2012).

Neutrosophic Diagram and Classes of Neutrosophic Paradoxes, or To The Outer-Limits of Science

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Abstract: These paradoxes are called "neutrosophic"since they are based on indeterminacy (or neutrality, i.e. neither true nor false), which is the third component in neutrosophic logic. We generalize the Venn Diagram to a Neutrosophic Diagram, which deals with vague, inexact, ambiguous, ill defined ideas, statements, notions, entities with unclear borders. We define the neutrosophic truth table and introduce two neutrosophic operators (neuterization and antonymization operators) give many classes of neutrosophic paradoxes.

Key Words: neuterization operator, antonymization operator, neutrosophic paradoxes

AMS(2010): 51M15, 53B15, 53B40, 57N16

§1. Introduction

$< A >$ be an idea, or proposition, statement, attribute, theory, event, concept, entity, and $< nonA >$ what is not $< A >$. Let $< antiA >$ be the opposite of $< A >$. We have introduced a new notation [1998], $< neutA >$, which is neither $< A >$ nor $< antiA >$ but in between. $< neutA >$ is related with $< A >$ and $< antiA >$. Let' s see an example for vague (not exact) concepts: if $< A >$ is "tall" (an attribute), then $< antiA >$ is "*short*" , and $< neutA >$ is "*medium*" , while $< nonA >$ is "*not tall*" (which can be "*medium or short*"). Similarly for other $< A >, < neutA >, < antiA >$ such as: $< good >$, $< soso >$, $< bad >$, or $< perfect >, < average >, < imperfect >$, or $< high >, < medium >, < small >$, or respectively $< possible >, < sometimespossibleandothertimesimpossible >, < impossible >$, etc.

Now, let' s take an exact concept/statement: if $< A >$ is the statement "$1 + 1 = 2$ *in base* 10" , then $< antiA >$ is "$1 + 1 \neq 2$ *in base* 10" , while $< neutA >$ is undefined (doesn' t exist) since it is not possible to have a statement in between "$1 + 1 = 2$ *in base* 10" and "$1 + 1 \neq 2$ *in base* 10" because in base 10 we have $1 + 1$ is either equal to 2 or $1 + 1$ is different from 2. $< nonA >$ coincides with $< antiA >$ in this case, $< nonA >$ is "$1 + 1 \neq 2$ *in base* 10" .

Neutrosophy is a theory the author developed since 1995 as a generalization of dialectics. This theory considers every notion or idea $< A >$ together with its opposite or negation $< antiA >$, and the spectrum of *neutralities* in between them and related to them, noted by

$< neutA >$.

The neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

Its Fundamental Thesis:

Any idea $< A >$ is $T\%$ true, $I\%$ indeterminate (i.e. neither true nor false, but neutral, unknown), and $F\%$ false.

Its Fundamental Theory:

Every idea $< A >$ tends to be neutralized, diminished, balanced by $< nonA >$ ideas (not only by ¡antiA¿ as Hegel asserted) - as a state of equilibrium.

In between $< A >$ and $< antiA >$ there may be a continuous spectrum of particular $< neutA >$ ideas, which can balance $< A >$ and $< antiA >$. To neuter an idea one must discover all its three sides: of sense (truth), of nonsense (falsity), and of undecidability (indeterminacy) - then reverse/combine them. Afterwards, the idea will be classified as neutrality.

There exists a *Principle of Attraction* not only between the opposites $< A >$ and $< antiA >$ (as in dialectics), but also between them and their neutralities $< neutA >$ related to them, since $< neutA >$ contributes to the *Completeness of Knowledge*. Hence, neutrosophy is based not only on analysis of oppositional propositions as dialectic does, but on analysis of these contradictions together with the neutralities related to them.

Neutrosophy was extended to *Neutrosophic Logic*, *Neutrosophic Set*, *Neutrosophic Probability* and *Neutrosophic Statistics*, which are used in technical applications.

In the Neutrosophic Logic (which is a generalization of fuzzy logic, especially of intuitionistic fuzzy logic) every logical variable x is described by an ordered triple $x = (T, I, F)$, where T is the degree of truth, F is the degree of falsehood, and I the degree of indeterminacy (or neutrality, i.e. neither true nor false, but vague, unknown, imprecise), with T, I, F standard or non-standard subsets of the non-standard unit interval $]-0, 1+[$. In addition, these values may vary over time, space, hidden parameters, etc.

Neutrosophic Probability (as a generalization of the classical probability and imprecise probability) studies the chance that a particular event $< A >$ will occur, where that chance is represented by three coordinates (variables): $T\%$ chance the event will occur, $I\%$ indeterminate (unknown) chance, and $F\%$ chance the event will not occur.

Neutrosophic Statistics is the analysis of neutrosophic probabilistic events.

Neutrosophic Set (as a generalization of the fuzzy set, and especially of intuitionistic fuzzy set) is a set such that an element belongs to the set with a neutrosophic probability, i.e. T degree of appurtenance (membership) to the set, I degree of indeterminacy (unknown if it is appurtenance or non-appurtenance to the set), and F degree of non-appurtenance (non-membership) to the set.

There exist, for each particular idea: PRO parameters, CONTRA parameters, and NEUTER parameters which influence the above values. Indeterminacy results from any hazard which may occur, from unknown parameters, or from new arising conditions. This resulted from practice.

§2. Applications of Neutrosophy

Neutrosophic logic/set/probability/statistics are useful in artificial intelligence, neural networks, evolutionary programming, neutrosophic dynamic systems, and quantum mechanics.

§3. Examples of Neutrosophy Used in Arabic philosophy

While Avicenna promotes the idea that the world is contingent if it is necessitated by its causes, Averroes rejects it, and both of them are right from their point of view. Hence $< A >$ and $< antiA >$ have common parts.

- Islamic dialectical theology (kalam) promoting creationism was connected by Avicenna in an extraordinary way with the opposite Aristotelian-Neoplatonic tradition.

Actually a lot of work by Avicenna falls into the frame of neutrosophy.

- Averroes's religious judges (qadis) can be connected with atheists' believes.

- al-Farabi's metaphysics and general theory of emanation vs. al-Ghazali's Sufi writings and mystical treatises [we may think about a coherence of al-Ghazali's "Incoherence of the Incoherence" book].

- al-Kindi's combination of Koranic doctrines with Greek philosophy.

- Islamic Neoplatonism + Western Neoplatonism.

- Ibn-Khaldun's statements in his theory on the cyclic sequence of civilizations, says that:

Luxury leads to the raising of civilization (because the people seek for comforts of life) but also Luxury leads to the decay of civilization (because its correlation with the corruption of ethics).

- On the other hand, there's the method of absent-by-present syllogism in jurisprudence, in which we find the same principles and laws of neutrosophy.

- In fact, we can also function a lot of Arabic aphorisms, maxims, Koranic miracles (Ayat Al- Qur'ãn) and Sunna of the prophet, to support the theory of neutrosophy.

Take the colloquial proverb that "*The continuance of state is impossible*" too, or "*Everything, if it's increased over its extreme, it will turn over to its opposite*" !

§4. The Venn Diagram

In a Venn Diagram we have with respect to a universal set U shown in Fig.1. Therefore, there are no common parts amongst $< A >, < neutA >$ and $< antiA >$, and all three of them are (completely) contained by the universal set U. Also, all borders of these sets $< A >, < neutA >, < antiA >$, and U are clear, exact. All these four sets are well-defined. While $< neutA >$ means neutralities related to $< A >$ and $< antiA >$, what is outside of $< A > \bigcup < neutA > \bigcup < antiA >$ but inside of U are other neutralities, not related to $< A >$ or to $< antiA >$.

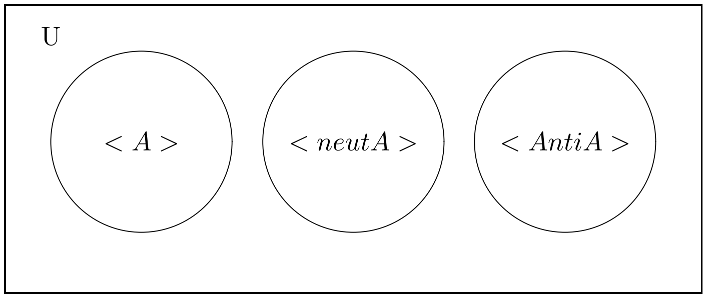

Fig.1

Given $< A >$, there are two types of neutralities: those related to $< A >$ (and implicitly related to $< antiA >$), and those not related to $< A >$ (and implicitly not related to $< antiA >$).

§5. The Neutrosophic Diagram as Extension of the Venn Diagram

Yet, for ambiguous, vague, not-well-known (or even unknown) imprecise ideas/notions/ statements/ entities with unclear frontiers amongst them the below relationships may occur because between an approximate idea noted by $< A >$ and its opposite $< antiA >$ and their neutralities $< neutA >$ there are not clear delimitations, not clear borders to distinguish amongst what is $< A >$ and what is not $< A >$. There are buffer zones in between $< A >$ and $< antiA >$, and $< neutA >$, and an element x from a buffer zone between $< A >$ and $< antiA >$ may or may not belong to both $< A >$ and $< antiA >$ simultaneously. And similarly for an element y in a buffer zone between $< A >$ and $< neutA >$, or an element z in the buffer zone between $< neutA >$ and $< antiA >$. We may have a buffer zone where the confusion of appurtenance to $< A >$, or to $< neutA >$, or to $< antiA >$ is so high, that we can consider that an element w belongs to all of them simultaneously (or to none of them simultaneously).

We say that all four sets $< A >, < neutA >, < antiA >$, and the neutrosophic universal set U are illdefined, inexact, unknown (especially if we deal with predictions. For example if $< A >$ is a statement with some degree of chance of occurring, with another degree of change of not occurring, plus an unknown part). In the general case, none of the sets $< A >$, $< neutA >$, $< antiA >$, $< nonA >$ are completely included in U, and neither U is completely known. For example, if U is the neutrosophic universal set of some specific given events, *what about an unexpected event that might belong to* U? That's why an approximate U (with vague borders) leaves room for expecting the unexpected.

The neutrosophic diagram in the general case is shown in Fig.2. The borders of $< A >$, $< antiA >$, and $< neutA >$ are dotted since they are unclear. Similarly, the border of the neutrosophic universal set U is dotted, meaning also unclear, so U may not completely contain $< A >$, nor $< neutA >$ or $< antiA >$, but U "*approximately*" contains each of them. Therefore, there are elements in $< A >$ that may not belong to U, and the same thing for $< neutA >$

and $< antiA >$. Or elements, in the most ambiguous case, there may be elements in $< A >$ and in $< neutA >$ and in $< antiA >$ which are not contained in the universal set U. Even the neutrosophic universal set is ambiguous, vague, and with unclear borders.

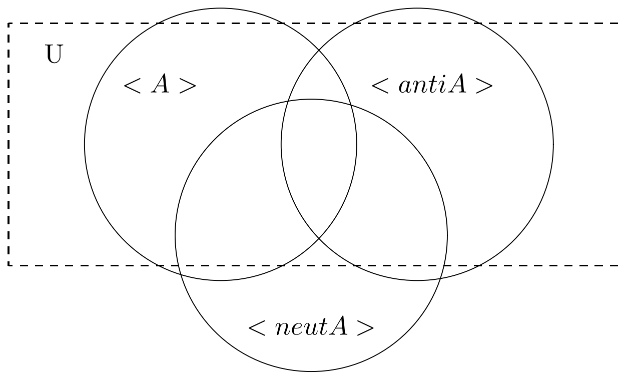

Fig.2

Of course, the intersections amongst $< A >, < neutA >, < antiA >$, and U may be smaller or bigger or even empty depending on each particular case.

See Fig.3 an example of a particular neutrosophic diagram, when some intersections are contained by the neutrosophic universal set.

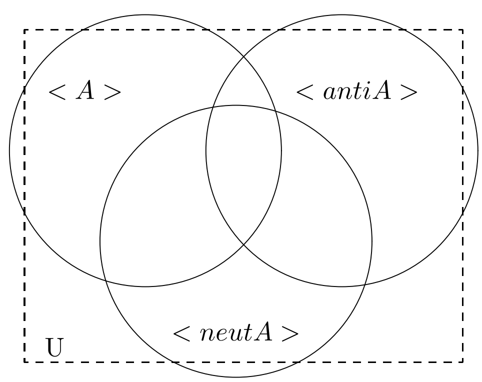

Fig.3

A neutrosophic diagram is different from a Venn diagram since the borders in a neutrosophic diagram are vague. When all borders are exact and all intersections among $< A >, < neutA >$ and $< antiA >$ are empty, and all $< A >, < neutA >$ and $< antiA >$ are included in the neutrosophic universal set U, then the neutrosophic diagram becomes a Venn diagram.

The neutrosophic diagram, which complies with the neutrosophic logic and neutrosophic set, is an extension of the Venn diagram.

§6. Classes of Neutrosophic Paradoxes

The below classes of neutrosophic paradoxes are not simply word puzzles. They may look absurd or unreal from the classical logic and classical set theory perspective.

If $< A >$ is a precise / exact idea, with well-defined borders that delimit it from others, then of course the below relationships do not occur. $< nonA >$ means what is not $< A >$, and $< antiA >$ means the opposite of $< A >$. $< neutA >$ means the neutralities related to $< A >$ and $< antiA >$, neutralities which are in between them.

When $< A >, < neutA >, < antiA >, < nonA >, U$ are uncertain, imprecise, they may be selfcontradictory. Also, there are cases when the distinction between a set and its elements is not clear. Although these neutrosophic paradoxes are based on "*pathological sets*" (those whose properties are considered atypically counterintuitive), they are not referring to the theory of Meinongian objects (Gegenstandstheorie) such as round squares, unicorns, etc. Neutrosophic paradoxes are not reported to objects, but to vague, imprecise, unclear ideas or predictions or approximate notions or attributes from our everyday life.

§7. Two New Neutrosophic Operators

Let' s introduce for the first time two new neutrosophic operators following.

1) An operator that "*neuterizes*" an idea. To neuterize [neuter+ize, transitive verb; from the Latin word neuter = neutral, in neither side], $n(\ .\ )$, means to map an entity to its neutral part. (We use the Segoe Print for "$n(\ .\ )$" .)

"*To neuterize*" is different from "*to neutralize*" [from the French word neutraliser] which means to declare a territory neutral in war, or to make ineffective an enemy, or to destroy an enemy. $n(< A >) =< neutA >$. By definition $n(< neutA >) =< neutA >$. For example, if $< A >$ is "*tall*" , then $n(tall) = medium$, also $n(short) = medium, n(medium) = medium$. But if $< A >$ is "$1 + 1 = 2$ *in base* 10" then $n(< 1 + 1 = 2 \text{ in base } 10 >)$ is undefined (does not exist), and similarly $n(< 1 + 1 \neq 2 \text{ in base } 10 >)$ is undefined.

2) And an operator that "*antonymizes*" an idea. To antonymize [antonym+ize, transitive verb; from the Greek work antönymia = instead of, opposite], $a(\ .\ )$, means to map an entity to its opposite. (We use the Segoe Print for "$a(\ .\ )$" .) $a(< A >) =< antiA >$. For example, if $< A >$ is "*tall*", then $a(tall) = short$, also $a(short) = tall$, and $a(medium) = tall$ or $short$. But if $< A >$ is "$1+1 = 2$ *in base* 10" then $a(< 1+1 = 2 \text{ in base } 10 >) =< 1+1 \neq 2 \text{ in base } 10 >$ and reciprocally $a(< 1 + 1 \neq 2 \text{ in base } 10 >) =< 1 + 1 = 2 \text{ in base } 10 >$.

The classical operator for negation/complement in logics respectively in set theory, "*to negate*" ($\neg$), which is equivalent in neutrosophy with the operator "*to nonize*" (i.e. to non+ize) or nonization (i.e. non+ization), means to map an idea to its neutral or to its opposite (a union of the previous two neutrosophic operators: neuterization and antonymization): $\neg < A >=< nonA >=< neutA > \bigcup < antiA >= n(< A >) \bigcup a(< A >)$.

Neutrosophic Paradoxes result from the following neutrosophic logic / set connectives following all apparently impossibilities or semi-impossibilities of neutrosophically connecting $< A >, < antiA >, < neutA >, < nonA >$, and the neutrosophic universal set U.

§8. Neutrosophic Truth Tables

For $< A >=$ "*tall*" ,

$< A >$	$a(< A >)$	$n(< A >)$	$\neg < A >$
tall	*short*	*medium*	*short or medium*
medium	*short or tall*	*medium*	*short or tall*
short	*tall*	*medium*	*tall or medium*

To remark that $n(< medium >) = medium$. If $< A >= tall$, then $< neutA >= medium$, and $< neut(neutA) >=< neutA >$, or $n(< n(< A >) >) = n(< A >)$.

For $< A >=$ "$1 + 1 = 2$ *in base* 10" we have $< antiA >=< nonA >=$ "$1 + 1 \neq 2$ *in base 10*" , while $< neutA >$ is undefined (N/A) - whence the neutrosophic truth table becomes:

$< A >$	$a(< A >)$	$n(< A >)$	$\neg < A >$
$True$	$False$	N/A	$False$
$False$	$True$	N/A	$True$

In the case when a statement is given by its neutrosophic logic components $< A >= (T, I, F)$, i.e. $< A >$ is $T\%$ true, $I\%$ indeterminate, and $F\%$ false, then the neutrosophic truth table depends on the defined neutrosophic operators for each application.

§9. Neutrosophic Operators and Classes of Neutrosophic Paradoxes

9.1 Complement/Negation

$\neg(\neg < A >) \neq< A >$;
$\neg(\neg < antiA >) \neq< antiA >$;
$\neg(\neg < nonA >) \neq< nonA >$;
$\neg(\neg < neutA >) \neq< neutA >$;
$\neg(\neg < U >) \neq< U >$, where U is the neutrosophic universal set,
$\neg(\neg < \emptyset >) \neq< \emptyset >$, where $\emptyset$ is the neutrosophic empty set.

9.2 Neuterization

$< n(< A >) \neq< neutA >$;
$< n(< antiA >) \neq< neutA >$;
$< n(< nonA >) \neq< neutA >$;
$< n(n(< A >)) \neq< A >$.

9.3 Antonymization

$< a(< A >) \neq< antiA >$;
$< a(< antiA >) \neq< A >$;

$< a(< nonA >) \not< A >$;
$< a(a(< A >)) \neq < A >$.

9.4 Intersection/Conjunction

$< A > \bigcap < nonA > \neq \Box$ (neutrosophic empty set), symbolically $(\exists x)(x \in A \wedge x \in \neg A$, or even more $< A > \bigcap < antiA > \neq ?$, symbolically $(\exists x)(x \in A \wedge x \in a(A))$, similarly $< A > \bigcap < neutA > \neq ?$ and $< antiA > \bigcap < neutA > \neq ?$, up to $< A > \bigcap < neutA > \bigcap < antiA > \neq ?$. The symbolic notations will be in a similar way.

This is Neutrosophic Transdisciplinarity, which means to find common features to uncommon entities. For examples:

There are things which are good and bad in the same time.

There are things which are good and bad and medium in the same time (because from one point of view they may be god, from other point of view they may be bad, and from a third point of view they may be medium).

9.5 Union/Weak Disjunction

$A > \bigcup < neutA > \bigcup < antiA > \neq U$;
$< antiA > \bigcup < neutA > \neq < nonA >, \cdots$, etc.

9.6 Inclusion/Conditional

$< A > \subset < antiA >$ $(\forall x)(x \in A \to x \in a(A))$.
All is $< antiA >$, the $< A >$ too.
All good things are also bad. All is imperfect, the perfect too.

$< antiA > \subset < A >$ $(\forall x)(x \in a(A) \to x \in A)$.
All is $< A >$, the $< antiA >$ too.
All bad things have something good in them this is rather a fuzzy paradox. All is perfect things are imperfect in some degree.

$< nonA > \subset < A >$ $(\forall x)(x \in \neg A \to x \in A)$.
All is $< A >$, the $< nonA >$ too.
All bad things have something good and something medium in them (this is a neutrosophic paradox, since it is based on good, bad, and medium).
All is perfect things have some imperfectness and mediocrity in them at some degree.

$< A > \subset < neutA >$ $(\forall x)(x \in A \to x \in n(A))$.
All is $< neutA >$, the $< A >$ too.

$< nonA > \subset < neutA >$ (partial neutrosophic paradox of inclusion) $(\forall x)(x \in \neg A \to x \in n(A))$.
All is $< neutA >$, the $< nonA >$ too.

$< nonA > \subset < antiA >$ (partial neutrosophic paradox of inclusion) $(\forall x)(x \in \neg A \to x \in a(A))$.

All is $< antiA >$, the $< nonA >$ too.

$< antiA > \subset < neutA >$ $(\forall x)(x \in a(A) \to x \in n(A))$.
All is $< neutA >$, the $< antiA >$ too.

$< A > \bigcup < antiA > \subset < neutA >$ $(\forall x)((x \in A \vee x \in a(A)) \to x \in n(A))$.
All is $< neutA >$, the $< A >$ and $< antiA >$ too.

Paradoxes of Some Neutrosophic Arguments

$< A > \to < B >$
$< B > \to < antiA >$

$\therefore < A > \to < antiA >$

Example too much work produces sickness; sickness produces less work (absences from work, low efficiency); therefore, too much work implies less work (this is a Law of Self-Equilibrium).

$< A > \to < B >$
$< B > \to < nonA >$

$\therefore < A > \to < nonA >$

$< A > \to < B >$
$< B > \to < neutA >$

$\therefore < A > \to < neutA >$

9.7 Equality/Biconditional

Unequal Equalities:

$< A > \neq < A >$, which symbolically becomes $(\exists x)(x \in A \leftrightarrow x \notin A)$ or even stronger inequality $(\forall x)(x \in A \leftrightarrow x \notin A)$.

Nothing is $< A >$, nor even $< A >$.

$< antiA > \neq < antiA >$, which symbolically becomes $(\exists x)(x \in \neg A \leftrightarrow x \notin \neg A)$ or even stronger inequality $(\forall x)(x \in \neg A \leftrightarrow x \notin \neg A)$.

$< neutA > \neq < neutA >$, which symbolically becomes $(\exists x)(x \in vA \leftrightarrow x \notin vA)$ or even stronger inequality $(\forall x)(x \in vA \leftrightarrow x \notin vA)$.

$< nonA > \neq < nonA >$, which symbolically becomes $(\exists x)(x \in \neg A \leftrightarrow x \notin \neg A)$ or even stronger inequality $(\forall x)(x \in \neg A \leftrightarrow x \notin \neg A)$.

Equal Inequalities:

$< A > = < antiA >$ $(\forall x)(x \in A \leftrightarrow x \in a(A))$.

All is $< A >$, the $< antiA >$ too; and reciprocally, all is $< antiA >$, the $< A >$ too. Or, both combined implications give: All is $< A >$ is equivalent to all is $< antiA >$.

And so on:

$< A >=< neutA >$;
$< antiA >=< neutA >$;
$< nonA >=< A >$.

Dilations and Absorptions:

$< antiA >=< nonA >$, which means that $< antiA >$ is dilated to its neutrosophic superset $< nonA >$, or $< nonA >$ is absorbed to its neutrosophic subset $< antiA >$.

Similarly for $< neutA >=< nonA >$, $< A >= U$, $< neutA >= U$, $< antiA >= U$ and $< nonA >= U$.

9.8 **Combinations**

Combinations of the previous single neutrosophic operator equalities and/or inequalities, resulting in more neutrosophic operators involved in the same expression. For examples,

$< neutA > \bigcap (< A > \bigcup < antiA >) \neq \emptyset$ (two neutrosophic operators).

$< A > \bigcup < antiA > \neq \neg < neutA >$ and reciprocally $\neg(< A > \bigcup < antiA >) \neq < neutA >$.

$< A > \bigcup < neutA > \neq \neg < antiA >$ and reciprocally. $\neg < A > \bigcup < neutA > \bigcup < antiA > \neq \Box$ and reciprocally, $\cdots$, etc.

i) We can also take into consideration other logical connectors, such as strong disjunction (we previously used the weak disjunction), Shaffer' s connector, Peirce' s connector, and extend them to the neutrosophic form.

j) We may substitute $< A >$ by some entities, attributes, statements, ideas and get nice neutrosophic paradoxes, but not all substitutions will work properly.

§10. **Some Particular Paradoxes**

A Quantum Semi-Paradox

Let's go back to 1931 Schröinger's paper. Saul Youssef writes (flipping a quantum coin) in arXiv.org at quant-ph/9509004:

"The situation before the observation could be described by the distribution $(1/2, 1/2)$ and after observing heads our description would be adjusted to $(1, 0)$. The problem is, what would you say to a student who then asks: Yes, but what causes $(1/2, 1/2)$ to evolve into $(1, 0)$? How does it happen?"

http://god-does-not-play-dice.net/Adler.html.

It is interesting. Actually we can say the same for any probability different from 1: If at the beginning, the probability of a quantum event, $P(quantumevent) = p$ with $0 < p < 1$, and if later the event occurs, we get to $P(quantumevent) = 1$; but if the event does not occur, then we get $P(quantumevent) = 0$, so still a kind of contradiction.

Torture's Paradox

An innocent person P, who is tortured, would say to the torturer T whatever the torturer wants to hear, even if P doesn't know anything. So, T would receive incorrect information that will work against him/her. Thus, the torture returns against the torturer.

Paradoxist Psychological Behavior

Instead of being afraid of something, say $< A >$, try to be afraid of its opposite $< antiA >$, and thus - because of your fear - you'll end up with the $< anti < antiA >>$, which is $< A >$.

Paradoxically, negative publicity attracts better than positive one (enemies of those who do negative publicity against you will sympathize with you and become your friends).

Paradoxistically (word coming etymologically from paradoxism, paradoxist), to be in opposition is more poetical and interesting than being opportunistic. At a sportive, literary, or scientific competition, or in a war, to be on the side of the weaker is more challenging but on the edge of chaos and, as in Complex Adoptive System, more potential to higher creation.

Law of Self-Equilibrium

(Already cited above at the Neutrosophic Inclusion / Conditional Paradoxes.)

$< A > \to < B >$ and $< B > \to < antiA >$, therefore $< A > \to < antiA >$!

Example too much work produces sickness; sickness produces less work (absences from work, low efficiency); therefore, too much work implies less work.

References

[1] Ashbacher Charles, The most paradoxist mathematician of the world, by Charles T. Le, review in *Journal of Recreational Mathematics*, USA, Vol. 28(2), 130, 1996-7.

[2] Begay Anthony, The Smarandache semantic paradox, *Humanistic Mathematics Network Journal*, Harvey Mudd College, Claremont, CA, USA, Issue #17, 48, May 1998.

[3] Burton Emil, *On Smarandache Class of Paradoxes*, Mss., 1995.

[4] Carol Horn Greenstein, *Dictionary of Logical Terms and Symbols*, Van Nostrand Reinhold Co., 1978.

[5] Crăciun, Gheorghe, Florentin Smarandache / Creatorul Miscării Literare Paradoxiste / The Smarandache Semantic Paradox: All is possible, the impossible too!, in Timpul, Melbourne, Australia, Anul III, Nr. 32, p. 2, 1994.

[6] Dr.Devaraj Ramasamy, Florentin Smarandache set up the paradoxist literary movement, in *Parnassus of World Poets 1994*, Madras, India, September 1994.

[7] Hintikka Jaako, The Logic of Epistemology and the Epistemology of Logic.

[8] Holden editor, The Smarandache semantic paradox: All is possible, the impossible too!, in *Short Fuse*, Santa Barbara, CA, USA, issue 60, p. 1, 1995.

[9] Bernd Hutschenreuther, Die Smarandache' sche Klasse von Paradoxien [German], http://portal.acm.org/citation.cfm?id=1006537.

[10] Jacquette Dale, *Logic: The Semantics of Existence and Nonexistence*, Berlin: de Gruyter, 1996.

[11] Le Charles T., The Smarandache class of paradoxes, *Bulletin of the Transylvania University of Brasov*, Vol. 1 (36), New Series, Series B, 7-8, 1994.

[12] Le Charles T., The Smarandache class of paradoxes, *Bulletin of Pure and Applied Sciences*, Delhi, India, Vol. 14E (No. 2), 109-110, 1995.

[13] Le Charles T., The most paradoxist mathematician of the World: Florentin Smarandache, *Bulletin of Pure and Applied Sciences*, Delhi, India, Vol. 15E(Maths & Statistics), No.1, 81-100, January-June 1996.

[14] Le Charles T., The Smarandache class of paradoxes, *Journal of Indian Academy of Mathematics*, Indore, Vol.18, No.1, 53-55, 1996.

[15] Le Charles T., The Smarandache class of paradoxes / (mathematical poem), Henry C. Bunner / An Anthology in Memoriam, Bristol Banner Books, Bristol, IN, USA, 94, 1996.

[16] Le C., Clasa de paradoxuri Smarandache, in Tempus, Bucharest, Anul III, Nr. 2(5), p. 4, 1994.

[17] Mirescu Marian, *Totul e posibil ⋯ chiar si imposibilul* (Paradox de Florentin Smarandache), Abracadabra, Salinas, CA, USA, p. 19, March 1995.

[18] Mitroiescu I., The Smarandache class of paradoxes applied in computer sciences, *Abstracts of Papers Presented to the American Mathematical Society*, New Jersey, USA, Vol. 16, No.3, 651, Issue 101, 1995.

[19] Mitroiescu Ilie, The Smarandache class of paradoxes, *The Mathematical Gazette*, Vol. 79, No. 484, p.125, 1995.

[20] Rapauzu Mircea, Tara Paradoxurilor, Scrisoare către Autor, Bucureşti, 22 Nov. 1994.

[21] Mudge Michael R., A paradoxist mathematician: his function, paradoxist geometry, and class of paradoxes, *Smarandache Notions Journal*, Vail, AZ, USA, Vol. 7, No. 1-2-3, 127-129, 1996.

[22] Popescu Marian, A model of the Smarandache paradoxist geometry, *Abstracts of Papers Presented to the American Mathematical Society*, New Providence, RI, USA, Vol. 17, No. 1, Issue 103, 96T-99-15, 265, 1996.

[23] Popescu Titu, *Estetica Paradoxismului*, Editura Tempus, Bucarest, 26, 27-28, 1995.

[24] Rotaru Ion, *Din nou despre Florentin Smarandache*, Vatra, Tg. Mures, Romania, Nr. 2 (299), 93-94, 1996.

[25] Seagull Larry, *Clasa de Paradoxuri Semantice Smarandache* (translation), Abracadabra, Salinas, CA, USA, Anul 2, Nr.20, 2, June 1994.

[26] Smarandache Florentin, *Neutrosophy/Neutrosophic Probability, Set, and Logic*, American Research Press, Rehoboth, USA,105 p.,1998; Republished in 2000, 2003, 2005, as *A Unifying Field in Logics: Neutrosophic Logic Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics* (fourth edition), American Research Press, 156 p.; Chinese translation by F. Liu, *A Unifying Field in Logics: Neutrosophic Logic/Neutrosophy, Neutrosophic Set, Neutrosophic Probability and statistics*, Xiquan Chinese Branch, 121 p., 2003.

[27] Smarandache F., *Mixed Non-Euclidean Geometries*, Arhivele Statului, Filiala Valcea, Rm. Valcea, 1969.

[28] Smarandache F., Osman S., *Neutrosophy in Arabic Philosophy* [English version], Renaissance High Press (Ann Arbor), 291 pp., 2007; translated into Arabic language by Dr.Osman Salah, 418 pp., Munsha't al-Ma'arif Publ. Hse., Jalal Huzie & Partners, Alexandria, Egypt, 2007.

[29] Smarandache Florentin, Mathematical Fancies & Paradoxes, *The Eugene Strens Memorial on Intuitive and Recreational Mathematics and its History*, University of Calgary, Alberta, Canada, 27 July - 2 August, 1986.

[30] Florentin Smarandache, Paradoxul vie.ii noastre (poem matematic), in *Poesis*, Satu Mare, Romania, Nr. 9(69), September 1995.

[31] Vasiliu Florin, Paradoxism's main roots, Translated from Romanian by Ştefan Benea, Xiquan Publishing House, Phoenix, USA, 64 p., 1994; review in Zentralblatt für Mathematik, Berlin, Germany, No. 5, 830 - 17, 03001, 1996.

[32] Tilton Homer B., *Smarandache's Paradoxes*, Math Power, Tucson, AZ, USA, Vol. 2, No. 9, 1-2, September 1996.

[33] Weisstein Eric W., Smarandache paradox, *CRC Concise Enciclopedia of Mathematics*, CRC Press, Boca Raton, FL, 1661, 1998;
http://mathworld.wolfram.com/SmarandacheParadox.html.

[34] Zitarelli David E., Le Charles T./the most paradoxist mathematician of the world, *Historia Mathematica*, PA, USA, Vol. 22, No. 4,# 22.4.110, 460, November 1995.

[35] Zitarelli David E., Mudge, Michael R./A paradoxist mathematician: his function, paradoxist geometry and class of paradoxes, *Historia Mathematica*, PA, USA, Vol. 24, No. 1, #24.1.119, 114, February 1997.

Neutrosophic Groups and Subgroups

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Abstract: This paper is devoted to the study of neutrosophic groups and neutrosophic subgroups. Some properties of neutrosophic groups and neutrosophic subgroups are presented. It is shown that the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup even if neither is contained in the other. It is also shown that all neutrosophic groups generated by the neutrosophic element I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups. The partitioning of neutrosophic groups is also presented.

Key Words: Neutrosophy, neutrosophic, neutrosophic logic, fuzzy logic, neutrosophic group, neutrosophic subgroup, pseudo neutrosophic subgroup, Lagrange neutrosophic group, Lagrange neutrosophic subgroup, pseudo Lagrange neutrosophic subgroup, weak Lagrange neutrosophic group, free Lagrange neutrosophic group, weak pseudo Lagrange neutrosophic group, free pseudo Lagrange neutrosophic group, smooth left coset, rough left coset, smooth index.

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§1. Introduction

In 1980, Florentin Smarandache introduced the notion of neutrosophy as a new branch of philosophy. Neutrosophy is the base of neutrosophic logic which is an extension of the fuzzy logic in which indeterminancy is included. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminancy in a subset I, and the percentage of falsity in a subset F. Since the world is full of indeterminancy, several real world problems involving indeterminancy arising from law, medicine, sociology, psychology, politics, engineering, industry, economics, management and decision making, finance, stocks and share, meteorology, artificial intelligence, IT, communication etc can be solved by neutrosophic logic.

Using Neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache introduced the concept of neutrosophic algebraic structures in [1,2]. Some of the neutrosophic algebraic structures introduced and studied include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. In [5], Agboola et al studied the structure of neutrosophic polynomial. It was shown that Division Algorithm is generally not true for neutrosophic polynomial rings and it was also shown that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain. Also in [5], it was shown that $\langle R \cup I \rangle [x]$ cannot be a Unique Factorization Domain even if R is a unique factorization domain and it was also shown that every non-zero neutrosophic principal ideal in a neutrosophic polynomial ring is not a neutrosophic prime ideal. In [6], Agboola et al studied ideals of neutrosophic rings. Neutrosophic quotient rings were also studied. In the present paper, we study neutrosophic group and neutrosophic subgroup. It is shown that the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup even if neither is contained in the other. It is also shown that all neutrosophic groups generated by I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups. The partitioning of neutrosophic groups is also studied. It is shown that the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) is a partition of the neutrosophic group (resp. Lagrange neutrosophic group).

§2. Main Results

Definition 2.1 *Let $(G, *)$ be any group and let $\langle G \cup I \rangle = \{a + bI : a, b \in G\}$. $N(G) = (\langle G \cup I \rangle, *)$ is called a neutrosophic group generated by G and I under the binary operation $*$. I is called the neutrosophic element with the property $I^2 = I$. For an integer n, $n{+}I$, and nI are neutrosophic elements and $0.I = 0$. I^{-1}, the inverse of I is not defined and hence does not exist.*

N(G) is said to be commutative if $ab = ba$ for all $a, b \in N(G)$.

Theorem 2.2 *Let $N(G)$ be a neutrosophic group.*

(i) $N(G)$ in general is not a group;

(ii) $N(G)$ always contain a group.

Proof (i) Suppose that $N(G)$ is in general a group. Let $x \in N(G)$ be arbitrary. If x is a neutrosophic element then $x^{-1} \notin N(G)$ and consequently N(G) is not a group, a contradiction.

(ii) Since a group G and an indeterminate I generate $N(G)$, it follows that $G \subset N(G)$ and N(G) always contain a group. □

Definition 2.3 *Let $N(G)$ be a neutrosophic group.*

(i) A proper subset $N(H)$ of $N(G)$ is said to be a neutrosophic subgroup of $N(G)$ if $N(H)$ is a neutrosophic group such that is $N(H)$ contains a proper subset which is a group;

(ii) $N(H)$ is said to be a pseudo neutrosophic subgroup if it does not contain a proper

subset which is a group.

Example 2.4 (i) $(N(\mathcal{Z}), +)$, $(N(\mathcal{Q}), +)$ $(N(\mathcal{R}), +)$ and $(N(\mathcal{C}), +)$ are neutrosophic groups of integer, rational, real and complex numbers respectively.
(ii) $(\langle\{\mathcal{Q}-\{0\}\}\cup I\rangle, .)$, $(\langle\{\mathcal{R}-\{0\}\}\cup I\rangle, .)$ and $(\langle\{\mathcal{C}-\{0\}\}\cup I\rangle, .)$ are neutrosophic groups of rational, real and complex numbers respectively.

Example 2.5 Let $N(G) = \{e, a, b, c, I, aI, bI, cI\}$ be a set where $a^2 = b^2 = c^2 = e$, $bc = cb = a, ac = ca = b, ab = ba = c$, then N(G) is a commutative neutrosophic group under multiplication since $\{e, a, b, c\}$ is a Klein 4-group. $N(H) = \{e, a, I, aI\}$, $N(K) = \{e, b, I, bI\}$ and $N(P) = \{e, c, I, cI\}$ are neutrosophic subgroups of N(G).

Theorem 2.6 *Let N(H)be a nonempty proper subset of a neutrosophic group $(N(G), \star)$. N(H)is a neutrosophic subgroup of N(G) if and only if the following conditions hold:*

(i) $a, b \in N(H)$ *implies that* $a \star b \in N(H)$ $\forall$ $a, b \in N(H)$;
(ii) *there exists a proper subset A of N(H) such that $(A, \star)$ is a group.*

Proof Suppose that N(H) is a neutrosophic subgroup of $((N(G), \star)$. Then $(N(G), \star)$ is a neutrosophic group and consequently, conditions (i) and (ii) hold.

Conversely, suppose that conditions (i) and (ii) hold. Then $N(H) = \langle A \cup I\rangle$ is a neutrosophic group under $\star$. The required result follows. □

Theorem 2.7 *Let N(H) be a nonempty proper subset of a neutrosophic group (N(G),*). N(H) is a pseudo neutrosophic subgroup of N(G) if and only if the following conditions hold:*

(i) $a, b \in N(H)$ *implies that* $a * b \in N(H)$ $\forall$ $a, b \in N(H)$;
(ii) *N(H) does not contain a proper subset A such that (A,*) is a group.*

Definition 2.8 *Let N(H) and N(K) be any two neutrosophic subgroups (resp. pseudo neutrosophic subgroups) of a neutrosophic group N(G). The product of N(H) and N(K) denoted by N(H).N(K) is the set* $N(H).N(K) = \{hk : h \in N(H), k \in N(K)\}$.

Theorem 2.9 *Let N(H) and N(K) be any two neutrosophic subgroups of a commutative neutrosophic group N(G). Then:*

(i) $N(H) \cap N(K)$ *is a neutrosophic subgroup of N(G);*
(ii) $N(H).N(K)$ *is a neutrosophic subgroup of N(G);*
(iii) $N(H) \cup N(K)$ *is a neutrosophic subgroup of N(G) if and only if* $N(H) \subset N(K)$ *or* $N(K) \subset N(H)$.

Proof The proof is the same as the classical case. □

Theorem 2.10 *Let N(H) be a neutrosophic subgroup and let N(K) be a pseudo neutrosophic subgroup of a commutative neutrosophic group N(G). Then:*

(i) $N(H).N(K)$ *is a neutrosophic subgroup of N(G);*

(ii) $N(H) \cap N(K)$ *is a pseudo neutrosophic subgroup of N(G);*

(iii) $N(H) \cup N(K)$ *is a neutrosophic subgroup of N(G) even if* $N(H) \not\subseteq N(K)$ *or* $N(K) \not\subseteq N(H)$.

Proof (i) Suppose that N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic subgroup of N(G) respectively. Let $x, y \in N(H).N(K)$. Then $xy \in N(H).N(K)$. Since $N(H) \subset N(H).N(K)$ and $N(K) \subset N(H).N(K)$, it follows that N(H).N(K) contains a proper subset which is a group. Hence N(H).N(K) is a neutrosophic of N(G).

(ii) Let $x, y \in N(H) \cap N(K)$. Since N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic of N(G) respectively, it follows that $xy \in N(H) \cap N(K)$ and also since $N(H) \cap N(K) \subset N(H)$ and $N(H) \cap N(K) \subset N(K)$, it follows that $N(H) \cap N(K)$ cannot contain a proper subset which is a group. Therefore, $N(H) \cap N(K)$ is a pseudo neutrosophic subgroup of N(G).

(iii) Suppose that N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic subgroup of N(G) respectively such that $N(H) \not\subseteq N(K)$ or $N(K) \not\subseteq N(H)$. Let $x, y \in N(H) \cup N(K)$. Then $xy \in N(H) \cup N(K)$. But then $N(H) \subset N(H) \cup N(K)$ and $N(K) \subset N(H) \cup N(K)$ so that $N(H) \cup N(K)$ contains a proper subset which is a group. Thus $N(H) \cup N(K)$ is a neutrosophic subgroup of N(G). This is different from what is obtainable in classical group theory. □

Example 2.11 $N(G) = \langle \mathcal{Z}_{10} \cup I \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, I, 2I, 3I, 4I, 5I, 6I, 7I, 8I, 9I, 1 + I, 2+I, 3+I, 4+I, 5+I, 6+I, 7+I, 8+I, 9+I, \cdots, 9+9I\}$ is a neutrosophic group under multiplication modulo 10. $N(H) = \{1, 3, 7, 9, I, 3I, 7I, 9I\}$ and $N(K) = \{1, 9, I, 9I\}$ are neutrosophic subgroups of N(G) and $N(P) = \{1, I, 3I, 7I, 9I\}$ is a pseudo neutrosophic subgroup of N(G). It is easy to see that $N(H) \cap N(K)$, $N(H) \cup N(K)$, $N(H).N(K)$, $N(P) \cup N(H)$, $N(P) \cup N(K)$, $N(P).N(H)$ and $N(P).N(K)$ are neutrosophic subgroups of N(G) while $N(P) \cap N(H)$ and $N(P) \cup N(K)$ are pseudo neutrosophic subgroups of N(G).

Definition 2.12 *Let N(G) be a neutrosophic group. The center of N(G) denoted by Z(N(G)) is the set* $Z(N(G)) = \{g \in N(G) : gx = xg \ \forall \ x \in N(G)\}$.

Definition 2.13 *Let g be a fixed element of a neutrosophic group N(G). The normalizer of g in N(G) denoted by N(g) is the set* $N(g) = \{x \in N(G) : gx = xg\}$.

Theorem 2.14 *Let N(G) be a neutrosophic group. Then*

(i) $Z(N(G))$ *is a neutrosophic subgroup of N(G);*

(ii) $N(g)$ *is a neutrosophic subgroup of N(G);*

Proof (i) Suppose that $Z(N(G))$ is the neutrosophic center of N(G). If $x, y \in Z(N(G))$, then $xy \in Z(N(G))$. Since $Z(G)$, the center of the group G is a proper subset of $Z(N(G))$, it follows that $Z(N(G))$ contains a proper subset which is a group. Hence $Z(N(G))$ is a neutrosophic subgroup of N(G).

(ii) The proof is the same as (i). □

Theorem 2.15 *Let N(G) be a neutrosophic group and let $Z(N(G))$ be the center of N(G) and $N(x)$ the normalizer of x in N(G). Then*

(*i*) *N(G) is commutative if and only if $Z(N(G)) = N(G)$;*

(*ii*) *$x \in Z(N(G))$ if and only if $N(x) = N(G)$.*

Definition 2.16 *Let $N(G)$ be a neutrosophic group. Its order denoted by $o(N(G))$ or $\mid N(G) \mid$ is the number of distinct elements in $N(G)$. $N(G)$ is called a finite neutrosophic group if $o(N(G))$ is finite and infinite neutrosophic group if otherwise.*

Theorem 2.17 *Let N(H)and N(K)be two neutrosophic subgroups (resp. pseudo neutrosophic subgroups) of a finite neutrosophic group N(G). Then $o(N(H).N(K)) = \frac{o(N(H)).o(N(K))}{o(N(H)\cap N(K))}$.*

Definition 2.18 *Let N(G)and N(H)be any two neutrosophic groups. The direct product of N(G) and N(H) denoted by $N(G) \times N(H)$ is defined by $N(G) \times N(H) = \{(g, h) : g \in N(G), h \in N(H)\}$.*

Theorem 2.19 *If $(N(G), *_1)$ and $(N(H), *_2)$ are neutrosophic groups, then $(N(G) \times N(H), *)$ is a neutrosophic group if $(g_1, h_1) * (g_2, h_2) = (g_1 *_1 g_2, h_1 *_2 h_2)\ \forall\ (g_1, h_1), (g_2, h_2) \in N(G) \times N(H)$.*

Theorem 2.20 *Let N(G)be a neutrosophic group and let H be a classical group. Then $N(G)\times H$ is a neutrosophic group.*

Definition 2.21 *Let $N(G)$ be a finite neutrosophic group and let $N(H)$ be a neutrosophic subgroup of $N(G)$.*

(*i*) *$N(H)$ is called a Lagrange neutrosophic subgroup of $N(G)$ if $o(N(H)) \mid o(N(G))$;*

(*ii*) *$N(G)$ is called a Lagrange neutrosophic group if all neutrosophic subgroups of $N(G)$ are Lagrange neutrosophic subgroups;*

(*iii*) *$N(G)$ is called a weak Lagrange neutrosophic group if $N(G)$ has at least one Lagrange neutrosophic subgroup;*

(*iv*) *$N(G)$ is called a free Lagrange neutrosophic group if it has no Lagrange neutrosophic subgroup.*

Definition 2.22 *Let $N(G)$ be a finite neutrosophic group and let $N(H)$ be a pseudo neutrosophic subgroup of $N(G)$.*

(*i*) *$N(H)$ is called a pseudo Lagrange neutrosophic subgroup of $N(G)$ if $o(N(H)) \mid o(N(G))$;*

(*ii*) *$N(G)$ is called a pseudo Lagrange neutrosophic group if all pseudo neutrosophic subgroups of $N(G)$ are pseudo Lagrange neutrosophic subgroups;*

(*iii*) *$N(G)$ is called a weak pseudo Lagrange neutrosophic group if $N(G)$ has at least one pseudo Lagrange neutrosophic subgroup;*

(*iv*) *$N(G)$ is called a free pseudo Lagrange neutrosophic group if it has no pseudo Lagrange neutrosophic subgroup.*

Example 2.23 (*i*) Let N(G) be the neutrosophic group of Example 2.5. The only neutrosophic subgroups of N(G)are $N(H) = \{e, a, I, aI\}$, $N(K) = \{e, b, I, bI\}$ and $N(P) = \{e, c, I, cI\}$. Since $o(N(G)) = 8$ and $o(N(H)) = o(N(K)) = o(N(P)) = 4$ and $4 \mid 8$, it follows that N(H), N(K)and N(P)are Lagrange neutrosophic subgroups and N(G) is a Lagrange neutrosophic group.

(*ii*) Let $N(G) = \{1, 3, 5, 7, I, 3I, 5I, 7I\}$ be a neutrosophic group under multiplication modulo 8. The neutrosophic subgroups $N(H) = \{1, 3, I, 3I\}$, $N(K) = \{1, 5, I, 5I\}$ and $N(P) = \{1, 7, I, 7I\}$ are all Lagrange neutrosophic subgroups. Hence N(G) is a Lagrange neutrosophic group.

(*iii*) $N(G) = N(\mathcal{Z}_2) \times N(\mathcal{Z}_2) = \{(0,0), (0,1), (1,0), (1,1), (0,1+I), (1,I), \cdots, (1+I, 1+I)\}$ is a neutrosophic group under addition modulo 2. N(G) is a Lagrange neutrosophic group since all its neutrosophic subgroups are Lagrange neutrosophic subgroups.

(*iv*) Let $N(G) = \{e, g, g^2, g^3, I, gI, g^2I, g^3I\}$ be a neutrosophic group under multiplication where $g^4 = e$. $N(H) = \{e, g^2, I, g^2I\}$ and $N(K) = \{e, I, g^2I\}$ are neutrosophic subgroups of N(G). Since $o(N(H)) \mid o(N(G))$ but $o(N(K))$ does not divide $o(N(G))$ it shows that N(G) is a weak Lagrange neutrosophic group.

(*v*) Let $N(G) = \{e, g, g^2, I, gI, g^2I\}$ be a neutrosophic group under multiplication where $g^3 = e$. N(G) is a free Lagrange neutrosophic group.

Theorem 2.24 *All neutrosophic groups generated by I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups.*

Definition 2.25 *Let $N(H)$ be a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a neutrosophic group $N(G)$. For a $g \in N(G)$, the set $gN(H) = \{gh : h \in N(H)\}$ is called a left coset (resp. pseudo left coset) of N(H) in N(G). Similarly, for a $g \in N(G)$, the set $N(H)g = \{hg : h \in N(H)\}$ is called a right coset (resp. pseudo right coset) of N(H) in N(G). If N(G) is commutative, a left coset (resp. pseudo left coset) and a right coset (resp. pseudo right coset) coincide.*

Definition 2.26 *Let N(H)be a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group N(G). A left coset xN(H)of N(H) in N(G) determined by x is called a smooth left coset if $\mid xN(H) \mid = \mid N(H) \mid$. Otherwise, xN(H) is called a rough left coset of N(H) in N(G).*

Definition 2.27 *Let N(H)be a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). The number of distinct left cosets of N(H) in N(G)denoted by [N(G):N(H)] is called the index of N(H) in N(G).*

Definition 2.28 *Let N(H)be a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group N(G). The number of distinct smooth left cosets of N(H) in N(G) denoted by [N(H):N(G)] is called the smooth index of N(H) in N(G).*

Theorem 2.29 *Let X be the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Then X is a partition of N(G).*

Proof Suppose that $X = \{X_i\}_{i=1}^{n}$ is the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Since $o(N(H)) \mid o(N(G))$ and $\mid xN(H) \mid=\mid N(H) \mid \forall\, x \in N(G)$, it follows that X is not empty and every member of N(G) belongs to one and only one member of X. Hence $\cap_{i=1}^{n} X_i = \emptyset$ and $\cup_{i=1}^{n} X_i = N(G)$. Consequently, X is a partition of N(G). □

Corollary 2.30 *Let $[N(H) : N(G)]$ be the smooth index of a Lagrange neutrosophic subgroup in a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Then $\mid N(G) \mid=\mid N(H) \mid [N(H) : N(G)]$.*

Proof The proof follows directly from Theorem 2.29. □

Theorem 2.31 *Let X be the set of distinct left cosets of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). Then X is not a partition of N(G).*

Proof Suppose that $X = \{X_i\}_{i=1}^{n}$ is the set of distinct left cosets of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). Since N(H) is a non-Lagrange pseudo neutrosophic subgroup, it follows that $o(N(H))$ is not a divisor of $o(N(G))$ and $\mid xN(H) \mid\neq\mid N(H) \mid \forall\, x \in N(G)$. Clearly, X is not empty and every member of N(G) can not belongs to one and only one member of X. Consequently, $\cap_{i=1}^{n} X_i \neq \emptyset$ and $\cup_{i=1}^{n} X_i \neq N(G)$ and thus X is not a partition of N(G). □

Corollary 2.32 *Let $[N(G) : N(H)]$ be the index of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) in a finite neutrosophic group N(G). Then $\mid N(G) \mid\neq\mid N(H) \mid [N(G) : N(H)]$.*

Proof The proof follows directly from Theorem 2.31. □

Example 2.33 Let N(G)be a neutrosophic group of Example 2.23(iv).

(a) Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{e, g^2, I, g^2I\}$ are: $X_1 = \{e, g^2, I, g^2I\}$, $X_2 = \{g, g^3, gI, g^3I\}$, $X_3 = \{I, g^2I\}$, $X_4 = \{gI, g^3I\}$. X_1, X_2 are smooth cosets while X_3, X_4 are rough cosets and therefore $[N(G) : N(H)] = 4$, $[N(H) : N(G)] = 2$.
$\mid N(H) \mid [N(G) : N(H)] = 4 \times 4 \neq\mid N(G) \mid$ and $\mid N(H) \mid [N(H) : N(G)] = 4 \times 2 =\mid N(G) \mid$.
$X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = N(G)$ and hence the set $X = \{X_1, X_2\}$ is a partition of N(G).

(b) Distinct left cosets of the pseudo non-Lagrange neutrosophic subgroup $N(H) = \{e, I, g^2I\}$ are: $X_1 = \{e, I, g^2I\}$, $X_2 = \{g, gI, g^3I\}$, $X_3 = \{g^2, I, g^2I\}$, $X_4 = \{g^3, gI, g^3I\}$, $X_5 = \{I, g^2I\}$, $X_6 = \{gI, g^3I\}$. X_1, X_2, X_3, X_4 are smooth cosets while X_5, X_6 are rough cosets. $[N(G) : N(H)] = 6$, $[N(H) : N(G)] = 4$, $\mid N(H) \mid [N(G) : N(H)] = 3 \times 6 \neq\mid N(G) \mid$ and $\mid N(H) \mid [N(H) : N(G)] = 3 \times 4 \neq\mid N(G) \mid$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are not mutually disjoint and hence do not form a partition of N(G).

Example 2.34 Let $N(G) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$ be a neutrosophic group under multiplication modulo 5. Distinct left cosets of the non-Lagrange neutrosophic subgroup $N(H) = \{1, 4, I, 2I, 3I, 4I\}$ are $X_1 = \{1, 4, I, 2I, 3I, 4I\}$, $X_2 = \{2, 3, I, 2I, 3I, 4I\}$, $X_3 = \{I, 2I, 3I, 4I\}$. X_1, X_2 are smooth cosets while X_3 is a rough coset and therefore $[N(G) : N(H)] = 3$,

$[N(H) : N(G)] = 2$, $| N(H) | [N(G) : N(H)] = 6 \times 3 \neq | N(G) |$ and $| N(H) | [N(H) : N(G)] = 6 \times 2 \neq | N(G) |$. Members of the set $X = \{X_1, X_2\}$ are not mutually disjoint and hence do not form a partition of N(G).

Example 2.35 Let N(G)be the Lagrange neutrosophic group of Example 2.5. Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{e, a, I, aI\}$ are: $X_1 = \{e, a, I, aI\}$, $X_2 = \{b, c, bI, cI\}$, $X_3 = \{I, aI\}$, $X_4 = \{bI, cI\}$. X_1, X_2 are smooth cosets while X_3, X_4 are rough cosets and thus $[N(G) : N(H)] = 4$, $[N(H) : N(G)] = 2$, $| N(H) | [N(G) : N(H)] = 4 \times 4 = 16 \neq | N(G) |$ and $| N(H) | [N(H) : N(G)] = 4 \times 2 = 8 = | N(G) |$. Members of the set $X = \{X_1, X_2\}$ are mutually disjoint and $N(G) = X_1 \cup X_2$. Hence X is a partition of N(G).

Example 2.36 Let N(G)be the Lagrange neutrosophic group of Example 2.23(iii).

(a) Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{(0, 0), (0, 1), (0, I), (0, 1+I)\}$ are respectively $X_1 = \{(0, 0), (0, 1), (0, I), (0, 1+I)\}$, $X_2 = \{(1, 0), (1, 1), (1, I), (1, 1+I)\}$, $X_3 = \{(I, 0), (I, 1), (I, I), (I, 1+I)\}$, $X_4 = \{(I+I, 0), (1+I, 1), (1+I, I), (1+I, 1+I)\}$, $X_5 = \{(1+I, 0), (1+I, 1), (1+I, 1+I)\}$. X_1, X_2, X_3, X_4 are smooth cosets while X_5 is a rough coset. Thus, $[N(G) : N(H)] = 5$, $[N(H) : N(G)] = 4$, $| N(H) | [N(G) : N(H)] = 4 \times 5 = 20 \neq | N(G) | = 16$ and $| N(H) | [N(H) : N(G)] = 4 \times 4 = 16 = | N(G) |$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are mutually disjoint and $N(G) = X_1 \cup X_2 \cup X_3 \cup X_4$ so that X is a partition of N(G).

(b) Distinct left cosets of the pseudo Lagrange neutrosophic subgroup $N(H) = \{(0, 0), (0, I), (I, 0), (I, I)\}$ are respectively $X_1 = \{(0, 0), (0, I), (I, 0), (I, 1)\}$, $X_2 = \{(0, 1), (0, 1+I), (I, 1), (I, 1+I)\}$, $X_3 = \{(1, 0), (1, I), (1+I, 0), (1+I, I)\}$, $X_4 = \{(1, 1), (1, 1+I), (1+I, 1), (1+I, 1+I)\}$. X_1, X_2, X_3, X_4 are smooth cosets and $[N(G) : N(H)] = [N(H) : N(G)] = 4$. Consequently, $| N(H) | [N(G) : N(H)] = | N(H) | [N(H) : N(G)] = 4 \times 4 = 16 = | N(G) |$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are mutually disjoint, $N(G) = X_1 \cup X_2 \cup X_3 \cup X_4$ and hence X is a partition of N(G).

References

[1] Vasantha Kandasamy W.B. and Florentin Smarandache, *Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures*, Hexis, Phoenix, Arizona, 2006.

[2] Vasantha Kandasamy W.B. and Florentin Smarandache, *Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models*, Hexis, Church Rock, 2004.

[3] Florentin Smarandache, *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, (3rd edition), American Research Press, Rehoboth, 2003.

[4] Agboola A.A.A. and Akinola L.S., On the bicoset of a bivector space, *International J. Math. Combin.* Vol.4, 1-8, 2009.

[5] Agboola A.A.A., Akinola A.D. and Oyebola O.Y., Neutrosophic rings I, *International J. Math. Combin.* Vol.4, 1-14, 2011.

[6] Agboola A.A.A., Adeleke E.O. and Akinleye S.A., Neutrosophic rings II, *International J. Math. Combin.*, Vol.2, 1-8, 2012.

[7] Herstein I.N., *Topics in Algebra*, Wiley Eastern Limited, 1975.

[8] Fraleigh J.B., *A First Course in Abstract Algebra* (5th edition), Addison-Wesley Publishing Company, 1994.

[9] Marshall H., *Theory of Groups*, The Macmillan Company, New York, 1994.

[10] Lang S., *Algebra*, Addison Wesley, 1984.

Neutrosophic Rings I

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Abstract: In this paper, we present some elementary properties of neutrosophic rings. The structure of neutrosophic polynomial rings is also presented. We provide answers to the questions raised by Vasantha Kandasamy and Florentin Smarandache in [1] concerning principal ideals, prime ideals, factorization and Unique Factorization Domain in neutrosophic polynomial rings.

Key Words: Neutrosophy, neutrosophic, neutrosophic logic, fuzzy logic, neutrosophic ring, neutrosophic polynomial ring, neutrosophic ideal, pseudo neutrosophic ideal, neutrosophic R-module.

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§1. Introduction

Neutrosophy is a branch of philosophy introduced by Florentin Smarandache in 1980. It is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics. While neutrosophic set generalizes the fuzzy set, neutrosophic probability generalizes the classical and imprecise probabilty, neutrosophic statistics generalizes classical and imprecise statistics, neutrosophic logic however generalizes fuzzy logic, intuitionistic logic, Boolean logic, multi-valued logic, paraconsistent logic and dialetheism. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminancy in a subset I, and the percentage of falsity in a subset F. The use of neutrosophic theory becomes inevitable when a situation involving indeterminancy is to be modeled since fuzzy set theory is limited to modeling a situation involving uncertainty.

The introduction of neutrosophic theory has led to the establishment of the concept of neutrosophic algebraic structures. Vasantha Kandasamy and Florentin Smarandache for the first time introduced the concept of neutrosophic algebraic structures in [2] which has caused a paradigm shift in the study of algebraic structures. Some of the neutrosophic algebraic structures introduced and studied in [2] include neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. The study of neutrosophic rings was introduced for the first time by Vasantha Kandasamy and Florentin Smarandache in [1]. Some of the neutrosophic rings studied in [1] include neutrosophic polynomial rings, neutrosophic matrix rings, neutrosophic direct product rings, neutrosophic integral domains, neutrosophic unique factorization domains, neutrosophic division rings, neutrosophic integral quaternions, neutrosophic rings of real quarternions, neutrosophic group rings and neutrosophic semigroup rings.

In Section 2 of this paper, we present elementary properties of neutrosophic rings. Section 3 is devoted to the study of structure of neutrosophic polynomial rings and we present algebraic operations on neutrosophic polynomials. In section 4, we present factorization in neutrosophic polynomial rings. We show that Division Algorithm is generally not true for neutrosophic polynomial rings. We show that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain and also we show that $\langle R \cup I \rangle [x]$ cannot be a Unique Factorization Domain even if R is a Unique Factorization Domain. In section 5 of this paper, we present neutrosophic ideals in neutrosophic polynomial rings and we show that every non-zero neutrosophic principal ideal is not a neutrosophic prime ideal.

§2. **Elementary Properties of Neutrosophic Rings**

In this section we state for emphasis some basic definitions and results but for further details about neutrosophic rings, the reader should see [1].

Definition 2.1([1]) *Let $(R, +, .)$ be any ring. The set*

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I under the operations of R.

Example 2.2 $\langle \mathcal{Z} \cup I \rangle$, $\langle \mathcal{Q} \cup I \rangle$, $\langle \mathcal{R} \cup I \rangle$ and $\langle \mathcal{C} \cup I \rangle$ are neutrosophic rings of integer, rational, real and complex numbers respectively.

Theorem 2.3 *Every neutrosophic ring is a ring and every neutrosophic ring contains a proper subset which is just a ring.*

Definition 2.4 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if $\forall\ x, y \in \langle R \cup I \rangle$, $xy = yx$.*

If in addition there exists $1 \in \langle R \cup I \rangle$ such that $1.r = r.1 = r$ for all $r \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 2.5 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring of $\langle R \cup I \rangle$ if $P = \langle S \cup nI \rangle$ where S is a subring of R and n an integer. P is said to be generated by S and nI under the operations of R.*

Definition 2.6 *Let $\langle R \cup I \rangle$ be a neotrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$ which is just a ring. Then P is called a subring.*

Definition 2.7 *Let T be a non-empty set together with two binary operations $+$ and $.$ T is said to be a pseudo neutrosophic ring if the following conditions hold:*

(i) *T contains elements of the form (a+bI), where a and b are real numbers and* $b \neq 0$ *for at least one value;*

(ii) *(T,+) is an Abelian group;*

(iii) $(T, .)$ *is a semigroup;*

(iv) $\forall\ x, y, z \in T$, $x(y+z) = xy + xz$ *and* $(y+z)x = yx + zx$.

Definition 2.8 *Let* $\langle R \cup I \rangle$ *be any neutrosophic ring. A non-empty subset P of* $\langle R \cup I \rangle$ *is said to be a neutrosophic ideal of* $\langle R \cup I \rangle$ *if the following conditions hold:*

(i) *P is a neutrosophic subring of* $\langle R \cup I \rangle$*;*

(ii) *for every* $p \in P$ *and* $r \in \langle R \cup I \rangle$, $rp \in P$ *and* $pr \in P$.

If only $rp \in P$, we call P a left neutrosophic ideal and if only $pr \in P$, we call P a right neutrosophic ideal. When $\langle R \cup I \rangle$ is commutative, there is no distinction between rp and pr and therefore P is called a left and right neutrosophic ideal or simply a neutrosophic ideal.

Definition 2.9 *Let* $\langle R \cup I \rangle$ *be a neutrosophic ring and let P be a pseudo neutrosophic subring of* $\langle R \cup I \rangle$*. P is said to be a pseudo neutrosophic ideal of* $\langle R \cup I \rangle$ *if* $\forall\ p \in P$ *and* $r \in \langle R \cup I \rangle$, $rp, pr \in P$.

Theorem 2.10([1]) *Let* $\langle \mathcal{Z} \cup I \rangle$ *be a neutrosophic ring. Then* $\langle \mathcal{Z} \cup I \rangle$ *has a pseudo ideal P such that*

$$\langle \mathcal{Z} \cup I \rangle \cong \mathcal{Z}_n.$$

Definition 2.11 *Let* $\langle R \cup I \rangle$ *be a neutrosophic ring.*

(i) $\langle R \cup I \rangle$ *is said to be of characteristic zero if* $\forall\ x \in R$, $nx = 0$ *implies that* $n = 0$ *for an integer n;*

(ii) $\langle R \cup I \rangle$ *is said to be of characteristic n if* $\forall x \in R$, $nx = 0$ *for an integer n.*

Definition 2.12 *An element x in a neutrosophic ring* $\langle R \cup I \rangle$ *is called a left zero divisor if there exists a nonzero element* $y \in \langle R \cup I \rangle$ *such that* $xy = 0$.

A right zero divisor can be defined similarly. If an element $x \in \langle R \cup I \rangle$ is both a left and a right zero divisor, it is then called a zero divisor.

Definition 2.13 *Let* $\langle R \cup I \rangle$ *be a neutrosophic ring.* $\langle R \cup I \rangle$ *is called a neutrosophic integral domain if* $\langle R \cup I \rangle$ *is commutative with no zero divisors.*

Definition 2.14 *Let* $\langle R \cup I \rangle$ *be a neutrosophic ring.* $\langle R \cup I \rangle$ *is called a neutrosophic division ring if* $\langle R \cup I \rangle$ *is non-commutative and has no zero divisors.*

Definition 2.15 *An element x in a neutrosophic ring* $\langle R \cup I \rangle$ *is called an idempotent element if* $x^2 = x$.

Example 2.16 In the neutrosophic ring $\langle \mathcal{Z}_2 \cup I \rangle$, 0 and 1 are idempotent elements.

Definition 2.17 *An element* $x = a + bI$ *in a neutrosophic ring* $\langle R \cup I \rangle$ *is called a neutrosophic* *idempotent element if $b \neq 0$ and $x^2 = x$.*

Example 2.18 In the neutrosophic ring $\langle \mathcal{Z}_3 \cup I \rangle$, I and 1+2I are neutrosophic idempotent elements.

Definition 2.19 *Let $\langle R \cup I \rangle$ be a neutrosophic ring. An element $x = a + bI$ with $a \neq \pm b$ is said to be a neutrosophic zero divisor if there exists $y = c + dI$ in $\langle R \cup I \rangle$ with $c \neq \pm d$ such that $xy = yx = 0$.*

Definition 2.20 *Let $x = a + bI$ with $a, b \neq 0$ be a neutrosophic element in the neutrosophic ring $\langle R \cup I \rangle$. If there exists an element $y \in R$ such that $xy = yx = 0$, then y is called a semi neutrosophic zero divisor.*

Definition 2.21 *An element $x = a + bI$ with $b \neq 0$ in a neutrosophic ring $\langle R \cup I \rangle$ is said to be a neutrosophic nilpotent element if there exists a positive integer n such that $x^n = 0$.*

Example 2.22 In the neutrosophic ring $\langle \mathcal{Z}_4 \cup I \rangle$ of integers modulo 4, 2+2I is a neutrosophic nilpotent element.

Example 2.23 Let $\langle M_{2\times 2} \cup I \rangle$ be a neutrosophic ring of all 2×2 matrices. An element $A = \begin{bmatrix} 0 & 2I \\ 0 & 0 \end{bmatrix}$ is neutrosophic nilpotent since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition 2.24 *Let Let r be a fixed element of the neutrosophic ring $\langle R \cup I \rangle$. We call the set*

$$N(r) = \{x \in \langle R \cup I \rangle : xr = rx\}$$

the normalizer of r in $\langle R \cup I \rangle$.

Example 2.25 Let M be a neutrosophic ring defined by

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \langle \mathcal{Z}_2 \cup I \rangle \right\}.$$

It is clear that M has 16 elements.

(i) The normalizer of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix} \right\}.$$

(ii) The normalizer of $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ in M is obtained as

$$N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \right) =$$

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix}\right\}.$$

It is clear that $N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ and $N\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right)$ are pseudo neutrosophic subrings of M and in fact they are pseudo neutrosophic ideals of M. These emerging facts are put together in the next proposition.

Proposition 2.26 *Let N(r) be a normalizer of an element in a neutrosophic ring $\langle R \cup I \rangle$. Then*

(*i*) *$N(r)$ is a pseudo neutrosophic subring of $\langle R \cup I \rangle$;*

(*ii*) *$N(r)$ is a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.*

Definition 2.27 *Let P be a proper subset of the neutrosophic ring $\langle R \cup I \rangle$. The set*

$$Ann_l(P) = \{x \in \langle R \cup I \rangle : xp = 0 \ \ \forall\ p \in P\}$$

is called a left annihilator of P and the set

$$Ann_r(P) = \{y \in \langle R \cup I \rangle : py = 0 \ \ \forall\ p \in P\}$$

is called a right annihilator of P. If $\langle R \cup I \rangle$ is commutative, there is no distinction between left and right annihilators of P and we write Ann(P).

Example 2.28 Let M be the neutrosophic ring of Example 2.25. If we take

$$P = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1+I & 1+I \\ 0 & 0 \end{bmatrix}\right\},$$

then, the left annihilator of P is obtained as

$$Ann_l(P) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+I \\ 0 & 0 \end{bmatrix}\right\}$$

which is a left pseudo neutrosophic ideal of M.

Proposition 2.29 *Let $\langle R \cup I \rangle$ be a neutrosophic ring and let P be a proper subset of $\langle R \cup I \rangle$. Then the left(right) annihilator of P is a left(right) pseudo neutrosophic ideal of $\langle R \cup I \rangle$.*

Example 2.30 Consider $\langle \mathcal{Z}_2 \cup I \rangle = \{0, 1, I, 1+I\}$ the neutrosophic ring of integers modulo 2. If $P = \{0, 1+I\}$, then $Ann(P) = \{0, I\}$.

Example 2.31 Consider $\langle \mathcal{Z}_3 \cup I \rangle = \{0, 1, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ the neutrosophic ring of integers modulo 3. If $P = \{0, I, 2I\}$, then $Ann(P) = \{0, 1+2I, 2+I\}$ which is a pseudo nuetrosophic subring and indeed a pseudo neutrosophic ideal.

Proposition 2.32 *Let $\langle R\cup I\rangle$ be a commutative neutrosophic ring and let P be a proper subset of $\langle R\cup I\rangle$. Then Ann(P) is a pseudo neutrosophic ideal of $\langle R\cup I\rangle$.*

Definition 2.33 *Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R\cup I\rangle$. The set*

$$\langle R\cup I\rangle/P = \{r+P : r\in\langle R\cup I\rangle\}$$

is called the neutrosophic quotient ring provided that $\langle R\cup I\rangle/P$ is a neutrosophic ring.

To show that $\langle R\cup I\rangle/P$ is a neutrosophic ring, let $x=r_1+P$ and $y=r_2+P$ be any two elements of $\langle R\cup I\rangle/P$ and let $+$ and . be two binary operations defined on $\langle R\cup I\rangle/P$ by:

$$\begin{aligned} x+y &= (r_1+r_2)+P, \\ xy &= (r_1r_2)+P, \quad r_1,r_2\in\langle R\cup I\rangle. \end{aligned}$$

It can easily be shown that

(i) the two operations are well defined;

(ii) $(\langle R\cup I\rangle/P,+)$ is an abelian group;

(iii) $(\langle R\cup I\rangle/P,.)$ is a semigroup, and

(iv) if $z=r_3+P$ is another element of $\langle R\cup I\rangle/P$ with $r_3\in\langle R\cup I\rangle$, then we have $z(x+y)=zx+zy$ and $(x+y)z=xz+yz$. Accordingly, $\langle R\cup I\rangle/P$ is a neutrosophic ring with P as an additive identity element.

Definition 2.34 *Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a neutrosophic ideal of $\langle R\cup I\rangle$. $\langle R\cup I\rangle/P$ is called a false neutrosophic quotient ring if $\langle R\cup I\rangle/P$ is just a ring and not a neutrosophic ring.*

Definition 2.35 *Let $\langle R\cup I\rangle$ be a neutrosophic ring and let P be a pseudo neutrosophic ideal of $\langle R\cup I\rangle$. $\langle R\cup I\rangle/P$ is called a pseudo neutrosophic quotient ring if $\langle R\cup I\rangle/P$ is a neutrosophic ring. If $\langle R\cup I\rangle/P$ is just a ring, then we call $\langle R\cup I\rangle/P$ a false pseudo neutrosophic quotient ring.*

Definition 2.36 *Let $\langle R\cup I\rangle$ and $\langle S\cup I\rangle$ be any two neutrosophic rings. The mapping ϕ : $\langle R\cup I\rangle\rangle\langle S\cup I\rangle$ is called a neutrosophic ring homomorphism if the following conditions hold:*

(i) *ϕ is a ring homomorphism;*

(ii) *$\phi(I)=I$.*

The set $\{x\in\langle R\cup I\rangle : \phi(x)=0\}$ is called the kernel of ϕ and is denoted by Kerϕ.

Theorem 2.37 *Let $\phi : \langle R\cup I\rangle\rangle\langle S\cup I\rangle$ be a neutrosophic ring homomorphism and let $K=$ Kerϕ be the kernel of ϕ. Then:*

(i) *K is always a subring of $\langle R\cup I\rangle$;*

(ii) *K cannot be a nuetrosophic subring of $\langle R\cup I\rangle$;*

(iii) *K cannot be an ideal of $\langle R\cup I\rangle$.*

Proof (i) It is Clear. (ii) Since $\phi(I)=I$, it follows that $I\notin K$ and the result follows. (iii) Follows directly from (ii). □

Example 2.38 Let $\langle R\cup I\rangle$ be a nuetrosophic ring and let $\phi : \langle R\cup I\rangle\rangle \langle R\cup I\rangle$ be a mapping defined by $\phi(r) = r \;\; \forall\, r \in \langle R\cup I\rangle$. Then ϕ is a neutrosophic ring homomorphism.

Example 2.39 Let P be a neutrosophic ideal of the neutrosophic ring $\langle R\cup I\rangle$ and let ϕ : $\langle R\cup I\rangle\rangle \langle R\cup I\rangle /P$ be a mapping defined by $\phi(r) = r+P,\ \forall\, r \in \langle R\cup I\rangle$. Then $\forall\, r, s \in \langle R\cup I\rangle$, we have

$$\phi(r+s) = \phi(r)+\phi(s), \qquad \phi(rs) = \phi(r)\phi(s),$$

which shows that ϕ is a ring homomorphism. But then,

$$\phi(I) = I + P \neq I.$$

Thus, ϕ is not a neutrosophic ring homomorphism. This is another marked difference between the classical ring concept and the concept of netrosophic ring.

Proposition 2.40 *Let $(\langle R\cup I\rangle, +)$ be a neutrosophic abelian group and let $Hom(\langle R\cup I\rangle, \langle R\cup I\rangle)$ be the set of neutrosophic endomorphisms of $(\langle R\cup I\rangle, +)$ into itself. Let + and . be addition and multiplication in $Hom(\langle R\cup I\rangle, \langle R\cup I\rangle)$ defined by*

$$\begin{aligned}(\phi+\psi)(x) &= \phi(x)+\psi(x),\\ (\phi.\psi)(x) &= \phi(\psi(x)), \forall\ \phi, \psi \in Hom(\langle R\cup I\rangle, \langle R\cup I\rangle), x \in \langle R\cup I\rangle.\end{aligned}$$

Then $(Hom(\langle R\cup I\rangle, \langle R\cup I\rangle), +, .)$ is a neutrosophic ring.

Proof The proof is the same as in the classical ring. □

Definition 2.41 *Let R be an arbitrary ring with unity. A neutrosophic left R-module is a neutrosophic abelian group $(\langle M\cup I\rangle, +)$ together with a scalar multiplication map . : $R\times \langle M\cup I\rangle\rangle \langle M\cup I\rangle$ that satisfies the following conditions:*

(i) $r(m+n) = rm + rn$;

(ii) $(r+s)m = rm + sm$;

(iii) $(rs)m = r(sm)$;

(iv) $1.m = m$, *where* $r, s \in R$ *and* $m, n \in \langle M\cup I\rangle$.

Definition 2.42 *Let R be an arbitrary ring with unity. A neutrosophic right R-module is a neutrosophic abelian group $(\langle M\cup I\rangle, +)$ together with a scalar multiplication map . : $\langle M\cup I\rangle\times R\rangle \langle M\cup I\rangle$ that satisfies the following conditions:*

(i) $(m+n)r = mr + nr$;

(ii) $m(r+s) = mr + ms$;

(iii) $m(rs) = (mr)s$;

(iv) $m.1 = m$, *where* $r, s \in R$ *and* $m, n \in \langle M\cup I\rangle$.

If R is a commutative ring, then a neutrosophic left R-module $\langle M\cup I\rangle$ becomes a neutrosophic right R-module and we simply call $\langle M\cup I\rangle$ a neutrosophic R-module.

Example 2.43 Let $(\langle M \cup I\rangle, +)$ be a nuetrosophic abelian group and let $\mathcal{Z}$ be the ring of integers. If we define the mapping $f : \mathcal{Z} \times \langle M \cup I\rangle\rangle \langle M \cup I\rangle$ by $f(n, m) = nm, \forall\ n \in \mathcal{Z}, m \in \langle M \cup I\rangle$, then $\langle M \cup I\rangle$ becomes a neutrosophic $\mathcal{Z}$-module.

Example 2.44 Let $\langle R \cup I\rangle [x]$ be a neutrosophic ring of polynomials where R is a commutative ring with unity. Obviously, $(\langle R \cup I\rangle [x], +)$ is a neutrosophic abelian group and the scalar multiplication map $. : R \times \langle R \cup I\rangle [x]\rangle \langle R \cup I\rangle [x]$ satisfies all the axioms of the neutrosophic R-module. Hence, $\langle R \cup I\rangle [x]$ is a neutrosophic R-module.

Proposition 2.45 *Let $(\langle R \cup I\rangle, +)$ be a neutrosophic abelian group and let $Hom(\langle R \cup I\rangle, \langle R \cup I\rangle)$ be the neutrosophic ring obtained in Proposition (2.40). Let $. : Hom(\langle R \cup I\rangle, \langle R \cup I\rangle) \times \langle R \cup I\rangle\rangle \langle R \cup I\rangle$ be a scalar multiplication defined by $.(f, r) = fr,\ \forall\ f \in Hom(\langle R \cup I\rangle, \langle R \cup I\rangle)$, $r \in \langle R \cup I\rangle$. Then $\langle R \cup I\rangle$ is a neutrosophic left $Hom(\langle R \cup I\rangle, \langle R \cup I\rangle)$-module.*

Proof Suppose that $\mathrm{Hom}(\langle R \cup I\rangle, \langle R \cup I\rangle)$ is a neutrosophic ring. Then by Theorem (2.3), it is also a ring. It is clear that $.(f, r) = fr$ is the image of r under f and it is an element of $\langle R \cup I\rangle$. It can easily be shown that the scalar multiplication "." satisfies the axioms of a neutrosophic left R-module. Hence, $\langle R \cup I\rangle$ is a neutrosophic left $\mathrm{Hom}(\langle R \cup I\rangle, \langle R \cup I\rangle)$-module. $\square$

Definition 2.46 *Let $\langle M \cup I\rangle$ be a neutrosophic left R-module. The set $\{r \in R : rm = 0\ \ \forall\ m \in \langle M \cup I\rangle\}$ is called the annihilator of $\langle M \cup I\rangle$ and is denoted by $Ann(\langle M \cup I\rangle)$. $\langle M \cup I\rangle$ is said to be faithful if $Ann(\langle M \cup I\rangle) = (0)$. It can easily be shown that $Ann(\langle M \cup I\rangle)$ is a pseudo neutrosophic ideal of $\langle M \cup I\rangle$.*

§3. Neutrosophic Polynomial Rings

In this section and Sections 4 and 5, unless otherwise stated, all neutrosophic rings will be assumed to be commutative neutrosophic rings with unity and x will be an indetrminate in $\langle R \cup I\rangle [x]$.

Definition 3.1 (i) *By the neutrosophic polynomial ring in x denoted by $\langle R \cup I\rangle [x]$ we mean the set of all symbols $\sum_{i=1}^{n} a_i x^i$ where n can be any nonnegative integer and where the coefficients $a_i,\ i = n, n-1, \ldots, 2, 1, 0$ are all in $\langle R \cup I\rangle$.*

(ii) *If $f(x) = \sum_{i=1}^{n} a_i x^i$ is a neutrosophic polynomial in $\langle R \cup I\rangle [x]$ such that $a_i = 0,\ \forall\ i = n, n-1, \ldots, 2, 1, 0$, then we call f(x) a zero neutrosophic polynomial in $\langle R \cup I\rangle [x]$.*

(iii) *If $f(x) = \sum_{i=1}^{n} a_i x^i$ is a nonzero neutrosophic polynomial in $\langle R \cup I\rangle [x]$ with $a_n \neq 0$, then we call n the degree of f(x) denoted by deg f(x) and we write $deg f(x) = n$.*

(iv) *Two neutrosophic polynomials $f(x) = \sum_{i=1}^{n} a_i x^i$ and $g(x) = \sum_{j=1}^{m} b_j x^j$ in $\langle R \cup I\rangle [x]$ are said to be equal written $f(x) = g(x)$ if and only if for every integer $i \geq 0$, $a_i = b_i$ and $n = m$.*

(v) *A neutrosophic polynomial $f(x) = \sum_{i=1}^{n} a_i x^i$ in $\langle R \cup I\rangle [x]$ is called a strong neutrosophic polynomial if for every $i \geq 0$, each a_i is of the form $(a + bI)$ where $a, b \in R$ and $b \neq 0$.*

$f(x) = \sum_{i=1}^{n} a_i x^i$ *is called a mixed neutrosophic polynomial if some* $a_i \in R$ *and some* a_i *are of the form* $(a + bI)$ *with* $b \neq 0$. *If every* $a_i \in R$ *then* $f(x) = \sum_{i=1}^{n} a_i x^i$ *is called a polynomial.*

Example 3.2 $\langle \mathcal{Z} \cup I \rangle [x], \langle \mathcal{Q} \cup I \rangle [x], \langle \mathcal{R} \cup I \rangle [x], \langle \mathcal{C} \cup I \rangle [x]$ are neutrosophic polynomial rings of integers, rationals, real and complex numbers respectively each of zero characteristic.

Example 3.3 Let $\langle \mathcal{Z}_n \cup I \rangle$ be the neutrosophic ring of integers modulo n. Then $\langle \mathcal{Z}_n \cup I \rangle [x]$ is the neutrosophic polynomial ring of integers modulo n. The characteristic of $\langle \mathcal{Z}_n \cup I \rangle [x]$ is n. If $n = 3$ and $\langle \mathcal{Z}_3 \cup I \rangle [x] = \{ax^2 + bx + c : a, b, c \in \langle \mathcal{Z}_3 \cup I \rangle\}$, then $\langle \mathcal{Z}_3 \cup I \rangle [x]$ is a neutrosophic polynomial ring of integers modulo 3.

Example 3.4 Let $f(x), g(x) \in \langle \mathcal{Z} \cup I \rangle [x]$ such that $f(x) = 2Ix^2 + (2 + I)x + (1 - 2I)$ and $g(x) = x^3 - (1 - 3I)x^2 + 3Ix + (1 + I)$. Then f(x) and g(x) are strong and mixed neutrosophic polynomials of degrees 2 and 3 respectively.

Definition 3.5 *Let* α *be a fixed element of the neutrosophic ring* $\langle R \cup I \rangle$. *The mapping* $\phi_\alpha : \langle R \cup I \rangle [x]\rangle \langle R \cup I \rangle$ *defined by*

$$\phi_\alpha \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \right) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$$

is called the neutrosophic evaluation map. It can be shown that ϕ_α *is a neutrosophic ring homomorphism. If* $R = \mathcal{Z}$ *and* $f(x) \in \langle \mathcal{Z} \cup I \rangle [x]$ *such that* $f(x) = 2Ix^2 + x - 3I$, *then* $\phi_{1+I}(f(x)) = 1 + 6I$ *and* $\phi_I(f(x)) = 0$. *The last result shows that* $f(x)$ *is in the kernel of* ϕ_I.

Theorem 3.6([1]) *Every neutrosophic polynomial ring* $\langle R \cup I \rangle [x]$ *contains a polynomial ring R[x].*

Theorem 3.7 *The neutrosophic ring* $\langle R \cup I \rangle$ *is not an integral domain (ID) even if R is an ID.*

Proof Suppose that $\langle R \cup I \rangle$ is an ID. Obviously, $R \subset \langle R \cup I \rangle$. Let $x = (\alpha - \alpha I)$ and $y = \beta I$ be two elements of $\langle R \cup I \rangle$ where α and β are non-zero positive integers. Clearly, $x \neq 0$ and $y \neq 0$ and since $I^2 = I$, we have $xy = 0$ which shows that x and y are neutrosophic zero divisors in $\langle R \cup I \rangle$ and consequently, $\langle R \cup I \rangle$ is not an ID. $\square$

Theorem 3.8 *The neutrosophic polynomial ring* $\langle R \cup I \rangle [x]$ *is not an ID even if R is an ID.*

Proof Suppose that R is an ID. Then $R[x]$ is also an ID and $R[x] \subset \langle R \cup I \rangle [x]$. But then by Theorem 3.7, $\langle R \cup I \rangle$ is not an ID and therefore $\langle R \cup I \rangle [x]$ cannot be an ID. $\square$

Example 3.9 Let $\langle \mathcal{Z} \cup I \rangle [x]$ be the neutrosophic polynomial ring of integers and let f(x), g(x), p(x)and q(x) be neutrosophic polynomials in $\in \langle \mathcal{Z} \cup I \rangle$ given by $f(x) = (2 - 2I)x^2 + 3Ix - I$, $g(x) = Ix + (1 + I)$, $p(x) = (8 - 8I)x^5$ and $q(x) = 7Ix^3$. Then $f(x)g(x) = (2 + I)x^2 + 5Ix - 2I$ and $p(x)q(x) = 0$. Now $deg\ f(x) + deg\ g(x) = 3$, $deg(f(x)g(x)) = 2 < 3$, $deg\ p(x) + deg\ q(x) = 8$ and $deg(p(x)q(x)) = 0 < 8$. The causes of these phenomena are the existence of neutrosophic zero divisors in $\langle \mathcal{Z} \cup I \rangle$ and $\langle \mathcal{Z} \cup I \rangle [x]$ respectively. We register these observations in the following theorem.

Theorem 3.10 *Let $\langle R\cup I\rangle$ be a commutative neutrosophic ring with unity. If $f(x)=\sum_{i=1}^{n}a_ix^i$ and $g(x)=\sum_{j=1}^{m}b_jx^j$ are two non-zero neutrosophic polynomials in $\langle R\cup I\rangle[x]$ with R an ID or not such that $a_n=(\alpha-\alpha I)$ and $b_m=\beta I$ where α and β are non-zero positive integers, then*

$$deg(f(x)g(x)) < deg\ f(x)+deg\ g(x).$$

Proof Suppose that $f(x)=\sum_{i=1}^{n}a_ix^i$ and $g(x)=\sum_{j=1}^{m}b_jx^j$ are two non-zero neutrosophic polynomials in $\langle R\cup I\rangle[x]$ with $a_n=(\alpha-\alpha I)$ and $b_m=\beta I$ where α and β are non-zero positive integers. Clearly, $a_n\neq 0$ and $b_m\neq 0$ but then $a_nb_m=0$ and consequently,

$$\begin{aligned} deg(f(x)g(x)) &= (n-1)+(m-1)\\ &= (n+m)-2<(n+m)\\ &= deg\ f(x)+deg\ g(x). \end{aligned}$$

□

§4. Factorization in Neutrosophic Polynomial Rings

Definition 4.1 *Let $f(x)\in\langle R\cup I\rangle[x]$ be a neutrosophic polynomial. Then*

(*i*) *f(x) is said to be neutrosophic reducible in $\langle R\cup I\rangle[x]$ if there exits two neutrosophic polynomials $p(x),q(x)\in\langle R\cup I\rangle[x]$ such that $f(x)=p(x).q(x)$.*

(*ii*) *f(x) is said to be semi neutrosophic reducible if $f(x)=p(x).q(x)$ but only one of p(x) or q(x) is a neutrosophic polynomial in $\langle R\cup I\rangle[x]$.*

(*iii*) *f(x) is said to be neutrosophic irreducible if $f(x)=p(x).q(x)$ but either p(x) or q(x) equals I or 1.*

Definition 4.2 *Let f(x) and g(x) be two neutrosophic polynomials in the neutrosophic polynomial ring $\langle R\cup I\rangle[x]$. Then*

(*i*) *The pair f(x) and g(x) are said to be relatively neutrosophic prime if the gcd $(f(x),g(x))=r(x)$ is not possible for a neutrosophic polynomial $r(x)\in\langle R\cup I\rangle[x]$.*

(*ii*) *The pair f(x) and g(x) are said to be strongly relatively neutrosophic prime if their gcd $(f(x),g(x))=1$ or I.*

Definition 4.3 *A neutrosophic polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\in\langle\mathcal{Z}\cup I\rangle[x]$ is said to be neutrosophic primitive if the gcd $(a_n,a_{n-1},\cdots,a_1,a_0)=1$ or I.*

Definition 4.3 *Let $f(x)=\sum_{i=1}^{n}a_ix^i$ be a neutrosophic polynomial in $\langle R\cup I\rangle[x]$. f(x) is said to be a neutrosophic monic polynomial if $a_n=1$.*

Example 4.5 Let us consider the neutrosophic polynomial ring $\langle\mathcal{R}\cup I\rangle[x]$ of all real numbers and let f(x) and d(x) be two neutrosophic polynomials in $\langle\mathcal{R}\cup I\rangle[x]$.

(*i*) If $f(x)=2Ix^2-(1+7I)x+6I$ and $d(x)=x-3I$, then by dividing f(x) by d(x) we obtain the quotient $q(x)=2Ix-(1+I)$ and the remainder $r(x)=0$ and hence $f(x)\equiv(2Ix-(1+I))(x-3I)+0$.

(ii) If $f(x) = 2Ix^3 + (1+I)$ and $d(x) = Ix + (2-I)$, then $q(x) = 2Ix^2 - 2Ix + 2I)$, $r(x) = 1 - I$ and $f(x) \equiv (2Ix^2 - 2Ix + 2I))(Ix + (2-I)) + (1-I)$.

(iii) If $f(x) = (2+I)x^2 + 2Ix + (1+I)$ and $d(x) = (2+I)x + (2-I)$, then $q(x) = x - \left(1 - \frac{4}{3}I\right)$, $r(x) = 3 - \frac{4}{3}I$ and $f(x) \equiv \left(x - \left(1 - \frac{4}{3}\right)\right)((2+I)x - (2-I)) + \left(3 - \frac{4}{3}I\right)$.

(iv) If $f(x) = Ix^2 + x - (1+5I)$ and $d(x) = x - (1+I)$, then $q(x) = Ix + (1+2I)$, $r(x) = 0$ and $f(x) \equiv (Ix + (1+2I))(x - (1+I)) + 0$.

(v) If $f(x) = x^2 - Ix + (1+I)$ and $d(x) = x - (1-I)$, then $q(x) = x + (1-2I)$, $r(x) = 2$ and $f(x) \equiv (x + (1-2I))(x - (1-I)) + 2$.

The examples above show that for each pair of the neutrosophic polynomials f(x) and d(x) considered there exist unique neutrosophic polynomials $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ where $deg\ r(x) < deg\ d(x)$. However, this is generally not true. To see this let us consider the following pairs of neutrosophic polynomials in $\langle \mathcal{R} \cup I \rangle [x]$:

(i) $f(x) = 4Ix^2 + (1+I)x - 2I, d(x) = 2Ix + (1+I)$;

(ii) $f(x) = 2Ix^2 + (1+I)x + (1-I), d(x) = 2Ix + (3-2I)$;

(iii) $f(x) = (-8I)x^2 + (7+5I)x + (2-I), d(x) = Ix + (1+I)$;

(iv) $f(x) = Ix^2 - 2Ix + (1+I), d(x) = Ix - (1-I)$.

In each of these examples, it is not possible to find $q(x), r(x) \in \langle \mathcal{R} \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ with $deg\ r(x) < deg\ d(x)$. Hence Division Algorithm is generally not possible for neutrosophic polynomial rings. However for neutrosophic polynomial rings in which all neutrosophic polynomials are neutrosophic monic, the Division Algorithm holds generally. The question of wether Division Algorithm is true for neutrosophic polynomial rings raised by Vasantha Kandasamy and Florentin Smarandache in [1] is thus answered.

Theorem 4.6 *If $f(x)$ and $d(x)$ are neutrosophic polynomials in the neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ with f(x) and d(x) neutrosophic monic, there exist unique neutrosophic polynomials $q(x), r(x) \in \langle R \cup I \rangle [x]$ such that $f(x) = q(x)d(x) + r(x)$ with $deg\ r(x) < deg\ d(x)$.*

Proof The proof is the same as the classical case. □

Theorem 4.7 *Let $f(x)$ be a neutrosophic monic polynomial in $\langle R \cup I \rangle [x]$ and for $u \in \langle R \cup I \rangle$, let $d(x) = x - u$. Then $f(u)$ is the remainder when $f(x)$ is divided by $d(x)$. Furthermore, if $f(u) = 0$ then $d(x)$ is a neutrosophic factor of $f(x)$.*

Proof Since $f(x)$ and $d(x)$ are neutrosophic monic in $\langle R \cup I \rangle [x]$, there exists $q(x)$ and $r(x)$ in $\langle R \cup I \rangle [x]$ such that $f(x) = q(x)(x-u) + r(x)$, with $r(x) = 0$ or $deg\ r(x) < deg\ d(x) = 1$. Hence $r(x) = r \in \langle R \cup I \rangle$. Now, $\phi_u(f(x)) = 0 + r(u) = r(u) = r \in \langle R \cup I \rangle$. If $f(u) = 0$, it follows that $r(x) = 0$ and consequently, $d(x)$ is a neutrosophic factor of $f(x)$. □

Observation 4.8 Since the indeterminancy factor I has no inverse, it follows that the neutrosophic rings $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ cannot be neutrosophic fields and consequently neutrosophic equations of the form $(a + bI)x = (c + dI)$ are not solvable in $\langle \mathcal{Q} \cup I \rangle, \langle \mathcal{R} \cup I \rangle, \langle \mathcal{C} \cup I \rangle$ except $b \equiv 0$.

Definition 4.9 *Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I \rangle [x]$ with $deg\ f(x) \geq 1$. An* *element $u \in \langle R \cup I \rangle$ is said to be a neutrosophic zero of f(x) if $f(u) = 0$.*

Example 4.10 (*i*) Let $f(x) = 6x^2 + Ix - 2I \in \langle \mathcal{Q} \cup I \rangle [x]$. Then f(x) is neutrosophic reducible and (2x-I) and (3x+2I) are the neutrosophic factors of $f(x)$. Since $f(\frac{1}{2}I) = 0$ and $f(-\frac{2}{3}I) = 0$, then $\frac{1}{2}I, -\frac{2}{3}I \in \langle \mathcal{Q} \cup I \rangle$ are the neutrosophic zeroes of $f(x)$. Since $f(x)$ is of degree 2 and it has two zeroes, then the Fundamental Theorem of Algebra is obeyed.

(*ii*) Let $f(x) = 4Ix^2 + (1 + I)x - 2I \in \langle \mathcal{Q} \cup I \rangle [x]$. $f(x)$ is neutrosophic reducible and $p(x) = 2Ix + (1 + I)$ and $q(x) = (1 + I)x - I$ are the neutrosophic factors of $f(x)$. But then, $f(x)$ has no neutrosophic zeroes in $\langle \mathcal{Q} \cup I \rangle$ and even in $\langle \mathcal{R} \cup I \rangle$ and $\langle \mathcal{C} \cup I \rangle$ since I^{-1}, the inverse of I does not exist.

(*iii*) $Ix^2 - 2$ is neutrosophic irreducible in $\langle \mathcal{Q} \cup I \rangle [x]$ but it is neutrosophic reducible in $\langle \mathcal{R} \cup I \rangle [x]$ since $Ix^2 - 2 = (Ix - \sqrt{2})(Ix + \sqrt{2})$. However since $\langle \mathcal{R} \cup I \rangle$ is not a field, $Ix^2 - 2$ has no neutrosophic zeroes in $\langle \mathcal{R} \cup I \rangle$.

Theorem 4.11 *Let $f(x)$ be a neutrosophic polynomial of degree > 1 in the neutrosophic polynomial ring $\langle R \cup I \rangle [x]$. If $f(x)$ has neutrosophic zeroes in $\langle R \cup I \rangle$, then $f(x)$ is neutrosophic reducible in $\langle R \cup I \rangle [x]$ and not the converse.*

Theorem 4.12 *Let $f(x)$ be a neutrosophic polynomial in $\langle R \cup I \rangle [x]$. The factorization of $f(x)$ if possible over $\langle R \cup I \rangle [x]$ is not unique.*

Proof Let us consider the neutrosophic polynomial $f(x) = 2Ix^2 + (1 + I)x + 2I$ in the neutrosophic ring of polynomials $\langle \mathcal{Z}_3 \cup I \rangle [x]$. $f(I) = 0$ and by Theorem 4.11, $f(x)$ is neutrosophic reducible in $\langle \mathcal{Z}_3 \cup I \rangle [x]$ and hence f(x) can be expressed as $f(x) = (2Ix + 1)(x - I) = (2Ix + 1)(x + 2I)$. However, $f(x)$ can also be expressed as $f(x) = [(1 + I)x + I][Ix + (1 + I)]$. This shows that the factorization of $f(x)$ is not unique in $\langle \mathcal{Z}_3 \cup I \rangle [x]$. We note that the first factorization shows that $f(x)$ has $I \in \langle \mathcal{Z}_3 \cup I \rangle$ as a neutrosophic zero but the second factorization shows that f(x) has no neutrosophic zeroes in $\langle \mathcal{Z}_3 \cup I \rangle$. This is different from what obtains in the classical rings of polynomials. □

Observation 4.13 Let us consider the neutrosophic polynomial ring $\langle R \cup I \rangle [x]$. It has been shown in Theorem 3.8 that $\langle R \cup I \rangle [x]$ cannot be a neutrosophic ID even if R is an ID. Also by Theorem 4.12, factorization in $\langle R \cup I \rangle [x]$ is generally not unique. Consequently, $\langle R \cup I \rangle [x]$ cannot be a neutrosophic Unique Factorization Domain (UFD) even if R is a UFD. Thus Gauss's Lemma, which asserts that R[x] is a UFD if and only if R is a UFD does not hold in the setting of neutrosophic polynomial rings. Also since $I \in \langle R \cup I \rangle$ and I^{-1}, the inverse of I does not exist, then $\langle R \cup I \rangle$ cannot be a field even if R is a field and consequently $\langle R \cup I \rangle [x]$ cannot be a neutrosophic UFD. Again, the question of wether $\langle R \cup I \rangle [x]$ is a neutrosophic UFD given that R is a UFD raised by Vasantha Kandasamy and Florentin Smarandache in [1] is answered.

§5. Neutrosophic Ideals in Neutrosophic Polynomial Rings

Definition 5.1 *Let $\langle R \cup I \rangle [x]$ be a neutrosophic ring of polynomials. An ideal J of $\langle R \cup I \rangle [x]$*

is called a neutrosophic principal ideal if it can be generated by an irreducible neutrosophic polynomial $f(x)$ *in* $\langle R \cup I \rangle [x]$.

Definition 5.2 *A neutrosophic ideal* P *of a neutrosophic ring of polynomials* $\langle R \cup I \rangle [x]$ *is called a neutrosophic prime ideal if* $f(x)g(x) \in P$, *then* $f(x) \in P$ *or* $g(x) \in P$ *where* $f(x)$ *and* $g(x)$ *are neutrosophic polynomials in* $\langle R \cup I \rangle [x]$.

Definition 5.3 *A neutrosophic ideal* M *of a neutrosophic ring of polynomials* $\langle R \cup I \rangle [x]$ *is called a neutrosophic maximal ideal of* $\langle R \cup I \rangle [x]$ *if* $M \neq \langle R \cup I \rangle [x]$ *and no proper neutrosophic ideal* N *of* $\langle R \cup I \rangle [x]$ *properly contains* M *that is if* $M \subseteq N \subseteq \langle R \cup I \rangle [x]$ *then* $M = N$ *or* $N = \langle R \cup I \rangle [x]$.

Example 5.4 Let $\langle \mathcal{Z}_2 \cup I \rangle [x] = \{ax^2 + bx + c : a, b, c \in \langle \mathcal{Z}_2 \cup I \rangle\}$ and consider $f(x) = Ix^2 + Ix + (1 + I) \in \langle \mathcal{Z}_2 \cup I \rangle [x]$. The neutrosophic ideal $J =< f(x) >$ generated by $f(x)$ is neither a neutrosophic principal ideal nor a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$. This is so because f(x) is neutrosophic reducible in $\langle \mathcal{Z}_2 \cup I \rangle [x]$ eventhough it does not have zeroes in $\langle \mathcal{Z}_2 \cup I \rangle$. Also, $(Ix + (1 + I))(Ix + 1) \in J$ but $(Ix + (1 + I)) \notin J$ and $(Ix + 1) \notin J$. Hence J is not a neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$. However, $< 0 >$ is the only neutrosophic prime ideal of $\langle \mathcal{Z}_2 \cup I \rangle [x]$ which is not a neutrosophic maximal ideal.

Theorem 5.5 *Let* $\langle R \cup I \rangle [x]$ *be a neutrosophic ring of polynomials. Every neutrosophic principal ideal of* $\langle R \cup I \rangle [x]$ *is not prime.*

Proof Consider the neutrosophic polynomial ring $\langle \mathcal{Z}_3 \cup I \rangle [x] = \{x^3 + ax + b : a, b \in \langle \mathcal{Z}_3 \cup I \rangle\}$ and Let $f(x) = x^3 + Ix + (1 + I)$. It can be shown that $f(x)$ is neutrosophic irreducible in $\langle \mathcal{Z}_3 \cup I \rangle [x]$ and therefore $< f(x) >$, the neutrosophic ideal generated by f(x) is principal and not a prime ideal. We have also answered the question of Vasantha Kandasamy and Florentin Smarandache in [1] of wether every neutrosophic principal ideal of $\langle R \cup I \rangle [x]$ is also a neutrosophic prime ideal. □

References

[1] Vasantha Kandasamy W.B. and Florentin Smarandache, *Neutrosophic Rings*, Hexis, Phoenix, Arizona, 2006.

[2] Vasantha Kandasamy W.B. and Florentin Smarandache, *Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures*, Hexis, Phoenix, Arizona, 2006.

[3] Vasantha Kandasamy W.B. and Florentin Smarandache, *Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models*, Hexis, Church Rock, 2004.

[4] Vasantha Kandasamy W.B., Gaussian polynomial rings, *Octogon*, Vol.5(1997), 58-59.

[5] Vasantha Kandasamy W.B., Inner associative rings, *J. of Math. Res. and Expo.*, Vol.18(1998), 217-218.

[6] Vasantha Kandasamy W.B., *CN rings*, *Octogon*, Vol.9(2001), 343-344.

[7] Vasantha Kandasamy W.B., On locally semi unitary rings, *Octogon*, Vol.9(2001), 260-262.

[8] Vougiouklis Thomas, On rings with zero divisors strong V-groups, *Comment Math. Univ. Carolin J.*, Vol.31(1990), 431-433.

[9] Wilson John S., A note on additive subgroups of finite rings, *J. Algebra*, Vol.234(2000), 362-366.

[10] Florentin Smarandache, *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*(3rd edition), American Research Press, Rehoboth, 2003.

[11] Agboola A.A.A. and Akinola L.S., On the Bicoset of a Bivector Space, *Int. J. of Math. Comb.* Vol.4(2009), 1-8.

[12] Fraleigh, J.B., *A First Course in Abstract Algebra*(5th Edition), Addison-Wesley, 1994.

[13] Lang S., *Algebra*, Addison Wesley, 1984.

[14] Atiyah M.F. and MacDonald I.G., *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[15] Herstein I.N., *Topics in Algebra*, John Wiley and Sons, 1964.

[16] Herstein I.N., *Topics in Ring Theory*, University of Chicago Press, 1969.

Neutrosophic Degree of a Paradoxicity

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§1. Definition of a Paradox

A paradox is called a statement $< P >$ which is true and false in the same time. Therefore, if we suppose that statement $< P >$ is true, it results that $< P >$ is false; and reciprocally, if we suppose that $< P >$ is false, it results that $< P >$ is true.

§2. Semi-Paradox

There are statements that do not completely obey the definition in Section 1. We call a *semi-paradox* a statement $< SP >$ such that either supposing that $< SP >$ is true it results that $< SP >$ is false (but not reciprocally), or supposing that $< SP >$ is false it results that $< SP >$ is true (but not reciprocally). So, the statement has a degree of 0.50 (50%) of a paradox, and 0.50 of a non-paradox.

§3. Three-Quarters Paradox

3.1 Definition

There are cases when a statement $< QP >$ can be between a paradox and a semi-paradox. For example:

a) If we suppose that the statement $< QP >$ is true, it results that $< QP >$ is false, but reciprocally if we suppose that the statement $< QP >$ is false, it may be possible resulting that $< QP >$ is true. Therefore, the second implication (conditional) does not always occur.

b) Or, if we suppose that the statement $< QP >$ is false, it results that $< QP >$ is true, but reciprocally if we suppose that the statement $< QP >$ is true, it may be possible resulting that $< QP >$ is false. Therefore, the second implication (conditional) does not always occur.

In this case we may have a degree of paradoxicity in between 0.50 and 1, actually in a neighborhood of 0.75.

These types of fuzzy and especially neutrosophic implications are derived from the fuzzy or neutrosophic logic connectives.

3.2 Examples of Three-Quarters Paradoxes

Social Three-Quarters Paradox

In a democracy, should the non-democratic ideas be allowed?

a) If no, i.e. other ideas are not allowed - even those non-democratic -, then one not has a democracy, because the freedom of speech is restricted.

b) If yes, i.e. the non-democratic ideas are allowed, then one might end up to a non-democracy (because the non-democratic ideas could overthrow the democracy as, for example, it happened in Nazi Germany, in totalitarian countries, etc.).

Three-Quarters Paradox of Freedom of Speech & Religion (I)

As a freedom of speech do we have the right to insult religion?

a) If not, then we don't have freedom of speech.

b) If yes, i.e. we have the right to insult religion, then we don't respect the freedom of faith.

Devine Three-Quarters Paradox (II)

Can God prove He can commit suicide?

a) If not, then it appears that there is something God cannot do, therefore God is not omnipotent.

b) If God can prove He can commit suicide, then God dies - because He has to prove it, therefore God is not immortal.

Devine Three-Quarters Paradox (III)

Can God prove He can be atheist, governed by scientific laws?

a) If God cannot, then again He's not omnipotent.

b) If God can prove He can be atheist, then God doesn't believe in Himself, therefore why should we believe in Him?

Devine Three-Quarters Devine Paradox (IV)

Can God prove He can do bad things?

a) If He cannot, then He is not omnipotent, therefore He is not God.

b) If He can prove He can do bad things, again He's not God, because He doesn't suppose to do bad things.

Devine Three-Quarters Paradox (V)

Can God create a man who is stronger than him?

a) If not, then God is not omnipotent, therefore He is not God.

b) If yes, i.e. He can create someone who is stronger than Him, then God is not God any longer since such creation is not supposed to be possible, God should always be the strongest. (God was egocentric because he didn't create beings stronger than Him.)

Devine Three-Quarters Paradox (VI)

Can God transform Himself in his opposite, the Devil?

a) If not, then God is not omnipotent, therefore He is not God.

b) If yes, then God is not God anymore since He has a dark side: the possibility of transforming Himself into the Devil (God doesn't suppose to be able to do that).

§4. **Degree of a Paradox**

Let's consider a statement $< DP >$.

(α) If we suppose that the statement $< DP >$ is true it may result that $< DP >$ is false, and reciprocally

(β) if we suppose that the statement $< DP >$ is false it may result that $< DP >$ is true. Therefore, both implications (conditionals) depend on other factors in order to occur or not, or partially they are true, partially they are false, and partially indeterminate (as in neutrosophic logic).

§5. **Discussion**

This is the general definition of a statement with some degree of paradoxicity.

a) If both implications (α) and (β) are true 100%, i.e. the possibility **"***it may result***"** is replaced by the certitude **"***it results***"** we have a 100% paradox.

b) If one implication is 100% and the other is 100% false, we have a semiparadox (50% of a paradox).

c) If both implications are false 100%, then we have a non-paradox (normal logical statement).

d) If one condition is $p\%$ true and the other condition $q\%$ true (truth values measured with the fuzzy logic connectives or neutrosophic logic connectives), then the degree of paradoxicity of the statement is the average $\frac{p+q}{2}\%$.

e) Even more general from the viewpoint of the neutrosophic logic, where a statement is $T\%$ true, $I\%$ indeterminate, and $F\%$ false, where T, I, F are standard or non-standard subsets of the non-standard unit interval $]-0, 1+[$.

If one condition has the truth value (T_1, I_1, F_1) and the other condition the truth value (T_2, I_2, F_2), then the neutrosophic degree of paradoxicity of the statement is the average of the component triplets:

$$\left(\frac{T_1 + T_2}{2}, \frac{I_1 + I_2}{2}, \frac{F_1 + F_2}{2}\right),$$

where the addition of two sets A and B (in the case when T, I, or F are sets) is simply defined as:

$$A + B = \{x | x = a + b \text{ with } a \in A \text{ and } b \in B\}.$$

§6. **Comment**

When T, I, F are crisp numbers in the interval $[0, 1]$, and $I = 0$, while $T + F = 1$, then the neutrosophic degree of paradoxicity coincides with the (fuzzy) degree of paradoxicity from d).

References

[1] Smarandache Florentin, *Neutrosophy/Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, 1998.

我与重空间的故事

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摘要： 什么是重空间？什么又是数学组合？这两者都是科学认识的方法，同时又是推动科学发展的重要思想。这里，数学组合是内蕴空间拓扑结构的 Smarandache 重空间，所以更有利于采用数学方法确定事物行为。本文回顾了我所认识的重空间，特别是赋有空间拓扑结构的重空间 – 数学组合，阐释了其对对人类认识自然的重要意义，以及筹备“首届 Smarandache 重空间与重结构国际学术交流会”过程中的一些趣事。

Abstract: *What is a multispace*? *And what is the mathematical combinatorics*? Both of them are the philosophic notions for scientific research. In where, the mathematical combinatorics is such maltispaces underlying combinatorial structures in topological spaces. So these notions are useful approaches for determining the behavior of things in the world by mathematics. I review the multispaces that I known, particularly, the mathematical combinatorics in this paper, and explain why it is important for one understanding things in the world. Some interesting things in preparing the *First international Conference on Smarandache multispace and Multistructure* are also included.

我是学习组合学与图论，特别是图在曲面上的嵌入和组合地图理论的。虽然在博士、博士后阶段也曾按照经典组合学思想，研究过一些组合问题，如图的 hamiltonian 性质、图的离心率、Cayley 图的泛因子，以及标根与不标根地图计数问题等，但从 2003 年到 2004 年这 2 年时间，我的科研工作陷入了瓶颈而停滞不前，因为看不清组合学的终极目标是什么。这两年我也一直在思考这样两个哲学问题；

组合学对认识自然并适应其发展的作用是什么？它对数学发展又有哪些贡献？

这两个问题对完成我的博士后报告显得日益重要，因为如不能从思想上解决问题，最终我的博士后研究不会有大进展，它们涉及人类认识自然的过程。好在我博士阶段还有很多科研工作没发表，将其推广至 Riemann 曲面、Klein 曲面，并计算一些新的组合结果不是一件难事。

按照这一思路，2004 年 11 月至 2005 年 2 月，我完成了中国科学院的博士后报告《Automorphism Groups of Maps and Klein Surfacs》(地图与 Klein 曲面的自同构群)，其出发点是将图在曲面上的一些嵌入结果，以及计数推广至 Riemann 曲面、Klein 曲面，并由此采用组合方法解决其上的一些几何问题。报告给人的直观感觉是关于代数曲面的，属于纯数学领域。我自己很欣赏这篇报告，也很想在数学上留下一点痕迹，特别是其中提出的组合学对数学科学发展作用构想——数学组合化猜想 (CC-Conjecture) 即：

任何一门数学学科可以进行组合重建或组合推广。

于是，这份博士后报告经过我几个月的改写和扩充，于 2005 年 3 月将其发在美国一家出版社出版。审稿过程中，编辑提出，报告中的 Riemann 几何，实际上可以在更广的范围，即 Smarandache 几何中研究，从而把一个组合问题转化为一个具有实际几何意义的问题，并建议我在报告中添加 Smarandache 几何有关内容，并与报告风格一致。这就是 2005 年我在美国 American Research Press 出版的《Automorphism Groups of Maps, Surfaces and Smarandache Geometries》(地图、曲面与 Smarandache 几何自同构群) 那本著作。

Smarandache 几何思想实际上就是一种重空间思想，特别是其中一个几何公理可以同时成立或不成立，或是以多种方式不成立思想，是我在多年接受的数学教育中没有出现的，因为经典数学中从来不讨论一个有矛盾的系统，一直认为它违背矛盾律，即数学仅研究无矛盾系统，但实际上，矛盾在自然界中无处不在，这也是数学方法不能成为解决所有人类认识自然和适应自然问题的原因，因为人类认识未知世界等同于“盲人摸象”蕴含的哲学道理，即对未知事物的全面认识和把握，唯一方法是综合所有局部认识结果，即认识集合的并集，这就是 Smarandache 重空间。

那么，*这种含有矛盾的系统应怎样刻画，它又有哪行为值得关注？*抽象研究 Smarandache 重空间不过是一个集合问题，不会得到太多有价值的定量结果。实际上，如果一个重空间对应着一个具体事物，那么集合元之间一定存在某种联系，即其内蕴一种空间拓扑结构。为此，我在 Smarandache 重空间基础上提出了*数学组合*，即给定空间拓扑结构的 Smarandache 重空间思想，并于 2006 年在美国出版《Smarandache Multi-Space Theory》(Smarandache 重空间理论）一书，对经典数学系统进行了组合推广。这是国际上第一本系统总结 Smarandache 重空间理论的著作。2006 年，在“*全国第二届组合学与图论学术交流会*”上，我利用大会组织者提供的 15 分钟报告时间，把我的数学组合化猜想进行了系统阐述，并随后将这次报告“Combinatorial speculation and combinatorial conjecture for mathematics”(组合思想及数学组合化猜想）发给一些国际网站刊发，受到学术界普遍关注。这篇报告后来成为了国际互联网百科全书用匈牙利语解释“*组合学*”一词的引用文献。而我的《Smarandache Multi-Space Theory》一书经过进一步扩充，则于 2011 年在美国一家教育出版社出版，成为了数学研究生教材。

值得注意的是，Smarandache 重空间思想反映的是哲学整体观，而我的数学组合化猜想反映的是哲学联系观，即事物之间存在普遍联系。这两者的有机结合才是人类认识自然界和事物发展应有的思想，也是数学科学的发展方向，因为它体现的是哲学整体观和联系观，这当中，空间拓扑结构，无疑将成为描写事物行为的基本工具，这也是我近 10 年一直在从事数学组合研究的一个主要原因。

人的多面性使得他人常不能准确把握一个人，这也造成他人错误理解一个人，比如公众人物，其台前与台后表现的巨大反差常常让人惊讶，殊不知，这正是人在认识上的片面性使然。讲到 Smarandache 重空间，我本人就是一个恰当例子。我第一个专业是工业与民用建筑，于是在中国一家大型建筑施工企业从事了 10 多年技术管理工作，但个人一直致力于数学研究，因为个人志向是数学，在博士与博士后都是从事的数学。很奇怪的现象是在谷歌上搜我的名字“*Linfan Mao*”，看到的基本上都是一些我与数学有关的事项，包括我在国外用英文出版的一些专著、发表的论文等，但在百度上搜“*毛林繁*”，看到的几乎都是招标事项，因为我同时担任中国招标投标协会副秘书长一职，这可能也是中国特色吧！于是我一个人就出现了两种面孔给公众：*在国外人眼里，我是一位数学工作者；在国人眼里，我是一位招标采购理论工作者*，所以我本身就是一个“*重空间*”。为此，2011 年，一些美国朋友建议我于 2013 年 28-30 日组织“*首届 Smarandache 重空间及重结构国际学术研讨会议*”，并在美国数学会

网站上登出会议通知如下：

First International Conference
On Smarandache Multispace and Multistructure

Month: June 2013

Date: June 28–30

Name: First International Conference on Smarandache Multispace and Multistructure

Location: Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing 100190, People's Republic of China.

Description: The notion of multispace was introduced by F. Smarandache in 1969 under his idea of hybrid mathematics: combining different fields into a unifying field, which is closer to our real life, since we don.t have a homogeneous space, but many heterogeneous ones. Today, this idea is widely accepted by the world of sciences. S-Multispace is a qualitative notion and includes both metric and non-metric spaces. It is believed that the smarandache multispace with its multistructure is the best candidate for 21st century Theory of Everything in any domain. It unifies many knowledge fields. In a general definition, a smarandache multi-space is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap. A such multispace can be used, for example, in physics for the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak and strong interactions. Other applications: multi-groups, multi-rings, geometric multispace.

Information: *http://fs.gallup.unm.edu/multispace.htm*

同时在美国新墨西哥大学网站登出进一步会议信息：

Date and Location: 28-30 June 2013, Chinese Academy of Sciences, Beijing, P. R. China

Organizer: Dr.Linfan Mao, Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing 100190, P. R. China, [email: *maolinfan@163.com*]

American Mathematical Society's Calendar website:

http://www.ams.org/meeting/calendar/2013_jun28-30_beijing100190.html

The notion of multispace was introduced by F. Smarandache in 1969 under his idea of hybrid mathematics: combining different fields into a unifying field, which is closer to our real life, since we don't have a homogeneous space, but many heterogeneous ones. Today, this idea is widely accepted by the world of sciences.

S-Multispace is a qualitative notion, since it is too large and includes both metric and non-metric spaces.

It is believed that the smarandache multispace with its multistructure is the best candidate for 21st century Theory of Everything in any domain. It unifies many knowledge fields.

In a general definition, a smarandache multi-space is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap.

A such multispace can be used for example in physics for the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak and strong interactions. Or in the parallel quantum computing and in the mu-bit theory, in multi-entangled states or particles and up to multi-entangles objects.

As applications we also mention: the algebraic multispaces (multi-groups, multi-rings, multi-vector spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold, etc.), geometric multispaces (combinations of Euclidean and Non-Euclidean geometries into one space as in Smarandache geometries), theoretical physics, including the relativity theory, M-theory and cosmology, then multi-space models for p-branes and cosmology, etc.

Papers will be published in the Proceedings of the Conference.

既然是国际会议，本应该邀请一些国外学者到会。此前，也确有美国、墨西哥、印度、尼日利亚、伊朗等国 10 多位学者与我联系，想参加这次会议，但苦于我组织这次会议没有经费，于是将会议邀请函发给这些学者后，只能实情相告费用自理。好在这些学者比较理解，把会议论文电子版发给我，算是他们参加了这次会议而不用亲自到会，其中，印度研究代数重结构的 Vasantha Kandasamy 教授除给我发了一封不能到会的致歉信外，还让美国的 Smarandache 教授转发给我一封她的信，进一步表达不能到会报告的歉意；墨西哥有两位教授是夫妻俩，来信表明到会是为学习 Smarandache 重空间与重结构理论，因为他们刚体会到这一领域对科学发展的巨大推动作用。

“*首届 Smarandache 重空间及重结构国际学术研讨会议*”于 2013 年 6 月 28 日在北京建筑大学召开，应该说，没有这些国际学者的关心和帮助，想在中国举办一次这样的会议，并得到数学工作者响应是不可能的一件事。而这次会议的组织，包括会场布置与安排、印刷海报、接待与会代表等事项，更是得到了北京建筑大学经济与管理工程学院院长*姜军教授*、*张俊副教授*，以及在校研究生*李帅锋*、*要翠玲*和*雷雨*等同学大力支持。没有这几位同志的辛勤工作，这次会议也难于在北京建筑大学如期举办。在此，特向这些朋友表示感谢。

实际上，Smarandache 重空间思想与中国哲学“*天人合一*”思想一致，又与老子的思想相通，这也是我这些年极力推崇的科学研究思想，即：*科学研究需要哲学，特别是中国哲学*，因为中国哲学不是那种“*头疼医头，脚疼医脚*”，而是一种系统认识自然的思想。之所以有这样的认识，是因为 2004 年我在中国科学院从事数学研究时遇到的*研究方向迷惑*，是通过研读《道德经》，并体会其中的科学认识过程得到启发而解决的。

我在美国出版的另一本著作《Combinatorial Geometry with Applications to Field Theory》(组合几何及其在场论中的应用，2009 年) 中有一节，专门讨论《*道德经*》中几段关于科学认识的思想，并将拓扑图作为事物内蕴结构，对拓扑学、微分几何中的空间模型和引力场、规范场进行组合并分析其行为，而这正是我的数学组合思想在科学研究中的应用实例。所以，从事科学研究，不懂中国哲学是不行的，因为只有中国人的思想是系统思想，研究事物考虑其方方面面。有道是：*哲学给人以智慧，数学给人以精准*，二者有机结合，就在重空间基础上产生了研究事物多面性的工具 – *数学组合*，这是一种科学研究的升华，对于人类认识自然不能不说是一件十分有益的事情，这也是组织“*首届 Smarandache 重空间及重结构国际学术研讨会议*”的直接动因。

Summary: In any domain of knowledge, a Smarandache multispace (or S-multispace) with its multistructure is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap. The notions of multispace (also spelt multi-space) and multistructure (also spelt multi-structure) were introduced by Smarandache in 1969 under his idea of hybrid science: combining different fields into a unifying field, which is closer to our real life world since we live in a heterogeneous space. Today, this idea is widely accepted by the world of sciences. S-multispace is a qualitative notion, since it is too large and includes both metric and non-metric spaces. It is believed that the smarandache multispace with its multistructure is the best candidate for 21st century Theory of Everything in any domain. It unifies many knowledge fields.

Applications

A such multispace can be used for example in physics for the *Unified Field Theory* that tries to unite the gravitational, electromagnetic, weak and strong interactions. Or in the parallel quantum computing and in the mu-bit theory, in multi-entangled states or particles and up to multi-entangles objects. We also mention: the algebraic multispaces (multi-groups, multi-rings, multi-vector spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold, etc.), geometric multispaces (combinations of Euclidean and Non-Euclidean geometries into one space as in Smarandache geometries), theoretical physics, including the relativity theory, the M-theory and the cosmology, then multi-space models for p-branes and cosmology, etc.

- The multispace and multistructure were first used in the Smarandache geometries (1969), which are combinations of different geometric spaces such that at least one geometric axiom behaves differently in each such space.

- In paradoxism (1980), which is a vanguard in literature, arts, and science, based on finding common things to opposite ideas [i.e. combination of contradictory fields].

- In neutrosophy (1995), which is a generalization of dialectics in philosophy, and takes into consideration not only an entity $< A >$ and its opposite $< antiA >$ as dialectics does, but also the neutralities $< neutA >$ in between. Neutrosophy combines all these three $< A >, < antiA >$ and $< neutA >$ together. Neutrosophy is a metaphilosophy.

- Then in neutrosophic logic (1995), neutrosophic set (1995), and neutrosophic probability (1995), which have, behind the classical values of truth and falsehood, a third component called indeterminacy (or neutrality, which is neither true nor false, or is both true and false simultaneously - again a combination of opposites: true and false in indeterminacy).

- Also used in Smarandache algebraic structures (1998), where some algebraic structures are included in other algebraic structures.

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