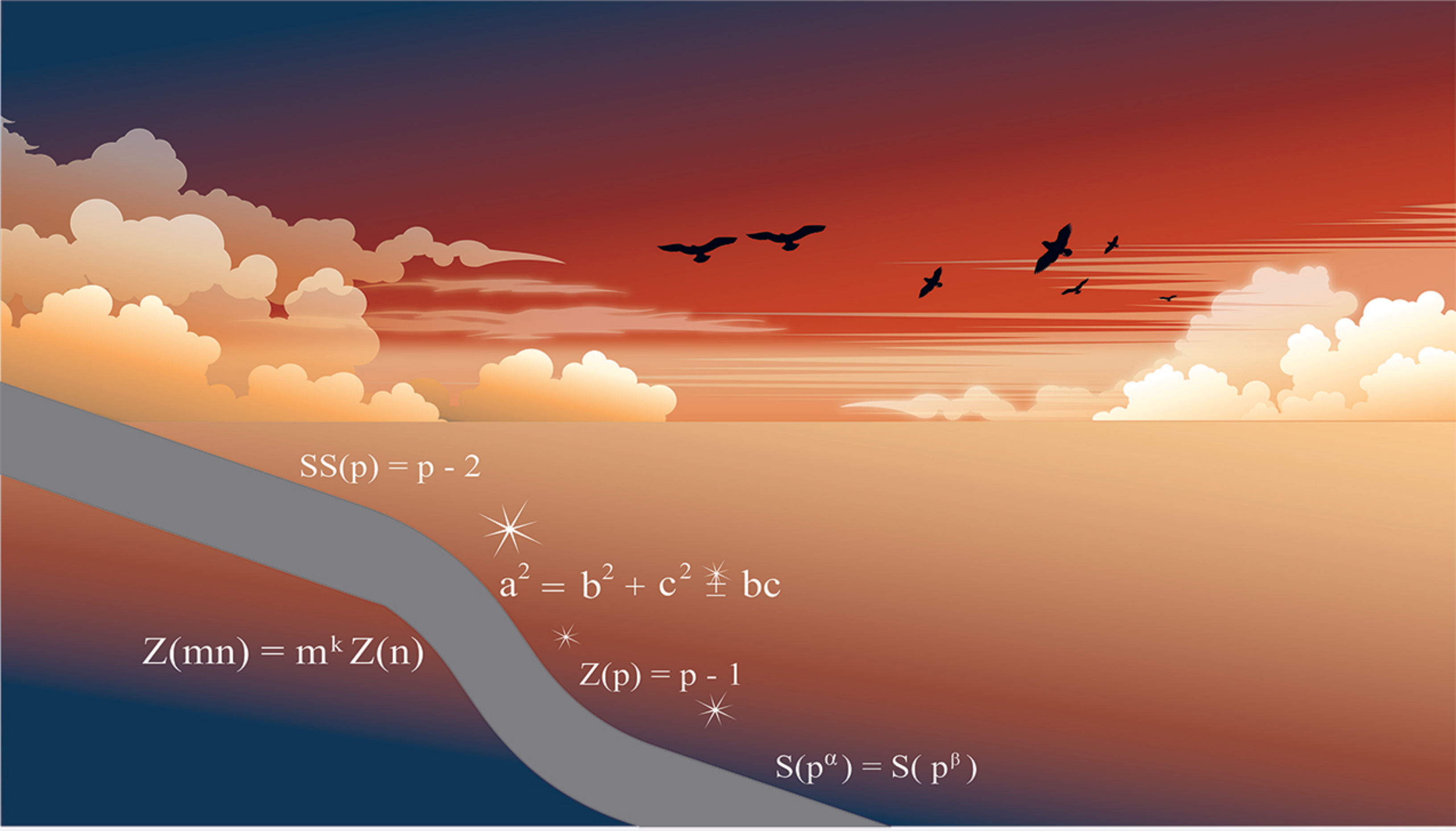
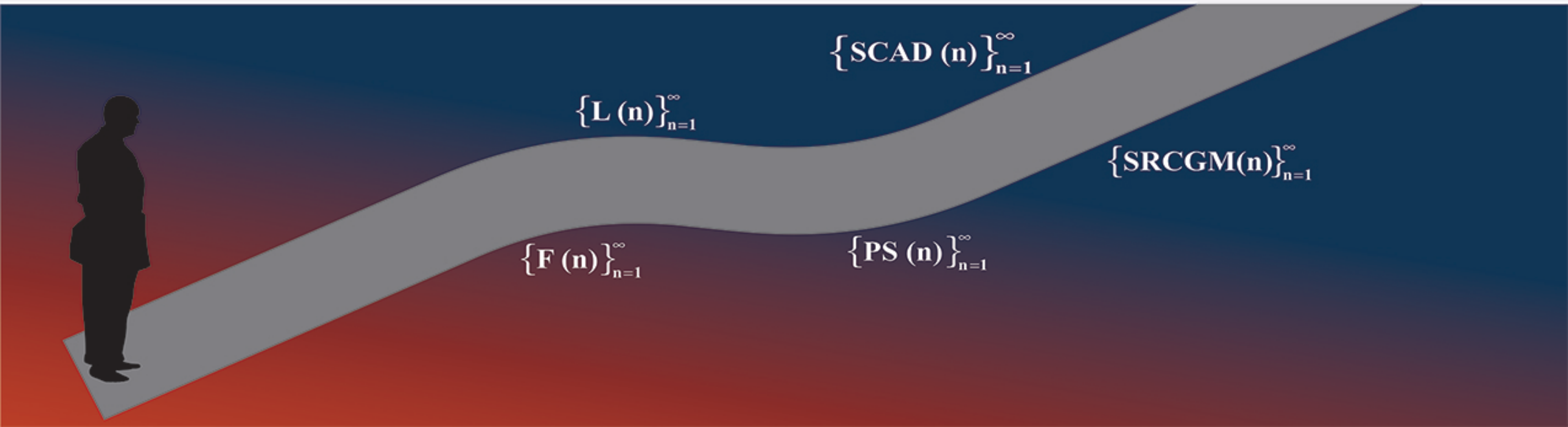


Smarandache Numbers Revisited

A.A.K. Majumdar



SMARANDACHE NUMBERS REVISITED

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PREFACE

The primary source of all
mathematics is the integers

More than seven years ago, my first book on some of the Smarandache notions was published. The book consisted of five chapters, and the topics covered were as follows : (1) some recursive type Smarandache sequences, (2) Smarandache determinant sequences, (3) the Smarandache function, (4) the pseudo Smarandache function, and (5) the Smarandache function related and the pseudo Smarandache function related triangles.

Since then, new and diversified results have been published by different researchers. The aim of this book to update some of the contents of my previous book, and add some new results.

In Chapter 1, some recurrence type of Smarandache sequences are considered. These are : The Smarandache odd sequence, the Smarandache even sequence, the Smarandache circular seqence, the Smarandache square product sequences, the Smarandache permutation sequence, the Smarandache reverse sequence, the Smarandache symmetric sequence, and the Smarandache prime product sequence. It has been conjectured that none of these sequences contain infinitely many Fibonacci or Lucas numbers. In my earlier book, it has been shown that none of these sequences satisfies the recurrence relations of the Fibonacci and Lucas numbers. Here, we show that this result, in fact, follow from the common characteristic of these recursive sequences. Chapter 2 deals with two types of geometric type Smarandache determinant sequences; it is also shown that the results of some particular type of Smarandache determinant sequences are simplified with the introduction of the circulant matrices. In Chapter 3, some new expressions of the Smarandache function S(n) are given. Chapter 4 gives some new results on the pseudo Smarandache function, Z(n), including solutions of the Diophantine equations $Z(n) + SL(n) = n$ and $Z(n) = SL(n)$, where SL(n) is the Smarandache LCM function. In Chapter 4, we also consider the equation $Z(mn) = m^k Z(n)$, where m, n and k are positive integers. In connection with the Smarandache number related triangles, it is known that, if a, b and c are the sides of the 60-degree and 120-degree triangles, then the Diophantine equations $c^2 = a^2 + b^2 \pm ab$ are satisfied by a, b and c. Chapter 5 gives partial solutions to these Diophantine equations. Finally, in Chapter 6, some miscellaneous topics are treated. The five topics covered in Chapter 6 are (1) the triangular numbers and the Smarandache T-sequence, (2) the Smarandache friendly numbers, (3) the Smarandache reciprocal partition sets of unity, (4) the Smarandache LCM ratio, and (5) the Sandor-Smarandache function. Most of the results appeared before, but some results, particularly some in Chapter 2, Chapter 5 and Chapter 6, are new. Particular mention must be made of Section 6.5 dealing with the Sandor-Smarandache function. In writing the book, I took the freedom of including the more recent results, found by other researchers, to keep the expositions up-to-date. In the previous book, several open problems / conjectures / questions were listed, most of which still remain unsolved. In this book, we add some new open problems and conjectures at the end of Chapter 1, Chapter 3, Chapter 4, Chapter 5 and Chapter 6.

I would like to take this opportunity to thank the Department of Mathematics, Jahangirnagar University, Bangladesh, for hosting me as a guest Professor during the Academic Development Leave from the Ritsumeikan Asia-Pacific University, Japan, from April to September, 2018.

A.A.K. Majumdar
Ritsumeikan Asia-Pacific University, Japan

Dedicated to the Memory

of

My

Departed Parents

who were like umbrellas and canopies enveloping us

TABLE OF CONTENTS

Notations and Symbols

$F_n = F(n)$: The n-th term of the sequence of Fibonacci numbers

$L_n = L(n)$: The n-th term of the sequence of Lucas numbers

$A = (a_{ij})$: The matrix A (of order $I \times J$) whose entries are a_{ij}, $1 \le i \le I$, $1 \le j \le J$

$C_i \leftrightarrow C_j$: The columns C_i and C_j (of the matrix $A = (a_{ij})$) are interchanged

$R_i \leftrightarrow R_j$: The rows R_i and R_j (of the matrix $A = (a_{ij})$) are interchanged

$C_i \to C_i + kC_j$: The column C_j is multiplied by the constant k and is added to the column C_i

$R_i \to R_i + kR_j$: The row R_j is multiplied by the constant k and is added to the row R_i

$D = \left| d_{ij} \right|$: The determinant D with entries d_{ij}, $1 \le i, j \le n$

$\lfloor x \rfloor$: The integer part of the real number $x > 0$ (the *floor* of x)

$\mathbb{Z}^+$: The set of positive integers

$m \mid n$: The integer m divides the integer n

S(.) : The Smarandache function

Z(.) : The pseudo Smarandache function

SL(n) : The Smarandache LCM function

$(N_1, N_2, \ldots, N_n)$: GCD (Greatest Common Divisor) of the n (positive) integers $N_1, N_2, \ldots, N_n$

$[N_1, N_2, \ldots, N_n]$: LCM (Least Common Multiple) of the n (positive) integers $N_1, N_2, \ldots, N_n$

$(m, n) = 1$: The integers m and n are relatively prime

T(n, r) : The Smarandache LCM ratio function of degree r

SL(n, r) : The Smarandache LCM ratio function of the second type

SRRPS(n) : The Smarandache repeatable reciprocal function of unity with n arguments

$F_{RP}(n)$: The order of the set SRRPS(n)

SDRPS(n) : The Smarandache distinct reciprocal partition of unity with n integers

$f_{DP}(n)$: The order of the set SDRPS(n)

$\varnothing$: The empty set

$\binom{n}{k}$: The binomial coefficient $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$; $0 \le k \le n$

Chapter 0 Introduction

Eight Smarandache sequences were considered in Majumdar(*). They are : (1) the Smarandache odd sequence, (2) the Smarandache even sequence, (3) the Smarandache circular sequence, (4) the Smarandache square product sequences, (5) the Smarandache permutation sequence, (6) the Smarandache reverse sequence, (7) the Smarandache symmetric sequence, and (8) the Smarandache prime product sequence. These sequences share the common characteristic that they are all recurrence type, that is, in each case, the n–th term can be expressed in terms of one or more of the preceding terms. In case of the Smarandache odd, even, circular and symmetric sequences, we showed that none of these sequences satisfies the recurrence relationship for Fibonacci or Lucas numbers. We proved further that none of the Smarandache prime product and reverse sequences contains Fibonacci or Lucas numbers (in a consecutive row of three or more). In Chapter 1, we show the general result that the recurrence type sequences do not satisfy the recurrence relations of the Fibonacci or Lucas numbers.

We recall that the sequence of *Fibonacci numbers*, $\{F(n)\}_{n=1}^{\infty}$, and the sequence of *Lucas numbers* $\{L(n)\}_{n=1}^{\infty}$, are defined through the following recurrence relations :

$$F(1) = 1, \quad F(2) = 1; \quad F(n+2) = F(n+1) + F(n), \quad n \geq 1, \tag{0.1}$$

$$L(1) = 1, \quad L(2) = 3; \quad L(n+2) = L(n+1) + L(n), \quad n \geq 1. \tag{0.2}$$

From the recurrence relation (0.1), we see that F(n) is increasing in $n \geq 1$; in fact, F(n) is strictly increasing in $n \geq 2$, since

$$F(n+1) - F(n) = F(n-1) > 0 \text{ for all } n \geq 2.$$

Moreover, we have the following result, which shows that F(n) is strictly convex in the sense of the inequality.

Lemma 0.1 : For $n \geq 1$,

$$F(n+2) - F(n+1) > F(n+1) - F(n).$$

Proof : Since for $n \geq 2$,

$$F(n+2) - F(n+1) = F(n) > F(n-1) = F(n+1) - F(n),$$

the result follows. ■

In a similar manner, from (0.2), we see that L(n) is strictly increasing in $n \geq 1$, with

$$L(n+2) - L(n+1) > L(n+1) - L(n) \text{ for all } n \geq 1.$$

Lemmas 0.2 – 0.4 give some properties satisfied by the terms of $\{F(n)\}_{n=1}^{\infty}$ and $\{L(n)\}_{n=1}^{\infty}$.

Lemma 0.2 : In the sequence of Fibonacci numbers, $\{F(n)\}_{n=1}^{\infty}$, the terms $F(3n-2)$ and $F(3n-1)$ are odd, and the terms $F(3n)$ are even, for all $n \geq 1$.

Proof : is by induction on n. From (0.1), we see that the result is true for $n = 1$. So, we assume that the result is true for some integer n. Now, since

$$F(3n+1) = F(3n) + F(3n-1),$$

$$F(3n+2) = F(3n+1) + F(3n),$$

$$F(3(n+1)) = F(3n+2) + F(3n+1).$$

it follows that the result is true for $n+1$ as well, completing induction. ■

Lemma 0.3 : In the sequence of Lucas numbers, $\{L(n)\}_{n=1}^{\infty}$, the terms $L(3n-2)$ and $L(3n-1)$ are odd, and the terms $L(3n)$ are even, for all $n \geq 1$.

***Proof* :** is by induction on n, similar to that of Lemma 0.2, and is omitted here. ■

***Lemma 0.4* :** For all $n \geq 1$,

(1) 3 divides $F(4n)$,

(2) 5 divides $F(5n)$,

(3) 4 divides $F(6n)$.

***Proof* :** Since

$$F(4) = 3,\ F(5) = 5,\ F(6) = 8,$$

we see that the result is true for $n = 1$. To proceed by induction on n, we assume that the result is true for some integer n. Now, since

$$\begin{aligned} F(4(n+1)) &= F(4n+3) + F(4n+2) \\ &= [F(4n+2) + F(4n+1)] + F(4n+2) \\ &= 2\,F(4n+2) + F(4n+1) \\ &= 2[F(4n+1) + F(4n)] + F(4n+1) \\ &= 3\,F(4n+1) + 2\,F(4n), \end{aligned}$$

$$\begin{aligned} F(5(n+1)) &= F(5n+4) + F(5n+3) \\ &= [F(5n+3) + F(5n+2)] + F(5n+3) \\ &= 2\,F(5n+3) + F(5n+2) \\ &= 2[F(5n+2) + F(5n+1)] + F(5n+2) \\ &= 3[F(5n+1) + F(5n)] + 2\,F(5n+1) \\ &= 5\,F(5n+1) + 3\,F(5n), \end{aligned}$$

$$\begin{aligned} F(6(n+1)) &= F(6n+5) + F(6n+4) \\ &= [F(6n+4) + F(6n+3)] + F(6n+4) \\ &= 2\,F(6n+4) + F(6n+3) \\ &= 2[F(6n+3) + F(6n+2)] + F(6n+3) \\ &= 3\,F(6n+3) + 2\,F(6n+2) \\ &= 3[\,F(6n+2) + F(6n+1)] + 2\,F(6n+2) \\ &= 5\,F(6n+2) + 3\,F(6n+1)] \\ &= 5[\,F(6n+1) + F(6n)] + 3\,F(6n+1) \\ &= 8\,F(6n+1) + 5\,F(6n), \end{aligned}$$

we see that the result is true for $n + 1$ as well, thereby completing induction. ■

Let $\{a_n\}_{n=1}^{\infty}$ be the sequence such that

$$a_1 = A,\ a_{n+1} > 10\,a_n \text{ for all } n \geq 1\ (A > 0). \tag{0.3}$$

***Lemma 0.5* :** For the sequence $\{a_n\}_{n=1}^{\infty}$ (defined in (0.3)),

(1) $a_{n+1} > a_n,\ n \geq 1$,

(2) $a_{n+2} - a_{n+1} > a_{n+1} - a_n,\ n \geq 1$,

(3) $a_{n+2} > a_{n+1} + a_n,\ n \geq 1$.

***Proof* :** Clearly, all the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are positive.
Part (1) of the lemma follows from the fact that

$$a_{n+1} - a_n > 9\,a_n > 0 \text{ for all } n \geq 1.$$

Since

$$a_{n+2} - a_{n+1} > 9a_{n+1} > a_{n+1} - a_n,$$

part (2) of the lemma follows.

Part (3) follows by virtue of the following chain of inequalities :

$$a_{n+2} - a_n > (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n) > 9(a_{n+1} + a_n) > a_{n+1}.$$

All these complete the proof of the lemma. ■

From the proof of Lemma 0.5, we see that the inequality in part (3) holds for any sequence satisfying the condition (0.3). However, it should be kept in mind that the inequality in part (3) of Lemma 0.5 may hold true even for a sequence which does not satisfy the condition (0.3).

***Lemma 0.6* :** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ (defined in (0.3)) with $A \geq 1$. Then,

(1) $a_{n+1} > F(n+1)$ for all $n \geq 1$,

(2) $a_{n+1} > L(n+1)$ for all $n \geq 1$.

***Proof* :** First, note that

$$a_1 \geq F(1) = L(1),\ a_2 > 9A > F(2),\ a_2 > L(2).$$

The proof is now by induction on n. So, we assume that the result is true for some n (including all numbers less than n). Since (by virtue of part (3) of Lemma 0.5, together with the recurrence relation (0.1))

$$a_{n+2} - F(n+2) > [a_{n+1} - F(n+1)] + [a_n - F(n)],$$

by the induction hypothesis, it follows that

$$a_{n+2} - F(n+2) > 0,$$

so that the result is true for n + 1 as well.

The proof of part (2) is similar and is left to the reader. ■

The proof of Lemma 0.6 shows that the inequalities therein depend on the inequality given in part (3) of Lemma 0.5. Thus, the result in Lemma 0.6 may be true for a sequence not satisfying the condition in (0.3).

***Lemma 0.7* :** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ (defined in (0.3)).

(1) If $a_n \geq F(n+m)$ for some integers $n \geq 1$ and $m \geq 1$, then

$$a_{n+1} > F(n+m+1),$$

(2) If $a_n \geq L(n+m)$ for some integers $n \geq 1$ and $m \geq 1$, then

$$a_{n+1} > L(n+m+1).$$

***Proof* :** The proofs of part (1) and part (2) are similar, and we prove part (1) only.
To prove part (1), we observe that

$$2F(n+m) \geq F(n+m) + F(n+m-1) = F(n+m+1) \text{ for all } n \geq 1,\ m \geq 1.$$

Now,

$$a_{n+1} > 10a_n \geq 10\,F(n+m) \geq 5\,F(n+m+1) > F(n+m+1),$$

and we get the result desired. ■

Lemma 0.8 **:** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ (defined in (0.3)). Let

$$a_{n+2} = F(n+m+2) \text{ for some integers } n \geq 1 \text{ and } m \geq 1.$$

Then

(1) $a_{n+1} < F(n+m+1)$,

(2) $a_n < F(n+m)$.

Proof **:** From the recurrence relation in (0.1) and part (3) of Lemma 0.5,

$$0 = a_{n+2} - F(n+m+2) > a_{n+1} - F(n+m+1) + a_n - F(n+m).$$

The proof is by contradiction. So, let $a_{n+1} > F(n+m+1)$. But then, by Lemma 0.7,

$$a_{n+2} > F(n+m+2),$$

leading to the contradiction to the assumption. This contradiction establishes part (1) of the lemma. Again, if $a_n \geq F(n+m)$, then $a_{n+1} > F(n+m+1)$, contradicting part (a).
Thus, we get the result desired. ■

The proofs of Lemma 0.7 and Lemma 0.8 show that, in each case, the results mentioned are valid only for sequences satisfying the condition in (0.3). Lemma 0.7 states that, if the n-th term of the sequence $\{a_n\}_{n=1}^{\infty}$ is greater than (or, equal to) the $(n+m)$-th term of the sequence of Fibonacci numbers for some integer m (≥ 1), then the $(n+1)$-st term of the sequence $\{a_n\}_{n=1}^{\infty}$ is greater than the $(n+m+1)$-st term of the sequence of the Fibonacci numbers. Similar result holds for the sequence of Lucas numbers as well. Again, Lemma 0.8 states that, if the $(n+2)$-nd term of the sequence $\{a_n\}_{n=1}^{\infty}$ equals the $(n+m+2)$-nd term of the sequence of Fibonacci numbers (or, Lucas numbers) for some integer m (≥ 1), then the n-th term of the sequence $\{a_n\}_{n=1}^{\infty}$ must be less than the $(n+m)$-th term of the sequence of Fibonacci numbers (or, Lucas numbers), and the $(n+1)$-st term must be less than the $(n+m+1)$-st term of the sequence of Fibonacci (or, Lucas) numbers.

Chapter 1 deals with eight recursive type Smarandache sequences, and it has been shown that none of these sequences satisfies the recurrence relation of the Fibonacci or Lucas numbers.

Chapter 2 focuses on two types of Smarandache geometric determinant sequences, nanely, the bisymmetric geometric determinant sequence, and the cyclic geometric determinant sequence. The n-th terms of these sequences are derived.

The Smarandache function S(n) and the pseudo Smarandache function Z(n), are the subject matters of Chapter 3 and Chapter 4 respectively, where we give some new results.

Chapter 5 derives partial solutions of the Diophantine equations which arise in connection with the Smarandache number related (S-related and Z-related) 60-degree and 120-degree triangles.

The final Chapter 6 gives some miscellaneous topics, such as the Smarandache T-Sequence, the Smarandache friendly numbers, the Smarandache reciprocal partition sets of unity, the Smarandache LCM ratio functions of two types, and the Sandor-Smarandache function.

Some of the materials in Chapter 2, Chapter 3, Chapter 4, Chapter 5 and Chapter 6 are based on our previous papers, published in different journals. In the meantime, some new results have been found, which are also included in this book. Moreover, to keep the book up-to-date, we have included the results of other researchers as well, generally with simpler proofs.

***** Majumdar, A.A.K. *Wandering in the World of Smarandache Numbers*, InProQuest, U.S.A., 2010.

Chapter 1 Some Recursive Smarandache Sequences

In Majumdar[1], the following eight recurrence type Smarandache sequences were treated :

(1) Smarandache odd sequence
(2) Smarandache even sequence
(3) Smarandache circular sequence
(4) Smarandache square product sequences
(5) Smarandache permutation sequence
(6) Smarandache reverse sequence
(7) Smarandache symmetric sequence
(8) Smarandache prime product sequence

It is conjectured that there are no Fibonacci or Lucas numbers in any of the Smarandache odd, even, circular and symmetric sequences. We showed that none of these sequences satisfies the recurrence relationship for Fibonacci or Lucas numbers. We proved further that none of the Smarandache prime product and reverse sequences contains Fibonacci or Lucas numbers (in a consecutive row of three or more).

Definition 1.1 **:** The *Smarandache odd sequence*, denoted by $\{OS(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by repeatedly concatenating the odd positive integers (Ashbacher[2]).

The first few terms of the sequence are

$$1,\ 13,\ 135,\ 1357,\ 13579,\ 1357911,\ 135791113,\ 13579111315,\ \ldots\ .$$

In general, the n–th term of the sequence is given by

$$OS(n) = \overline{135\ \ldots\ (2n-1)},\ n \geq 1.$$

The Smarandache odd sequence satisfies the following inequality.

Lemma 1.1 **:** For any integer $n \geq 1$, $OS(n+1) > 10 \times OS(n)$.

Definition 1.2 **:** The *Smarandache even sequence*, denoted by $\{ES(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by repeatedly concatenating the even positive integers (Ashbacher[2]).

The n–th term of the sequence is given by

$$ES(n) = \overline{24\ \ldots\ (2n)},\ n \geq 1.$$

The first few terms of the sequence are

$$2,\ 24,\ 246,\ 2468,\ 246810,\ 24681012,\ 2468101214,\ \ldots\ .$$

In connection with the Smarandache even sequence, we have the result below.

Lemma 1.2 **:** For any integer $n \geq 1$, $ES(n+1) > 10 \times ES(n)$.

Definition 1.3 : The *Smarandache circular* (also called *consecutive*) *sequence*, denoted by $\{CS(n)\}_{n=1}^{\infty}$, is obtained by repeatedly concatenating the positive integers (Dumitrescu and Seleacu[3]).

The n–th tem of the circular sequence is given

$$CS(n) = \overline{123\ldots(n-1)(n)},\ n \geq 1.$$

The first few terms of the sequence are

1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, … .

For the Smarandache circular sequence, the result below holds true.

Lemma 1.3 **:** For any integer $n \geq 1$, $CS(n + 1) > 10 \times CS(n)$.

Definition 1.4 **:** The *Smarandache square product sequence of the first kind*, denoted by $\{SPS_1(n)\}_{n=1}^{\infty}$, and the *Smarandache square product sequence of the second kind*, denoted by $\{SPS_2(n)\}_{n=1}^{\infty}$, are defined by (Russo[(4)])

$$SPS_1(n) = 1^2 \times 2^2 \times \ \ldots\ \times n^2 + 1 = (n!)^2 + 1,\ n \geq 1,$$
$$SPS_2(n) = 1^2 \times 2^2 \times \ \ldots\ \times n^2 - 1 = (n!)^2 - 1,\ n \geq 1.$$

We then have the following results.

Lemma 1.4 **:** The following relationships hold :

(1) $SPS_1(n + 1) > 10 \times SPS_1(n),\ n \geq 3,$

(2) $SPS_2(n + 1) > 10 \times SPS_2(n),\ n \geq 2.$

Proof **:** Using the definition, we see that

$$SPS_1(n+1) \equiv \left[(n+1)!\right]^2 + 1 > 10\left[(n!)^2 + 1\right] \equiv 10 \times SPS_1(n)$$

if and only if

$$(n!)^2 \left[(n+1)^2 - 10\right] > 9,$$

which is true for $n \geq 3$. This proves part (1) of the lemma.

The proof of part (2) is similar, and is left as an exercise. ■

Definition 1.5 **:** The *Smarandache permutation sequence*, denoted by $\{PS(n)\}_{n=1}^{\infty}$, is defined by (Dumitrescu and Seleacu[(3)])

$$PS(n) = \overline{135\ \ldots\ (2n-1)(2n)(2n-2)\ \ldots\ 42}\ ,\ n \geq 1.$$

The first few terms of the sequence are 12, 1342, 135642, 13578642, 13579108642, … .

In connection with the Smarandache permutation sequence, we define the sequence below.

Definition 1.6 **:** The *reverse even sequence*, and denoted by $\{RES(n)\}_{n=1}^{\infty}$, as follows :

$$RES(n) = \overline{(2n)(2n-2)\ \ldots\ 42},\ n \geq 1.$$

The first few terms of the above sequence are

2, 42, 642, 8642, 108642, 12108642, 1412108642, 161412108642, … .

The terms of the sequence satisfy the following inequality.

Lemma 1.5 **:** $RES(n+2) > RES(n+1) + RES(n)$ for any integer $n \geq 1$.

Note that the sequences $\{PS(n)\}_{n=1}^{\infty}$ and $\{RES(n)\}_{n=1}^{\infty}$ are related through the relationship

$$PS(n) = 10^s \times OS(n) + RES(n) \text{ for some integer } s \geq n.$$

The lemma below gives the inequality satisfied by the Smarandache permutation sequence.

***Lemma 1.6* :** For any integer $n \geq 1$, $PS(n+2) > PS(n+1) + PS(n)$.

***Definition 1.7* :** The *Smarandache reverse sequence*, denoted by $\{RS(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by concatenating the consecutive integers on the left side, starting with $RS(1) = 1$ (Ashbacher[(2)]).

The first few terms of the sequence are

1, 21, 321, 4321, 54321, 654321, 7654321, 87654321, …,

and in general, the n– th term is given by

$$RS(n) = \overline{n(n-1)\ldots 21}, \; n \geq 1.$$

The Smarandache reverse sequence satisfies the inequality below.

***Lemma 1.7* :** For any integer $n \geq 1$, $RS(n+2) > RS(n+1) + RS(n)$.

***Definition 1.8* :** The *Smarandache symmetric sequence*, denoted by $\{SS(n)\}_{n=1}^{\infty}$, is defined by (Ashbacher[(2)])

1, 11, 121, 12321, 1234321, 123454321, 12345654321, … .

More precisely, the n–th term of the Smarandache symmetric sequence is

$$SS(n) = \overline{12 \ldots (n-2)(n-1)(n-2) \ldots 21}, \; n \geq 3;$$

$$SS(1) = 1, \; SS(2) = 11.$$

The Smarandache symmetric sequence can be expressed in terms of the Smarandache circular sequence and the Smarandache reverse sequence as follows : For all $n \geq 3$,

$$SS(n) = 10^s \times CS(n-1) + RS(n-2) \text{ for some integer } s \geq 1.$$

***Lemma 1.8* :** For any integer $n \geq 1$, $SS(n+2) > SS(n+1) + SS(n)$.

***Definition 1.9* :** Let $\{p_n\}_{n=1}^{\infty}$ be the (infinite) sequence of primes in their natural order, that is,

$$p_1 = 2, \; p_2 = 3, \; p_3 = 5, \; p_4 = 7, \; p_5 = 11, \; p_6 = 13, \; \ldots .$$

The *Smarandache prime product sequence*, denoted by $\{PPS(n)\}_{n=1}^{\infty}$, is defined as follows (Smarandache[(5)]) :

$$PPS(n) = p_1 p_2 \ldots p_n + 1, \; n \geq 1.$$

The following lemma gives a relation in connection with the prime product sequence.

***Lemma 1.9* :** $PPS(n+2) > PPS(n+1) + PPS(n)$ for all $n \geq 1$.

We now state and prove the following result.

***Theorem 1.1* :** None of the Smarandache odd sequence, even sequence, circular sequence, square product sequences, permutation sequence, reverse sequence, symmetric sequence, and prime product sequence satisfies the recurrence relation of the Fibonacci (and Lucas) numbers.

***Proof* :** From Lemma 1.1, Lemma 1.2, Lemma 1.3 and Lemma 1.4, we see that the Smarandache odd sequence, even sequence, circular sequence and the square product sequences of two types each satisfies the condition (0.3). Thus, from Lemma 0.5, none of these four sequences satisfies the recurrence relations (0.1) and (0.2). For each of the Smarandache permutation sequence, reverse sequence, symmetric sequence and prime product sequence, the result follows from Lemma 1.6, Lemma 1.7, Lemma 1.8 and Lemma 1.9 respectively. ■

This chapter gives the common properties satisfied by eight Smarandache recurrence type sequences. We see that, the Smarandache odd sequence, the even sequence, the circular sequence and the square product sequences of two types, each satisfies the inequality (0.3), while each of the Smarandache permutation sequence, the reverse sequence, the symmetric sequence and the prime product sequence, satisfies the inequality below :

$$a_{n+2} > a_{n+1} + a_n.$$

The consequence of this fact is given in Theorem 1.1.

In case of the Smarandache odd sequence, even sequence, prime product sequence, square product sequences of two types, permutation sequence, circular sequence and the reverse sequence, we have shown that none of these sequences satisfy the recurrence relation of the Fibonacci or Lucas numbers. This shows that, none of these sequences can contain three or more consecutive Fibonacci numbers or Lucas numbers in a row. The same result follows very simply from Lemma 0.2, which shows that the sequence of Fibonacci numbers F(n) (and Lucas numbers L(n)) are of the form

O, O, E, O, O, E, O, O, E, …,

(where the letter O stands for **O**dd number, and E denotes **E**ven number). Thus, for example, all the terms of the Smarandache odd sequence are odd, and hence, this sequence cannot contain three (or, more) consecutive Fibonacci (or, Lucas) numbers in a row. Since all the terms are odd for each of the Smarandache square product sequences, reverse sequence, symmetric sequence and the prime product sequence, it follows that in each of these cases, there cannot be three (or, more) consecutive Fibonacci (or, Lucas) numbers in a row for any of these sequences. Again, all the terms of the Smarandache sequence are even, and by the same reasoning, this sequence cannot contain three (or, more) consecutive Fibonacci (or, Lucas) numbers in a row. The same argument holds for the Smarandache permutation sequence as well, whose all the terms are even. Finally, the terms of the Smarandache circular sequence are alternately odd and even, and so, this sequence also cannot contain three (or, more) consecutive Fibonacci (or, Lucas) numbers in a row.

It should be mentioned here that our theorems don't rule out the possibility of appearance of Fibonacci or Lucas numbers, scattered here and there, in any of these sequences.

We conclude this chapter with the following conjecture.

***Conjecture 1.1* :** For all $n \geq 1$, $PS(n+1) > 100 \times PS(n)$, $RS(n+1) > 10 \times RS(n)$.

References

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Chapter 2 Smarandache Determinant Sequences

Murthy[1] introduced the Smarandache cyclic arithmetic determinant sequence (SCADS) and the Smarandache bisymmetric arithmetic determinant sequence (SBADS), defined as follows.

Definition 2.1 **:** *The Smarandache cyclic arithmetic determinant sequence*, denoted by $\{SCADS(n)\}_{n=1}^{\infty}$, is

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d & a+3d \\ a+d & a+2d & a+3d & a \\ a+2d & a+3d & a & a+d \\ a+3d & a & a+d & a+2d \end{vmatrix}, \dots \right\},$$

and the *Smarandache bisymmetric arithmetic determinant sequence* $\{SBADS(n)\}_{n=1}^{\infty}$ is

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \dots \right\},$$

where a and d are any real numbers.

The first few terms of the Smarandache cyclic arithmetic seterminant sequence are

$$a, -(2ad+d^2), -9(a+d)d^2, \dots,$$

and the first few terms of the Smarandache bisymmetric arithmetic determinant sequence are

$$a, -(2ad+d^2), -4(a+d)d^2, \dots,$$

The n-th terms of these sequences, derived by Majumdar[2] and independently by Maohua[3] (in a slightly different form), are reproduced in the following theorem.

Theorem 2.1 **:** For any integer $n \geq 1$,

(1) $SCAD(n) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\left(a + \frac{n-1}{2}d\right)(nd)^{n-1}$,

(2) $SBADS(n) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\left(a + \frac{n-1}{2}d\right)(2d)^{n-1}$.

In the particular case with a = 1, d = 1, we have respectively the *Smarandache cyclic determinant natural sequence* and the *Smarandache bisymmetric determinant natural sequence*.

Bueno[4] extended the concept of the Smarandache cyclic arithmetic determinant sequence to the Smarandache cyclic geometric sequences of right circulant form, and derived formulas for the nth terms of the sequences. Later, Bueno[5] extended the concept of the Smarandache bisymmetric arithmetic determinant sequence to the Smarandache bisymmetric geometric determinant sequence. Section 2.1 treats the Smarandache bisymmetric geometric determinant sequence, followed by the Smarandache circulant geometric determinant sequence in Section 2.2. Section 2.3 introduces the Smarandache circulant arithmetic determinant sequence. In each case, the n-th term of the sequence is found.

2.1 Smarandache Bisymmetric Geometric Determinant Sequence

In this section, we consider the Smarandache bisymmetric geometric determinant sequence.

We start with the following definition, which differs slightly from the definition given in Bueno[5].

Definition 2.1.1 **:** The *Smarandache bisymmetric geometric determinant sequence*, denoted by $\{SBGDS(n)\}_{n=1}^{\infty}$, is

$$\left\{ \begin{vmatrix} 1 \end{vmatrix}, \begin{vmatrix} 1 & r \\ r & 1 \end{vmatrix}, \begin{vmatrix} 1 & r & r^2 \\ r & r^2 & r \\ r^2 & r & 1 \end{vmatrix}, \ldots \right\}.$$

The first few terms of the Smarandache bisymmetric geometric determinant sequence are

$$1,\ 1-r^2,\ -r^2(r^2-1)^2,\ r^6(r^2-1)^3,\ r^{12}(r^2-1)^4,\ -r^{20}(r^2-1)^5,\ \ldots\ .$$

The n-th term of the sequence is given in Theorem 2.1.1. To prove the theorem, we need the following results.

Lemma 2.1.1 **:** Let $D \equiv \left| d_{ij} \right|$ $(1 \le i, j \le n)$ be the square determinant of order n (≥ 2) with

$$d_{ij} = \begin{cases} 1, & \text{if } j = n - i + 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$D \equiv \begin{vmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}.$$

Proof **:** First, let n be odd, say,

$$n = 2m + 1 \text{ for some integer } m \ge 1.$$

Then, using the m column operations $C_j \leftrightarrow C_{2m-j+2}$ $(1 \le j \le m)$, that is, interchanging the j-th column C_j and the $(2m-j+2)$-nd column C_{2m-j+2}, we get

$$D \equiv (-1)^m \begin{vmatrix} 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = (-1)^m .$$

Next, let $n = 2m$ $(m \ge 1)$. Then, using the column operations $C_j \leftrightarrow C_{2m-j+1}$ $(1 \le j \le m)$, we get

$$D = (-1)^m.$$

Since, in either case, $m = \left\lfloor \frac{n}{2} \right\rfloor$, the lemma is established. ■

***Lemma 2.1.2* :** Let $D \equiv \left| d_{ij} \right|$ be the determinant of order $2m+2$ $(m \geq 1)$ with

$$d_{ij} = \begin{cases} 1, & \text{if } j = 2m - i + 3 \\ r, & \text{if } i = m+1, j = m+2, \text{ or } i = m+2, j = m+1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$D = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & r & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & r & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = (-1)^{m+1} (r^2 - 1).$$

***Proof* :** Using the column operations $C_j \leftrightarrow C_{2m-j+1}$ $(1 \leq j \leq m)$, we get

$$D = (-1)^{m+1} \left| \begin{array}{c:c:c} I_m & 0_{m\times 2} & 0_{m\times m} \\ \hdashline 0_{2\times m} & \begin{matrix} r & 1 \\ 1 & r \end{matrix} & 0_{2\times m} \\ \hdashline 0_{m\times m} & 0_{m\times 2} & I_m \end{array} \right|$$

where I_m is the identity matrix of order m, $\mathbf{0}_{m\times 2}$ is the zero matrix of order $m\times 2$, $\mathbf{0}_{2\times m}$ is the zero matrix of order $2\times m$, and $\mathbf{0}_{m\times m}$ is the zero matrix of order $m\times m$. Therefore,

$$D = (-1)^{m+1} \left| I_m \right| \left| \begin{matrix} r & 1 \\ 1 & r \end{matrix} \right| \left| I_m \right| = (-1)^{m+1} (r^2 - 1). \blacksquare$$

***Theorem 2.1.1* :** The n^{th} term, SBGDS(n), of the Smarandache bisymmetric geometric determinant sequence is given by

$$\text{SBGDS}(n) = \begin{vmatrix} 1 & r & r^2 & \cdots & r^{n-1} & r^{n-2} & r^{n-1} \\ r & r^2 & r^3 & \cdots & r^{n-2} & r^{n-1} & r^{n-2} \\ r^2 & r^3 & r^4 & \cdots & r^{n-1} & r^{n-2} & r^{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ r^{n-3} & r^{n-2} & r^{n-1} & \cdots & r^4 & r^3 & r^2 \\ r^{n-2} & r^{n-1} & r^{n-2} & \cdots & r^3 & r^2 & r \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r^2 & r & 1 \end{vmatrix}$$

$$= (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} r^{(n-1)(n-2)} (1 - r^2)^{n-1}.$$

***Proof* :** Let

$$\mathrm{SBGDS}(n) = \left| a_{ij} \right|,$$

where the matrix (a_{ij}) is bisymmetric in the sense that

$$a_{ij} = a_{ji} \text{ for all } 1 \le i, j \le n,$$

$$a_{ij} = a_{n-i+1,jn-j+1} \text{ for all } 1 \le i, j \le \left\lfloor \frac{n}{2} \right\rfloor.$$

Note that

$$a_{1j} = a_{n,n-j+1} = r^{j-1},\ a_{n-j+1,j} = r^{n-1};\ 1 \le j \le n,$$

and for $2 \le i \le n-1$,

$$a_{ij} = \begin{cases} r^{i+j-2}, & \text{if } 1 \le j \le n-i+1 \\ r^{2n-i-j}, & \text{if } n-i+2 \le j \le n \end{cases}$$

We consider separately the following two possible cases :

Case 1. When n is odd, say, $n = 2m+1$ (for some integer $m \ge 1$).
In this case,

$$\mathrm{SBGDS}(2m+1)$$

$$= \begin{array}{|ccccccccccccc|l}
1 & r & r^2 & \ldots & r^{m-1} & r^m & r^{m+1} & \ldots & r^{2m-2} & r^{2m-1} & r^{2m} & & \\
r & r^2 & r^3 & \ldots & r^m & r^{m+1} & r^{m+2} & \ldots & r^{2m-1} & r^{2m} & r^{2m-1} & & \\
r^2 & r^3 & r^4 & \ldots & r^{m+1} & r^{m+2} & r^{m+3} & \ldots & r^{2m} & r^{2m-1} & r^{2m-2} & & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & & \\
r^{m-1} & r^m & r^{m+1} & \ldots & r^{2m-2} & r^{2m-1} & r^{2m} & \ldots & r^{m+3} & r^{m+2} & r^{m+1} & & \\
r^m & r^{m+1} & r^{m+2} & \ldots & r^{2m-1} & r^{2m} & r^{2m-1} & \ldots & r^{m+2} & r^{m+1} & r^m & & \longleftarrow i = m+1 \\
r^{m+1} & r^{m+2} & r^{m+3} & \ldots & r^{2m} & r^{2m-1} & r^{2m-2} & \ldots & r^{m+1} & r^m & r^{m-1} & & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & & \\
r^{2m-2} & r^{2m-1} & r^{2m} & \ldots & r^{m+3} & r^{m+2} & r^{m+1} & \ldots & r^4 & r^3 & r^2 & & \\
r^{2m-1} & r^{2m} & r^{2m-1} & \ldots & r^{m+2} & r^{m+1} & r^m & \ldots & r^3 & r^2 & r & & \\
r^{2m} & r^{2m-1} & r^{2m-2} & \ldots & r^{m+1} & r^m & r^{m-1} & \ldots & r^2 & r & 1 & & \\
\multicolumn{1}{c}{} & & & & & \uparrow & & & & & & \multicolumn{1}{c}{} & \\
\multicolumn{1}{c}{} & & & & & j = m+1 & & & & & & \multicolumn{1}{c}{} &
\end{array}$$

$$= \left| \alpha_{ij} \right|, \text{ say,}$$

where for $1 \le i \le 2m+1$,

$$\alpha_{ij} = \begin{cases} r^{i+j-2}, & \text{if } 1 \le j \le 2m-i+2 \\ r^{4m-i-j+2}, & \text{if } 2m-i+3 \le j \le 2m+1 \end{cases}$$

Note that

$$\alpha_{ii} = \begin{cases} r^{2(i-1)}, & \text{if } 1 \le i \le m+1 \\ r^{2(2m-i+1)}, & \text{if } m+1 \le i \le 2m+1 \end{cases}$$

$$\alpha_{ij} = \alpha_{2m-i+2,2m-j+2} \text{ for all } 1 \le i, j \le m;\ \alpha_{2m-j+2,j} = r^{2m} \text{ for all } 1 \le j \le 2m+1.$$

Then, the common factor in the column C_j $(1 \le j \le 2m+1)$ is

$$\begin{cases} r^{j-1}, & \text{if } 1 \le j \le m+1 \\ r^{2m-j+1}, & \text{if } m+2 \le j \le 2m+1 \end{cases}$$

We now take out the common factor r from the 2^{nd} and the $(2m)^{th}$ columns, the common factor r^2 from the 3^{rd} and the $(2m-1)^{st}$ columns, …, the common factor r^{m-1} from the m^{th} and $(m+2)^{nd}$ columns, and the common factor r^m from the $(m+1)^{st}$ column, so that the total common factor is

$$r^{2(1+2+...+m-1)+m} = r^{m^2}.$$

Therefore,

$$\text{SBGDS}(2m+1)$$

$$= r^{m^2} \begin{array}{cc} \left|\begin{array}{cccccccccc} 1 & 1 & 1 & \ldots & 1 & 1 & r^2 & \ldots & r^{2m-4} & r^{2m-2} & r^{2m} \\ r & r & r & \ldots & r & r & r^3 & \ldots & r^{2m-3} & r^{2m-1} & r^{2m-1} \\ r^2 & r^2 & r^2 & \ldots & r^2 & r^2 & r^4 & \ldots & r^{2m-2} & r^{2m-2} & r^{2m-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ r^{m-1} & r^{m-1} & r^{m-1} & \ldots & r^{m-1} & r^{m-1} & r^{m+1} & \ldots & r^{m+1} & r^{m+1} & r^{m+1} \\ r^m & r^m & r^m & \ldots & r^m & r^m & r^m & \ldots & r^m & r^m & r^m \\ r^{m+1} & r^{m+1} & r^{m+1} & \ldots & r^{m+1} & r^{m-1} & r^{m-1} & \ldots & r^{m-1} & r^{m-1} & r^{m-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ r^{2m-2} & r^{2m-2} & r^{2m-2} & \ldots & r^4 & r^2 & r^2 & \ldots & r^2 & r^2 & r^2 \\ r^{2m-1} & r^{2m-1} & r^{2m-3} & \ldots & r^3 & r & r & \ldots & r & r & r \\ r^{2m} & r^{2m-2} & r^{2m-4} & \ldots & r^2 & 1 & 1 & \ldots & 1 & 1 & 1 \end{array}\right| & \begin{array}{c} \\ \\ \\ \\ \\ \leftarrow i = m+1 \\ \\ \\ \\ \\ \\ \end{array} \\ \begin{array}{ccccccccccc} & & & & & \uparrow & & & & & \\ & & & & & j = m+1 & & & & & \end{array} & \end{array}$$

Let the above determinant be denoted by $|\beta_{ij}|$. Then,

$$\beta_{ij} = \beta_{2m-i+2,2m-j+2} \text{ for all } 1 \le i, j \le m,$$

for $1 \le i \le m+1$,

$$\beta_{ij} = r^{i-1}, \text{ if } 1 \le j \le m+1,$$

and for $m+2 \le i \le 2m+1$,

$$\beta_{ij} = \begin{cases} r^{i-1}, & \text{if } 1 \le j \le 2m-i+2 \\ r^{2(2m-i+1)}, & \text{if } 2m-i+3 \le j \le m+1 \end{cases}$$

Note that

$$\beta_{m+3,j} = \begin{cases} r^{m+2}, & \text{if } 1 \le j \le m-1 \\ r^m, & \text{if } j = m \\ r^{m-2}, & \text{if } m+1 \le j \le 2m+1 \end{cases}$$

Therefore, the common factor in the row R_i $(1 \le i \le 2m+1)$ is

$$\begin{cases} r^{i-1}, & \text{if } 1 \le i \le m+1 \\ r^{2m-i+1}, & \text{if } m+2 \le i \le 2m+1 \end{cases}$$

Now, taking out the common factor r from the 2^{nd} and $(2m)^{th}$ rows, r^2 from the 3^{rd} and $(2m-1)^{st}$ rows, …, r^{m-1} from the m^{th} and $(m+2)^{nd}$ rows, and r^m from the $(m+1)^{st}$ row (so that, as before, the common factors total to

$$r^{2(1+2+...+m-1)+m} = r^{m^2}).$$

we get

$$\text{SBGDS}(2m+1)$$

$$= r^{2m^2}\begin{array}{|ccccccccccc|l}
1 & 1 & 1 & \ldots & 1 & 1 & r^2 & \ldots & r^{2m-4} & r^{2m-2} & r^{2m} & \\
1 & 1 & 1 & \ldots & 1 & 1 & r^2 & \ldots & r^{2m-4} & r^{2m-2} & r^{2m-2} & \\
1 & 1 & 1 & \ldots & 1 & 1 & r^2 & \ldots & r^{2m-4} & r^{2m-4} & r^{2m-4} & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
1 & 1 & 1 & \ldots & 1 & 1 & r^2 & \ldots & r^2 & r^2 & r^2 & \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \longleftarrow i=m+1 \\
r^2 & r^2 & r^2 & \ldots & r^2 & 1 & 1 & \ldots & 1 & 1 & 1 & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
r^{2m-4} & r^{2m-4} & r^{2m-4} & \ldots & r^2 & 1 & 1 & \ldots & 1 & 1 & 1 & \\
r^{2m-2} & r^{2m-2} & r^{2m-4} & \ldots & r^2 & 1 & 1 & \ldots & 1 & 1 & 1 & \\
r^{2m} & r^{2m-2} & r^{2m-4} & \ldots & r^2 & 1 & 1 & \ldots & 1 & 1 & 1 & \\
\multicolumn{1}{c}{} & & & & & \uparrow & & & & & \multicolumn{1}{c}{} & \\
\multicolumn{1}{c}{} & & & & & j=m+1 & & & & & \multicolumn{1}{c}{} &
\end{array}$$

Denoting by $\left|\gamma_{ij}\right|$ the above determinant, we have

$$\gamma_{ij} = \gamma_{2m-i+2,2m-j+2} \text{ for all } 1 \le i, j \le m,$$

$$\gamma_{ij} = \begin{cases} 1, & \text{if } \; 1 \le i \le m+1,\ 1 \le j \le m+1 \\ r^{2(i-m-1)}, & \text{if } \; m+2 \le i \le 2m+1,\ 1 \le j \le 2m-i+2 \\ r^{2(m-j+1)}, & \text{if } \; m+2 \le i \le 2m+1,\ 2m-i+3 \le j \le m+1 \end{cases}$$

We now perform the m row operations $R_i \to R_i - R_{i+1}\ (1 \le i \le m)$ (the row R_{i+1} is subtracted from the row R_i), and then the m row operations $R_{m+i+1} \to R_{m+i+1} - R_{m+i}\ (1 \le i \le m)$, to get

$$\text{SBGDS}(2m+1)$$

$$= r^{2m^2}\begin{array}{r|ccccccccc|}
 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & r^{2m-2}(r^2-1) \\
 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & r^{2m-4}(r^2-1) & r^{2m-4}(r^2-1) \\
 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & r^{2m-6}(r^2-1) & r^{2m-6}(r^2-1) \\
 & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
 & 0 & 0 & \ldots & 0 & 0 & r^2-1 & \ldots & r^2-1 & r^2-1 \\
i=m+1 \longrightarrow & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 \\
 & r^2-1 & r^2-1 & \ldots & r^2-1 & 0 & 0 & \ldots & 0 & 0 \\
 & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
 & r^{2m-6}(r^2-1) & r^{2m-6}(r^2-1) & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
 & r^{2m-4}(r^2-1) & r^{2m-4}(r^2-1) & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
 & r^{2m-2}(r^2-1) & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
 & \multicolumn{1}{c}{} & & & & \uparrow & & & & \multicolumn{1}{c}{} \\
 & \multicolumn{1}{c}{} & & & & j=m+1 & & & & \multicolumn{1}{c}{}
\end{array}$$

In the above determinant, we have the common factors $r^{2m-2}(r^2-1)$ in the 1st and the $(2m+1)$st rows, $r^{2m-4}(r^2-1)$ in the 2nd and the $(2m)$th rows, …, and (r^2-1) in the mth and the $(m+2)$nd rows, so that, the total common factor is

$$r^{2[(2m-2)+(2m-4)+\ldots+2]} = r^{2m(m-1)}.$$

Taking out the common factors out, we get,

$$\mathrm{SBGDS}(2m+1)$$

$$= r^{2m^2+2m(m-1)}(r^2-1)^{2m}\begin{array}{|ccccccccccc|l}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \leftarrow i=m+1 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\
1 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
\multicolumn{5}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{5}{c}{} & \\
\multicolumn{5}{c}{} & \multicolumn{1}{c}{j=m+1} & \multicolumn{5}{c}{} &
\end{array}$$

Let the above determinant be denoted by $|\delta_{ij}|$. Then,

$\delta_{ij} = \delta_{2m-i+2,2m-j+2}$ for all $1 \le i, j \le m$,

$\delta_{ij} = 0$ if $1 \le i \le m$, $1 \le j \le m+1$,

$\delta_{m+1,j} = 1$ for all $1 \le j \le 2m+1$.

Finally, we perform the m column operations $C_j \to C_j - C_{j+1}\ (1 \le j \le m)$, followed by the m column operations $C_{m+j+1} \to C_{m+j+1} - C_{m+j}\ (1 \le j \le m)$, to get

$$\mathrm{SBGDS}(2m+1)$$

$$= r^{2m^2+2m(m-1)}(r^2-1)^{2m}\begin{array}{|ccccccccccc|l}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \leftarrow i=m+1 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
\multicolumn{5}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{5}{c}{} & \\
\multicolumn{5}{c}{} & \multicolumn{1}{c}{j=m+1} & \multicolumn{5}{c}{} &
\end{array}$$

Now, appealing to Lemma 2.1.1, we get

$$\mathrm{SBGDS}(2m+1) = (-1)^m r^{2m(2m-1)}(r^2-1)^{2m}.$$

Case 2. When n is even, say, $n = 2m + 2$ (for some integer $m \geq 1$).
In this case,

$$\text{SBGDS}(2m+2)$$

$$= \begin{array}{|cccccccccc|l}
1 & r & r^2 & \ldots & r^m & r^{m+1} & \ldots & r^{2m-1} & r^{2m} & r^{2m+1} & \\
r & r^2 & r^3 & \ldots & r^{m+1} & r^{m+2} & \ldots & r^{2m} & r^{2m+1} & r^{2m} & \\
r^2 & r^3 & r^4 & \ldots & r^{m+2} & r^{m+3} & \ldots & r^{2m+1} & r^{2m} & r^{2m-1} & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
r^m & r^{m+1} & r^{m+2} & \ldots & r^{2m} & r^{2m+1} & \ldots & r^{m+3} & r^{m+2} & r^{m+1} & \longleftarrow i = m+1 \\
r^{m+1} & r^{m+2} & r^{m+3} & \ldots & r^{2m+1} & r^{2m} & \ldots & r^{m+2} & r^{m+1} & r^m & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
r^{2m-1} & r^{2m} & r^{2m+1} & \ldots & r^{m+3} & r^{m+2} & \ldots & r^4 & r^3 & r^2 & \\
r^{2m} & r^{2m+1} & r^{2m} & \ldots & r^{m+2} & r^{m+1} & \ldots & r^3 & r^2 & r & \\
r^{2m+1} & r^{2m} & r^{2m-1} & \ldots & r^{m+1} & r^m & \ldots & r^2 & r & 1 & \\
\multicolumn{1}{c}{} & & & & \uparrow & & & & & \multicolumn{1}{c}{} & \\
\multicolumn{1}{c}{} & & & & j = m+1 & & & & & \multicolumn{1}{c}{} &
\end{array}$$

Denoting by $\left| x_{ij} \right|$ the above determinant, we have

$$x_{ij} = x_{ji} \text{ for all } 1 \leq i, j \leq 2m+2;\ x_{ij} = x_{2m-i+3,2m-j+3},\ 1 \leq i, j \leq m+1.$$

Taking out the common factor r from the 2nd and $(2m+1)^{\text{st}}$ columns, r^2 from the 3rd and $(2m)^{\text{th}}$ columns, …, r^m from the $(m+1)^{\text{st}}$ and $(m+2)^{\text{nd}}$ columns (so that the total common factor is

$$r^{2(1+2+\ldots+m)} = r^{m(m+1)}),$$

we get

$$\text{SBGDS}(2m+2)$$

$$= r^{m(m+1)} \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 & r & \ldots & r^{2m-3} & r^{2m-1} & r^{2m+1} \\
r & r & r & \ldots & r & r^2 & \ldots & r^{2m-2} & r^{2m} & r^{2m} \\
r^2 & r^2 & r^2 & \ldots & r^2 & r^3 & \ldots & r^{2m-1} & r^{2m-1} & r^{2m-1} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
r^m & r^m & r^m & \ldots & r^m & r^{m+1} & \ldots & r^{m+1} & r^{m+1} & r^{m+1} \\
r^{m+1} & r^{m+1} & r^{m+1} & \ldots & r^{m+1} & r^m & \ldots & r^m & r^m & r^m \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
r^{2m-1} & r^{2m-1} & r^{2m-1} & \ldots & r^3 & r^2 & \ldots & r^2 & r^2 & r^2 \\
r^{2m} & r^{2m} & r^{2m-2} & \ldots & r^2 & r & \ldots & r & r & r \\
r^{2m+1} & r^{2m-1} & r^{2m-3} & \ldots & r & 1 & \ldots & 1 & 1 & 1
\end{vmatrix}$$

$$= r^{2m(m+1)} \begin{array}{|cccccccccc|l}
1 & 1 & 1 & \ldots & 1 & r & \ldots & r^{2m-3} & r^{2m-1} & r^{2m+1} & \\
1 & 1 & 1 & \ldots & 1 & r & \ldots & r^{2m-3} & r^{2m-1} & r^{2m-1} & \\
1 & 1 & 1 & \ldots & 1 & r & \ldots & r^{2m-3} & r^{2m-3} & r^{2m-3} & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
1 & 1 & 1 & \ldots & 1 & r & \ldots & r & r & r & \longleftarrow i = m+1 \\
r & r & r & \ldots & r & 1 & \ldots & 1 & 1 & 1 & \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
r^{2m-3} & r^{2m-3} & r^{2m-3} & \ldots & r & 1 & \ldots & 1 & 1 & 1 & \\
r^{2m-1} & r^{2m-1} & r^{2m-3} & \ldots & r & 1 & \ldots & 1 & 1 & 1 & \\
r^{2m+1} & r^{2m-1} & r^{2m-3} & \ldots & r & 1 & \ldots & 1 & 1 & 1 &
\end{array}$$

where the last determinant above is obtained from the preceding one after taking out the common factor r from the 2^{nd} and $(2m+1)^{st}$ rows, r^2 from the 3^{rd} and $(2m)^{th}$ rows, …, r^m from the $(m+1)^{st}$ and $(m+2)^{nd}$ rows (so that the total common factor is $r^{2(1+2+\ldots+m)} = r^{m(m+1)}$).

Now, performing in succession, the two row operations, $R_i \to R_i - R_{i+1}\ (1 \le i \le m)$ and $R_{m+i+1} \to R_{m+i+1} - R_{m+i}\ (1 \le i \le m)$, we get

$$\text{SBGDS}(2m+2)$$

$$= r^{2m(m+1)} \begin{array}{r} \\ \\ \\ i = m+1 \longrightarrow \\ \\ \\ \\ \\ \end{array} \begin{vmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & r^{2m-1}(r^2-1) \\ 0 & 0 & 0 & \ldots & 0 & 0 & r^{2m-3}(r^2-1) & r^{2m-3}(r^2-1) \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & r & r & r \\ r & r & r & \ldots & r & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ r^{2m-3}(r^2-1) & r^{2m-3}(r^2-1) & 0 & \ldots & 0 & 0 & 0 & 0 \\ r^{2m-1}(r^2-1) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\uparrow$$
$$j = m+1$$

Denoting by $|y_{ij}|$ the determinant above, we have

$$y_{ij} = y_{2m-i+3,2m-j+3} \text{ for all } 1 \le i, j \le m+1.$$

In the above determinant, we may take out the common factors $r^{2m-1}(r^2-1)$ from the 1^{st} and the $(2m+2)^{nd}$ rows, $r^{2m-3}(r^2-1)$ from the 2^{nd} and the $(2m+1)^{st}$ rows, …, and $r(r^2-1)$ from the m^{th} and $(m+3)^{rd}$ rows, and the total common factor is

$$r^{2[(2m-1)+(2m-3)+\ldots+1]} = r^{2m^2}.$$

Therefore,

$$\text{SBGDS}(2m+2)$$

$$= r^{2m(m+1)}(r^2-1)^{2m} \begin{vmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & r & \ldots & r & r & r \\ r & r & r & \ldots & r & 1 & \ldots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 \end{vmatrix} \begin{array}{l} \\ \\ \\ \\ \longleftarrow i = m+1 \\ \\ \\ \\ \\ \\ \end{array}$$

Letting $|z_{ij}|$ be the determinant above, we have

$$z_{ij} = z_{2m-i+3,2m-j+3} \text{ for all } 1 \le i, j \le m+1.$$

Note that

$$z_{ij} = 0 \text{ if } 1 \le i \le m,\ 1 \le j \le m+1.$$

Now, we perform successively the m column operations $C_j \to C_j - C_{j+1}\ (1 \le j \le m)$ and the m column operations $C_{m+j+2} \to C_{m+j+2} - C_{m+j+1}\ (1 \le j \le m)$, to get

$$\text{SBGDS}(2m+2)$$
$$= r^{2m(m+1)}(r^2-1)^{2m}\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & r & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & r & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \end{vmatrix}$$
$$= (-1)^{m+1} r^{2m(2m+1)}(r^2-1)^{2m+1},$$

where the last result follows from Lemma 2.1.2.

All these complete the proof of the theorem. ■

2.2 Smarandache Circulant Geometric Determinant Sequence

Bueno[4] introduced the concept of the Smarandache right circulant geometric matrix with geometric sequence, defined as follows :

Definition 2.2.1 **:** The *Smarandache right circulant matrix* (*of order n*) *with geometric sequence*, denoted by SRCGM(n), is a matrix of the form

$$\text{SRCGM}(n) = \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \dots & r^{n-3} & r^{n-2} \\ r^{n-2} & r^{n-1} & 1 & \dots & r^{n-4} & r^{n-3} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \dots & 1 & r \\ r & r^2 & r^3 & \dots & r^{n-1} & 1 \end{pmatrix}.$$

Bueno[4] found an explicit form of the determinant, $\det(\text{SRCGM}(n))$. However, using the known results of the circulant matrix, the expression of $\det(\text{SRCGM}(n))$ follows readily, as Lemma 2.2.1 shows. After defining the circulant matrix, the expression of its determinant is given in Lemma 2.2.1. For a proof of the lemma, see Geller, Kra, Popescu and Simanca[8].

Definition 2.2.2 **:** The *circulant matrix* with the vector $C = (c_0, c_1, \dots, c_{n-1})$, denoted by C(n), is the matrix of the form

$$C(n) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{pmatrix}.$$

The determinant of the circulant matrix above is given in the following lemma.

***Lemma 2.2.1* :** For any integer $n \geq 2$,

$$\begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \ldots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \ldots & c_{n-1} & c_0 \end{vmatrix} = \prod_{j=0}^{n-1} (c_0 + c_1\omega_j + c_2\omega_j^2 + \ldots + c_{n-1}\omega_j^{n-1}).$$

where $\omega_j = e^{\frac{2\pi i}{n} j}$ $(0 \leq j \leq n-1)$ are the n n^{th} roots of unity (with $\omega_0 \equiv 1$).

***Lemma 2.2.2* :** For $n \geq 1$,

$$\det\big(SRCGM(n)\big) = (1 - r^n)^{n-1}.$$

***Proof* :** From Lemma 2.2.1 with $c_j = r^j$ $(0 \leq j \leq n-1)$, we see that

$$\det\big(SRCGM(n)\big) = \prod_{j=0}^{n-1} \big(1 + r\omega_j + r^2\omega_j^2 + \ldots + r^{n-1}\omega_j^{n-1}\big).$$

But, for any j with $0 \leq j \leq n-1$,

$$1 + r\omega_j + r^2\omega_j^2 + \ldots + r^{n-1}\omega_j^{n-1} = \frac{1 - (r\omega_j)^n}{1 - r\omega_j} = \frac{1 - r^n}{1 - r\omega_j}. \tag{1}$$

Since

$$x^n - 1 = (x - \omega_0)(x - \omega_1)(x - \omega_2) \ldots (x - \omega_{n-1}),$$

with $x = \frac{1}{r}$, we get

$$\frac{1 - r^n}{r^n} = \frac{(1 - r\omega_0)(1 - r\omega_1)(1 - r\omega_2) \ldots (1 - r\omega_{n-1})}{r^n},$$

so that

$$(1 - r\omega_0)(1 - r\omega_1)(1 - r\omega_2) \ldots (1 - r\omega_{n-1}) = 1 - r^n. \tag{2}$$

The lemma now follows by virtue of (1) and (2). ■

2.3 Smarandache Circulant Arithmetic Determinant Sequence

Generalizing the concept of the Smarandache right circulant matrix with geometric sequence to the case of arithmetic sequence, we have the following definition.

***Definition 2.3.1* :** The *Smarandache right circulant matrix* (*of order n*) *with arithmetic sequence*, denoted by SRCAM(n), is a matrix of the form

$$SRCAM(n) = \begin{pmatrix} a & a+d & a+2d & \ldots & a+(n-2)d & a+(n-1)d \\ a+(n-1)d & a & a+d & \ldots & a+(n-3)d & a+(n-2)d \\ a+(n-2)d & a+(n-1)d & a & \ldots & a+(n-4)d & a+(n-3)d \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ a+2d & a+3d & a+4d & \ldots & a & a+d \\ a+d & a+2d & a+3d & \ldots & a+(n-1)d & a \end{pmatrix}.$$

The determinant of the above matrix is given in the Lemma 2.3.1 below, making use of Lemma 2.2.1.

***Lemma 2.3.1* :** For any integer $n \geq 1$,

$$\det(\text{SRCAM}(n)) = (-1)^{n-1}\left(a + \frac{n-1}{2}d\right)(nd)^{n-1}.$$

***Proof* :** From Lemma 2.2.1 with

$$c_j = a + jd \; (0 \leq j \leq n-1),$$

we have

$$\det(\text{SRCAM}(n)) = \prod_{j=0}^{n-1}\left[a + (a+d)\omega_j + (a+2d)\omega_j^2 + \ldots + \{a+(n-1)d\}\omega_j^{n-1}\right].$$

Now, when $j = 0$,

$$a + (a+d) + \ldots + [a+(n-1)d] = n\left(a + \frac{n-1}{2}d\right),$$

and for j with $1 \leq j \leq n-1$,

$$\begin{aligned} &a + (a+d)\omega_j + (a+2d)\omega_j^2 + \ldots + [a+(n-1)d]\omega_j^{n-1} \\ &= a\left(1 + \omega_j + \omega_j^2 + \ldots + \omega_j^{n-1}\right) + d\left[\omega_j + 2\omega_j^2 + \ldots + (n-1)\omega_j^{n-1}\right]. \end{aligned} \quad (3)$$

But,

$$1 + \omega_j + \omega_j^2 + \ldots + \omega_j^{n-1} = \frac{1-\omega_j^n}{1-\omega_j} = 0,$$

and using the formula

$$x + 2x^2 + \ldots + (n-1)x^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{1+(n-1)x^n}{1-x}; \; n \geq 2,$$

we get, noting that $\omega_j^n = 1$,

$$\omega_j + 2\omega_j^2 + \ldots + (n-1)\omega_j^{n-1} = -\frac{n}{1-\omega_j}.$$

Therefore, from (3), we get

$$a + (a+d)\omega_j + (a+2d)\omega_j^2 + \ldots + [a+(n-1)d]\omega_j^{n-1} = -\frac{nd}{1-\omega_j}, \quad 1 \leq j \leq n-1.$$

And hence,

$$\det(\text{SRCAM}(n)) = \left[n\left(a + \frac{n-1}{2}d\right)\right]\prod_{j=1}^{n-1}\left(-\frac{nd}{1-\omega_j}\right). \quad (4)$$

Since

$$x^n - 1 = (x-\omega_0)(x-\omega_1)(x-\omega_2)\ldots(x-\omega_{n-1}),$$

differentiating both sides with respect to x and then putting $x = \omega_0 = 1$, we get

$$(1-\omega_1)(1-\omega_2)\ldots(1-\omega_{n-1}) = n. \quad (5)$$

We now get the desired expression from (4) and (5). ■

The first few terms of the Smarandache circulant arithmetic determinant sequence are

$$a,\; -d(2a+d),\; 9(a+d)d^2,\; \ldots,$$

while the first few terms of the Smarandache cyclic arithmetic determinant sequence are

$$a,\; -d(2a+d),\; -9(a+d)d^2,\; \ldots\,.$$

Thus, the Smarandache cyclic arithmetic determinant sequence, defined in Definition 2.1, is a variant of the Smarandache circulant arithmetic determinant sequence, and the determinant SCAD(n) may be obtained from $\det(\text{SRCAM}(n))$ by appropriate interchange of rows.

***Remark 2.3.1* :** In proving Lemma 2.3.1, we have made use of the result that

$$x + 2x^2 + \ldots + (n-1)x^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{1+(n-1)x^n}{1-x};\ n \geq 2.$$

The proof is as follows : Let

$$S \equiv x + 2x^2 + 3x^3 + \ldots + (n-2)x^{n-2} + (n-1)x^{n-1}. \qquad (6)$$

Then,

$$xS = \quad x^2 + 2x^3 + \ldots + (n-3)x^{n-2} + (n-2)x^{n-2} + (n-1)x^n. \qquad (7)$$

Subtracting (7) from (6), term-by-term, we get

$$(1-x)S = x(1 + x + x^2 + \ldots + x^{n-2}) - (n-1)x^n$$

$$= x\frac{1-x^{n-1}}{1-x} - (n-1)x^n$$

$$= \frac{1-x^n}{1-x} - 1 - (n-1)x^n,$$

which gives the desired expression after simplification.

References

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There are several papers, giving the application of the circulant matrices. The interested readers are referred to the following :

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Chapter 3 The Smarandache Function

The arithmetical function, introduced by Florentin Smarandache[1] is defined as follows ($\mathbb{Z}^+$ being the set of positive integers).

Definition 3.1 **:** For any integer $n \geq 1$, the *Smarandache function*, denoted by S(n), is the smallest positive integer m such that $1.2. \ldots .m \equiv m!$ is divisible by n. That is,

$$S(n) = \min\{ m : m \in \mathbb{Z}^+,\ n \mid m! \};\ n \geq 1.$$

Several researchers, including Ashbacher[2, 3], Sandor[4], Stuparu and Sharpe[5] and Farris and Mitchell[6], have studied some of the elementary properties satisfied by S(n). These are summarized below.

Lemma 3.1 **:** For any composite number $n \geq 4$, $S(n) \geq \max\{S(d) : d \mid n\}$.

Lemma 3.2 **:** For any integers $n_1, n_2, \ldots, n_k \geq 1$, $S(n_1 n_2 \ldots n_k) \leq S(n_1) + S(n_2) + \ldots + S(n_k)$.

Lemma 3.3 **:** Let $n_1, n_2, \ldots, n_k \geq 1$ be k integers with $(n_1, n_2, \ldots, n_k) = 1$. Then,

$$S(n_1 n_2 \ldots n_k) = \max\{S(n_1), S(n_2), \ldots, S(n_k)\}.$$

Lemma 3.4 **:** p divides $S(p^k)$ for any prime $p \geq 2$, and any integer $k \geq 1$.

Corollary 3.1 **:** For any prime $p \geq 2$, and any integer $k \geq 1$,

$$S(p^k) = \alpha p \text{ for some integer } \alpha \geq 1.$$

Lemma 3.5 **:** For any integer n and integers k, r with $k \geq r$, $S(n^k) \geq S(n^r)$.

Lemma 3.6 **:** For any integers n and m with $n \geq m$, $S(n^k) \geq S(m^k)$ for any $k \geq 1$.

Lemma 3.7 **:** For any prime $p \geq 2$, $S(p^{\alpha+\beta}) \leq S(p^\alpha) S(p^\beta)$ for any integers $\alpha, \beta \geq 1$.

Lemma 3.8 **:** $S(n^2) \leq n-1$ for any integer $n \neq p, 2p, 8, 9$ (p being a prime).

Theorem 3.1 **:** Let n be represented in terms of its distinct prime factors $p, p_1, p_2, \ldots, p_k$ as

$$n = p^\alpha \ p_1^{\alpha_1} \ p_2^{\alpha_2} \ \ldots, \ p_k^{\alpha_k}$$

Then,

$$S(n) = \max\left\{ S(p^\alpha),\ S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \ldots,\ S(p_k^{\alpha_k}) \right\}.$$

Thus, in order to find

$$S(n) = S(p^\alpha \ p_1^{\alpha_1} \ p_2^{\alpha_2} \ \ldots p_k^{\alpha_k}),$$

it is necessary and sufficient to know the values of

$$S(p^\alpha),\quad S(p_1^{\alpha_1}),\quad S(p_2^{\alpha_2}),\quad \ldots,\quad S(p_k^{\alpha_k}),$$

which are not available till now. However, we have expressions in some particular cases.

Section 3.1 derives explicit forms of $S\left(p^{p^2+kp+k}\right)$, $S\left(p^{2p^2+kp+k}\right)$ and $S\left(p^{kp^2+(k+1)p+k+1}\right)$. Some remarks and conjectures are given in Section 3.2.

3.1 Some Explicit Expressions for S(n)

In this section, we derive some explicit expressions for S(n). First, we mention the following well-known major results, giving the expressions for S(n) in some particular cases, which are available in the current literature.

Lemma 3.1.1 **:** For any prime $p \geq 2$, $S(p) = p$.

Lemma 3.1.2 **:** For any integer $n \geq 1$, $S(n!) = n$.

Lemma 3.1.3 **:** For any prime $p \geq 2$,
(1) $S(p^k) = kp$, if $1 \leq k \leq p$, **(2)** $S(p^{p+1}) = p^2$.

Lemma 3.1.4 **:** For any prime $p \geq 2$,
(1) $S(p^{p+k+1}) = p^2 + kp$, if $1 \leq k \leq p$, **(2)** $S(p^{2(p+1)}) = 2p^2$,
(3) $S\left(2^{p-1}(2^p - 1)\right) = 2^p - 1$ ($2^p - 1$ being the *Mercenne prime*).

Lemma 3.1.5 **:** For any prime $p \geq 2$, $S\left(p^{p^k - k}\right) = (p-1)p^k$ for any integer $k \geq 1$.

Some more expressions of S(n), due to Majumdar[7], are given in the lemmas below.

Lemma 3.1.6 **:** If $S(p^\alpha) = N$ (for some integers $\alpha \geq 1$ and $N \equiv N(p)$, $p \geq 2$ being a prime) such that $p^{\alpha+1}$ does not divide N!, then

$$S(p^{\alpha+1}) = N + p.$$

Proof **:** Since $S(p^\alpha) = N$, it follows that p^α divides N!. Also, by Lemma 3.4, p divides N. Thus, p divides $(N + p)$, but none of $(N + p - 1)$, $(N + p - 2)$ … $(N + 1)$. Therefore, $p^{\alpha+1}$ divides $(N + p)! = N!\,(N + 1)(N + 2) \dots (N + p)$. ■

Corollary 3.1.1 **:** If $S(p^\alpha) = N$ (for some integers $\alpha \geq 1$ and $N \equiv N(p)$, $p \geq 2$ being a prime) such that $p^{\alpha+1}$ does not divide N!, then

$$S(p^{\alpha+\nu}) = N + \nu p \text{ for } \nu = 1, 2, ..., p - 1.$$

Proof **:** follows by repeated application of Lemma 3.1.6. ■

Lemma 3.1.7 **:** Let, for some integers $s \geq 1$, $t \geq 1$ and $a_0, a_1, \dots, a_t$,

$$S(p^\alpha) = p^s(a_0 + a_1 p + \dots + a_t p^t) \equiv P(p),$$

(where $a_0 \neq 0$ is not divisible by the prime $p \geq 2$), such that $S(p^{\alpha-1}) \neq P$. Then,

$$S(p^{\alpha+k}) = P(p) \text{ for } k = 1, 2, \dots, s - 1.$$

Proof **:** Since $S(p^{\alpha-1}) \neq P$, it follows that $S(p^{\alpha-1}) = P - p$. Now,

$$p^{\alpha-1} \,|\, (P - p)! \;\Rightarrow\; p^\alpha \,|\, (P - p)!\,p$$

$$\Rightarrow\; p^\alpha . p^k \,|\, (P - p)!\,p^s \text{ for all } k = 1, 2, \dots, s - 1.$$

This shows that $p^{\alpha+k} \,|\, P!$ for all $k = 1, 2, \dots, s - 1$, which we intended to prove. ■

***Lemma 3.1.8* :** For any prime $p \geq 2$,

$$S(p^{kp}) = kp^2 - (k-1)p \quad \text{for } k = 1, 2, \ldots, p-1.$$

***Proof* :** The proof is by induction on k. By virtue of Lemma 3.1.4, the result is true for $k = 1$. So, we assume that the result is true for some k. Then, by Corollary 3.1.1,

$$S(p^{kp+\nu}) = kp^2 - (k-\nu-1)p, \quad \nu = 1, \ 2, \ \ldots, \ p-1.$$

In particular, we have

$$S(p^{kp+k-2}) = \{kp^2 - (k-1)p\} + (k-2)p = kp^2 - p,$$

$$S(p^{kp+k-1}) = kp^2.$$

Now,

$$p^{kp+k-2} \mid (kp^2 - p)! \;\Rightarrow\; p^{kp+k} \mid (kp^2 - p)!\,p^2 \;\Rightarrow\; p^{kp+k} \mid (kp^2)!,$$

so that

$$S(p^{kp+k}) = kp^2.$$

By Corollary 3.1.1,

$$S(p^{kp+k+\nu}) = kp^2 + \nu p \quad \text{for } \nu = 1, 2, \ldots, p-1.$$

Choosing $\nu = p - k$, we get

$$S(p^{kp+p}) = S\left(p^{(k+1)p}\right) = kp^2 + (p-k)p = (k+1)p^2 - kp,$$

which shows that the result is true for $k + 1$ as well.
This completes induction, thereby establishing the lemma. ■

In course of proving Lemma 3.1.8, we also found the following expressions :

***Corollary 3.1.2* :** For any prime $p \geq 2$,

$$S\left(p^{kp+k-1}\right) = kp^2 = S\left(p^{kp+k}\right) \quad \text{for } k = 1, 2, \ldots, p-1.$$

From Corollary 3.1.2, we get, in the particular case with $k = p - 1$,

$$S\left(p^{p^2-2}\right) = (p-1)p^2 = S\left(p^{p^2-1}\right), \tag{3.1.1}$$

which, together with Corollary 3.1.1, gives

$$S\left(p^{p^2+\nu}\right) = (p-1)p^2 + (\nu+1)p; \quad \nu = 1, \ 2, \ \ldots, \ p-1. \tag{3.1.2}$$

In (3.1.2) above, the case $\nu = p - 1$ follows from the case $\nu = p - 2$ by virtue of Lemma 3.1.6. With $\nu = p - 1$, coupled with Lemma 3.1.7, we have

$$S\left(p^{p^2+p-1}\right) = p^3 = S\left(p^{p^2+p}\right) = S\left(p^{p^2+p+1}\right). \tag{3.1.3}$$

Using Corollary 3.1.1 once more, we get from (3.1.3),

$$S\left(p^{p^2+p+\nu}\right) = p^3 + (\nu-1)p; \quad \nu = 2, \ 3, \ \ldots, \ p. \tag{3.1.4}$$

From (3.1.4) with $\nu = p$, together with Lemma 3.1.7, we have, for $p \geq 2$,

$$S\left(p^{p^2+2p+1}\right) = p^3 + p^2 = S\left(p^{p^2+2p+2}\right). \tag{3.1.5}$$

We now state and prove the general result.

Lemma 3.1.9 **:** For any prime $p \geq 2$,

$$S\left(p^{p^2+kp+k-1}\right) = p^3 + (k-1)p^2 = S\left(p^{p^2+kp+k}\right) \text{ for } k = 2, 3, \ldots, p.$$

Proof **:** is by induction on k. The validity of the result for $k = 2$ follows from (3.1.5). So, we assume that the result is true for some k. Then, by Corollary 3.1.1,

$$S\left(p^{p^2+kp+k+\nu}\right) = p^3 + (k-1)p^2 + \nu p; \quad \nu = 1, \ 2, \ \ldots, \ p-1. \tag{3.1.6}$$

With $\nu = p - 1$ in (3.1.6), we have

$$S\left(p^{p^2+(k+1)p+k-1}\right) = p^3 + kp^2 - p,$$

so that, by Lemma 3.1.6,

$$S\left(p^{p^2+(k+1)p+k}\right) = p^3 + kp^2,$$

and hence, by Lemma 3.1.7,

$$S\left(p^{p^2+(k+1)p+k+1}\right) = p^3 + kp^2.$$

Thus, the result is also true for $k + 1$, completing induction. ■

Lemma 3.1.9 with $k = p$ gives

$$S\left(p^{2p^2+p-1}\right) = 2p^3 - p^2 = S\left(p^{2p^2+p}\right),$$

and so, by Corollary 3.1.1,

$$S\left(p^{2p^2+p+\nu}\right) = 2p^3 - p^2 + \nu p; \quad \nu = 1, \ 2, \ \ldots, \ p-1. \tag{3.1.7}$$

From (3.1.7) with $\nu = p - 1$, we get

$$S\left(p^{2p^2+2p-1}\right) = 2p^3 - p, \tag{3.1.8}$$

which, by Lemma 3.1.6 and Lemma 3.1.7, gives

$$S\left(p^{2p^2+2p}\right) = 2p^3 = S\left(p^{2p^2+2p+1}\right) = S\left(p^{2p^2+2p+2}\right). \tag{3.1.9}$$

From (3.1.9), by virtue of Corollary 3.1.1, we get

$$S\left(p^{2p^2+2p+\nu}\right) = 2p^3 + (\nu-2)p; \quad \nu = 3, \ 4, \ \ldots, \ p+1, \tag{3.1.10}$$

and in particular,

$$S\left(p^{2p^2+3p+1}\right) = 2p^3 + (p-1)p, \tag{3.1.11}$$

so that, by Lemma 3.1.7,

$$S\left(p^{2p^2+3p+2}\right) = 2p^3 + p^2 = S\left(p^{2p^2+3p+3}\right). \qquad (3.1.12)$$

Continuing in this way, we get

$$S\left(p^{2p^2+4p+3}\right) = 2p^3 + 2p^2 = S\left(p^{2p^2+4p+4}\right),$$

and in general, we have the following lemma.

***Lemma 3.1.10* :** For any prime $p \geq 3$,

$$S\left(p^{2p^2+kp+k-1}\right) = 2p^3 + (k-2)p^2 = S\left(p^{2p^2+kp+k}\right) \text{ for } k = 3, 4, \ldots, p.$$

***Proof* :** The validity of the result for $k = 3$ follows from (3.1.12). To proceed by induction on k, we assume that the result is true for some k. Then, by Corollary 3.1.1,

$$S\left(p^{2p^2+kp+k+\nu}\right) = 2p^3 + (k-2)p^2 + \nu p; \quad \nu = 1, \ 2, \ \ldots, \ p. \qquad (3.1.13)$$

From (3.1.13) with $\nu = p$, we get

$$S\left(p^{2p^2+(k+1)p+k}\right) = 2p^3 + (k-1)p^2,$$

so that, by Lemma 3.1.7,

$$S\left(p^{2p^2+(k+1)p+k+1}\right) = 2p^3 + (k-1)p^2.$$

Thus, the result is also true for $k + 1$, thereby completing induction. ■

In the particular case when $k = p$ (≥ 3) in Lemma 3.1.10, we have

$$S\left(p^{3p^2+p-1}\right) = 3p^3 - 2p^2 = S\left(p^{3p^2+p}\right), \qquad (3.1.14)$$

so that, by Corollary 3.1.1,

$$S\left(p^{3p^2+p+\nu}\right) = 3p^3 - 2p^2 + \nu p; \quad \nu = 1, \ 2, \ \ldots, \ p. \qquad (3.1.15)$$

Corresponding to $\nu = p$ ($p \geq 3$), (3.1.15) gives

$$S\left(p^{3p^2+2p}\right) = 3p^3 - p^2 = S\left(p^{3p^2+2p+1}\right), \qquad (3.1.16)$$

where the r.h.s. expression in (3.1.16) follows by virtue of Lemma 3.1.7. From (3.1.16), by Corollary 3.1.1, we have

$$S\left(p^{3p^2+2p+\nu}\right) = 3p^3 - p^2 + (\nu-1)p; \quad \nu = 2, \ 3, \ \ldots, \ p+1, \qquad (3.1.17)$$

which, in the particular case when $\nu = p + 1$ ($p \geq 5$), gives

$$S\left(p^{3p^2+3p+1}\right) = 3p^3 = S\left(p^{3p^2+3p+2}\right) = S\left(p^{3p^2+3p+3}\right), \qquad (3.1.18)$$

where we have used Lemma 3.1.7 to get the expressions of $S\left(p^{3p^2+3p+2}\right)$ as well as $S\left(p^{3p^2+3p+3}\right)$ on the r.h.s. of (3.1.18). Then, for $p \geq 5$, by Corollary 3.1.1, we have

$$S\left(p^{3p^2+3p+\nu}\right) = 3p^3 + (\nu-3)p; \quad \nu = 4, \ 5, \ \ldots, \ p+3. \qquad (3.1.19)$$

With $\nu = p + 3$ in (3.1.19), we get

$$S\left(p^{3p^2+4p+3}\right) = 3p^3 + p^2 = S\left(p^{3p^2+4p+4}\right), \qquad (3.1.20)$$

where the r.h.s. of (3.1.20) follows by virtue of Lemma 3.1.7.

We now state and prove the general case in the lemma below.

***Lemma 3.1.11* :** For $k = 4, 5, \ldots, p+2$; ($p \geq 5$ is a prime),

$$S\left(p^{3p^2+kp+k-1}\right) = 3p^3 + (k-3)p^2 = S\left(p^{3p^2+kp+k}\right).$$

***Proof* :** is by induction on k. From (3.1.20), we see that the result is true for $k = 4$. So, we assume that the result is true for some k. Then, by Corollary 3.1.1, we have

$$S\left(p^{3p^2+kp+k+\nu}\right) = 3p^3 + (k-3)p^2 + \nu p; \quad \nu = 1, \; 2, \; \ldots, \; p. \qquad (3.1.21)$$

With $\nu = p$ in (3.1.21), we have

$$S\left(p^{3p^2+(k+1)p+k}\right) = 3p^3 + (k-2)p^2,$$

so that, by Lemma 3.1.7,

$$S\left(p^{3p^2+(k+1)p+k+1}\right) = 3p^3 + (k-2)p^2,$$

which shows that the result is true for $k + 1$. This completes the proof by induction, thereby establishing the lemma. ■

***Lemma 3.1.12* :** For $k = 1, 2, \ldots, p-1$,

(1) $S\left(p^{kp^2+kp+k-2}\right) = kp^3 = S\left(p^{kp^2+kp+k-1}\right) = S\left(p^{kp^2+kp+k}\right)$,

(2) $S\left(p^{kp^2+(k+1)p+k}\right) = kp^3 + p^2 = S\left(p^{kp^2+(k+1)p+k+1}\right)$.

where p is a prime.

***Proof* :** In either case, the proof is by induction on k.

From (3.1.3), we see that part (1) of the lemma is true for $k = 1$; and the validity of part (2) for $k = 1$ follows from (3.1.5). So, we assume that the results are true for some k.

(1) By Corollary 3.1.1, together with the induction hypothesis,

$$S\left(p^{kp^2+(k+1)p+k+\nu+1}\right) = kp^3 + p^2 + \nu p; \quad \nu = 1, \; 2, \; \ldots, \; p, \qquad (3.1.22)$$

which, for $\nu = p$, gives

$$S\left(p^{kp^2+(k+1)p+k+p+1}\right) = kp^3 + 2p^2 = S\left(p^{kp^2+(k+1)p+k+p+2}\right). \qquad (3.1.23)$$

In (3.1.23), the right-hand side expression follows by virtue of Lemma 3.1.7.

Again, from (3.1.23), by virtue of Corollary 3.1.1, we get

$$S\left(p^{kp^2+(k+1)p+k+p+\nu+2}\right) = kp^3 + 2p^2 + \nu p; \quad \nu = 1, \; 2, \; \ldots, \; p, \qquad (3.1.24)$$

which, with $\nu = p$, gives

$$S\left(p^{kp^2+(k+1)p+k+2p+2}\right) = kp^3 + 3p^2 = S\left(p^{kp^2+(k+1)p+k+2p+3}\right). \qquad (3.1.25)$$

Note that, the right-hand expression in (3.1.25) follows by virtue of Lemma 3.1.7.

Continuing in this way, we get

$$S\left(p^{kp^2+(k+1)p+k+(\nu-1)p+\nu-1}\right)=kp^3+\nu p^2=S\left(p^{kp^2+(k+1)p+k+(\nu-1)p+\nu}\right), \quad (3.1.26)$$

which can be proved by induction on ν for ν = 1, 2, …, p. The case ν = 1 follows from the induction hypothesis. Then, (3.1.23) proves the validity of the result for ν = 2. To proceed by induction on ν, we assume that (3.1.26) is valid for some ν. Then, by Corollary 3.1.1,

$$S\left(p^{kp^2+(k+1)p+k+(\nu-1)p+\nu+m}\right)=kp^3+\nu p^2+mp; \quad m=1, \ 2, \ \ldots, \ p.$$

With m = p, we get

$$S\left(p^{kp^2+(k+1)p+k+\nu p+\nu}\right)=kp^3+(\nu+1)p^2=S\left(p^{kp^2+(k+1)p+k+\nu p+\nu+1}\right).$$

Note that, the expression of $S\left(p^{kp^2+(k+1)p+k+\nu p+\nu+1}\right)$ on the right-hand side above follows by virtue of Lemma 3.1.7. Thus, the result is true for ν + 1.

Now, (3.1.26) with ν = p gives

$$S\left(p^{(k+1)p^2+(k+1)p+k-1}\right)=(k+1)p^3=S\left(p^{(k+1)p^2+(k+1)p+k}\right), \quad (3.1.27a)$$

and then, by Lemma 3.1.7,

$$S\left(p^{(k+1)p^2+(k+1)p+k+1}\right)=(k+1)p^3. \quad (3.1.27b)$$

The validity of part (1) of the lemma for k + 1 follows from (3.1.27).

(2) From (3.1.27b), we get

$$S\left(p^{(k+1)p^2+(k+1)p+k+\nu+1}\right)=(k+1)p^3+\nu p; \quad \nu=1, \ 2, \ \ldots, \ p, \quad (3.1.28)$$

which, for ν = p, gives

$$S\left(p^{(k+1)p^2+(k+2)p+k+1}\right)=(k+1)p^3+p^2, \quad (3.1.29a)$$

so that, by Lemma 3.1.7,

$$S\left(p^{(k+1)p^2+(k+2)p+k+1}\right)=(k+1)p^3+p^2. \quad (3.1.29b)$$

(3.1.29) show that part (2) of the lemma is true for k + 1.

All these complete the proof of the lemma. ■

From part (1) of Lemma 3.1.12 with k = p – 1, we get

$$S\left(p^{p^3-3}\right)=(p-1)p^3=S\left(p^{p^3-2}\right)=S\left(p^{p^3-1}\right). \quad (3.1.30)$$

From (3.1.27) with k = p – 1, we get

$$S\left(p^{p^3+p^2+p-1}\right)=p^4=S\left(p^{p^3+p^2+p}\right)=S\left(p^{p^3+p^2+p+1}\right). \quad (3.1.31)$$

In the next section, we make some observations and conjectures related to the Smarandache function.

3.2 Some Remarks

From Lemma 3.1.3, $S(p^k)$ is linear in p for $1 \le k \le p-1$. By Lemma 3.5 and Lemma 3.1.3,

$$k > p \quad \Rightarrow \quad S(p^k) \ge S(p^p) = p^2.$$

It then follows that $S(p^k)$ is linear in p if and only if $1 \le k \le p-1$.

We conjecture the following, based on Lemma 3.1.5.

Conjecture 3.2.1 **:** $S\left(p^{p^k - k + \mu}\right) = (p-1)p^k$ for all $\mu = 1, 2, ..., k-1$, $k \ge 2$; (p is a prime).

The conjecture above follows from Lemma 3.1.5 by repeated application of Lemma 3.1.7, provided that $S\left(p^{p^k - k - 1}\right) \ne (p-1)p^k$. For $k = 3$, we refer the reader to (3.1.30).

Also, based on the expressions of $S\left(p^{p+1}\right)$, $S\left(p^{p^2+p+1}\right)$ and $S\left(p^{p^3+p^2+p+1}\right)$, given by Lemma 3.1.3, (3.1.3) and (3.1.31) respectively, we make the following conjecture.

Conjecture 3.2.2 **:** For any prime $p \ge 3$, $S\left(p^{p^k + p^{k-1} + \,...\, + p + 1}\right) = p^{k+1}$ for $k \ge 1$.

We conclude this chapter with the following conjecture.

Conjecture 3.2.3 **:** Let p and q be primes with $q > p$. Then, the equation

$$S(p^\alpha) = S(q^\beta) \tag{3.2.1}$$

has an infinite number of solutions.

Since $S(2^4) = 6 = S(3^2)$, we see that the equation (3.2.1) indeed possesses a solution. To find more solutions, we proceed as follows : Let the primes p (≥ 3) and q be such that $q = 2p-1$. Then, by Lemma 3.1.8,

$$S(p^{2p}) = 2p^2 - p = p(2p-1).$$

Now, by Lemma 3.1.3,

$$S(q^p) = pq,$$

so that $S(p^{2p}) = S(q^p)$ for all such primes p and q. Then, a second solution of the equation (3.2.1) is : $p = 3$, $\alpha = 6$, $q = 5$, $\beta = 3$, with

$$S(3^6) = 15 = S(5^3).$$

It is conjectured that the equation

$$S(n) = S(n+1) \tag{3.2.2}$$

has no solution. Assuming that $n = M p^\alpha$, $n + 1 = N q^\beta$ for some primes p and q such that $(M, p) = 1$, $(N, q) = 1$, $(M, N) = 1$ with $S(n) = p^\alpha$ and $S(n+1) = q^\beta$, we see that a necessary condition that the equation (3.2.2) has a solution is that the equation (3.2.1) possesses a solution. Thus, in order to look for solutions of the equation (3.2.2), it is necessary to concentrate our attention to the solution of the equation (3.2.1). As has been noted above, the minimum solution of the equation (3.2.1) is $p^\alpha = 2^4$, $q^\beta = 3^2$, and it is easy to check that the Diophantine equation below has no solution :

$$16M - 9N = \pm 1,$$

where $(M, N) = 1$, $M < 16$ with $(M, 16) = 1$, $N < 9$ with $(N, 9) = 1$.

We conclude this chapter with the following equation, proposed by Muller[8], and subsequently, taken up by Maohua[9] :

$$S(mn) = m^k S(n); \text{ m, n and k are positive integers.} \qquad (3.2.3)$$

Note that $m = 1$, $n = 1$, $k = 1$ is always a solution of the equation (3.2.3), called its trivial solution. To find non-trivial solutions of equation (3.2.3), first note that, there is nothing to prove if $m = 1$. Again, with $n = 1$, (3.2.3) reads as

$$S(m) = m^k, \qquad (3.2.4)$$

and we must have $k = 1$ (since, by Lemma 3.7 in Majumdar[10], $S(m) \le m$ for all integer $m \ge 1$). In this case, by Lemma 3.3.5 in Majumdar[10], the only solutions of the corresponding equation $S(m) = m$ are $m = 1, 4, p$ (where $p \ge 2$ is a prime). If $m = 2$, then (3.2.3) takes the form

$$S(2n) = 2^k S(n), \qquad (3.2.5)$$

which has no solution if $n \ge 3$ (since, by Lemma 3.9 in Majumdar[10], $S(2n) \le n$ for $n \ge 3$). Clearly, when $m = 2$ and $n = 2$ in (3.2.3), we must have $k = 1$. With $n = 2$, the equation (3.2.3) becomes

$$S(2m) = 2m^k,$$

which does not have any solution in integers $m \ge 3$ and $k \ge 1$. So, let $n \ge 3$. Then, since

$$m^k S(n) = S(mn) \le S(m) + S(n),$$

we get

$$m^k \le \frac{S(m)}{S(n)} + 1 \le \frac{m}{3} + 1,$$

and we are led to a contradiction. Thus, we must have $n \le 2$, and corresponding to $n = 2$, the only solution of the equation (3.2.3) is $m = 2$, $n = 2$, $k = 1$.

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Chapter 4 The Pseudo Smarandache Function

This chapter is devoted to the pseudo Smarandache function, denoted by Z(n).

The pseudo Smarandache function, introduced by Kashihara(1), is as follows :

***Definition 4.1* :** For any integer $n \geq 1$, the *pseudo Smarandache function* Z(n) is the smallest positive integer m such that $1+2+\ldots+m \equiv \frac{m(m+1)}{2}$ is divisible by n. Thus,

$$Z(n) = \min \left\{ m : m \in \mathbb{Z}^+ ,\ n \,|\, \frac{m(m+1)}{2} \right\};\ n \geq 1,$$

(where $\mathbb{Z}^+$ is the set of all positive integers).

Kashihara(1), Ibstedt(2), Ashbacher(3) and Pinch(4) studied some of the elementary properties satisfied by Z(n). Their findings are summarized below.

***Lemma 4.1* :** $3 \leq Z(n) \leq 2n-1$ for all integer $n \geq 4$.

***Lemma 4.2* :** $Z(n) \leq n-1$ for any odd integer $n \geq 3$.

***Lemma 4.3* :** For any composite number $n \geq 4$, $Z(n) \geq \max \{ Z(d) : d \,|\, n \}$.

***Lemma 4.4* :** For any integer $n \geq 4$, $Z(n) \geq \frac{1}{2}(\sqrt{1+8n}-1) > \sqrt{n}$.

This chapter gives some new results related to the pseudo Smarandache function Z(n).

In Section 4.1, we give some explicit forms for Z(n), which are available in the current literature. The values of $Z(p.2^k)$, $p = 5, 7, 11, 13, 17, 19, 31$ are available In the current literature. A closed-form expression of $Z(23.2^k)$ has been derived in Lemma 4.1.9 in Section 4.1 below. An expression for Z(pq) is given in Theorem 4.1.2, which shows that the method of finding the value of Z(pq) involves the solution of two Diophantine equations in p and q. Some particular cases of Theorem 4.1.2 are given in Corollariy 4.1.5 – Corollary 4.1.8. Section 4.2 is devoted to the study of some miscellaneous topics, where the solutions of the three Diophantine equations, Z(n) + SL(n) = n, Z(n) = SL(n), and $Z(mn) = m^k S(n)$, are given, SL(n) being the Smarandache LCM function.

4.1 Some Explicit Expressions for Z(n)

Some closed-form expressions of Z(n), available in the current literature, are summarized in the following lemmas.

***Lemma 4.1.1* :** $Z\left(\frac{k(k+1)}{2}\right) = k$ for any integer $k \geq 1$.

***Lemma 4.1.2* :** For any integer $k \geq 1$, $Z(2^k) = 2^{k+1}-1$.

***Lemma 4.1.3* :** For any prime $p \geq 3$ and any integer $k \geq 1$, $Z(p^k) = p^k - 1$.

***Lemma 4.1.4* :** If $p \geq 3$ is a prime and $n \geq 2$ is an integer (not divisible by p), then

$$Z(np) = \begin{cases} p-1, & \text{if } 2n \,|\, (p-1) \\ p, & \text{if } 2n \,|\, (p+1) \end{cases}$$

Lemma 4.1.5 **:** If $p \geq 3$ is a prime and $n \geq 2$ is an integer not divisible by p, then

$$Z(np^2) = \begin{cases} p^2 - 1, & \text{if } 2n \mid (p^2 - 1) \\ p^2, & \text{if } 2n \mid (p^2 + 1) \end{cases}$$

Lemma 4.1.6 **:** If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(2p^k) = \begin{cases} p^k, & \text{if } 4 \mid (p+1) \text{ and k is odd} \\ p^k - 1, & \text{otherwise} \end{cases}$$

In particular, the expressions for Z(np), are available when $n = 4(1)13$, 16, 32, 24, 48, 96, (where p is a prime). Also, available are the explicit forms of $Z(p.2^k)$, p = 5, 7, 11, 13, 17, 19, 31. These values are given in Majumdar[5].

Lemma 4.1.9 gives an expression of $Z(23.2^k)$. To prove the lemma, we need the results below.

Lemma 4.1.7 **:** If $p = 2P + 1$ is a prime, then

(1) p divides $2^P - 1$ if $p = 8\ell \pm 1$ (for some integer ℓ), **(2)** p divides $2^P + 1$ if $p = 8\ell \pm 3$.

Proof **:** See, for example, Theorem 18 in Daniel Shanks[6]. ■

Corollary 4.1.1 **:** Given any prime $p \geq 3$, $p = 2P + 1$, there is an integer n such that p divides $2^n - 1$.

Proof **:** If p is of the form $2^n - 1$ for some integer $n \geq 2$, there is nothing to prove. So, let p be not of the form $2^n - 1$. Now, if $p = 8\ell \pm 1$, by Lemma 4.1.7, p divides $2^P - 1$, so that the result is true with $n = P$. On the other hand, if $p = 8\ell \pm 3$, then

$$p \mid 2^P + 1 \quad \Rightarrow \quad p \mid (2^P + 1)(2^P - 1) = 2^{2P} - 1,$$

and the result holds true with $n = 2P$. ■

Lemma 4.1.8 **:** If p divides $2^n - 1$ for some integer $n \geq 1$, then p divides $2^{mn} - 1$ for any integer $m \geq 1$.

Proof **:** The proof is by induction on m. When $m = 1$, the result is clearly true. So, let the result be true for some integer m.
Now, since

$$2^{(m+1)n} - 1 = 2^n(2^{mn} - 1) + 2^n - 1,$$

it follows that the result is true for $m + 1$ as well, completing induction. ■

The expression of $Z(23.2^k)$, due to Majumdar[7], is given in the following lemma.

Lemma 4.1.9 **:** For any integer $k \geq 0$,

$$Z(23.2^k) = \begin{cases} 11.2^{k+1}, & \text{if } 11 \mid k \\ 3.2^{k+2} - 1, & \text{if } 11 \mid (k-1) \\ 3.2^{k+1} - 1, & \text{if } 11 \mid (k-2) \\ 5.2^{k+2}, & \text{if } 11 \mid (k-3) \\ 5.2^{k+1}, & \text{if } 11 \mid (k-4) \\ 9.2^{k+1} - 1, & \text{if } 11 \mid (k-5) \\ 7.2^{k+1}, & \text{if } 11 \mid (k-6) \\ 2^{k+4} - 1, & \text{if } 11 \mid (k-7) \\ 2^{k+3} - 1, & \text{if } 11 \mid (k-8) \\ 2^{k+2} - 1, & \text{if } 11 \mid (k-9) \\ 2^{k+1} - 1, & \text{if } 11 \mid (k-10) \end{cases} = \begin{cases} 11.2^{k+1}, & \text{if } k = 11a \\ 3.2^{k+2} - 1, & \text{if } k = 11a+1 \\ 3.2^{k+1} - 1, & \text{if } k = 11a+2 \\ 5.2^{k+2}, & \text{if } k = 11a+3 \\ 5.2^{k+1}, & \text{if } k = 11a+4 \\ 9.2^{k+1} - 1, & \text{if } k = 11a+5 \\ 7.2^{k+1}, & \text{if } k = 11a+6 \\ 2^{k+4} - 1, & \text{if } k = 11a+7 \\ 2^{k+3} - 1, & \text{if } k = 11a+8 \\ 2^{k+2} - 1, & \text{if } k = 11a+9 \\ 2^{k+1} - 1, & \text{if } k = 11a+10 \end{cases}$$

***Proof* :** By definition,

$$Z(23.2^k) = \min\left\{m : 23.2^k \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : 23.2^{k+1} \mid m(m+1)\right\}. \qquad (1)$$

Here, 2^{k+1} must divide one of m and $m+1$, and 23 must divide the other.
We now consider all the possible cases below :

***Case 1* :** When k is of the form $k = 11a$ for some integer $a \geq 1$.

By Corollary 4.1.1, 23 divides $2^{11a} - 1 = 2^k - 1$. Now,

$$23 \mid (11.2^{k+1} + 1) = 22(2^k - 1) + 23 \quad \Rightarrow \quad 23.2^{k+1} \mid (11.2^{k+1}) \times (11.2^{k+1} + 1).$$

Therefore, the minimum m in (1) can be taken as 11.2^{k+1}.

***Case 2* :** When k is of the form $k = 11a + 1$ for some integer $a \geq 0$.
Here, since

$$3.2^{k+2} - 1 = 24(2^{k-1} - 1) + 23,$$

by Corollary 4.1.1, 23 divides $3.2^{k+2} - 1$ and hence, in this case, the minimum m in (1) may be taken as $3.2^{k+1} - 1$.

***Case 3* :** When k is of the form $k = 11a + 2$ for some integer $a \geq 0$.

In this case, since

$$3.2^{k+1} - 1 = 24(2^{k-2} - 1) + 23,$$

it follows that, 23 divides $3.2^{k+2} - 1$, and hence, $Z(23.2^k) = 2^{k+3}$.

***Case 4* :** When k is of the form $k = 11a + 3$ for some integer $a \geq 0$.

Here, since

$$5.2^{k+2} + 1 = 160(2^{k-3} - 1) + 161,$$

and since 23 divides both $2^{k-3} - 1 = 2^{11a} - 1$ (by Corollary 4.1.1) and 161, it follows that 23 divides $5.2^{k+2} - 1$, and consequently, $Z(23.2^k) = 5.2^{k+2}$.

***Case 5* :** When k is of the form $k = 11a + 4$ for some integer $a \geq 0$.

In this case,

$$23 \mid (5.2^{k+1} + 1) = 160(2^{k-4} - 1) + 161 \quad \Rightarrow \quad Z(23.2^k) = 5.2^{k+1}.$$

***Case 6* :** When k is of the form $k = 11a + 5$ for some integer $a \geq 0$.
Here, note that

$$9.2^{k+1} - 1 = 576(2^{k-5} - 1) + 575 \quad \Rightarrow \quad 23 \mid (9.2^{k+1} - 1),$$

and hence, $Z(23.2^k) = 9.2^{k+1} - 1$.

***Case 7* :** When k is of the form $k = 11a + 6$ for some integer $a \geq 0$.

Here, the result follows from the following chain of implications :

$$7.2^{k+1} + 1 = 896(2^{k-6} - 1) + 897 \quad \Rightarrow \quad 23 \mid (7.2^{k+1} + 1) \quad \Rightarrow \quad Z(23.2^k) = 7.2^{k+1}.$$

***Case 8* :** When k is of the form $k = 11a + 7$ for some integer $a \geq 0$.

In this case,

$$2^{k+4} - 1 = 2^{11}(2^{k-7} - 1) + 2^{11} - 1.$$

Now, since 23 divides $2^{k-7} - 1 = 2^{11a} - 1$ and $2^{11} - 1$ (by Corollary 4.1.1), it follows that 23 divides $2^{k+4} - 1$, and consequently, $Z(23.2^k) = 2^{k+4} - 1$.

***Case 9* :** When k is of the form $k = 11a + 8$ for some integer $a \geq 0$.

Here, by Corollary 4.1.1,

$$23 \mid (2^{k+3} - 1) = 2^{11(a+1)} - 1 \quad \Rightarrow \quad Z(23.2^k) = 2^{k+3} - 1.$$

***Case 10* :** When k is of the form $k = 11a + 9$ for some integer $a \geq 0$.
In this case, since 23 divides $2^{k+2} - 1 = 2^{11(a+1)} - 1$, it follows that $Z(23.2^k) = 2^{k+2} - 1$.

***Case 11* :** When k is of the form $k = 11a + 10$ for some integer $a \geq 0$.
Here, since 23 divides $2^{k+1} - 1 = 2^{11(a+1)} - 1$, it follows that $Z(23.2^k) = 2^{k+1} - 1$.

All these complete the proof. ■

The following lemma gives a closed-form expression of $Z(p.2^k)$ for $k \geq 1$ in the particular case when p is a prime of the form $2^q - 1$, that is, when p is a Mercenne prime. The result is due to Majumdar[8].

***Lemma 4.1.10* :** Let p be a prime of the form $p = 2^q - 1$, $q \geq 2$. Then

$$Z(p.2^k) = \begin{cases} (p-1)2^k, & \text{if } \ q \text{ divides } k \\ 2^{k+q-i}, & \text{if } \ q \text{ divides } k-i, \ 1 \leq i \leq q-1 \end{cases}$$

***Proof* :** First note that, if $p = 2^q - 1$ is prime, then by the Cataldi-Fermat Theorem, q must be a prime (see, for example, Theorem 4 in Daniel Shanks[6]).
Now, by definition,

$$Z(p.2^k) = \min\left\{m : p.2^k \mid \frac{m(m+1)}{2}\right\} = \min\{m : p.2^{k+1} \mid m(m+1)\}. \qquad (2)$$

Here, p must divide one of m and $m + 1$, and 2^{k+1} must divide the other.
We now consider all the possible cases below :

***Case (1)* :** When k is of the form $k = qa$ for some integer $a \geq 1$.
Let $p = 2P + 1$. Now, since

$$P.2^{k+1} + 1 = 2P(2^k - 1) + (2P + 1) = 2P(2^{qa} - 1) + p,$$

it follows that p divides $P.2^{k+1} + 1$ (by Lemma 4.1.8), so that $p.2^{k+1}$ divides $P.2^{k+1}(P.2^{k+1} + 1)$.
Therefore, the minimum m in (2) can be taken as $P.2^{k+1}$, and hence,

$$Z(p.2^k) = P.2^{k+1} = (p - 1)2^k.$$

***Case (2)* :** When k is of the form $k = qa + 1$ for some integer $a \geq 0$.
Here,

$$2^{q-2}.2^{k+1} - 1 = 2^q(2^{qa} - 1) + 2^q - 1,$$

so that, p divides $2^{k+q-1} - 1$ and hence, $p.2^{k+1}$ divides $2^{k+q-1}(2^{k+q-1} - 1)$. Thus, in this case, the minimum m in (2) may be taken as $2^{k+q-1} - 1$, so that $Z(p.2^k) = 2^{k+q-1} - 1$.

***Case (3)* :** When k is of the form $k = qa + 2$ for some integer $a \geq 0$.
In this case, since

$$2^{q-3}.2^{k+1} - 1 = 2^q(2^{qa} - 1) + 2^q - 1,$$

it follows that, $p.2^{k+1}$ divides $2^{k+q-2}(2^{k+q-2} - 1)$, and hence, $Z(p.2^k) = 2^{k+q-2} - 1$.

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***Case (q)* :** When k is of the form $k = qa + q - 1$ for some integer $a \geq 0$.
Here,

$$2^{k+1} - 1 = 2^{q(a+1)} - 1,$$

so that $p.2^{k+1}$ divides $2^{k+1}(2^{k+1} - 1)$, and consequently, $Z(p.2^k) = 2^{k+1} - 1$.

All these complete the proof of the lemma. ■

Some particular cases of Lemma 4.1.10 are $Z(3.2^k)$ (corresponding to $q = 2$), $Z(7.2^k)$ (corresponding to $q = 3$), and $Z(31.2^k)$ (corresponding to $q = 5$). The explicit forms of $Z(3.2^k)$, $Z(7.2^k)$ and $Z(31.2^k)$ are given below. These results have been established by Majumdar[5] in detail by alternative methods.

Corollary 4.1.2 **:** For any integer $k \geq 1$,

$$Z(3.2^k) = \begin{cases} 2^{k+1} - 1, & \text{if k is odd} \\ 2^{k+1}, & \text{if k is even} \end{cases}$$

Proof **:** Since this case corresponds to $q = 2$, q divides k if and only if k is even. The result then follows from Lemma 4.1.10 immediately. ■

Corollary 4.1.3 **:** For any integer $k \geq 1$,

$$Z(7.2^k) = \begin{cases} 3.2^{k+1}, & \text{if } 3 \mid k \\ 2^{k+2} - 1, & \text{if } 3 \mid (k-1) \\ 2^{k+1} - 1, & \text{if } 3 \mid (k-2) \end{cases}$$

Proof **:** This case corresponds to $q = 3$, and so, there are three possibilities, namely, k is one of the three forms $k = 3a$, $3a + 1$, $3a + 2$. Then, appealing to Lemma 4.1.10, we get the desired expression for $Z(7.2^k)$. ■

Corollary 4.1.4 **:** For any integer $k \geq 1$,

$$Z(31.2^k) = \begin{cases} 15.2^{k+1}, & \text{if } 5 \mid k \\ 2^{k+4} - 1, & \text{if } 5 \mid (k-1) \\ 2^{k+3} - 1, & \text{if } 5 \mid (k-2) \\ 2^{k+2} - 1, & \text{if } 5 \mid (k-3) \\ 2^{k+1} - 1, & \text{if } 5 \mid (k-4) \end{cases}$$

Proof **:** Here, k can be one of the five forms $k = 5a$, $5a + 1$, $5a + 2$, $5a + 3$, $5a + 4$. When $k = 5a$, by Lemma 4.1.10, $Z(31.2^k) = 30.2^k = 15.2^{k+1}$. The other four cases also follow readily from Lemma 4.1.10. ■

The following result, due to Ibstedt[2], gives an expression for $Z(pq)$, where p and q are two distinct primes.

Theorem 4.1.1 **:** Let p and q be two primes with $q > p$. Let $g = q - p$. Then,

$$Z(pq) = \min \left\{ \frac{p(qk+1)}{g}, \frac{q(pk-1)}{g} \right\},$$

where both $qk + 1$ and $pk - 1$ are divisible by g.

An alternative expression for $Z(pq)$, due to Majumdar[5], is given below.

Theorem 4.1.2 **:** Let p and q be two primes with $q > p \geq 5$. Then,

$$Z(pq) = \min \{ qy_o - 1, px_o - 1 \},$$

where

$$y_o = \min \{ y : x, y \in \mathbb{Z}^+, qy - px = 1 \},$$

$$x_o = \min \{ x : x, y \in \mathbb{Z}^+, px - qy = 1 \}.$$

To apply Theorem 4.1.2 to find $Z(pq)$, it is convenient, from the computational point of view, the two possible cases, considered in Remark 4.1.1 below.

***Remark 4.1.1* :** Let p and q be two primes with $q > p \geq 5$. Let

$$q = kp + \ell \text{ for some integers k and } \ell \text{ with } k \geq 1 \text{ and } 1 \leq \ell \leq p-1.$$

We now consider the two cases given in Theorem 4.2.2 :

Case 1 : When p divides m and q divides $(m+1)$. In this case,

$m = px$ for some integer $x \geq 1$,

$m+1 = qy = (kp+\ell)y$ for some integer $y \geq 1$.

From the above two equations, we get

$$\ell y - (x - ky)p = 1. \tag{4.1.1}$$

Case 2 : When p divides $(m+1)$ and q divides m. Here,

$m+1 = px$ for some integer $x \geq 1$,

$m = (kp+\ell)y$ for some integer $y \geq 1$.

These two equations lead to

$$(x - ky)p - \ell y = 1. \tag{4.1.2}$$

Thus, to find Z(pq), it is necessary to solve the Diophantine equations (4.1.1) and (4.1.2) for minimum x or y.

Some particular cases are given in Corollary 4.1.5 – Corollary 4.1.8 below, which illustrate the application of Theorem 4.1.2.

When $q = kp + 9$, $k \geq 2$, the Diophantine equations (4.1.1) and (4.1.2) become

$$(kp+9)y - px = 1,$$

$$px - (kp+9)y = 1,$$

that is,

$$9y - (x - ky)p = 1, \tag{4.1.3}$$

$$(x - ky)p - 9y = 1. \tag{4.1.4}$$

***Corollary 4.1.5* :** Let p and $q > p$ be two primes; moreover, let q be of the form $q = kp + 9$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{9}, & \text{if } 9 \mid (p-1) \\ \dfrac{q(4p+1)}{9} - 1, & \text{if } 9 \mid (p-2) \\ \dfrac{q(2p+1)}{9} - 1, & \text{if } 9 \mid (p-4) \\ \dfrac{q(2p-1)}{9}, & \text{if } 9 \mid (p-5) \\ \dfrac{q(4p-1)}{9}, & \text{if } 9 \mid (p-7) \\ \dfrac{q(p+1)}{9} - 1, & \text{if } 9 \mid (p-8) \end{cases}$$

***Proof* :** We consider the following cases that may arise.

Case 1. When $p = 9a + 1$ for some integer $a \geq 2$.

In this case, the Diophantine equations (4.1.3) and (4.1.4) take the following forms :

$$1 = 9y - (x - ky)(9a+1)p = 9[y - (x - ky)a] - (x - ky),$$

$$1 = (x - ky)(9a+1) - 9y = (x - ky) - 9[y - (x - ky)a].$$

The minimum solution is then obtained from the second of the above two equations with

$$x - ky = 1,\ y - (x - ky)a = 0.$$

Therefore, $y = a$, and consequently, the minimum m is given by

$$m = qy = \frac{q(p-1)}{9}.$$

Case 2. When $p = 9a + 2$ for some integer $a \geq 1$.

Here, (4.1.3) and (4.1.4) take the forms

$$1 = 9y - (x - ky)(9a + 2)p = 9[y - (x - ky)a] - 2(x - ky),$$

$$1 = (x - ky)(9a + 2) - 9y = 2(x - ky) - 9[y - (x - ky)a].$$

The minimum solution is then obtained from the first of the above two Diophantine equations with

$$x - ky = 4,\ y - (x - ky)a = 1,$$

so that $y = 4a + 1$, and consequently, the minimum m is

$$m = qy - 1 = q(4a + 1) - 1 = \frac{q(4p+1)}{9} - 1.$$

Case 3. When $p = 9a + 4$ for some integer $a \geq 1$.

In this case, from (4.1.3) and (4.1.4), the Diophantine equations satisfied are

$$1 = 9y - (x - ky)(9a + 4)p = 9[y - (x - ky)a] - 4(x - ky),$$

$$1 = (x - ky)(9a + 4) - 9y = 4(x - ky) - 9[y - (x - ky)a],$$

and the minimum solution, obtained from the first equation, is

$$x - ky = 2,\ y - (x - ky)a = 1.$$

Therefore, $y = 2a + 1$, and the minimum m is

$$m = qy - 1 = q(2a + 1) - 1 = \frac{q(2p+1)}{9} - 1.$$

Case 4. When $p = 9a + 5$ for some integer $a \geq 0$.

Here, the Diophantine equations (4.1.3) and (4.1.4) become

$$1 = 9y - (x - ky)(9a + 5)p = 9[y - (x - ky)a] - 5(x - ky),$$

$$1 = (x - ky)(9a + 5) - 9y = 5(x - ky) - 9[y - (x - ky)a].$$

Thus, the minimum solution is obtained from the second equation, with

$$x - ky = 2,\ y - (x - ky)a = 1.$$

Thus, $y = 2a + 1$, and the minimum m is given by

$$m = qy = q(2a + 1) = \frac{q(2p-1)}{9}.$$

Note that, in this case, when $a = 0$, we get

$$Z(5q) = q \text{ for } q = 5k + 9;\ k = 2, 4, \ldots,$$

which is true (by Lemma 4.2.17 in Majumdar[(5)]).

Case 5. When $p = 9a + 7$ for some integer $a \geq 0$.

In this case, the Diophantine equations (4.1.3) and (4.1.4) reduce to

$$1 = 9y - (x - ky)(9a + 7)p = 9[y - (x - ky)a] - 7(x - ky),$$

$$1 = (x - ky)(9a + 7) - 9y = 7(x - ky) - 9[y - (x - ky)a],$$

and the minimum solution is obtained from the second one as follows :

$$x - ky = 4, \; y - (x - ky)a = 3.$$

Then, $y = 4a + 3$, and the minimum m is given by

$$m = qy = q(4a+3) = \frac{q(4p-1)}{9}.$$

When $a = 0$, we get

$$Z(7q) = 3q, \; q = 7k + 9, \; k = 2, 4, 6, \ldots,$$

which is valid (by Lemma 4.2.19 in Majumdar[(5)]).

Case 6. When $p = 9a + 8$ for some integer $a \geq 1$.

Here, the Diophantine equations (4.1.3) and (4.1.4) take the form

$$1 = 9y - (x - ky)(9a+8)p = 9[y - (x - ky)a] - 8(x - ky),$$

$$1 = (x - ky)(9a+8) - 9y = 8(x - ky) - 9[y - (x - ky)a].$$

Clearly, the minimum solution is obtained from the first equation as follows :

$$x - ky = 1, \; y - (x - ky)a = 1.$$

Therefore, $y = a + 1$, and hence, the minimum m is

$$m = qy - 1 = q(a+1) - 1 = \frac{q(p+1)}{9} - 1.$$

All these complete the proof of the corollary. ■

With $q = (k+1)p - 9$, the Diophantine equations (4.1.1) and (4.1.2) read as

$$[(k+1)p - 9]y - px = 1,$$

$$px - [(k+1)p - 9]y = 1,$$

that is,

$$1 = [(k+1)y - x]p - 9y, \tag{4.1.5}$$

$$1 = 9y - [(k+1)y - x]p. \tag{4.1.6}$$

Corollary 4.1.6 : Let p and $q > p$ be two primes with $q = (k+1)p - 9$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{9} - 1, & \text{if } 9 \mid (p-1) \\ \dfrac{q(4p+1)}{9}, & \text{if } 9 \mid (p-2) \\ \dfrac{q(2p+1)}{9}, & \text{if } 9 \mid (p-4) \\ \dfrac{q(2p-1)}{9} - 1, & \text{if } 9 \mid (p-5) \\ \dfrac{q(4p-1)}{9} - 1, & \text{if } 9 \mid (p-7) \\ \dfrac{q(p+1)}{9}, & \text{if } 9 \mid (p-8) \end{cases}$$

Proof : We consider the six possibilities that may arise :

Case 1. When $p = 9a + 1$ for some integer $a \geq 2$.

In this case, the Diophantine equations (4.1.5) and (4.1.6) read respectively as

$$1 = [(k+1)y - x](9a+1) - 9y = [(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+1) = 9[y - \{(k+1)y - x\}a] - [(k+1)y.$$

The first of the above two equations give the minimum solution, namely,

$$(k+1)y - x = 1,\ y - \{(k+1)y - x\}a = 0.$$

Therefore, $y = a$, and the minimum m is

$$m = qy - 1 = \frac{q(p-1)}{9} - 1.$$

Case 2. When $p = 9a + 2$ for some integer $a \geq 1$.

In this case, (4.1.5) and (4.1.6) become

$$1 = [(k+1)y - x](9a+2) - 9y = 2\,[(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+2) = 9[y - \{(k+1)y - x\}a] - 2[(k+1)y - x].$$

The minimum solution is then obtained from the second of the above two equations :

$$(k+1)y - x = 4,\ y - \{(k+1)y - x\}a = 1.$$

Then, $y = 4a + 1$, and the minimum m is

$$m = qy = \frac{q(4p+1)}{9}.$$

Case 3. When $p = 9a + 4$ for some integer $a \geq 1$.

Here, from (4.1.5) and (4.1.6), we have

$$1 = [(k+1)y - x](9a+4) - 9y = 4\,[(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+4) = 9[y - \{(k+1)y - x\}a] - 4[(k+1)y - x].$$

The minimum solution is then obtained from the second of the above two equation as follows :

$$(k+1)y - x = 2,\ y - \{(k+1)y - x\}a = 1.$$

Thus, $y = 2a + 1$, and the minimum m is

$$m = qy = \frac{q(2p+1)}{9}.$$

Case 4. When $p = 9a + 5$ for some integer $a \geq 0$.

From (4.1.5) and (4.1.6), we get

$$1 = [(k+1)y - x](9a+5) - 9y = 5\,[(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+5) = 9[y - \{(k+1)y - x\}a] - 5[(k+1)y - x].$$

The minimum solution, obtained from the first equation, is as follows :

$$(k+1)y - x = 2,\ y - \{(k+1)y - x\}a = 1.$$

This gives $y = 2a + 1$, and the minimum m is

$$m = qy - 1 = \frac{q(2p-1)}{9} - 1.$$

It may be noted here that $a = 0$ gives

$$Z(5q) = q - 1;\ q = 5k - 9,\ k = 4, 6, \ldots,$$

which is true (by Lemma 4.2.17 in Majumdar(5)).

Case 5. When $p = 9a + 7$ for some integer $a \geq 0$.

In this case, from (4.1.5) and (4.1.6), we have

$$1 = [(k+1)y - x](9a+7) - 9y = 7\,[(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+7) = 9[y - \{(k+1)y - x\}a] - 7[(k+1)y - x].$$

The first equation gives the minimum solution as follows :

$$(k+1)y - x = 4,\ y - \{(k+1)y - x\}a = 3.$$

Therefore, $y = 4a + 3$, and the minimum m is

$$m = qy - 1 = \frac{q(4p-1)}{9} - 1.$$

In this case, for $a = 0$, we get

$$Z(7q) = 3q - 1;\ q = 7k - 9,\ k = 4, 6, \ldots,$$

which is true (by virtue of Lemma 4.2.19 in Majumdar[5]).

Case 6. When $p = 9a + 8$ for some integer $a \geq 1$.
From (4.1.5) and (4.1.6), we have

$$1 = [(k+1)y - x](9a+8) - 9y = 8\,[(k+1)y - x] - 9[y - \{(k+1)y - x\}a],$$

$$1 = 9y - [(k+1)y - x](9a+8) = 9[y - \{(k+1)y - x\}a] - 8[(k+1)y - x].$$

The minimum solution is then obtained from the second equation, with

$$(k+1)y - x = 1,\ y - \{(k+1)y - x\}a = 1.$$

That is, $y = a + 3$, and the minimum m is

$$m = qy = \frac{q(p+1)}{9}.$$

All these complete the proof of the corollary. ■

When $q = kp + 10$, $k \geq 2$, the Diophantine equations (4.1.1) and (4.1.2) become

$$(kp + 10)y - px = 1,$$

$$px - (kp + 10)y = 1,$$

that is,

$$10y - (x - ky)p = 1, \tag{4.1.7}$$

$$(x - ky)p - 10y = 1. \tag{4.1.8}$$

The corollary below gives the closed-form expression of Z(pq), where $q = kp + 10$.

Corollary 4.1.7 : Let p and $q > p$ be two primes; moreover, let q be of the form $q = kp + 10$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{10}, & \text{if } 10 \mid (p-1) \\ \dfrac{q(3p+1)}{10} - 1, & \text{if } 10 \mid (p-3) \\ \dfrac{q(3p-1)}{10}, & \text{if } 10 \mid (p-7) \\ \dfrac{q(p+1)}{10} - 1, & \text{if } 10 \mid (p-9) \end{cases}$$

Proof : We consider the following four cases that may arise.

Case 1. When $p = 10a + 1$ for some integer $a \geq 1$.
In this case, the Diophantine equations (4.1.7) and (4.1.8) read as

$$1 = 10y - (x - ky)(10a + 1)p = 10[y - (x - ky)a] - (x - ky),$$

$$1 = (x - ky)(10a + 1) - 10y = (x - ky) - 10[y - (x - ky)a].$$

Clearly, the minimum solution is obtained from the second of the above two equations with

$$x - ky = 1,\ y - (x - ky)a = 0.$$

Then, the minimum solution is $y = a$, and hence, the minimum m is given by

$$m = qy = \frac{q(p-1)}{10}.$$

Case 2. When $p = 10a + 3$ for some integer $a \geq 1$.
Here, the Diophantine equations (4.1.7) and (4.1.8) become

$$1 = 10y - (x - ky)(10a + 3)p = 10[y - (x - ky)a] - 3(x - ky),$$

$$1 = (x - ky)(10a + 3) - 10y = 3(x - ky) - 10[y - (x - ky)a].$$

The minimum solution is obtained from the first of the above two Diophantine equations with

$$x - ky = 3,\ y - (x - ky)a = 1.$$

Therefore, the minimum y is $y = 3a + 1$, and consequently, the minimum m is

$$m = qy - 1 = q(3a + 1) - 1 = \frac{q(3p+1)}{10} - 1.$$

Case 3. When $p = 10a + 7$ for some integer $a \geq 1$.
Here, the Diophantine equations satisfied are

$$1 = 10y - (x - ky)(10a + 7)p = 10[y - (x - ky)a] - 7(x - ky),$$

$$1 = (x - ky)(10a + 7) - 10y = 7(x - ky) - 10[y - (x - ky)a],$$

For which the minimum solution is obtained from the second equation as

$$x - ky = 3,\ y - (x - ky)a = 2.$$

Thus, the minimum solution is $y = 3a + 2$, and the minimum m is

$$m = qy = q(3a + 2) = \frac{q(3p-1)}{10}.$$

Case 4. When $p = 10a + 9$ for some integer $a \geq 1$.
In this case, the Diophantine equations (4.1.7) and (4.1.8) take the forms

$$1 = 10y - (x - ky)(10a + 9)p = 10[y - (x - ky)a] - 9(x - ky),$$

$$1 = (x - ky)(10a + 9) - 9y = 10(x - ky) - 9[y - (x - ky)a].$$

Clearly, the minimum solution is obtained from the first equation, with

$$x - ky = 1,\ y - (x - ky)a = 1.$$

Thus, $y = a + 1$, and the minimum m is given by

$$m = qy - 1 = q(a + 1) - 1 = \frac{q(p+1)}{10} - 1.$$

All these complete the proof of the corollary. ■

When $q = (k + 1)\,p - 10$, the Diophantine equations (4.1.1) and (4.1.2) become

$$[(k + 1)\ p - 10]y - px = 1,$$

$$px - [(k + 1)\ p - 10]y = 1,$$

that is,

$$1 = [(k + 1)y - x]p - 10y, \tag{4.1.9}$$

$$1 = 10y - [(k + 1)y - x]p. \tag{4.1.10}$$

We now prove the following result, which gives an expression of Z(pq) when p and q ($> p$) are primes with $q = (k + 1)p - 10$, $k \geq 2$.

***Corollary 4.1.8* :** Let p and $q > p$ be two primes with $q = (k+1)p - 10$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{10} - 1, & \text{if } 10 \mid (p-1) \\ \dfrac{q(3p+1)}{10}, & \text{if } 10 \mid (p-3) \\ \dfrac{q(3p-1)}{10} - 1, & \text{if } 10 \mid (p-7) \\ \dfrac{q(p+1)}{10}, & \text{if } 10 \mid (p-9) \end{cases}$$

***Proof* :** We consider below separately the four cases that may arise :

Case 1. When $p = 10a + 1$ for some integer $a \geq 1$.

In this case, the Diophantine equations (4.1.9) and (4.1.10) may be rewritten as

$$1 = [(k+1)y - x](10a + 1) - 10y = [(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a + 1) = 10[y - \{(k+1)y - x\}a] - [(k+1)y - x].$$

The first of these two give the minimum solution, namely,

$$(k+1)y - x = 1,\ y - \{(k+1)y - x\}a = 0.$$

The minimum solution is thus $y = a$, and consequently, the minimum m is

$$m = qy - 1 = \frac{q(p-1)}{10} - 1.$$

Case 2. When $p = 10a + 3$ for some integer $a \geq 1$.

Here, (4.1.9) and (4.1.10) become

$$1 = [(k+1)y - x](10a + 3) - 10y = 3\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a + 3) = 10[y - \{(k+1)y - x\}a] - 3[(k+1)y - x].$$

The minimum solution, obtained from the second of the above two equations, is

$$(k+1)y - x = 3,\ y - \{(k+1)y - x\}a = 1.$$

Then, the minimum solution is $y = 3a + 1$, and the minimum m is

$$m = qy = \frac{q(3p+1)}{10}.$$

Case 3. When $p = 10a + 7$ for some integer $a \geq 1$.

In this case, from (4.1.9) and (4.1.10), we have

$$1 = [(k+1)y - x](10a + 7) - 10y = 7\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a + 7) = 10[y - \{(k+1)y - x\}a] - 7[(k+1)y - x].$$

The minimum solution is then obtained from the first equation as follows :

$$(k+1)y - x = 3,\ y - \{(k+1)y - x\}a = 2.$$

Thus, $y = 3a + 2$, and the minimum m is

$$m = qy - 1 = \frac{q(p-1)}{10} - 1.$$

Case 4. When $p = 10a + 9$ for some integer $a \geq 1$.

From the Diophantine equations (4.1.9) and (4.1.10), we have

$$1 = [(k+1)y - x](10a + 9) - 10y = 9\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a + 9) = 10[y - \{(k+1)y - x\}a] - 9[(k+1)y - x].$$

Clearly, the minimum solution is obtained from the second equation, which is

$$(k+1)y - x = 1,\ y - \{(k+1)y - x\}a = 1.$$

This gives the minimum solution $y = a + 1$, and the minimum m is

$$m = qy = \frac{q(p+1)}{10}.$$

All these complete the proof. ■

The following lemma derives an expression of $Z(5.3^k)$ for any integer $k \geq 1$.

Lemma 4.1.11 **:** For any integer $k \geq 1$,

$$Z(5.3^k) = \begin{cases} 3^k, & \text{if } k = 4n+2 \\ 3^k - 1, & \text{if } k = 4(n+1) \\ 2.3^k - 1, & \text{if } k = 4n+1 \\ 2.3^k, & \text{if } k = 4n+3 \end{cases}$$

Proof **:** By definition,

$$Z(5.3^k) = \min\left\{m : 5.3^k \mid \frac{m(m+1)}{2}\right\},$$

where 3^k must divide one of m and $m + 1$.

We first prove that, for any integer $n \geq 1$, 10 divides $3^{2(2n+1)} + 1$. The proof is by induction on n. Clearly, the result is true for $n = 1$, as can easily be verified. So, we assume that the result is true for some integer m. Now, writing

$$3^{2(2n+3)} + 1 = 81(3^{2(2n+1)} + 1) - 80,$$

we see, by virtue of the induction hypothesis, that 10 divides $3^{2(2n+3)} + 1$ as well. This in turn shows that the result is true for $n + 1$, which we intended to prove.

The above analysis shows that $10.3^{2(2n+1)}$ divides $3^{2(2n+1)}[3^{2(2n+1)} + 1]$, and consequently,

$$Z(5.3^{2(2n+1)}) = 3^{2(2n+1)},\ n \geq 1.$$

Next, we show that, 10 divides $3^{4(n+1)} - 1$ for any integer $n \geq 1$. Here also, the proof is by induction on n. The proof for $n = 1$ is straight-forward. So, we assume that the result is true for some integer n. Then, since

$$3^{4(n+2)} - 1 = 81(3^{4(n+1)} - 1) + 80,$$

this, together with the induction hypothesis, shows that the result is true for $n + 1$ as well. Thus, 10 divides $3^{4(n+1)} - 1$, and hence,

$$Z(5.3^{4(n+1)}) = 3^{4(n+1)} - 1,\ n \geq 1.$$

To find $Z(5.3^{4n+1})$, we write $2.3^{4n+1} - 1$ as follows :

$$2.3^{4n+1} - 1 = 6.3^{4n} = 5.3^{4n} + (3^{4n} - 1),$$

which shows that 5 divides $2.3^{4n+1} - 1$. Thus, 10.3^{4n+1} divides $2.3^{4n+1}(2.3^{4n+1} - 1)$, and hence,

$$Z(5.3^{4n+1}) = 2.3^{4n+1} - 1,\ n \geq 1.$$

Finally, since

$$2.3^{4n+3} + 1 = 6.3^{4n+2} + 1 = 5.3^{4n+2} + (3^{4n+2} - 1),$$

it follows that 5 divides $2.3^{4n+3} + 1$. Consequently,

$$Z(5.3^{4n+3}) = 2.3^{4n+3},\ n \geq 1.$$

All these complete the proof of the lemma. ■

It may be mentioned here that, the above result is valid for $n = 0$ as well, since

$$Z(45) = 9,\ Z(405) = 80,\ Z(15) = 5,\ Z(135) = 54.$$

4.2 Miscellaneous Topics

In this section, we consider two Diophantine equations involving Z(n). The first one is the equation $Z(n) + SL(n) = n$, posed by Xin Wu(9), while the second one is $Z(n) + SL(N) = n$. In this section, we also introduce the Diophantine equation $Z(mn) = m^k Z(n)$ for which $m = 1$, $n = 1$, $k = 1$ is the trivial solution, and the problem is to find non-trivial solutions of the equation. The analysis shows that the equation $Z(mn) = m^k Z(n)$ is rather interesting.

Xin Wu(9) considered the Diophantine equation

$$Z(n) + SL(n) = n,$$

where SL(n) is the Smarandache LCM function, defined below.

Definition 4.2.1 **:** The *Smarandache LCM function*, denoted by SL(n), is defined as

$$SL(n) = \min \{k \geq 1 : n \,|\, [1, 2, ..., k]\},$$

where [1, 2, ..., k] is the least common multiple of the integers 1, 2, ..., k.

Then, we have the following result, which gives the expression of SL(n).

Lemma 4.2.1 **:** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ be the representation of the integer n in terms of its r prime factors $p_1, p_2, ..., p_r$. Then,

$$SL(n) = \max \{ p_1^{\alpha_1}, p_2^{\alpha_2}, ..., p_r^{\alpha_r} \}.$$

The solution of the Diophantine equation $Z(n) + SL(n) = n$ is given in Theorem 4.2.1. To prove the theorem, we need the following result, which gives the expression of $Z(2^k p^\alpha)$ for some particular cases.

Lemma 4.2.2 **:** Let n be of the form

$$n = 2^k p^\alpha,$$

where $p \geq 3$ is a prime, and $k \geq 1$ and $\alpha \geq 1$ are integers. Then,

(1) if 2^{k+1} divides $(p^\alpha + 1)$, then $Z(n) = p^\alpha$,
(2) if 2^{k+1} divides $(p^\alpha - 1)$, then $Z(n) = p^\alpha - 1$,
(3) if 2^k divides $(p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$, then $Z(n) = p^\alpha (2^k - 1)$,
(4) if p^α divides $(2^k - 1)$, then $Z(n) = 2^k (p^\alpha - 1)$.

Proof **:** By definition,

$$Z(n) = Z(2^k p^\alpha) = \min \left\{ m : 2^k p^\alpha \,\Big|\, \frac{m(m+1)}{2} \right\}.$$

(1) If 2^{k+1} divides $(p^\alpha + 1)$, then

$$2^{k+1} p^\alpha \text{ divides } p^\alpha (p^\alpha + 1),$$

and consequently, $Z(n) = p^\alpha$.

Now, to prove the remaining cases, we consider the two possibilities that may arise :

Case 1. When 2^{k+1} divides m, and p^α divides $(m + 1)$.
In this case,

$$m = 2^{k+1} x,\ m + 1 = p^\alpha y \text{ for some integers } x \geq 1, y \geq 1.$$

Then, we have the following Diophantine equation :

$$p^\alpha y - 2^{k+1} x = 1. \tag{3}$$

Case 2. When 2^{k+1} divides $(m+1)$, and p^{α} divides m.
Here,

$$m+1=2^{k+1}x,\ m=p^{\alpha}y \text{ for some integers } x\geq 1,\ y\geq 1,$$

resulting in the Diophantine equation

$$2^{k+1}x-p^{\alpha}y=1. \tag{4}$$

(2) Let 2^{k+1} divide $(p^{\alpha}-1)$, so that

$$p^{\alpha}-1=2^{k+1}a \text{ for some integer } a\geq 1.$$

Since, $p^{\alpha}=2^{k+1}a+1$, plugging in the equations (3) and (4), we get

$$(2^{k+1}a+1)y-2^{k+1}x=1,\ 2^{k+1}x-(2^{k+1}a+1)y=1,$$

that is,

$$2^{k+1}(ay-x)+y=1, \tag{5}$$

$$2^{k+1}(x-ay)-y=1. \tag{6}$$

Clearly, the minimum solution is obtained from (5) with

$$y=1,\ ay-x=0.$$

Thus, the minimum m is given by

$$m=p^{\alpha}-1.$$

(3) Let 2^{k} divide $(p^{\alpha}-1)$ but 2^{k+1} does not divide $(p^{\alpha}-1)$. Then,

$$p^{\alpha}-1=2^{k}b \text{ for some integer } b\geq 1,\ b\neq 2.$$

Therefore, the equations (3) and (4) take the forms

$$(2^{k}a+1)y-2^{k+1}x=1,\ 2^{k+1}x-(2^{k}a+1)y=1,$$

that is,

$$2^{k}(ay-2x)+y=1, \tag{7}$$

$$2^{k}(2x-ay)-y=1. \tag{8}$$

Then, (8) gives the minimum solution as follows :

$$2x-ay=1,\ y=2^{k}-1.$$

Thus, the minimum m is

$$m=p^{\alpha}y=p^{\alpha}(2^{k}-1).$$

(4) Let p^{α} divide $(2^{k}-1)$.
Then,

$$2^{k}-1=p^{\alpha}c \text{ for some integer } c\geq 1,$$

and the Diophantine equations (3) and (4) become

$$p^{\alpha}y-2(p^{\alpha}c+1)x=1,\ 2(p^{\alpha}c+1)x-p^{\alpha}y=1,$$

that is,

$$p^{\alpha}(y-2cx)-2x=1, \tag{9}$$

$$p^{\alpha}(2cx-y)+2x=1. \tag{10}$$

The minimum solution, obtained from (9), is

$$y-2cx=1,\ 2x=p^{\alpha}-1.$$

Consequently, the minimum m is

$$m=2^{k+1}x=2^{k}(p^{\alpha}-1).$$

All these complete the proof of the lemma. ■

Theorem 4.2.1 below gives the solution of the Diophantine equation $Z(n) + SL(n) = n$.

***Theorem 4.2.1* :** The Diophantine equation $Z(n) + SL(n) = n$ has the solution

$$n = 2^k p^\alpha,$$

where 2^k and p^α are such that either $p^\alpha \mid (2^k - 1)$, or $2^k \mid (p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$.

***Proof* :** If p^α divides $2^k - 1$, then by part (3) of Lemma 4.2.2, $Z(n) = Z(2^k p^\alpha) = 2^k(p^\alpha - 1)$. Now, since $SL(n) = SL(2^k p^\alpha) = 2^k$, we get

$$Z(n) + SL(n) = n.$$

On the other hand, if 2^k divides $(p^\alpha - 1)$ (but 2^{k+1} does not divide $(p^\alpha - 1)$), then (by part (4) of Lemma 4.2.2) $Z(n) = Z(2^k p^\alpha) = p^\alpha(2^k - 1)$. In this case, $SL(n) = SL(2^k p^\alpha) = p^\alpha$, and hence,

$$Z(n) + SL(n) = n. \blacksquare$$

Theorem 4.2.1 proves that $n = 2^k p^\alpha$ (satisfying either of the two conditions (3) and (4) of Lemma 4.2.2) is a solution of the Diophantine equation $Z(n) + SL(n) = n$. The question is

***Question 4.2.1* :** Are there other solutions of the Diophantine equation $Z(n) + SL(n) = n$, besides those given in Theorem 4.2.1?

It may be mentioned here that, in finding $Z(2^k p^\alpha)$ in Lemma 4.2.2, we have considered only those cases that ate relevant to the solution of the Diophantine equation $Z(n) + SL(n) = n$. In Lemma 4.2.2, there are other cases that remain to be considered. When $\alpha = 1$ in Lemma 4.2.2, we get $Z(p.2^k)$, some of which have already been found (the cases $p = 3, 5, 7, 11, 13, 17, 19, 31$ have been derived in Lemma 4.2.6, Lemma 4.2.7, Lemma 4.2.8, Lemma 4.2.9, Lemma 4.2.10, Lemma 4.2.11, Lemma 4.2.12 and Lemma 4.2.13 in Majumdar(5), while the explicit expression of $Z(23.2^k)$ appears in Lemma 4.1.9 in this book). We have the following open problem.

***Open Problem 4.2.1* :** Find $Z(2^k p^\alpha)$, where $p \geq 3$ is a prime, and $k \geq 1$ and $\alpha \geq 1$ are integers.

Though Lemma 4.2.2 provides only a partial expression for $Z(2^k p^\alpha)$, nevertheless, we can derive some important and interesting results from it. For example, in part (1) of Lemma 4.2.2, let $\alpha = 1$, $p = 2^q - 1$, where $q \geq 2$ is a prime (so that p is a Mercenne prime). Now, choosing $k + 1 = q$, we get the following formula :

$$Z\big(2^{q-1}(2^q - 1)\big) = 2^q - 1, \; q \geq 2 \text{ is a prime.} \tag{4.2.1}$$

In particular, we get the following expressions :

$$Z(2.3) = 3, \; Z(2^2.7) = 7, \; Z(2^4.31) = 31, \; Z(2^6.127) = 127.$$

Again, note that, in part (2) of Lemma 4.2.2 (with $p = 3$, $\alpha = 2$), both 2^2 and 2^3 divide $p^\alpha - 1 = 8$, so that

$$Z(2.3^2) = 8 = Z(2^2.3^2).$$

On the other hand, since 2^3 divides $p^2 - 1 = 3^2 - 1$ but 2^4 does not divide $p^2 - 1 = 8$, it follows by part (3) of Lemma 4.2.2 that

$$Z(2^3.3^2) = 3^2(2^3 - 1) = 63.$$

Finally, to demonstrate the result of part (4) of Lemma 4.2.2, note that 3^2 divides $2^6 - 1$ and 3^3 divides $2^{18} - 1$. Therefore,

$$Z(2^6.3^2) = 512, \; Z(2^{18}.3^3) = 6815744.$$

From part (4) of Lemma 4.2.2, we immediately get the following

***Corollary 4.2.1* :** Let p (≥ 3) be a prime, and let k (> 0) be an integer such that p divides $2^k - 1$. Then,

$$Z(p.2^k) = (p - 1)2^k.$$

In Example 4.2.1, we consider in detail some straight-forward and elementary implications of Corollary 4.2.1.

Example 4.2.1 **:** We consider several applications of Corollary 4.2.1 below.

(1) We first prove the following result : 3 divides $2^{2m} - 1$ for any integer $m \geq 1$.
The proof is by induction on m. Clearly, the result is true for $m = 1$. So, we assume that the result holds for some integer m. Now, writing

$$2^{2(m+1)} - 1 = 4(2^m - 1) + 3,$$

we see, by virtue of the induction hypothesis, that 3 divides $2^{2(m+1)} - 1$, showing that the result is true for $m + 1$ as well. This completes the proof by induction.

Alternatively, since $2^m - 1$, 2^m and $2^m + 1$ are three consecutive integers, it follows that one of $2^m - 1$ and $2^m + 1$ is divisible by 3, and hence, so also is $(2^m - 1)(2^m + 1) = 2^{2m} - 1$.
Having proved the above result, we get, by Corollary 4.2.1,

$$Z(3.2^{2m}) = 2^{2m+1}, \ m \geq 1.$$

For an alternative proof, see Corollary 4.1.2 in Majumdar(5).
Some particular cases of the above result are the following :

$$Z(12) = 8, \ Z(48) = 32, \ Z(192) = 128, \ Z(768) = 512.$$

(2) Before finding an explicit expression form of $Z(5.2^{4m})$, $m \geq 1$, we prove that, 5 divides $2^{4m} - 1$ for any integer $m \geq 1$. This can be proved by induction on m. The case $m = 1$ is trivial. So, we assume that the result is true for some integer m. Then, since

$$2^{4(m+1)} - 1 = 16(2^{4m} - 1) + 15,$$

we find, appealing to the induction hypothesis, that $2^{4(m+1)} - 1$ is divisible by 5. Thus, the result is true for $m + 1$ as well. This proves the desired assertion.
Hence, by Corollary 4.2.1,

$$Z(5.2^{4m}) = 2^{2(2m+1)}, \ m \geq 1.$$

The same result has been established in Lemma 4.2.7 in Majumdar(5).
Using the above result, we get the following values :

$$Z(80) = 64, \ Z(1280) = 1024, \ Z(20480) = 16384.$$

(3) In order to find $Z(7.2^{3m})$, $m \geq 1$, we need the following result : 7 divides $2^{3m} - 1$ for any integer $m \geq 1$. To prove the result by induction on m, we assume its validity for some integer m. Then, since

$$2^{3(m+1)} - 1 = 8(2^{3m} - 1) + 7,$$

we see that $2^{3(m+1)} - 1$ is divisible by 7. This completes induction.
By Corollary 4.2.1,

$$Z(7.2^{3m}) = 3.2^{3m+1}, \ m \geq 1,$$

which matches with the result given in Corollary 4.1.3 in Majumdar(5).
In particular, we get the following values :

$$Z(56) = 48, \ Z(448) = 384, \ Z(3584) = 3072.$$

(4) Since 11 divides $2^5 + 1 = 33$, it follows that 11 divides $(2^5 + 1)(2^5 - 1) = 2^{10} - 1$. It then follows that 11 divides $2^{10m} - 1$ for any integer $m \geq 1$. The proof is by induction on m. Assuming that $2^{10m} - 1$ is divisible by 11 for some integer m, and then writing

$$2^{10(m+1)} - 1 = 2^{10} (2^{10m} - 1) + (2^{10} - 1),$$

we see that 11 divides $2^{10(m+1)} - 1$, which we intended to prove to complete induction.
By Corollary 4.2.1,

$$Z(11.2^{10m}) = 5.2^{10m+1}, \ m \geq 1.$$

See also Lemma 4.2.9 in Majumdar(5) for an alternative proof.

(5) To find an expression of $Z(13.2^{12m})$, $m \geq 1$, we note that 13 divides $2^6 + 1 = 65$, so that 13 divides $(2^6 + 1)(2^6 - 1) = 2^{12} - 1$. We now prove that 13 divides $2^{12m} - 1$ for any integer $m \geq 1$. The proof is by induction on m. So, we assume that $2^{12m} - 1$ is divisible by 13 for some integer m. Then, since

$$2^{12(m+1)} - 1 = 2^{12}(2^{12m} - 1) + (2^{12} - 1),$$

it follows that 13 divides $2^{12(m+1)} - 1$, so that the result holds for $m + 1$ as well, thereby completing induction.

By Corollary 4.2.1,

$$Z(13.2^{12m}) = 3.2^{2(6m+1)},\ m \geq 1.$$

For an alternative proof, see also Lemma 4.2.10 in Majumdar(5).

(6) It is easy to see that 17 divides $2^4 + 1$. Thus, 17 divides $(2^4 + 1)(2^4 - 1) = 2^8 - 1$, and hence, by induction on m, 17 divides $2^{8m} - 1$ for any integer $m \geq 1$. To do so, let $2^{8m} - 1$ be divisible by 17 for some integer m. Then, since

$$2^{8(m+1)} - 1 = 2^8(2^{8m} - 1) + (2^8 - 1),$$

it follows that 17 divides $2^{8(m+1)} - 1$, so that the result holds true for $m + 1$ as well. This completes induction.

By Corollary 4.2.1,

$$Z(17.2^{8m}) = 2^{4(2m+1)},\ m \geq 1.$$

For an alternative proof, see also Lemma 4.2.11 in Majumdar(5).

(7) In order to apply Corollary 4.2.1, for a given prime p, we have to find the (smallest) integer k such that p divides $2^k - 1$. In simple cases, we may find such a k by inspection. In other cases, it may not be simple. In such a case, we may refer to Lemma 4.1.7. For example, consider the prime $p = 19$. Writing $19 = 2\times 9 + 1$, and noting that 19 is of the form $8\ell + 3$, by part (2) of Lemma 4.1.7, 19 divides $2^9 + 1$, and hence, by Corollary 4.1.1, 19 divides $(2^9 + 1)(2^9 - 1) = 2^{18} - 1$. We can then prove by induction on m that 19 divides $2^{18m} - 1$ for any integer $m \geq 1$. Then, in a similar fashion, we can deduce that

$$Z(19.2^{18m}) = 9.2^{18m+1},\ m \geq 1.$$

Example 4.2.1 shows that, the equation

$$Z(p.2^k) = (p - 1)2^k$$

has an infinite number of solutions (both in p and k), which in turn proves that, the Diophantine equation $Z(n) + SL(n) = n$ has an infinite number of solutions. Moreover, we may put the result in Corollary 4.2.1 as follows : The equation

$$Z(p.2^k) = 2^k Z(p) \ (p \geq 3 \text{ is a prime})$$

has an infinite number of solutions.

***Corollary 4.2.2* :** For any prime $q \geq 2$,

$$Z\big(2^q(2^q - 1)\big) = 2^{q+1}(2^{q-1} - 1).$$

***Proof* :** follows readily from Corollary 4.2.1, choosing $k = q$, $p = 2^q - 1$. ■

From Corollary 4.2.2, we get

$$Z(3.2^2) = 8,\ Z(7.2^3) = 48,\ Z(31.2^5) = 960.$$

The problem below is related to the Diophantine equation involving both the functions Z(n) and SL(n).

***Problem 4.2.1* :** Find all the solutions of the Diophantine equation $Z(n) = SL(n)$.

If $p \geq 3$ is a prime and n is such that 2n divides $p + 1$, then (by Lemma 4.2.4 in Majumdar[5]),

$$Z(np) = p = SL(np);$$

and for any prime $p \geq 3$, and n such that 2n divides $p^2 + 1$, (by Lemma 4.2.5 in Majumdar[5]),

$$Z(np^2) = p^2 = SL(np^2).$$

This shows that the Diophantine equation $Z(n) = SL(n)$ possesses solutions. So, the question is : Are there other solutions as well?

In Section 3.2 (in Chapter 3), we have seen that the only solutions of the equation

$$S(mn) = m^k S(n)$$

are **(a)** $(m, n, k) = (1, 1, 1)$, **(b)** $(m, n, k) = (4, 1, 1)$, **(c)** $(m, n, k) = (p, 1, 1)$ (where $p\ (\geq 2)$ is a prime), and **(d)** $(m, n, k) = (2, 2, 1)$. However, the situation is different in case of the equation

$$Z(mn) = m^k Z(n), \tag{4.2.1}$$

where m, n and k are positive integers.

There is nothing to prove if $m = 1$ in (4.2.1). Moreover, we would ignore the trivial solution (corresponding to $m = 1, n = 1, k = 1$). When $n = 1$, (4.2.1) becomes

$$Z(m) = m^k, \tag{4.2.2}$$

which has a solution only when $k = 1$ (since $Z(m) \leq 2m - 1$ for all $m \geq 4$), with the solution $m = 1$. Recall that $Z(m) = m$ has no solution for $m \geq 2$. When $n = 2$ in (4.2.1), the resulting equation $Z(2m) = 3m^k$ has no solution in m and k. However, we have the following result.

Lemma 4.2.3 **:** The equation $Z(mn) = m^k Z(n)$ always has a solution, and has, in fact, an infinite number of solutions.

Proof **:** Each of the nine examples in Example 4.2.1 shows that the equation has an infinite number of solutions when $k = 1$. ■

The (infinite number of) solutions of the equation $Z(mn) = m^k Z(n)$ corresponding to the primes $p = 3, 5, 7, 11, 13, 17, 19$ are given in Example 4.2.1. We found no solution of the equation (4.2.1) when $n = 4$, but for $n = 6$, the corresponding equation is

$$Z(6m) = 3m^k, \tag{4.2.3}$$

which possesses solutions. Appealing to Lemma 4.2.18 in Majumdar[5], we see that the equation (4.2.3) admits solutions when $k = 1$. In such a case, the solutions are given by

$$(m, n, k) = (p, 6, 1), \text{ p is a prime such that 12 divides } p - 5.$$

Thus, for example,

$$Z(6\times5) = 15,\ Z(6\times17) = 51,\ Z(6\times29) = 87,\ Z(6\times41) = 123,\ Z(6\times89) = 267.$$

In this case, by Lemma 4.1.6 (with $p = 3$), the second solution of (4.2.3) (with $k = 1$) is $m = 3^{2t}$:

$$Z(6.3^{2t}) = 3.3^{2t},\ t \geq 1.$$

When $n = 10$, the equation (4.2.2) reads as

$$Z(10m) = 4m^k. \tag{4.2.4}$$

For $k = 1$, by Lemma 4.2.22 in Majumdar[5], $m = p$ is a solution of (4.2.4), where p is a prime such that 20 divides $p + 9$. Thus, for example,

$$Z(110) = 44,\ Z(310) = 124,\ Z(710) = 284,\ Z(1310) = 524,\ Z(1510) = 604.$$

However, Lemma 4.2.7 in Majumdar[5] provides a second solution of the equation (4.2.1), namely, $k = 1$, $m = 2^{s-1}$, $n = 5$, where 4 divides $s - 1$ (that is, $m = 2^{4m}$, $m \geq 1$). Thus, for example,

$$Z(10.2^4) = 4.2^4 = 64,\ Z(10.2^8) = 4.2^8 = 1024.$$

We prove the result below involving $Z(2p^p)$ and $Z(4p^p)$.

***Theorem 4.2.2* :** Let $p \ge 3$ be a prime. Then,

(1) If p is of the form $p = 8k + 1$, then $Z(4p^p) = p^p - 1$,

(2) If p is of the form $p = 8k - 1$, then $Z(4p^p) = p^p$,

(3) If p is of the form $p = 8k + 3$, then $Z(2p^p) = p^p - 1$,

(4) If p is of the form $p = 8k + 5$, then $Z(2p^p) = p^p$.

***Proof* :** To prove part (1) of the theorem, note that, by the Binomial expansion,

$$(8k+1)^{2m+1} = \sum_{i=1}^{2m+1} \binom{2m+1}{i} (8k)^i + 1, \text{ for any integers } m \ge 1, k \ge 1.$$

This shows that 8 divides $(8k + 1)^{2m+1} - 1$ for all integers $m \ge 1$, $k \ge 1$. Hence,

$$Z(4p^p) = p^p - 1.$$

(2) Expanding $(8k - 1)^{2m+1}$ by Binomial expansion, we see that 8 divides $(8k - 1)^{2m+1} + 1$, and so

$$Z(4p^p) = p^p.$$

(3) First note that, for any integers $n \ge 1$ and $m \ge 1$, $(4n - 1)^{2m+1} + 1$ is divisible by 4. Thus,

$$4 \text{ divides } (8k+3)^{2m+1} - 1 = \sum_{i=1}^{2m+1} \binom{2m+1}{i} (8k)^i 3^{2m-i+1} - 1, \text{ for any integers } m \ge 1, k \ge 1.$$

This gives the result desired.

(4) Since $(4n + 1)^{2m+1} - 1$ is divisible by 4 for any integers $n \ge 1$ and $m \ge 1$, we see that

$$4 \text{ divides } (8k+5)^{2m+1} + 1 = \sum_{i=1}^{2m+1} \binom{2m+1}{i} (8k)^i 5^{2m-i+1} + 1, \text{ for any integers } m \ge 1, k \ge 1.$$

All these complete the proof of the theorem. ■

***Open Problem 4.2.2* :** Solve completely the equation $Z(p.2^k) = (p - 1)2^k$.

***Open Problem 4.2.3* :** Solve the equation $Z(mn) = m^k Z(n)$ with $k \ge 2$.

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Chapter 5 Smarandache Number Related Triangles

This chapter is devoted to the Smarandache function related and pseudo Smarandache function related triangles. The concept of the Smarandache function related triangles was introduced by Sastry[1], which was extended by Ashbacher[2] to include the pseudo Smarandache function related triangles as well.

Let ΔABC be a triangle with sides $a = BC$, $b = AC$, $c = AB$. Following Sastry[1], we denote by $T(a, b, c)$ the triangle ΔABC. Let $T(a', b', c')$ be a second triangle with sides of lengths a', b', c'. Recall that the two triangles $T(a, b, c)$ and $T(a', b', c')$ are similar if and only if their three sides are proportional (in any order). Thus, for example, if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \ \text{(or, if } \frac{a}{b'} = \frac{b}{c'} = \frac{c}{a'}, \ \text{ or if } \frac{a}{c'} = \frac{b}{a'} = \frac{c}{b'}\text{)},$$

then the two triangles $T(a, b, c)$ and $T(a', b', c')$ are similar.

The following definition is due to Sastry[1] and Ashbacher[2].

Definition 5.1 **:** Given are two triangles $T(a, b, c)$ and $T(a', b', c')$ (where a, b, c and a', b', c' are all positive integers).

(1) $T(a, b, c)$ and $T(a', b', c')$ are said to be *Smarandache function related* (or, *S–related*) if

$$S(a) = S(a'),\ S(b) = S(b'),\ S(c) = S(c'),$$

(where $S(.)$ is the Smarandache function, defined in Chapter 3);

(2) $T(a, b, c)$ and $T(a', b', c')$ are said to be *pseudo Smarandache function related* (or, *Z–related*) if

$$Z(a) = Z(a'),\ Z(b) = Z(b'),\ Z(c) = Z(c'),$$

(where $Z(.)$ is the pseudo Smarandache function, treated in Chapter 4).

A different way of relating two triangles has been proposed by Sastry[1] : The triangle ΔABC, with angles α, β and γ (α, β and γ being positive integers), is denoted by $T(\alpha, \beta, \gamma)$. Then, we have the following definition, due to Sastry[1] and Ashbacher[2].

Definition 5.2 **:** Given two triangles, $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$, with

$$\alpha + \beta + \gamma = 180 = \alpha' + \beta' + \gamma', \tag{5.1}$$

(1) they are said to be *Smarandache function related* (or, *S–related*) if

$$S(\alpha) = S(\alpha'),\ S(\beta) = S(\beta'),\ S(\gamma) = S(\gamma');$$

(2) they are said to be *pseudo Smarandache function related* (or, *Z–related*) if

$$Z(\alpha) = Z(\alpha'),\ Z(\beta) = Z(\beta'),\ Z(\gamma) = Z(\gamma').$$

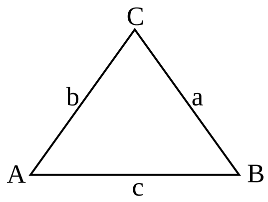
Note that, in Definition 5.1, the sides of the pair of triangles are S–related / Z–related, while their angles, measured in degrees, are S–related / Z–related in Definition 5.2.

Section 5.1 reproduces the results related to the 60-degree and 120-degree triangles, which show that the sides of such a triangle satisfies a Diophantine equation. A set of partial solutions of the Diophantine equation is derived in Section 5.2. Some remarks are given in the final Section 5.3.

5.1 60-Degree and 120-Degree Triangles

Let $T(a, b, c)$ be the triangle with sides a, b and c, and angles $\angle A$, $\angle B$ and $\angle C$, as shown in the figure. Then,

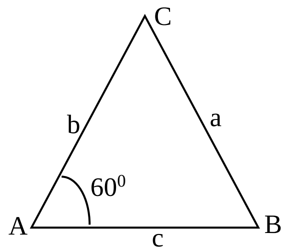
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Of particular interest is the triangle whose one angle is 60^0. The lemma below gives the Diophantine equation satisfied by the sides of such a triangle.

Lemma 5.1.1 **:** Let $T(a, b, c)$ be the triangle with sides a, b and c, whose $\angle A = 60^0$ (as shown in the figure). Then,

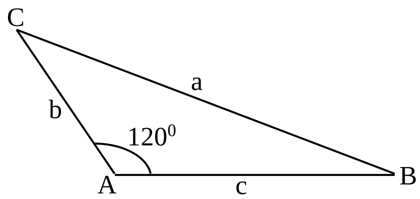
$$4a^2 = (2c - b)^2 + 3b^2. \qquad (5.1.1)$$



The following lemma gives the Diophantine equation satisfied by the sides a, b and c of the triangle whose $\angle A = 120^0$.

Lemma 5.1.2 **:** Let $T(a, b, c)$ be the triangle with sides a, b and c, whose $\angle A = 120^0$ (as shown in the figure). Then,

$$4a^2 = (2c + b)^2 + 3b^2. \qquad (5.1.2)$$



When $\angle A = 60^0$, then a is in between the smallest and the largest sides of the triangle $T(a, b, c)$; and if $\angle A = 120^0$, then a is the largest side of the triangle. Formally, we have

Lemma 5.1.3 **:** If (a_0, b_0, c_0) is a non–trivial solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (5.1.1)$$

then $\min\{b_0, c_0\} < a_0 < \max\{b_0, c_0\}$.

By Lemma 5.1.3, if (a_0, b_0, c_0) is a (non–trivial) solution of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$ (so that a, b, c are all positive and distinct), then, without loss of generality,

$$b_0 > a_0 > c_0.$$

If one solution of the Diophantine equation (5.1.1) is known, then we can find a second independent solution of it. This is given in the following lemma.

Lemma 5.1.4 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (5.1.1)$$

then $(a_0, b_0, b_0 - c_0)$ is also a solution of (5.1.1).

Lemma 5.1.4 shows that the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$ possesses (positive integer) solutions in pairs, namely, (a_0, b_0, c_0) and $(a_0, b_0, b_0 - c_0)$, which are independent. Note that, by symmetry, (a_0, c_0, b_0) and $(a_0, b_0 - c_0, b_0)$ are also solutions of the Diophantine equation.

Next, we confine our attention to the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2.$$

In this case, we have the following result.

Lemma 5.1.5 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2, \tag{5.1.2}$$

then $a_0 > \max\{b_0, c_0\}$.

The following lemma shows how the solutions of (5.1.1) and (5.1.2) are related.

Lemma 5.1.6 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \tag{5.1.1}$$

with $b_0 > c_0$, then $(a_0, b_0 - c_0, c_0)$ is a solution of the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2. \tag{5.1.2}$$

In Section 5.2, partial solutions of the Diophantine equations $a^2 = b^2 + c^2 \pm bc$ are given. Note that, by Lemma 5.1.6, it is sufficient to consider the Diophantine equation $a^2 = b^2 + c^2 - bc$ only.

5.2 Partial Solutions of $a^2 = b^2 + c^2 \pm bc$

First, we consider the Diophantine equation

$$a^2 = b^2 + c^2 - bc, \tag{5.2.1}$$

where, without loss of generality, we may assume that $b > c > 0$. One set of solutions of (5.2.1) is given below.

Proposition 5.2.1 **:** The Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$ has the following solutions :

(1) For $m \geq 1$,

$$a = 3m^2 + 3m + 1,$$

$$b = 3m^2 + 4m + 1 = a + m,$$

$$c = m(3m + 2) = a - m - 1.$$

(2) For $m \geq 2$,

$$a = m^2 + m + 1,$$

$$b = m(m + 2),$$

$$c = m^2 - 1.$$

Proof **:** In the Diophantine equation (5.2.1), substituting

$$b = X + Y,\ c = X - Y, \tag{5.2.2}$$

so that

$$X = \frac{1}{2}(b + c),\ \ Y = \frac{1}{2}(b - c),$$

we get

$$a^2 = (X + Y)^2 + (X - Y)^2 - (X + Y)(X - Y) = X^2 + 3Y^2,$$

that is,

$$(a + X)(a - X) = 3Y^2. \tag{5.2.3}$$

We now consider the following cases that may arise :

Case 1 **:** When $a + X = 3Y^2$, $a - X = 1$.
Here,

$$a = \frac{1}{2}(3Y^2 + 1), \; X = \frac{1}{2}(3Y^2 - 1).$$

Then, Y must be odd, say, $Y = 2m + 1$ for some integer $m \geq 1$. Substituting this, we get

$$a = \frac{1}{2}\left[3(2m+1)^2 + 1\right] = 2(3m^2 + 3m + 1),$$

$$X = \frac{1}{2}\left[3(2m+1)^2 - 1\right] = 6m^2 + 6m + 1,$$

which give

$$b = X + Y = 2(3m^2 + 4m + 1),$$
$$c = X - Y = 2m(3m + 2).$$

We thus get the set of solutions (discarding the common factor 2) mentioned in part (1) of the proposition.

Case 2 **:** When $a + X = Y^2$, $a - X = 3$.
In this case,

$$a = \frac{1}{2}(Y^2 + 3), \; X = \frac{1}{2}(Y^2 - 3).$$

Then, Y must be odd, say, $Y = 2m + 1$ for some integer $m \geq 1$. Substituting this, we get

$$a = \frac{1}{2}\left[(2m+1)^2 + 3\right] = 2(m^2 + 3m + 1),$$

$$X = \frac{1}{2}\left[(2m+1)^2 - 3\right] = 2m^2 + 2m - 1,$$

which give

$$b = X + Y = 2m(m + 2),$$

$$c = X - Y = 2(m^2 - 1).$$

Therefore, corresponding to this case, we get the set of solutions (ignoring the common factor 2), stated in part (2) of the proposition.

Case 3 **:** When $a + X = 3Y$, $a - X = Y$.

In this case, $a = 2Y$, $b = 2Y$, $c = 0$, and hence, we exclude this case.

This completes the proof. ■

Proposition 5.2.1 gives only partial solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$. More solutions are given in Proposition 5.2.2 and Proposition 5.2.3.

Proposition 5.2.2 **:** A second set of solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$ is :

(1) For $m \geq 2$,

$$a = 3m^2 + 1,$$

$$b = 3m^2 + 2m - 1, \; c = 3m^2 - 2m - 1.$$

(2) For $m \geq 4$,

$$a = m^2 + 3,$$

$$b = m^2 + 2m - 3, \; c = m^2 - 2m - 3.$$

***Proof* :** Let, in the Diophantine equation (5.2.3), Y be a multiple of 2, say,

$$Y = 2Z \text{ for some integer } Z \geq 1.$$

Then, (5.2.3) takes the form

$$(a + X)(a - X) = 12Z^2. \tag{5.2.4}$$

We now consider the following two cases :

***Case 1* :** When $a + X = 3Z^2$, $a - X = 4$.
In this case,

$$a = \frac{1}{2}(3Z^2 + 4), \ X = \frac{1}{2}(3Z^2 - 4).$$

Substituting $Z = 2m$, we get

$$a = 2(3m^2 + 1), \ X = 2(3m^2 - 1),$$

and hence,

$$b = X + Y = 2(3m^2 + 2m - 1), \ c = X - Y = 2(3m^2 - 2m - 1).$$

***Case 2* :** When $a + X = Z^2$, $a - X = 12$.
Here,

$$a = \frac{1}{2}(Z^2 + 12), \ X = \frac{1}{2}(Z^2 - 12).$$

We now substitute $Z = 2m$ to get

$$a = 2(m^2 + 3), \ X = 2(m^2 - 3),$$

so that

$$b = X + Y = 2(m^2 + 2m - 3), \ c = X - Y = 2(m^2 - 2m - 3).$$

In each case, disregarding the common factor 2, we get the desired results. ∎

***Proposition 5.2.3* :** The Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$ has the following sets of solutions ($p \geq 3$ being a prime) :

(1) For $m \geq 1$,

$$a = 3(m^2 + m) + \frac{1}{4}(p^2 + 3),$$

$$b = 3m^2 + (p + 3)m - \frac{1}{4}(p^2 - 2p - 3), \ c = 3m^2 - (p - 3)m - \frac{1}{4}(p^2 + 2p - 3).$$

(2) For $m \geq 1$,

$$a = p^2(m^2 + m) + \frac{1}{4}(p^2 + 3),$$

$$b = p^2 m^2 + p(p + 1)m + \frac{1}{4}(p^2 + 2p - 3), \ c = p^2 m^2 + p(p - 1)m + \frac{1}{4}(p^2 - 2p - 3).$$

(3) For $m \geq 1$,

$$a = 3p^2(m^2 + m) + \frac{1}{4}(3p^2 + 1),$$

$$b = 3p^2 m^2 + p(3p + 1)m + \frac{1}{4}(3p^2 + 2p - 1), \ c = 3p^2 m^2 + p(3p - 1)m + \frac{1}{4}(3p^2 - 2p - 1).$$

(4) For $m \geq 1$,

$$a = (m^2 + m) + \frac{1}{4}(3p^2 + 1),$$

$$b = m^2 + (p + 1)m - \frac{1}{4}(3p^2 - 2p - 1), \ c = m^2 - (p - 1)m - \frac{1}{4}(3p^2 + 2p - 1).$$

***Proof* :** Let, in (5.2.3), Y be a multiple of the prime $p \geq 3$, that is,

$Y = pZ$ for some integer $Z \geq 1$.

Then, (5.2.3) becomes

$$(a + X)(a - X) = 3p^2Z^2. \qquad (5.2.5)$$

We now consider the following four cases only, which give different sets of solutions of the equation :

***Case 1* :** When $a + X = 3Z^2$, $a - X = p^2$.
Then,

$$a = \frac{1}{2}(3Z^2 + p^2), \; X = \frac{1}{2}(3Z^2 - p^2).$$

Substituting $Z = 2m + 1$, we get

$$a = 6(m^2 + m) + \frac{p^2 + 3}{2}, \; X = 6(m^2 + m) - \frac{p^2 - 3}{2},$$

and hence,

$$b = X + Y = 6(m^2 + m) - \frac{p^2 - 3}{2} + p(2m + 1) = 2[3m^2 + (p + 3)m] - \frac{p^2 - 2p - 3}{2},$$

$$c = X - Y = 6(m^2 + m) - \frac{p^2 + 3}{2} - p(2m + 1) = 2[3m^2 - (p - 3)m] - \frac{p^2 + 2p - 3}{2}.$$

***Case 2* :** When $a + X = p^2Z^2$, $a - X = 3$.
In this case,

$$a = \frac{1}{2}(p^2 Z^2 + 3), \; X = \frac{1}{2}(p^2 Z^2 - 3).$$

Substituting $Z = 2m + 1$, we get

$$a = 2p^2(m^2 + m) + \frac{p^2+3}{2}, \; X = 2p^2(m^2 + m) + \frac{p^2-3}{2},$$

which, in turn, gives

$$b = X + Y = 2p^2(m^2 + m) + \frac{p^2-3}{2} + p(2m + 1) = 2[p^2 m^2 + p(p + 1)m] + \frac{p^2+2p-3}{2},$$

$$c = X - Y = 2p^2(m^2 + m) + \frac{p^2-3}{2} - p(2m + 1) = 2[p^2 m^2 + p(p - 1)m] + \frac{p^2-2p-3}{2}.$$

***Case 3* :** When $a + X = 3p^2Z^2$, $a - X = 1$.
Here,

$$a = \frac{1}{2}(3p^2 Z^2 + 1), \; X = \frac{1}{2}(3p^2 Z^2 - 1).$$

We substitute $Z = 2m + 1$, to get

$$a = 6p^2(m^2 + m) + \frac{3p^2+1}{2}, \; X = 6p^2(m^2 + m) + \frac{3p^2-1}{2},$$

so that

$$b = X + Y = 6p^2(m^2 + m) + \frac{3p^2-1}{2} + p(2m + 1) = 2[3p^2 m^2 + p(3p + 1)m] + \frac{3p^2+2p-1}{2},$$

$$c = X - Y = 6p^2(m^2 + m) + \frac{3p^2-1}{2} - p(2m + 1) = 2[3p^2 m^2 + p(3p - 1)m] + \frac{3p^2-2p-1}{2}.$$

Case 4 **:** When $a + X = Z^2$, $a - X = 3p^2$.
Here,

$$a = \frac{1}{2}(Z^2 + 3p^2), \ X = \frac{1}{2}(Z^2 - 3p^2).$$

We substitute $Z = 2m + 1$, to get

$$a = 2(m^2 + m) + \frac{3p^2 + 1}{2}, \ X = 2(m^2 + m) - \frac{3p^2 - 1}{2},$$

$$b = X + Y = 2(m^2 + m) - \frac{3p^2 - 1}{2} + p(2m + 1) = 2[m^2 + (p + 1)m] - \frac{3p^2 - 2p - 1}{2},$$

$$c = X - Y = 2(m^2 + m) - \frac{3p^2 - 1}{2} - p(2m + 1) = 2[m^2 - (p - 1)m] - \frac{3p^2 + 2p - 1}{2}.$$

All these complete the proof. ■

Note that, in Proposition 5.2.3, we have made use of the fact that, $p^2 + 3$, $3p^2 + 1$, $p^2 \pm 2p - 3$ and $3p^2 \pm 2p - 1$ are all divisible by 4 when $p \geq 3$ is a prime. The proof is simple : For example, since $p = 2q + 1$ for some $q \geq 1$, $p^2 + 3 = 4(q^2 + q + 1)$, and then writing $3p^2 + 1 = 3(p^2 + 3) - 8$, the proof follows for $p^2 + 3$ and $3p^2 + 1$. The proofs of the other cases are similar.

Some particular cases of Proposition 5.2.3 are given in the following five corollaries, for some particular values of p.

Corollary 5.2.1 **:** Corresponding to $p = 3$, there are two sets of (independent and distinct) solutions of the Diophantine equation (5.2.1), which are given below :

(1) For $m \geq 1$,

$$a = 27m^2 + 27m + 7,$$
$$b = 27m^2 + 30m + 8, \ c = 27m^2 + 24m + 5.$$

(2) For $m \geq 5$,

$$a = m^2 + m + 7,$$
$$b = m^2 + 4m - 5, \ c = m^2 - 2m - 8.$$

Corollary 5.2.2 **:** Corresponding to the prime $p = 5$, the four sets of (independent) solutions of the Diophantine equation (5.2.1) are as follows :

(1) For $m \geq 3$,

$$a = 3m^2 + 3m + 7,$$
$$b = 3m^2 + 8m - 3, \ c = 3m^2 - 2m - 8.$$

(2) For $m \geq 1$,

$$a = 25m^2 + 25m + 7,$$
$$b = 25m^2 + 30m + 8, \ c = 25m^2 + 20m + 3.$$

(3) For $m \geq 1$,

$$a = 75m^2 + 75m + 19,$$
$$b = 75m^2 + 80m + 21, \ c = 75m^2 + 70m + 16.$$

(4) For $m \geq 8$,

$$a = m^2 + m + 19,$$
$$b = m^2 + 6m - 16, \ c = m^2 - 4m - 21.$$

Corollary 5.2.3 **:** When $p = 7$, there are four sets of (independent) solutions of the Diophantine equation (5.2.1), which are as follows :

(1) For $m \ge 4$,

$$a = 3m^2 + 3m + 13,\ b = 3m^2 + 10m - 8,\ c = 3m^2 - 4m - 15.$$

(2) For $m \ge 1$,

$$a = 49m^2 + 49m + 13,\ b = 49m^2 + 56m + 15,\ c = 49m^2 + 42m + 8.$$

(3) For $m \ge 1$,

$$a = 147m^2 + 147m + 37,\ b = 147m^2 + 154m + 40,\ c = 147m^2 + 70m + 33.$$

(4) For $m \ge 11$,

$$a = m^2 + m + 37,\ b = m^2 + 8m - 33,\ c = m^2 - 6m - 40.$$

Corollary 5.2.4 **:** When $p = 11$, the four sets of (independent) solutions of the Diophantine equation (5.2.1) are

(1) For $m \ge 6$,

$$a = 3m^2 + 3m + 31,\ b = 3m^2 + 14m - 24,\ c = 3m^2 - 8m - 35.$$

(2) For $m \ge 1$,

$$a = 121m^2 + 121m + 31,\ b = 121m^2 + 132m + 35,\ c = 121m^2 + 110m + 24.$$

(3) For $m \ge 1$,

$$a = 363m^2 + 363m + 91,\ b = 363m^2 + 374m + 96,\ c = 363m^2 + 352m + 85.$$

(4) For $m \ge 17$,

$$a = m^2 + m + 91,\ b = m^2 + 12m - 85,\ c = m^2 - 10m - 96.$$

Corollary 5.2.5 **:** When $p = 13$, the four sets of (independent) solutions of the Diophantine equation (5.2.1) are given below.

(1) For $m \ge 6$,

$$a = 3m^2 + 3m + 43,$$
$$b = 3m^2 + 16m - 35,\ c = 3m^2 - 10m - 48.$$

(2) For $m \ge 1$,

$$a = 169m^2 + 169m + 43,$$
$$b = 169m^2 + 182m + 48,\ c = 169m^2 + 156m + 35.$$

(3) For $m \ge 1$,

$$a = 507m^2 + 507m + 127,$$
$$b = 507m^2 + 520m + 133,\ c = 507m^2 + 494m + 120.$$

(4) For $m \ge 19$,

$$a = m^2 + m + 127,$$
$$b = m^2 + 14m - 120,\ c = m^2 - 12m - 133.$$

Eight sets of solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$ are given in Proposition 5.2.1 – Proposition 5.2.3, which are supplemented by the solutions, given as special cases, in Corollary 5.2.1 – Corollary 5.2.5. Note that, the solutions found in the three propositions and five corollaries are not exhaustive; moreover, some of the cases are overlapping. For example, when $p = 3$, the solutions given in parts (1) and (2) of Proposition 5.2.3 coincide with those given by Proposition 5.2.1.

***Remark 5.2.1* :** It may be mentioned here that, unrestricting m in Propositions 5.2.1 – 5.2.3, we get the solutions of the unrestricted Diophantine equation $a^2 = b^2 + c^2 - bc$ (with no restriction on the signs of a, b and c), and we have put restrictions on m only to guarantee that a, b, c are all positive (integers). Note that, if $(a_0, -b_0, -c_0)$ is a solution of the Diophantine equation (5.2.1), so also is (a_0, b_0, c_0). We now state and prove the following result.

***Lemma 5.2.1* :** Let $(a_0, b_0, -c_0)$ be a solution of the Diophantine equation

$$a^2 = b^2 + c^2 - bc.$$

Then, $(a_0, b_0 + c_0, c_0)$ is also a solution.

***Proof* :** Since $(a_0, b_0, -c_0)$ be a solution of the Diophantine equation, we have

$$a_0^2 = b_0^2 + c_0^2 + b_0c_0.$$

Now, since

$$(b_0 + c_0)^2 + c_0^2 - (b_0 + c_0)c_0 = b_0^2 + c_0^2 + b_0c_0,$$

we get the desired result. ■

For example, in part (3) of Corollary 5.2.1, $m = 0$ gives the solution (7, –5, –8) of the unrestricted solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$, so that (7, 5, 8) is also a solution; again, $m = 2$ gives the solution (13, 7, –8), so that by Lemma 5.2.1, (13, 15, 8) is also a solution. Again, from part (1) of Corollary 5.2.4, (91, 80, –19) is a solution corresponding to $m = 4$, and hence, by Lemma 5.2.1, so also is (91, 99, 19).

Next, we consider the Diophantine equation

$$a^2 = b^2 + c^2 + bc. \tag{5.2.2}$$

From the solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$, we can find the solution of the Diophantine equation $a^2 = b^2 + c^2 + bc$, using Lemma 5.1.6. Thus, using Proposition 5.2.1, we get the solution of (5.2.2) as follows.

***Proposition 5.2.4* :** The following two sets of solutions are the solutions of the Diophantine equation $a^2 = b^2 + c^2 + bc$:

(1) For $m \geq 1$,

$$a = 3m^2 + 3m + 1,$$
$$b = 2m + 1, \ c = m(3m + 2).$$

(2) For $m \geq 2$,

$$a = m^2 + m + 1,$$
$$b = 2m + 1, \ c = m^2 - 1.$$

Similarly, from Proposition 5.2.2, we get the following result.

***Proposition 5.2.5* :** The Diophantine equation $a^2 = b^2 + c^2 + bc$ has the following solutions :

(1) For $m \geq 2$,

$$a = 3m^2 + 1,$$
$$b = 4m, \ c = 3m^2 - 2m - 1.$$

(2) For $m \geq 4$,

$$a = m^2 + 3,$$
$$b = 4m, \ c = m^2 - 2m - 3.$$

Finally, we have the following result, by virtue of Proposition 5.2.3.

***Proposition 5.2.6* :** The Diophantine equation $a^2 = b^2 + c^2 + bc$ has the following sets of solutions :

(1) For $m \geq 1$,

$$a = 3(m^2 + m) + \frac{1}{4}(p^2 + 3),$$
$$b = p(2m+1),\ c = 3m^2 - (p-3)m - \frac{1}{4}(p^2 + 2p - 3).$$

(2) For $m \geq 1$,

$$a = p^2(m^2 + m) + \frac{1}{4}(p^2 + 3),$$
$$b = p(2m+1),\ c = p^2 m^2 + p(p-1)m + \frac{1}{4}(p^2 - 2p - 3).$$

(3) For $m \geq 1$,

$$a = 3p^2(m^2 + m) + \frac{1}{4}(3p^2 + 1),$$
$$b = p(2m+1),\ c = 3p^2 m^2 + p(3p-1)m + \frac{1}{4}(3p^2 - 2p - 1).$$

(4) For $m \geq 1$,

$$a = (m^2 + m) + \frac{1}{4}(3p^2 + 1),$$
$$b = p(2m+1),\ c = \left[m^2 - (p-1)m\right] - \frac{1}{4}(3p^2 + 2p - 1).$$

Proposition 5.2.4 – Proposition 5.2.6 gives, in total, eight sets of solutions of the Diophantine equation $a^2 = b^2 + c^2 + bc$. The following five corollaries are particular cases of Proposition 5.2.6 for $p = 3, 5, 7, 11, 13$.

***Corollary 5.2.6* :** Corresponding to $p = 3$, the two sets of solutions of $a^2 = b^2 + c^2 + bc$ are

(1) For $m \geq 1$,

$$a = 27m^2 + 27m + 7,$$
$$b = 6m + 38,\ c = 27m^2 + 24m + 5.$$

(2) For $m \geq 5$,

$$a = m^2 + m + 7,$$
$$b = 6m + 3,\ c = m^2 - 2m - 8.$$

***Corollary 5.2.7* :** Corresponding to the prime $p = 5$, the four sets of (independent) solutions of the Diophantine equation (5.2.2) are as follows :

(1) For $m \geq 3$,

$$a = 3m^2 + 3m + 7,$$
$$b = 5(2m+1),\ c = 3m^2 - 2m - 8.$$

(2) For $m \geq 1$,

$$a = 25m^2 + 25m + 7,$$
$$b = 5(2m+1),\ c = 25m^2 + 20m + 3.$$

(3) For $m \geq 1$,

$$a = 75m^2 + 75m + 19,$$
$$b = 5(2m+1),\ c = 75m^2 + 70m + 16.$$

(4) For $m \geq 8$,

$$a = m^2 + m + 19,$$
$$b = 5(2m+1),\ c = m^2 - 4m - 21.$$

Corollary 5.2.8 **:** When p = 7, there are four sets of (independent) solutions of the Diophantine equation (5.2.2), which are as follows :

(1) For $m \geq 4$,

$$a = 3m^2 + 3m + 13,$$
$$b = 7(2m+1),\ c = 3m^2 - 4m - 15.$$

(2) For $m \geq 1$,

$$a = 49m^2 + 49m + 13,$$
$$b = 7(2m+1),\ c = 49m^2 + 42m + 8.$$

(3) For $m \geq 1$,

$$a = 147m^2 + 147m + 37,$$
$$b = 7(12m+1), c = 147m^2 + 70m + 33.$$

(4) For $m \geq 11$,

$$a = m^2 + m + 37,$$
$$b = 7(2m+7),\ c = m^2 - 6m - 40.$$

Corollary 5.2.9 **:** When $p = 11$, the four sets of (independent) solutions of the Diophantine equation (5.2.2) are

(1) For $m \geq 6$,

$$a = 3m^2 + 3m + 31,$$
$$b = 11(2m+1),\ c = 3m^2 - 8m - 35.$$

(2) For $m \geq 1$,

$$a = 121m^2 + 121m + 31,$$
$$b = 11(2m+1),\ c = 121m^2 + 110m + 24.$$

(3) For $m \geq 1$,

$$a = 363m^2 + 363m + 91,$$
$$b = 11(2m+1),\ c = 363m^2 + 352m + 85.$$

(4) For $m \geq 17$,

$$a = m^2 + m + 91,$$
$$b = 11(2m+1),\ c = m^2 - 10m - 96.$$

Corollary 5.2.10 **:** When $p = 13$, the four sets of (independent) solutions of the Diophantine equation (5.2.2) are given below.

(1) For $m \geq 6$,

$$a = 3m^2 + 3m + 43,\ b = 13(2m+1),\ c = 3m^2 - 10m - 48.$$

(2) For $m \geq 1$,

$$a = 169m^2 + 169m + 43,\ b = 13(2m+1),\ c = 169m^2 + 156m + 35.$$

(3) For $m \geq 1$,

$$a = 507m^2 + 507m + 127,\ b = 13(2m+1),\ c = 507m^2 + 494m + 120.$$

(4) For $m \geq 17$,

$$a = m^2 + m + 127,\ b = 13(2m+1),\ c = m^2 - 12m - 133.$$

In total, eight sets of solutions of the Diophantine equation $a^2 = b^2 + c^2 + bc$ are given in Proposition 5.2.4 – Proposition 5.2.6, which follow directly from the corresponding results for the Diophantine equation $a^2 = b^2 + c^2 - bc$, by virtue of Lemma 5.1.6.

5.3 Some Remarks

Note that, if (a_0, b_0, c_0) is a solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$, so also is $k(a_0, b_0, c_0)$ for any integer $k \geq 1$. Thus, in finding the solutions (of any of the two Diophantine equations), it seems reasonable to find the *primitive solution*, which corresponds to the case when a is a prime. Then, any other solution is a constant multiple of one of these solutions. Note that, the Diophantine equations have solutions only for certain primes; restricting a to $0 < a < 100$, we get solutions only for a = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97. As has been mentioned in Majumdar[(4)], a computer search with $0 < a < 100$ revealed that the Diophantine equations have solutions only for these primes. Also note that, for the Diophantine equation $a^2 = b^2 + c^2 - bc$, corresponding to such primes, there are two independent solutions, given in Lemma 5.1.4. Restricting a to $100 < a < 200$, Propositions 5.2.1 – 5.2.3 and Corollaries 5.2.1 – 5.2.5 give solutions only for the primes a = 103, 109, 127, 139, 151, 157, 163, 181, 193, 199. These primitive solutions are given in Table 5.3.1 below. In the table, we also show the solutions when a = 169 and a = 133.

Table 5.3.1 : The primitive solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ $(b > c > 0)$ for $100 < a < 200$

a	b	c	a	b	c	a	b	c
103	117	77	163	187	112	169	195	91
	117	40		187	75		195	104
109	119	95	181	209	104		176	161
	119	24		209	105		176	15
127	133	120	193	207	175	133	152	57
	133	13		207	32		152	95
139	160	91	199	221	165		147	35
	160	69		221	56		147	112
151	171	56					143	120
	171	115					143	23
157	168	143					153	88
	168	25					153	65

Recall from Majumdar[(4)] that, for $a = 49 = 7^2$, there are four independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$; two are $7(7, 8, 5) = (49, 56, 35)$ and $7(7, 8, 3) = (49, 56, 21)$, and the third one is (49, 55, 39), given by part (1) of Proposition 5.2.2. And when a = 91, there are eight independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$. Since $91 = 7\times13$, four solutions are

$$13(7, 8, 5) = (91, 104, 65),\ 13(7, 8, 3) = (91, 104, 39),$$

$$7(13, 15, 8) = (91, 105, 56),\ 7(13, 15, 7) = (91, 105, 49).$$

The fifth one is (91, 96, 85), given by part (1) of Proposition 5.2.1 (with (91, 96, 11) as the sixth solution), and the seventh solution is (91, 99, 80), which can be obtained from part (2) of Proposition 5.2.1. From Table 5.3.1, we see that, when $a = 169 = 13^2$, there are four independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$; two are

$$13(13, 15, 8) = (169, 195, 104),\ 13(13, 15, 7) = (169, 195, 71),$$

and the third one is (169, 176, 161), which may be obtained from part (1) of Corollary 5.2.1. And when $a = 133 = 7\times19$, there are eight independent solutions of the Diophantine equation.

Thus, we see that, in certain cases, the Diophantine equation (5.2.1) has more than two independent solutions. To find these solutions, we start with the Diophantine equation (5.1.1) in the form

$$4a^2 = (2b - c)^2 + 3c^2, \tag{5.1.1}$$

and consider the Diophantine equation

$$x^2 = y^2 + 3z^2, \tag{5.3.1}$$

where

$$x = 2a,\ y = 2b - c,\ z = c. \tag{5.3.2}$$

When $b < c$, we may still consider the above equation with the roles of b and c interchanged.

For example, $x = 7$, $y = 1$, $z = 4$ is a solution of (5.3.1), and corresponding to $x = 7$, it is the only solution; however, when $x = 14 = 2\times7$, there are three (independent) solutions, as shown below :

$$14^2 = 13^2 + 3.3^2 = 11^2 + 3.5^2 = 2^2 + 3.8^2. \tag{1}$$

Again,

$$13^2 = 11^2 + 3.4^2,$$

but

$$26^2 = 23^2 + 3.7^2 = 22^2 + 3.8^2 = 1^2 + 3.15^2. \tag{2}$$

Finally,

$$49^2 = 47^2 + 3.8^2,$$

but

$$98^2 = 71^2 + 3.39^2 = 23^2 + 3.55^2 = 94^2 + 3.16^2. \tag{3}$$

The above examples show that the Diophantine equation (5.1.1) and the Diophantine equation (5.3.1) are not equivalent; unlike (5.3.1), (5.1.1) admits only two independent solutions.

Given a solution (x_0, y_0, z_0) of the Diophantine equation (5.3.1), we can find the corresponding solution of the Diophantine equation (5.2.1), using the lemma below.

***Lemma 5.3.1* :** Let (x_0, y_0, z_0) be a solution of the Diophantine equation

$$x^2 = y^2 + 3z^2. \tag{5.3.1}$$

Then, (a_0, b_0, c_0) is a solution of the Diophantine equation

$$a^2 = b^2 + c^2 - bc, \tag{5.2.1}$$

where

$$a_0 = \frac{x_0}{2},\ b_0 = \frac{1}{2}(y_0 + z_0),\ c_0 = z_0.$$

***Proof* :** follows from (5.3.2), noting that $y_0 = 2b_0 - c_0 > 0$ when $b_0 > c_0$. ■

In applying Lemma 5.3.1 when $c_0 > b_0$, we have to interchange the roles of b_0 and c_0.

Now, given a solution (x_0, y_0, z_0) of the Diophantine equation (5.3.1), we can find a solution of it when $x = x_0^2$. This is given in the following lemma.

***Lemma 5.3.2* :** Let (x_0, y_0, z_0) be a solution of the Diophantine equation

$$x^2 = y^2 + 3z^2. \tag{5.3.1}$$

Then, $\left(x_0^2, |y_0^2 - 3z_0^2|, 2y_0 z_0\right)$ is also a solution of (5.3.1).

***Proof* :** We write $(y^2 + 3z^2)^2$ as follows :

$$(y^2 + 3z^2)^2 = (y^2 - 3z^2)^2 + 12y^2 z^2 = (y^2 - 3z^2)^2 + 3(2yz)^2,$$

that is,

$$(x^2)^2 = (y^2 - 3z^2)^2 + 3(2yz)^2.$$

Hence, the lemma follows. ■

If (x_0, y_0, z_0) is a solution of $x^2 = y^2 + 3z^2$, then obviously $(x_0^2, x_0 y_0, x_0 z_0)$ is also its one solution; Lemma 5.3.2 gives another independent solution of $x^2 = y^2 + 3z^2$. For example, starting with the solution, given in (1), we get, by virtue of Lemma 5.3.2, the following three solutions when $x = 14^2$:

$$x_1 = 14^2,\ y_1 = 142,\ z_1 = 78,$$

$$x_1 = 14^2,\ y_1 = 46,\ z_1 = 110,$$

$$x_1 = 14^2,\ y_1 = 188,\ z_1 = 32,$$

the last one giving the solution

$$x_1 = 7^2,\ y_1 = 47,\ z_1 = 8.$$

Thus, corresponding to $x = 14^2$, there are, in total, six solutions of the Diophantine equation (5.3.1)

Again, from the three solutions corresponding to $x = 26$ (given by (2)), by Lemma 5.3.2, we get the following three solutions when $x = 26^2$:

$$x_1 = 26^2,\ y_1 = 382,\ z_1 = 322,$$

$$x_1 = 26^2,\ y_1 = 292,\ z_1 = 352,$$

$$x_1 = 26^2,\ y_1 = 674,\ z_1 = 30,$$

the second one giving the solution

$$x_1 = 13^2,\ y_1 = 73,\ z_1 = 88.$$

***Proposition 5.3.1* :** Let (a_0, b_0, c_0) be a solution of the Diophantine equation

$$a^2 = b^2 + c^2 - bc. \tag{5.2.1}$$

Then, $\left(a_0^2, b_0^2 - c_0^2, c_0(2b_0 - c_0)\right)$ is also a solution of (5.2.1).

***Proof* :** By Lemma 5.3.1,

$$a_0 = \frac{x_0}{2},\ b_0 = \frac{1}{2}(y_0 + z_0),\ c_0 = z_0.$$

Now, by Lemma 5.3.2, if (a_1, b_1, c_1) is a solution of (5.2.1) corresponding to $a = a_1 = a_0^2$, then

$$a_1 = \frac{x_0^2}{2},\ 2b_1 - c_1 = \left|y_0^2 - 3z_0^2\right|,\ c_1 = 2y_0 z_0.$$

Therefore, if $y_0^2 - 3z_0^2 > 0$, then

$$2b_1 = 2y_0 z_0 + (y_0^2 - 3z_0^2) = (z_0 + y_0)^2 - 4z_0^2 = 4(b_0^2 - c_0^2),$$

so that

$$b_1 = 2(b_0^2 - c_0^2);$$

also,

$$a_1 = 2a_0^2,\ c_1 = 2c_0(2b_0 - c_0).$$

Now, disregarding the common factor 2, the result follows. ■

If the solution (x_0, y_0, z_0) of the Diophantine equation (5.3.1) is known, then using Lemma 5.3.1, we may find the corresponding solution of the Diophantine equation (5.2.1). And if a solution (a_0, b_0, c_0) of the Diophantine equation (5.2.1) is known, Proposition 5.3.1 may be exploited to find the solution of (5.3.1) corresponding to $a = a_0^2$. For example, from the first two solutions given in (1), we get (by Lemma 5.3.1)

$$a_0 = 7, b_0 = 8, c_0 = 3, \tag{4}$$

$$a_0 = 7, b_0 = 8, c_0 = 5, \tag{5}$$

while the third one in (1) gives, after interchanging the roles of b_0 and c_0,

$$a_0 = 7, b_0 = 3, c_0 = 8.$$

Now, applying Proposition 5.3.1 to the solution (4), we get

$$a_1 = 7^2, b_1 = 55, c_1 = 39, \tag{6}$$

while (5) gives

$$a_1 = 7^2, b_1 = 39, c_1 = 55,$$

which is just the solution (6) with the roles of b_1 and c_1 interchanged.

Thus, from the three solutions of the Diophantine equation (5.3.1), given in (1), we get only one (distinct) solution of the Diophantine equation (5.2.1), by applying Proposition 5.3.1.

Again, from the first two solutions given in (2), we get

$$a_0 = 13, b_0 = 15, c_0 = 7, \tag{7}$$

$$a_0 = 13, b_0 = 15, c_0 = 8, \tag{8}$$

which give

$$a_1 = 13^2, b_1 = 176, c_1 = 161, \tag{9}$$

$$a_1 = 13^2, b_1 = 161, c_1 = 176.$$

Thus, applying Proposition 5.3.1 to the solution (2), we get only one solution of the Diophantine equation (5.2.1) corresponding to $a = 13^2$.

The lemma below considers the case when two independent solutions of the Diophantine equation (5.3.1) are known.

Lemma 5.3.3 **:** Let (X, Y, Z) and (A, B, C) be two independent solutions of the Diophantine equation

$$x^2 = y^2 + 3z^2. \tag{5.3.1}$$

Then, $(XA, |BY - 3CZ|, BZ + CY)$ and $(XA, BY + 3CZ, |BZ - CY|)$ are also solutions of (5.3.1).

Proof **:** Note that $X^2A^2 = (Y^2 + 3Z^2)(B^2 + 3C^2)$ can be expressed in two ways as follows :

$$(Y^2 + 3Z^2)(B^2 + 3C^2) = (BY - 3CZ)^2 + 3(BZ + CY)^2 = (BY + 3CZ)^2 + 3(BZ - CY)^2.$$

Thus, we get the desired result. ■

If two independent solutions of $x^2 = y^2 + 3z^2$ are known, Lemma 5.3.3 enables us to find two more. For example, from the solutions given in (1) and (2), by Lemma 5.3.2, we get the six independent solutions, given below :

$$(14\times26)^2 = 236^2 + 3.160^2 = 362^2 + 3.22^2$$
$$= 148^2 + 3.192^2 = 358^2 + 3.38^2$$
$$= 122^2 + 3.198^2 = 214^2 + 3.170^2.$$

Thus, corresponding to $x = 7\times13$, there are only two independent solutions of (5.3.1), namely,

$$(7\times13)^2 = 59^2 + 3.40^2 = 37^2 + 3.48^2.$$

***Proposition 5.3.2* :** Let (a_0, b_0, c_0) and (a_1, b_1, c_1) be two independent solutions of the Diophantine equation

$$a^2 = b^2 + c^2 - bc. \tag{5.2.1}$$

Then, $\left(a_0 a_1,\ b_0 b_1 - c_0 c_1,\ c_0(b_1 - c_1) + b_0 c_1\right)$ and $\left(a_0 a_1,\ b_0(b_1 - c_1) + c_0 c_1,\ b_1 c_0 - b_0 c_1\right)$ are also solutions of (5.2.1), where in the latter case, $b_1 c_0 - b_0 c_1 > 0$; if $b_1 c_0 - b_0 c_1 < 0$, interchange the roles of the first and the second solutions.

***Proof* :** By assumption, we have, in view of Lemma 5.3.1,

$$a_0 = \frac{X}{2},\ b_0 = \frac{1}{2}(Y+Z),\ c_0 = Z,$$

$$a_1 = \frac{A}{2},\ b_1 = \frac{1}{2}(B+C),\ c_1 = C.$$

Now, by Lemma 5.3.2,

$$a_2 = \frac{XA}{2},\ b_2 = \frac{1}{2}\left[(BY-3CZ)+(BZ+CY)\right],\ c_2 = BZ+CY,$$

$$a_3 = \frac{XA}{2},\ b_3 = \frac{1}{2}\left[(BY+3CZ)+(BZ-CY)\right],\ c_3 = BZ-CY.$$

We now simplify as follows :

$$a_2 = 2a_0 a_1,\ c_2 = (2b_1 - c_1)c_0 + (2b_0 - c_0)c_1 = 2(b_0 c_1 + b_1 c_0 - c_0 c_1),$$

$$BY - 3CZ = (2b_1 - c_1)(2b_0 - c_0) - 3c_1 c_0 = 2(2b_0 b_1 - b_0 c_1 - b_1 c_0 - c_0 c_1),$$

so that

$$b_2 = (2b_0 b_1 - b_0 c_1 - b_1 c_0 - c_0 c_1) + (b_0 c_1 + b_1 c_0 - c_0 c_1) = 2(b_0 b_1 - c_0 c_1).$$

Again,

$$a_3 = 2a_0 a_1,\ c_3 = (2b_1 - c_1)c_0 - (2b_0 - c_0)c_1 = 2(b_1 c_0 - b_0 c_1),$$

$$BY + 3CZ = (2b_1 - c_1)(2b_0 - c_0) + 3c_1 c_0 = 2(2b_0 b_1 + 2c_0 c_1 - b_0 c_1 - b_1 c_0),$$

so that

$$b_3 = (2b_0 b_1 + 2c_0 c_1 - b_0 c_1 - b_1 c_0) + (b_1 c_0 - b_0 c_1) = 2(b_0 b_1 + c_0 c_1 - b_0 c_1).$$

Now, ignoring the common factor 2, we get the desired result. ■

If (a_0, b_0, c_0) and (a_1, b_1, c_1) are two independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$, then obviously $(a_0 a_1, b_0 a_1, c_0 a_1)$ and $(a_0 a_1, a_0 b_1, a_0 c_1)$ are its two independent solutions. Proposition 5.3.2 gives two more (independent) solutions corresponding to $a = a_0 a_1$. For example, the two solutions (5) and (7) together gives, by virtue of Proposition 5.3.2, the two independent solutions

$$a_2 = 7\times 13,\ b_2 = 85,\ c_2 = 96,$$

$$a_2 = 7\times 13,\ b_2 = 99,\ c_2 = 19.$$

If we want to apply the second part of Proposition 5.3.2, we see that, with the two solutions (4) and (7) (in this order), $c_3 = -11 < 0$. Thus, we rewrite them as

$$a_0 = 13,\ b_0 = 15,\ c_0 = 7,$$

$$a_1 = 7,\ b_1 = 8,\ c_1 = 3,$$

and then, by Proposition 5.3.2,

$$a_2 = 7\times 13,\ b_2 = 99,\ c_2 = 80,$$

$$a_2 = 7\times 13,\ b_2 = 96,\ c_2 = 11.$$

It can be seen easily that the two solutions obtained from (5) and (8) are not distinct from the two solutions obtained above. Thus, the four independent solutions of the Diophantine equation (5.2.1), obtained by the application of Proposition 5.3.2, are

$$a_2 = 7\times 13,\ b_2 = 96,\ c_2 = 85,$$

$$a_2 = 7\times 13,\ b_2 = 96,\ c_2 = 11,$$

$$a_2 = 7\times 13,\ b_2 = 99,\ c_2 = 80,$$

$$a_2 = 7\times 13,\ b_2 = 96,\ c_2 = 19.$$

This explains why the Diophantine equation (5.2.1) possesses eight independent solutions corresponding to $a = 7\times 13$.

The primitive solutions of the Diophantine equation $a^2 = b^2 + c^2 + bc$ $(100 < a < 200)$, obtained from those of $a^2 = b^2 + c^2 - bc$ by applying Lemma 5.1.6, are given in Table 5.3.2.

Table 5.3.2 **:** The positive primitive solutions of the Diophantine equation $a^2 = b^2 + c^2 + bc$ for $100 \le a \le 200$

a	b	c	a	b	c	a	b	c
103	40	77	163	75	112	169	91	104
109	24	95	181	104	105		15	161
127	13	120	193	32	175	133	57	95
139	69	91	199	56	165		35	112
151	56	115					23	120
157	25	143					65	88

We conclude the chapter with the following questions.

Question 5.3.1 **:** Restricting a to the primes on the range $100 < a < 200$, are the solutions (of the Diophantine equation $a^2 = b^2 + c^2 - bc$) given in Table 5.3.1 exhaustive?

We conjecture that the solutions in Table 5.3.1 list all when a is a prime with $100 < a < 200$. Thus, restricting a to the primes on $1 < a < 100$, there are 11 solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$, and the number is 10 when $100 < a < 200$.

We have already seen that, for certain primes p, the Diophantine equation (5.2.1) possesses two (distinct) independent solutions when $a = p$, given by (p, b_0, c_0) and $(p, b_0, b_0 - c_0)$ (assuming that $b_0 > c_0$). Proposition 5.3.1 shows that, corresponding to $a = p^2$, there are four independent solutions of (5.2.1). Thus, for example, when $p = 7$, there are two independent solutions, and for $p = 7^2$, there are four independent solutions, namely, $(7^2, 55, 39)$, $(7^2, 55, 16)$, $(7^2, 56, 35)$ and $(7^2, 56, 21)$. So, the question arises :

Question 5.3.2 **:** Let the Diophantine equation $a^2 = b^2 + c^2 - bc$ possess a solution when $a = p$. How many solutions are there when $a = p^3$? $a = p^4$?

Corresponding to $a = 7^3$, there are six independent solutions of the Diophantine equation (5.2.1), namely, $(7^3, 360, 323)$, $(7^3, 360, 37)$, $(7^3, 385, 273)$, $(7^3, 385, 112)$, $(7^3, 392, 245)$ and $(7^3, 392, 147)$, and corresponding to $a = 7^4$, there are eight independent solutions. These are $(7^4, 2769, 1504)$, $(7^4, 2769, 1265)$, $(7^4, 2695, 1911)$, $(7^4, 2695, 784)$, $(7^4, 2744, 1715)$, $(7^4, 2744, 1715)$, $(7^4, 2520, 2261)$, and $(7^4, 2520, 259)$.

Question 5.3.3 **:** How many solutions are there of the Diophantine equation $a^2 = b^2 + c^2 - bc$ when $a = p \times q \times r$ (where p, q and r are distinct primes, and (5.2.1) has solutions in all the three cases corresponding to each of $a = p$, $a = q$ and $a = r$)?

We find that, corresponding to $a = 7\times13\times19$, there are as many as 26 independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$. These are as follows (in increasing values of b) :

(7×13×19, 1775, 1679), (7×13×19, 1775, 96), (7×13×19, **1824, 1615**),

(7×13×19, **1824, 209**), (7×13×19, 1840, 1591), (7×13×19, 1840, 249),

(7×13×19, **1859, 1560**), (7×13×19, **1859, 299**), (7×13×19, **1881, 1520**),

(7×13×19, **1881, 361**), (7×13×19, 1911, 1456), (7×13×19, 1911, 455),

(7×13×19, **1925, 1421**), (7×13×19, **1925, 504**), (7×13×19, **1960, 1309**),

(7×13×19, **1960, 651**), (7×13×19, 1961, 1305), (7×13×19, 1961, 656),

(7×13×19, 1976, 1235), (7×13×19, 1976, 741), (7×13×19, 1984, 1185),

(7×13×19, 1984, 799), (7×13×19, **1989, 1144**), (7×13×19, **1989, 845**),

(7×13×19, 1995, 1064) and (7×13×19, 1995, 931).

These values have been calculated by repeated application of Proposition 5.3.2, considering, in succession, $a_0 = 7\times13$, $a_1 = 19$; $a_0 = 7\times19$, $a_1 = 13$; and $a_0 = 13\times19$, $a_1 = 7$. For example, to apply Proposition 5.3.2 when $a_0 = 7\times13$, $b_0 = 96$; $c_0 = 11$ and $a_1 = 19$; $b_1 = 21$, $c_1 = 5$, we rewrite them as follows (to avoid negative signs) :

$a_0 = 19$; $b_0 = 21$, $c_0 = 5$,

$a_1 = 7\times13$, $b_b = 96$; $c_1 = 11$.

Now, applying Proposition 5.3.1, we get two additional solutions of the the Diophantine equation $a^2 = b^2 + c^2 - bc$, namely, (7×13×19, 1961, 656) and (7×13×19, 1840, 249).

The values shown bold are particular cases, not shared by the other two cases.

Question 5.3.4 **:** What is the general solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$?

The above problem is a challenging one, particularly since it admits different number of solutions depending on the value of a.

Question 5.3.5 **:** What is the general solution of the Diophantine equation $x^2 = y^2 + 3z^2$?

We have considered the Diophantine equation $x^2 = y^2 + 3z^2$ in connection with the Diophantine equation (5.1.1), but the Diophantine equation $x^2 = y^2 + 3z^2$ deserves closer study by its own right. The examples in the equations (1), (2) and (3) show that, if (x_0, y_0, z_0) is a solution of the Diophantine equation $x^2 = y^2 + 3z^2$, then it possesses three (independent) solutions corresponding to $x = 2x_0$. In fact, we can prove the following result.

Lemma 5.3.4 **:** Let (x_0, y_0, z_0) be a solution of the Diophantine equation

$$x^2 = y^2 + 3z^2. \tag{5.3.1}$$

Then,

(1) corresponding to $x = 2x_0$, there are three independent solutions of (5.3.1), namely,

$$4x_0^2 = (2y_0)^2 + 3(2z_0)^2 = (y_0 + 3z_0)^2 + 3(y_0 - z_0)^2 = (y_0 - 3z_0)^2 + 3(y_0 + z_0)^2,$$

(2) corresponding to $x = 4x_0$, the solution of (5.3.1) is two times one of the three solutions of part (1) above.

***Proof* :** By assumption,

$$x_0^2 = y_0^2 + 3z_0^2.$$

Now, since

$$4x_0^2 = (2y_0)^2 + 3(2z_0)^2 = (y_0 + 3z_0)^2 + 3(y_0 - z_0)^2 = (y_0 - 3z_0)^2 + 3(y_0 + z_0)^2,$$

part (1) of the lemma follows.

To prove part (2), let

$$4x^2 = Y^2 + 3Z^2,$$

where, by part (1), Y and Z satisfy one of the following three conditions :

(i) $Y = 2y$, $Z = 2z$,

(ii) $Y = y + 3z$, $Z = |y - z|$,

(iii) $Y = |y - 3z|$, $Z = y + z$.

Now, let

$$16x^2 = A^2 + 3B^2.$$

If $A = 2Y$, $B = 2Z$, there is nothing to prove. So, we need to consider the following two cases :

Case 1. When $A = Y + 3Z$, $B = |Y - Z|$.

If $Y = 2y$, $Z = 2z$, then

$$A = 2(y + 3z),\ B = 2|y - z|.$$

On the other hand, if $Y = y + 3z$, $Z = |y - z|$, there are two possibilities : If $y > z$, then

$$A = (y + 3z) + 3(y - z) = 4y,\ B = |(y + 3z) - (y - z)| = 4z,$$

and if $y < z$, then

$$A = |(y + 3z) + 3(z - y) = 2|y - 3z|,\ B = |(y + 3z) - (z - y)| = 2(y + z).$$

Again, if $Y = |y - 3z|$, $Z = y + z$, then

$$A = (y - 3z) + 3(y + z) = 4y,\ B = |(y - 3z) - (y + z)| = 4z,$$

if $y > 3z$, and if $y < 3z$, then

$$A = (3z - y) + 3(y + z) = 2(y + 3z),\ B = |(3z - y) - (y + z)| = 2|y - z|.$$

Case 2. When $A = |Y - 3Z|$, $B = Y + Z$.

If $Y = 2y$, $Z = 2z$, then

$$A = 2|y - 3z|,\ B = 2(y + z).$$

If $Y = y + 3z$, $Z = |y - z|$, then

$$A = |(y + 3z) - 3(y - z)| = 2|y - 3z|,\ B = (y + 3z) + (y - z) = 2(y + z),$$

if $y > z$, and if $y < z$, then

$$A = |(y + 3z) - 3(z - y) = 4y,\ B = (y + 3z) + (z - y) = 4z$$

Finally, if $Y = |y - 3z|$, $Z = y + z$, then

$$A = |(y - 3z) - 3(y + z)| = 2(y + 3z),\ B = (y - 3z) + (y + z) = 2(y + z),$$

if $y > 3z$, and if $y < 3z$, then

$$A = |(3z - y) - 3(y + z)| = 4y,\ B = |(3z - y) + (y + z)| = 4z.$$

All these complete the proof of the lemma. ■

By Lemma 5.3.4, from (1), we have

$$28^2 = 26^2 + 3.6^2 = 22^2 + 3.10^2 = 4^2 + 3.16^2.$$

In Chapter 3 in Majumdar[4], it has been shown that, the study of the 60-degree and 120-degree triangles involves the two Diophantine equations $a^2 = b^2 + c^2 \pm bc$. Of particular interest is the Diophantine equation $a^2 = b^2 + c^2 - bc$ (which arises for the 60-degree triangles). It is indeed interesting to find that the solution of the Diophantine equations $a^2 = b^2 + c^2 + bc$ can be obtained from the solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$. It has been found that the solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ have interesting features, some of which are given in Proposition 5.3.1 and Proposition 5.3.2. It is also interesting to find that, in general, we get the solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$ in pairs corresponding to particular value of a. This chapter gives partial solutions of the Diophantine equation

$$a^2 = b^2 + c^2 - bc,$$

where the sides a, b, c of the triangle T(a, b, c) are all unequal. Note that, any equilateral triangle is necessarily a 60-degree triangle. Then, considering the three dissimilar triangles

$$T(7p, 3p, 8p), T(7p, 5p, 8p) \text{ and } T(p, p, p),$$

where $p \geq 11$ is a prime, we have a triplet of triangles, any two of which are S-related. And if we wish, we may enlarge the list; for example, the dissimilar triangles

$$T(49p, 55p, 16p), T(49p, 55p, 39p), T(49p, 56p, 21p), T(49p, 56p, 35p),$$

are pair-wise S-related for any prime $p \geq 17$.

When p is a prime of the form $p = 30n - 1$, $n \geq 1$, the two dissimilar triangles T(7p, 3p, 8p) and T(7p, 5p, 8p) are Z-related. Is it possible to find three (or, more) dissimilar 60-degree and 120-degree triangles which are pair-wise Z-related?

Another problem of interest is the the study of similar S-related or Z-related triangles which are 60-degree or 120-degree or Pythagorean. Trivially, for any prime $p \geq 17$, the 60-degree similar triangles

$$T(p, p, p), T(2p, 2p, 2p), \ldots, T(10p, 10p, 10p),$$

are S-related. Again, the 120-degree triangles $\{T(7), T(3), T(5)\}$ and $\{T(14), T(6), T(10)\}$ are similar and S-related. An example of a pair of 60-degree similar triangles which are Z-related is $\{T(31), T(11), T(15)\}$ and $\{T(155), T(65), T(105)\}$, since

$$Z(31) = 30 = Z(155), Z(11) = 10 = Z(65), Z(35) = 14 = Z(105).$$

It remains open to study in more detail the problems on similar S-related/Z-related triangles.

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Chapter 6 Miscellaneous Topics

In this chapter, we consider some topics related to Smarandache notions. The topics covered are (1) the triangular numbers and the Smarandache T-sequence, (2) the Smarandache friendly numbers, (3) the Smarandache reciprocal partition sets, (4) the Smarandache LCM ratio of two types and (5) the Sandor-Smarandache function. They are treated in Section 6.1, Section 6.2, Section 6.3, Section 6.4 and Section 6.5 respectively.

6.1 Triangular Numbers and Smarandache T-Sequence

The n-th *triangular number*, denoted by T_n, is defined as follows :

$$T_n = \frac{n(n+1)}{2}, \quad n \geq 1.$$

The first few triangular numbers are

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, …,

and may be obtained recursively from the recurrence relation

$$T_{n+1} = T_n + (n+1),\ n \geq 1;\ T_1 = 1.$$

Many interesting properties of the triangular numbers are well-known, some of which are given in Beiler[1] and Pickover[2].

We start with the following results (see, for example, Hardy and Wright[3]), related to a particular type of *Pell's equation*.

***Lemma 6.1.1* :** The solutions of the Diophantine (Pell's) equation $x^2 - 2y^2 = 1$ are given by

$$x_n + \sqrt{2}\, y_n = \left(1+\sqrt{2}\right)^{2n}, n \geq 1,$$

or, recursively by

$$x_{n+1} = 3x_n + 4y_n,\ y_{n+1} = 2x_n + 3y_n,\ n \geq 1;$$

$$x_1 = 3,\ y_1 = 2.$$

***Corollary 6.1.1* :** For $n \geq 1$, x_n is odd and y_n is even.

***Proof* :** is by induction on n, and is left to the reader. ■

We now state and prove the following result.

***Theorem 6.1.1* :** An infinite number of terms of $\{T_n\}_{n=1}^{\infty}$ are perfect squares.

***Proof* :** Let

$$\frac{n(n+1)}{2} = m^2 \quad \text{for some integers } n \geq 1,\ m \geq 1. \tag{1}$$

Substituting

$$n = X - \frac{1}{2}, \tag{2}$$

(1) takes the form

$$4X^2 - 8m^2 = 1.$$

Now, letting $x = 2X$, $y = 2m$, we get the Diophantine equation $x^2 - 2y^2 = 1$, which possesses an infinite number of solutions, given by Lemma 6.1.1. Also, since

$$n = \frac{1}{2}(x-1),\ m = \frac{y}{2},$$

by Corollary 6.1.1, both n and m are (positive) integers. Thus, the theorem is established. ■

Theorem 6.1.1 proves that there is an infinite number of perfect squares in the sequence of triangular numbers. It is interesting to observe that, the first perfect square in the sequence of triangular numbers is T_1, the next one is T_8, the third one is T_{49}, but the fourth one is T_{288}.

Next, we prove the result below.

Theorem 6.1.2 **:** Any triangular number T_n, $n \geq 2$, can be expressed as the difference of two other triangular numbers.

Proof **:** We prove the result by actually constructing three such triangular numbers. So, let

$$T_n = T_\ell - T_m \text{ for some (positive) integers n, } \ell \text{ and m with } \ell > m.$$

Then, n, ℓ and m must satisfy the following relationship :

$$n(n+1) = (\ell - m)(\ell + m + 1) \tag{3}$$

In (3), let

$$\ell - m = k, \tag{4}$$

so that (3) reads as

$$n(n+1) = k(k+2m+1). \tag{5}$$

Letting $k = 1$, we have

$$n(n+1) = 2(m+1) \text{ (and } \ell = m+1). \tag{6}$$

Now, for any $n \geq 2$ fixed, (6) has a (unique) solution in m. This proves the theorem. ■

From the proof of Theorem 6.1.2 above, we see that, for any $n \geq 2$,

$$T_n = T_{T_n} - T_{T_n - 1}.$$

This shows that the triangular number T_n can, in fact, be expressed as the difference of two consecutive triangular numbers. Thus, in particular,

$$T_2 = T_3 - T_2,\ T_3 = T_6 - T_5,\ T_4 = T_{10} - T_9,\ T_5 = T_{15} - T_{14},\ \ldots\ .$$

A consequence of Theorem 6.1.2 is the following

Corollary 6.1.2 **:** There is an infinite number of triangular numbers, each of which can be expressed as the sum of two triangular numbers.

In (5), choosing $k = 2$, we get

$$n(n+1) = 2(2m+3);\ n \geq 3,\ m \geq 1 \text{ (with } \ell = m+2). \tag{7}$$

It can easily be shown that (7) permits solutions for m if and only if n is either of the forms $n = 4r + 1$ and $n = 2(2r + 1)$, and the solutions are as follows :

if $n = 4r + 1$, then $m = 4r^2 + 3r - 1$ ($\ell = m + 2$),
if $n = 2(2r + 1)$, then $m = r(4r + 5)$ ($\ell = m + 2$).

This shows that, in some cases, T_n can be expressed as the difference of two triangular numbers in more than one way. Thus, for example,

$$T_5 = T_8 - T_6,\ T_6 = T_{11} - T_9,\ T_9 = T_{23} - T_{21},\ T_{10} = T_{28} - T_{26},\ \ldots .$$

We may continue this process, choosing $k = 3, 4, \ldots$ in succession, and we would have more cases of expressing some triangular numbers as the difference of other triangular numbers in more than two ways. For example, $k = 3$ in (5) leads to the equation

$$n(n+1) = 6(m+2);\ n \geq 4,\ m \geq 1 \quad (\text{with } \ell = m+3). \tag{8}$$

It can easily be shown that (8) has solutions for m if and only if n is of the form $n = 3r - 1$ or $n = 3r$, with the solutions

$$m = \frac{r(3r-1)}{2} - 2 \quad (\ell = m+3), \text{ if } n = 3r - 1,$$

$$m = \frac{r(3r+1)}{2} - 2 \quad (\ell = m+3), \text{ if } n = 3r.$$

Thus, for example,

$$T_5 = T_6 - T_3,\ T_6 = T_8 - T_5,\ T_8 = T_{13} - T_{10},\ T_9 = T_{16} - T_{13},\ T_{11} = T_{23} - T_{20},\ \ldots .$$

Again, choosing $k = 4$, (5) reads as

$$n(n+1) = 4(2m+5);\ n \geq 6,\ m \geq 1 \quad (\text{with } \ell = m+4).$$

which has a solution if and only if n is of the form $n = 4(2r + 1)$ with $m = r(8r + 9)$.

The above analysis shows that, in some cases, a triangular number can be expressed as the difference of two other triangular numbers in more than one way. For example, each of T_5, T_6 and T_9 can be expressed as the difference of two other triangular numbers in three ways.

Shyam Sunder Gupta[(4)] introduced the Smarandache T-sequence as follows :

Definition 6.1.1 **:** The *Smarandache T-Sequence*, denoted by $\{ST(n)\}_{n=1}^{\infty}$, is defined by

$$ST(n) = \overline{T_1\, T_2 \ldots T_n}\,,\ n \geq 1,$$

which is obtained by successively concatenating the triangular numbers $T_1, T_2, \ldots, T_n$.

The first few terms of the Smarandache T-Sequence are

1, 13, 136, 13610, 1361015, 136101521, 13610152128, 1361015212836, … .

Note that, $ST(n + 1)$ can be expressed in terms of $ST(n)$ as follows :

$$ST(n+1) = \overline{T_1\, T_2\, \ldots\, T_n T_{n+1}} = 10^s\ \overline{T_1\, T_2\, \ldots\, T_n} + T_{n+1} = 10^s\, ST(n) + T_{n+1},$$

where $s\ (\geq 1)$ is the number of digits in T_{n+1}.

In the Smarandache T-sequence, the second and the sixth terms are primes. In fact, as has been reported by Shyam Sunder Gupta(4), these are the only primes in the first 1000 terms of the Smarandache T-Sequence. Furthermore, as has been observed by Shyam Sunder Gupta(4), (except for the first term) the third term of the Smarandache T-Sequence is the only triangular number in the first 1000 terms of the Smarandache T-Sequence.

Note that, for $n \ge 3$, T_n are composite numbers; in fact

$$\begin{cases} n \text{ divides } T_n, & \text{if n is odd} \\ (n+1) \text{ divides } T_n, & \text{if n is even} \end{cases}$$

Now, since

$$T_{3n-1} = \frac{3n(3n-1)}{2}, \; T_{3n} = \frac{3n(3n+1)}{2}; \; n \ge 1.$$

it follows that, both T_{3n-1} and T_{3n} are divisible by 3, so that each of

$$1 + T_{3n+1} + T_{3n+4} = T_{3n} + T_{3n+3} + 6(n+1),$$

and

$$T_{3n+1} + T_{3n+4} + T_{3n+7} = T_{3n} + T_{3n+3} + T_{3n+6} + 3(2n+3),$$

is divisible by 3. This, in turn, shows that $1 + T_{3n+1} + T_{3n+4} + T_{3n+7} + T_{3n+10} + T_{3n+13}$ is divisible by 3. Again, since

$$\begin{aligned} & T_{9n+8} + T_{9n+9} + T_{9n+10} + T_{9n+11} + T_{9n+12} + T_{9n+13} + T_{9n+14} + T_{9n+15} + T_{9n+16} \\ &= T_{9n+9} - (9n+9) + T_{9n+9} + T_{9n+9} + (9n+10) \\ &\quad + T_{9n+11} - (9n+12) + T_{9n+12} + T_{9n+12} + (9n+13) \\ &\quad + T_{9n+15} - (9n+15) + T_{9n+15} + T_{9n+15} + (9n+16) \\ &= 3(T_{9n+9} + T_{9n+12} + T_{9n+15} + 1), \end{aligned}$$

We see that 3 divides $T_{9n+8} + T_{9n+9} + T_{9n+10} + T_{9n+11} + T_{9n+12} + T_{9n+13} + T_{9n+14} + T_{9n+15} + T_{9n+16}$.

We now state and prove the following result.

***Lemma 6.1.2* :** For $n \ge 0$, 3 divides $ST(9n + 7)$.

***Proof* :** It can easily be checked that 3 divides $ST(7) = 13610152128$. This proves the result for $n = 0$. To proceed by induction on n, we assume that the result is true for some $n > 0$.

Now,

$$ST(9n+16) = 10^s \, \overline{T_1 T_2 \ldots T_n T_{9n+7}} + 10^a T_{9n+8} + 10^b T_{9n+9} + 10^c T_{9n+10}$$
$$+ 10^d T_{9n+11} + 10^e T_{9n+12} + 10^f T_{9n+13} + 10^g T_{9n+14} + 10^h T_{9n+15} + T_{9n+16},$$

(for some non-negative integers s, a, b, c, d, e, f, g and h) so that, by virtue of the induction hypothesis, 3 divides $ST(9n + 16)$. Thus, the result is true for $n + 1$, completing induction. ■

Since, for $n \ge 1$,

$$ST(4n-1) = 10^s \, ST(4n-2) + T_{4n-1} \text{ for some integer } s \ge 1,$$

$$ST(4n) = 10^t \, ST(4n-1) + T_{4n} \text{ for some integer } t \ge 1,$$

it follows that both ST(4n – 1) and ST(4n) are even. It is thus sufficient to look for the primes among the terms of the forms ST(4n + 1) and ST(4n + 2). Similarly, since 5 divides each of ST(5n – 1) and ST(5n), it is sufficient to check the terms of the forms ST(5n + 1), ST(5n + 2) and ST(5n + 3) for possible primes.

Shyam Sunder Gupta[4] also introduced the *Smarandache reverse T-Sequence*, denoted by $\{SRT(n)\}_{n=1}^{\infty}$, and is defined as follows :

$$SRT(n) = \overline{T_n\ T_{n-1} \ldots T_1},\ n \geq 1.$$

Note that, all the terms of the Smarandache reverse T-Sequence are odd. Shyam Sunder Gupta[4] reports that, among the first 1000 terms of the Smarandache reverse T-Sequence, there are only six prime numbers, namely, SRT(2), SRT(3), SRT(4), SRT(10), SRT(12) and SRT(14); but there is no triangular number (except the first term).

For the Smarandache reverse T-Sequence, the following result holds true.

***Lemma 6.1.3* :** For n ≥ 0, 3 divides SRT(9n + 7).

***Proof* :** follows immediately from Lemma 6.1.2. ■

In passing, we mention the following problems :

***Open Problem 6.1.1* :** Given any integer $k \geq 2$, are there triangular numbers T_m and T_n such that $T_n = k\,T_m$?

In the equation $T_n = k\,T_m$, using the transformation $n = X - \frac{1}{2}$, $m = Y - \frac{1}{2}$, we get the Diophantine equation $A^2 - kB^2 = -(k-1)$, where $A = 2X$, $B = 2Y$.

***Open Problem 6.1.2* :** What is the general solution of the Diophantine equation (5)?

In (5), fixing $k = k_0$ (≥ 2), we see that it has a solution if and only if $1 + 4k_0(k_0 + 2m + 1)$ is a perfect square.

***Open Problem 6.1.3* :** Is it possible to find the pattern of the triangular numbers which can be expressed as the difference of two other triangular numbers in more than one way?

***Open Problem 6.1.4* :** How many terms of the Smarandache T-Sequence are primes? And how many terms are triangular numbers?

***Open Problem 6.1.5* :** How many terms of the Smarandache reverse T-Sequence are primes? And how many of them are triangular numbers?

Another problem of interest is the *palindromic triangular number*, that is, the triangular number that reads the same backward and forward. Pickover[2] mentions that

$$T_{1111} = 617716,$$

$$T_{111111} = 6172882716,$$

are palindromic numbers. Here, in each case, the index is also palindromic. We find that

$$T_{11111111} = 61728399382716$$

is also palindromic, and is the maximum in this series, the minimum being $T_{11} = 66$.

6.2 Smarandache Friendly Numbers

Murthy[5] defined the Smarandache friendly numbers as follows.

Definition 6.2.1 : A pair of positive integers (m, n) (with $n > m$) is called the *Smarandache friendly numbers* if and only if

$$m + (m + 1) + \ldots + n = mn.$$

For example, (3, 6) is a Smarandache friendly pair, since

$$3 + 4 + 5 + 6 = 18 = 3\times 6.$$

Khainar, Vyawahare and Salunke[6] considered the problem of finding Smarandache friendly pairs and Smarandache friendly primes, using computer search.

In this section, we show that the problem of finding the Smarandache friendly pairs can be reduced to the problem of solving a particular type of *Pell's equation*.

The following result, related to the Diophantine equation $x^2 - 2y^2 = -1$, is well-known (see, for example, Hardy and Wright[3]). The equation is a particular type of Pell's equation.

Lemma 6.2.1 : The general solution of the Diophantine (Pell`s) equation $x^2 - 2y^2 = -1$ is

$$x + \sqrt{2}\ y = (1 + \sqrt{2})^{2\nu+1};\ \nu \ge 0. \tag{6.2.1}$$

The lemma below gives a recurrence relation to find the solution of $x^2 - 2y^2 = -1$ recursively, starting with the *fundamental solution* $x_1 = 7$, $y_1 = 5$.

Lemma 6.2.2 : Denoting by (x_ν, y_ν) the ν-th solution of the Diophantine equation $x^2 - 2y^2 = -1$, (x_ν, y_ν) satisfies the following recurrence relation :

$$x_{\nu+1} = 3x_\nu + 4y_\nu,\ y_{\nu+1} = 2x_\nu + 3y_\nu;\ \nu \ge 1. \tag{6.2.2}$$

Proof : Since

$$\begin{aligned} x_{\nu+1} + \sqrt{2}\ y_{\nu+1} &= (1 + \sqrt{2})^{2\nu+3} \\ &= (x_\nu + \sqrt{2}\ y_\nu)(1 + \sqrt{2})^2 \\ &= (x_\nu + \sqrt{2}\ y_\nu)(3 + 2\sqrt{2}) \\ &= (3x_\nu + 4\ y_\nu) + \sqrt{2}(2x_\nu + 3\ y_\nu), \end{aligned}$$

the result follows. ■

We now consider the problem of finding the pair of integers (m, n), with $n > m > 0$, such that

$$m + (m + 1) + \ldots + n = mn. \tag{6.2.3}$$

Writing

$$n = m + k \text{ for some integer } k > 0, \tag{6.2.4}$$

(6.2.3) takes the form

$$m + (m + 1) + \ldots + (m + k) = m(m + k)$$

that is, $$m(k + 1) + \frac{k(k + 1)}{2} = m(m + k)$$

which, after some algebraic manipulations, gives

$$k(k + 1) = 2m(m - 1). \tag{6.2.5}$$

In (6.2.5), we substitute

$$k = K - \frac{1}{2},\ m = M + \frac{1}{2} \tag{6.2.6}$$

to get

$$K^2 - \frac{1}{4} = 2(M^2 - \frac{1}{4})$$

that is, $4K^2 - 8M^2 = -1$

that is, $$x^2 - 2y^2 = -1, \tag{6.2.7}$$

where

$$x = 2K,\ y = 2M. \tag{6.2.8}$$

Note that, in (6.2.8), though K and M are not integers, each of x and y is a positive integer.

Lemma 6.2.3 **:** The sequence of Smarandache friendly pair of numbers, $\{(m_\nu, n_\nu\}_{\nu=1}^{\infty}$, is given by

$$m_\nu = M_\nu + \frac{1}{2} = \frac{1}{2}(y_\nu + 1),$$

$$k_\nu = K_\nu - \frac{1}{2} = \frac{1}{2}(x_\nu - 1), \tag{6.2.9}$$

$$n_\nu = m_\nu + k_\nu = \frac{1}{2}(x_\nu + y_\nu),$$

where

$$x_\nu + \sqrt{2}\ y_\nu = (1+\sqrt{2})^{2\nu+1}, \nu \geq 0. \tag{6.2.10}$$

Proof **:** Since x and y satisfy (6.2.7), (6.2.10) follows from Lemma 6.2.2. ■

Lemma 6.2.4 **:** The sequence of Smarandache friendly pair of numbers $\{(m_\nu, n_\nu\}_{\nu=1}^{\infty}$ satisfies the following recurrence relation :

$$m_{\nu+1} = m_\nu + 2n_\nu,\ n_{\nu+1} = 2m_\nu + 5n_\nu - 1;\ \nu \geq 1.$$

with

$$m_1 = 3,\ n_1 = 6.$$

Proof **:** By Lemma 6.2.2,

$$x_{\nu+1} = 3x_\nu + 4y_\nu,\ y_{\nu+1} = 2x_\nu + 3y_\nu;\ \nu \geq 1.$$

Now,

$$m_{\nu+1} = \frac{1}{2}(y_{\nu+1} + 1) = \frac{1}{2}(2x_\nu + 3y_\nu + 1) = x_\nu + y_\nu + \frac{1}{2}(y_\nu + 1) = 2n_\nu + m_\nu,$$

$$n_{\nu+1} = \frac{1}{2}(x_{\nu+1} + y_{\nu+1}) = \frac{1}{2}\left[(3x_\nu + 4y_\nu) + (2x_\nu + 3y_\nu)\right]$$

$$= \frac{1}{2}(5x_\nu + 7y_\nu)$$

$$= \frac{5}{2}(x_\nu + y_\nu) + y_\nu$$

$$= 5n_\nu + (2m_\nu - 1),$$

and we get the desired results. ■

Lemma 6.2.4 shows that, if the pair (of numbers) (x_ν, y_ν) is Smarandache friendly, so also is the pair $(m_\nu + 2n_\nu, 2m_\nu + 5n_\nu - 1)$, which is, in fact, the next pair. Thus, starting with the first Smarandache friendly pair (3, 6), Lemma 6.2.4 may be employed to find the others recursively. Lemma 6.2.4 shows that there are infinite number of Smarandache friendly pairs.

The following table gives the first nine Smarandache pairs, which are obtained from the recurrence relation given in Lemma 6.2.4.

Table 6.2.1 **:** The first nine Smarandache friendly pairs

ν	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
m_ν	3	15	85	493	2871	16731	97513	568345	3312555
n_ν	6	35	204	1189	6930	40391	235416	1372105	799724

From Table 6.2.1 above, we observe that the Smarandache friendly pairs grow quite rapidly. When $\nu = 10$,

$$m_{10} = 19306983, \; n_{10} = 46611179.$$

Similar to the Smarandache friendly pair of numbers, we have the Smarandache friendly pair of primes, defined as follows (see Murthy[(5)]) :

Definition 6.2.2 **:** Let p and q be two primes with $q > p \geq 3$. Then, the pair (p, q) is called a *Smarandache pair of friendly primes* if the sum of all the primes from p through q (including p and q) is equal to the product pq.

For example, (2, 5) is a pair of Smarandache pair of friendly primes, since

$$2 + 3 + 5 = 10 = 2\times5.$$

The next pair of Smarandache friendly primes is (3, 13), since

$$3 + 5 + 7 + 11 + 13 = 39 = 3\times13,$$

followed by the pairs (5, 31) and (7, 53). This observation leads to the following open question.

Question 6.2.1 **:** Given any prime p (≥ 3), is it possible to find a prime q ($> p$) such that (p, q) is a Smarandache pair of friendly primes?

Let $p_1, p_2, .., p_n, \ldots$ be any sequence of consecutive primes with $p_1 \geq 3$ such that

$$p_1 + p_2 + p_3 + \ldots + p_n = p_1 p_n \text{ for some primes } p_1 \text{ and } p_n. \tag{1}$$

Since the primes $p_1, p_2, ..., p_n, \ldots$ are all odd, it follows that the right-hand side product in the relationship (1) above is odd, and hence, n must be odd. Now, let

$$p_2 + p_3 + \ldots + p_n + p_{n+1} + \ldots + p_{n+r} = p_2 p_{n+r} \text{ for some prime } p_{n+r}. \tag{2}$$

Then, r must be odd; moreover, $r \geq 3$. To prove that $r \geq 3$, first note that, by virtue of (1),

$$p_2 + p_3 + \ldots + p_n + p_{n+1} = p_1 p_n - p_1 + p_{n+1} < p_1 p_n + p_{n+1}.$$

Now,

$$p_2 p_{n+1} \geq (p_1 + 2)p_{n+1} = p_1 p_{n+1} + 2p_{n+1}.$$

Therefore,

$$p_2 p_{n+1} \geq p_1 p_{n+1} + 2p_{n+1} > p_1 p_n + p_{n+1} > p_2 + p_3 + \ldots + p_n + p_{n+1},$$

which shows that $r \neq 1$. Again, if $p_2 \geq p_1 + 4$, then

$$p_2 p_{n+1} \geq (p_1 + 4)p_{n+1} > p_2 + p_3 + \ldots + p_n + p_{n+1} + p_{n+2} + p_{n+3},$$

which shows that $r \geq 5$.

6.3 Smarandache Reciprocal Partition Sets of Unity

The idea of the sets of Smarandache reciprocal partition of unity was introduced by Murthy[7], and was studied in a systematic manner by Murthy and Ashbacher[8]. This section treats some properties related to the sets of the Smarandache reciprocal partition of unity.

We start with the following definition.

Definition 6.3.1 : The *Smarandache repeatable reciprocal partition of unity in n partitions*, denoted by SRRPS(n), is defined as follows :

$$\mathrm{SRRPS}(n) = \left\{ \left(a_1, a_2, \ldots, a_n\right) : a_1, a_2, \ldots, a_n \text{ are integers; } \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1 \right\}.$$

The order (the number of elements) of the set SRRPS(n) is denoted by $f_{RP}(n)$.

Note that, in the above definition, without any loss of generality, we may assume that the n integers $a_1, a_2, \ldots, a_n$ are arranged in increasing (non-decreasing) order, that is

$$0 < a_1 \le a_2 \le \ldots \le a_n.$$

With this convention, we get

$$\mathrm{SRRPS}(1) = \{1\}, \text{ the singleton set,}$$

$$\mathrm{SRRPS}(2) = \{(2, 2)\}, \text{ the singleton set,}$$

$$\mathrm{SRRPS}(3) = \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\},$$

with

$$f_{RP}(1) = 1,\ f_{RP}(2) = 1,\ f_{RP}(3) = 3.$$

In Definition 6.3.1, the n integers $a_1, a_2, \ldots, a_n$ are not necessarily distinct; if they are distinct, then we have the following definition.

Definition 6.3.2 : The *Smarandache distinct reciprocal partition of unity in n partitions*, denoted by SDRPS(n), is defined as follows :

$$\mathrm{SDRPS}(n) = \left\{ \left(a_1, a_2, \ldots, a_n\right) : 0 < a_1 < a_2 < \ldots < a_n;\ \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1 \right\}.$$

The order of the set SDRPS(n) is denoted by $f_{DP}(n)$.

Note that, SDRPS(2) = ∅ (the empty set). Thus, in studying the sets SDRPS(n), it is sufficient to confine the attention to the case when $n \ge 3$.

Lemma 6.3.1 : In the set SDRPS(n), $n \ge 3$, $2 \le a_1 \le n - 1$.

Proof : Let, on the contrary $a_1 \ge n$, so that by the condition of Definition 6.3.2,

$$a_i > n \text{ for all } i = 2, 3, \ldots, n.$$

But then

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 1,$$

contradicting the condition of the definition. ■

As a consequence of the above lemma, we can prove the following

Corollary 6.3.1 : In the set SRRPS(n), $n \ge 3$, if $a_1 = n$, then $a_i = n$ for all $1 \le i \le n$.

***Proof* :** By assumption,

$$a_i \geq a_1 = n \text{ for all } i = 2, 3, \ldots, n.$$

Now, if $a_j > n$ for some j, $2 \leq j \leq n$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 1,$$

and we reach to a contradiction.

This contradiction establishes the desired result. ■

***Lemma 6.3.2* :** For $n \geq 3$, there always exist integers $a_1, a_2, \ldots, a_n$, satisfying the condition

$$2 = a_1 < a_2 < \ldots < a_n$$

such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

***Proof* :** is by induction on n.

When $n = 3$, we choose

$$a_2 = 3, a_3 = 6,$$

so that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1.$$

Thus, the result is true for $n = 3$. To proceed by induction on n, we assume that the result is true for some integer n, that is, we assume that there are n integers $2 = a_1, a_2, \ldots, a_n$ with

$$a_1 < a_2 < \ldots < a_n$$

such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

We now define the integers $b_1, b_2, \ldots, b_n, b_{n+1}$ as follows :

$$b_1 = a_1,$$
$$b_2 = a_2,$$
$$\vdots$$
$$b_{n-1} = a_{n-1},$$
$$b_n = a_n + 1,$$
$$b_{n+1} = a_n(a_n + 1).$$

The numbers so defined satisfy the following two conditions :

$$b_1 < b_2 < \ldots < b_n < b_{n+1},$$

$$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n} + \frac{1}{b_{n+1}} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

Thus, the result also holds for $n + 1$, completing induction. ■

To find SDRPS(3), we first note that, by Lemma 6.3.1, we must have $a_1 = 2$.

We first prove the following result.

***Lemma 6.3.3* :** The only (positive) integer solutions of the Diophantine equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \qquad (6.3.1)$$

are **(1)** $a = b = 4$, **(2)** $a = 3, b = 6$.

***Proof* :** From (6.3.1), we get

$$2(a+b)=ab.$$

Then, a must divide b, so that

$$b=ka \text{ for some integer } k \geq 1.$$

Therefore, we get

$$2(1+k)=ka.$$

Since k does not divide $1+k$, it follows that a must divide $1+k$.

Now, when $k=1$, then $a=2(1+k)=4$ (so that $b=4$), and when $k=2$, $a=k+1=3$ (and $b=6$). Then, we get the desired result. ■

From Lemma 6.3.3, we see that, under the restriction that $b>a$, the equation (6.3.1) has the unique solution $a=3$, $b=6$.

***Lemma 6.3.4* :** SDRPS(3) is the singleton set SDRPS(3) = $\{(2, 3, 6)\}$.
***Proof* :** follows by virtue of Lemma 6.3.3. ■

Next, we find SDRPS(4). To do so, we need the following intermediate results.

***Lemma 6.3.5* :** The only (positive) solutions of the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{6} \tag{6.3.2}$$

are **(1)** $a=7$, $b=42$, **(2)** $a=8, b=24$, **(3)** $a=9$, $b=18$, **(4)** $a=12=b$.

***Proof* :** We rewrite (6.3.2) as

$$6(a+b)=ab.$$

Now, a must divide b, say

$$b=ka \text{ for some integer } k \geq 1.$$

Then

$$6(1+k)=ka,$$

so that a must divide $1+k$ (since k does not divide $1+k$).

Now, when $k=1$, then $a=6(1+k)=12$ (so that $b=12$), and when $k=2$, $a=3(k+1)=9$ (and $b=18$), when $k=3$, then $a=2(1+k)=8$ (and $b=24$), and when $k=6$, $a=1+k=7$ (so that $b=42$). Hence, we get the desired result. ■

***Lemma 6.3.6* :** The only (positive) solutions of the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{4} \tag{6.3.3}$$

are **(1)** $a=5$, $b=20$, **(2)** $a=6, b=12$, **(3)** $a=8=b$.

***Proof* :** We recast (6.3.2) in the form below :

$$6(a+b)=ab.$$

Now, a must divide b, so that

$$b=ka \text{ for some integer } k \geq 1.$$

Then

$$4(1+k)=ka,$$

and hence, a must divide $1+k$ (since k does not divide $1+k$).

Now, when $k=1$, then $a=4(1+k)=8$ (so that $b=8$), and when $k=2$, $a=2(k+1)=6$ (and $b=12$), and when $k=4$, then $a=1+k=5$ (and $b=20$). Hence, we get the desired result. ■

***Corollary 6.3.2* :** Consider the Diophantine equation

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}=1;\ 0<a_1<a_2<a_3<a_4<1. \tag{6.3.4}$$

(1) Under the condition that $a_1 = 2$, $a_2 = 3$, the equation (6.3.4) has three solutions, namely, **(i)** $a_3 = 7$, $a_4 = 42$, **(ii)** $a_3 = 8$, $a_4 = 24$, and **(iii)** $a_3 = 9$, $a_4 = 18$,

(2) With $a_1 = 2$, $a_2 = 4$, (6.3.4) has two solutions, namely, **(i)** $a_3 = 5$, $a_4 = 20$, and **(ii)** $a_3 = 6$, $a_4 = 12$.

***Proof* :** Part (1) follows from Lemma 6.3.5, while part (2) is an immediate consequence of Lemma 6.3.6. ■

***Lemma 6.3.7* :** Consider the following Diophantine equation :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}. \tag{5.3.5}$$

Under the condition that $c > b > a \geq 5$, (5.3.5) has no solution.

***Proof* :** First, let $a = 5$. Then,

$$\frac{1}{6}+\frac{1}{7}>\frac{3}{10},$$

but

$$\frac{1}{6}+\frac{1}{8}<\frac{3}{10}.$$

This shows that $a \neq 5$.

Next, let $a \geq 6$. Then, for $c > b > a \geq 6$,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{1}{6}+\frac{1}{7}+\frac{1}{8}<\frac{3}{10}.$$

Thus, a cannot be greater than 6.

All these complete the proof of the lemma. ■

***Lemma 6.3.8* :** Consider the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{3}. \tag{6.3.6}$$

Under the condition that $c > b > a \geq 4$, (6.3.6) has no solution.

***Proof* :** follows from the fact that, for $c > b > a \geq 4$,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{1}{4}+\frac{1}{5}+\frac{1}{6}<\frac{2}{3}.\ ■$$

***Lemma 6.3.9* :** SDRPS(4) has exactly six elements, namely,

$$\text{SDRPS}(4) = \{(2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 4, 5, 20), (2, 4, 6, 12)\}.$$

***Proof* :** To find SDRPS(4), first note that, by Lemma 6.3.1, a_1 is either 2 or 3. By virtue of Lemma 6.3.7, $a_1 \neq 3$. Hence, a_1 must be 2. Again, by Lemma 6.3.7, a_2 cannot be greater than 5. The result now follows by Corollary 6.3.2. ■

Now, given the set SDRPS(n) (with $f_{DP}(n)$ elements), we consider the problem of extending it to get some of the elements of the set SDRPS(n + 1). Here, the question is

***Question 6.3.1* :** In how many ways the elements of SDRPS(n) can be extended to get the elements of SDRPS(n + 1)? And what is the number of elements of SDRPS(n)?

Murthy and Ashbacher[8] suggest different methods of extending the elements of SDRPS(n) to get the elements of SDRPS(n + 1). One such method is stated in the lemma below.

***Lemma 6.3.10* :** Let $(a_1, a_2, \ldots, a_n) \in \text{SDRPS}(n)$. Then, $(2, 2a_1, 2a_2, \ldots, 2a_n) \in \text{SDRPS}(n + 1)$.

***Proof* :** By assumption, $n-1 \le a_1 < a_2 < \dots < a_n$ with

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = 1.$$

Then,

$$\frac{1}{2} + \frac{1}{2a_1} + \frac{1}{2a_2} + \dots + \frac{1}{2a_n} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) = 1.$$

Hence, the lemma. ■

***Lemma 6.3.11* :** If $(a_1, a_2, \dots, a_n) \in SDRPS(n)$, then $(2, 3, 6a_1, 6a_2, \dots, 6a_n) \in SDRPS(n+2)$.

***Proof* :** By assumption,

$$2 \le a_1 < a_2 < \dots < a_n,$$

with

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = 1.$$

Therefore,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6a_1} + \frac{1}{6a_2} + \dots + \frac{1}{6a_n} = \frac{5}{6} + \frac{1}{6}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) = \frac{5}{6} + \frac{1}{6} = 1.$$

Thus, the lemma is established. ■

Since $(2, 3, 6) \in SDRPS(3)$, it follows, by Lemma 6.3.11, that $(2, 3, 12, 18, 36) \in SDRPS(5)$.

Now, let $(a_1, a_2, \dots, a_i, \dots, a_n) \in SDRPS(n)$. Then, replacing a_i $(1 \le i \le n)$ by b_{i1} and b_{i2}, where

$$b_{i1} = a_i + 1,\ b_{i2} = a_i(a_i + 1), \tag{6.3.7}$$

we see that $(a_i, a_2, \dots, a_{i-1}, b_{i1}, b_{i2}, a_{i+1}, \dots, a_n)$ (rearranging the numbers, if necessary), we get an element of $SDRPS(n+1)$. The proof follows from the fact that

$$\frac{1}{b_{i1}} + \frac{1}{b_{i2}} = \frac{1}{a_i+1} + \frac{1}{a_i(a_i+1)} = \frac{a_i+1}{a_i(a_i+1)} = \frac{1}{a_i}.$$

The third method mentioned in Murthy and Ashbacher(8) is as follows : Let d_{i1} and d_{i2} be two distinct divisors of a_i, so that $a_i = d_{i1} \times d_{i2}$. Then, replacing a_i by c_{i1} and c_{i2}, where

$$c_{i1} = d_{i1}(d_{i1} + d_{i2}),\ c_{i2} = d_{i2}(d_{i1} + d_{i2}), \tag{6.3.8}$$

(and rearranging the terms, if necessary), we get an element of $SDRPS(n+1)$. Note that

$$\frac{1}{c_{i1}} + \frac{1}{c_{i2}} = \frac{1}{d_{i1}(d_{i1}+d_{i2})} + \frac{1}{d_{i2}(d_{i1}+d_{i2})} = \frac{1}{d_{i1}d_{i2}} = \frac{1}{a_i}.$$

In connection with the third method above, we have the following result.

***Lemma 6.3.12* :** In the set $SDRPS(n)$, $n \ge 3$, a_n is not a prime.

***Proof* :** The proof is by contradiction. So, let a_n be a prime, say, $a_n = p$. Then,

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} = \frac{1-p}{p}. \tag{6.3.9}$$

Let

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} = \frac{A}{a_1 a_2 \dots a_{n-1}},$$

so that from (6.3.9), we get

$$\frac{A}{a_1 a_2 \dots a_{n-1}} = \frac{1-p}{p},$$

that is,

$$pA = (p-1)\, a_1 a_2 \dots a_n,$$

and we reach to a contradiction, since none of $a_1, a_2, \dots, a_n$ is divisible by p. ■

Since a_n is not a prime, it follows that a_n (if not a square) has at least two divisors (excepting the trivial ones 1 and a_n). Note that, if a_i is a prime, say, $a_i = p$, then by the third method (6.3.8), p is replaced by $p + 1$ and $p(p + 1)$ to get the corresponding element of SDRPS(n + 1).

Starting with the single element $(2, 3, 6) \in$ SDPRS(3), by applying the three methods outlined above, we get the following three elements of SDRPS(4) :

$$(2, 4, 6, 12), (2, 3, 7, 42), (2, 3, 10, 15).$$

Note that, the three methods are overlapping, and do not generate all the elements of SDRPS(4).

Let the sequence of integers $\{a_{nj}\}_{j=3}^{n}$, $n \geq 3$, be defined as follows :

$$a_{3j} = \begin{cases} 2, & \text{if } \ j = 1 \\ 3, & \text{if } \ j = 2 \\ 6, & \text{if } \ j = 3 \end{cases}$$

and for $n \geq 4$,

$$a_{nj} = \begin{cases} 2, & \text{if } \ j = 1 \\ a_{n-1, j-1}, & \text{if } \ 2 \leq j \leq n \end{cases}$$

Let

$$A_n = (a_{n1}, a_{n2}, ..., a_{nn}), \ n \geq 3,$$

where

$$2 = a_{n1} < a_{n2} < ... < a_{nn}.$$

Then, as has been pointed out by Maohua[9], we have an infinite sequence of sets, $\{A_n\}_{n=3}^{\infty}$, the elements of each of which satisfy the condition of Definition 6.3.2.

To find a lower bound for $f_{DR}(n)$, we confine our attention to SDRPS(4). A closer look at the elements of SDRPS(4) shows that the second procedure, outlined in (6.3.7), can be applied to only three (of the four components) in each of the first five elements of SDRSP(4), while its last element gives only two elements of SDRPS(5). Thus, by the second method, we get the 17 elements of SDRPS(5) as follows : (2, 4, 7, 12, 42), (2, 3, 8, 42, 56), (2, 3, 7, 43, 1806), (2, 4, 8, 12, 24), (2, 3, 9, 24, 72), (2, 3, 8, 25, 600), (2, 4, 9, 12, 18), (2, 3, 10, 18, 90), (2, 3, 9, 19, 342), (2, 4, 10, 12, 15), (2, 3, 11, 15, 110), (2, 3, 10, 16, 240), (3, 4, 5, 6, 20), (2, 4, 6, 20, 30), (2, 4, 5, 21, 420), (2, 4, 6, 12, 20) and (2, 4, 6, 13, 156). Applying the third method, outlined in (6.3.8), the following eleven elements of SDRPS(5) are obtained : (2, 3, 7, 78, 91), (2, 3, 8, 33, 88), (2, 3, 8, 40, 60), (2, 3, 8, 28, 168), (2, 3, 9, 22, 99), (2, 3, 9, 27, 54), (2, 3, 14, 15, 35), (2, 3, 10, 24, 40), (2, 4, 5, 24, 120), (2, 4, 5, 36, 45), (2, 4, 6, 21, 28). And finally, by Lemma 6.3.10, the following four elements of SDRPS(5) result : (2, 4, 6, 14, 84), (2, 4, 6, 16, 48), (2, 4, 6, 18, 36) and (2, 4, 8, 10, 40). Thus, the suggested three methods together give only 32 elements of SDRPS(5), while Murthy and Ashbacher[8] reports that there are 72 elements in the set SDRPS(5).

Thus, the analysis with the elements of SDRPS(4) leads to the following estimate of SDRPS(n + 1) :

$$f_{DR}(n + 1) \geq (n - 1)[f_{DR}(n) - 1] + n - 2 + f_{DR}(n) = n\, f_{FD}(n) - 1.$$

In the above inequality, the number of elements of SDRPS(n + 1), obtained from the elements of SDRPS(n) by the method of (6.3.7) is $(n - 1)[f_{DR}(n) - 1] + n - 2$, and since a_n is not a prime, we can safely say that the number of elements of SDRPS(n + 1), arising from the elements of SDRPS(n), by the third method outlined in (6.3.8), is $f_{DR}(n)$. With $n = 4$, we get $f_{DR}(5) \geq 23$, which is a better and more reasonable estimate than the one given in Murthy and Ashbacher[8].

6.4 Smarandache LCM Ratio

In this section, we consider the Smarandache LCM ratio function, introduced by Murthy(10). As we shall see later, there are, in fact, two types of Smarandache LCM ratio functions in the literature.

Given any number k of positive integers $n_1, n_2, \ldots, n_k$, we denote by $(n_1, n_2, \ldots, n_k)$ the *greatest common divisor* (GCD) of $n_1, n_2, \ldots, n_k$, and by $[n_1, n_2, \ldots, n_k]$ their *least common multiple* (LCM). Then, we have the following results, whose proofs may be found, for example, in Apostol(11).

Lemma 6.4.1 **:** For any positive integers n and m, $[n, m] = \frac{nm}{(n, m)}$.

Lemma 6.4.2 **:** For any integers $n_1, n_2, \ldots, n_k$,

(1) $[n_1, n_2, \ldots, n_k] = [[n_1, n_2, \ldots, n_t], [n_{t+1}, n_{t+2}, \ldots, n_k]]$, $t < k$,

(2) $(n_1, n_2, \ldots, n_k) = ((n_1, n_2, \ldots, n_t), (n_{t+1}, n_{t+2}, \ldots, n_k))$, $t < k$.

The Smarandache LCM ratio function, proposed by Murthy(10), is as follows :

Definition 64.1 **:** The *Smarandache LCM ratio function of degree r*, denoted by $T(n, r)$, is given by

$$T(n,\ r) = \frac{[n,\ n+1,\ n+2, \ldots,\ n+r-1]}{[1,\ 2,\ 3, \ldots,\ r]};\quad n, r \in N.$$

The explicit expressions for $T(n, 1)$ and $T(n, 2)$ are given in Murthy(10), and are reproduced in the following two lemmas. The proof of Lemma 6.4.3 is immediate from the definition, while Lemma 6.4.4 follows from the fact that $(n, n+1) = 1$ for any integer $n \geq 1$.

Lemma 6.4.3 **:** $T(n, 1) = n$ for all $n \geq 1$.

Lemma 6.4.4 **:** For $n \geq 1$, $T(n, 2)$ is given by

$$T(n, 2) = \frac{n(n+1)}{2} .$$

The following two lemmas, due to Maohua(12), give explicit expressions for $T(n, 3)$ and $T(n, 4)$ respectively.

Lemma 6.4.5 **:** For $n \geq 1$,

$$T(n, 3) = \begin{cases} \dfrac{n(n+1)(n+2)}{6}, & \text{if n is odd} \\ \dfrac{n(n+1)(n+2)}{12}, & \text{if n is even} \end{cases}$$

Lemma 6.4.6 **:** For $n \geq 1$,

$$T(n,\ 4) = \begin{cases} \dfrac{n(n+1)(n+2)(n+3)}{72}, & \text{if 3 divides n} \\ \dfrac{n(n+1)(n+2)(n+3)}{24}, & \text{if 3 does not divide n} \end{cases}$$

Finally, the expression for T(n, 4), given by Wang Ting[13], is

***Lemma 6.4.7* :** For $n \geq 1$,

$$T(n,\ 5) = \begin{cases} \dfrac{n(n+1)(n+2)(n+3)(n+4)}{1440}, & \text{if } n = 12m,\ 12m+8 \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{120}, & \text{if } n = 12m+1,\ 12m+7 \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{720}, & \text{if } n = 12m+2,\ 12m+6 \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{360}, & \text{if } n = 12m+3,\ 12m+5,\ 12m+9,\ 12m+11 \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{480}, & \text{if } n = 12m+4 \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{240}, & \text{if } n = 12m+10 \end{cases}$$

Khairnar, Vyawahare and Salunke[14] treated a variant of the Smarandache LCM ratio function, defined as follows :

***Definition 6.4.2* :** The *Smarandache LCM ratio function*, denoted by $SL(n, r)$ is

$$SL(n, r) = \frac{[n,\ n-1,\ n-2,\ ...,\ n-r+1]}{[1,\ 2,\ 3,\ ...,\ r]},\ r \leq n;\quad n,\ r \in N.$$

The function $SL(n, r)$, given in Definition 6.4.2 above, may be called the *Smarandache LCM ratio function of the second type*.

***Lemma 6.4.8* :** For any integer $n \geq 1$,

$$SL(n, n) = 1.$$

***Proof* :** is trivial from Definition 6.4.2. ■

The following lemma gives the relationship between $T(n, r)$ and $SL(n, r)$.

***Lemma 6.4.9* :** For any integer $n \geq 1$ and r with $1 \leq r \leq n$,

$$SL(n, r) = T(n-r+1, r).$$

***Proof* :** is evident from Definition 6.4.1 and Definition 6.4.2. ■

Note that, in Lemma 6.4.9 above, the condition $n-r+1 \geq 1$ requires that $SL(n, r)$ is defined only for $r \leq n$.

***Lemma 6.4.10* :** If p is a prime, then p divides $SL(p, r)$ for all $r < p$.

***Proof* :** By definition,

$$SL(p,\ r) = \frac{[p,\ p-1,\ p-2,\ ...,\ p-r+1]}{[1,\ 2,\ 3,\ ...,\ r]},\ r \leq p.$$

Now, p divides $[p,\ p-1,\ p-2,\ \ldots,\ p-r+1]$ for all $r < p$, while p does not divide $[1, 2, 3, \ldots, r]$. Thus, p divides $SL(p, r)$. ■

***Lemma 6.4.11* :** If p is a prime, then p does not divide $SL(2p,\ r)$ for any $p \leq r \leq 2p$. Moreover, if q is the prime next to p, then q divides $SL(2p, r)$ for all $p \leq r \leq q-1$.

***Proof* :** If p is a prime, then p divides $[2p, 2p-1, \ldots, 2p-r+1]$ for all $r \le 2p$, but p^2 does not divide $[2p, 2p-1, \ldots, 2p-r+1]$; also, p divides $[1, 2, \ldots, r]$ for all $r \ge p$. Hence, p does not divide

$$SL(2p, r) = \frac{[2p, 2p-1, \ldots, 2p-r+1]}{[1, 2, \ldots, r]}, \; p \le r \le 2p. \tag{6.4.1}$$

To prove the remaining part of the lemma, first note that, by Bertrand's postulate (see, for example, Hardy and Wright[3]), there is at least one prime, say, q, such that $p < q < 2p$. Now, from (6.4.1), q divides the numerator if $p \le r \le q-1$, but q does not divide the denominator. As such, q divides $SL(2p, r)$ for all $p \le r \le q-1$. ■

The explicit expressions for the functions $SL(n, 1)$, $SL(n, 2)$, $SL(n, 3)$, $SL(n, 4)$ and $SL(n, 5)$ are given in Theorem 6.4.1 – Theorem 6.4.5 below, which make use of Lemma 6.4.9, together with the expressions of $T(n, 1)$, $T(n, 2)$, $T(n, 3)$, $T(n, 4)$ and $T(n, 5)$, given in the five lemmas, Lemma 6.4.3 – Lemma 6.4.7. An alternative method of derivations of these functions is given in the Appendix, where the definition of $SL(n, r)$ is used.

***Theorem 6.4.1* :** For any $n \ge 1$, $SL(n, 1) = n$.
***Proof* :** $SL(n, 1) = T(n, 1) = n$. ■

***Theorem 6.4.2* :** For any $n \ge 2$, $SL(n, 2) = \frac{n(n-1)}{2}$.
***Proof* :** By Lemma 6.4.4 and Lemma 6.4.9,

$$SL(n, 2) = T(n-1, 2) = \frac{(n-1)n}{2}. \quad ■$$

***Theorem 6.4.3* :** For any $n \ge 3$,

$$SL(n, 3) = \begin{cases} \dfrac{n(n-1)(n-2)}{6}, & \text{if n is odd} \\[2mm] \dfrac{n(n-1)(n-2)}{12}, & \text{if n is even} \end{cases}$$

***Proof* :** Using Lemma 6.4.5, together with Lemma 6.4.9,

$$SL(n, 3) = T(n-2, 3) = \begin{cases} \dfrac{(n-2)(n-1)n}{6}, & \text{if } n-2 \text{ is odd} \\[2mm] \dfrac{(n-2)(n-1)n}{12}, & \text{if } n-2 \text{ is even} \end{cases}$$

Now, since n is odd if and only if $n-2$ is odd, the result follows. ■

***Theorem 6.4.4* :** For any $n \ge 4$,

$$SL(n, 4) = \begin{cases} \dfrac{n(n-1)(n-2)(n-3)}{72}, & \text{if 3 divides n} \\[2mm] \dfrac{n(n-1)(n-2)(n-3)}{24}, & \text{if 3 does not divide n} \end{cases}$$

***Proof* :** By Lemma 6.4.6 and Lemma 6.4.9,

$$SL(n, 4) = T(n-3, 4) = \begin{cases} \dfrac{(n-3)(n-2)(n-1)n}{72}, & \text{if 3 divides } n-3 \\[2mm] \dfrac{(n-3)(n-2)(n-1)n}{24}, & \text{if 3 does not divide } n-3 \end{cases}$$

Noting that, 3 divides $n-3$ if and only if 3 divides n, the theorem follows. ■

***Theorem 6.4.5* :** For any $n \geq 5$,

$$SL(n,5) = \begin{cases} \frac{n(n-1)(n-2)(n-3)(n-4)}{1440}, & \text{if } \ n = 12m, 12m+4 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{360}, & \text{if } \ n = 12m+1, 12m+3, 12m+7, 12m+9 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{240}, & \text{if } \ n = 12m+2 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{120}, & \text{if } \ n = 12m+5, 12m+11 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{720}, & \text{if } \ n = 12m+6, 12m+10 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{480}, & \text{if } \ n = 12m+8 \end{cases}$$

***Proof* :** By virtue of Lemma 6.4.7 and Lemma 6.4.9,

$$SL(n,\ 5) = T(n-4,\ 5)$$

$$= \begin{cases} \frac{(n-4)(n-3)(n-2)(n-1)n}{1440}, & \text{if } n-4 = 12m,\ 12m+8 \\ \frac{(n-4)(n-3)(n-2)(n-1)n}{120}, & \text{if } n-4 = 12m+1,\ 12m+7 \\ \frac{(n-4)(n-3)(n-2)(n-1)n}{720}, & \text{if } n-4 = 12m+2,\ 12m+6 \\ \frac{(n-4)(n-3)(n-2)(n-1)n}{360}, & \text{if } n-4 = 12m+3,\ 12m+5,\ 12m+9,\ 12m+11 \\ \frac{(n-4)(n-3)(n-2)(n-1)n}{480}, & \text{if } n-4 = 12m+4 \\ \frac{(n-4)(n-3)(n-2)(n-1)n}{240}, & \text{if } n-4 = 12m+10 \end{cases}$$

Now, since $n-4$ is of the form $12m$ if and only if n is of the form $12m+4$, $n-4$ is of the form $12m+8$ if and only if n is of the form $12m$, $n-4$ is of the form $12m+1$ if and only if n is of the form $12m+5$, $n-4$ is of the form $12m+7$ if and only if n is of the form $12m+11$, $n-4$ is of the form $12m+2$ if and only if n is of the form $12m+6$, $n-4$ is of the form $12m+6$ if and only if n is of the form $12m+10$, $n-4$ is of the form $12m+3$ if and only if n is of the form $12m+7$, $n-4$ is of the form $12m+5$ if and only if n is of the form $12m+9$, $n-4$ is of the form $12m+9$ if and only if n is of the form $12m+1$, $n-4$ is of the form $12m+11$ if and only if n is of the form $12m+3$, $n-4$ is of the form $12m+4$ if and only if n is of the form $12m+8$, and $n-4$ is of the form $12m+10$ if and only if n is of the form $12m+2$, the result follows. ■

Using the values of $SL(n,r)$, $n \geq 1$, $1 \leq r \leq n$, the following table, due to Murthy[10], and called the *Smarandache-Amar LCM triangle*, is formed as follows :

The 1st column contains the elements of the sequence $\{SL(n,\ 1)\}_{n=1}^{\infty}$, the 2nd column is formed with the elements of the sequence $\{SL(n,\ 2)\}_{n=2}^{\infty}$, and so on, and in general, the k-th column contains the elements of the sequence $\{SL(n,\ k)\}_{n=k}^{\infty}$.

Note that, the 1st column contains the natural numbers, and the 2nd column contains the triangular numbers.

The Smarandache-Amar LCM triangle of size 13×10 (13 rows and 10 columns) is given below.

	1-st column SL(n, 1)	2-nd column SL(n, 2)	3-rd column SL(n, 3)	4-th column SL(n, 4)	5-th column SL(n, 5)	6-th column SL(n, 6)	7-th column SL(n, 7)	8-th column SL(n, 8)	9-th column SL(n, 9)	10-th column SL(n, 10)
1-st row	1									
2-nd row	2	1								
3-rd row	3	3	1							
4-th row	4	6	2	1						
5-th row	5	10	10	5	1					
6-th row	6	15	10	5	1	1				
7-th row	7	21	35	35	7	7	1			
8-th row	8	28	28	70	14	14	2	1		
9-th row	9	36	84	42	42	42	6	3	1	
10-th row	10	45	60	210	42	42	6	3	1	1
11-th row	11	55	165	330	462	462	66	33	55	55
12-th row	12	66	440	165	66	462	66	33	11	11
13-th row	13	78	286	715	429	858	858	429	143	143

Note that, by Lemma 6.4.8, the leading diagonal contains all unity.

***Lemma 6.4.12* :** If p is a prime, then sum of the elements of the p-th row $\equiv 1 \pmod{p}$.
***Proof* :** The sum of the p-th row is

$$SL(p, 1) + SL(p, 2) + \ldots + SL(p, p-1) + SL(p, p)$$

$$= \left[SL(p, 1) + SL(p, 2) + \ldots + SL(p, p-1)\right] + 1$$

$$\equiv 1 \pmod{p},$$

by virtue of Lemma 6.4.10. ■

***Open Problem 6.4.1* :** Find a congruence property for the sum of the elements of the k-th row when k is a composite?

***Open Problem 6.4.2* :** Find the sum of the elements of the k-th row?

Note that, by Lemma 6.4.10 and Lemma 6.4.11, some of the elements of the (2p)-th row is divisible by p, and some elements are not divisible by p but are divisible by q, where q is the next larger prime to p.

Looking at the 9^{th} row of the triangle, we observe that the number 42 appears in three consecutive places. Note that, 42 is divisible by the prime next to p in the interval $(p, 2p)$ with $p = 5$.

***Open Problem 6.4.3* :** In the Smarandache-Amar triangle, is it possible to find (in some row) repeating values of arbitrary length?

Note that, the above problem is related to the problem of finding the solutions of the equation

$$SL(n, r) = SL(n, r + 1). \tag{6.4.2}$$

A necessary and sufficient condition that (6.4.2) holds is

$$([n,\ n-1,\ ...,\ n-r+1],\ n-r)(r+1) = ([1,\ 2,\ ...,\ r],\ r+1)(n-r). \tag{6.4.3}$$

The proof is as follows : The equation (6.4.2) holds for some n and r if and only if

$$\frac{[n,\ n-1,\ ...,\ n-r+1]}{[1,\ 2,\ ...,\ r]} = \frac{[n,\ n-1,\ ...,\ n-r]}{[1,\ 2,\ ...,\ r,\ r+1]}$$

that is, if and only if

$$[n,\ n-1,\ ...,\ n-r+1].[1,\ 2,\ ...,\ r,\ r+1] = [n,\ n-1,\ ...,\ n-r].[1,\ 2,\ ...,\ r]$$

that is, if and only if

$$[n,\ n-1,\ ...,\ n-r+1].\frac{[1,\ 2,\ ...,\ r](r+1)}{([1,\ 2,\ ...,\ r],\ r+1)} = \frac{[n,\ n-1,\ ...,\ n-r+1](n-r)}{([n,\ n-1,\ ...,\ n-r+1],\ n-r)}.[1,\ 2,\ ...,\ r],$$

which reduces to (6.4.3) after simplification. ■

***Lemma 6.4.13* :** If n is an odd (positive) integer, then the equation (6.4.2) has always a solution.

***Proof* :** We show that

$$SL(2r+1, r) = SL(2r+1, r+1) \text{ for any integer } r \geq 1.$$

In this case, the necessary and sufficient condition (6.4.3) takes the form

$$([2r+1,\ 2r,\ ...,\ r+2],\ r+1)(r+1) = ([1,\ 2,\ ...,\ r],\ r+1)(r+1).$$

Now, since

$$([2r+1,\ 2r,\ ...,\ r+2],\ r+1) = ([1,\ 2,\ ...,\ r],\ r+1) \text{ for any integer } r \geq 1,$$

we see that (6.4.3) is satisfied, which, in turn, establishes the result. ■

If n is an even integer, the equation (6.4.2) may not have a solution. A counter-example is the case when $n = 4$. However, we have the following result.

***Lemma 6.4.14* :** If n is an integer of the form $n = 2p+1$, where p is a prime, then

$$SL(2p, p) = SL(2p, p+1)$$

if and only if

$$([1,\ 2,\ ...,\ p],\ p+1) = p+1.$$

***Proof* :** If $n = 2p+1$, then the l.h.s. of the condition (6.4.3) is

$$([2p,\ 2p-1,\ ...,\ p+1],\ p)(p+1) = p(p+1),$$

which, together with the r.h.s. of (6.4.3), gives the desired condition. ■

***Conjecture 6.4.1* :** The equation $SL(n, r) = SL(n, r+1)$ has always a solution for any $n \geq 5$.

In the worst case, $SL(n, n-1) = SL(n, n) = 1$, and the necessary and sufficient condition is that n divides $[1, 2, \ldots, n-1]$.

Another interesting problem is to find the solution of the equation

$$SL(n+1, r) = SL(n, r). \tag{6.4.4}$$

The equation (6.4.4) holds for some n and r if and only if

$$\frac{[n,\ n-1,\ ...,\ n-r+1]}{[1,\ 2,\ ...,\ r]} = \frac{[n+1,\ n,\ ...,\ n-r+2]}{[1,\ 2,\ ...,\ r]}$$

that is, if and only if

$$\frac{[n,\ n-1,\ ...,\ n-r+2].(n-r+1)}{([n,\ n-1,\ ...,\ n-r+2],\ n-r+1)} = \frac{(n+1).[n,\ n-1,\ ...,\ n-r+2]}{([n,\ n-1,...,\ n-r+2],\ n+1)},$$

which, after simplification, leads to

$$(n-r+1).([n,\ n-1,...,\ n-r+2],\ n+1) = (n+1).([n,\ n-1,\ ...,\ n-r+2],\ n-r+1), \tag{6.4.5}$$

which is the necessary and sufficient condition for the equation (6.4.4) to hold.

From (6.4.5), we observe the following facts :

(1) $n+1$ cannot be prime, for otherwise,

$$([n,\ n-1, ...,\ n-r+2],\ n+1) = 1,$$

which leads to a contradiction.

(2) In (6.4.5),

$$[n,\ n-1, ...,\ n-r+2],\ n+1) = n+1 \quad \Leftrightarrow \quad ([n,\ n-1,\ ...,\ n-r+2],\ n-r+1) = n-r+1.$$

(3) In (6.4.5), if $n-r+1=2$, then

$$([n,\ n-1,\ ...,\ n-r+2],\ n-r+1) = 2 \quad \Rightarrow \quad [n,\ n-1, ...,\ n-r+2],\ n+1) = n+1.$$

(4) If $n-r+1 \neq 2$ is prime, then

$$([n,\ n-1,\ ...,\ n-r+2],\ n-r+1) = 1$$

$$\Rightarrow \quad n+1 = (n-r+1).([n,\ n-1, ...,\ n-r+2],\ n+1) \tag{6.4.6}$$

$$\Rightarrow \quad n+1 = \frac{([n,\ n-1, ...,\ n-r+2],\ n+1)}{([n,\ n-1, ...,\ n-r+2],\ n+1)-1} r,$$

after simplification, showing that $([n,\ n-1, ...,\ n-r+2],\ n+1)-1$ must divide r.

(5) In (6.4.5), if $([n,\ n-1, ...,\ n-r+2],\ n+1) = n+1,$ then $n-r+1$ cannot be an odd prime, for otherwise, by (6.4.6),

$$n+1 = \frac{n+1}{(n+1)-1} r \quad \Rightarrow \quad n = r,$$

which leads to a contradiction.

Conjecture 6.4.2 **:** The equation $SL(n+1, r) = SL(n, r)$ has always a solution for any $r \geq 3$.

In the worst case, $SL(r+1, r) = SL(r, r) = 1$, and the necessary and sufficient condition is that $([1,\ 2, ...,\ r],\ r+1) = r+1$.

Remark 6.4.1 **:** Khairnar, Vyawahare and Salunke(14) mentioned some identities involving the ratio and sum of reciprocals of two consecutive LCM ratios. The validity of these results depends on the fact that $SL(n, r)$ can be expressed as

$$SL(n,\ r) = \frac{n(n-1) \,....\, (n-r+1)}{r!}. \tag{6.4.7}$$

If SL(n, r) can be represented as in (6.4.7), it can be deduced that

$$\frac{SL(n,\ r+1)}{SL(n,\ r)} = \frac{n-r}{r+1}, \quad \frac{1}{SL(n,\ r)} + \frac{1}{SL(n,\ r+1)} = \frac{n+1}{(r+1).SL(n,\ r+1)}.$$

However, the above results are valid only under certain conditions on n and r. For example, for $r=2$, the above two identities are valid only for odd (positive) integers n. Thus, the next question is : What are the conditions on n and r for (6.4.7)?

If $r=p$, where p is a prime, then $SL(p!-1, p)$ can be expressed as in (6.4.7), because in such a case

$$SL(p!\ -1,\ p) = \frac{[p!\ -1,\ p!\ -2,\ ...,\ p!\ -p]}{[1,\ 2,\ ...,\ p]} = \frac{(p!\ -1)(p!\ -2) \,...\, (p!\ -p)}{p!}.$$

In the Appendix, the expressions of SL(n, 1), SL(n, 2), SL(n, 3), SL(n, 4) and SL(n, 5) are derived directly from the definition. Some identities involving SL(n, 1), SL(n, 2), SL(n, 3), SL(n, 4) and SL(n, 5) are also given.

APPENDIX

The derivations of the expressions for SL(n, 1), SL(n, 2), SL(n, 3), SL(n, 4) and SL(n, 5) are given in Theorem A.1 – Theorem A.5 below.

***Theorem A.1* :** For any $n \geq 1$, $SL(n, 1) = n$.

***Proof* :** follows immediately from Definition 6.4.2. ■

***Theorem A.2* :** For any $n \geq 2$, $SL(n, 2) = \frac{n(n-1)}{2}$.

***Proof* :** Since for any positive integer n, $(n, n-1) = 1$, it follows from Lemma 6.4.1 that $[n, n-1] = n(n-1)$. Thus, the result follows from Definition 6.4.2. ■

***Theorem A.3* :** For any $n \geq 3$,

$$SL(n,3) = \begin{cases} \frac{n(n-1)(n-2)}{6}, & \text{if n is odd} \\ \frac{n(n-1)(n-2)}{12}, & \text{if n is even} \end{cases}$$

***Proof* :** By Definition 6.4.2,

$$SL(n,3) = \frac{[n,\ n-1,\ n-2]}{[1,\ 2,\ 3]} = \frac{[n,\ n-1,\ n-2]}{6};\ n \geq 3. \tag{1}$$

To find $[n, n-1, n-2]$, we consider the two possible cases separately below :

***Case 1* :** When n is odd, say $n = 2m + 1$ for some integer $m \geq 1$. In this case,

$$[n, n-1, n-2] = [2m+1, 2m, 2m-1] = \big[[2m+1, 2m], 2m-1\big].$$

But,

$$[2m+1, 2m] = 2m(2m+1),$$

$$\big[[2m+1, 2m], 2m-1\big] = [2m(2m+1), 2m-1] = 2m(2m+1)(2m-1).$$

Now, inserting the above expression in (1), the result follows.

***Case 2* :** When n is even, say $n = 2m$ for some integer $m \geq 2$. In this case, since

$$(n, n-2) = (2m, 2m-2) = 2,$$

it follows that

$$\begin{aligned} [n, n-1, n-2] &= \big[[2m, 2m-2], 2m-1\big] \\ &= \Big[\frac{2m(2m-2)}{2}, 2m-1\Big] \\ &= m(2m-2)(2m-1). \end{aligned}$$

The above expression now gives the desired result. ■

***Theorem A.4* :** For any $n \geq 4$,

$$SL(n,4) = \begin{cases} \frac{n(n-1)(n-2)(n-3)}{72}, & \text{if 3 divides n} \\ \frac{n(n-1)(n-2)(n-3)}{24}, & \text{if 3 does not divide n} \end{cases}$$

***Proof* :** Appealing to Definition 6.4.2, we get

$$SL(n,4) = \frac{[n,\ n-1,\ n-2, n-3]}{[1,\ 2,\ 3,\ 4]} = \frac{[n,\ n-1,\ n-2,\ n-3]}{12};\ n \geq 4. \tag{2}$$

To find $[n, n-1, n-2, n-3]$, we consider the three possibilities separately below :

***Case 1* :** When n is of the form $n = 3m$ for some integer $m \geq 2$.
In this case,

$$(n, n-3) = (3m, 3m-3) = 3, \ (n-1, n-2) = 1,$$

so that

$$[n, n-3] = [3m, 3m-3] = \frac{3m(3m-3)}{3},$$

$$[n-1, n-2] = [3m-1, 3m-2] = (3m-1)(3m-2).$$

Therefore,

$$\begin{aligned}[n, n-1, n-2, n-3] &= [3m, 3m-1, 3m-2, 3m-3] \\ &= \big[[3m, 3m-3], [3m-1, 3m-2]\big] \\ &= \Big[\frac{3m(3m-3)}{3}, (3m-1)(3m-2)\Big] \\ &= \frac{3m(3m-1)(3m-2)(3m-3)}{3\times 2} \end{aligned} \tag{3}$$

where the last expression follows by virtue of the fact that

$$\big(3m(m-1), (3m-1)(3m-2)\big) = 2.$$

Now, (2) and (3) give the desired result.

***Case 2* :** When n is of the form $n = 3m+1$ for some integer $m \geq 1$.
In this case,

$$(n, n-1) = (3m+1, 3m) = 1 = (3m-1, 3m-2) = (n-2, n-3).$$

Therefore,

$$\begin{aligned}[n, n-1, n-2, n-3] &= [3m+1, 3m, 3m-1, 3m-2] \\ &= \big[[3m+1, 3m], [3m-1, 3m-2]\big] \\ &= [3m(3m+1), (3m-1)(3m-2)] \\ &= \frac{3m(3m-1)(3m-2)(3m-3)}{2} \end{aligned} \tag{4}$$

using the fact that

$$\big(3m(3m+1), (3m-1)(3m-2)\big) = 2.$$

Substituting the expression of $[n, n-1, n-2, n-3]$ in (2), we get the result.

***Case 3* :** When n is of the form $n = 3m+2$ for some integer $m \geq 1$.
In this case, the proof is similar to that of Case 2 above, and is left with the reader. ■

In course of proving Theorem A.4, we found explicit expressions of $[n, n-1, n-2, n-3]$, which are summarized in the following lemma. These values would be required later for the proof of Theorem A.5.

***Lemma A.1* :** For any $n \geq 4$,

$$[n, n-1, n-2, n-3] = \begin{cases} \dfrac{n(n-1)(n-2)(n-3)}{6}, & \text{if 3 divides n} \\ \dfrac{n(n-1)(n-2)(n-3)}{2}, & \text{if 3 does not divide n} \end{cases}$$

Theorem A.5 : For any $n \geq 5$,

$$SL(n,5) = \begin{cases} \frac{n(n-1)(n-2)(n-3)(n-4)}{1440}, & \text{if} \quad n = 12m, 12m+4 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{360}, & \text{if} \quad n = 12m+1, 12m+3, 12m+7, 12m+9 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{240}, & \text{if} \quad n = 12m+2 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{120}, & \text{if} \quad n = 12m+5, 12m+11 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{720}, & \text{if} \quad n = 12m+6, 12m+10 \\ \frac{n(n-1)(n-2)(n-3)(n-4)}{480}, & \text{if} \quad n = 12m+8 \end{cases}$$

Proof : By Definition 6.4.2,

$$SL(n,5) = \frac{[n,\ n-1,\ n-2, n-3,\ n-4]}{[1,\ 2,\ 3,\ 4, 5]} = \frac{[n,\ n-1,\ n-2,\ n-3,\ n-4]}{60};\ n \geq 5. \qquad (5)$$

Now, by Lemma 6.4.1,

$$[n, n-1, n-2, n-3, n-4] = \big[[n, n-1, n-2, n-3], n-4\big]$$
$$= \frac{[n,\ n-1,\ n-2,\ n-3](n-4)}{([n,\ n-1,\ n-2,\ n-3],\ n-4)}. \qquad (6)$$

To find $\big([n, n-1, n-2, n-3], n-4\big)$, we consider the following 12 possibilities that may arise :

Case 1 : When n is of the form $n = 12m$ for some integer $m \geq 1$.

In this case, by Lemma A.1,

$$[n, n-1, n-2, n-3] = [12m, 12m-1, 12m-2, 12m-3]$$
$$= \frac{12m(12m-1)(12m-2)(12m-3)}{6}$$
$$= 12m(12m-1)(6m-1)(4m-1),$$

so that,

$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12m(12m-1)(6m-1)(4m-1), 12m-4\big) = 4.$$

Therefore, from (6), using Lemma A.1, we get

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{6 \times 4}.$$

Plugging in this expression in (5), we get the desired result.

Case 2 : When n is of the form $n = 12m+1$ for some integer $m \geq 1$.

Here,

$$[n, n-1, n-2, n-3] = [12m+1, 12m, 12m-1, 12m-2]$$
$$= \frac{12m(12m+1)(12m-1)(12m-2)}{2}$$
$$= 12m(12m+1)(12m-1)(6m-1),$$

so that,

$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12m(12m+1)(12m-1)(6m-1), 12m-3\big) = 3,$$

and hence

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \times 3}.$$

Case 3 **:** When n is of the form $n = 12m + 2$ for some integer $m \geq 1$.

In this case,

$$[n, n-1, n-2, n-3] = [12m+2,\ 12m+1,\ 12m,\ 12m-1]$$
$$= \frac{12m(12m+1)(12m+2)(12m-1)}{2}$$
$$= 12m(12m+1)(6m+1)(12m-1),$$

and hence,

$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12m(12m+1)(6m+1)(12m-1),\ 12m-2\big) = 2,$$

and

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2\times 2}.$$

Case 4 **:** When n is of the form $n = 12m + 3$ for some integer $m \geq 1$.

In this case, $[n, n-1, n-2, n-3]$ and $\big([n, n-1, n-2, n-3], n-4\big)$ are given as follows :

$$[n, n-1, n-2, n-3] = [12m+3,\ 12m+2,\ 12m+1,\ 12m]$$
$$= \frac{12m(12m+1)(12m+2)(12m+3)}{6}$$
$$= 12m(12m+1)(6m+1)(4m+1),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12m(12m+1)(6m+1)(4m+1),\ 12m-1\big) = 1.$$

Consequently,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{6}.$$

Case 5 **:** When n is of the form $n = 12m + 4$ for some integer $m \geq 1$.

Corresponding to this case, $[n, n-1, n-2, n-3]$ and $\big([n, n-1, n-2, n-3], n-4\big)$ are

$$[n, n-1, n-2, n-3] = [12m+4,\ 12m+3,\ 12m+2,\ 12m+1]$$
$$= \frac{(12m+1)(12m+2)(12m+3)(12m+4)}{2}$$
$$= 12(12m+1)(6m+1)(4m+1)(3m+1),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(12m+1)(6m+1)(4m+1)(3m+1),\ 12m\big) = 12,$$

and so,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2\times 12}.$$

Case 6 **:** When n is of the form $n = 12m + 5$ for some integer $m \geq 0$.

In this case, since

$$[n, n-1, n-2, n-3] = [12m+5,\ 12m+4,\ 12m+3,\ 12m+2]$$
$$= \frac{(12m+2)(12m+3)(12m+4)(12m+5)}{2}$$
$$= 12(6m+1)(4m+1)(3m+1)(12m+5),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(6m+1)(4m+1)(3m+1)(12m+5),\ 12m+1\big) = 1,$$

we get,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2}.$$

***Case 7* :** When n is of the form $n = 12m + 6$ for some integer $m \geq 0$.

In this case, the expressions for $[n, n-1, n-2, n-3]$ and $\big([n, n-1, n-2, n-3], n-4\big)$ are

$$[n, n-1, n-2, n-3] = [12m+6, 12m+5, 12m+4, 12m+3]$$
$$= \frac{(12m+3)(12m+4)(12m+5)(12m+6)}{6}$$
$$= 12(4m+1)(3m+1)(12m+5)(2m+1),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(4m+1)(3m+1)(12m+5)(2m+1), 12m+2\big) = 2,$$

and hence,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{6 \times 2}.$$

***Case 8* :** When n is of the form $n = 12m + 7$ for some integer $m \geq 0$.

Here,

$$[n, n-1, n-2, n-3] = [12m+7, 12m+6, 12m+5, 12m+4]$$
$$= \frac{(12m+4)(12m+5)(12m+6)(12m+7)}{2}$$
$$= 12(3m+1)(12m+5)(2m+1)(12m+7),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(3m+1)(12m+5)(2m+1)(12m+7), 12m+3\big) = 3,$$

and consequently,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \times 3}.$$

***Case 9* :** When n is of the form $n = 12m + 8$ for some integer $m \geq 0$.

Here,

$$[n, n-1, n-2, n-3] = [12m+8, 12m+7, 12m+6, 12m+5]$$
$$= \frac{(12m+5)(12m+6)(12m+7)(12m+8)}{2}$$
$$= 12(12m+5)(2m+1)(12m+7)(3m+2),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(12m+5)(2m+1)(12m+7)(3m+2), 12m+4\big) = 4,$$

and consequently,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \times 4}.$$

***Case 10* :** When n is of the form $n = 12m + 9$ for some integer $m \geq 0$.

In this case, the expressions for $[n, n-1, n-2, n-3]$ and $\big([n, n-1, n-2, n-3], n-4\big)$ are

$$[n, n-1, n-2, n-3] = [12m+9, 12m+8, 12m+7, 12m+6]$$
$$= \frac{(12m+6)(12m+7)(12m+8)(12m+9)}{6}$$
$$= 12(2m+1)(12m+7)(3m+2)(4m+3),$$
$$\big([n, n-1, n-2, n-3], n-4\big) = \big(12(2m+1)(12m+7)(3m+2)(4m+3), 12m+5\big) = 1,$$

and hence,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{6}.$$

Case 11 **:** When n is of the form $n = 12m + 10$ for some integer $m \ge 0$.

Here, the expressions for $[n, n-1, n-2, n-3]$ and $([n, n-1, n-2, n-3], n-4)$ are as follows :

$$\begin{aligned}[n, n-1, n-2, n-3] &= [12m+10, 12m+9, 12m+8, 12m+7] \\ &= \frac{(12m+7)(12m+8)(12m+9)(12m+10)}{2} \\ &= 12(12m+7)(3m+2)(4m+3)(6m+5),\end{aligned}$$

$$([n, n-1, n-2, n-3], n-4) = (12(12m+7)(3m+2)(4m+3)(6m+5), 12m+6) = 6,$$

Therefore,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2\times 6}.$$

Case 12 **:** When n is of the form $n = 12m + 11$ for some integer $m \ge 0$.

In this case, the expressions for $[n, n-1, n-2, n-3]$ and $([n, n-1, n-2, n-3], n-4)$ are

$$\begin{aligned}[n, n-1, n-2, n-3] &= [12m+11, 12m+10, 12m+9, 12m+8] \\ &= \frac{(12m+8)(12m+9)(12m+10)(12m+11)}{2} \\ &= 12(3m+2)(4m+3)(6m+5)(12m+11),\end{aligned}$$

$$([n, n-1, n-2, n-3], n-4) = (12(3m+2)(4m+3)(6m+5)(12m+11), 12m+7) = 1,$$

and hence,

$$[n, n-1, n-2, n-3, n-4] = \frac{n(n-1)(n-2)(n-3)(n-4)}{2}.$$

In each of the above cases, substituting the expression of $[n, n-1, n-2, n-3, n-4]$ in (1), the result follows. ■

In proving Theorem A.5, we also found the expression of $[n, n-1, n-2, n-3, n-4]$, which is summarized below. These values would be necessary to find the expression of $SL(n, 6)$.

Lemma A.2 **:** For any $n \ge 5$,

$$[n, n-1, n-2, n-3, n-4] = \begin{cases} \dfrac{n(n-1)(n-2)(n-3)(n-4)}{1440}, & \text{if } n = 12m,\ 12m+4 \\ \dfrac{n(n-1)(n-2)(n-3)(n-4)}{120}, & \text{if } n = 12m+1,\ 12m+3,\ 12m+7,\ 12m+9 \\ \dfrac{n(n-1)(n-2)(n-3)(n-4)}{720}, & \text{if } n = 12m+2 \\ \dfrac{n(n-1)(n-2)(n-3)(n-4)}{360}, & \text{if } n = 12m+5,\ 12m+11 \\ \dfrac{n(n-1)(n-2)(n-3)(n-4)}{480}, & \text{if } n = 12m+6,\ 12m+10 \\ \dfrac{n(n-1)(n-2)(n-3)(n-4)}{240}, & \text{if } n = 12m+8 \end{cases}$$

Some identities of the forms $SL(n, r) - SL(n-1, r)$ are given in the five lemmas below.

Lemma A.3 **:** $SL(n, 3)$ satisfies the following recurrence relations :

(1) $SL(n, 3) - SL(n-1, 3) = \begin{cases} \dfrac{(n-1)(n-2)(n+3)}{12}, & \text{if } n \ge 5 \text{ is odd} \\ \dfrac{(n-1)(n-2)(6-n)}{12}, & \text{if } n \ge 4 \text{ is even} \end{cases}$

(2) $SL(n, 3) - SL(n-2, 3) = \begin{cases} (n-2)^2, & \text{if } n \ge 5 \text{ is odd} \\ \dfrac{(n-2)^2}{2}, & \text{if } n \ge 6 \text{ is even} \end{cases}$

Proof **:** To prove the lemma, we make use of Theorem A.3.

(1) If $n \geq 5$ is odd, then

$$SL(n, 3) - SL(n-1, 3) = \frac{n(n-1)(n-2)}{6} - \frac{(n-1)(n-2)(n-3)}{12},$$

which gives the desired result after some algebraic manipulation.

Again, if $n \geq 4$ is even, then

$$SL(n, 3) - SL(n-1, 3) = \frac{n(n-1)(n-2)}{12} - \frac{(n-1)(n-2)(n-3)}{6} = \frac{(n-1)(n-2)(6-n)}{6},$$

(2) If $n \geq 5$ is odd, then

$$SL(n, 3) - SL(n-2, 3) = \frac{n(n-1)(n-2)}{6} - \frac{(n-2)(n-3)(n-4)}{6} = \frac{(n-2)(6n-12)}{6},$$

from which the result is immediate.

On the other hand, if $n \geq 4$ is even, then

$$SL(n, 3) - SL(n-2, 3) = \frac{n(n-1)(n-2)}{12} - \frac{(n-2)(n-3)(n-4)}{12} = \frac{(n-2)(6n-12)}{12}.$$

All these complete the proof of the lemma. ■

Lemma A.4 **:** $SL(n, 4)$ satisfies the recurrence relations below :

(1) $SL(3n, 4) - SL(3n-1, 4) = -\frac{(3n-1)(3n-2)(n-1)(n-2)}{4}, \quad n \geq 2,$

(2) $SL(3n+1, 4) - SL(3n, 4) = \frac{n(n+1)(3n-1)(3n-2)}{4}, \quad n \geq 2,$

(3) $SL(3n+2, 4) - SL(3n+1, 4) = \frac{n(3n-1)(3n+1)}{2}, \quad n \geq 2.$

Proof **:** To prove part (1) of the lemma, using Theorem A.4, we get

$$SL(3n, 4) - SL(3n-1, 4) = \frac{3n(3n-1)(3n-2)(3n-3)}{72} - \frac{(3n-1)(3n-2)(3n-3)(3n-4)}{24},$$

which then gives the desired result after some algebraic manipulation.

To prove part (2), note that, by Theorem A.4,

$$SL(3n+1, 4) - SL(3n, 4) = \frac{3n(3n+1)(3n-1)(3n-2)}{24} - \frac{3n(3n-1)(3n-2)(3n-3)}{72},$$

giving the desired result after simplification.

Finally, since

$$SL(3n+2, 4) - SL(3n+1, 4) = \frac{3n(3n-1)(3n+1)(3n+2)}{24} - \frac{3n(3n+1)(3n-1)(3n-2)}{24},$$

part (3) of the lemma follows. ■

The following lemma involves three identities of the forms $SL(3n+1, 4) - SL(3n-1, 4)$, $SL(3n+2, 4) - SL(3n, 4)$ and $SL(3n+3, 4) - SL(3n+1, 4)$.

Lemma A.5 **:** $SL(n, 4)$ satisfies the following recurrence formulas :

(1) $SL(3n+1, 4) - SL(3n-1, 4) = \frac{(3n-1)(3n-2)(2n-1)}{2}, \quad n \geq 2,$

(2) $SL(3n+2, 4) - SL(3n, 4) = \frac{n^2(3n-1)(3n+7)}{4}, \quad n \geq 2,$

(3) $SL(3n+3, 4) - SL(3n+1, 4) = \frac{n^2(3n+1)(3n-7)}{4}, \quad n \geq 3.$

***Proof* :** Since

(1) $SL(3n+1,\ 4) - SL(3n-1,\ 4) = \frac{3n(3n+1)(3n-1)(3n-2)}{24} - \frac{(3n-1)(3n-2)(3n-3)(3n-4)}{24}$,

(2) $SL(3n+2,\ 4) - SL(3n,\ 4) = \frac{3n(3n+2)(3n+1)(3n-1)}{24} - \frac{3n(3n-1)(3n-2)(3n-3)}{72}$,

(3) $SL(3n+3,\ 4) - SL(3n+1,\ 4) = \frac{3n(3n+3)(3n+2)(3n+1)}{72} - \frac{3n(3n+1)(3n-1)(3n-2)}{24}$,

the result follows after simplifications. ■

***Lemma A.6* :** If $n \geq 7$ is not divisible by 3, then

$$SL(n,\ 4) - SL(n-3,\ 4) = \frac{(n-3)(n^2-6n+10)}{2}.$$

***Proof* :** If $n \geq 7$ is not divisible by 3, then by Theorem A.4,

$$SL(n,\ 4) - SL(n-3,\ 4) = \frac{n(n-1)(n-2)(n-3)}{24} - \frac{(n-3)(n-4)(n-5)(n-6)}{24},$$

which, after simplification, gives the result desired. ■

***Lemma A.7* :** $SL(n, 5)$ satisfies the following recurrence relations :

(1) $SL(12n+1,\ 5) - SL(12n,\ 5) = \frac{1}{5}n(12n-1)(6n-1)(4n-1)(9n+2);\ n \geq 1,$

(2) $SL(12n+2,\ 5) - SL(12n+1,\ 5) = \frac{2}{5}n(12n+1)(12n-1)(6n-1)(n+1);\ n \geq 1,$

(3) $SL(12n+3,\ 5) - SL(12n+2,\ 5) = -\frac{2}{5}n(6n+1)(12n+1)(12n-1)(n-1);\ n \geq 1,$

(4) $SL(12n+4,\ 5) - SL(12n+3,\ 5) = -\frac{1}{5}n(4n+1)(6n+1)(12n+1)(9n-2);\ n \geq 1,$

(5) $SL(12n+5,\ 5) - SL(12n+4,\ 5) = \frac{1}{5}(3n+1)(4n+1)(6n+1)(12n+1)(11n+5);\ n \geq 1,$

(6) $SL(12n+6,\ 5) - SL(12n+5,\ 5) = -2n(3n+1)(4n+1)(6n+1)(12n+5);\ n \geq 0,$

(7) $SL(12n+7,\ 5) - SL(12n+6,\ 5) = \frac{6}{5}(2n+1)(3n+1)(4n+1)(12n+5)(n+1);\ n \geq 0,$

(8) $SL(12n+8,\ 5) - SL(12n+7,\ 5) = -\frac{1}{5}(2n+1)(3n+1)(12n+5)(12n+7)(n-1);\ n \geq 0,$

(9) $SL(12n+9,\ 5) - SL(12n+8,\ 5) = \frac{1}{5}(2n+1)(3n+2)(12n+5)(12n+7)(n+2);\ n \geq 0,$

(10) $SL(12n+10,\ 5) - SL(12n+9,\ 5) = -\frac{7}{5}n(2n+1)(3n+2)(4n+3)(12n+7);\ n \geq 0,$

(11) $SL(12n+11,\ 5) - SL(12n+10,\ 5) = 2(3n+2)(4n+3)(6n+5)(12n+7)(n+1);\ n \geq 0,$

(12) $SL(12(n+1),\ 5) - SL(12n+11,\ 5) = -\frac{1}{5}(3n+2)(4n+3)(6n+5)(12n+11)(11n+6);\ n \geq 0.$

***Proof* :** follows from Theorem A.5. ■

In parts (6) – (12) of Lemma A.7 above, the case $n = 0$ can easily be verified using the values of the sequence $\{SL(n,\ 5)\}_{n=5}^{\infty}$, the first few terms of which are given by

1, 1, 7, 14, 42, 42, 462, 66, 429, 1001, 1001, 364, 6188, 1428, … .

It is indeed interesting to find that the LCM ratio function $SL(n, r)$ is related to function $SS(n)$, called the Sandor-Smarandache function, treated in the next section. The function $SS(n)$ itself is related to the binomial coefficient $\binom{n}{k}$.

6.5 Sandor-Smarandache Function

Sandor[19] introduced a new Smarandache type function, involving the binomial coefficient. The new function would be called the Sandor-Smarandache function.

Given any integer n (≥ 1), the binomial coefficient, denoted by C(n, k), is defined by

$$C(n, k) \equiv \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad 0 \leq k \leq n,$$

where, by definition,

$$C(n, 0) = 1 \text{ for all } n \geq 1.$$

Then, the Sandor-Smarandache function, denoted by SS(n) is defined as follows :

Definition 6.5.1 **:** For $n \geq 3$ fixed, the *Sandor-Smarandache function*, SS(n), is defined by

$$SS(n) = \max\left\{k : 1 \leq k \leq n-2, \text{ n divides } \binom{n}{k}\right\},$$

where, by convention,

$$SS(1) = 1, \; SS(2) = 1.$$

Clearly, for any n (≥ 2), n divides $\binom{n}{1} = \binom{n}{n-1}$, so that, as Sandor[19] has pointed out, the condition $k < n-1$ is introduced to make the problem non-trivial. However, observe that 6 does not divide any of the three numbers $\binom{6}{2} = \binom{6}{4} = 15$, $\binom{6}{3} = 20$, and it divides only the two trivial ones, namely, $\binom{6}{1} = \binom{6}{5}$. Similar case happens with $n = 4$. Thus,

$$SS(4) = 1, \; SS(6) = 1.$$

More generally,

$$SS(n) \geq 1 \text{ for any integer } n \geq 1.$$

Since

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

we have the following theorem.

Theorem 6.5.1 **:** For some n (≥ 2) and k, n divides $\binom{n}{k}$ if and only if k divides $\binom{n-1}{k-1}$.

Moreover, we have the following result, proved by Sandor[19].

Theorem 6.5.2 **:** If $(n, k) = 1$, then n divides $\binom{n}{k}$.

Since $(57, 4) = 1$, the above Theorem 6.5.2 guarantees that 57 divides $\binom{57}{4}$. By virtue of Theorem 6.5.1, 4 divides $\binom{56}{3}$. Note that 57 divides $\binom{57}{6}$, though $(57, 6) = 3$. This example shows that the condition in Theorem 6.5.2 is sufficient but not necessary.

***Remark 6.5.1* :** Since, for any integer $n \geq 2$ fixed and $1 \leq k \leq n-1$, we have $\binom{n}{k} = \binom{n}{n-k}$,

Definition 6.5.1 is equivalent to the following one :

$$SS(n) = \min\left\{k : 1 \leq k \leq n-2, \text{ n divides } \binom{n}{n-k}\right\}.$$

In calculating the binomial coefficient $C(n, k) = \binom{n}{k}$ when $k < 2n$, the (computationally) useful formula is the following :

$$C(n, k) = \binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!},\ 0 \leq k \leq n.$$

For finding SS(n), the above formula has been employed.

In view of Remark 6.5.1 above, we may redefine SS(n) as follows :

$$SS(n) = \max\left\{k : 1 \leq k \leq n-2, \text{ n divides } \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}\right\},$$

***Lemma 6.5.1* :** If n (≥ 3) is an odd integer, then $SS(n) = n-2$.

***Proof* :** Let $n = 2a+1$ for some integer $a \geq 1$. Then, since

$$C(n, n-2) \equiv \binom{2a+1}{2a-1} = \binom{2a+1}{2} = \frac{2a(2a+1)}{2},$$

obviously $n = 2a+1$ divides $C(n, n-2)$. This proves the lemma. ■

An immediate consequence of the above lemma are the following two corollaries.

***Corollary 6.5.1* :** $SS(p) = p-2$ for any prime $p \geq 3$.

***Corollary 6.5.2* :** $SS(pq) = pq-2$ for any two primes p and q with $q > p > 2$.

Corollary 6.5.2 may be extended to any number of odd primes as follows :

***Corollary 6.5.3* :** If $p_1, p_2, \ldots, p_n$ be any n number of odd primes, then

$$SS(p_1\, p_2 \ldots p_n) = p_1\, p_2 \ldots p_n - 2.$$

***Corollary 6.5.4* :** The function SS(n) is not multiplicative.

***Proof* :** Let p and q be two distinct odd primes. Then, by Corollary 6.5.2,

$$SS(pq) = pq-2 \neq (p-2)(q-2) = SS(p)\, SS(q). \ ■$$

***Lemma 6.5.2* :** For some integer n (≥ 3), $SS(n) = n-2$ if and only if n is odd.

***Proof* :** The "if" part of the lemma follows from Lemma 6.5.1. To prove the converse, let $SS(n) = n-2$, so that $n-2$ ($< n-1$) is the maximum number such that n divides

$$\binom{n}{n-2} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Then, n must be odd, for otherwise, we are led to a contradiction. ■

***Corollary 6.5.5* :** $SS(n) > n-2$ for any composite number n (≥ 4).

***Lemma 6.5.3* :** Let n be an integer of the form 2a, where a is not a multiple of 3. Then,

$$SS(n) = n-3.$$

***Proof* :** Let $n=2a$ $(a\geq 1)$ be an integer not divisible by 3. Clearly, 2a does not divide

$$\binom{2a}{2a-2}=\frac{2a(2a-1)}{2}.$$

Now, consider

$$\binom{2a}{2a-3}=\frac{2a(2a-1)(2a-2)}{2\times 3}=\frac{2a(2a-1)(a-1)}{3}.$$

Here, since $2(a-1)$, $2a-1$ and $2a$ are three consecutive integers, one of them must be divisible by 3; and hence, one of $2(a-1)$ and $2a-1$ is divisible by 3 (since, a is not a multiple of 3). Thus, 6 divides $2(a-1)(2a-1)$. Hence, $n=2a$ divides $\binom{2a}{2a-3}$. ■

***Corollary 6.5.6* :** $SS(2p)=2p-3$ for any prime $p\geq 5$.
***Proof* :** follows from Lemma 6.5.3. ■

Generalizing Corollary 6.5.6 above to any number of odd primes, we get the following

***Corollary 6.5.7* :** $SS(2p_1 p_2 \ldots p_n)=2p_1 p_2 \ldots p_n-3$ for any odd primes $p_1, p_2, \ldots, p_n$.

***Lemma 6.5.4* :** Let p (≥ 2) be a prime and n (≥ 1) be any integer. Then,

$$SS(p^n)=\begin{cases} p^n-2, & \text{if } \; p\geq 3 \\ p^n-3, & \text{if } \; p=2,\ n\geq 3 \end{cases}$$

***Proof* :** By Lemma 6.5.2, it remains to prove the second part only. To do so, consider

$$2^n\left[\frac{(2^n-1)(2^n-2)}{2\times 3}\right],$$

where the term $(2^n-1)(2^n-2)$ is obviously divisible by 6. ■

***Lemma 6.5.5* :** Let p (≥ 3) be a prime and n (≥ 1) be any integer. Then,

$$SS(2p^n)=\begin{cases} 2p^n-3, & \text{if } \; p\neq 3 \\ 2p^n-4, & \text{if } \; p=3 \text{ and n is even} \\ 2p^n-5, & \text{if } \; p=3 \text{ and n is odd} \end{cases}$$

***Proof* :** First, let $p\neq 3$. Then, the first part is a consequence of Lemma 6.5.3. Next, let $p=3$. Here, we have to consider separately the two possibilities :
Case (A) : When n is even, say, $n=2b$ for some integer $b\geq 1$.
Here, consider the expression below :

$$2.3^{2b}\left[\frac{(2.3^{2b}-1)(2.3^{2b}-2)(2.3^{2b}-3)}{2\times 3\times 4}\right]$$
$$=2.3^{2b}\left[\frac{(2.3^{2b}-1)(3^{2b}-1)(2.3^{2b-1}-1)}{4}\right].$$

Since $3^{2b}-1=(3^b-1)(3^b+1)$ is divisible by 4, the second part of the lemma follows.
Next, let n be odd, say, $n=2c+1$ for some integer $c\geq 1$. Now, consider the expression

$$2.3^{2c+1}\left[\frac{(2.3^{2c+1}-1)(2.3^{2c+1}-2)(2.3^{2c+1}-3)(2.3^{2c+1}-4)}{2\times 3\times 4\times 5}\right]$$
$$=2.3^{2c+1}\left[\frac{(2.3^{2c+1}-1)(3^{2c+1}-1)(2.3^{2c}-1)(3^{2c+1}-2)}{2\times 5}\right].$$

Clearly, 10 divides $(2.3^{2c+1}-1)(3^{2c}-1)(2.3^{2c}-1)(3^{2c+1}-2)$ (since 5 does not divide 3^{2c+1}).
All these complete the proof of the lemma. ■

***Lemma 6.5.6* :** Let n be an integer of the form $n = 6m$, $m \geq 1$. Then,

$SS(n) = n - 4$, if m is of the form $4s + 3$ for any integer $s \geq 0$.

***Proof* :** Clearly, $n = 6m$ does not divide any of $\binom{6m}{6m-2}$ and $\binom{6m}{6m-3}$. Consider

$$\binom{6m}{6m-4} = 6m\left[\frac{(6m-1)(6m-2)(6m-3)}{2\times3\times4}\right] = 6m\left[\frac{(6m-1)(3m-1)(2m-1)}{4}\right].$$

Now, if $m = 4s + 3$ for some integer $s \geq 0$, then $3m - 1 = 4(3s + 2)$ is divisible by 4, and hence, $n = 6m$ divides $\binom{6m}{6m-4} = \binom{n}{n-4}$.

This proves the lemma. ■

Observe that, in Lemma 6.5.6, $m = 4s + 3$ for some integer $s \geq 0$ if and only if $3m = 4t + 1$ for some integer $t \geq 2$. The "only if" part is immediate. To prove the other part, let $3m = 4t + 1$. Then, t must be of the form $t = 3s + 2$. Also, note that, Lemma 6.5.6 is valid for $m = p$, where p is a prime of the form $p = 4s + 3$ $(s \geq 0)$. Thus, for example,

$SS(6\times3) = 14$, $SS(6\times7) = 38$, $SS(6\times11) = 62$, $SS(6\times19) = 110$, $SS(6\times23) = 134$.

***Lemma 6.5.7* :** Let m be an integer of the form $m = 4s + 3$, $s \geq 0$. Then,

$SS(30m) = 30m - 4$.

***Proof* :** We start with

$$30m\left[\frac{(30m-1)(30m-2)(30m-3)}{2\times3\times4}\right] = 30m\left[\frac{(30m-1)(15m-1)(10m-1)}{4}\right].$$

Now, note that, if $m = 4s + 3$, then $15m - 1 = 4(15s + 11)$, so that 4 divides the term inside the square bracket. ■

Note that, the lemma above is valid for $m = p$, where $p \geq 7$ is a prime of the form $p = 4s + 3$.

In Lemma 6.5.7 above, observe that 4 divides $15m - 1$ if and only if $m = 4s + 3$ for some integer $s \geq 0$. The proof is as follows : Let $15m - 1 = 4a$ for some integer a, so that $15m = 4a + 1$. The solution of the equation is $a = 15s + 11$, $s \geq 0$. Then, $4a + 1 = 15(4s + 3)$, and we get the desired result. As an illustration of Lemma 6.5.7, we mention the following :

$SS(210) = 206$, $SS(330) = 326$, $SS(450) = 446$, $SS(570) = 566$, $SS(690) = 686$, $SS(810) = 806$.

***Lemma 6.5.8* :** Let $n = 6m$, where 5 does not divide m. Then, $SS(n) \neq n - 6$.

***Proof* :** We start with

$$\binom{6m}{6m-6} = \binom{6m}{6} = 6m\left[\frac{(6m-1)(6m-2)(6m-3)(6m-4)(6m-5)}{2\times3\times4\times5\times6}\right],$$

which simplifies to

$$\binom{6m}{6m-6} = 6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)(6m-5)}{2\times5\times6}\right].$$

Now, inside the square bracket, only one of $3m - 1$ and $3m - 2$ is even. Moreover, if 5 does not divide m, then the last term (that is, $6m - 5$) is inactive. ■

***Lemma 6.5.9* :** Let p be a prime of the form $p = 6u + 1$, $u \geq 1$. Then,

$SS(p + 1) = p - 2$.

***Proof* :** If $p = 6u + 1$, then clearly $p + 1$ divides

$$\binom{p+1}{3} = \frac{(p+1)p(p-1)}{3!}. \quad ■$$

***Lemma 6.5.10* :** Let p be a prime of the form $p = 6u + 5$ for some integer $u \geq 1$. Then,

$$SS(p + 1) = p - 3$$

if and only if

$$u = 2(2s + 1),\ s \geq 0 \text{ being an integer.}$$

***Proof* :** We start with the expression

$$\frac{(p+1)p(p-1)(p-2)}{4!}.$$

When $p = 6u + 5$, the above expression simplifies to

$$(p+1)p\left[\frac{(3u+2)(2u+1)}{4}\right].$$

Thus, in order that the expression inside the square bracket is an integer, 4 must divide $3u + 2$, so that

$$3u = 4a - 2 \text{ for some integer } a \geq 1.$$

The solution of the above Diophantine equation is

$$u = 2(2s + 1) \text{ for any integer } s \geq 0.$$

This completes the proof of the lemma. ■

Applying Lemma 6.5.10 for small values of u, we get the following functions :

$$SS(18) = 14,\ SS(42) = 38,\ SS(90) = 86,\ SS(114) = 110.$$

From Lemma 6.5.10, we see that, if $p = 6u + 5$ with $u \neq 2(2s + 1)$ (for any integer $s \geq 0$), then

$$SS(p + 1) \geq p - 4.$$

We now state and prove the following lemma.

***Lemma 6.5.11* :** Let p be a prime of the form $p = 6u + 5$, for some integer $u \geq 1$, with $u \neq 2(2s + 1)$ for any integer $s \geq 0$. Then,

$$SS(p + 1) = p - 4,$$

if and only if u is of any of the three forms : $u = 5x + 1$ $(x \geq 0)$, $u = 5y + 2$ $(y \geq 1)$, $u = 5z + 3$ $(z \geq 0)$.

***Proof* :** With $p = 6u + 5$, we have

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\left[\frac{(3u+2)(2u+1)(3u+1)}{2\times 5}\right].$$

Since one of the two numbers, $3u + 1$ and $3u + 2$, is even, it is sufficient to find the condition such that 5 divides one of the three factors, $3u + 2$, $2u + 1$ and $3u + 1$.

Now, if 5 divides $3u + 2$, then

$$3u = 5a - 2 \text{ for some integer } a \geq 1,$$

with the solution

$$u = 5x + 1 \text{ for any integer } x \geq 0.$$

Again, if 5 divides $2u + 1$, then

$$2u = 5b - 1 \text{ for some integer } b \geq 1,$$

whose solution is

$$u = 5y + 2,\ y \geq 1 \text{ being an integer.}$$

Finally, if 5 divides $3u + 1$, then

$$3u = 5c - 1 \text{ for some integer } c \geq 1.$$

The solution of the above Diophantine equation is

$$u = 5z + 3 \text{ for any integer } z \geq 0.$$

All these complete the proof of the lemma. ■

Using Lemma 6.5.11, we get the following functions :

$SS(12) = 7$, $SS(72) = 67$, $SS(102) = 97$, $SS(132) = 125$, $SS(192) = 187$, $SS(252) = 247$,

$SS(48) = 43$, $SS(108) = 103$, $SS(168) = 163$, $SS(198) = 193$, $SS(228) = 223$,

$SS(24) = 19$, $SS(54) = 49$, $SS(84) = 79$, $SS(174) = 169$.

Lemma 6.5.10 and Lemma 6.5.11 are concerned with the function $SS(p + 1)$, when p is a prime of the form $p = 6u + 5$ (for some integer $u \geq 1$). Lemma 6.5.11 gives the condition that must be satisfied for $SS(p + 1) = p - 4$. It can readily be shown that, the three conditions given in Lemma 6.5.11 are satisfied if and only if p is of one of the three forms $p = 5u + 1$, $p = 5u + 2$ and $p = 5u + 3$ respectively : When $u = 5x + 1$, then $p = 6(5x + 1) + 5 = 5(6x + 2) + 1$, when $u = 5y + 2$, then $p = 6(5y + 2) + 5 = 5(6y + 3) + 2$, and with $u = 5z + 3$, $p = 6(5z + 3) + 5 = 5(6z + 4) + 3$. where p is a prime of the form $p = 5u + 3$. In the next five lemmas, we consider the problem of finding $SS(p + 1)$ when the prime p is of the form $p = 5u + 4$.

***Lemma 6.5.12* :** Let p be a prime of the form $p = 5u + 4$. Then,

$$SS(p + 1) = p - 3,$$

if u is of one of the following three forms :

(1) $u = 24s + 17$, $s \geq 0$ being an integer,

(2) $u = 72s + 41$, $s \geq 0$ is any integer,

(3) $u = 48s + 41$, $s \geq 0$ being an integer.

***Proof* :** With $p = 5u + 4$, the expression

$$(p + 1)p\frac{(p - 1)(p - 2)}{4!},$$

takes the form

$$(p + 1)p\left[\frac{(5u + 3)(5u + 2)}{2\times3\times4}\right].$$

(1) We consider the case when 8 divides $5u + 3$ and 3 divides $5u + 2$. Then,

$$5u = 8a - 3, \; 5u = 3b - 2 \text{ for some integers a } (\geq 1) \text{ and b } (\geq 1),$$

with the respective solutions

$$u = 8c + 1, \; u = 3d + 2; \; c\ (\geq 0) \text{ and } d\ (\geq 0) \text{ being integers.}$$

Then, solving the combined Diophantine equation (that is, $8c = 3d + 1$), we get

$$c = 3s + 2, \; s \geq 0.$$

Therefore,

$$u = 8(3s + 2) + 1 = 24s + 17,$$

which is the desired condition.

(2) We now consider the case when 8 divides $5u + 3$ and 9 divides $5u + 2$, so that

$$5u = 8a - 3, \; 5u = 9b - 2 \text{ for some integers a } (\geq 1) \text{ and b } (\geq 1),$$

with the respective solutions

$$u = 8c + 1, \; u = 9d + 5 \text{ for any integers c } (\geq 0) \text{ and d } (\geq 0).$$

Then, solving the combined Diophantine equation, (namely, $8c = 9d + 4$), we get

$$c = 9s + 5, \; s \geq 0.$$

Therefore, finally

$$u = 8(9s + 5) + 1 = 72s + 41,$$

which is the condition desired.

(3) We consider the case when 16 divides $5u+3$ and 3 divides $5u+2$. In this case, we have

$$5u=16a-3,\ 5u=3b-2 \text{ for some integers } a\ (\geq 1) \text{ and } b\ (\geq 1),$$

whose solutions are

$$u=16c+9,\ u=3d+2 \text{ for any integers } c\ (\geq 0) \text{ and } d\ (\geq 0).$$

Now, solving the combined Diophantine equation, in the form $16c=3d-7$, we get

$$c=3s+2 \text{ for any integer } s\ (\geq 0).$$

Hence, finally

$$u=16(3s+2)+9=48s+41.$$

All these complete the proof of the lemma. ■

Lemma 6.5.13 **:** Let p be a prime of the form $p=5u+4$. Then, $SS(p+1)\neq p-4$.

Proof **:** Considering the expression

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!}=(p+1)p\left[\frac{(5u+3)(5u+2)(5u+1)}{2\times3\times4\times5}\right],$$

the result follows since 5 does not divide any of $5u+3$, $5u+2$ and $5u+1$. ■

Lemma 6.5.14 **:** Let p be a prime of the form $p=5u+4$, $u=72s+59$ ($s\geq 0$ being an integer). Then,

$$SS(p+1)=p-5.$$

Proof **:** With $p=5u+4$, the expression

$$(p+1)p\frac{(p-1)(p-2)(p-3)(p-4)}{6!},$$

becomes

$$(p+1)p\left[\frac{(5u+3)(5u+2)(5u+1)u}{2\times3\times4\times6}\right].$$

Clearly, the product term inside the square bracket is an integer if the numerator is divisible by 16×9. Here, one possibility is that 9 divides $5u+2$ and 8 divides $5u+1$, leading to the equations

$$5u=9a-2,\ 5u=8b-1 \text{ for some integers } a\ (\geq 1) \text{ and } b\ (\geq 1),$$

with the respective solutions

$$u=9c+5,\ u=8d+3;\ c\ (\geq 0) \text{ and } d\ (\geq 0) \text{ being any integers.}$$

Combining these two together, we have to solve the Diophantine equation

$$9c=8d-2,$$

whose solution is

$$c=8s+6,\ s\geq 0 \text{ being an integer.}$$

Therefore,

$$u=9(8s+6)+5=72c+59,$$

which we intended to establish. ■

Lemma 6.5.15 **:** Let p be a prime of the form $p=5u+4$, $u=24s+11$ ($s\geq 0$ being an integer), such that $s\neq 3y+2$ for any integer y (≥ 0) and 7 does not divide $p+1$. Then,

$$SS(p+1)=p-6.$$

Proof **:** Substituting in

$$(p+1)p\frac{(p-1)(p-2)(p-3)(p-4)(p-5)}{7!},$$

$p=5u+4$, we get

$$(p+1)p\left[\frac{(5u+3)(5u+2)(5u+1)(5u-1)u}{2\times3\times4\times6\times7}\right].$$

Now, we consider the case when 3 divides $5u+2$ and 8 divides $5u+1$, so that

$$5u=3a-2,\ 5u=8b-1 \text{ for some integers } a\ (\geq 1) \text{ and } b\ (\geq 1).$$

The solutions of the above equations are

$$u=3c+2,\ u=8d+3 \text{ (for any integers } c\ (\geq 0) \text{ and } d\ (\geq 0))$$

respectively. This, in turn, leads to the Diophantine equation below :

$$3c=8d+1,$$

with the solution

$$c=8s+3,\ s\geq 0 \text{ being an integer.}$$

Thus,

$$u=3(8s+3)+2.$$

Now, if $s = 3y + 2$, then $u = 24s + 11 = 72y + 59$, and by Lemma 6.5.14, $SS(p + 1) = p - 5$; Otherwise, since 7 does not divide $p+1$, $SS(p+1)=p-6$. ■

***Lemma 6.5.16* :** Let p be a prime of the form $p=5u+4$, $u=12s+5$ ($s\geq 0$ being an integer). Then,

$$SS(p+1)=\begin{cases} p-3, & \text{if } s \text{ is odd} \\ p-6, & \text{if } s \text{ is even, } 7 \text{ does not divide } p+1 \end{cases}$$

***Proof* :** We start with

$$(p+1)p\frac{(p-1)(p-2)(p-3)(p-4)(p-5)}{7!}$$

$$=(p+1)p\left[\frac{(5u+3)(5u+2)(5u+1)u(5u-1)}{2\times 3\times 4\times 6\times 7}\right]. \qquad (1)$$

Now, we consider the case when 12 divides $5u-1$, so that

$$5u=12a+1 \text{ for some integer } a\ (\geq 1).$$

The solutions of the above equation is

$$u=12s+5,\ s\geq 0 \text{ being an integer.}$$

Now, note that

$$5u+3=4(15s+7),\ 5u+2=3(20s+9),\ 5u+1=2(30s+23),\ 5u-1=12(5s+2).$$

Thus, if s odd, then $5u+3$ is divisible by 8, and thus, in (1), the term $(5u+3)(5u+2)$ is divisible by $2\times 3\times 4$, so that

$$SS(p+1)=p-3.$$

Otherwise, the term inside the square bracket in (1) is divisible by 7, and hence,

$$SS(p+1)=p-6.$$

All these complete the proof of the lemma. ■

***Example 6.5.1* :** The first few instances of part (1) Lemma 6.5.12 are as follows :

$$SS(90)=86,\ SS(450))=446,\ SS(570)=566,\ SS(810)=806.$$

Part (2) of Lemma 6.5.12 gives the following expressions of SS(n) :

$$SS(570)=566,\ SS(930)=926,\ SS(1290)=1286,\ SS(1650)=1646;$$

and from part (3) of Lemma 6.5.12, we get the following expressions :

$$SS(450)=446,\ SS(930)=926,\ SS(1410)=1406.$$

Lemma 6.5.14 gives the functions $SS(660)=654$, $SS(1020)=1014$, $SS(2100)=2094$.

Using Lemma 6.5.15, we get the first two functions as $SS(180)=173$, $SS(1500)=1493$.

The first few functions obtained from Lemma 6.5.16 are as follows :

$$SS(30)=23,\ SS(90)=86,\ SS(150)=143,\ SS(270)=263.$$

***Remark 6.5.2* :** The eight lemmas, Lemma 6.5.9 – Lemma 6.5.16, are concerned with the problem of finding SS(p + 1), where p is a prime. Since the pattern of the primes is unknown, we have to consider the possible cases. Needless to say, these eight lemmas do not cover all the primes; moreover, some of the cases may be overlapping (in Lemma 6.5.9 – Lemma 6.5.16). Note that, in case of Lemmas 6.5.9 – 6.5.12 as well as in Lemmas 6.5.14 – 6.5.16, both p + 1 and p(p + 1) divide $\binom{p+1}{k}$.

***Remark 6.5.3* :** If u = 5 in Lemma 6.5.16, we get the prime p = 29. Then, by virtue of the lemma, SS(30) = 23. This shows that Lemma 6.5.16 is valid when s = 0 as well. Also, corresponding to u = 17, we get the prime p = 89, with SS(90) = 86; and corresponding to s = 7, we get u = 89 in Lemma 6.5.16, we get the prime p = 449, and hence, SS(450) = 446. In Lemma 6.5.16, we assume that p + 1 is not a multiple of 7 to guarantee that one of the five terms inside the square bracket in (1) is divisible by 7. However, searching for a prime p with p + 1 being divisible by 7, we found that the first such prime is $p = 349 = 5\times69 + 4$, with SS(350) = 346. Note that, in Lemma 6.5.16, the problem of finding p such that p + 1 is a multiple of 7 reduces to the problem of solving the Diophantine equation

$$u - 7a = 1.$$

***Lemma 6.5.17* :** Let n = 6m, where $m \neq 4s + 3$ for any integer $s \geq 0$. Then,

$$SS(6m) = 6m - 5, \text{ if m is not divisible by 5.}$$

***Proof* :** We start with

$$6m\left[\frac{(6m-1)(6m-2)(6m-3)(6m-4)}{2\times3\times4\times5}\right] = 6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)}{2\times5}\right].$$

Clearly, one of the two numbers, 3m – 1 and 3m – 2, is divisible by 2; moreover, the terms in the numerator inside the square bracket is divisible by 5, if m is not a multiple of 5.
Thus, the lemma is proved. ■

The corollary below follows from Lemma 6.5.17 in case of primes.

***Corollary 6.5.8* :** SS(6p) = 6p – 5 for any prime $p \neq 4s + 3$ for any integer $s \geq 0$.

***Lemma 6.5.18* :** Let n = 6m, where $m \neq 4s + 3$ for any integer $s \geq 0$. Then,

$$SS(6m) = \begin{cases} 6m-6, & \text{if } m = 10(6s+5),\ s \geq 0 \\ 6m-7, & \text{if } m \neq 10(6s+5) \text{ for any } s \geq 0, \text{ m is not divisible by 7} \\ 6m-8, & \text{if } m = 35(8s+3),\ s \geq 0 \\ 6m-11, & \text{if } m = 35(12s+7),\ s = 2, 6, 10, \ldots \end{cases}$$

***Proof* :** To find the condition such that SS(6m) = 6m – 6, we start with

$$6m\left[\frac{(6m-1)(6m-2)(6m-3)(6m-4)(6m-5)}{2\times3\times4\times5\times6}\right]$$
$$= 6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)(6m-5)}{2\times5\times6}\right].$$

Now, the numerator inside the square bracket must be divisible by $2\times5\times6$; moreover, the last term 6m – 5 (which is divisible by neither 2 nor 3) must be divisible by 5. Note that, of the five terms, three are odd, and only one of the two terms 3m – 1 and 3m – 2 is even. Thus, the three conditions below must be satisfied simultaneously :

(1) 2m – 1 must be a multiple of 3, (2) 4 must divide 3m – 2, and (3) m must be divisible by 5.
Now, from first condition, we get

$$m = 3a + 2 \text{ for any integer } a \geq 0.$$

Then, since $3m - 2 = 9a + 4$, the second condition requires that $a = 4b$ (for some integer $b \geq 1$), so that

$$m = 2(6b + 1).$$

Again, by the third condition, $m = 5c$ for some integer $c \geq 1$. Combined together, we get

$$2(6b + 1) = 5d,$$

with the solution

$$b = 5s + 4,\ s \geq 0.$$

Thus, finally

$$m = 2\left[6(5s + 4) + 1\right] = 10(6s + 5),\ s \geq 1,$$

which is the desired condition on m such that

$$SS(6m) = 6m - 6.$$

Next, let $m \neq 10(6s + 5)$ for any integer $s \geq 0$. We consider

$$6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)(6m-5)(m-1)}{2\times5\times7}\right].$$

Clearly, the term $(6m - 1)(3m - 1)(2m - 1)(3m - 2)(6m - 5)(m - 1)$ in the numerator inside the square bracket is divisible by $2\times5\times7$, if m is not a multiple of 7. Thus,

$$SS(6m) = 6m - 7, \text{ if 7 does not divide m.}$$

Next, we find the condition such that $SS(6m) = 6m - 8$. To do so, we start with the reduced simplified expression :

$$6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)(6m-5)(m-1)(6m-7)}{2\times5\times7\times8}\right].$$

Clearly, the numerator inside the square bracket on the right is an integer if the following three conditions are satisfied simultaneously : (1) m is a multiple of 5, (2) m is divisible by 7, and (3) 8 divides $m - 1$.

Now, from the first two conditions, we see that 35 must divide m (so that, $m = 35a,\ a \geq 1$); moreover, from the third condition, $m = 8b + 1,\ b \geq 1$. Thus, we have to solve the equation

$$35a = 8b + 1.$$

The solution of the above equation is $a = 8s + 3$, $s \geq 0$ being any integer. Therefore,

$$m = 35(8s + 3),\ s \geq 0.$$

Thus, we get the condition such that

$$SS(6m) = 6m - 8.$$

Finally, we find the condition such that $SS(6m) = 6m - 11$ as follows : Consider the reduced, simplified expression

$$6m\left[\frac{(6m-1)(3m-1)(2m-1)(3m-2)(6m-5)(m-1)(6m-7)(3m-4)(2m-3)(3m-5)}{3\times4\times5\times7\times10\times11}\right].$$

Now, one of $3m - 1$ and $3m - 2$ is even, so that 4 must divide $m - 1$ (so that $m = 4a + 1$ for some integer $a \geq 1$). Also, 35 must divide m (that is, $m = 35b,\ b \geq 1$). Moreover, 3 must divide $2m - 1$. Thus, we have to solve the three simultaneous Diophantine equations :

$$70b = 3c + 1 = 8a + 2.$$

The left-hand side equation gives the solution $b = 3d + 1$, $d \geq 0$, and so, it remains to solve

$$m - 1 = 105d + 34 = 4a,$$

whose solution is $d = 4s + 2$, $s \geq 0$. Hence, finally, we get

$$m = 35(12s + 7),\ s = 2, 6, 10, \ldots,$$

which is the condition we intended to find.

All these complete the proof of the lemma. ■

In the example below, we illustrate Lemma 6.5.17 and Lemma 6.5.18 with the help of some simple examples. Note that, it remains to find SS(6m), when m is divisible by $5\times7\times11$.

Example 6.5.2 **:** Since the number $n = 54$ is of the form $6(4s+1)$, by Lemma 6.5.17 above, $SS(54) = 49$. Similarly, $SS(78) = 73$. Again, the number $n = 30$ is divisible by 5 but not by 7, so that, by Lemma 6.5.18, $SS(30) = 23$. Similarly, $SS(150) = 143$. The number 105 is the first number of the form $35(8s+3)$, and hence, by Lemma 6.5.18, $SS(6\times105) = SS(630) = 622$. Since $2310 = 35\times66$, $SS(2310) = 2302$. Finally, since $n = 1470 = 6\times245 = 6\times(7\times35)$, by Lemma 6.5.18, $SS(1470) = 1459$. By the application of Lemma 6.5.17, we have

$$SS(12) = 7,\ SS(24) = 19,\ SS(36) = 31,\ SS(48) = 43,\ SS(84) = 79,\ SS(324) = 319.$$

We now concentrate our attention to the expression SS(60m). Since

$$60m\frac{(60m-1)(60m-2)}{3!} = 60m\frac{(60m-1)(30m-1)}{3},$$

$$60m\frac{(60m-1)(60m-2)(60m-3)}{4!} = 60m\frac{(60m-1)(30m-1)(20m-1)}{4},$$

$$60m\frac{(60m-1)(60m-2)(60m-3)(60m-4)}{5!} = 60m\frac{(60m-1)(30m-1)(20m-1)(15m-1)}{5},$$

we see that $SS(60m) \neq 60m-3$, $SS(60m) \neq 60m-4$, $SS(60m) \neq 60m-5$ for any integer m.

Lemma 6.5.19 **:** For $m \geq 1$,

$$SS(60m) = \begin{cases} 60m-6, & \text{if} \quad m = 6s+5,\ s \geq 0 \\ 60m-7, & \text{if} \quad m \neq 6s+5 \text{ and 7 does not divide m} \\ 60m-8, & \text{if} \quad m = 7(8s+9),\ s \geq 0 \end{cases}$$

Proof **:** The first part is just a restatement of the first part of Lemma 6.5.18. To prove the second part, consider

$$60m\left[\frac{(60m-1)(30m-1)(20m-1)(15m-1)(12m-1)(10m-1)}{7}\right].$$

Now, $(60m-1)(60m-2)(60m-3)(60m-4)(60m-5)(60m-6)$ is divisible by 7, if m is not a multiple of 7.

Next, we consider

$$60m\frac{(60m-1)(60m-2)(60m-3)(60m-4)(60m-5)(60m-6)(60m-7)}{8!}$$

$$= 60m\left[\frac{(60m-1)(30m-1)(20m-1)(15m-1)(12m-1)(10m-1)(60m-7)}{7\times8}\right].$$

Here, we have to find the condition such that the term inside the square bracket is an integer. So, we consider the case when 7 divides m and 8 divides $15m-1$. Thus,

$$m = 7a,\ 15m = 8b+1 \text{ for some integers } a \geq 1,\ b \geq 1.$$

The solution of the second equation is

$$m = 8b+7 \text{ for any integer } b \geq 0.$$

Now, considering the combined equation

$$7a = 8b+7,$$

we get the solution

$$a = 8s+9,\ s \geq 0 \text{ being an integer},$$

and hence finally,

$$m = 7(8s+9).$$

All these complete the proof of the lemma. ■

***Lemma 6.5.20* :** Let either $m = 7(9s + 4)$, $s \geq 0$, or $m = 7(9t + 2)$, $t \geq 0$. Then,

$$SS(60m) = 60m - 9.$$

***Proof* :** To prove the lemma, we start with

$$60m\frac{(60m-1)(60m-2)(60m-3)(60m-4)(60m-5)(60m-6)(60m-7)(60m-8)}{9!},$$

$$= 60m\left[\frac{(60m-1)(30m-1)(20m-1)(15m-1)(12m-1)(10m-1)(60m-7)(15m-2)}{2\times7\times9}\right].$$

Now, we find the condition such that the term inside the square bracket is an integer.
First, note that, one of the two terms, $15m-1$ and $15m-2$ is even. It is thus sufficient to find the condition that the term inside the square bracket is divisible by 7×9. Now, note that m must be divisible by 7. Thus,

$$m = 7a \text{ for some integer } a \geq 1.$$

Next, note that, if 3 divides $10m - 1$, so that

$$10m = 3\alpha + 1 \text{ for some integer } \alpha \geq 1,$$

then $20m - 1 = 6\alpha + 1$, so that 3 does not divide $20m - 1$. Thus, we have only two possibilities :
Case 1 : When 9 divides $10m - 1$.
Then,

$$10m = 9b + 1 \text{ for some integer } b \geq 1.$$

Thus, we are lead with the two Diophantine equations :

$$m = 7a,\ 10m = 9b + 1.$$

Combining together, we have, $70a = 9b + 1$, whose solution is

$$a = 9s + 4,\ s \geq 0 \text{ being an integer.}$$

Hence,

$$m = 7(9s + 4),\ s \geq 0.$$

Case 2 : When 9 divides $20m - 1$.
In this case,

$$20m = 9c + 1 \text{ for some integer } c \geq 1,$$

so that we are faced with the two Diophantine equations :

$$m = 7a,\ 20m = 9c + 1.$$

Writing the second equation as $140a = 9c + 1$, we get the solution

$$a = 9t + 2 \text{ for any integer } t \geq 0.$$

Consequently,

$$m = 7(9s + 4),\ t \geq 0,$$

which is the desired condition.
Thus, the lemma is established. ■

***Lemma 6.5.21* :** $SS(60m) = 60m - 10$ if $m = 42(10s + 9)$, $s \geq 0$ being any integer.
***Proof* :** we start with

$$60m\frac{(60m-1)(60m-2)(60m-3)(60m-4)(60m-5)(60m-6)(60m-7)(60m-8)(60m-9)}{10!}$$

$$= 60m\left[\frac{(60m-1)(30m-1)(20m-1)(15m-1)(12m-1)(10m-1)(60m-7)(15m-2)}{2\times3\times4\times5\times7}\right].$$

We now have to find the condition such that the term inside the square bracket is an integer. To do so, we consider the case when 7 divides m, 3 divides $20m-3$, 5 divides $12m-1$, and 4 divides $15m-2$, so that, we have,

$$m=7a,\ 20m=3b+3,\ 12m=5c+1,\ 15m=4d+2$$

for some integers $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d \geq 1$. The solutions of the last three equations are

$$m=3\alpha+3,\ m=5\beta+3,\ m=4\gamma+2 \text{ for any integers } \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0.$$

Considering the two equations

$$7a=3\alpha+3,\ 5\beta=4\gamma-1,$$

we get the respective solutions

$$a=3(r+1) \text{ (so that, } m=21(r+1) \text{ for any integer } r \geq 0),$$

and

$$\beta=4w+3 \text{ (so that, } m=5(4w+3)+3=2(10w+9) \text{ for any integer } w \geq 0).$$

Next, we consider the combined Diophantine equation

$$21r=20w-3,$$

which shows that w must be a multiple of 3, say,

$$w=3t \text{ for some integer } t \geq 1.$$

Thus, we need to consider the equation

$$7r=20t-1,$$

whose solution is

$$t=7s+6 \text{ for any integer } s \geq 0.$$

Therefore,

$$w=3t=3(7s+6).$$

Hence, finally

$$m=20w+18=60t+18=60(7s+6)+18,$$

which gives the desired result after some algebraic manipulation. ■

To find SS(60m) when m is a multiple of 7, let $m=7a$ for some integer $a \geq 1$. We first state and prove the lemma below.

Lemma 6.5.22 : $SS(420a)=420a-6$ if $a=6s+5$, $s \geq 0$.
Proof : We start with

$$420a\left[\frac{(420a-1)(420a-2)(420a-3)(420a-4)(420a-5)}{6!}\right]$$

$$=420a\left[\frac{(420a-1)(210a-1)(140a-1)(105a-1)(84a-1)}{6}\right].$$

Then, we have to find the condition such that the term inside the square bracket is an integer. To do so, we consider the case when 3 divides $140a-1$ and 2 divides $105a-1$, so that

$$140a=3\alpha+1,\ 105a=2\beta+1 \text{ for some integers } \alpha \text{ and } \beta.$$

The solutions of the above equations are

$$a=3b+2,\ a=2c+1;\ b\ (\geq 0) \text{ and } c\ (\geq 0) \text{ being integers.}$$

And the solution of the combined equation $3b=2c-1$ is

$$b=2s+1 \text{ for any integer } s \geq 0.$$

Therefore, we finally get

$$a=3(2s+1)+2=6s+5,\ s \geq 0.$$

This establishes the lemma. ■

However, note that, $SS(420a) \neq 420a - 7$, which is evident from the expression below :

$$420a\left[\frac{(420a-1)(420a-2)(420a-3)(420a-4)(420a-5)(420a-6)}{7!}\right]$$

$$= 420a\left[\frac{(420a-1)(210a-1)(140a-1)(105a-1)(84a-1)(70a-1)}{7}\right].$$

Now, consider

$$420a\left[\frac{(420a-1)(210a-1)(140a-1)(105a-1)(84a-1)(70a-1)(60a-1)}{8}\right].$$

When a is of the form

$$a = 8b + 1, \ b \geq 1,$$

then $105a - 1 = 8(105b + 13)$ is divisible by 8, and consequently,

$$SS(420a) = 420a - 8 \text{ when } a = 8b + 1, \ b \geq 0.$$

Next, consider

$$420a\left[\frac{(420a-1)(210a-1)(140a-1)(105a-1)(84a-1)(70a-1)(60a-1)(105a-2)}{2\times 9}\right].$$

Here, one of the two numbers $105a - 1$ and $105a - 2$ is even, depending on whether a is odd or even. Now, there are two possibilities : Either 9 divides $140a - 1$, or else, 9 divides $70a - 1$. It can easily be be verified that $140a - 1$ is divisible by 9 if a is of the form $a = 9b + 2$, $b \geq 0$, and $70a - 1$ is divisible by 9 if $a = 9c + 4$, $c \geq 0$. Thus,

$$SS(420a) = 420a - 9 \text{ when } a = 9b + 2, \ b \geq 0; \text{ or when } a = 9c + 4, \ c \geq 0.$$

Now, consider

$$420a\left[\frac{(420a-1)(210a-1)(140a-1)(105a-1)(84a-1)(70a-1)(60a-1)(105a-2)(140a-3)(42a-1)}{2\times 3\times 11}\right].$$

Here, one of the ten terms in the numerator inside the square bracket is divisible by 11, provided that a is not a multiple of 11. Also, note that, one of $(105a - 1)$ and $(105a - 2)$ is even; and one of the three numbers $140a - 1$, $70a - 1$ and $140a - 3$ is divisible by 3. Thus, if a is not a multiple of 11, then

$$SS(420a) = 420a - 11, \text{ if a is an integer, not of the form } 8b + 1, \text{ or } 9c + 2.$$

The above discussions are summarized in the lemma below, where part (4) is just a restatement of Lemma 6.5.21 :

Lemma 6.5.23 **:** We have the following result :

(1) $SS(420a) = 420a - 8$, when $a = 8b + 1$, $b \geq 0$, $a \neq 6s + 5$ for any $s \geq 0$,

(2) $SS(420a) = 420a - 9$, when $a = 9b + 2$, $b \geq 0$; or when $a = 9c + 4$, $c \geq 0$,

(3) $SS(420a) = 420a - 11$, if a is not of any of the forms $8b + 1$, $9c + 2$, $9d + 4$,

(4) $SS(420a) = 420a - 10$, if $a = 6(10s + 9)$ for any integer $s \geq 0$.

Example 6.5.3 **:** By (the first part of) Lemma 6.5.19, $SS(300) = 294$, $SS(660)=654$; and by the second part, $SS(60) = 53$, $SS(120) = 113$, $SS(180) = 173$, $SS(240) = 233$, $SS(360) = 353$. Also, (by the third part of Lemma 6.5.19), $SS(3780) = 3772$. By Lemma 6.5.20, $SS(1680) = 1671$, $SS(840) = 831$. By Lemma 6.5.21, $SS(22680) = 22670$. By Lemma 6.5.22, $SS(2100) = 2094$. And by Lemma 6.5.23, $SS(420) = 412$, $SS(840) = 831$.

From Lemmas 6.5.1 – 6.5.23, we see that, given any integer k with $2 \leq k \leq 11$, there is an integer n such that $SS(n) = n - k$. Some generalizations are given in the following lemmas.

Lemma 6.5.24 : $SS(2mp) = 2mp - 3$ if the prime $p \neq 3$ and 3 does not divide the integer m.

Proof : Consider the expression below :

$$2mp\left[\frac{(2mp-1)(2mp-2)}{2\times 3}\right] = 2mp\left[\frac{(2mp-1)(mp-1)}{3}\right].$$

If $p \neq 3$ and 3 does not divide m, then the term inside the square bracket is an integer. ■

Lemma 6.5.25 : $SS(6mp) = 6mp - 4$ if one of m and p is of the form $4a+1$, and the other one is of the $4b+3$ (for some integers $a \geq 0$ and $b \geq 0$).

Proof : Consider the expression

$$6mp\left[\frac{(6mp-1)(6mp-2)(6mp-3)}{2\times 3\times 4}\right] = 6mp\left[\frac{(6mp-1)(3mp-1)(2mp-1)}{4}\right].$$

Now, if, for example, $p = 4a+1$, $m = 4b+3$ for some integers $a \geq 0$ and $b \geq 0$, then

$$3mp - 1 = 3(4a+1)(3b+3) - 1 = 4(4ab + 3a + b + 2),$$

which is divisible by 4. ■

Lemma 6.5.26 : Let p (≥ 3) be a prime and m (≥ 1) be an integer such that the condition of the above Lemma 6.5.25 is not satisfied. Let $p \neq 5$, and 5 do not divide m. Moreover, let p and m satisfy any one of the following sets of conditions (for some integers a (≥ 0) and b (≥ 0)) :

Set 1 : $m = 5a+1$, $p = 5b+4$,
$m = 5a+2$, $p = 5b+2$,
$m = 5a+3$, $p = 5b+3$,
$m = 5a+4$, $p = 5b+1$;

Set 2 : $m = 5a+1$, $p = 5b+3$,
$m = 5a+2$, $p = 5b+4$,
$m = 5a+3$, $p = 5a+1$,
$m = 5a+4$, $p = 5b+2$;
$mp \neq 20s+3$, $s \geq 0$;

Set 3 : $m = 5a+1$, $p = 5b+2$,
$m = 5a+2$, $p = 5a+1$
$m = 5a+3$, $p = 5b+4$,
$m = 5a+4$, $p = 5b+3$;
$mp \neq 20s+7$, $s \geq 0$;

Set 4 : $m = 5a+1$, $p = 5b+1$,
$m = 5a+2$, $p = 5b+3$,
$m = 5a+3$, $p = 5b+2$,
$m = 5a+4$, $p = 5b+4$;
$3mp \neq 20s+13$, $s \geq 1$.

Then,

$$SS(6mp) = 6mp - 5.$$

Proof : We start with

$$6mp\left[\frac{(6mp-1)(6mp-2)(6mp-3)(6mp-4)}{2\times 3\times 4\times 5}\right] = 6mp\left[\frac{(6mp-1)(3mp-1)(2mp-1)(3mp-2)}{2\times 5}\right].$$

Now, one of $3mp-1$ and $3mp-2$ is even. Consequently, to prove the lemma, it is sufficient to find the conditions such that any of $6mp-1$, $3mp-1$, $2mp-1$ and $3mp-2$, is divisible by 5.
It can easily be verified that $3mp-2$ is divisible by 5 whenever any of the four conditions in the first Set 1 is satisfied. For example, when $m = 5a+1$ and $p = 5b+4$, then

$$3mp - 2 = (5a + 1)(5b + 4) - 2 = 5(15ab + 12a + 3b + 2).$$

Again, if any of the four conditions in Set 2 is satisfied, then $2mp - 1$ is divisible by 5. For example, with the first condition in Set 2, we have

$$2mp - 1 = 2(5a + 1)(5b + 3) - 2 = 5(10ab + 6a + 2b + 1).$$

However, note that, in this case, even if 5 divides $2mp - 1$, we need the extra condition that 4 does not divide $3mp - 1$ (otherwise, $SS(6mp) \neq 6mp - 5$). Thus, we find the condition such that 4 divides $3mp - 1$, and hence, we get the following two Diophantine equations :

$$2mp - 1 = 5x,\ 3mp - 1 = 4y \text{ for some integers } x\ (\geq 1) \text{ and } y\ (\geq 1).$$

These two equations, when combined together, leads to the Diophantine equation

$$15x = 8y - 1$$

with the solution

$$x = 8s + 1,\ y = 15t + 2 \ (s\ (\geq 0) \text{ and } t\ (\geq 0) \text{ being any integers}).$$

Therefore,

$$2mp = 5x + 1 = 5(8s + 1) + 1 = 2(20s + 3).$$

Thus, the condition

$$mp \neq 20s + 3,\ s \geq 0,$$

guarantees that 4 does not divide $3mp - 1$.

Now, if any of the four conditions in Set 3 is satisfied, then 5 divides $3mp - 1$. For example, with $m = 5a + 1$ and $p = 5b + 2$, we have

$$3mp - 1 = 3(5a + 1)(5b + 2) - 1 = 5(15ab + 6a + b + 1).$$

In this case also, we require the additional condition that 4 does not divide $3mp - 1$. To do so, we consider the following two Diophantine equations

$$3mp - 1 = 5x,\ 3mp - 1 = 4y,$$

whose solutions are

$$x = 4(a + 1),\ y = 5(t + 1) \text{ for any integers } a\ (\geq 0) \text{ and } t\ (\geq 0).$$

Therefore,

$$3mp = 5x + 1 = 20a + 21,$$

which shows that a must be a multiple of 3, say, $a = 3s$ for some integer $s\ (\geq 1)$. Then,

$$mp = 20s + 7,\ s \geq 0.$$

Finally, as can easily be checked, any of the four conditions in Set 4 guarantees that 5 divides $6mp - 1$. As an example, consider the first condition in Set 4; here, we get

$$6mp - 1 = 6(5a + 1)(5b + 1) - 1 = 5(20ab + 6a + 6b + 1).$$

In this case, the condition that ensures that $3mp - 1$ is not a multiple of 4 is :

$$3mp \neq 20s + 13,\ s \geq 1.$$

All these complete the proof of the lemma. ∎

Example 6.5.4 **:** We give below some examples illustrating each set in Lemma 6.5.26. Note that, Lemma 6.4.26 is to be applied in conjunction with Lemma 6.4.25.

$SS(114) = 109$, $SS(84) = 79$, $SS(54) = 49$, $SS(264) = 260$ (Set 1),

$SS(78) = 73$, $SS(228) = 223$, $SS(198) = 193$, $SS(168) = 163$ (Set 2),

but $SS(18) = 14$ (why?), $SS(1218) = 1214$ (why?), $SS(2418) = 2414$ (why?), $SS(378) = 374$ (why?),

$SS(252) = 247$, $SS(132) = 127$, $SS(342) = 337$, $SS(72) = 67$,

but $SS(42) = 38$ (why?), $SS(1722) = 1718$ (why?), $SS(522) = 518$ (why?), $SS(162) = 158$ (why?),

$SS(396)=391$, $SS(156)=151$, $SS(126)=121$, $SS(456)=451$.

***Remark 6.5.4* :** Here, we give an alternative method of proof of Lemma 6.5.26. To do so, we have to consider the following four possible cases :

Case 1. When $m = 5a + 1$ for some integer $a \geq 1$. Then,

$$3mp - 2 = 15ap + (3p - 2).$$

We now want to find the condition such that 5 divides $3p - 2$. Thus, we get

$$3p - 2 = 5x \text{ for some integer } x \geq 1.$$

The above equation has the solution $p = 5b + 4$ for any integer $b \geq 1$.

Case 2. When $m = 5a + 2$ for some integer $a \geq 1$. In this case,

$$3mp - 2 = 15ap + (6p - 2),$$

and the condition that 5 divides $3mp - 2$ leads to the Diophantine equation

$$6p - 2 = 5y \text{ for some integer } y \geq 1.$$

The solution is found to be $p = 5c + 2$, $c \geq 1$ being an integer.

Case 3. When $m = 5a + 3$ for some integer $a \geq 1$. Here,

$$3mp - 2 = 15ap + (9p - 2).$$

We now want to find the condition that 5 divides $9p - 2$. Thus, we have

$$9p - 2 = 5z \text{ for some integer } z \geq 1,$$

whose solution is $p = 5d + 3$, $d \geq 1$ being an integer.

Case 4. When $m = 5a + 4$ for some integer $a \geq 1$. Here,

$$3mp - 2 = 15ap + (12p - 2),$$

and the condition that 5 divides $3mp - 2$ gives rise to the Diophantine equation

$$12p - 2 = 5u \text{ for some integer } u \geq 1,$$

with the solution $p = 5e + 1$ for any integer $e \geq 1$.

We thus get the four conditions in Set 1, any of which guarantees that 5 divides $3mp - 2$.

Proceeding in the same way, we get the four conditions in Set 2, any of which ensures that 5 divides $2mp - 1$. However, in this case, in addition, we have to ensure that 4 does not divide $3mp - 1$. To do so, we solve the simultaneous Diophantine equations :

$$2mp - 1 = 5x,\ 3mp - 1 = 4y \text{ for some integers } x \geq 1,\ y \geq 1.$$

The combined Diophantine equation is $15x = 8y - 1$, with the solutions

$$x = 8s + 1,\ y = 15t + 2 \text{ for any integers } x \geq 1 \text{ and } y \geq 1.$$

Then,

$$mp = 20s + 3,\ s \geq 1.$$

Hence, if the above relationship is satisfied, we have $SS(6mp) = 6mp - 4$.

Any condition in Set 3 guarantees that 5 divides $3mp - 1$. Here also, we have to ensure that 4 does not divide $3mp - 1$, and so, we consider the two simultaneous equations

$$3mp - 1 = 5x,\ 3mp - 1 = 4y \text{ for some integers } x \geq 1 \text{ and } y \geq 1,$$

whose solutions are

$$x = 4(s + 1),\ y = 5(t + 1);\ s \geq 1 \text{ and } t \geq 1 \text{ being any integers.}$$

Now,

$$3mp = 5x + 1 = 20s + 21,$$

and hence, 3 must divide s. Let $s = 3a$ for some integer $a \geq 1$. Thus, finally, we get

$$mp = 20a + 7,\ a \geq 0,$$

and the violation of the above condition ensures that $SS(6mp) = 6mp - 5$.

Finally, to find the conditions such that 5 divides $6mp - 1$ but 4 does not divide $3mp - 1$, we solve

$$6mp = 5x + 1,\ 3mp = 4y + 1 \text{ for some integers } x \geq 1 \text{ and } y \geq 1.$$

The first equation gives the four conditions of Set 4, while the combined equations give

$$3mp = 20s + 13,\ s \geq 1,$$

the violation of which guarantees that $SS(6mp) = 6mp - 5$.

We now consider the case when $m = 5$, that is, the case of SS(30p), p (≥ 3) being a prime.

***Corollary 6.5.9* :** $SS(30p) = 30p - 4$, if the prime p is of the form $p = 4s + 3$, $s \geq 0$.
***Proof* :** follows from Lemma 6.5.25. ■

However, we have the following lemma.

***Lemma 6.5.27* :** For any prime p, **(1)** $SS(30p) \neq 30p - 5$, **(2)** $SS(30p) \neq 30p - 6$.
***Proof* :** From the expression

$$30p\left[\frac{(30p-1)(30p-2)(30p-3)(30p-4)}{2\times3\times4\times5}\right],$$

part (1) is evident. The proof of part (2) is similar. ■

***Lemma 6.5.28* :** $SS(30p) = 30p - 7$, if the prime p $(\neq 7)$ is of the form $p = 4s + 1$, $s \geq 1$.
***Proof* :** Consider the expression

$$30p\left[\frac{(30p-1)(30p-2)(30p-3)(30p-4)(30p-5)(30p-6)}{2\times3\times4\times5\times6\times7}\right]$$

$$= 30p\left[\frac{(30p-1)(15p-1)(10p-1)(15p-2)(6p-1)(5p-1)}{2\times7}\right].$$

Now, one of of the two numbers $15p - 1$ and $15p - 2$ is even. Therefore, since $p \neq 7$, the term inside the square bracket is an integer. ■

Next, we confine our attention to the function SS(60p), where p is a prime.

***Lemma 6.5.29* :** For any prime p, **(1)** $SS(60p) \neq 60p - 4$, **(2)** $SS(60p) \neq 60p - 5$.
***Proof* :** Part (1) is evident from the expression below :

$$60p\left[\frac{(60p-1)(60p-2)(60p-3)}{2\times3\times4}\right] = 60p\left[\frac{(60p-1)(30p-1)(20p-1)}{4}\right].$$

The proof of the remaining part is similar. ■

***Lemma 6.5.30* :** $SS(60p) = 60p - 6$, if the prime p is of the form $p = 3s + 2$, $s \geq 1$.
***Proof* :** We start with

$$60p\left[\frac{(60p-1)(60p-2)(60p-3)(60p-4)(60p-5)}{2\times3\times4\times5\times6}\right]$$

$$= 60p\left[\frac{(60p-1)(30p-1)(20p-1)(15p-1)(12p-1)}{6}\right].$$

Now, $15p - 1$ is even when p is an odd prime. It is thus sufficient to find the condition that 3 divides $20p - 1$, that is, we have to solve the Diophantine equation

$$20p = 3a + 1 \text{ for some integer } a \geq 1.$$

The solution of the above equation is $p = 3s + 2$, $s \geq 1$, which is the desired condition such that

$$SS(60p) = 60p - 6.$$

All these complete the proof of the lemma. ■

***Lemma 6.5.31* :** Let p (≥ 11) be a prime such that $p \neq 3s + 2$, $s \geq 1$. Then, $SS(60p) = 60p - 7$.
***Proof* :** We consider

$$60p\left[\frac{(60p-1)(60p-2)(60p-3)(60p-4)(60p-5)(60p-6)}{2\times3\times4\times5\times6\times7}\right],$$

$$= 60p\left[\frac{(60p-1)(30p-1)(20p-1)(15p-1)(12p-1)(10p-1}{7}\right].$$

The result is now evident from the above expression (since 7 does not divide p). ■

Lemma 6.5.26 gives an expression of SS(6mp) under the restrictive condition that none of m and p is divisible by 5. We now consider the case of SS(30mp), where m (≥ 1) is an integer and p is a prime. The following lemma is simple, and the proof is left for the reader.

***Lemma 6.5.32* :** For any integer m (≥ 1) and for any prime p (≥ 3),

$$SS(30mp) \neq 30mp - 3.$$

***Proof* :** is evident from the following expression :

$$30mp\left[\frac{(30mp-1)(30mp-2)}{2\times 3}\right] = 30mp\left[\frac{(30mp-1)(15mp-1)}{3}\right],$$

since the term in the numerator inside the square bracket on the right side is not divisible by 3. ■

However, we have the following lemma.

***Lemma 6.5.33* :** Let m (≥ 1) be an integer and p (≥ 3) be a prime. Then,

$$SS(30mp) = 30mp - 4,$$

if and only if $mp = 4s + 3$ for some integer $s \geq 0$.

***Proof* :** We consider the expression

$$30mp\left[\frac{(30mp-1)(30mp-2)(30mp-3)}{2\times 3\times 4}\right] = 30mp\left[\frac{(30mp-1)(15mp-1)(10mp-1)}{4}\right].$$

Then, the term inside the square bracket is an integer if and only if 4 divides $15mp - 1$, that is,

$$15mp = 4a + 1 \text{ for some integer a } (\geq 1).$$

The solution of the above Diophantine equations is

$$mp = 4s + 3, \; s \geq 0 \text{ being an integer.}$$

We thus get the desired condition to be satisfied by m and p such that $SS(30mp) = 30mp - 4$. ■

From the condition of Lemma 6.5.33, we see that, m must be odd. When m = 1, the condition of Lemma 6.5.33 becomes

$$p = 4s + 3 \text{ for some integer } s \geq 0.$$

When $p = 3$ (corresponding to $s = 0$), we get, by Lemma 6.5.33, $SS(30\times 1\times 3) = SS(90) = 86$. The next prime is p = 7 (for s = 1), with SS(210) = 206. A few more solutions are p = 11 with SS(330) = 326, p = 19 with SS(570) = 566, p = 23 with SS(690) = 686, p = 31 with SS(930) = 926, and p = 43 with SS(1290) = 1286. When m = 3, the condition takes the form $3p = 4s + 3$, with the solution $p = 4a + 1$ ($a \geq 1$ being an integer). The first few functions in this case are SS(450) = 446, SS(1170) = 1166, SS(1530) = 1526, SS(2610) = 2606, SS(3330) = 3326, SS(3690) = 3686, SS(4770) = 4766 and SS(5490) = 5486. When m = 5, the condition is $5p = 4s + 3$, which gives the solution $p = 4b + 3$ ($b \geq 0$ being an integer). In this case, for small values of p, we get the functions SS(1050) = 1046, SS(1650) = 1646 and SS(2850) = 2846. For m = 7, the condition becomes $7p = 4s + 3$ with the solution $p = 4c + 1$ ($c \geq 1$ is an integer). Some of the functions obtained in this case are SS(2730) = 2726, SS(3570) = 3566, SS(6090) = 6086, SS(7770) = 7766 and SS(8610) = 8606.

***Lemma 6.5.34* :** For any integer m (≥ 1) and for any prime p (≥ 3), $SS(30mp) \neq 30mp - 5$.

***Proof* :** Considering the expression

$$30mp\left[\frac{(30mp-1)(30mp-2)(30mp-3)(30mp-4)}{2\times 3\times 4\times 5}\right]$$

$$= 30mp\left[\frac{(30mp-1)(15mp-1)(10mp-1)(15mp-2)}{2\times 5}\right],$$

the result follows readily. ■

***Lemma 6.5.35* :** $SS(30mp) = 30mp - 6$ if $mp = 2(6s + 5)$, $s \geq 0$ being an integer.

***Proof* :** We start with

$$30mp\left[\frac{(30mp-1)(30mp-2)(30mp-3)(30mp-4)(30mp-5)}{2\times 3\times 4\times 5\times 6}\right]$$

$$=30mp\left[\frac{(30mp-1)(15mp-1)(10mp-1)(15mp-2)(6mp-1)}{2\times 6}\right].$$

Now, we have to find the condition such that the term inside the square bracket is an integer. To do so, we consider the case when 3 divides $10mp-1$ and 4 divides $15mp-2$. Thus, we have the following two Diophantine equations :

$$10mp = 3a+1,\ 15mp = 4b+2 \text{ for some integers } a \ge 1,\ b \ge 1.$$

The solutions of the above equations are

$$mp = 3c+1,\ mp = 4d+2 \ (c \ge 1 \text{ and } d \ge 1 \text{ being integers})$$

respectively. Next, we consider the combined equation, namely, $3c = 4d+1$, whose solution is

$$c = 4s+3,\ s \ge 0 \text{ being any integer.}$$

Hence, finally, we get

$$mp = 3(4s+3)+1,$$

which gives the desired condition after simplification. ■

It may be remarked here that, another possibility in the proof of Lemma 6.5.35 is that 3 divides $10mp-1$ and 4 divides $15mp-1$; but then, $SS(30mp) = 30mp-4$. From Lemma 6.5.35, we see that m must be an even integer in order that the condition therein is satisfied; in fact, letting $m = 2x$, the condition becomes $xp = 6s+5$, which has a solution if and only if $(x, 6) = 1$. For example, when $m = 2$, the condition reads as $p = 6s + 5$, and we get successively the functions $SS(300) = 294$ (when $p = 5$), $SS(660) = 654$ (when $p = 11$), $SS(1020) = 1014$ (when $p = 17$), $SS(1380) = 1374$ (when $p = 23$), and $SS(1740) = 1734$. However, when $m = 4$, the condition becomes $2p = 6s+5$, which has no solution. Again, for $m = 6$, the condition takes the form $3p = 6s+5$, which has no solution. There is no solution when $m = 8$ (since the condition is $4p = 6s + 5$). But, for $m = 10$, the condition becomes $5p = 6s + 5$, which possesses a solution, namely, $p = 6a+7$ ($a \ge 0$ being an integer). The first few functions corresponding to this case are $SS(2100) = 2094$, $SS(3900) = 3894$ and $SS(5700) = 5694$. For $m = 12$, the condition becomes $6p = 6s+5$, which clearly has no solution. But when $m = 14$, the condition can be rewritten as $7p = 6s+5$, with the solution $p = 6b+5$ ($b \ge 0$ being an integer). Some of the functions in this case are $SS(4620) = 4614$, $SS(7140) = 7134$, $SS(9660) = 9656$ and $SS(12180) = 12174$.

***Lemma 6.5.36* :** If none of m and p is divisible by 7, $mp \ne 4s+3$, and $mp \ne 2(6t+5)$ (for any integers $s \ge 1$ and $t \ge 1$), then

$$SS(30mp) = 30mp-7.$$

***Proof* :** is left as an exercise. ■

The following lemma is simple, and the proof is left for the reader.

***Lemma 6.5.37* :** For any integer m (≥ 1) and for any prime p (≥ 3),

(1) $SS(60mp) \ne 60mp-4$, **(2)** $SS(60mp) \ne 60mp-5$.

***Lemma 6.5.38* :** Let m (≥ 1) be an integer and p (≥ 5) be a prime such that $mp = 6s+5$ (for some integer $s \ge 1$). Then, $SS(60mp) = 60mp-6$.

***Proof* :** follows from Lemma 6.5.35. ■

***Lemma 6.5.39* :** If none of m and p is divisible by 7, and $mp \ne 6s+5$ (for any $s \ge 1$), then

$$SS(60mp) = 60mp-7.$$

***Proof* :** is left as an exercise. ■

We now focus our attention to the function SS(210mp). The proof of the lemma below is easy, and is left as an exercise.

***Lemma 6.5.40* :** $SS(210mp) \neq 210mp - 3$ for any integer m (≥ 1) and any prime p (≥ 3).

But we have the following result.

***Lemma 6.5.41* :** $SS(210mp) = 210mp - 4$, if $mp = 4s + 1$, $s \geq 1$ being an integer.
***Proof* :** We start with

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)}{2\times3\times4}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)}{4}\right].$$

Now, noting that the term inside the square bracket is an integer if and only if $105mp - 1$ is divisible by 4, we get the equation :

$$105mp = 4a + 1 \text{ for some integer } a \geq 1.$$

The solution of the above Diophantine equation is $mp = 4s + 1$. ■

For any odd prime p, the equation $mp = 4s + 1$ ($s \geq 1$) has a solution if and only if m is an odd integer. For example, when $m = 1$, the condition is simply $p = 4s + 1$, and the first few primes of this form are $p = 5, 13, 17, 29$, with $SS(1050) = 1046$, $SS(2730) = 2726$, $SS(3570) = 3566$ and $SS(6090) = 6086$. When $m = 3$, the condition becomes $3p = 4s + 1$, with the solution $p = 4t + 3$ ($t \geq 1$). Corresponding to this case, we get the functions $SS(1890) = 1886$, $SS(4410) = 4406$, $SS(6930) = 6926$. When $m = 5$, the condition is $5p = 4s + 1$ with the solution $p = 4v + 1$ ($v \geq 1$). The first few functions corresponding to this case are $SS(5250) = 5246$, $SS(13650) = 13646$ and $SS(17850) = 17846$.

The proof of the following lemma is left to the reader.

***Lemma 6.5.42* :** $SS(210mp) \neq 210mp - 5$ for any integer m (≥ 1) and any prime p (≥ 3).

***Lemma 6.5.43* :** $SS(210mp) = 210mp - 6$ if $mp = 2(6s + 5)$, $s \geq 0$ being an integer.
***Proof* :** Consider the expression below :

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)(210mp-4)(210mp-5)}{2\times3\times4\times5\times6}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)(105mp-2)(42mp-1)}{3\times4}\right].$$

We now consider the case when 3 divides $70mp - 1$ and 4 divides $105mp - 2$. Then,

$$70mp = 3a + 1,\ 105mp = 4b + 2 \text{ for some integers } a \geq 1,\ b \geq 1.$$

The solutions of the above Diophantine equations are respectively

$$mp = 3c + 1,\ mp = 4d + 2 \ (c \geq 1 \text{ and } d \geq 1 \text{ being integers}).$$

Combining together, we have the equation $3c = 4d + 1$, which gives the desired condition. ■

From Lemma 6.5.43, we see that, the condition $mp = 2(6s + 5)$ is satisfied only if m is even. More specifically, letting $m = 2y$, the condition has a solution if and only if $(y, 6) = 1$. For example, when $m = 2$, the condition becomes $p = 6s + 5$ ($s \geq 0$). In this case, we successively get the primes $p = 5, 11, 17$, with $SS(2100) = 2094$, $SS(4620) = 4614$ and $SS(7140) = 7134$. The next value of m for which there is a solution is $m = 10$. In this case, the solution is $p = 6t + 7$ ($t \geq 0$ being an integer). The first few primes corresponding to this case are $p = 7, 13, 19$, with $SS(14700) = 14694$, $SS(27300) = 27294$, $SS(39900) = 39894$.

The following lemma is easy to prove, and is left as an exercise for the reader.

***Lemma 6.5.44* :** $SS(210mp) \neq 210mp - 7$ for any integer m (≥ 1) and any prime p (≥ 3).

However, we have the result below.

***Lemma 6.5.45* :** $SS(210mp) = 210mp - 8$ if one of the following conditions is satisfied :
(1) $mp = 2(8s+1)$ ($s \geq 1$ being an integer), **(2)** $mp = 16t + 11$ ($t \geq 0$ being an integer).
***Proof* :** We start with the following expression :

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)(210mp-4)(210mp-5)(210mp-6)(210mp-7)}{2\times3\times4\times5\times6\times7\times8}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)(105mp-2)(42mp-1)(35mp-1)(30mp-1)}{16}\right].$$

We now want to find the condition such that the term inside the square bracket is an integer. To do so, we have to consider the following two possible cases :
Case 1. When 16 divides $105mp - 2$.
In this case, we have the Diophantine equation

$$105mp = 16a + 2 \text{ for some integer } a \geq 1,$$

whose solution is $mp = 2(8s+1)$, $s \geq 1$ being an integer.
Case 2. When 16 divides $35mp - 1$.
Here, we have

$$35mp = 16b + 1 \text{ for some integer } b \geq 1,$$

with the solution $mp = 16t + 11$, $t \geq 0$ being an integer.
Thus, the proof is complete. ■

From part (1) of Lemma 6.5.45, we see that, if the prime p is odd, m must be even. In fact, letting $m = 2\mu$ (for some integer $\mu \geq 1$), the condition takes the form $\mu p = 8s + 1$, which admits solutions only when $(\mu, 8) = 1$. For $m = 2$, the condition is $p = 8s + 1$, and the first few primes corresponding to this case are p = 17, 41, 73, 89, which give the functions SS(7140) = 7132, SS(17220) = 17212, SS(30660) = 30652 and SS(37380) = 37372. For m = 6, the condition becomes $3p = 8s + 1$, with the solution $p = 8\nu + 3$ ($\nu \geq 0$ being an integer). The first few functions corresponding to this case are SS(3780) = 3772 (for p = 3), SS(13860) = 13852 (for p = 11) and SS(23940) = 23932 (for p = 19). When m = 10 (for which the condition is $5p = 8s + 1$), there is a solution, namely, $p = 8u + 5$ ($u \geq 0$ being an integer). In this case, to mention a few, we get the functions SS(10500) = 10492, SS(27300) = 27292, SS(60900) = 60892 and SS(77700) = 77692. When m = 14, there is a solution. In this case, the condition takes the form $7p = 8s + 1$, which gives $p = 8v + 7$ ($v \geq 0$ being an integer). Here, we get the functions SS(20580) = 20572, SS(67620) = 67612 and SS(91140) = 91132. From part (2), we see that the condition admits solutions for p only when m is odd. For m = 1, the condition is simply $p = 16t + 11$, and we get successively the primes p = 11, 43, 59, 107, with the respective functions SS(2310) = 2302, SS(9030) = 9022, SS(12390) = 12382, SS(22470) = 22462. When m = 3, the condition takes the form $3p = 16t + 11$ with the solution $p = 16w + 9$ ($w \geq 2$ being an integer). A few functions in this case are SS(25830) = 25822, SS(45990) = 45982 and SS(56070) = 56062. When m = 5, the condition is $5p = 16t + 11$, with the solution $p = 16r + 15$ ($r \geq 1$ being an integer). Some of the functions in this case are SS(32550) = 32542, SS(49350) = 49342 and SS(82950) = 82942.

From Lemma 6.5.33, Lemma 6.5.35, Lemma 6.5.38, Lemma 6.5.41, Lemma 6.5.43 and Lemma 6.5.45, we see that we need to solve equations like $mp = ax + b$. It may be recalled here that, letting $(m, a) = d$, the equation has a solution if and only if d divides b.

Lemma 6.5.46 **:** $SS(210mp) = 210mp - 9$ if
(1) $mp = 4(9s+11)$ ($s \geq 0$ being an integer), **(2)** $mp = 2(18t+13)$, $t \neq 4\nu + 2$ for any $\nu \geq 0$.

Proof **:** We consider the expression below :

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)(210mp-4)(210mp-5)(210mp-6)(210mp-7)(210mp-8)}{2\times3\times4\times5\times6\times7\times8\times9}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)(105mp-2)(42mp-1)(35mp-1)(105mp-4)}{8\times9}\right],$$

where we have to find the condition that the term in the numerator inside the square bracket is an integer. We consider the following two possibilities :

Case 1. When 9 divides $35mp - 1$ and 4 divides $105mp - 4$. In this case

$$35mp = 9a + 1,\ 105mp = 4b + 4 \text{ for some integers } a \geq 1,\ b \geq 1,$$

with the respective solutions

$$mp = 9c + 8,\ mp = 4d + 4 \text{ for any integers } c \geq 0,\ d \geq 0.$$

Then, the combined Diophantine equation is $9c = 4d - 4$, whose solution is

$$c = 4(s+1),\ s \geq 0 \text{ being an integer.}$$

Therefore,

$$mp = 36(s+1) + 8 = 4(9s+11).$$

Case 2 . When 9 divides $35mp - 1$ and 4 divides $105mp - 2$. Here,

$$105mp = 4e + 2 \text{ for some integer } e \geq 0,$$

with the solution

$$mp = 4w + 2 \text{ for any integer } w \geq 0.$$

Then, considering the combined equation, $9c = 4w - 6$, we get the solution

$$c = 4t + 2,\ t \geq 0 \text{ being an integer,}$$

with

$$mp = 9(4t+2) + 8 = 2(18t+13).$$

However, note that, when $t = 4\nu + 2$, $\nu \geq 0$, then

$$mp = 2[8(9\nu+6)+1],$$

which shows, by part (1) of Lemma 6.5.45, that $SS(210mp) = 210mp - 8$. Otherwise,

$$SS(210mp) = 210mp - 9. \blacksquare$$

Lemma 6.5.47 **:** $SS(210mp) = 210mp - 10$ if $mp = 12(10s+9)$ ($s \geq 0$ being an integer).

Proof **:** We consider the following expression :

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)(210mp-4)(210mp-5)(210mp-6)(210mp-7)(210mp-8)(210mp-9)}{2\times3\times4\times5\times6\times7\times8\times9\times10}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)(105mp-2)(42mp-1)(35mp-1)(105mp-4)(70mp-3)}{3\times5\times16}\right].$$

We consider the case when 3 divides $70mp - 3$, 5 divides $42mp - 1$ and 8 divides $105mp - 4$. Then,

$$70mp = 3a + 3,\ 42mp = 5b + 1,\ 105mp = 8c + 4 \text{ for some integers } a \geq 1,\ b \geq 1,\ c \geq 1,$$

with the respective solutions

$$mp = 3x + 3,\ mp = 5y + 3,\ mp = 8z + 4 \text{ for any integers } x \geq 0,\ y \geq 0,\ z \geq 0.$$

The solution of the first two equations is $x = 5(d+1)$, which gives

$$mp = 15(d+1) + 3 = 3(5d+6),\ d \geq 0 \text{ being an integer.}$$

Combining with the third equation, we get $15d = 8z - 14$, whose solution is $d = 2(4s+3)$, $s \geq 0$. Then,

$$mp = 30(4s+3) + 18 = 12(10s+9),$$

which we intended to find. $\blacksquare$

***Lemma 6.5.48* :** Let mp be not be divisible by 11. Then,

$$SS(210mp) = 210mp - 11 \text{ if } mp = 12(s+1) \ (s \geq 0 \text{ being an integer}).$$

***Proof* :** To prove the lemma, we start with the expression :

$$210mp\left[\frac{(210mp-1)(210mp-2)(210mp-3)(210mp-4)(210mp-5)(210mp-6)(210mp-7)(210mp-8)(210mp-9)(210mp-10)}{2\times3\times4\times5\times6\times7\times8\times9\times10\times11}\right]$$

$$= 210mp\left[\frac{(210mp-1)(105mp-1)(70mp-1)(105mp-2)(42mp-1)(35mp-1)(105mp-4)(70mp-3)(21mp-1)}{3\times8\times11}\right].$$

We now consider the case when 3 divides $70mp - 3$ and 4 divides $105mp - 4$. Then,

$$70mp = 3a + 3, \ 105mp = 4b + 4 \text{ for some integers } a \geq 1, \ b \geq 1,$$

with the solutions

$$mp = 3(c+1), \ mp = 4(d+1), \text{ for any integers } c \geq 0, \ d \geq 0,$$

respectively. Now, the combined equation is $3c = 4d + 1$, whose solution is $c = 4s + 3$, $s \geq 0$. Then,

$$mp = 3(4s+3) + 3,$$

which gives the desired condition. ■

In the examples following Lemma 6.5.41, Lemma 6.5.43 and Lemma 6.5.45, we have seen that, in each case, for the admissible values of the integer m, the prime p can be found from the equation involving mp. But, the situation is different in case of Lemma 6.5.46, Lemma 6.5.47 and Lemma 6.5.48, in the sense that, in each case, though the possible values of mp are known, it might be complicated to solve the equation in mp for p. From part (1) of Lemma 6.5.46, we get the functions :

$$SS(9240) = 9231, \ SS(16800) = 16791, \ SS(24360) = 24351, \ SS(31920) = 31911,$$

while part (2) gives the functions below :

$$SS(5460) = 5451, \ SS(13020) = 13011, \ SS(28140) = 28131, \ SS(35700) = 35691.$$

Recall that, part (2) is valid if $t \neq 4\nu + 2$ for any $\nu \geq 0$; otherwise, we get the functions below :

$$SS(20580) = 20572, \ SS(50820) = 50812, \ SS(81060) = 81052.$$

Lemma 6.5.47 gives the following functions :

$$SS(22680) = 22670, \ SS(47880) = 47870, \ SS(73080) = 73070, \ SS(98280) = 98270.$$

Some of the functions obtained from Lemma 6.5.48 are

$$SS(2520) = 2509, \ SS(5040) = 5029, \ SS(10080) = 10069.$$

Lemma 6.5.48 assumes that 11 does not divide mp. What happens if 11 divides mp? It thus remains an open problem to find $SS(11\times210mp)$, where 11 does not divide mp. In this context, the lemma below gives a partial answer.

***Lemma 6.5.49* :** Let $mp = 11(4s+4)$ for any integer $s \geq 0$. Then, $SS(210mp) = 210mp - 4$.

***Proof* :** Writing mp in the form $mp = 4(11s+8)+1$, and then appealing to Lemma 6.5.41, we result follows immediately. ■

After studying the function SS(210mp) (to some extent), it is also of interest to study the behavior of the function SS(420mp). The following lemma is straight-forward to prove.

***Lemma 6.5.50* :** For any integer $m \geq 1$ and any prime $p \geq 3$,

(1) $SS(420mp) \neq 420mp - 4$, **(2)** $SS(420mp) \neq 420mp - 5$, **(3)** $SS(420mp) \neq 420mp - 7$.

Part (1) of Lemma 6.5.50 makes the difference between SS(420mp) and SS(210mp) (see Lemma 6.5.41). Using the corresponding results for SS(210mp), we can readily deduce the conditions such that $SS(420mp) = 420mp - 6$, $SS(420mp) = 420mp - 8$, $SS(420mp) = 420mp - 9$, $SS(420mp) = 420mp - 10$ and $SS(420mp) = 420mp - 11$.

The following four corollaries give equations involving the function SS(n).

***Corollary 6.5.10* :** The equation $SS(n) = n-2$ has an infinite umber of solutions.

***Proof* :** Let n be an odd integer of the form $n = 2m+1$ for all $m \geq 3$. Then, by Lemma 6.5.1, $SS(n) = n-2$, which clearly possesses an infinite number of solution. ■

***Corollary 6.5.11* :** The equation $SS(n) = n-3$ has an infinite number of solutions.

***Proof* :** Let n be an even integer of the form $n = 2m$, where $m \geq 4$ is such that 3 does not divide m. Then, by Lemma 6.5.3, such an n satisfies the given equation. ■

It may be mentioned here that, by virtue of Lemma 6.5.2, the solution of the equation given in Corollary 6.5.8 (that is, the equation $SS(n) = n-2$) is unique, while the equation $SS(n) = n-3$ has other solutions as well (see, for example, Corollary 6.6.7 and Lemma 6.5.4).

***Corollary 6.5.12* :** The equation $SS(n) = SS(n+1)$ has an infinite number of solutions.

***Proof* :** Let n and $n+1$ be two integers, not divisible by 3. Then, with n odd,

$$SS(n) = n-2 = SS(n+1).\ ■$$

The following corollary involves three Smarandache type arithmetic functions.

***Corollary 6.5.13* :** The equation $S(n) + SS(n) = 2\,Z(n)$ has an infinite number of solutions.

***Proof* :** Since $S(p) = p$, $SS(p) = p-2$, $Z(p) = p-1$ for any odd prime p, the result follows. ■

Given any integer n (≥ 1), there are $n+1$ binomial coefficients, namely, $\binom{n}{k}$, $0 \leq k \leq n$; and for n (≥ 3) fixed, in general, there are two integers k such that n divides $\binom{n}{k}$, namely, $k = 1$ and $k = n-1$. If n is an odd prime, then there are exactly two such numbers. But, in some cases, there are more than two. Consider the following example, which is one of the very extreme cases.

***Example 6.5.5* :** Given $SS(n) = n-k$ for some integers n (≥ 3) and k ($1 \leq k \leq n-2$), it is understood that k divides $\binom{n}{k}$ (and k is the largest such integer, so that $k+1$ does not divide $\binom{n}{k}$). But is it possible that $k-1$ divides $\binom{n}{k}$? Consider the case $n = 57$. Clearly, 57 divides each of $\binom{57}{1}$ and $\binom{57}{2}$, but 57 does not divide $\binom{57}{3}$. In fact, $SS(57) = 54$. It can easily be checked that 57 divides $\binom{57}{4}$; In fact, 57 divides each of the next 14 binomial coefficients (from $\binom{57}{5}$ through $\binom{57}{19}$). Actually, of the possible 57 binomial coefficients, 51 of them is divided by 57 (excepting $\binom{57}{54} = \binom{57}{3}$, $\binom{57}{19} = \binom{57}{38}$, $\binom{57}{27} = \binom{57}{30}$). Note that 57 divides $\binom{57}{6}$, though $(57, 6) \neq 1$.

The above example leads us to pose the problem below.

***Problem 6.5.1* :** For a given integer n, let $SS(n) = n-k$. Find the number of integers k such that n divides $\binom{n}{k}$. Note that, in the worst situation, there are 2 elements.

Remark 6.5.5 **:** Since n = 42 divides $\binom{42}{4}=\frac{42\times41\times40\times39}{4!}$, it follows that SS(42) = 38. Note that, $k=4$ divides $\binom{41}{3}=\frac{41\times40\times39}{3!}$. In this particular case, n divides $\binom{n}{k}$ and (n – 1) divides $\binom{n-1}{k-1}$. Note that, $SS(41)=39$, $SS(43)=41$. In Example 6.5.5 above also, we would find several instances where n divides both $\binom{n}{k}$ and $\binom{n}{k+1}$.

Like other Smarandache type arithmetic functions, particularly, the Smarandache function and the pseudo Smarandache function, the Sandor-Smarandache function also has many open problems which remain to be solved. We close this chapter, and therefore stop writing further, with the following comments, questions and open problems.

Open Problem 6.5.1 **:** Find SS(6m), $m\neq 4s+3$, when 11 divides m (see Lemma 6.5.18).

Question 6.5.1 **:** Given any integer k (≥ 2), is it always possible to find an integer n such that $SS(n)=n-k$?

If $k=2$, then by virtue of Lemma 6.5.1 (and Lemma 6.5.2), for any odd integer n, we have $SS(n)=n-2$; and if $k=3$, then by Lemma 5.5.3, $SS(n)=n-3$ for any even integer n, not divisible by 3. In fact, the proof of Lemma 6.5.3 shows that, $SS(n)=n-3$ for some integer n if and only if n is of the form $n=2m$, where m (≥ 4) is an integer not divisible by 3. Observe that, Lemma 6.5.9 gives a condition such that $SS(p+1)=(p+1)-3$, where p is a prime of the form $p=6u+1$. But what happens when $k\geq 4$? To find an integer n such that $SS(n)=n-k$, $k\geq 4$, we have to keep in mind two points : The first condition is that, such an n must be even, and secondly, 3 must divide n, that is, such an n must be of the form $n=6m$ (for some integer $m\geq 1$). Lemma 6.5.6 deals with the case when $SS(n)=n-4$. The proof of Lemma 6.5.6 shows that $SS(6m)=6m-4$ if and only if $m=4s+3$ for some integer s (≥ 0). Lemma 6.5.7 gives a second instance, such that $SS(30m)=30m-4$ under the condition specified. Lemma 6.5.10 shows that, under the given condition, we can find n such that $SS(n)=n-4$, while Lemma 6.5.11 gives the conditions on n such that $SS(n)=n-5$. Lemma 6.5.14 finds the conditions on the prime p such that $SS(p+1)=(p+1)-6$. Lemma 6.5.18 gives the conditions on n such that $SS(n)=n-7$, $SS(n)=n-8$, and $SS(n)=n-11$. Lemma 6.5.20 finds the condition such that $SS(n)=n-9$, and the condition for $SS(n)=n-10$ is given in Lemma 6.5.21. We thus see that, given any integer k with $2\leq k\leq 11$, we can find n such that $SS(n)=n-k$. In this context, we make the following conjectures.

Conjecture 6.5.1 **:** Given any integer k, there is an integer n such that $SS(n)=n-k$.

Conjecture 6.5.2 **:** Given any integer k, there is an infinite number of integers n such that $SS(n)=n-k$.

In passing, we point out that the binomial coefficient can be expressed as

$$\binom{n}{k}=SL(n,k)\left\{\frac{(n,(n-1),\ldots,(n-k+1))}{(1,2,\ldots,k)}\right\},$$

where SL(n, k) ($k\leq n$) is the Smarandache LCM ratio function (of the second type), defined in Definition 6.4.2. This formula offers an alternative method of finding SL(n, k). For example, for $k=2$, since $(1, 2)=1$, $(n, n-1)=2$, we get

$$SL(n,2)=\frac{n(n-1)}{2}.$$

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SUBJECT INDEX

This book deals with some selected topics of Smarandache notions. Some of the materials are already published in different journals, but some are new and appear for the first time in this book. Except for the trivial and simple ones, all the results are provided with proofs.

- Chapter 1 gives the common characteristics shared by eight recursive type Smarandache sequences, namely, the Smarandache Odd, Even, Circular, Square Product (of two types), Permutation, Reverse, Symmetric and Prime Product sequences
- Chapter 2 deals with the Smarandache bisymmetric geometric determinant sequence, the Smarandache circulant geometric determinant sequence, and the Smarandache circulant arithmetic determinant sequence
- Chapter 3 treats the Smarandache function S(n)
- Chapter 4 considers the pseudo Smarandache function Z(n)
- Chapter 5 gives partial solutions of the Diophantine equations $a^2 = b^2 + c^2 \pm bc$ which arise in connection with the Smarandache function related 60-degree and 120-degree triangles
- And the final Chapter 6 contains miscellaneous topics, six in number. They are : the Smarandache T-sequence, the Smarandache friendly numbers, the Smarandache reciprocal partition sets of unity, the Smarandache LCM ratio, and the Sandor-Smarandache function.The book contains a rather extensive study on the Sandor-Smarandache function.Several open problems are given.

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