SMARANDACHE SPECIAL ELEMENTS IN MULTISET SEMIGROUPS

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PREFACE

Authors in this book study the notion of Smarandache element in multiset semigroups. It is important to keep on record that we define four operations on multisets viz. +, X, union and intersection in a free way. Thus all sets finite or infinite order contribute to infinite order multisets and the semigroup under any of these operations is of infinite order.

We in this book define a new notion called n- multiplicity multiset using any set S, denoted by n-M(S). This n-multiplicity multiset contains all set got using S where the number of times an element in the multiset M(S) cannot repeat more than n-times. On n-multiplicity multisets we cannot define + or × or ∪ as in case of multisets, for closure axiom fails in all cases. We overcome this problem by the method of levelling. We have defined l(+) where l denotes the levelling of addition or levelled addition. It is proved that n-multiplicity of M(S) under l(+) is a semigroup of finite order if S is finite and if S is infinite {n-multiplicity M(S) , l(+)} is a commutative semigroup of infinite order. Similarly in case of ∪ and × we use levelling.

Several interesting properties are derived and one of the innovations made here is the defining special Smarandache elements on these semigroups, like Smarandache special idempotents, Smarandache special zero divisors and so on.

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Chapter One

BASIC CONCEPTS

In this chapter we just proceed onto define some basic concepts which are essential for an easy understanding of the other chapters and the algebraic structures defined on multisets. We first list out the properties of power set, semilattices and lattices.

A power set of a nonempty set S is the collection of all subsets of S together with S and ϕ denoted by P(S); P(S) = {All subsets of S including S and the empty set ϕ}.

Example 1.1. Let S = $\{a_1, a_2, a_3, a_4\}$ be set of four elements the power set of S is P(S). $P(\{a_1, a_2, a_3, a_4\}) = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \phi, \{a_1, a_2\}, \{a_1, a_3\} \{a_1, a_4\}, \{a_2, a_3\} \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_4\}\}$. Clearly $|P(S)| = 2^4$ so if $|S| = n$ then $|P(S)| = 2^n$ and the power set of a finite set is always finite.

It is well know P(S) under union of set '$\cup$' is a semilattice.

Further P(S) is a partially ordered set {P(S), $\cup$} is a commutative idempotent semigroup of finite order.

Similarly P(S) under intersection of set $\cap$ is a semilattice or {(P(S), $\cap$} is again a finite idempotent semigroup which is commutative.

Further it has been proved that {P(S), $\cup$, $\cap$} is a lattice and this lattice is complemented and distributive hence a Boolean algebra of order $2^{|S|}$. We would be using this concept as the multisets in general under the unrestricted union and intersection is not a Boolean algebra. Further the order of multisets of any set S finite or infinite is of infinite order.

For instance if S = {a_1, a_2, a_3} we see P(S) = {ϕ, {a_1}, {a_2}, {a_3}, {a_1, a_2}, {a_1, a_3}, {a_2, a_3}, {a_1, a_2, a_3}} is the power set associated with S and |S| = 8 = 2^3. The Boolean algebra B associated with P(S) is as follows.

B =

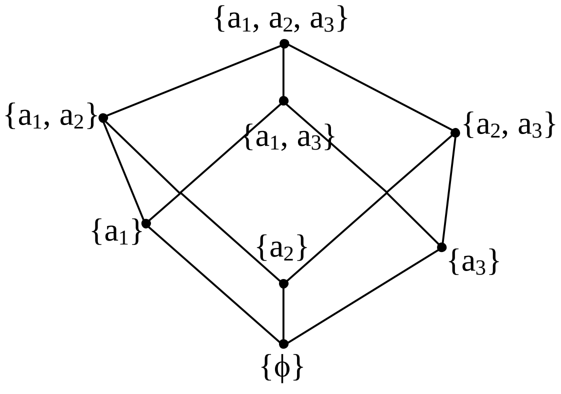

Figure 1.1

We define B as a Boolean algebra of order 8.

Now we proceed onto describe multisets and the operations which we have used in this book.

Definition 1.1: *Let S be a finite or infinite set. Multiset of S denoted by M(S) = {collection of all subsets of S where in the subsets any element of S is repeated from one time to infinitely many times)}.*

Thus if S ={a}; a singleton set, the multiset of S denoted by M(S) = {ϕ, {a}, {a, a}, {a, a, a} … , {a, a, a, ….-infinite number of times}. Hence M(S) is always infinite immaterial of whether S is finite or infinite.

(i). In this first place a multiset of any set S finite or infinite is always infinite and is denoted by M(S).

(ii) We illustrate how $\cup$ operation is defined on M(S).

If A and B $\in$ M(S), A $\cup$ B contains all elements of A and all elements of B; no restrictions on the repetitions of the elements.

Thus if A = {a_1, a_1, a_1, a_2, a_2, a_3, a_8, a_8, a_8} and B = {a_1, a_1, a_2, a_2, a_2, a_2, a_2, a_2, a_2, a_9, a_9} $\in$ M(S) where S = {a_1, a_2, …, a_9} A $\cup$ B = {a_1, a_1, a_1, a_2, a_2, a_3, a_8, a_8, a_8, a_1, a_1, a_2, a_2, a_2, a_2, a_2, a_2, a_2, a_2, a_9, a_9}.

This is the way we have define '$\cup$' operation on multisets.

Hence if we take infinite union of many sets then the resulting set is infinite though |S| = 9.

iii) We define intersection as the common elements occurring in A and B so A ∩ B = {a_1, a_1, a_2, a_2} and nothing more, that is intersection is the usual intersection.

iv) Since M(S) is a partially ordered set we see {M(S), ∪, ∩} is a lattice.

We will show that in general {M(S), ∪, ∩} the lattice of M(S) is not a distributive lattice. It cannot be a Boolean algebra.

We now give an example where the distributive law fails.

Let A = {0, 0, 0, 1, 1, 2}, B = {0, 0, 3, 1, 4} and C = {0, 1, 1, 5, 3, 7} ∈ M(S) where S = {0, 1, 2, …, 8}.

Consider A ∩ (B ∪ C) = {0, 0, 0, 1, 1, 2} ∩ [{0, 0, 3, 1, 4} ∪ {0, 1, 1, 5, 3, 7}] = {0, 0, 0, 1, 1, 2} ∩ {0, 0, 0, 1, 1, 1, 3, 3, 5, 7, 4} = {0, 0, 0, 1, 1} I

Consider (A ∩ B) ∪ (B ∩ C) = [{0, 0, 0, 1, 1, 2} ∩ {0, 0, 3, 1, 4}] ∪ [{0, 0, 0, 1, 1, 2} ∩ {0, 1, 1, 5, 3, 7}] = {0, 0, 1} ∪ {0, 1, 1} = {0, 0, 0, 1, 1, 1} II

Clearly I and II are different hence A ∩ (B ∪ C) ≠ (A ∩ B) ∪ (A ∩ C) in general for A, B, C ∈ M(S).

Thus we see ∪ and ∩ do not in general follow the distributive law hence {P, ∪, ∩} cannot be a distributive lattice.

We recall the definition a Smarandache lattice in the following.

Definition 1.2. *Let {L, ∪, ∩} be a lattice. We call L a Smarandache lattice if there exists a ≠ B ⊆ L such that {B, ∪, ∩} is a Boolean algebra of order four.*

For more refer [39].

Next we proceed onto define and describe the properties of Smarandache structures.

Definition 1.3. *Let {S, ×} be a semigroup {H, ×} ⊆ {S,×} be a subsemigroup of S.*

If {H, ×} is a group we define {S, ×} to be a Smarandache semigroup or in short S-semigroup.

We give an example or two.

Example 1.2. Let {Z, ×} be the semigroup under product. P = {{1, – 1}, ×} ⊆ {Z, ×} is a cyclic group of order two given by the following example.

×	–1	1
–1	1	–1
1	–1	1

Thus {Z, ×} is a S-semigroup.

Example 1.3. Let S(3) = {Symmetric semigroup got from {1, 2, 3} by taking all mappings from {1, 2, 3} to {1, 2, 3}} S(3) under composition of mappings is a semigroup known as the symmetric semigroup [17]. Clearly S_3 is the group of permutations on {1, 2, 3}; $S_3 \subseteq S(3)$. Hence {S(3), composition maps} is a S-semigroup. For more [17].

We now proceed onto define Smarandache zero divisors and Smarandache weak zero divisors in semigroups.

Definition 1.4. *Let $\{S, \times\}$ be a semigroup with $0 \in S$. We say x, $y \in S \setminus \{0\}$ is a zero divisor if $x \times y = 0$.*

x, $y \in S \setminus \{0\}$ is defined as a Smarandache zero divisor (S-zero divisor) if there exists a, $b \in S \setminus \{x, y, 0\}$ such that

$x \times a = 0$ *or* $a \times x = 0$

$y \times b = 0$ *or* $b \times y = 0$

and $a \times b = 0$.

Example 1.4. Let S = $\{Z_{12}, \times\}$ be the semigroup under product modulo 12.

We have

$4 \times 3 = 0$, $4 \times 6 = 0$

$6 \times 10 = 0$, $2 \times 6 = 0$, $8 \times 3 = 0$ and $8 \times 6 = 0$

x = 8 and y = 3 then $x \times y = 0$.

Choose a = 4 and b = 6 we see $6 \times 8 = 0$ $4 \times 3 = 0$

and $4 \times 6 = 0$ so {x, y} is a Smarandache zero divisor in S.

Now we proceed onto give an example of a Smarandache zero divisor.

Consider 6, 4 $\in Z_{12}$ we see 6 × 4 = 0, to prove 6, 4 $\in Z_{12}$ is a S-weak zero divisor we have to show there exists a, b $\in Z_{12}$ \ {6, 4, 10} with a × b ≠ 0 a × 6 = 0 and b × 4 = 0. Take 3, 2 $\in Z_{12}$. We see 3 × 4 = 0 and 6 × 2 = 0 but 3 × 2 = 6 ≠ 0. Hence {6, 4} is a S-weak zero divisor of Z_{12}.

Results in this direction for rings can be had from [16].

Now we proceed onto define idempotents and Smarandache idempotents in case of semigroups with idempotents.

Definition 1.5. *Let {S, ×} be a semigroup suppose x ∈ S is an idempotent; that is $x^2 = x$; we call x a Smarandache idempotent if there exists a ∈ S with $a^2 = x$ and ax = x or ax = a.*

Example 1.5. Let S = {Z_{10}, ×} be the semigroup under product modulo 10.

We see 6 $\in Z_{10}$ is such that $6^2 = 6$ is an idempotent of Z_{10}.

Consider 4 $\in Z_{10}$, 4 × 4 ≡ 6 (mod 10) and 6 × 4 = 4 (mod 10). Hence 6 is a Smarandache idempotent of Z_{10}.

Example 1.6. Let S = {Z_{12}, ×} be a semigroup. Clearly 4 $\in Z_{12}$ is such that 4 × 4 ≡ 4 (mod 12). We see 8 $\in Z_{12}$ is such that 8 × 8 ≡ 4 (mod 12) and 8 × 4 = 8 (mod 12) so 4 is a S-idempotent of Z_{12}.

More properties can be had from [17]

We have used in this book the notion of natural product of matrices.

Definition 1.6. *Let N and M be two m × n matrices m ≠ n where M = (m_{ij}) and N = (n_{ij}). We define the natural product of M with N denoted by $\times_n$ as M $\times_n$ N = ($m_{ij} \times n_{ij}$); $1 \leq i \leq n$ and $1 \leq j \leq m$, that is component multiplication is done.*

Clearly the classical matrix product can be defined only if there exists a compatibility of multiplication; that is if A is a n × m matrix if A × B is to be defined B must only be a m × t matrix if n ≠ t. B × A is not defined. For more refer [30].

We will just give an example or two of this situation.

Example 1.7. Let

$$A = \begin{bmatrix} 3 & 0 & 1 & -4 & 8 \\ 0 & 5 & 6 & 9 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 4 & 9 & 9 & 2 \\ 9 & 9 & 9 & -1 & 9 \end{bmatrix}$$

be two 2 × 5 matrices. Under usual or classical product A × B is not defined. But the natural product $\times_n$ is defined for A and B as well as B and A.

$$A \times_n B = \begin{bmatrix} 3 & 0 & 1 & -4 & 8 \\ 0 & 5 & 6 & 9 & 1 \end{bmatrix} \times_n \begin{bmatrix} 2 & 4 & 0 & 9 & 2 \\ 9 & 0 & 9 & -1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2 & 0\times 4 & 1\times 0 & -4\times 9 & 8\times 2 \\ 0\times 9 & 5\times 0 & 6\times 9 & 9\times -1 & 1\times 9 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 & -36 & 16 \\ 0 & 0 & 54 & -9 & 9 \end{bmatrix}.$$

It is easily seen B $\times_n$ A = A $\times_n$B.

Thus under natural product the matrix multiplication is commutative.

For more about natural product $\times_n$ and their related properties refer [30].

Next we just keep on record that multisets defined in this book are not the usual classical ones used by [1, 2].

We define '$\cup$' as free one so the multisets are of infinite cardinality immaterial of the basic set.

Further $\cap$ happens to be as that of the classical one.

All multisets under '$\cup$' and '$\cap$' happen to be lattice. In fact they are Smarandache lattices as they contain the Boolean algebra of order 2^2. Though X is the basic set used and M(X) is the multiset of X then P(X) the power set of X is always a subset of M(X), but we see {P(X), $\cup$, $\cap$} is not a Boolean algebra of order $2^{|x|}$ and M(X) can be finite or infinite.

Thus we see {M(X), $\cup$, $\cap$} is a lattice which does not contain the Boolean algebra of P(X) but contains always a Boolean algebra of order four, hence the multiset lattice {M(X), $\cup$, $\cap$} is a Smarandache multiset lattice.

Next one the innovative portion of this book is that we define multisets on a semigroup under product or sum and we induce the product or sum operation on the multisets and those multisets also become a semigroup under product or sum respectively.

We study mainly the special Smarandache elements present in them.

We just enumerate some of the possible applications of these new algebraic structures introduced in [2].

Some properties of query languages for bags can be had from [6].

Various applications of multisets are becoming fundamental in combinatorics. Multisets have become an important tool in databases.

Certainly with algebraic structures imposed on them and some innovative new techniques in defining the very multisets and operations on them, they in due course of time will find much more applications in database and data storages.

Chapter Two

SMARANDACHE ELEMENTS IN MULTISET SEMIGROUPS

In this chapter we proceed onto describe the other type of operations on multisets other than '∪' (union), '∩' (intersection) and \ existing in the literature which are briefly discussed in chapter I of this book.

Now before we proceed to define operations + and × on multisets, we discuss some of the vital properties associated with them.

Consider the subsets of the reals R which are multisets. Clearly the collection of all multisets of R are of infinite cardinality. In fact some of the multiset can be even of infinite cardinality. We will indicate some operations on them.

Let A = {1, 1, 2} be a multiset

A + A = {1, 1, 2} + {1, 1, 2}

= {2, 2, 3, 2, 2, 3, 3, 3, 4}.

Thus o(A) = 3 and o(A + A) = 9.

A × A = {1, 1, 2} × {1, 1, 2}

= {1, 1, 2, 1, 1, 2, 2, 2, 4}.

We see o(A × A) = 9.

However A × A ≠ A + A.

Consider (A + A) + A = {2, 2, 3, 2, 2, 3, 3, 3, 4} + {1, 1, 2}

= {3, 3, 4, 3, 3, 4, 4, 4, 5, 3, 3, 4, 3, 3, 4, 4, 4, 5, 5, 5, 6, 4, 4, 5, 4, 4, 5}.

o(A + A + A) = 27.

(A × A) × A = {1, 1, 2, 1, 1, 2, 2, 2, 4} × {1, 1, 2}

= {1, 1, 2, 1, 1, 2, 2, 2, 4, 1, 1, 2, 1, 1, 2, 2, 2, 4, 2, 2, 4, 2, 2, 4, 4, 4, 8}.

(A × A × A) = 27.

However (A + A) + A ≠ (A × A) × A and (A + A) + A = A + (A + A) and (A × A) × A = A × (A × A).

Now we will check will (A + B) + C = A + (B + C)

Where A, B and C are three distinct sets.

Let A = {3, 1, 1, 4}, B = {0, 2, 0, 1, 6, 5} and C = {7, 5, 5, 10} be three multisets.

We find (A + B) + C = ({3, 1, 1, 4} + {0, 0, 2, 1, 6, 5}) + {7, 5, 5, 10}

= {3, 1, 1, 4, 3, 1, 1, 4, 5, 3, 3, 6, 4, 2, 2, 5, 9, 7, 7, 10, 8, 6, 6, 9} + {7, 5, 5, 10} = {10, 8, 8, 11, 10, 8, 8, 11, 12, 10, 10, 13, 11, 9, 9, 12, 16, 14, 14, 17, 15, 13, 13, 16, 8, 6, 6, 9, 8, 6, 6, 9, 10, 8, 8, 11, 9, 7, 7, 10, 14, 12, 12, 15, 13, 11, 11, 14, 8, 6, 6, 9, 8, 6, 6, 9, 10, 8, 8, 11, 9, 7, 7, 10, 14, 12, 12, 15, 13, 11, 11, 14, 13, 11, 11, 14, 15, 13, 13, 16, 14, 12, 12, 15, 19, 17, 17, 20, 18, 16, 16, 19}

o(A + B + C) = 96

= o(A) × o(B) × o(C)

It is easily verified A × (B × C) = (A × B) × C and (A + B) + C

= A + (B + C) with o(A + B + C) = o(A × B × C) = o(A) × o(B) × o(C).

When finite order multisets are taken we see the sum or product defined on them finite number of times using only finite number of elements from the multisets, yields only a finite order multiset.

Thus if M(R) = {collection of multisets from the set of real} then M(R) is of infinite order and M(R) contains both finite and infinite multisets.

Further M(R) contains P(R) the power set of R which is also of infinite order but P(R) contains no multiset. In fact we make it as a convention to accept the power set of the set of reals R to be included as a multiset, however if a researcher wishes to make a proper difference he/she can call the multiset to be a collection not including the power set of R.

It is upto the needs of the researcher.

Further the sum of multiset A with B from P(R), the power set of R yields only a multiset.

However it is important to keep on record whether the set S is finite or infinite the multiple set M(R) (or M(S)) is always infinite.

We can define freely the four operations +, ×, ∪ and ∩.

Under each of these M(S) is a semigroup of infinite order which is commutative.

We will illustrate these situations by some examples.

Example 2.1. Let S = {0, 1, 2, 3} = Z_4. Define M(S) to be the multiset of S. Clearly M(S) is of infinite order

Consider x = {0, 2, 2, 2, 0, 0, 0} ∈ M(S). We see x + x = {0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 2, 2, 2, 0, 0, 0, 2, 2, 2, 2, 0, 0 , 0, 2, 2, 2, 2} ∈ M(S).

o(x + x) = 49 whereas o(x) = 7.

Thus we see $|\underbrace{x + x + ... + x}_{\text{n-times}}| = 7^n$; as n → ∞ o(x + x + … + x) → ∞. It is easily prove {M(S), +} is an infinite commutative semigroup of infinite order.

Next consider

x ∩ x = {0, 0, 0, 0, 2, 2, 2} ∩ {0, 0, 0, 0, 2, 2, 2}

= {0, 0, 0, 0, 2, 2, 2} ∈ M(S) = x.

Thus {M(S), $\cap$} can be easily proved to be a commutative semigroup of infinite order.

Infact {M(S), $\cap$} has subsemigroups of finite order and every x $\in$ M(S) is such that x $\cap$ x = x is an idempotent.

Thus {M(S), $\cap$} is a semilattice of infinite order.

Likewise {M(S), $\cup$} is not a semilattice for

if x = {2, 2, 2, 0, 0, 0, 0} $\in$ M(S)

x $\cup$ x = {2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0} $\in$ M(S).

x $\cup$ x $\neq$ x and if |x| = 7 then | x $\cup$ x| = 14 that |x $\cup$ x $\cup$ x| = 3 × 7 and further $\underbrace{|x \cup x \cup ... \cup x|}_{\text{n-times}}$ = 7n.

Consider M(S) under the operation +

x + x = {2, 2, 2, 0, 0, 0, 0} + {2, 2, 2, 0, 0, 0, 0}

= {0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 2, 2, 2, 2,
0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2,
0, 0, 0, 0, 2, 2, 2} $\in$ M(S)

o(x + x) = 7 x 7 = 49.

{M(S), +} is a commutative semigroup of infinite order.

We consider the product operation on M(S). For the same x $\in$ M(S). We see x × x = {0, 0, 0, 0, 2, 2, 2} × {0, 0, 0, 0, 2, 2, 2} = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, … ,0}.

Thus we see there are special multiset nilpotents, which we choose call x as special because it is such that

$$x \times x = \underbrace{\{0,\ldots,0\}}_{49\text{-times}}$$

We have x ∈ $M(Z_4)$ such that x × x = {0} or {0, 0}, {0, 0, 0}, {0, 0, 0, 0} and so on $\underbrace{\{0,0,0,\ldots,0\}}_{n\text{-times}}$.

These will be called only as special multiset nilpotents.

Further we wish to record that we have in multisets infinite number of zeros.

Next let x = {2, 2, 0, 0, 0, 0, 2, 0, 2, 2, 2} and y = {2, 2, 2} ∈ $M(Z_4)$. We see x × y = {2, 2, 2, 2, 2, 2, 0, 0, 0, 0 } × {2, 2, 2} = $\underbrace{\{0,0,0,0\ldots,0,0,0\}}_{30\text{-times}}$.

Thus x, y in $M(Z_4)$ is a special multiset zero divisor.

We have infinitely many special multiset zero divisors in $M(Z_4)$.

In fact {$M(Z_4)$, ×} is an infinite semigroup which has infinite number of special multiset nilpotents and special multiset zero divisors.

Also all multiset subsemigroups are of infinite order. Further {$M(Z_4)$, ×} has no nontrivial multiset idempotents.

It is easily verified that {$M(Z_4)$, ×} can contain ideals of infinite order also. It can have infinite order subsemigroups.

We see M({0, 2}) ⊆ $M(Z_4)$ is an ideal of infinite order under ×.

Infact when we say multiset zero ideal it will contain {{0}, {0, 0}, {0, 0, 0}, …, {0, 0, 0, …, 0} and soon}. It is pertinent to keep on record that in case of semigroup under product the zero ideal is just the element ({0}).

But in case of multiset semigroups under product the zero ideal is an infinite collection of zeros given by {{0}, {0, 0}, {0, 0, 0}, …, {0, 0, 0, …, 0} and so on}.

Now we show the existence of non zero ideals in case of infinite set S and a finite set P.

Example 2.2. Let Z be the set of integers M(Z) be the multiset of Z. {M(Z), ×} be the semigroup.

Clearly P = {M(2n) / n ∈ Z)} be the multiset of even integers. We see P is an ideal of infinite order.

Infact for every prime p ∈ Z we have {M(px) / x ∈ Z} ⊆ M(Z) is an ideal of {M(Z), ×}.

Thus {M(Z), ×} has finite number of ideals all of which are of infinite order.

Example 2.3. Let R be the field of reals. M(R) be the multiset of R. {M(R), ×} is an infinite semigroup.

Clearly {M(R), ×} has no nontrivial ideal. The only trivial proper ideal being {{0}, {0, 0}, {0, 0, 0}, …, {0, 0, 0, …, 0}}.

Thus if F is a field the multisets of F under product does not yield any proper special nilpotents or special zero divisors or any nontrivial ideals.

Example 2.4. Let S = Z_{11} and $M(Z_{11})$ be the collection of multisets of Z_{11} = S. We see {$M(Z_{11})$, ×} is a semigroup which has no nontrivial ideals only zero ideal {{0}, {0, 0}, {0, 0, 0}, …, {0, 0, …,0}}. Further $M(Z_{11})$ has no special zero divisors or nilpotents.

Thus M(S) has no special idempotents what ever be the set S.

In view of all these we put forth the following theorem.

Theorem 2.1. *Let {S, ×} be a semigroup of finite or infinite order. M(S) be the multiset of S.*

i) *{M(S), ×} is always a semigroup of infinite order*

ii) *{M(S), ×} has special multiset nilpotents and special multiset zero divisors if and only if {S, ×} has nilpotents and zero divisors respectively.*

iii) *{M(S), ×} has multiset nontrivial ideals if and only if {S, ×} has nontrivial ideals.*

iv) *{0} is the zero ideal of {S, ×} then {{0}, {0, 0}, {0, 0, 0}, …, {0, 0, 0, …, 0} and so on} is the special zero ideals of {M(S), ×}.*

v) *Whatever be S, {M(S), ×} has no nontrivial multiset idempotents.*

Proof is left as an exercise to the reader.

We now show how the set {S, ×} has idempotents but {M(S), ×} has no idempotents.

Example 2.5. Let {Z_{12}, ×} be the semigroup under product modulo 12. {M(Z_{12}, ×} be the multiset semigroup on {Z_{12}, ×}.

We see 4 and 9 are idempotents of Z_{12} but any multiset with 4 and 9 or 4 or 9 does not contribute for an idempotent. However trivial idempotents of $M(Z_{12})$ which are not multisets are {4} and {9}.

For {4} × {4} = {4} and {9} × {9} = {9}. Trivial ones being {1} and {0}.

However even trivial idempotents does not occur in case of $M(Z_{12})$. If $\{0, 0\} \in M(Z_{12})$, $\{0, 0\} \times \{0, 0\} = \{0, 0, 0, 0\} \neq \{0, 0\}$.

Similar if $\{1, 1, 1\} \in M(Z_{12})$ $\{1, 1, 1\} \times \{1, 1, 1\}$

$= \{1, 1, 1, 1, 1, 1, 1, 1, 1\} \neq \{1, 1, 1\}$.

Thus $\{M(Z_{12}), \times\}$ has nontrivial multiset idempotents.

Next we proceed onto give some multiset zero divisors of $\{M(Z_{12}), \times\}$.

Consider x = {6, 3, 9, 0, 0, 6, 6, 3, 9, 9, 3} and y = {4, 4, 4, 8, 8, 8, 8, 4, 0, 0, 0} $\in M(Z_{12})$

x × y = {6, 3, 9, 0, 0, 6, 6, 3, 9, 9, 3} × {4, 4, 8, 4, 8, 8, 8, 4, 0, 0, 0} = $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots, \underbrace{0, 0, \ldots, 0\}}_{121\text{-times}}$.

Thus x and y are special multiset zero divisors of $M(Z_{12})$ as 6, 3 and 9 are such that

$6 \times 4 \equiv 0 \pmod{12}$; $3 \times 4 \equiv 0 \pmod{12}$,

$6 \times 8 = 0 \pmod{12}$; $3 \times 8 \equiv 0 \pmod{12}$ and

$9 \times 4 \equiv 0 \pmod{12}$ and $9 \times 8 \equiv 0 \pmod{12}$

are zero divisors of Z_{12}.

However the idempotents 9 and 4 in Z_{12} cannot contribute to any form of idempotent multisets in $M(Z_{12})$ as if order of the multiset x is n then that of x × x will be n × n hence we cannot get any multiset idempotents other than {9}, {4}, {0} and {1}.

But nontrivial zero divisors Z_{12} contribute to multiset special zero divisors of $\{M(Z_{12}), \times\}$.

Hence if {S, ×} has zero divisors or nilpotents then {M(S), ×} will also have special multiset zero divisors and special multiset nilpotents.

We in this book are not interested in studying the multiset with the operations defined by [2, 6].

We further can prove that if $\{Z_n, \times\}$ has Smarandache zero divisors then $\{M(Z_n), \times\}$ will have infinite number of Smarandache special zero divisors.

To this end we will supply an example or two.

Example 2.6. Let $\{M(Z_4), \times\}$ be the multiset of Z_4. Clearly $M(Z_4)$ has infinite number of zero divisors and special Smarandache zero divisors and no Smarandache special weak multiset zero divisors.

Example 2.7. Let $\{Z_6, \times\}$ be the semigroup $\{M(Z_6), \times\}$ be the multiset semigroup $\{M(Z_6), \times\}$ has special multiset Smarandache zero divisors and no special multiset S-weak zero divisors. This is true since $\{Z_6, \times\}$ has no S-zero divisors or S-weak zero divisors. However {M(Z_6), ×} has infinite collection of special multiset zero divisor. {Z_6, ×} also has zero divisor.

We see if x = {3, 3, 0, 0, 3, 0, 3, 3} and y = {2, 2, 4, 4, 4, 4, 0, 0, 0} ∈ M(Z_6).

Clearly x × y = {0, 0, 0, 0, 0, $\underbrace{,...,0,0,0}_{\text{72-times}}$} is a special multiset zero divisors.

However as {Z_6, ×} has no S-zero divisors and S-weak zero divisors. {Z_6, ×} has idempotent by {M(Z_6), ×} has no nontrivial multiset idempotents but {Z_6, ×} has 3 ∈ Z_6; 3 × 3 = 3 ≡ (mod 6).

Example 2.8: Let {Z_{12}, ×} be the semigroup under product modulo 12. {M(Z_{12}), ×} be the multiset semigroup.

The zero divisors of Z_{12} are

$3 \times 4 \equiv 0 \pmod{12}$, $2 \times 6 \equiv 0 \pmod{12}$;
$3 \times 8 \equiv 0 \pmod{12}$, $6 \times 4 \equiv 0 \pmod{12}$;
$6 \times 8 \equiv 0 \pmod{12}$, $9 \times 4 \equiv 0 \pmod{12}$
and $9 \times 8 \equiv 0 \pmod{12}$.

The S-zero divisors of {Z_{12}, ×} are

$6 \times 8 \equiv 0 \pmod{12}$ and $3 \times 8 \equiv 0 \pmod{12}$
$6.2 \equiv 0 \pmod{12}$ but $3.2 \not\equiv 0 \pmod{12}$

Thus {Z_{12}, ×} has both S-zero divisors and S-weak zero divisors.

So if x = {0, 6, 0, 6, 6} and 9 = {8, 8, 8, 8, 0} we see

$x \times y \equiv \underbrace{\{0,0,0,...,0\}}_{25\text{-times}}$. z = {3, 0, 3, 3, 3} ∈ $M(Z_{12})$ is such that

$z \times y \equiv \underbrace{\{0,0,0,...,0\}}_{25\text{-times}}$ but z × x ≠ {0, 0, …, 0}.

Thus we can prove $\{Z_{12}, \times\}$ has S-zero divisors and S-weak zero divisors then $\{M(Z_{12}), \times\}$ also has Smarandache special multiset zero divisors and Smarandache special multiset weak zero divisors.

Infact it is left as an exercise for the reader to prove the following theorem.

Theorem 2.2. *Let $\{Z_n, \times\}$ be the semigroup under product modulo n. $\{M(Z_n), \times\}$ be the multiset semigroup.*

i) *$\{M(Z_n), \times\}$ has special multiset S-zero divisors if $\{Z_n, \times\}$ has S-zero divisors.*

ii) *$\{M(Z_n), \times\}$ has Smarandache special multiset weak zero divisors if and only if $\{Z_n, \times\}$ S-weak zero divisors.*

Next we proceed onto prove and describe for what values of n; $\{M(Z_n), \times\}$ has nontrivial S-zero divisors and S-weak divisors.

Example 2.9. Let S = $\{Z_{23}, \times\}$ be the semigroup. $\{M(Z_{23}), \times\}$ be the multiset semigroup. S has no zero divisors, S-zero divisors and S-weak zero divisors. Hence $\{M(Z_{23}),\times\}$ has no nontrivial multiset zero divisors, S-multiset special zero divisors and S-multiset special weak zero divisors.

In view of this we leave the following result as an exercise for the reader.

Theorem 2.3. *Let S = {Z_p, ×} be the semigroup under product. B = {M(Z_p), ×} be the multiset semigroup under product. B has no special multiset zero divisors, S-multiset special zero divisors and S-multiset special weak zero divisors.*

We define these as semidomain group or semidomains.

Definition 2.1. *Let S = {P, ×} be semigroup. We define S as a semidomains or semidomain group or domain semigroup if*

i) S has zero
ii) S has no zero divisors
iii) S has no idempotents
iv) S has no units

We first provide some examples of them.

Example 2.10. Let S = {Z_{47}, ×} be the semigroup under product modulo 47.

B = {M(Z_{47}), ×} be the multiset semigroup under product modulo 47.

We see S has no idempotents, zero divisors or S-zero divisors and S-idempotents but has $0 \in Z_{47}$ such that $0 \times x = 0$ for all $x \in Z_{47}$.

Hence P = {M(Z_{47}), ×} has no special S-zero divisors or S-special weak zero divisors or special zero divisors.

But P has infinite number of zeros and has no special idempotents S-special idempotent.

In view of all these we have the following result.

Theorem 2.4: *Let S = {Z_p, ×} be the semigroup under product modulo p, p a prime. B = {M(Z_p), ×} be the multiset semigroup under product. B has no special multiset zero divisors, multiset idempotents, S-special multiset zero divisors and S-special multiset weak zero divisors.*

Proof. Follows from the very fact {Z_p, ×} has no zero divisors or idempotents or S-Zero divisors or S-weak zero divisors so B has none of the above said elements mentioned in the theorem.

Example 2.11. Let S = {Z_{10}, ×} be the semigroup under product B = {M(Z_{10}), ×} be the multiset semigroup of Z_{10}. Z_{10} has only zero divisors hence M(Z_{10}) has multiset special zero divisors which are infinite in number.

However S has no S-zero divisors or S-weak zero divisors so B has no special multiset S-zero divisors or special multiset S-weak zero divisors.

In view of this we have the following theorem.

Theorem 2.5. *Let S = {S_{2p}, ×} be the semigroup under product modulo 2p, p a prime B = {M(Z_{2p}, ×} be the multiset semigroup. B has special S-multiset zero divisors and no special S-multiset weak zero divisors. B has infinite number of special subset zero divisors.*

Proof. Follows from the fact Z_{2p} has zero divisors only of the form $p \times x \equiv 0$; where $x \in 2Z_{2p}$. Now in the multisets B we see for all x and y $\in$ B where x has only elements of the form p and 0 repeating and y has only elements of the form {2, 4, …, 2n} repeating we see $x \times y = \{0, \ldots, 0\}$ hence has special multiset zero divisors.

Thus we have infinitely a big collection of special multiset zero divisors.

If we take a, b $\in$ B \ {x, y} where a has only elements 0's and p's and b has elements only from $2Z_{2p}$ then with x have only elements from 0's and p's with $|a| \neq |x|$ and y having only elements from $2Z_{2p}$, with $|b| \neq |y|$ we get $a \times y \equiv 0$ and $b \times x = 0$ with $a \times b \equiv 0$. Thus this collection a, b, x, y $\in M(Z_{2p})$ is a special Smarandache multiset zero divisor.

We have infinite such collection for in a and x we can also change the number of zeros and number of p so that even if $|a| = |x|$ still we see a and x are distinctly different.

Thus the result of the theorem.

We will illustrate this situation before we give the abstract definitions.

Example 2.12. Let $S = \{Z_{22}, \times\}$ be the semigroup under product × modulo 22. $B = \{M(Z_{22}), \times\}$ be the multiset semigroup under product ×.

$2 \times 11 \equiv 0 \pmod{22}$; $x \times 11 \equiv 0 \pmod{22}$

$x \in 2Z_{22}$.

Now let x = {0, 0, 11, 11, 11} and y = {2, 2, 0, 4} $\in$ M(Z_{22}).

$$x \times y = \underbrace{\{0,0,0,...,0\}}_{\text{20-times}}$$

Clearly x, y is a special multiset zero divisors of B.

Consider a = {11, 11, 11, 11, 11, 0, 0} and b = {8, 8, 8, 8, 8, 8, 0, 0, 0} $\in$ M(Z_{22}).

We see $x \times b \equiv \underbrace{\{0,0,0,0,...,0,0\}}_{\text{45-times}}$ $a \times y = \underbrace{\{0,0,0,0,...,0,0\}}_{\text{28-times}}$

Further $a \times b = \underbrace{\{0,0,0,0,...,0,0\}}_{\text{63-times}}$

Thus (x, y) is a special multiset Smarandache zero divisor of B.

Infact it can be easily proved that B has infinite number of special multiset Smarandache zero divisors.

We cannot find S-weak zero divisors in S.

Example 2.13. Let S = {Z_{86}, ×} be the semigroup under product 43.

{M(Z_{86}), ×} = P be the multiset semigroup.

The zero divisors of S are 43 × x ≡ 0 (mod 86); where x $\in$ {0, 2, 4, 6, 8, …, 84}. Thus S has only finite number zero divisors. S has S-zero divisors but S has no S-weak zero divisors.

Now we see P has infinite number of special multiset S-zero divisors but no special multiset S-weak zero divisors.

We just give the abstract definition of both.

Definition 2.2. *Let {S, ×} be a semigroup under ×. {M(S), ×} be the multiset semigroup.*

Let x, y ∈ M(S) such that x × y = {0} (A multiset special zero divisor. If there exist a, b ∈ M(S) \ {x, y} such that

a × x = {0} or x × a = {0}
b × y = {0} or y × b = {0}
a × b = {0} or b × a = {0}

then (x, y) is defined as the special multiset Smarandache zero divisor of {M(S), ×}.

We call x, y ∈ M(S) with x × y = {0} a special multiset zero divisor of M(S) to a multiset special Smarandache weak zero divisor of M(S) if there exist a, b ∈ M(S) \ {x, y} with

ax = {0} or xa = {0}
by = {0} or yb = {0}
and ab ≠ {0} or ba ≠ {0}

We will supply some more examples of them.

Example 2.14. Let S = {Z_{12}, ×} be semigroup under product modulo 12.

P = {M(Z_{12}), ×} be the multiset semigroup

3.4 ≡ 0 mod 12.	4.9 ≡ 0 (mod 12)
3.8 ≡ 0 (mod 12)	8.9 ≡ 0 (mod 12)

$6.8 \equiv 0 \pmod{12}$ and $3.8 \equiv 0 \pmod{12}$

$6.2 \equiv 0 \pmod{12}$ $3.2 \equiv 0 \pmod{12}$

Thus $\{Z_{12}, \times\}$ has S-zero divisors and S-weak zero divisors.

We will show $\{M(Z_{12}), \times\}$ has special multiset S-zero divisors and special multiset S-weak zero divisors and special multiset S-weak zero divisors which are infinitely many.

Consider x = {6, 6, 6, 6, 0, 0, 0} and y = {8, 8, 8, 0, 8, 0} $\in$ B.

Clearly $x \times y = \underbrace{\{0,0,0,\ldots,0,0,0\}}_{42\text{-times}}$.

In fact we can show there are infinitely many special multiset zero divisors in $M(Z_{12})$.

Consider special multiset zero divisor x = {3, 0, 3, 3, 3, 0, 0} and y = {4, 4, 4, 0, 0} $\in M(Z_{12})$ $x \times y = \underbrace{\{0,0,0,0,...,0\}}_{42\text{-times}}$.

Now {8, 8, 8, 0, 0} = a and b = {9, 9, 9, 9, 0} $\in M(Z_{12})$ $a \times b = \underbrace{\{0,0,0,...,0\}}_{25\text{-times}}$ is a special multiset zero divisor.

Further $a \times x = \underbrace{\{0,0,0,...,0\}}_{35\text{-times}}$

$y \times b = \underbrace{\{0,0,...,0\}}_{25\text{-times}}$ is again a special multiset zero divisor.

Thus a, b $\in M(Z_{12}) \setminus \{x, y\}$ so {x, y} is a S-special multiset zero divisor of $M(Z_{12})$.

We can find infinitely many such S-special multiset in zero divisors.

Consider 6, 8, 3, 2 $\in Z_{12}$ we see $6 \times 8 \equiv 0 \pmod{12}$

For the pair 3, 2 $\in Z_{12} \setminus \{6, 8\}$ we get $3 \times 8 \equiv 0 \pmod{12}$ and $6 \times 2 \equiv 0 \pmod{12}$ but $3 \in 2 \neq 0 \pmod{12}$.

Thus (6, 8) is a S-weak zero divisor of Z_{12}.

Take x = {6, 6, 6, 6, 0, 0} and y = {8, 8, 0, 0, 0, 0, 0, 0} $\in M(Z_{12})$. We get x × y = $\underbrace{\{0,0,0,...,0\}}_{48\text{-times}}$ is a multiset special zero divisor of $M(Z_{12})$.

Let a = {3, 3, 3, 0} and b = {2, 2, 2, 2, 0, 0} $\in M(Z_{12}) \setminus$ {x, y}. Using this a, b $\in M(Z_{12})$ we get x × b = $\underbrace{\{0,0,0,...,0\}}_{36\text{-times}}$ is a special multiset zero divisor of $M(Z_{12})$.

y × a = {8, 8, 0, 0, 0, 0, 0, 0} × {3, 3, 3, 0}

= $\underbrace{\{0,0,0,...,0\}}_{32\text{-times}}$ is also a special multiset zero divisor of $M(Z_{12})$.

But a × b = {3, 3, 3, 0} × {2, 2, 2, 2, 0, 0} = {6, 6, 6, 0, 6, 6, 6, 0, 6, 6, 6, 0, 6, 6, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0} ≠ {0, 0, …, 0} so x, y $\in$ M (Z_{12}) is a S-special multiset weak zero divisor of $M(Z_{12})$.

Infact $M(Z_{12})$ has infinite number of S-multiset special zero divisors and S-multiset special weak zero divisors.

Example 2.15. Let S = {Z_{16}, ×} be the semigroup under product modulo 16.

We see $8 \times 2 \equiv 0$,

$8 \times 6 \equiv 0$, $8 \times 8 \equiv 0$,

$8 \times 10 \equiv 0$ $8 \times 12 = 0$, $8 \cup 4 = 0$ and

$8 \times 14 \equiv 0$ are zero divisors contributed by the value 8 in Z_{16}.

Clearly S has S-zero divisors but has no S-weak zero divisors.

However M(Z_{16}) has special multiset zero divisors which are infinitely many. M($_{16}$) has special multiset S-zero divisors and special multiset S-weak zero divisors.

Next we proceed onto define and describe the new notion of special multiset pseudo idempotents.

Definition 2.3. *Let (S,×) be a semigroup under product. {M(S), ×} be the multiset semigroup under product. We say x ∈ M(S) is a pseudo multiset idempotent of M(S), if x × x = {only elements in x} but the number of times these elements repeat is different from that of x.*

We will first illustrate this situation by some examples.

Example 2.16. Let S = {Z_6, ×} be the semigroup under product B = {M(Z_6), ×} be the multiset semigroup.

Consider x_1 = {0, 1, 1, 0, 0, 0, 1, 1, 1} ∈ H. We see

$x_1 \times x_1$ = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1} $\neq x_1$; yet elements of $x_1 \times x_1$ are the same as that of x_1 and $x_1 \times x_1$ has no new elements only number of times they repeat are more or to be more precise $|x_1 \times x_1| = 81$ whereas that of x_1 is 9.

We call x_1 as the pseudo multiset idempotent of B. In fact we have infinitely many such pseudo multiset idempotents. These are also known as trivial pseudo multiset idempotents as $\{0\} \times \{0\}$ and $\{1\} \times \{1\} = \{1\}$ are trivial idempotents in the reals and modulo integers $3 \in Z_6$ is such that $3 \times 3 = 3$ is a nontrivial idempotents of Z_6, 0 and 1 are trivial idempotents of Z_6.

Consider the multiset $x = \{3, 3, 1, 3\} \in B$. $x \times x = \{3, 3, 3, 1\} \times \{3, 3, 3, 1\} = \{3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1\}$.

Thus x is a nontrivial pseudo multiset idempotent of M.

We have infinitely many nontrivial pseudo multiset idempotents in M though Z_6 on which the multiset is built is only a finite set.

Let $a = \{3, 3, 3\} \in M(Z_6)$ $a \times a = \{3, 3, 3, 3, 3, 3, 3, 3, 3\}$ is a nontrivial pseudo multiset idempotent of M.

In view of all these we have the following results.

Theorem 2.6. *Let R or C or Q of Z be the reals or complex or rationals or integers respectively under product ×. {M(R), ×} (or M(C) or M(Q) or M(Z)) be the multisets M(R) (or M(C) or M(Q) or M(Z)) has infinitely many trivial multiset pseudo* *idempotents although R or C or Q or Z are only fields or domains.*

Proof is direct and left as an exercise to the reader.

Theorem 2.7. *Let Z_n be the modulo integer and $\{Z_n, \times\}$ be the semigroup of finite order $\{M(Z_n), \times\}$ be the multiset semigroup.*

i) *$\{M(Z_n), \times\}$ has infinitely many pseudo multiset idempotents for all n including n = p a prime.*

ii) *When n is not a prime and $n = p_1^{\alpha_1}, \ldots, p_t^{\alpha_t}$; p_i's are distinct primes $1 \leq i \leq t$ $(t \geq 2)$ and $\alpha_i \leq 1$; $1 \leq i \leq t$; then $\{M(Z_n), \times\}$ has infinitely many pseudo multiset idempotents.*

Proof is direct and left for the reader to prove.

Example 2.17. Let $\{Z_{180}, \times\}$ be the semigroup under ×. $M(Z_{180})$ be the multiset semigroup. This has nontrivial idempotents hence $\{M(Z_{180}), \times\}$ has pseudo multiset special idempotents which are infinitely many in $M(Z_{180})$.

However $\{M(Z_{180}), \times\}$ has also infinitely many pseudo multiset idempotents. There are S-pseudo multiset special idempotents if and only if Z_{180} has S-idempotents.

Example 2.18. Consider $S = \{Z_{10}, \times\}$ be the semigroup under product modulo 10. $\{M(Z_{10}), \times\}$ be the multiset semigroup.

Clearly Z_{10} has 6 to be a S-idempotent as $6 \times 6 = 6$ and $4 \times 4 = 4$ with $6 \times 4 \equiv 4$ (modulo 10).

Consider the multisets x = {6, 6, 6, 6, 6, 6, 6, 6} and y = {4, 4, 4} $\in$ $M(Z_{10})$.

We see x × x = $\underbrace{\{6,6,6,...,6\}}_{64\text{-times}}$ y × y = {4, 4, 4, 4, 4, 4, 4, 4, 4} both x and y pseudo special multiset idempotents.

$$x \times y = \underbrace{\{4,4,4,4,4,4...,4\}}_{24\text{-times}}.$$

Thus x is a Smarandache pseudo special multiset idempotents of $M(Z_{10})$. We give yet another example of S-pseudo special multiset idempotents. We wish to keep in record there exists infinitely many S-special pseudo multiset idempotents in $M(Z_{10})$.

Consider the following example.

Example 2.19. Let S = {Z_{16}, ×} be the semigroup under product. {$M(Z_{16})$, ×} = B be the multiset semigroup. Clearly Z_{16} has no nontrivial idempotents so B cannot contain any special pseudo multiset idempotent or S-special pseudo multiset idempotent.

In view of all these we have the following results.

Theorem 2.8. *Let S = {Z_{2p}, ×} be the semigroup under product modulo 2p; p a prime. B = {$M(Z_{2p}$, ×} be the multiset semigroup.*

B has infinitely many special pseudo multiset idempotents and S-special pseudo multiset idempotents.

Proof. Following from the fact as Z_{2p} has idempotents and S-idempotents so will $M(Z_{2p})$.

But $M(Z_{2p})$ has infinitely many special pseudo idempotents and S-special pseudo idempotents.

In fact the theorem can be reformulated as the multiset $\{M(Z_{2p}), \times\}$ has infinitely many pseudo special idempotents if and only if Z_{2p} has idempotents.

We have another result which is as follows.

***Theorem 2.9**. Let $S = \{Z_{p^n}, \times\}$, p a prime, $n \geq 2$ be the semigroup under product modulo p^n. $B = \{M(Z_{p^n}), \times\}$ be the multiset semigroup under product modulo p^n. B has only infinitely many trivial pseudo special multiset idempotents.*

Proof is directed and hence left as an exercise to the reader.

We now give a few more examples of special pseudo multiset idempotents using Z_{3p}, Z_{18} and so on.

Example 2.20. Let $S = \{Z_{30}, \times\}$ be the semi group under product modulo 30.

6, 10, 15, 16, 21 and 25 are idempotents in S.

Clearly none of these elements are Smarandache idempotents.

Now $\{M(Z_{30}), \times\}$ has infinitely many nontrivial special pseudo multiset idempotents.

Example 2.21. Let S = {Z_{21}, ×} be the semigroup under product modulo 21. The idempotents of Z_{21} are 7 and 15 clearly we cannot find S-idempotents in S. Thus the multiset semgroup B = {M(Z_{21}), ×} cannot contain any S-pseudo multiset special idempotents.

In view of all these we have the following theorem.

Theorem 2.10. *Let S = {Z_{pq}, ×} (p and q two distinct primes) be the semigroup under product modulo pq.*

B = {(M(Z_{pq}), ×} be the multiset semigroup under modulo product pq.

i) *B = {M(Z_{pq}), ×} has infinitely many special multiset pseudo idempotents*

ii) *B = {M(Z_{pq}), ×} has no S-pseudo special multiset idempotents.*

Hint: Follows from the fact Z_{pq} has no S-idempotents which are nontrivial hence {M(Z_{pq}), ×} cannot contain S-special pseudo multiset idempotents.

In view o all these we leave it as an exercise to the reader to prove the following result.

Problem 2.1. Let S = { $Z_{p_1 p_2 \cdots p_t}$ | p_i are distinct primes $1 \le i \le t$, ×} be the semigroup under product.

B = {M($Z_{p_1 p_2}$, … p_t), ×} be the multiset semigroup.

Can B have S-multiset pseudo special idempotents?

In view of all these we study the following modulo semigroup.

Example 2.22. Let S = {Z_{18}, ×} be the semigroup modulo 18.

The idempotents of Z_{18} are 9, 10, B = {M(Z_{18}), ×} be the multiset semigroup. B has no S-multiset special pseudo idempotents.

In fact we leave the following problems open.

Problem 2.2. Let B = {M($Z_{p^t q}$) (p and q distinct primes) t ≥ 2, ×} be the semigroup under modulo $p^t q$.

Can B have nontrivial S-special pseudo multiset idempotents?

Problem 2.3. Let B = {M($Z_{p_1^{t_1} p_2^{t_2} \ldots p_r^{t_r}}$), ×} (p_i's distinct primes t_i > 2, 1 ≤ i ≤ r} be the multiset semigroup.

Can B have S-special pseudo multiset idempotents?

Next we proceed onto study nilpotents in multiset semigroups.

Example 2.23. Let S = {Z_{12}, ×} be the semigroup under product modulo 12. {M(Z_{12}), ×} = B be the multiset semigroup.

The nilpotents in S is 6.

Now let x = {6, 6} ∈ B.

x × x = {6, 6} × {6, 6} = {0, 0, 0, 0}.

This is a special case of nilpotents in multisets which we define as pseudo special multiset nilpotents.

In view of this we first define the new notion of special pseudo multiset nilpotents.

Definition 2.4. *Let B = P{M(S), ×} be a multiset collection on a nonempty set S under product (that is {S, ×} is a semigroup). A multiset x ∈ B is defined as a pseudo special multiset nilpotent if* $x \times x = \underbrace{\{0,0,\ldots,0\}}_{\text{t-times}}$; t > 2.

We just provide some more examples of them and give the results associated with this new notion.

Example 2.24. Let S = {Z_{16}, ×} be the semigroup under product modulo 16.

The nilpotents in S are 4, 8 and 12.

For we see 4 × 4 ≡ 0 (mod 16), 8 × 8 ≡ 0 (mod 16) and 12 × 12 ≡ 0 (mod 16).

Let B = {M(Z_{16}), ×} be the multiset semigroup.

Consider x = {4, 4, 4, 4, 0, 0, 0} ∈ B.

We see $x \times x = \underbrace{\{0,0,\ldots,0\}}_{\text{49-times}}$. Let a = {8, 4, 8, 12, 0, 4, 8, 4, 12, 8} ∈ B, we see $a \times a = \underbrace{\{0,0,\ldots,0\}}_{\text{100-times}}$.

Thus a and x are pseudo special multiset nilpotents.

In fact we can have infinitely many pseudo special multiset nilpotents which are also nontrivial.

Consider the following examples for the obtaining nontrivial special pseudo multiset idempotents.

Example 2.25. Let S = {Z_{43}, ×} be the semigroup under product modulo 43. Let B = {M(Z_{43}), ×} be the multiset semigroup Z_{43} has no idempotents, nilpotents and zero divisors. Hence B has no nontrivial multiset special nilpotents, idempotents and zero divisors.

In view of all these w have the following theorem.

Theorem 2.11. *Let S = {Z_p, ×} be the semigroup under product modulo p (p a prime).*

B = {M(Z_p), ×} be the multiset semigroup.

i) B has no non-trivial special pseudo multiset nilpotents.
ii) B has no non-trivial special pseudo multiset zero divisors.

Proof is direct and hence left as an exercise to the reader.

Theorem 2.12. *Let S = {Z_n, ×} be the semigroup under product modulo n. B = {M(Z_n), ×} be the multiset semigroup.*

i) B has nontrivial special pseudo multiset zero divisors if and only if S has nontrivial zero divisors.

ii) *B has nontrivial infinitely many pseudo multiset special nilpotents if and only if S has nontrivial nilpotents.*

iii) *B has nontrivial infinitely many special pseudo multiset idempotents if and only if S has non trivial idempotents.*

Proof is left as an exercise to the reader.

Thus we can now proceed on to describe the multiset subsemigroups of B = {$M(Z_n)$, ×}.

Example 2.26. Let S = {Z_{24}, ×} be the semigroup. B = {$M(Z_{24})$, ×} be the multiset semigroup of Z_{24}. Consider P = {0, 3, 6, 9, 12, 15, 18, 21} ⊆ Z_{24}. {M(P), ×} = C ⊆ B is a multiset subsemigroup of B as P is a subsemigroup of S.

We see C is again of infinite cardinality.

Consider T = {0, 23, 1} ⊆ Z_{24}, {T, ×} is a subsemigroup of S of order 3, however {M(T), ×} is a multiset subsemigroup of infinite order.

Example 2.27. Let S = {Z_{41}, ×} be the semigroup under product modulo 41. B = {$M(Z_{41})$, ×} be the semigroup. C = {1,40} ⊆ Z_{41} is a subsemigroup of order two. But D = {M({40, 1}), ×} ⊆ B is a multiset subsemigroup of infinite order.

Theorem 2.13. *Let S = {Z_n, ×} be the semigroup under product modulo n. D = {$M(Z_n)$, ×} be the multiset semigroup.*

All multiset subsemigroups of D are of infinite order.

Now we wish to study whether D can be Smarandache semigroup to this effect we define a new notion.

Definition 2.5. *Let S = {Z_n, ×} be a semigroup B = {M(Z_n), ×} be the multiset semigroup.*

We define for any subset P ⊆ Z_n where {P, ×} is a group to be associated with R = {M(P), ×} ⊆ B to a pseudo multiset group of B.

If B contains a pseudo multiset group which is nontrivial then we define B to be a Smarandache pseudo multiset semigroup.

We will provide some examples of this situations.

Example 2.28. Let S = {Z_{36}, ×} be the semigroup under product modulo 36.

B = {M(Z_{36}), ×} be the multiset semigroup. Let P = {0, 2, 4, …, 34} ⊆ Z_{36}, D = {M(P), ×} is a multiset subsemigroup of infinite order.

W = {1, 3, 5}; {W, ×} is a group of order two T = {M(W), ×} is defined as the pseudo special multiset group of B.

If x = {35, 35, 35, 35} ∈ T then x × x = $\underbrace{\{1,1,1,...,1\}}_{\text{16-times}}$.

Consider = {1, 35} ∈ T; we see y × y = {1, 35, 35}

Let Y = {{1}, {35}, {1, 1}, {35, 35}, {1, 1, 1}, {35, 35, 35}, {1, 1, 1, 1}, {35, 35, 35, 35}, …, $\underbrace{\{1,1,1,...,1\}}_{\text{n-times}}$, $\underbrace{\{35,35,...,35\}}_{\text{n-times}}$ n can be infinite. We define {Y, ×} is a multiset special pseudo cyclic group.

As B contains a multiset special pseudo cyclic group we call B as a Smarandache pseudo special multiset semigroup.

We have a large class of multisets which are S-pseudo special multiset semigroups.

We have seen examples of them.

In view of all these we have the following theorem.

Theorem 2.14. *Let S = {Z_n, ×} be a semigroup B = {M(Z_n), ×} be the multiset semigroup. B is a Smarandache pseudo special multiset semigroup.*

Proof follows from the fact that {1, (n – 1)} $\subseteq$ Z_n is a cyclic group of order two.

Consider H = {{1}, {n – 1}, {1, 1}, {n – 1, n – 1}, {1, 1, 1}, {n – 1, n – 1, n – 1}, …, $\underbrace{\{1,1,...,1\}}_{\text{m-times}}$ $\underbrace{\{n-1,n-1,...,n-1\}}_{\text{m-times}}$

1 ≤ m ≤ ∞; ×} $\subseteq$ B is a pseudo special multiset group. Hence B is a Smarandache pseudo special multiset semigroup.

We enlist the following problems for the interested reader.

Problems

1. Prove if S = {1, 2, 3, …, 8, 9} be a set of order nine. Thus M(S) the multiset under ∪ is an infinite semilattice.

2. Define (S) = {1, 2, 3, …, 9} ∩ operation and prove {M(S), ∩} is an infinite semilattice.

3. Let S = P{1 + I, –3I, 4I, 7 – 5I, –8 + 4I, 0, 1, 2, 4} be a set of 9. M(S) be the multiset of S.

   i) Prove M(S) under + is not closed.

   ii) Prove M(S) is not closed under ×.

   iii) Prove {M(S), ∪} and {M(S), ∩} are multiset semigroups of infinite order.

   iv) Can {M(S), ∩} have a finite order multiset subsemigroups?

   v) Prove {M(S), ∪} cannot have finite order multiset subsemigroups.

   vi) Obtain any other special feature enjoyed by M(S).

4. Let S = {Z_{10}} be the modulo integers {0, 1, 2, …, 9}

   i) Prove {M(Z_{10}), ∪} is an infinite order multiset semigroup.

   ii) Prove B = {M(Z_{10}), ∩} is an infinite order multiset semigroup which has multiset subsemigroups of finite order.

   iii) Prove T = {M(Z_{10}), +} is an infinite order multiset semigroup under + modulo 10.

iv) Can T contain multiset subsemigroups of finite order?

v) Prove R = {M(Z_{10}), ×} is a multiset semigroup under product modulo 10 of infinite order.

vi) Can R has finite order multiset subsemigroups?

vii) Can R have special multiset zero divisors?

viii) Can R have special multiset S-zero divisors?

ix) Can R have special multiset S-weak zero divisors?

5. Let S = {Z_{29}, ×} be the semigroup under product modulo 29. {M(Z_{29}), ×} = P be the multiset semigroup.

i) Study questions (i) to (ix) of problem 4 for this M(Z_{29}).

ii) Compare this P with R of problem 4.

6. Let P = {Z_{45}, +} be the group under + modulo 45. B = {M(Z_{45}), +} be the multiset semigroup under +.

i) Find all multiset subsemigroups of S.

ii) Prove or disprove all multiset subsemigroups are of infinite order

iii) Let W = {{0}, {1}, …, {44}} be the multiset subsemigroup.

a) What is the order of W?

b) Is W a group?

c) Is B a Smarandache multiset semigroup or just a Smarandache special pseudo subset semigroup?

Justify your claim.

iv) Obtain any other special feature enjoyed by B.

7. Let S = {Z_{29}, +} be the semigroup under + modulo 29. B = {M(Z_{29}), +} be the multiset semigroup.

i) Study questions (i) to (iv) of problem (6) for this B.

ii) Compare the results of problem 6 with this B.

iii) Hence or otherwise study multiset semigroup on Z_p and Z_n; p a prime and n a composite number.

8. Let S = {Z_{148}, ×} be a semigroup under product modulo 148 {M(Z_{148}), ×} = B be the multiset semigroup under product.

i) Prove o(B) is infinite

ii) Can B have multiset subsemigroups of finite order?

iii) Can B have multiset pseudo special zero divisors?

iv) Can B have special pseudo multiset S-special zero divisors?

v) Can B have multiset pseudo special idempotents?

vi) Can B have S-multiset pseudo special idempotents?

vii) Can B have multiset pseudo special nilpotents?

viii) Is B a S-pseudo special multiset semigroup?

ix) Prove/disprove B cannot have finite order multiset subsemigroups.

x) Obtain all the special features enjoyed by multiset semigroup.

9. Let S = $\{Z_{128}, \times\}$ be the semigroup under product modulo 128. B = $\{M(Z_{128}), \times\}$ be the multiset semigroup.

   Study questions (i) to (x) of problem 8 for this B.

10. Let S = $\{Z_{2310}, \times\}$ be the semigroup under product modulo 2310. D = $\{M(Z_{2310}), \times\}$ be the multiset semigroup.

    i) Study questions (i) to (x) of problem 8 for this D.

    ii) Compare B of problems 8 and 9 with this D.

11. Let S = $\{Z_{45}, \times\}$ be the semigroup under product modulo 45 and R = $\{M(Z_{48}), \times\}$ be the multiset semigroup.

i) Study questions (i) to (x) of problem 8 of this R.

ii) Compare this R with D of problem 10.

12. Let S = $\{Z_{10} \times Z_{45}\}$ be the semigroup under ×. R = $\{M(Z_{10} \times Z_{45}), \times\}$ be the multiset semigroup.

i) Study questions (i) to (x) of problem (8) for this D.

ii) Compare D of problem 11 with this R.

iii) Show that due to the presence of pairs in the multisets more number of pseudo multiset special zero divisors.

iv) If instead of $Z_{10} \times Z_{45}$ if are take $Z_{11} \times Z_{43}$ prove we can still have pseudo multiset special zero divisors.

13. Let S = $\{Z_{16} \times Z_{11} \times Z_{48}, \times\}$ be the semigroup under modulo product. T = $\{M(Z_{16} \times Z_{11} \cup Z_{48}), \times\}$ be the multiset semigroup.

i) Study questions (i) to (x) of problem (8) for this T.

ii) Compare this T with R of problem 2.

14. Let S = $\{Z_{12} \times Z_{12} \times Z_{12} \times Z_{12}, \times\}$ be the semigroup. V = $\{M(Z_{12} \times Z_{12} \times Z_{12} \times Z_{1}), \times\}$ be the multiset semigroup.

i) Study questions (i) to (x) of problem 8 for this V.

ii) Compare this V with T of problem 13.

Chapter Three

n-MULTIPLICITY- MULTISETS AND THEIR ALGEBRAIC PROPERTIES

Study of multisets started as early as (1498-1576) in the work of Marius Nizolium. Jean Prestet published a general rule for the multiset permutations in 1675, John Wallis explained this rule in more detail in 1685. A brief literature survey regarding multisets is given in chapter I of this book.

In this chapter we define only a special type of multisets which we define as n-multiplicity multiset where $1 \leq n < \infty$. Just for the sake of easy understanding before we proceed onto give the abstract definition provide some examples of them.

Example 3.1. Let X = {a} a singleton set. The two multiplicity of X is {{a}, {a, a}, ϕ}. Clearly one multicity multiset of X is {{a}, ϕ}. This it is the classical power set of X that is P(X).

The 3-multiplicity multiset of X is {{a}, {a, a}, {a, a, a}, ϕ}.

The 4-multiplicity multiset of X = {a} is {{a}, {a, a}, {a, a, a}, {a, a, a, a}, ϕ}.

Clearly we see the1-multiplicity multiset (P(X)) of X ⊆ 2-multiplicity multiset of X ⊆ 3-multiplicity multiset of X ⊆ … ⊆ n-multiplicity multiset of X = {{a}, {a, a}, {a, a, a}, {a, a, a, a}, … , $\underbrace{\{a, a, ..., a\}}_{(n-1)-\text{times}}, \underbrace{\{a, a, ..., a\}, \phi}_{n-\text{times}}\Big\}$ }.

We shall denote 1-M(X) = 1-multiplicity multiset of X = P (X) is the power set of X. 2-M(X) denotes the two multiplicity multiset of X, 3-M(X) denotes the 3-multiplicity multiset of X and so on.

Now we take in the following example X = {a, b}.

Example 3.2. Let X = {a, b} be the given set. The powerset of 1-multiplicity multiset of {a, b} = {{a, b}, {b}, {a}, ϕ} = P(X).

The 2-multiplicity multiset of {a, b} = {{a, a, b, b}, {a, a, b}, {a, b, b}, {a, a}, {b, b}, {a, b}, {a}, {b}, {ϕ}}.

The three multiplicity multiset of {a, b} = {{a, a, a, b, b, b}, {a, a, a, b, b}, {a, a, a, b}, {a, a, a}, {b, b, b, a, a}, {b, b, b, a}, {b, b, b}, {a, a, b, b}, {a, a, b}, {a, b, b}, {a, b}, {a}, {b}, {a, a}, {b, b}, {ϕ}} = 3-multiset of {a, b}.

The four multiplicity multiset of {a, b} = {{a, a, a, a, b, b, b, b}, {a, a, a, a, b, b, b}, {a, a, a, a, b, b} {a, a, a, a, b}, {a, b, b, b, b}, {a, a, b, b, b, b}, {a, a, a, b, b, b, b}, {a, a, a, a} {b, b, b, b}, {a, a, a, b, b, b}, {a, a, a}, {a, a, a, b, b}, {a, a, a, b}, {b, b, b, a, a}, {b, b, b, a}, {b, b, b}, {b, b, a, a} {b, b, a}, {a, a, b}, {b, b}, {a, a}, {a, b}, {a}, {b}, ϕ} = 4-M({a, b}).

Now we find for the same X {a, b} the five multiplicity multiset. 5-M(X) = {{ϕ}, {a}, {b}, {a, a}, {a, b}, {b, b}, {a, b, b}, {b, a, a}, {a, a, a}, {b, b, b}, {a, b, b, b}, {b, a, a, a}, {a, a, b, b}, {a, a, a, a}, {b, b, b, b}, {b, a, a, a, a}, {a, a, a, b, b, b}, {a, b, b, b, b}, {a, a, b, b, b}, {b, b, a, a, a}, {a, a, a, a, b}, {b, b, b, b, b}, {b, b, b, b, a}, {b, b, b, b, a, a}, {a, a, b, b, b}, {a, a, b, b, b}, {a, a, a, b, b, b, b}, {a, a, b, b, b, b, b} and so on. {a, a, a, a, a}, {a, a, a, a, a, b, b}, {a, a, a, a, a, b} {a, a, a, a, a, b, b, b, b}, {a, a, a, a, a, b, b, b}, {b, b, b, b, b, a, a, a, a, a,}, ϕ}.

Now we work out for 6-multiplicity multiple sets using {a, b} are {{ϕ}, {a}, {b}, {a, b}, {a, a,}, {b, b}, {a, b, a}, {a, b, b}, {a, a, a}, {b, b, b}, {a, a, b, b}, {a, a, a, b}, {b, b, b, a} {a, a, a, b, b}, {b, b, a, a, b}, {a, a, a, b, b, b}, {a, a, a, a}, {b, b, b, b}, {a, a, a, a, b}, {a, a, a, a, b, b}, {a, a, a, a, b, b, b}, {a, a, a, a, b, b, b, b}, {b, b, b, b, a}, {b, b, b, b, a, a}, {b, b, b, b, a, a, a}, {a, a, a, a, a}, {a, a, a, a, a, b}, {a, a, a, a, a, b, b}, {a, a, a, a, a, b, b, b}, {a, a, a, a, a, b, b, b, b}, {a, a, a, a, a, b, b, b, b}, {a, b, b, b, b, b}, {a, a, b, b, b, b, b}, {a, a, a, b, b, b, b, b}, {a, a, a, a, b, b, b, b, b}, {b, b, b, b, b}, {a, a, a, a, a, a}, {a, a, a, a, a, a, b}, {a, a, a, a, a, a, b, b}, {a, a, a, a, a, a, b, b, b}, {a, a, a, a, a, a, b, b, b} {b, b, b, b, b, b}, {a, a, a, a, a, a, b, b, b, b, b} {a, a, a, a, a, b, b, b, b, b, b}, {a, b, b, b, b, b, b}, {a, a, b, b, b, b, b, b}, {a, a, a, b, b, b, b, b, b}, {a, a, a, a, b, b, b, b, b, b}, {a, a, a, a, a, b, b, b, b, b, b} {a, a, a, a, a, a, b, b, b, b, b, b} and so on} |6-multiset} = 49.

In view of all these we have the following result.

Definition 3.1. *Let X = {a_1, …, a_m} with m-distinct elements. A n-multiplicity multiset of X denoted by n-M(X) is the collection* *of all multisets where no element in any of these multisets can have multiplicity greater than n. In other words in every multiset every element can have multiplicity n or repeat itself less than or equal to n number of times only. It is denoted by n-M(X).*

We will first illustrate this situation by some examples.

Example 3.3. Let X = $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a set of order 7.

We find the 5-multiplicity multisets of X. We give few of the multisets of 5- M(X).

A = $\{a_1, a_2, a_3, a_4, a_5, a_6, a_6, a_6, a_6, a_6, a_7, a_3, a_2, a_2, a_2, a_2\}$ is a 5-multiplicity multiset of X.

Clearly a_6 and a_2 have multiplicity 5 and the rest of the elements of A have multiplicity less than five.

B = $\{a_1, a_1, a_1, a_4, a_4, a_4, a_4, a_5, a_5, a_5, a_6, a_6, a_7, a_7, a_7\}$ is a multiset of 5-multiplicity multiset of X. It is pertinent to mention that A and B are 5-multiplicity multisets of X however the multiset A has two elements a_6 and a_2 of multiplicity five whereas none of the elements in B has multiplicity 5.

It is also important to note P(X) the powerset of X is always a subset of n-multiset of n-M(X).

$P(X) \subset n - M(X)$.

Finding cardinality is an interesting work.

We just describe the n-multiset with X = {a} $(1 \leq n < \infty)$.

We have n-M(X) = {ϕ, {a}, {a, a}, {a, a, a}, {a, a, a, a}, …, {a, a, a, …, a}, $\{\underbrace{a,a,...,a}_{n\text{-times}}\}$}.

Clearly n-M(X) is a chain lattice of order (n + 1) given by the following {ϕ} $\subset$ {a} $\subset$ {a, a} $\subset$ [a, a, a} $\subset$ {a, a, a, a}$\subset$ … $\subset$ $\{\underbrace{a,a,...,a}_{n\text{-times}}\}$.

When n = 1 we get the powerset of X, P(X) = {{ϕ}, {a}}.

Infact a chain lattice of order two, a basic Boolean algebra of order two given by

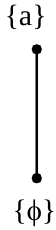

Figure 3.1

When n = 2 we get the 2-multiplicity multiset = {{ϕ}, {a}, {a, a}} where X = {a} which is a chain lattice of order three given the following

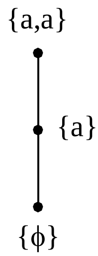

Figure 3.2

In case of r-multiplicity multiset of X = {a} is a chain lattice of order (r + 1) given the following figure

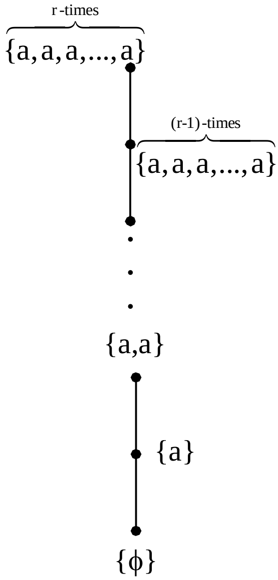

Figure 3.3

Now we find 3-multiplicity multisets of X = {a, b}.

3-M(X) = {{ϕ}, {a}, {b}, {a, b}, {a, a}, {b, b}, {a, a, b}, {b, b, b}, {a, b, b}, {a, a, b, b}, {a, a, a}, {b, b, b, a}, {b, b, b, a, a}, {b, b, b, a, a, a} {a, b, a, a}, {b, b, a, a, a}}}, |3-M(X)| = 16.

Now we give the lattice associated with 3-M(X). Clearly this has {ϕ} as the least element and {a, a, a, b, b, b} is the greatest element which is described by the following figure.

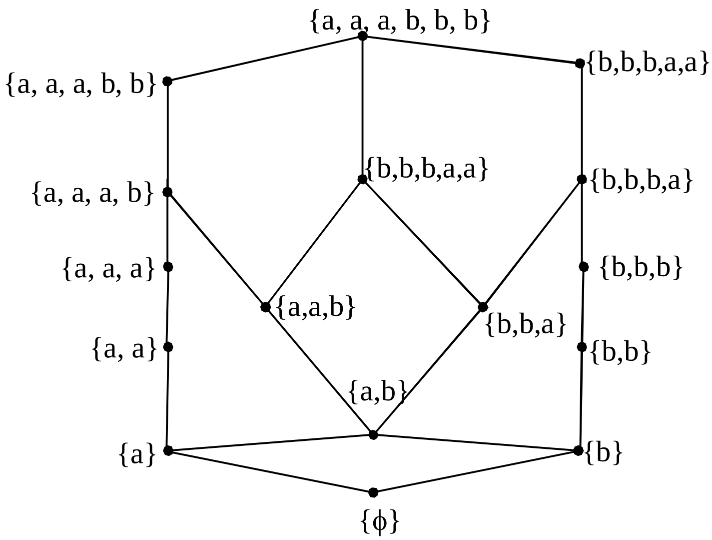

Figure 3.4

The lattice of a power set of a set X is always a Boolean algebra of order $2^{|X|}$ however when |X| = 2. However the lattice of a n-multiplicity multiset is not a Boolean algebra but it always contains a sublattice which is a Boolean algebra of order four.

We proceed onto provide some examples.

Example 3.4. The 2-multiplicity multiset of X = {a, b} is as follows 2-M ({a, b}) = {{ϕ}, {a}, {b}, {a, b}, {a, a}, {a, a, b}, {b, b}, {a, a, b, b}, {a, b, b}}.

The lattice associated with 2-M({a, b}) is as follows;

l({a, a, b} ∪ {a, a, b}) = {a, a, b, b} and not { a, a, a, b, b, b}. this is the way the ∪, union operation is leveled in case of 2-multiplicity multiset and l(∪) is defined as the leveled union on 2- M({a, b}).

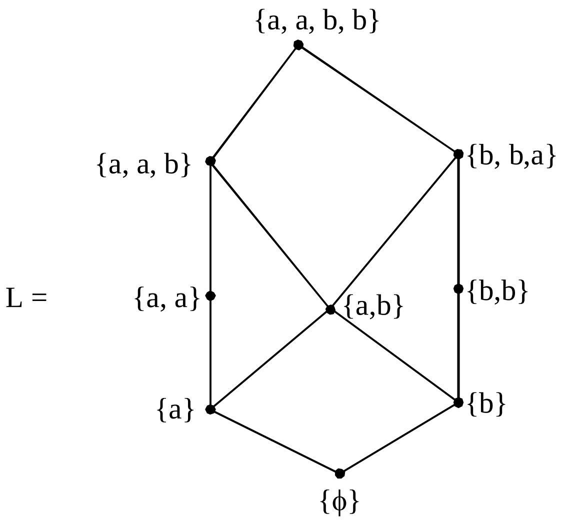

Figure 3.5

Clearly this lattice has {ϕ} to be the least element and {a, a, b, b} to be the largest element or the greatest element. Here U is the leveled union l(U). Further it is to be noted L contains a sublattice H which is a Boolean algebra of order four given in the following.

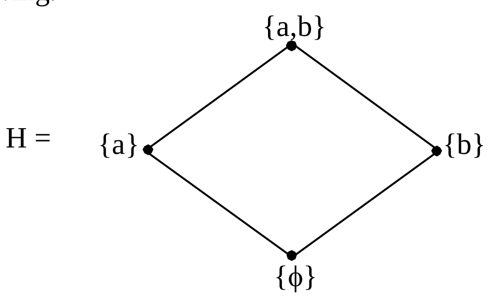

Figure 3.6

Now the maximal chains of L are of length five given by the following figures.

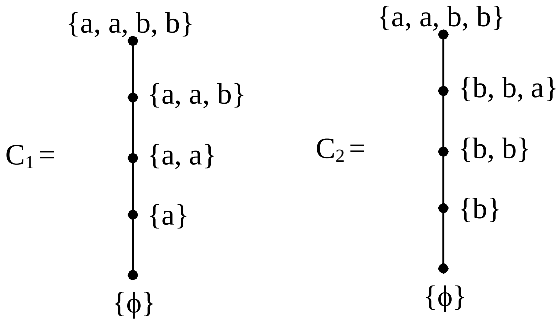

Figure 3.7

The other two maximal chains are given in the following figure 3.8.

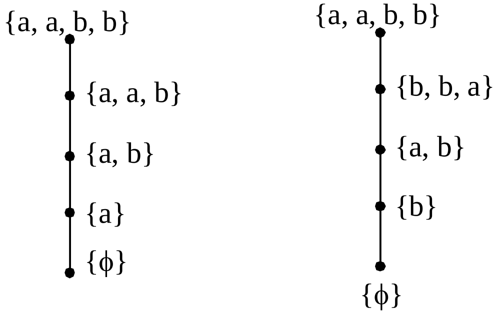

Figure 3.8

The reader is left with the task of finding whether the lattice L is modular or distributive.

Now we describe the 2 multiplicity multiset of X = {a, b, c} in the following.

Let X = {a, b, c}; 3M (X) = {{ϕ}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {a, a, c}, {a, a}, {a, a, b}, {b, b, c}, {b, b, a}, {c, c}, {c, c, a}, {c, c, b}, {a, a, b, b}, {a, a, c, c}, {b, b, c, c}, {b, b}, {a, a, b, c}, {a, b, b, c}, {a, c, c, b}, {a, a, b, b, c}, {a, a, c, c, b}, {b, b, c, c, a}, {a, a, c, c, b, b}}.

Clearly |2-M(X)| = 27 = 3^3. The lattice associated with 2-M(X) is as follows.

K =

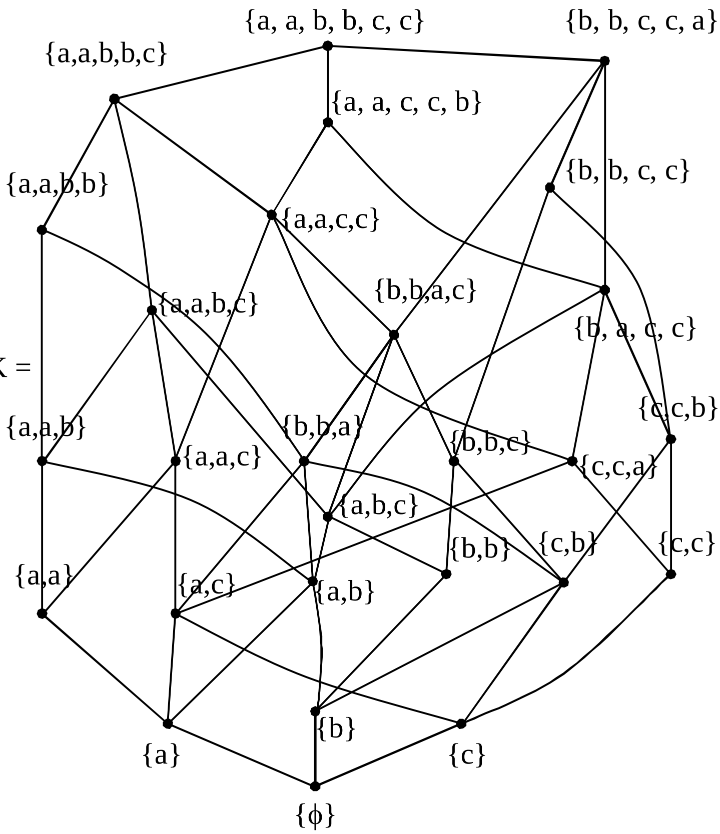

Figure 3.9

The reader is left with the task of finding the lattices of 3-M({a, b, c}) and 4-M({a, b}). Find all the maximal chains of K.

Clearly this K has a sublattice B which is a Boolean algebra of order 8 given by the following figure.

B =

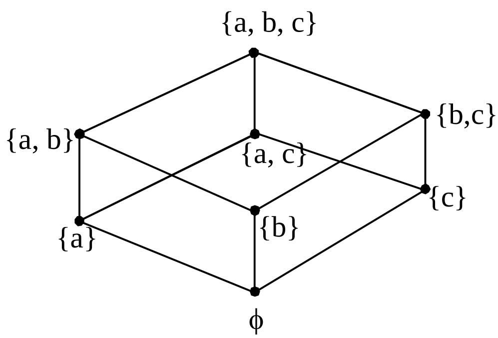

Figure 3.10

We can give one more example before we leave some open problems as exercise.

Example 3.5. Let X = {a, b} be a set of order two. 4-M(X) be the 4-multiplicity multiset of X given in the following. 4-M(X) = {{ϕ}, {a}, {b}, {a, b}, {a, a}, {b, b}, {a, a, a}, {b, b, b}, {a, a, a}, {b, b, a}, {a, a, a, a}, {a, a, b, b}, {b, b, b, b}, {a, a, a, b}, {b, b, b, a}, {a, a, a, a, b} {a, a, a, a, b, b}, {a, a, a, a, b, b, b} {a, a, a, a, b, b, b, b}, {b, b, b, b, a}, {b, b, b, b, a, a}, {b, b, b, b, a, a, a}, {a, a, a, b, b}, {a, a, a, b, b, b}, {a, a, b, b, b}}.

The reader is left with the task of finding the lattice associated with 4-M(X).

Recall [39] , a lattice L is a Smarandache lattice if it has a sublattice which is a Boolean algebra.

In view of this we have the following theorem.

***Theorem 3.1**. Let $X = \{a_1, \ldots, a_m\}$ n-M(X) be the n-multiplicity multiset of X. L be the lattice associated with the n-M(X) L is a Smarandache lattice and has a Boolean algebra of order 2^2 as a sublattice.*

Proof. Given $X = \{a_1, a_2, \ldots, a_m\}$ a set of order m. n-M(X) be the n-multiplicity multiset of X. We know by definition every multiset A has the order of multiplicity of any element in A to be less than or equal to n. Clearly the power set of X is a subcollection of n-M(X); that is $P(X) \subset$ n-M(X).

If L is a lattice of n-M(X) then the set P(X) in n-M(X) will contribute to a sublattice of L of order 2^2. The sublattice (lattice) associated with any power set of a set X is not a Boolean algebra of order $2^{|X|}$. Thus there is Boolean algebra of order four which is a sublattice of L so L is a Smarandache lattice. Hence the claim.

Thus n-multiplicity multisets contributes to a nontrivial class of Smarandache lattices.

It is left as an open conjecture to find the order of n-M(X) where $X = \{a_1, a_2, \ldots, a_m\}$.

Clearly when n = 1 we get n-M(X) = P(X) and |n-M(X)| = 2^m.

When n = 2 what is the order of n-M(X) $X = \{a_1, a_2, \ldots, a_m\}$ when n = 3 what is the order of n-M(X); $X = \{a_1, \ldots, a_m\}$. Thus when n = r what is the order of n-M(X), $X = \{a_1, \ldots, a_m\}$. It is left as an open problem for the reader to write an algorithm find the cardinality of n-M(X), when $X = \{a_1, \ldots, a_m\}$.

Further for any lattice associated with n-M(X) as itselements how many maximal chains exist between {ϕ} the least element and $\{\underbrace{a_1, a_1, \ldots, a_1}_{n\text{-times}}, \underbrace{a_2, a_2, \ldots, a_2}_{n\text{-times}}, \ldots, \underbrace{a_{m-1}, a_{m-1}, \ldots, a_{m-1}}_{n\text{-times}}\}$, the greatest element of the lattice.

Find whether the lattice L associated with n-M(X) is modular or distributive?

Thus we have used the already defined on n-M(X) the multiset with maximum multiplicity of any element to be n the notion of l(∪)and ∩ to obtain lattice structure using n-M(X) as n-M(X) is a partially ordered n-multiplicity mutiset. Further as the multiplicity of any element in the multiset is finite we see the o(n-M(X)) $< \infty$ as |X| $< \infty$.

These were examples of finite n-multiplicity multisets.

Now if we replace X by R or Z or Q even if we restrict the multiplicity to be finite we only get infinite order n-multiplicity multisets.

We give examples of them.

Example 3.6. Let Z be the set of integers positive and negative.

We define 2-multiplicity multiset of Z; i.e. 2-M(Z). Clearly |2-M(Z)| = ∞. Any X ∈ 2-M(Z) will be of the form (1, 1, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9, 16, 18, 18, 27, 27, 36).

Clearly B = {5, 5, 5, 6, 6, 6, 6, 6, 142, 142, 169, 999, 2018} is not a 2-multiplicity multiset as 5 in B is of multiplicity 3 and 6 in B is of multiplicity 5.

Now if A = {3, 3, 4, 4, 5, 9, 18, 27, 36, 48, 45, 45} and B = {3, 3, 4, 5, 9, 45} $\in$ 2-M(Z) then B $\subseteq$ A. It is easily verified 2-M(Z) is a partially ordered set under the usual containment ordering. Further on 2-M(Z) we can define l($\cup$) and $\cap$ in the usual way. Thus {2-M(Z), l($\cup$), $\cap$, $\subseteq$} is a lattice L, where by using the leveled union, l(U) an element can repeat itself here atmost two times.

This lattice is of infinite order. Infact L is Smarandache lattice as P(Z) the power set of Z is a proper subset n-M(Z).

Can we say the lattice L has infinite number of maximal chains?

Now on n-M(Z) we proceed onto define the two classical binary operations + and product ×.

First of all we know (Z, +) is an abelian group so closure operation n-M(Z) exists. Further {Z, ×} is only semi group of infinite order undr ×.

Consider A = {1, 1, 5, 5, 0, 2, –4, 11} and B = {2, 2, 0, –5} $\in$ 2-M(Z). We now show how A + B is obtained.

A + B = {1, 1, 5, 5, 0, 2, –4, 11} + {2, 2, 0, –5} = {1 + 2, 1 + 2, 5 + 2, 5 + 2, 0 + 2, 2 + 2, –4 + 2, 11 + 2,1 + 0, 1 + 0, 5 + 0, 5 + 0, 0 + 0, 2 + 0, –4 + 0, 11 + 0, 1 – 5, 1 – 5, –5 + 5, 5 – 5, 0 – 5, 2 – 5, –4 – 5, 11 – 5} = {3, 3, 7, 7, 2, 4, 2, 13, 3, 3, 7, 7, 2, 4, 2, 13, 1, 1, 5, 0, 2, –4, 11, –4, –4, 0, 0, –5, –3, –9, 6} $\notin$ 2-

M(Z) as there are some elements which has multiplicity greater than 2. Thus we level A + B to a 2-M(Z) element by replacing all higher than two multiplicity elements by multiplicity two. Thus modified or leveled l(A + B) = {3, 3, 7, 7, 2, 2, 4, 13, 4, 13, 1, 1, 5, 5, 0, 0, –4, –4, 11, –5, –3, –9, 6}

Thus in A + B multiplicity of 3, 7, 2 were four and 5 so it was replaced.

Clearly l(A+B) ∈ 2-M(Z). This is the way leveling + operations are performed on 2-M(Z). We call them as leveling as A + B did not belong to 2-M(Z) only l(A+B) ∈ 2-M(Z).

Is {2-M(Z), l(+)}a semigroup? It is permanent to record by l(+) we mean that usual + operation on 2-M(Z) followed by leveling is done.

We want to test whether l(+) on 2-M(Z) is associative.

Clearly {2-M(Z), l(+)} is a commutative closed structure.

Let A = {3, –2, 2, 2, 1, 1, 6, 6},

B = {0, 2, 4, 1, 0} and C = {1, 1, 2} ∈ 2-M(Z).

We find

l(A+B) = l({3, –2, 2, 2, 1, 1, 6, 6, 5, 0, 4, 4, 4, 3, 3, 8, 8, 7, 2, 6, 6, 5, 5, 10, 10, 4, -1, 3, 3, 2, 2, 7, 7, 3, –2, 2, 2, 1, 1, 6, 6})

= {3, –2, 2, 2, 1, 1, 6, 6, 5, 0, 4, 4, 3, 8, 8, 7, 7, 5, 10, 10,

–1, 3, –2} = D. l(D+C) = l (l(A + B) + C) = l (D + C)

= {4, –1, 3, 3, 2, 2, 7, 7, 6, 1, 5, 5, 4, 9, 9, 8, 8, 6, 11, 11, 0, 4, –1, 4, –1, 3, 3, 2, 2, 7, 7, 6, 1, 5, 5, 4, 9, 9, 8, 8, 6, 11, 11, 0, 4, –1, 5, 0, 4, 4, 3, 3, 8, 8, 7, 2, 6, 6, 5, 10, 10, 9, 9, 7, 12, 12, 1, 4, 0} = {4, 4, –1, –1, 3, 3, 2, 2, 7, 7, 6, 6, 1, 1, 5, 5, 9, 9, 8, 8, 11, 11, 0, 0, 12, 12, 10, 10} - I

Now we find l(B + C) = l({1, 3, 5, 2, 1, 0, 3, 5, 2, 1, 2, 4, 6, 3, 2}) = {1, 1, 3, 3, 5, 5, 2, 2, 0, 4, 6} = E

We find l(A + l(B + C)) = l (A + E) = l({4, –1, 3, 3, 2, 2, 7, 7, 4, –1, 3, 3, 2, 2, 7, 7, 6, 1, 5, 5, 4, 4, 9, 9, 6, 1, 5, 5, 4, 4, 9, 9, 8, 3, 7, 7, 6, 6, 11, 11, 5, 0, 4, 4, 3, 3, 8, 8, 8, 3, 7, 7, 6, 6, 1, 1, 1, 1, 5, 0, 4, 4, 3, 3, 8, 8, 3, –2, 2, 2, 1, 1, 6, 6, 7, 2, 6, 6, 7, 7, 10, 10, 9, 4, 8, 8, 3, 3, 12, 12}) = {–1, –1, 4, 4, 3, 3, 2, 2, 7, 7, 6, 6, 1, 1, 5, 5, 9, 9, 8, 8, 11, 11, 0, 0, –2, 10, 10, 12, 12} - II

Consider A = {1, 1, 0, 2}, B = {–1, 0, 2} and C = {3, 1} ∈ 2-M(Z).

We find l(l(A + B) + C) and l(A + l(B + C))

l (l(A + B) + C) = ({0, 0, –1, 1, 1, 1, 0, 2, 3, 3, 2, 4) + {3, 1}) = l({3, 3, 2, 4, 4, 4, 3, 5, 6, 6, 5, 7, 1, 1, 0, 2, 2, 2, 1, 3, 4, 4, 3, 5}) = {3, 3, 2, 2, 4, 4, 5, 5, 6, 6, 7, 1, 1, 0} I

We find l(A + l(B + C)) = l({1, 1, 0, 2} + {2, 3, 5, 0, 1, 3}) = l({3, 3, 2, 4, 4, 4, 3, 5, 6, 6, 5, 7, 1, 1, 0, 2, 2, 2, 1, 3, 4, 4, 3, 5}) = {3, 3, 2, 2, 4, 4, 5, 5, 6, 6, 7, 1, 1} II

Clearly I and II are equal so associative.

The reader is left with the task of proving (n-M(X), l(+)) is a commutative semigroup of infinite order.

Further it is important to mention only singleton sets under l(+) have inverse.

Next we proceed onto describe product on 2-M(×). We will first describe this situation by an example.

Let A = {1, 1, 0, 5, 5, 2} and B = {–2, –2, 0, 9, 9, –1, 3} ∈ 2-M(Z). We find A × B and see whether A × B ∈ 2-M(Z).

A × B = {1, 1, 0, 5, 5, 2} × {–2, –2, 0, 9, 9, –1, 3} = {–2, –2, 0, –10, –10, –4, –2, –2, 0, –10, –10, –4, 0, 0, 0, 0, 0, 9, 9, 0, 45, 45, 18, 9, 9, 0, 45, 45, 18, –1, –1, 0, –5, –5, –2, 3, 3, 0, 15, 15, 6} ∉ 2M(Z).

Thus we have to level or modify the set as 2-multiplicity multiset.

Hence l(A×B) = {0, 0, –2, –2, –10, –10, –4, –4, 9, 9, 45, 45, 18, 18, –1, –1, –5, –5, 3, 3, 15, 15, 6, 3, 3}∈ 2-M(Z). It is easily verified 2-M(Z) under product is not even a closed binary operation, however 2-M(Z) under leveled (or modified) product is a closed operation.

It is left as an exercise for the reader to verify l(×) on 2-M(Z) is both commutative and associative.

Thus we can prove the following theorem.

Theorem 3.2. *Let n-M(Z) (or R or C or Q) be the n-multiplicity multiset of Z (or R or C or Q).*

i) n-M(Z) is of infinite cardinality

ii) {n-M(Z), l(+)} is an infinite commutative semigroup which is not a monoid.

iii) {n-M(Z), l(×) is an infinite commutative semigroup which is a monoid; {1} acts as the multiplicative identity.

Proof is direct and hence left as an exercise to the reader.

Does l(×) distributive over l(+)?

That is will

l[A × l)(B + C)] = l[l(A × B) + l(A × C)]?

To this effect we first see some examples as it is difficult to prove the result in general unless otherwise one ventures to write a program on (algorithm) for the same.

Will be NP-hard or NP-complete remains as a open problem?

Let A = {1, 1, 0, 2} B = {–2, 1, 5} and C = {2, 2, –5, 5} ∈ 2-M(Z).

We find l[A × l(B + C)] and l[l (A × B) + l(A × C)].

l(A × l(B + C)) = l ({1, 1, 0, 2} × ({–2, 1, 5} + {2, 2, –5, 5})

= l ({1, 1, 0, 2} × l({0, 3, 7, 0, 3, 7, –7, –4, 0, 3, 6, 10}))

= l({1, 1, 0, 2} × {0, 0, 3, 3, 7, 7, 6, –7, –4, 10})

= l({0, 0, 3, 3, 7, 7, 6, –7, –4, 10, 0, 0, 3, 3, 7, 7, 6, –7, –4, 10, 0, 0, 0, 6, 6, 14, 14, 12, –14, –8, 20})

= {0, 0, 3, 3, 7, 7, 6, 6, –4, –4, –7, –7, 10, 10, 14, 14, 12, –14, –8, 20} - I

l(l (A × B) + l(A × C)} = l(l ({1, 1, 0, 2} × {–2, 1, 5}) + l({1, 1, 0, 2} × {2, 2, –5, 5})) = l(l({–2, –2, 0, –4, 1, 1, 0, 2, 5, 5, 0, 10}) + l ({2, 2, 0, 4, 2, 2, 0, 4, –5, –5, 0, –10, 5, 5, 0, 10})

= l({–2, –2, 0, 0, –4, 1, 1, 2, 5, 5, 10} + {2, 2, 0, 4, 4, 0, –5, –5, 5, 5, 10, –10})

= {0, 0, 2, 2, –2, 3, 3, 4, 7, 7, 12, –2, 4, 12, –4, 1, 1, 5, 5, 10, 10, –10, –10, –9, 9, 9, …} -II

Clearly I and II are not equal so

l(A × l(B + C)) ≠ l(l(A × C) + l(A × B)) in general.

Hence {n-M(Z), l(+), l(×)} cannot have a semiring structure.

n-M(Z) under ∪ operation is a is not closed. Just like we have leveled + and × we also level ∪. Then {n-M(Z), l(∪)} is a semigroup.

Similarly n-M(Z) under ∩ operation is a semilattice.

Thus {n-M(Z), l(∪), ∩} is a lattice of infinite order having infinite number of maximal chains.

It is left as an open problem for the reader to prove or disprove {n-M(Z), l(∪), ∩} is a distributive lattice or not.

One can write a program to prove or disprove l(+) and l(×) defined on n-M(Z) does not satisfy the distributive law.

Now if X is any arbitrary set it is not possible to define l(+) or l(×) on n-M(X).

But how to find for finite × which is compatible with l(+) and l(×), The only solution at present is replace X by Z_m or $C(Z_m)$ or $\langle Z_m \cup I\rangle$ or $\langle Z_m \cup g\rangle$ or $\langle Z_m \cup h\rangle$ or $\langle Z_m \cup k\rangle$ or $C(\langle Z_m \cup I\rangle) = \langle C(Z_m) \cup I\rangle$ then for any n we have n-M(Z) (or any of the above mentioned structures) is not only finite but also we can define on then l(+) and l(×).

First we will illustrate this situation by some examples.

Example 3.7. Let Z_3 = {0, 1, 2} be the ring of modulo integers. Consider 2-M(Z_3); that is the 2-multiplicity multiset on Z_3.

2-M(Z_3) = {ϕ, {1}, {0}, {2}, {1, 2}, {1, 0}, {2, 0}, {1, 1}, {0, 0}, {2, 2}, {1, 1, 0}, {1, 1, 2}, {2, 2, 0}, {2, 2, 1}, {1, 1, 2, 2, 0}, {0, 0, 1, 1, 2}, {2, 2, 0, 0, 1}, {0, 0, 1}, {0, 0, 2}, {0, 0, 1, 1, 2, 2} {2, 2, 1, 1}, {2, 2, 0, 0}, {1, 1, 0, 0}, {1, 1, 0, 2}, {0, 0, 1, 2}, {2, 2, 1, 0}}.

We find × and + on the set 2-M(Z_3).

Let x = {1, 1, 2, 2, 0} and y = {1, 2, 0, 1} ∈ 2-M(Z_3)

x + y = {1, 1, 2, 2, 0} + 1, 2, 0, 1} = {2, 2, 0, 0,1, 0, 0, 1, 1, 2, 1, 1, 2, 2, 0, 2, 2, 0, 0, 1} ∉ 2-M(Z_3).

We now level or modify x + y l(x + y) = l({2, 2, 0, 0, 1, 0, 0, 1, 1, 2, 1, 1, 2, 2, 0, 2, 2, 0, 0, 1}) = {2, 2, 1, 1, 0, 0} ∈ 2-M(Z_3)

We find x × y = {2, 2, 0, 1, 1} × {1, 2, 0, 1} = {2, 2, 0, 1, 1, 2, 2, 0, 1, 1, 0, 2, 2, 0, 1, 1} ∉ 2-M(Z_3)

l(x × y) = {2, 2, 0, 0, 1, 1} = l(x + y).

Only in this case we see l(x + y) = l(x × y).

Consider a = {2, 2, 1} and b = {1, 0, 2} ∈ 2-M(Z_3)

a + b = {2, 2, 1} + {0, 1, 2} = {2, 2, 1, 0, 0, 2, 1, 1, 0} ∉ 2-M(Z_3)

l(a+b) = {2, 2, 1, 1, 0, 0}.

Now we see {2-M(Z_3), l(+)} is closed set under l(+).

We test for associativity of l(+) on 2-M(Z_3).

Let a = {2, 2, 1}, b = {1, 2, 0} and c = {1, 1, 2, 2} ∈ 2-M(Z_3)

a + (b + c) = {2, 2, 1} + ({1, 2, 0} + {1, 2, 1, 2}) = l({2, 2, 1} + l{1, 2, 1, 2, 2, 0, 2, 0, 0, 1, 0, 1}) = l({2, 2, 1} + {1, 2, 0, 0, 1, 2}) = l({0, 1, 2, 2, 0, 1, 0, 1, 2, 2, 0, 2, 0, 1, 1, 2, 0}) = {0, 0, 1, 1, 2, 2} - I

l(l (A + B) + C) = l(l {2, 2, 1} + {0, 1, 2}) + {1, 1, 2, 2})

= l(l({2, 2, 1, 0, 0, 2, 1, 1, 0}) + {1, 1, 2, 2})

= l({2, 2, 1, 1, 0, 0} + {1, 1, 2, 2})

= l({0, 0, 2, 2, 1, 1, 0, 0, 2, 2, 1, 1, 1, 1, 0, 0, 2, 2, 1, 1, 0, 0, 2, 2})

= l (0, 0, 1, 1, 2, 2} -II

I and II are equal.

It is easily verified l(+) on 2-M(Z_3) is an associative operation and |2-M(Z_3)| = 25.

It is left as problem for the reader to prove {n-M(Z_m), l(+)} is a semigroup of finite order.

Now we define l(×) on 2-M(Z_3).

Let a = {2, 2, 1, 1} and b = {1, 2, 0, 1} ∈ 2-M)(Z_3).

We find l(a × b) = l({{2, 2, 1, 1} × {1, 2, 0, 1}) = l(2, 2, 1, 1, 1, 1, 2, 2, 0, 2, 2, 1, 1, 0})

= {0, 0, 1, 1, 2, 2} ∈ 2-M(Z_3).

Thus {2-M(Z_3), l(×)} is a commutative semigroup of finite order.

The reader is left with the task of proving l(×) on 2-M(Z_3) is associative.

Now we just enumerate all elements (multisets) of 3-M(Z_3) in the following 3-M(Z_3) = {{ϕ}, {0}, {1}, {2}, {1, 2}, {2, 0}, {1, 0}, {0, 0}, {1, 1}, {2, 2}, {0, 1, 2}, {0, 0, 1}, {1, 1, 0}, {0, 0, 2}, {1, 1, 2}, {2, 2, 0}, {2, 2, 1}, {0, 0, 1, 1}, {0, 0, 2, 2}, {1, 1, 2, 2}, {0, 1, 2, 1}, {0, 1, 2, 2}, {0, 1, 2, 0}, {1, 1, 1}, {2, 2, 2}, {0, 0, 0}, {1, 1, 1, 0}, {0, 0, 0, 2}, {1, 1, 1, 2} {0, 0, 0, 1}, {2, 2, 2, 0}, {2, 2, 2, 1}, {1, 1, 1, 0, 0}, {1, 1, 1, 0, 2}, {1, 1, 1, 2, 2}, {2, 2, 2, 0, 0}, {0, 0, 0, 1, 2}, {0, 0, 0, 1, 1}, {2, 2, 2, 1, 1}, {0, 0, 0, 2, 2}, {2, 2, 2, 0, 1}, {0, 0, 0, 1, 1, 1}, {0, 0, 0, 2, 2, 2}, {0, 0, 0, 2, 1, 1} {0, 0, 0, 1, 2, 2}, {0, 0, 2, 2, 2, 1}, {2, 2, 2, 0, 1, 1} {1, 1, 1, 0, 2, 2}, {1, 1, 1, 2, 0, 0} {1, 1, 1, 2, 2, 2}, {1, 1, 1, 2, 2, 2, 0} {1, 1, 1, 2, 2, 2, 0, 0} {1, 1,1, 2, 2, 2, 0, 0, 0} {2, 2, 2, 0, 0, 0, 1}, {2, 2, 2, 0, 0, 0, 1, 1}, {0, 0, 0, 1, 1, 1, 2} {0, 0, 0, 1, 1, 1, 2, 2}, {0, 0, 0, 22, 11}, {2, 2, 2, 0, 0, 11}, {2,2, 11, 0, 0}, {1, 1, 22, 0} {22, 0, 0, 1}, {1, 1, 0 0, 2}}

$|3\text{-}M(Z_3)| = 64 = 4^3$

$|2\text{-}M(Z_3)| = 27 = 3^3$

We will find what is 4-M(Z_3)?

4-M(Z_3) = {{ϕ}, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {1, 1},
{0, 0}, {2, 2}, {0, 0, 1}, {0, 0, 2}, {1, 1, 0}, {1, 1, 2}, {2, 2, 0},
{2, 2, 1}, {2, 2, 1, 1}, {1, 2, 0}, {2, 2, 0, 0}, {1, 1, 0, 0},
{11, 0, 0, 2}, {0, 0, 2, 2, 1}, {1, 1, 22, 0}, {1, 1, 22, 0, 0}
{2, 2, 1, 0} {1, 1, 0, 2}, {0, 0, 1, 2}, {1, 1, 1}, {0, 0, 0},
{2, 2, 2}, {2, 2, 2, 1}, {2, 2, 2, 0}, {1, 1, 1, 0}, {1, 1, 1, 2},
{1, 1, 1, 0, 0, 22}, {0, 0, 0, 2}, {0, 0, 0, 1}, {1, 1, 1, 2, 2, 0, 0},
{0, 0, 0, 1, 1, 2, 2}, {2, 2, 2, 1, 1, 0, 0},
{0, 0, 0, 0}, {1, 1, 1, 1}, {2, 2, 2, 2},
{0, 0, 0, 0, 1}, {0, 0, 0, 0, 2}, {2, 2, 2, 2, 1},
{2, 2, 2, 2, 0}, {1, 1, 1, 1, 0} {1, 1, 1, 1, 2},
{0, 0, 0, 1, 2}, {0, 0, 0, 1, 1}, {0, 0, 0, 2, 2},
{1, 1, 1, 0, 0}, {1, 1, 1, 2, 0}, {2, 2, 2, 1, 1},
{2, 2, 2, 0, 0}, {2, 2, 2, 1, 0}, {0, 0, 1, 1, 1, 1},
{1, 1, 1, 1, 0, 2} {1, 1, 1, 1, 2, 2}, {2, 2, 2, 2, 1, 1},
{2, 2, 2, 2, 0, 0}, {2, 2, 2, 2, 1, 0}, {0, 0, 0, 0, 1, 1},
{0, 0, 0, 0, 2, 2}, {0, 0, 0, 0, 2, 1}, {1, 1, 1, 0, 0, 0}
{0, 0, 0, 2, 2, 2}, {1, 1, 1, 2, 2, 2}
{1, 1, 1, 2, 2, 0}, {1, 1, 1, 0, 0, 2}, {0, 0, 0, 1, 1, 2}
{0, 0, 0, 2, 2, 1}, {0, 1, 1, 1, 1, 0, 2}
{1, 1, 1, 1, 2, 2, 0}, {0, 0, 0, 0, 1, 1, 2}

{0, 0, 0, 0, 2, 2, 1}, {2, 2, 2, 2, 1, 1, 0}
{2, 2, 2, 2, 0, 0, 1}, {0, 0, 0, 0, 11, 22}
{0, 0, 0, 0, 1, 1, 1, 2}, {0, 0, 0, 0, 2, 2, 2, 1}
{1, 1, 1, 1, 0, 0, 0, 2} {1, 1, 1, 1, 00, 2, 2}
{1, 1, 1, 1, 2, 2, 2, 0}, {0, 0, 0, 0, 1, 1, 1, 1}
{1, 1, 1, 1, 2, 2, 2, 2} {0, 0, 0, 0, 0, 2, 2, 2, 2}
{1, 1, 1, 2, 2, 2, 0, 0}, {1, 1, 1, 0, 0, 0, 2, 2}
{2, 2, 2, 0, 0, 0, 1, 1} {2, 2, 2, 2, 1, 1, 1, 1, 0}
{2, 2, 2, 2, 0, 0, 0, 0, 1} {1, 1, 1, 1, 0, 0, 0, 0, 2}
{1, 1, 1, 1, 2, 2, 2, 2, 0, 0}, {1, 1, 1, 1, 0, 0, 0, 0, 2, 2}
{2, 2, 2, 2, 0, 0, 0, 0, 1, 1}, {2, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0},
{2, 2, 2, 2, 0, 0, 0, 0, 1, 1, 1}, {1, 1, 1, 1, 0, 0, 0, 0, 2, 2, 2}
{1, 1, 1, 1, 2, 2, 2, 2, 0 0 0 0}, {2 2 2, 1, 1, 1, 0, 0, 0, 0},
{1, 1, 1, 0, 0, 0, 2, 2, 2, 2}, {1, 1, 1, 1, 0, 0, 0, 2, 2,2},
{1, 1, 1, 1, 0, 0, 0}, {2, 2, 2, 2, 0 0 0}, {2, 2, 2, 2, 1, 1, 1}
{1, 1, 1, 1, 2, 2, 2}, {0, 0, 0, 0, 2, 2, 2}
{0, 0, 0, 0, 1, 1, 1} {2, 2, 2, 1, 1, 1, 0 0, 0}
and so on}.

We see $|4\text{-}M(Z_3)|= 5^3 = 125$.

We find the cardinality of 2-M(Z_4). 4-M(Z_4) = {{ϕ}, {1}, {2}, {3}, {0}, {1, 1}, {0, 0}, {2, 2}, {3, 3}, {1, 2}, {1, 3}, {3, 2}, {1, 0}, {2, 0}, {3, 0}, {0, 0, 0}, {1, 1, 1}, {2, 2, 2}, {3, 3, 3}, {3, 3, 1}, {3, 3, 0}, {3, 3, 2}, {0, 0, 1}, {0, 0, 2}, {0, 0, 3}, {2, 2, 0}, {2, 2, 3}, {2, 2, 1}, {1, 1, 0}, {1, 1, 2}, {1, 1, 3}, {1, 2, 3}, {1, 2, 0}, {1, 3, 0}, {2, 3, 0}, {1, 1, 1, 1}, {0, 0, 0, 0}, {2, 2, 2, 2}, {3, 3, 3, 3}, {0, 0, 0, 0}, {0, 0, 0, 2}, {0, 0, 0,3}, {1, 1,1, 0}, {1, 1, 1, 2}, {1, 1, 1, 3}, {2, 2, 2, 0},

{2, 2, 2, 1}, {3, 3, 3, 1}, {2, 2, 2, 3}, {3, 3, 3, 0}, {3, 3, 3, 2}, {3, 3, 1, 1}, {3, 3, 2, 2}, {3, 3, 0, 0}, {2, 2, 0, 0}, {2, 2, 1, 1}, {1, 1, 0, 0},
{1, 2, 33}, {1 1, 2 3}, {0, 2, 3 3}, {1, 0, 3 3},
{1, 1, 0, 2}, {1, 1, 0, 3}, {2, 2, 1, 3}, {2, 2, 1, 0}, {2, 2, 3, 0},
{0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 3, 2}, {1, 2, 3, 0}, {1, 1, 1, 1, 0} and so on}.

The reader is left with the task of finding |4-M(Z_4)|.

Now we give some elements of 2-M(Z_4) in the following 2-M(Z_4) = {{ϕ}, {1}, {2}, {0}, {3}, {1, 1}, {2, 2}, {0, 0}, {3, 3}, {1, 0}, {1,3}, {1,2}, {0, 2}, {0, 3}, {3, 2}, {1, 1, 0}, {1, 1, 2}, {1,1, 3}, {2, 2, 0}, {2, 2, 3}, {2, 2, 1}, {3, 3, 1}, {3, 3, 0}, {3, 3, 2}, {0, 0, 1}, {0, 0, 2}, {0, 0, 3}, {1, 2, 0}, {1, 2, 3}, {1, 3, 0}, {3, 2, 0}, {1, 1,2, 2}, {2, 2, 0, 0}, {2, 2, 3 3}, {1, 1, 0 0}, {1, 1, 3 3}, {3, 3, 0, 0}, {2, 2, 1, 3}, {1, 2, 3, 0}, {1, 1, 2, 3}, {1, 1, 3 0}, {1, 1, 2, 0}, {2, 2, 1 0}, {2, 2, 3 0}, {3, 3, 0, 1}, {3, 3, 0, 2}, {3, 3, 1, 2}, {0, 0, 1, 2}, {0, 0, 1, 3}, {0, 0, 3, 2}, {0 0 1 1, 2}, {0 0 1 1, 3}, {1, 1, 2 2, 0}, {1, 1, 3, 3, 0}, {3, 3, 2, 2, 1}, {3 3, 2, 2, 0}, {1, 1, 2, 2, 3} {1, 1, 3, 3, 2}, {1, 2, 3, 0, 0}, {0, 1, 2, 3, 3}, {0, 1, 3, 2, 2}, {3, 2, 0, 1, 1} {1 1, 2 2, 0 0}, {1, 1, 2, 2, 3, 3}, {3, 3, 2, 2, 0, 0}, {3 3, 0, 0,1, 1}, {1 1 2 2 0 0 3}, {1 1 2 2 3 3 0} {3 3 0 0 1 1 2} {3 3 2 2 0 0 1}, {1, 1, 2, 2, 0, 0, 3 3}, {0 0 2 2 1 3}, {0 0 1 1 2 3} {1 1 2 2 3 0}, {2 2 3 3 1 -}, {1 1 3 3 2 0}, {0 0 3 3, 2 1}}.

Find the |2-M(Z_4)|.

It is left as an exercise for the reader to find the order of 2-M(Z_n); $2 \leq n < \infty$.

We can prove {n-M(Z_m), l(+)} is a commutative semigroup of finite order.

Likewise it is easily proved {n-M(Z_m), l(×)} is also a commutative semigroup of finite order.

Thus what we need to prove or disprove.

l(a×l(b+c) = l[l(a × b) + l(a × c)] the equality holds good or not in general.

To this effect consider the 2-multiplicity multiset, 2-M(Z_6). Let x = {5, 5, 4, 4, 2}, y = {3, 3, 1, 1} and z = {5, 5, 3, 3}.

Now l[l({x × y}) + l({x × z})}

= l[l({5, 5, 4, 4,2} × {3, 3, 1, 1})

+ l({5, 5, 4, 4, 2} × {5, 5, 3, 3})]

= l[l({3, 3, 0, 0, 0, 3, 3, 0, 0, 0, 5, 5, 4, 4, 2, 5, 5, 4, 4, 2}

+l{1, 1, 2, 2, 4, 4, 1, 1, 2, 2, 3, 3, 0, 0, 0, 3, 3, 0, 0, 0})]

=l({0, 0, 3, 3, 2, 2, 5, 5, 4, 4} + {1, 1, 0, 0,2, 2, 3, 3, 4, 4})

= l({1, 1, 4, 4, 3, 3, 0, 0, 5, 5, 1, 1, 4, 4, 3, 3, 0, 0, 2, 2, 5, 5, 4, 4, 1, 1, 0, 0, 2, 2, 5, 5, 4, 4, 1, 1, 0, 0, 3, 3, 0, 0, 5, 5, 2, 2, 1, 1, 3, 3, 0, 0, 5, 5, 2, 2, 1, 1, 4, 4, 1, 1, 0, 0, 3, 3, 2, 2})

= {0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5} -I

Consider l[(a) × l[b + c]] = l[(5, 5, 4, 4, 2)

× l[{3, 3, 1, 1} + {5, 5, 5, 3, 3}])

= l({5, 5, 4, 4, 2} × l[{2, 2, 0, 0, 2, 2, 0, 0, 0, 0, 4, 4}]

$= l[\{0, 0, 0, 0, 2, 2, 4, 4, 2, 2, 2, 4, 4, 2, 4, 4, 2, 2,$
$4, 4, 4, 4, 2, 2, 4\}]$
$= \{0, 0, 2, 2, 4,4\}$ -II

Clearly I and II are not equal hence the equality does not hold on 2-M(Z_6).

In view of this the reader is left with the task of finding or prove that

$l(a \times l[b + c]) \neq l[l[a \times b] \times l[a \times c]]$ in general in n-M(Z_m).

Thus on n-M(Z_m) we cannot define a semiring / semifield structure as the distributive law is not true.

Thus n-multiplicity multiset do not enjoy the rich algebraic structure at the maximum it can only be a semigroup under l(+) and l(×) that too only under leveled addition and multiplication as without levelling n-M(Z_n) is not even closed under the binary operations + and ×. Thus we can only conclude that n-M(Z_m) enjoys at most generalized algebraic structure viz a semigroup under l(×) and l(+). However it cannot be an algebraic structure under both the binary operations l(×) and l(+) as the distributive law in general is not true. Further {n-M(Z_n), l(∪), ∩} is a lattice infact a Smarandache lattice.

The task of proving or disproving that the lattice associated with n-(M(Z_m)) in general is not a distributive lattice is left as an exercise to the reader.

Thus there are lot of limitations when we try to define algebraic operations on them.

Now we consider the n-multiplicity multiset of $C(Z_n)$ and study the special features associated with them. Even {n-M(Z_m), l(×)} has both idempotents and nilpotents depending on the integer m of Z_m. Further n-M(Z_m) also can have zero divisors if Z_m has zero divisors.

Before we analyze n-M(C(Z_m)) we give some examples of multiset zero divisors, multiset idempoents and multiset units.

Definition 3.2 *Let n-M(Z_m) be a n-multiplicity multiset of Z_m. {n-M(Z_m), l(×)} be the n-multiset semigroup.*

Let x, y ∈ n-M(Z_m) we say xl(×)y is a zero divisor if

$$xl(\times)y = \underbrace{\{0,0,...,0\}}_{n\text{-zeros}}$$

If $xl(\times)y = \underbrace{\{0,0,...,0\}}_{r\text{-times}}$ *and if 0 < r < n then we say x and y are multiset partial zero divisors.*

To this effect we first provide some examples.

Example 3.8: Let 5-M(Z_6) be a 5-multiplicity multiset {5-M(Z_6), l(×)} be the 5-multiset semigroup.

Let x = {0, 0, 3, 3, 3}

and y = {4, 4, 4, 4, 2, 2, 2, 2, 2} ∈ 5-M(Z_6).

We find xl(×)y = l(x × y).

l(x×y) = l({0, 0, 3, 3, 3} × {4, 4, 4, 4, 2, 2, 2, 2, 2}) = l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,}) = {0, 0, 0, 0, 0}.

So x and y are multiset zero divisors of (5-M(Z_6), l(×)).

Consider a = {2, 2, 2} and b = {3} ∈ 5M(Z_6) l(a × b)

= al(×)b = l ({2, 2, 2} × {3})

= {0, 0, 0}. Thus a, b is a multiset partial zero divisor.

Now let x = {0, 0, 0, 0, 0, 3, 3, 3, 3, 3} ∈ 5-M(Z_6).

We find (x × x) = l({0, 0, 0, 0, 0, 3, 3, 3, 3, 3} × {0, 0, 0, 0, 0, 3, 3, 3, 3, 3}) = l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …, 0, …, 3, 3, 3, 3, 3, 0, …,0, 3 3 3, 3 3 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3}.

Thus l(x × x) = x.

The following observations are pertinent we see if

x = {3} n-M(Z_6) then l(x × x) = x.

If y = {3, 3} ∈ 5-M(Z_6) then l(y × y) = l({3, 3} × {3, 3}) = l({3, 3, 3, 3}) = {3, 3, 3, 3} so l (y × y) ≠ y so y is not as multiple set idempotent of 5-M(Z_6).

Let x = {3, 3, 3} ∈ 5(M(Z_6) l(x × x) = l({3, 3, 3} × {3, 3, 3})

= l({3, 3, 3, 3, 3, 3, 3, 3, 3}) = {3, 3, 3, 3, 3} ≠ x.

Let b = {3, 3, 4, 4} ∈ 5-M(Z_6)

l(b×b) = l({3, 3, 4, 4} × {3, 3, 4, 4})

= l({3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 4, 4, 0, 0, 4, 4}

= {3, 3, 3, 3,4, 4, 4, 4, 0, 0, 0, 0} ≠ b

Thus we see idempotent elements of Z_6 do not contribute to idempotents of 5-M(Z_6).

Consider x = {3, 3, 3} and y = {4, 4} $\in$ 5-M(Z_6).

l(x × y) = l({3, 3, 3} × {4, 4})

=l({0, 0, 0, 0, 0, 0)}

= {0, 0, 0, 0, 0}.

So x, y is a multiset zero divisor. However 5-M(Z_6) has no multiset nilpotents as Z_6 has no nilpotents.

Some of the properties of multisets depend on Z_6.

We know Z_8 has two nilpotents viz 2 and 4.

Now can 3-M(Z_8) the 3-multiplicity multiset have multiset nilpotents.

Consider x = {0, 2, 2, 2} $\in$ 3-M(Z_8)

l(x × x) = l({0, 2, 2, 2} × {0, 2, 2, 2})

= l({0, 0, 0, 0, 0, 4, 4, 4, 0, 4, 4, 4, 0, 4, 4, 4})

= {0, 0, 0, 4, 4, 4} = x^2.

We now find

l(x^2 × x) = l({0, 0, 0, 4, 4, 4} × {0, 2, 2, 2})

= l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0})

= {0, 0, 0}.

Thus as l(x^3) = {0, 0, 0}, x is a multiset nilpotent of order 3.

Consider x = {0, 4} $\in$ 3-M(Z_8)

l(x × x) = l({0, 4} × {0, 4})

$=l(\{0, 0, 0, 0\})$
$= \{0, 0, 0\}$.

Clearly x = {0, 4} is a nilpotent multiset of order two.

Thus only if Z_m has nilpotents we can expect n-M(Z_m) to have nilpotents. If n-M(Z_m) has nilpotent then Z_m should have nilpotents.

When we have larger n there is some problem which we illustrate in the following.

Example 3.9: Let 9-M(Z_{16}) be the 9 multiplicity multiset.

Let x = {0, 2, 4, 4} $\in$ 9-M(Z_{16}).

Consider $l(x \times x) \quad = l(\{0, 2, 4, 4) \times \{0, 2, 4, 4\})$

$= l(\{0, 0, 0, 0, 0, 4, 8, 8, 0, 8, 0, 0, 0, 8, 0, 0\})$

$= \{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0, 4, 8, 8\} = x^2$

$l(\{x^2 \times x\}) \quad = l(\{0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 8, 8\} \times \{0, 2, 4, 4\})$

$= l(\{0, 0, 0, 0, 0, 0, 0, \ldots, 8, 0, 0, 0, 0, 0, 0, 0)\}$

$= \{8, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

$l(\{x^3 \times x\}) \quad = l(\{0, 0, 0, 0, 0, 0, 0, 0, 0, 8\} \times \{0, 2, 4, 4\})$

$= l(\{0, 0, 0, \ldots, 0\})$

$= \{0, 0, 0, 0, 0, 0, 0, 0, 0\}$.

Thus x is a multiset nilpotent of order four.

Let y = {0, 4} $\in$ 9-M(Z_{16})

$l(\{0, 4\} \times \{0, 4\}) = l(\{0, 0, 0, 0\})$

$= \{0, 0, 0, 0\} \in$ 9-M(Z_{16})

However we do not call {0, 4} to be a multiset nilpotent as

$\{0, 0, 0, 0, 0\} \neq \{0, 0, \ldots, 0\}$

We call these types of multiset nilpotents as partial multiset nilpotents.

We have partial multiset zero divisors as well as partial multiset nilpotents in n-M(Z_m), provided m is not a prime and Z_m has nontrivial nilpotents.

Let us consider 8-M(Z_7). It is easily verified for no

$$x, y \in 8\text{-M}(Z_7) \setminus \begin{matrix}\{\{0, 0, 0, 0, 0, 0, 0, 0\} \\ \{0,0\}, \{0\}, \{0, 0, 0\} \\ \{0,0,0,0\}, \{0,0,0,0,0\}\} \\ \{0,0,0,0,0,0\},\{0,0,0,0,0,0,0\}\}\end{matrix}$$

l(x×y) = {0} or {0, 0} or so on.

Thus 8-M(Z_7) is a semigroup without multiset zero divisors.

Consider {3, 4, 3} ∈ 8-M(Z_7), l({3, 3, 4} × {{3, 3, 4}) =

l({2, 2, 5, 2, 5, 2, 5, 5, 2}) = {2, 2, 2, 2, 2, 5, 5, 5, 5}

l({2, 2, 2, 2, 2, 5, 5, 5, 5} × {3, 3, 4})

= l({6, 6, 6, 6, 6, 1, 1, 1, 1, 6, 6, 6, 6, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 6, 6, 6, 6}

= {6, 6, 6, 6, 6, 6, 6, 6, 1, 1, 1, 1, 1, 1, 1, 1}

l({6,6, 6, 6, 6, 6, 6, 6, 1, 1, 1, 1, 1, 1, 1, 1} × {3, 3, 4})

=l({4, 4, … 4, 3, 3, 3, …, 3, 4, 4, 4, …., 4, 3, 3, …,3})

= {4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3}.

Consider l({3, 3, 4} × {4, 4, 4, …, 4, 3, 3, …,3}) = {5, 5, 5, 5, 5, 5, 5, 5, 2, 2, 2, 2, 2, 2, 2, 2}

l({3, 3, 4} × {5, 5, 5, …, 5, 2, 2, …,2})

= {1, 1, 1, 1, 1, 1, 1, 1, 6, 6, 6, 6, 6, 6, 6, 6}

Thus we see if x = {3, 3, 4} there is an integer t, t > 1 such that $\underbrace{\{x \times, \ldots, \times x\}}_{t-\text{times}}$ = x^t is y and $x^t = x^{t+r} = \ldots x^{t+sr}$ = y.

Similarly the sequence of product has several such distinct y values

x^3 = {6, 6, 6, 6, 6, 6, 6, 6, 1, 1, 1, 1, 1, 1, 1, 1}
x^5 = {6, 6, 6, 6, 6, …, 6, 1, 1, …, 1, 1}
$x^3 = x^5 = x^7 = x^9 = x^{11} = x^{13} = x^{15}$ and so on
x^4 = {4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3}
x^6 = {3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4}
= x^8 = … .

If x = {6, 4} ∈ 8-M(Z_7) then l({x × x}) = l({6, 4} × {6, 4})
= {1, 3, 3, 2}

l(x^2 × x} = l({1, 3, 3, 2} × {6, 4})
= l({6, 4, 5, 4, 5, 5, 4, 1}) = {6, 4, 5, 4, 5, 5, 4, 1} = x^3

l({x^3 × x}) = l({5, 6, 4, 4, 5, 4, 1, 5} × {6, 4})
= {2, 1, 3, 3, 2, 3, 6, 2, 6, 3, 2, 2, 6, 2, 4, 6}
= x^4

l(x^4 × x) = l({1, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 6, 6, 6, 4, 6} × {6, 4}
= ({6, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 1, 1, 1, 1, 3, 4,
5, 5, 5, 5, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 2})
= {1, 1 ,1 , 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5,
3, 3, 3, 3, 3, 2, 4, 4, 4, 4, 6}
= x^5

l(x^5×x) = l({1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5,
3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 6, 1} × {6, 4}
= ({6, 6, 6, 6, 6, 6, 6, 6, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4,
3, 3, 3, 3, 3, 1, 6, 4, 4, 4, 4, 4, 4, 4, 4,
6, 6, 6, 6, 6, 6, 6, 6, 6, 5, 5, 5, 5, 2, 2, 2, 2, 3, 4})
= {6, 6, 6, 6, 6, 6, 6, 6, 2, 2, 2, 2, 2, 2, 2, 2,

4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3,
5, 5, 5, 5, 5, 1} = x^6

l(x^6 × x) = l({1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5,
3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 6,
3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2,
2, 2, 2, 2, 5, 5, 5, 6, 6, 6, 6, 6, 5, 5, 5, 4})
= {1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5, 3, 3, 3, 3,
3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2,
6, 6, 6, 6, 6,6} = x^7

l(x^7 × x) = {6, 6, 6, 6, 6, 6, 6, 6, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4,
4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5,5, 5,
1, 1, 1, 1, 1, 1, 3} = x^8

l(x^8 × x) = {1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5, 3, 3, 3,
3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2,
2, 2, 6, 6, 6, 6, 6, 6, 6, 6} = x^9

We see $x^{10} = x^9 = x^{11} = x^{12} = \ldots$.

Thus after 9^{th} power of x the product yields the fixed value viz x^9 only.

Let x = {4, 4, 4, 4, 2, 2, 2, 2} $\in$ 8 - M(Z_7)

l(x × x) = [({2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1,
1,1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1 1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,4,4
4, 4, 4, 4})
= {2, 2, 2, 2, 4, 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2,
4, 4, 4, 4} = x^2

$l(x \times x^2) = \{2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4\} = x^3 = x^2$

Clearly x^2 is an idempotent however x is not.

Let $y = \{1, 0, 1, 2\} \in 8\text{-}M(Z_7)$

$l(y \times y) = l(\{1, 1, 0, 2\} \times \{1, 1, 0, 2\}) = \{1, 1, 0, 2, 1, 1, 0, 2, 0, 0, 0, 0, 2, 2, 0, 4\} = y^2$

$l(y^2 \times y) = l(\{1, 1, 0, 2, 1, 1, 0, 2, 0, 0, 0, 0, 2, 2, 0, 4, 1, 1, 0, 2, 1, 1, 0, 2, 0, 0, 0, 0, 2, 2, 0, 4, 0, 0, \ldots, 0, 2, 2, 0, 4, 2, 2, 0, 4, 0, 0, 0, 0, 4, 4, 0, 8\})$

$= \{1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 8\}$

$= y^3$

$l(y^3 \times y) = (\{1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 1, 1, \ldots, 1, 0, \ldots 0, 2, \ldots, 2, 4, 4, 4, 4, 4, 4, 0, 2, \ldots, 2, 0, \ldots., 0, 4, 4, \ldots, 4, 8, 8, 8, 8, 8, 8, 8, 8\})$

$= \{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 0, 0, 0, 0, 0, 0, 0, 0\}$

$= y^4$

$l(y^4 \times y) = \{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 8, 8, 8, 8, 8, 8, 8, 8, 4, 4, 4, 4, 4, 4, 4, 4\}$

$= y^4$

Thus $y^4 = y^5 = y^6 = \ldots.$

We make the following observations.

Only the singleton sets {1}, {2}, {3}, {4}, {5}, {6} $\in$ 8-M(Z_4) can contribute to the partial units.

Is it even possible to find x $\in$ 8-M(Z_7) \ {1, 1, 1, 1, 1, 1, 1, 1} such that for some n, x^n = {1, 1, 1, 1, 1, 1, 1, 1} is a open problem.

However the following observations are mandatory.

i) In 8-M(Z_7) there is no non zero element in 8-M(Z_7) which can lead to zero divisors.

ii) Every x $\in$ 8-M(Z_7) is such that after a stage x^m = y = x^{m+1} = …. and so on; m a finite number

iii) 8-M(Z_7) has no nontrivial nilpotents partial or otherwise.

iv) 8-M(Z_7) has idempotents, other than zero sets and unit sets.

Before we put forth some open conjectures we give a few examples of this situation.

Example 3.10. Let S = 3-M(Z_3) be the three multiplicity multiset on Z_3.

Consider x = {2, 2, 1} $\in$ 3-M(Z_3)

l(x × x) = l({2, 2, 1} × {2, 2, 1})

= l({1, 1, 2, 1, 1, 2, 1, 2, 2})

$= \{1, 2, 2, 1, 1, 2\} = x^2$

$l(x^2 \times x) = l(\{1, 1, 1, 2, 2, 2\} \times \{1, 2, 2\})$

$= l(1, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1\})$

$= \{1, 1, 1, 2, 2, 2\} = x^3$

$x^3 = x^2.$

Thus if $y = \{1, 1, 1, 2, 2, 2\} \in 3\text{-}M(Z_3)$ then

$l(y \times y) = l(\{1, 1, 1, 2, 2, 2\} \times \{1, 1, 1, 2, 2, 2\})$

$= l(\{1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2,$
$2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1\})$

$= \{1, 1, 1, 2, 2, 2\} = y$

Thus $y^2 = y$

Consider $z = \{2, 2, 2, 0, 0, 0\} \in 3\text{-}M(Z_3))$.

$l(z \times z) = l(\{2, 2, 2, 0, 0, 0\} \times \{2, 2, 2, 0, 0, 0\})$

$= l(\{1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1,$
$0, 0, \ldots 0\})$

$= (1, 1, 1, 0, 0, 0) = z^2$

$l(z^2 \times z) = l(\{1, 1, 1, 0, 0, 0 \times \{2, 2, 2, 0, 0, 0\})$

$= l(\{2, 2, 2, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0\})$

$= \{2, 2, 2, 0, 0, 0\} = z$

Thus we see $z^3 = z$

Consider $s = \{1, 1, 1, 2, 2, 2\} \in 3\text{-}M(Z_3)$

$l(s \times s) = l(\{1, 1, 1, 2, 2, 2,\} \times \{1, 1, 1, 2, 2, 2\})$

$= l(\{1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2,$
$2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1\})$

$= \{1, 1, 1, 2, 2, 2\} = s$

Thus $s^2 = s$ is a multiset idempotent of $3\text{-}M(Z_3)$

We have $\{2, 2, 2, 1, 1, 1\}$, $\{1, 1, 1, 0, 0, 0\}$, $\{0, 0, 0, 2, 2, 2, 1, 1, 1\}$ to be nontrivial multiset idempotents of $3\text{-}M(Z_3)$.

Infact {0}, {1}, {0, 0} {1, 1}, {0, 0, 0}, {1, 1, 1} are all trivial multiset idempotents of 3-M(Z_3).

Now we work with the 3-M(Z_5) and study some more properties of n-multiplicity multiset of Z_m, m a prime integer.

Consider x = {1, 1, 1, 2, 2, 2, 4, 4, 4} $\in$ 3-M(Z_5).

We find

l(x × x) = l({1, 1, 1, 2, 2, 2, 4, 4, 4} × {1, 1, 1, 2, 2, 2, 4, 4, 4})
= l({1, 1, 1, 2, 2, 2, 4, 4, 4, 1, 1, 1, 2, 2, 2, 4, 4, 4,
1, 1, 1, 2, 2, 2, 4, 4, 4, 2, 2, 2, 4, 4, 4, 3, 3, 3, 2, 2, 2,
4, 4, 4, 3, 3, 3, 2, 2, 2, 4, 4, 4, 3, 3, 3, 4, 4, 4,
3, 3, 3, 1, 1, 1, 4, 4, 4, 3, 3, 3, 1, 1, 1, 4, 4, 4,
3, 3, 3, 1, 1, 1 })
= {1, 1, 1, 3, 3, 3, 4, 4, 4, 2, 2, 2} $\neq$ x.

Consider y = {1, 1, 1, 4, 4, 4} $\in$ 3-M(Z_5)

l(y × y) = l({1, 1, 1, 4, 4, 4} × {1, 1, 1, 4, 4, 4})
= l({1, 1, 1, 4, 4, 4, 1, 1, 1, 4, 4, 4, 1, 1, 1, 4, 4, 4,
4, 4, 4, 1, 1, 1, 4, 4, 4, 1, 1, 1, 4, 4, 4, 1, 1, 1})
={1, 1, 1, 4, 4, 4} = y^2 = y

Thus y^2 = y is a multiset idempotent of 3-M(Z_5).

Let z = {1, 1, 1, 2, 2, 2, 4, 4, 4, 3, 3, 3} $\in$ 3-M(Z_5)

Consider

l({z × z}) = {1, 1, 1, 2, 2, 2, 4, 4, 4, 3, 3, 3} = z^2.

Thus z^2 = z is again a multiset idempotent of 3-M(Z_5).

Clearly Z_5 has no idempotent but Z_5 has nontrivial multiset idempotents. This is marked deviation from Z_5 and 3-M(Z_5). However both Z_5 and 3-M(Z_5) has no zero divisors or nilpotents which are nontrivial.

In view of all these we have the following theorem.

***Theorem 3.3**. Let n-M(Z_m) be the n-multiplicity multiset of Z_m; m, a prime; ($2 \leq n < \infty$)*

i) *n-M(Z_m) has no nontrivial multiset zero divisors and no nontrivial multiset nilpotents.*

ii) *n-M(Z_m) has atleast five nontrivial multiset idempotents.*

iii) *N-M(Z_m) has multisets x such that $x^t = x^{t+1} = x^{t+2}$… and so on for some x.*

Proof (i) is true as Z_m is a field of order m

Clearly x_1 = {1, 1, 1, …, 1, 0, 0, …, 0}, x_2 = {1, 1, 1, …, 1, 0, 0, 0, …, 0 (m – 1), (m – 1), …., (m – 1)}, x_3 = {1, 1, …, 1, (m – 1), …, (m – 1)}, x_4 = {1, 1, …, 1, 2, 2, …, 2, …, (m – 1), …, (m – 1)} and x_5 = {0, 0, …,0, 1, 1, 1, …, 1, 2, 2, 2, …, 2, …, (m – 1), …, (m – 1)} are multiset idempotents where in each x_i ($1 \leq i \leq 5$) each element occurs exactly n-times.

Proof of (iii) is direct and hence left as an exercise to the reader.

Next we find some more properties of n-M(Z_m) m-any number.

Example 3.11. Let S = {4-M(Z_{24})} be the 4-multiplicity multiset of Z_{24}.

Consider x = {6, 6, 6, 6} ∈ S. We find l({x × x}) = l({6, 6, 6, 6} × {6, 6, 6, 6}) = $l(\underbrace{\{12,12,12,...,12\}}_{\text{16-times}})$ = {12, 12, 12, 12}

$= x^2$

$l(x^2 \times x)$ = l({12, 12, 12, 12} × {12, 12, 12, 12}) = {0, 0, 0, 0}

$= x^3$.

This x is a nilpotent multiset of order three.

Consider y = {6, 6} ∈ 4-M(Z_{24})

l(y × y) = l({6, 6} × {6, 6})

= l({12, 12, 12, 12})

$= y^2$

$l(y^2 \times y)$ = l({12, 12, 12, 12} × {6, 6}) = {0, 0, 0, 0} = y^3

y is also a nilpotent multiset of order three.

Clearly {6, 6} = y ⊆ {6, 6, 6, 6} = x.

Can we say all subsets of a nilpotent multiset be a nilpotent multiset? The answer is no in general.

Let z = {6} ⊆ {6, 6, 6, 6} = x

Consider l(z × z) = l({6} × {6}) = {12} = z^2

$l(z^2 \times z) = l(\{12\} \times \{6\})$

$= \{0\} \neq \{0, 0, 0, 0\}$.

Thus z is not a nilpotent multiset of 4-M(Z_{24}) but is only a partial nilpotent multiset of 4-M(Z_4).

Thus our claim, subset of a nilpotent multiset need not in general be a nilpotent multiset.

Let a = {6, 6, 6} $\in$ 4-M(Z_{24}).

Consider $l(a \times a) = l(\{6, 6, 6\} \times \{6, 6, 6\})$

$= \{12, 12, 12, 12\} = a^2$

$l(a^2 \times a) = [(\{12, 12, 12, 12\} \times \{6, 6, 6\})$

$= \{0, 0, 0, 0\} = a^3$.

We see a is a nilpotent multiset of order three.

Since Z_{24} has nilpotent so 4-M(Z_{24}) also has multiset nilpotents.

We see x = (5, 5, 5, 5) $\in$ 4-M(Z_{24}) is a multiset unit.

Y = {5, 5} $\in$ 4-M(Z_{24}) is a partial multiset unit as

$l(x \times x) = \{1, 1, 1, 1\}$ is a multiset unit.

$l(y \times y) = \{1, 1, 1, 1\}$ multiset unit.

z = {5} is such that $l(z \times z) = \{1\}$is a partial multiset unit

Let a = {5, 5, 5} is a multiset unit as

$l(a \times a) = \{1, 1, 1, 1\}$.

In view of all these we have the following results.

Theorem 3.4. *Let n-M(Z_m) be a n-multiplicity multiset of Z_m.*

i) *n-M(Z_m) has both multiset units as well as partial multiset units.*
(We define $\underbrace{\{1,1,...,1\}}_{n\text{-times}}$ as the multiset unit element of n-M(Z_m). {1}, {1, 1}, {1, 1, 1,}, {1, 1, 1, 1}, …, $\underbrace{\{1,1,...,1\}}_{(n-1)\text{times}}$ are partial multiset units of n-M(Z_m).

ii) *n-Z(m) has multiset nilpotents if and only if Z_m has nilpotents.*

iii) *n-Z(m) has multiset partial nilpotents if and only if Z_m has nilpotents.*

iv) *If m is a prime m-Z(m) has no nilpotents.*

Proof is left as an exercise to the reader.

We will illustrate these situations by some more examples.

Example 3.12. Let 2-M(Z_{32}) be the 2-multiplicity multiset x = {11, 11} and y = {3, 3} ∈ 2-M(Z_{32}) is such that l({x} × {y})
= l({11, 11} × {3, 3})
= ({1, 1, 1, 1} = {1, 1} is a 2-multiset unit.
x = {2, 2} ∈ 2-M(Z_{32}) is such that [({x × x × x × x × x}) = {0, 0} is a 2-multiset nilpotent.
y = {2} ∈ 2-M(Z_{32}) is a 2-multiset partial nilpotent.
Consider x = {2, 4} ∈ 2-M(Z_{32}); l(x × x)
= l({2, 4} × {2, 4})

$= l(\{4, 8, 8, 16\}) = \{4, 8, 8, 16\}$

$= l(\{x^2 \times x\}) = l(\{4, 8, 8, 16\} \times \{2, 4\})$

$= l(\{8, 16, 16, 0, 16, 0, 0, 0\})$

$= \{8, 16, 16, 0\} = x^3$

$l(x^3 \times x) = l(\{0, 16, 16, 8\} \times \{2, 4\})$

$= l(\{(0, 0, 0, 0, 16, 0, 0, 0\})$

$= \{0, 0, 16)\} = x^4$

$l(x^4 \times x) = l(\{0, 0, 16\} \times \{2, 4\})$

$= l(\{0, 0, 0, 0, 0, 0\})$

$= \{0, 0\}$.

Thus x is a multiset nilpotent of order 5.

Consider $y = \{16, 4, 8\} \in 2\text{-}M(Z_{32})$

$l(y \times y) = l(\{16, 4, 8\} \times \{16, 14, 8\})$

$=l(\{0, 0, 0, 0, 16, 0, 0, 0, 0\})$

$= \{0, 0, 16\} = y^2$

$l(y^2 \times y) = l(\{0, 0, 16\} \times \{16, 4, 8\})$

$= l(\{0, 0, 0, 0, 0, 0, 0, 0, 0\})$

$= \{0, 0\} = y^3$.

Thus y is a multiset nilpotent of order three.

Consider $x = \{16, 4\}$ and $y = \{8, 8\} \in 2\text{-}M(Z_{32})$

We see $l(x \times y) = l(\{0, 0, 0, 0\}) = \{0, 0\}$.

So x and y contribute to multiset zero divisors.

Let $a = \{8\}$ and $b = \{16\} \in 2\text{-}M(Z_{32})$.

$l(a \times b) = l(\{8\} \times \{16\}) = \{0\}$.

Thus a and b give way only to a partial multiset zero divisors.

Example 3.13. Let 3-M(Z_{11}) be a 3-multiplicity multiset. Let x = {3, 3} and y = {4, 4} $\in$ 3-M(Z_{11})

l({x × y}) = l({1, 1, 1, 1})

= {1, 1, 1}.

Thus x and y are unit multisets.

Consider x = {6} and y = {2} $\in$ 3-M(Z_{11}).

l({x × y) = {1}

Thus x and y contribute only to a partial multiset unit.

Consider x = {5, 5} and y = {9} $\in$ 3-M(Z_{11})

l(x × y) = l(x × y) = l({5, 5} × {9})

= {1, 1}.

Once again x and y yield only a partial multiset unit.

Another important feature which we wish to discuss is.

Is the power of every element in 3-M(Z_{11}) give way a unit or a partial unit or neither?

To this effect we find elements in 3-M(Z_{11}). Consdier x = {4, 2, 3} $\in$ 3-M(Z_{11}) l(x × x) = l({4, 2, 3} × {4, 2, 3})

= l({5, 8, 1, 8, 4, 6, 1, 6, 9})

= {5, 8, 1, 8, 4, 6, 6, 1, 9} = x^2

l(x^2 × x) = ({5, 1, 1, 8, 8, 6, 6, 9, 4} × {4, 2, 3})

= ({9, 4, 4, 10, 10, 3, 3, 3, 5, 10, 2, 2, 5, 5, 1, 1, 7, 8, 4, 3, 3, 2, 2, 7, 7, 5, 1})

= {1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 7, 7, 8, 9, 10, 10} = x^3

l(x^3 × x) = l({1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 7, 7, 8, 9, 10, 10} × {4, 2, 3})

$= l(\{4, 4, 8, 8, 1, 1, 5, 5, 9, 9, 6, 6, 10, 3, 7, 7, 2, 2,$
$4, 4, 6, 6, 8, 8, 10, 10, 3, 3, 5, 7, 9, 9, 3, 3, 6, 6, 9, 9,$
$1, 1, 4, 4, 10, 10, 2, 5, 8, 8\})$
$= \{1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9,$
$10, 10\} = x^4$

We see $x^5 = x^4 = x^6$, ….

Thus x does not lead to a multiset unit or multiset partial unit.

If x = {1} or {2} or {3} or any singleton set we see it gives the partial multiset unit.

Consider $x = \{2\} \in 3\text{-}M(Z_{11})$
$l(x \times x) = \{4\} = x^2$
$l(x^2 \times x) = \{8\} = x^3$
$l(x^3 \times x) = \{5\}= x^4$
$l(x^4 \times x) = \{10\} = x^5$
$l(x^5 \times x) = \{9\} = x^6$
$l(x^6 \times x) = \{7\}= x^7$
$l(x^7 \times x) = \{3\} = x^8$
$l(x^8 \times x) = \{6\} = x^9$
$l(x^9 \times x) = \{1\}= x^{10}$.

Thus $x^{10} = \{1\}$ so is only a partial multiset unit.

We can work with any of the singleton set and show all of them are only partial multiset units of $3\text{-}M(Z_{11})$,.

In view of this we have the following result.

Theorem 3.5. *Let S = {n-M(Z_m)} be a n-multiplicity multiset on Z_m, m a primie. {S, l(×)} be the n-multiplicity multiset semigroup under the level product.*

There exists at least {m – 1} singleton multiset sets {x_i} whose powers are bounded between 2 ≤ t ≤ m – 1 such that {x_i^t} = {1}; 1 ≤ i ≤ m – 1.

Proof is direct and hence left as an exercise to the reader.

There are elements from these {x_i}'s ∈ {{1}, {2}, {3}, …, {m – 1}} is a multiset partial unit, i ≠ j, 1 ≤ i, j ≤ m – 1.

However it is impossible to find in n-M(Z_m) if m is a prime any multiset nilpotents or multiset partial nilpotents.

Further if m is a prime them n-M(Z_m) cannot have multiset zero divisors or multiset partial zero divisors, however n-M(Z_m) can have multiset idempotents though Z_m has no idempotents. The next study is does the set of idempotent multisets enjoy any nice algebraic structure.

To this effect we first analyse some example.

Example 3.14. Let 2-M(Z_5) be the 5-multiplicity multiset of Z_5. We first enumerate the set of all trivial and nontrivial multiset idempotents of 2-M(Z_5).

I = {{0, 0}, {1, 1}, {0, 0, 1, 1}, {0}, {1}, {1, 1, 2, 2, 3, 3, 4, 4}, {1, 1, 4, 4}, {1, 1, 4, 4, 0, 0}, {1, 1, 2, 2, 3, 3, 4, 4, 0, 0}}.

We see {I, l(×)} is a set hence a subsemigroup of {2-M(Z_5), l(×)}. This will be also known as multiset idempotent subsemigroup of {2-M(Z_5), l(×)}.

In view of this we can prove the following result.

***Theorem 3.6**. Let S = {n-M(Z_m), l(×)} be a n-multiplicity multiset of Z_m under level product l(×) where m is a prime. The set of nontrivial strict multiset idempotents of S form a subsemigroup I where*

$$I = \{ \underbrace{\{0,0,...,0\}}_{n\text{-times}}, \underbrace{\{1,1,...,1\}}_{n\text{-times}}, \underbrace{\{0,0,0,..,0}_{n\text{-times}}, \underbrace{1,1,...,1\}}_{n\text{-times}}, \underbrace{\{0,0,0,..,0}_{n\text{-times}}, \underbrace{1,1,...,1}_{n\text{-times}}, \underbrace{(m-1),...,(m-1)\}}_{n\text{-times}}, \underbrace{\{1,1,...,1}_{n\text{-times}}, \underbrace{(m-1),...,(m-1)\}}_{n\text{-times}}.$$

$$\underbrace{\{0,0,...,0}_{n\text{-times}}, \underbrace{1,1,...,1}_{n\text{-times}}, \underbrace{2,2,...,2}_{n\text{-times}}, \underbrace{3,3,...,3}_{n\text{-times}}, \ldots,$$

$$\underbrace{m-1,m-1,...,m-1\}}_{n\text{-times}}, \underbrace{\{1,1,1,...,1}_{n\text{-times}}, \underbrace{2,2,...,2}_{n\text{-times}}, \underbrace{3,3,...,3}_{n\text{-times}}, \ldots,$$

$$\underbrace{m-1,m-1,...,m-1\}}_{n\text{-times}}\}; \quad o(I) = 7.$$

Proof is direct and left as an exercise to the reader.

If n is not a prime the working is very different.

We may have other multiset idempotent subsemigroups. To this effect we will give some examples.

Example 3.15. Let S = {2-M(Z_6), l(×)} be the 2-multiplicity multiset semigroup under level product l(×).

We see Z_6 has 3 and 4 to be idempotents.

Now the set of idempotent multisets of S are given in the following. We take only strict multiset idempotents when we say strict multiset we see to it that in each multiset any element is repeated 2 times (n-times if it is a n-multiplicity multiset).

The nontrivial strict idempotent multisets of S are

I = {{0, 0}, {1, 1}, {3, 3}, {4, 4}, {0, 0, 1, 1}, {0, 0, 3, 3}, {0, 0, 4, 4}, {1, 1, 4, 4}, {1, 1, 5, 5, 2, 2, 4, 4}, {1, 1, 3, 3}, {0, 0, 4, 4, 3, 3}, {1, 1, 4, 4, 3, 3, 0, 0}, {0, 0, 3, 3, 1, 1}, {1, 1, 5, 5}, {0, 0, 1, 1, 5, 5}, {1, 1, 5, 5, 3, 3}, {1, 1, 3, 3, 5, 5, 0, 0} {1, 1, 0, 0, 2, 2, 3, 3, 4,4, 5, 5}} I is not a subsemigroup for {0, 0, 4, 4}, {1, 1, 5, 5} ∈ I we see l({1, 1, 5, 5} × {0, 0, 4, 4}) = l({0, 0, 0, 0, 4, 4, 4, 2, 2}) = {0, 0, 2, 2, 4, 4} ∉ I.

Hence our claim.

However I has subsets which are multiset idempotent subsemigroups.

Take J = {{0, 0}, {1, 1}, {0, 0, 1, 1}, {0, 0, 3, 3}, {0, 0, 1, 1, 3, 3}, {1, 1, 3, 3}, {3, 3}} ⊂ I is a multiset idempotent subsemigroup contained in I.

J_2 = {{1, 1}, {3, 3}, {1, 1, 3, 3}} ⊆ I is again a multiset subset subsemigroup contained in I.

The table for {J_2, l(×)} is given in the following.

l(×)	{1, 1}	{3, 3}	{1, 1, 3, 3}
{1, 1}	{1, 1}	{3, 3,}	{1, 1, 3, 3}
{3, 3}	{3, 3}	{3, 3}	{3, 3}
{1, 1, 3, 3}	{1,1,3,3}	{3, 3}	{1,1, 3, 3}

Clearly {J_2, l(×)} is only a multset idempotent semigroup.

l({1, 1, 5, 5} × {1, 1, 4, 4})

= ({1, 1, 5, 5, 1, 1, 5, 5, 4, 4, 2, 2, 4, 4, 2, 2})

= {1, 1, 5, 5, 2, 2, 4, 4} = x l({x × x})

= l({1, 1, 5, 5, 2, 2, 4, 4} × {1, 1, 5, 5, 2, 2, 4, 4})

= l({1, 1, 5, 5, 2, 2, 4, 4, 1, 1, 5, 5, 2, 2, 4, 4, 5, 5, 1, 1, 4, 4, 2, 2, 5, 5, 1, 1, 4, 4, 5, 5, 1, 1, 2, 2, 4, 4, 4, 4, 2, 2, 4, 4, 2, 2, 2, 2, 4, 4})

= {1, 1, 5, 5, 2, 2, 4, 4} is also an idempotent.

Interested reader can find other multiset idempotent subsemigroups.

Further prove these multiset idempotent subsemigroups can never be a S-subsemigroup.

We now proceed onto define multiset Smarandache idempotents or Smarandache multiset idempotent.

Definition 3.3. *Let S = {n-M(Z_m), l(×)} be the n-multiplicity multiset on Z_m. We define a multiset 0 ≠ x ∈ S to be a Smarandache multiset idempotent if*

i) $x^2 = x$

ii) There exists y ∈ S \ {unitm zero and x} such that a) $y^2 = x$ *and (b) yx = y or yx = x or in (b) is in the mutually exclusive sense. y is defined us the Smarandache multiset co-idempotent of x.*

We will first illustrate this by same examples.

Example 3.16. Let S = {3-M(Z_6), l(×)} be the 3-multiplicity multiset semigroup on Z_6.

Consider x = {4, 4, 4} ∈ S we see l({4, 4, 4} × {4, 4, 4}) = {4, 4, 4} = x

Consider y = {2, 2, 2} ∈ S.

Clearly l(x × y) = l({4, 4, 4} × {2, 2, 2} = {2, 2, 2} ≡ y and l({2, 2, 2} × {2, 2, 2}) = {4, 4, 4}.

Thus {4, 4, 4} is a multiset Smarandache idempotent of S.

Consider

x = {1, 1, 1, 5, 5, 5, 2, 2, 2, 4, 4, 4} ∈ S

Clearly l(x × x) = x so x is an idempotent of S.

Let y = {1, 1, 1, 2, 2, 2, 5, 5, 5 } ∈ S.

Consider l(y × y) = l({1, 1, 1, 2, 2, 2, 5, 5, 5} × {1, 1, 1, 2, 2, 2, 5, 5, 5})

{1, 1, 1, 2, 2, 2, 5, 5, 5, 4, 4, 4] = x.

Further l(x × y) = x.

Thus x is a Smarandache multiset idempotent and y the Smarandache multiset coidempotent of x.

However can we say all multiset idempotents are not Smarandache multiset idempotents?

Consider x = {3, 3, 3} ∈ S.

We see l(x × x) = {3, 3, 3} = x.

Let y = {1, 1, 1, 3, 3, 3} ∈ S.

We see l(y × y) = y and l({x × y}) = x.

So x and y do not contribute to Smarandache multiset idempotents, however both x and y are multiset idempotents of S.

Hence the claim. In view of all these we can have the following theorem.

Theorem 3.7. *Let S = {n-M(Z_m), l(×)} be the n-multiplicity multiset (of Z_m) semigroup under level product l(×).*

i) All Smarandache multiset idempoents are multiset idempotents.

ii) *Every multiset idempotent of S need not in general be a Smarandache idempotent.*

Proof is left as an exercise to the reader.

Now if we continue the same example we get

x = {0, 0, 0, 1, 1, 1, 2, 2, 2, 4, 4, 4, 3, 3, 3, 5, 5, 5} $\in$ S is such that l(x $\times$ x) = x that is x is an multiset idempotent.

Consider y = {1, 1, 1, 3, 3, 3, 2, 2, 2, 5, 5, 5 } $\in$ S.

Clearly l(y $\times$ y) = {1, 1, 1, 0, 0, 0, 3, 3, 3, 5, 5, 5, 2, 2, 2, 4, 4, 4} = x and l(x $\times$ y) = x. Thus x is a Smarandache multiset idempotent.

Several open problems can be suggested.

Let S = {n-M(Z_m), l($\times$)} be the n-multiplicity multiset semigroup under level product.

i) Do the collection of a Smarandache multiset idempotents form a subsemigroup?

ii) Can any Smarandache multiset coidempotent be an idempotent of Smarandache idempotent?

Next we proceed onto discuss about the Smarandache mutliset zero divisors of S where S is a n-multiplicity multiset of Z_m.

Definition 3.4. *Let S = {n-M(Z_m), l(×)} be the n-multiplicity multiset semigroup under level product l(×). We say a non zero element x ∈ S is a Smarandache multiset zero divisor if there exists a nonzero multiset y in S such that (i) xl(×) y = {0} and non zero multisets a, b ∈ S \ {x, y} such that (i) a l(×)x = {0} (ii) b l(×)y = {0} (iii) a l(×)b ≠ 0.*

We will illustrate this situation by some examples.

Example 3.17. Let S = {4-M(Z_{12}), l(×) be a 4-multiplicity multiset on Z_{12}.

Consider x = {6, 6, 6, 6, 0, 0, 0, 0} ∈ S we see y = {8, 8, 8, 8, 0, 0, 0, 0} ∈ S is such that l(x × y) = {0, 0, 0, 0}.

We see a = {2, 2, 2, 2} and b = {3, 3, 3, 3} ∈ S is such that

l(x × a) = {0, 0, 0, 0},
l(y × b) = {0, 0, 0, 0) but
l(a × b) = {6, 6, 6, 6} ≠ {0, 0, 0, 0}.

Thus x = {0, 0, 0, 0, 6, 6, 6, 6} is a Smarandache multiset zero divisor of S.

We see in general all multiset zero divisors are not Smarandache multiset zero divisors.

For x = {4, 4, 4, 4, 0, 0, 0, 0} and y = {3, 3, 3, 3} ∈ S are such that l(x × y) = {0, 0, 0, 0}.

We do not have a, b ∈ S such that l(x × a) = {0, 0, 0, 0} and l(y × b) = {0, 0,0, 0} but l(a × b) ≠ {0, 0, 0, 0}.

It is easily verified that all multiset zero divisors in general are not Smarandache multiset zero divisors.

In view of all this we have the following theorem.

Theorem 3.8. *Let S = {n-M(Z_m), l(×)} be the n-multiplicity multiset semigorup using the level product.*

Every Smarandache multiset zero divisor of S is a multiset zero divisor of S and not vice versa

Proof follows from the definition of Smarandache multiset zero divisors. Converse or the other way result is proved by a counter example.

We give some more S-multiset zero divisors of S.

Consider x = {0, 0, 0, 0, 4, 4, 4, 4, 8, 8, 8, 8} and y = {0, 0, 0, 0, 6, 6, 6, 6} ∈ S. Clearly l(x × y) = {0, 0, 0, 0} is a multiset zero divisor of S.
Take b = {3, 3, 3, 3} and a = {2, 2, 2, 2} ∈ S.
We see l(b × x) = {0, 0, 0, 0} and l(a × y) = {0, 0, 0, 0}
but l(a × b) ≠ {0, 0, 0, 0}.

Hence the pair {x, y} is a Smarandache multiset zero divisor pair.

Thus we can define the notion of Smarandache multiset zero divisors and Smarandache multiset idempotents.

A natural question thrown open to the reader is that whether the set of all Smarandache multiset idempotents of a n-multiset semigroup S = {$\dot{n}$-M(Z_m), l(×)} under the level product be a subsemigroup of multiset idempotents or a S-multiset subsemigroup of idempotents.

However at the outset we are not in a position to define the notion of multiset units. For if in the above example if we take {1, 1, 1, 1} as a multiset unit them l({2, 2} × {1, 1, 1, 1}) = {2, 2, 2, 2} ≠ [2, 2}.

Also if we take {1, 1, 1} as a multiset unit
then l({2, 2} × {1, 1, 1}) ≠ {2, 2} as l({2, 2} × {1, 1, 1}) = {2, 2, 2, 2}.
Also {1, 1} is not a multiset unit as l({4, 4, 4} × {1, 1})
= {4, 4, 4, 4} ≠ {4, 4, 4}.

Obviously if we one accepts {1} to be a multiset unit as it is not a multiset.

So we define {1} to be the pseudo multiset unit of S then we can have for

x = {5, 5, 5, 5}, l(x × x) = {1, 1, 1, 1}.

Is this a multiset unit or not. So only we call {1} as multiset pseudo partial unit of S.

The reader is left with the task of specially defining the notion of multiset units in S.

In case of S-multiset zero divisors we can find the algebraic structure enjoyed by it.

It is left as an exercise for the reader to determine whether the set of all S-multiset zero divisors form a multiset subsemigroup of S or not.

However it is pertinent to record at this juncture that the notion of Smarandache multiset semigroup is not possible as we do not have the universal multiset unit which can be defined in $S = \{n\text{-}M(Z_m), l(\times)\}$.

Consider the finite complex number collection $C(Z_m)$, we can using the multiset $C(Z_m)$ get the n-multiplicity multiset, denoted by $n\text{-}M(C(Z_m))$.

As in case of $n\text{-}M(Z_m)$ we can derive all properties, but however to make the material self contained one or two examples will be given and their special features described.

Recall

$$C(Z_m) = \{a + bi_F \,/\, a, b \in Z_m,\ i_F^2 = m - 1\}.$$

Now using $C(Z_3)$ we will give an example.

Example 3.18. Let S = {2-M(C(Z_3)} = {{0}, {1}, {2}, {0, 0}, {0, 1}, {0, 2}, {1, 2}, {1, 1}, {2, 2}, {i_F}, {2i_F}, {i_F, i_F}, {0, i_F}, {1, i_F}, {2, i_F}, {2i_F, 2i_F}, {0, 2i_F}, {1, 2i_F}, {2, 2i_F}, {i_F, 2i_F}, {0, 0, 1}, {0, 0, 2}, {0, 0, i_F}, {0, 0, 2i_F}, {1 + i_F}, {2 + i_F}, {1 + 2i_F}, {2 + 2i_F}, {1 + i_F, 0}, {1 + i_F, 1}, {1 + i_F, 2}, {1 + i_F, i_F}, {1 + i_F, 2i_F}, {1 + i_F, 1 + i_F}, {1 + i_F, 2 + i_F}, {1 + i_F, 2 + 2i_F}, {1 + i_F, 1 + 2i_F}. {2 + i_F, 0}, {2 + i_F, 1}, {2 + i_F, 2}, {2 + i_F, i_F}, {2 + i_F, 2i_F}, {2 + i_F, 2 + i_F}, {2 + i_F, 2i_F + 1}, {1 + 2i_F, 2 + 2i_F}, {1 + 2i_F, 0}, {1 + 2i_F, 1}, {1 + 2i_F, 2}, {1 + 2i_F, i_F}, {1 + 2i_F, 2i_F}, {1 + 2i_F, 1 + 2i_F}, {1 + 2i_F, 2 + 2i_F}, {2 + 2i_F, 0}, {2 + 2i_F, 1}, {2 + 2i_F, 2}, {2 + 2i_F, 2 + 2i_F}, {0, 0, 1 + i_F}, {0, 0, 2 + i_F}, {0, 0, 2 + 2i_F}, {0, 0, 1 + 2i_F}, {1, 1, 0}, {1, 1, i_F}, {1, 1, 2}, {1, 1, 2i_F}, {1, 1, 1 + i_F}, {1, 1, 1 + 2i_F}, {1, 1, 2 + i_F}, {1 + 1, 2 + 2i_F}, {2, 2, 0}, {2, 2, i_F}, {2, 2, 1}, {2, 2, 2i_F}, {2, 2, 1 + i_F}, {2, 2, 2 + i_F}, {2, 2, 2 + 2i_F}, {2, 2, 1 + 2i_F}, {i_F, i_F, 0}, {i_F, i_F, 1}, {i_F, i_F, 2}, {i_F, i_F, 2i_F}, {i_F, i_F, 1+i_F}, {i_F, i_F, 1+ 2i_F}, {i_F, i_F, 2 + 2i_F}, {i_F, i_F, 2 + i_F}, {2i_F, 2i_F, 1}, {2i_F, 2i_F, 0}, …, {2i_F, 2i_F, 2 + 2i_F}, {1 + i_F, i_F + 1, 0}, {i_F + 1, 1 + i_F, 1}, …, {i_F + 1, i_F + 1, 2 + 2i_F}, …, {2 + 2i_F, 2 + 2i_F, 0}, {2 + 2i_F, 2 + 2i_F, 1}, …., {2 + 2i_F, 2 + 2i_F, 1+ 2i_F}, {1, 1, 0, 0}, {1, 1, 2, 2}, {1, 1, i_F, i_F}, …, {1, 1, 2 + 2i_F, 2 + 2i_F}, {0, 0, 2, 2}, {0, 0, i_F,i_F}, {0, 0,2i_F, 2i_F}, …., {0, 0, 2 + 2i_F, 2 + 2i_F}, …, {2 + 2_{iF}, 2 + 2i_F, 1 + 2i_F, 1 + 2i_F}, {0, 0, 1, 1, 2, 2}, {0, 0, 1, 1, i_F, i_F}, …, {0, 0, 1, 1, 2 + 2i_F, 2 + 2i_F}, {1 + i_F, I + i_F, 2 + i_F, 2 + i_F, 2 + 2i_F, 2 + 2i_f}, {0, 0, 1, 1, 2, 2, i_F, i_F}, {0, 0, 1, 1, 2, 2, 1 + i_F, 1 + i_F}, …, {0, 0, 1, 1, 2, 2, 2 + 2i_F, 2 + 2i_F}, {0, 0, i_F, 2}, {0, 0, 1, i_F}, …, {0, 0, 1 + 2i_F, 2 + 2i_F} and so on. {1, 1, 0, 0, 2, 2, i_F, i_F, 2i_F, 2i_F, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F, 2i_F + 1, 2i_F + 1, 2i_F + 2, 2i_F + 2}}.

The reader is left with the task of finding the order of {2-M(C(Z_3)} that is the order of the 2-multiplicity multiset of the finite complex modulo integer C(Z_3).

The reader can find the semilattices S_1 = {2-M(C(Z_3)), l(∪)}, S_2 = {2-M(C(Z_3)), ∩} and the lattice L = {2-M(C(Z_3)), l(∪), ∩}. Find sublattices of L and prove L is a S-lattice.

Now we can prove B_1 = {2-M(C(Z_3)), l(+)} is a commutative semigroup of finite order. The set {0, 0} ∈ B is the identity with respect to l(+).

However every x ∈ B_1 cannot have inverse only a few of them have inverse. Let a = {1} and b = {2}.

We see a l(+) b = {0}. We do not call a to be the inverse of b or vice versa.

However if a_1 = {1, 1} and b_1 = {2, 2} ∈ 2-M(C(Z_3)) then a l(+) b = {0, 0}.

Thus we may have inverses for some elements in 2-M(C(Z_3)) though not for all of them.

Consider x = {i_F, i_F, 0, 0, 1, 1} and y = {$2i_F$, $2i_F$, 0, 0, 2, 2}.

We find l(x + y)

= l({i_F, i_F, 0, 0, 1, 1} + {0, 0, $2i_F$, $2i_F$, 2, 2}) \

= l({i_F, i_F, 0, 0, 1, 1, i_F, i_F, 0, 0, 1, 1, 0, 0, $2i_F$, $2i_F$, 1 + $2i_F$, 1 + $2i_F$, 0, 0, $2i_F$, $2i_F$, 1 + $2i_F$, 1 + $2i_F$, 0, 0, 2, 2, 2 + i_F, 2 + i_F})

$= (0, 0, 1, 1, i_F, i_F, 2i_F, 2i_F, 2, 2, 1 + 2i_F, 1 + 2i_F,$
$2 + i_F, 2 + i_F\} \in$ 2-M(C(Z_3).

Suppose we find

$$\begin{aligned} l(x + x) &= l(\{0, 0, 1, 1, i_F, i_F\} + \{0, 0, 1, 1, i_F, i_F\}) \\ &= l(\{0, 0, 1, 1, i_F, i_F, 0, 0, 1, 1, i_F, i_F, 1, 1, 2, 2, \\ &\quad 1 + i_F, 1 + i_F, 2, 2, 1 + i_F, 1 + i_F, 1, 1, i_F, i_F, \\ &\quad 1 + i_F, 1 + i_F, 2i_F, 2i_F, i_F, i_F, 2i_F, 2i_F, 1 + i_F, \\ &\quad 1 + i_F\}) \\ &= \{0, 0, 1, 1, 2, 2, i_F, i_F, 2i_F, 2i_F, 1 + i_F, 1 + i_F\}. \end{aligned}$$

This is the way l(+) operations is performed on 2-MC(Z_3).

However we can only prove S = {2-M(C(Z_3)), l(+)} to be a semigroup. Finding other properties related with S is left as an exercise.

We call {2-M(C(Z_3))} as the 2-multiplicity complex valued (or complex) multiset of C(Z_3), if l(+) is defined on 2-M(C(Z_3)) then we define S = {2-M(C(Z_3)), l(+)} to be 2-multiplicity complex valued multiset semigroup under level plus or level addition and S is a finite commutative semigroup. We see S has no identity with respect to l(+).

For take $\{2 + i_F, 0, 2i_F, 1, 1\} \in$ S. $\{2 + i_F, 0, 2i_F, 1, 1\}$ l(+) {0}

$= \{2 + i_F, 0, 2i_F, 1, 1\}$.

Still {0} ∈ 2-M(C(Z_3)) cannot serve as the level additive identity of 2-M(C(Z_3)) as it is not a 2-multiset.

Suppose we take instead of {0} the element {0, 0} then

{$2i_F$ + 1, 0, $2i_F$, 1, 1} + {0, 0} = {0, 0, 1, 1,$2i_F$, $2i_F$, 1 + $2i_F$, 1 + $2i_F$} ≠ {0, 1,1, $2i_F$, 1 + $2i_F$}

Hence {0, 0} also cannot serve as the identity under the level addition.

Thus {2-M(C(Z_3)), l(+)} is only a finite order complex 2-multiplicity multiset semigroup under the level plus (or level addition) and is not a monoid.

Next we proceed onto illustrate level product operation (l(×)) on the 2-M(C(Z_3)) and show both l(+) and l(×) two distinct operations.

Consider x = {i_F, 1, 0, 0, 2 + i_F, 2 + i_F} and y = {1, 1, 0, $2i_F$, i_F + 1} ∈ 2-M(C(M_3)).

Now l(x × y)

= l({1, i_F, 0, 0, 2 + i_F, 2 + i_F} × {1, 1, 0, $2i_F$, 1 + i_F}) = l({1, i_F, 0, 0, 2 + i_F, 2 + i_F, 2, 1 + i_F, 1, 1, i_F, i_F, 2, 1 + i_F, 1, 1, i_F, i_F, $2i_F$ + 1, 0, $2i_F$, $2i_F$, 2, 2, 2 + i_F, 1 + $2i_F$, 1 + i_F, 1 + i_F, $2i_F$, $2i_F$})

= {1, 1, 0, 0, i_F, i_F, 2 + i_F, 2 + i_F, 1 + $2i_F$, 1 + $2i_F$, $2i_F$, $2i_F$, 2, 2, 1 + i_F, 1 + i_F} I

Now we find for the same x and y in 2-M(C(Z_3) the level addition l(+).

l(a + b) = l({i_F, 0, 0, 1, 2 + i_F, 2 + i_F} + {1, 1, 0, 2i_F, i_F + 1})

= l({1 + i_F, 1, 1, 2, i_F, i_F, 1 + i_F, 1, 1, 2, i_F, i_F, i_F, 0, 0, 1, 2 + i_F, 2 + i_F, 0, 2i_F, 2i_F, 2i_F, 2i_F, 1 + 2i_F, 2, 2, 2i_F + 1, 1 + i_F, 1 + i_F, 2 + i_F} = {1, 1, 2, 2, 0, 0, i_F, i_F, 2i_F, 2i_F, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F, 1 + 2i_F, 1 + 2i_F} II

In this case l(a + b) = l(a × b).

We call these are special elements pair or invariant pair. However it is important to record that all pairs of elements will not be invariant pairs.

We now find that pair a, b ∈ 2-M(C(Z_3)) where l(a +b) ≠ l(a × b).

Consider a = {0, 0, 1, 1, 2} and b = {1 + i_F, 1 + i_F, 0, 2} ∈ 2-M(C(Z_3); l(a + b) = l({0, 0, 1, 1, 2} + {0, 2, 1 + i_F, 1 + i_F}} = {0, 0, 2,1, 1, 2, 2, 0, 0, 1, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F, i_F, i_F, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F}

= {0, 0, 1, 1, 2, 2, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F, i_F, i_F} I

Consider
l(a × b) = l({0, 0, 1, 1, 2} × {0, 2, 1+ i_F, 1 + i_F})
= l ({0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1 + i_F, 1 + i_F, 0, 2, 1 + i_F, 1 + i_F, 0, 4, 2 + 2i_F, 2 + 2i_F})

= $\{0, 0, 2, 2, 1 + i_F, 1 + i_F, 2 + i_F, 2 + i_F, 1\}$ II

Clearly I and II are distinct. Hence $l(a \times b) \neq l(a \times b)$ thus we can define those elements of $S = \{n\text{-}M(Z_m)\}$ which has elements $a, b \in S$ such that $l(a + b) = l(a \times b)$.

This study will be both innovative and interesting. Our only open problem is will this collection be enjoining the special property that it is a multiset subsemigroup under both $l(+)$ and $l(\times)$.

Now as in case of $n\text{-}M(Z_m)$ we can in case of $n\text{-}M(C(Z_m))$ also define the n-multiset zero divisor and Smarandache n-multiset zero divisors.

Example 3.19. Let $S = \{3\text{-}M(C(Z_6)), l(\times)\}$ be the 3-multiplicity multiset on $C(Z_6)$.

Let $x = \{3 + 3i_F, 3 + 3i_F, 3 + 3i_F, 0, 0, 3, 3, 3, 3i_F, 3i_F, 3i_F\}$ and $y = \{2, 2, 2, 2i + 2, 2i_F + 2, 2 + 2i_F\} \in S$.

We see $l(x \times y) = \{ 0, 0, 0\}$. Thus x and y are multiset zero divisors of S.

Clearly x is not a S-multiset zero divisor for we cannot find non zero $a, b \in S \setminus \{x, y\}$ with $l(a \times b) \neq \{0, 0, 0\}$ but $l(x \times a) = \{0, 0, 0\}$ and $l(x \times b) = \{0, 0, 0\}$.

We see if we take a = {2 + $2i_F$} and b = {$3i_F$} $\in$ S then l(x × a) = {0, 0, 0} and l(y × b) = {0, 0, 0} however l(a × b) = {0} $\neq$ {0, 0,0} the {0} is the only partial zero and not the zero of S.

Thus we can have S multiset complex zero divisors in case of complex n-multiset semgroups.

Interested reader is left with the task of finding S-multiset complex zero divisors in the n-multiset complex semigroups under level product.

Now we find the S-complex multiset idempotents of {n-M(C(Z_m)), l(×)}.

We will illustrate this situation by some examples.

Example 3.20. Let S = {5-M(C(Z_{12})), l(×)} be the complex n-multiplicity multiset on C(Z_{12}).

Let x = {4, 4, 4, 4, 9, 9, 9, 9, 0, 0, 0, 0}. l(x × x) = x $\in$ S is a n-complex multiplicity multiset idempotent of S.

We see x is real. However finding finite complex idempotents in C(Z_m) itself is a difficult task. So finding Smarandache complex idempotents is difficult. It is still difficult to find complex n-multiplicity multiset idempotents and their Smarandache analogue.

The reader is left as an exercise with this task.

Next we proceed onto find idempotents in $C(Z_n)$ for some values of n, $2 \leq n < \infty$.

We see $C(Z_2) = \{1, 0, i_F, 1 + i_F\}$ has no nontrivial idempotoents $C(Z_3) = \{1, 2, 0, i_F, 2i_F, 1 + i_F, 2 + i_F, 1 + 2i_F, 2 + 2i_F / i_F^2 = 2\}$ has no nontrivial idempotents.

Let us consider $C(Z_4) = \{a + bi_F / a, b \in Z_4\ i_F^2 = 3\} = \{0, 1, 2, 3, i_F, 2i_F, 3i_F, 1 + i_F, 1 + 2i_F, 1 + 3i_F, 2 + i_F, 2 + 2i_F, 2 + 3i_F, 3 + i_F, 3 + 2i_F, 3 + 3i_F\}$ has no idempotents which are nontrivial.

In view of all these we throw open the following conjecture.

Let $C(Z_n) = \{a + bi_F / a, b \in Z_n, i_F^2 = (n - 1)\}$.

If $(a + bi_F)^2 = a + bi_F$ then $a^2 + b^2 (n - 1) = a$ and $2ab = b$.

If we have some 'n' for which there exists a, b, $0 < a, b < m$ such that $2ab = b$ and $a^2 + b^2 (n - 1) = a$.

Then we can have idempotents.

Thus we leave it as a open conjecture for the researchers to find $a, b \in Z_n \setminus \{0\}$ such that the two equations $2ab = b$ and $a^2 + b^2 (n - 1) = a$ are true simultaneously.

This problem can be also realized as a problems of solving equations in the modulo integers. If the solution exists we are sure there are idempotents in $C(Z_m)$ which will yield multiset idempotents in $n\text{-}M(C(Z_m))$.

Next we proceed onto study about n-multiplicity multisets using $\langle Z_m \cup I\rangle$ the set of neutrosophic modulo integers.

$$\langle Z_m \cup I\rangle = \{a + bI / a, b \in Z_m, I^2 = I\} \text{ where } 2 \leq m < \infty.$$

We will illustrate n-multiset $M(\langle Z_m \cup I\rangle)$ for various values of m and n.

Example 3.21. Let S = {2-M($\langle Z_2 \cup I\rangle$)} = {{0}, {1}, {I}, {1 + I}, {ϕ}, {0, 0}, {0, 1}, {0, I}, {0, 1 + I}, {1, 1}, {1, I}, {1, 1 + I}, {I, 1 + I}, {I, I}, {1 + I, 1 + I}, {0, 0, 1}, {0, 0, I}, {0, 0, 1 + I}, {0, 0, 1, I}, {0, 0, 1, 1 + I}, {0, 0, I, 1 + I}, {0, 0, 1, 1}, {0, 0, I, I}, {0, 0, 1 + I, 1 + I}, {0, 0, I, I}, {0, 0, 1 + I, 1 + I}, {0, 0, 1, I, 1 + I}, {0, 0, 1, 1, I}, {0, 0, 1, 1, 1 + I}, {0, 0, 1, 1, I, 1 + I}, {0, 0, 1, 1, I, I, 1 + I}, {0, 0, 1, 1, 1 + I, 1 + I, I}, {0, 0, 1, 1, 1 + I, 1 + I, I, I}, {1, 1, 0}, {1, 1, I}, {1, 1, 1 + I}, {1, 1, 0, I, 1 + I}, {1, 1, 0, I, I}, {1, 1, 0, 1 + I, 1 + I}, {1, 1, 0, I, I, 1 + I}, {1, 1, 0, 1 + I, 1 + I, I}, {1, 1, 1 + I, 1 + I, I, I, 0}, {1, 1, 1 + I, 1 + I, I, I}, {0, 0, 1, 1, I, I}, {0, 0, I, I, 1 + I, 1 + I}, {0, 0, 1, 1, I + 1, I + 1}, and so on}.

Finding the number of multisets in {2-M($\langle Z_2 \cup I\rangle$)} is itself a difficult task.

However we show how level addition and level product are performed on 2-M($\langle Z_2 \cup I\rangle$).

Let us now show how l(+) is performed on 2-M(⟨Z_2∪I⟩).

Consider x = {1, 1, I, I, 0, 1 + I} and y = {0, 0, 1 + I, 1} ∈ 2-M(⟨Z_2 ∪ I⟩),

l(x + y) = l({1, 1, I, I, 0, 1 + I} + {0, 0, 1 + I, I}) = l({1,1, I, I, 0, 1 + I, 1, 1, I, I, 0, 1 + I, I, I, 1, 1, 1 + I, 0, 0, 0, 1 + I, 1 + I, 1, I})
= {1, 1, 0, 0, I, I, 1 + I, 1 + I}.

This is the way level addition is performed on the 2-multiplicity multiset 2-M(⟨Z_2 ∪ I⟩) = S.

We call these 2-multiplicity multiset S as 2-multiplicity neutrosophic multiset.

Consider x = {1, 1, 0, 0, I, 1 + I} and y = {I, I + 1, 0, 1} ∈ S.

The level product l(×) is defined in the following way.
l(x × y) = l({1, 1, 0, 0, I, 1 + I} × {0, 1, 1 + I, I})
= l({0, 0, 0, 0, 0, 0. 1, 1, 0, 0, I, 1 + I, 1 + I, 1 + I, 0, 0, 0, 1 + I, I, I, 0, 0, I, 0})
= {0, 0, I, I, 1, 1, 1 + I, 1 + I}.

We see in this case l(x + y) = l(x × y). However in general l(x + y) ≠ l(x × y).

For take x = {0, 1, 1 + I} and y = {I, I, 0, 1 + I} ∈ S.
l(x + y) = l({0, 1, 1 + I} + {0, I, I, 1 + I})
= l({0, 1, 1 + I, I, 1 + I, 1, I, 1 + I, 1, 1 + I, I, 0})

$= \{0, 0, 1, 1, 1 + I, 1 + I, I, I\}$ I

$l(x \times y) = l(\{0, 1, 1 + I\} \times \{0, I, I, 1 + I\})$
$= \{0, 0, I, I, 1 + I, 1 + I\}$ II

It is clear I and II are distinct.

We now give one more example before we make the formal definition.

Example 3.22. Let S = {3-M($\langle Z_2 \cup I \rangle$) be the 3-multiplicity neutrosophic multiset of $\langle Z_2 \cup I \rangle$.

S = {{0}, {1}, {0, 1}|, {I}, {1 + I}, {0, 0}, {0, I}, {0, 1 + I}, {1, I}, {1, 1 + I}, {1, 1}, {I, I}, {I, 1 + I}, {1 + I, 1 + I}, {I, 0, 1}, {0, 1, 1 + I}, {0, I, 1 + I}, {1, I, 1 + I}, {1, 0, 1 + I, I}, {0, 0, 1}, {0, 0, I}, {0, 0, 1 + I}, {0, 0, 1, I}, {0, 0, I, 1 + I}, {0, 0, 1, 1 + I}, {0, 0, 1, I, 1 + I}, {0, 0, 1, 1}, {0, 0, I, I}, {0, 0, 1, 1}, {0, 0, I, I}, {0, 0, 1 + I, 1 + I}, {0, 0, 1, 1, I}, {0, 0, 1, 1, 1 + I}, {0, 0, 1, 1, I, 1 + I}, {0, 0, 0}, {0, 0, 0, 1}, {0, 0, 0, I}, {0, 0, 0, 1 + I}, {0, 0, 0, 1, 1}, {0, 0, 0, 1, I}, {0, 0, 0, I, 1 + I}, {0, 0, 0, 1, 1 + I}, {0, 0, 0, I, I} and so on {0, 0, 0, 1, 1, 1, I, I, I, 1 + I, 1 + I, 1 + I}}.

Even finding the cardinality of S happens to be a difficult problem. Now {S, l(+)} is a semigroup under the level addition l(+).

However {S, l(+)} is not a monoid for

{1, 1} + {0, 0, 0} = {1, 1, 1} ≠ {1, 1}. So {0, 0, 0} is not the level addition identity. Further {0, 0} is also not the identity for

{I} + {0, 0} = {I, I} ≠ {I}.

Likewise we do accept {0} as identity for {0} + {a} = {a} but as it is not a multiset or a trivial multiset we only call them as pseudo level additive identity.

For which of the zeros we can take as multiset zero {0} or {0, 0} or {0, 0, 0}?

We also see in case of level product l(×), {1} is not taken as the multiset unit. We have {1} or {1, 1} or {1, 1, 1}.

Now we study some more properties of n-M($\langle Z_m \cup I \rangle$), $2 \leq m < \infty$; the n-multiplicity neutrosophic multiset of $\langle Z_m \cup I \rangle$.

We see S = 2-M($\langle Z_3 \cup I \rangle$) = {{1}, {0}, {2}, {I}, {2I}, {1 + I}, {2 +I}, {1 + 2I}, {2 + 2I}, {0, I}, {0, 0, I}, {0, 1, I}, {0, 2, I}, …., {0, 0, 2}, {2, 0, 0, 2I}, {2 + 2I, 0, 0, 2 + I, 2I}, …, {2I, 2, I, I, I, 0, 0}, …, {0, 0, I, I, 2I, 2I, 1 + I, 1 + I}, …, {0, 0, 1, 1, I, I, 2I, 2I, 1 + I, 1 + I, 2 + I, 2 + I, 2 + 2I, 2 + 2I, 2, 2, 2I + 1, 2I + 1}}.

We see {S, l(+)} is a commutative semigroup of finite order which is not a monoid.

Now {S, l(×)} is also a finite order commutative semigroup which is not a monoid under level product. We will illustrate this situation by some examples.

Example 3.23. Let S = {4-M(⟨Z_4 ∪ I⟩), l(×)} be the 4-multiplicity multiset semigroup under the level product l(×) on ⟨Z_4∪I⟩.

Let x = {1, 1, 2 + 2I, I, 3I + 1} and y = {I, I, I + 1, 2 + 3I, 0} ∈ S.

We find l(x × y) = {1, 1, 2 + 2I, I, 3I + 1} × {I, I, 1 + I, 0, 2 + 3I}) = l({I, I, 0, I, 0, I, I, 0, I, 0, 1 + I, 1 + I, 2 + 2I, 2I, 3I + 1, 0, 0, 0, 0, 0, 2 + 3I, 2 + 3I, 0, I, 2 + 2I}).
= {I, I, I, I, 0, 0, 0, 0, 1 + I, 3I + 1, 1 + I, 2 + 2I, 2 + 2I, 2 + 3I, 2 + 3I}

Let x = {2 + 2I, 2 + 2I, 2, 2I, 2I, 2I} ∈ S.

We find l(x × x) = l({2 + 2I, 2 + 2I, 2, 2I, 2I, 2I} × {2 + 2I, 2 + 2I, 2, 2I, 2I, 2I}) = l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …, 0}) = {0, 0, 0, 0}.

Clearly x is of nilpotent neutrosophic multiset of order two.

Let x = {2 + 2I} ∈ S.

We find l(x × x) = {0}. Clearly x is a neutrosophic partial nilpotent multiset of order two.

Let x = {2 + 2I, 2, 2I, 0, 2I + 2} and y = {2 + 2I, 2, 2, 2, 0} ∈ S.

We find l(x × y) = {0, 0, 0, 0}, so x and y is a neutrosophic multiset zero divisor.

Let x = {2, 2 + 2I} and y = {2I}∈ S. l(x × y)

= l({2, 2 + 2I} × {2I}) = {0, 0} ∈ S.

Clearly x and y are only neutrosophic multiset partial zero divisors of S.

Let x = {1 + 3I} ∈ S. Clearly l(x × x) = l({1 + 3I} × {1 + 3I}) = {1 + 3I} so x is the neutrosophic multiset idempotent of S.

Consider y = {1 + 3I, 1 + 3I, 0, 0, 1 + 3I, 1 + 3I, 0, 0} ∈ S.

It is easily verified l(y × y) = y.

Thus y is a neutrosophic multiset idempotent of S.

Consider a = {I, I, I, I} ∈ S.

We see l(a × a) = a so a is multiset neutrosophic idempotent of S.

Consider x = {I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1, 0, 0, 0, 0, 1, 1, 1, 1}∈ S. It is easily verified l(x × x) = x is the neutrosophic multiset idempotent of S.

However if x = {I, I, I, I} and y = {3I + 1, 3I + 1, 3I + 1, 3I + 1} $\in$ S. We see both x and y are neutrosophic multiset idempotents of S. But we see l(x $\times$ y) = {0, 0, 0, 0} so we can say the two idempotents are orthogonal with each other.

Let r = {I, I, I, I, 1, 1, 1, 1} and s = {3I + 1, 3I + 1, 3I + 1, 3I + 1, 1, 1, 1, 1} $\in$ S.

We see both r and s are neutrosophic idempotent multisets but l(r $\times$ s) = {I, I, I, I, 1, 1, 1, 1, 0, 0, 0, 0, 3I + 1, 3I + 1, 3I + 1, 3I + 1} = p $\neq$ {0, 0, 0, 0}.

l(p $\times$ p) = p is also an neutrosophic idempotent of S.

Thus we see in general all neutrosophic multiset idempotents are not orthogonal.

It is also important to keep on record Z_4 has no idempotents but $\langle Z_4 \cup I \rangle$ has idempotents.

Let d = {1, 1, 1, 1, 0, 0, 0, 0, 2, 2, 2, 2, 3, 3, 3, 3} $\in$ S.

We see l(d $\times$ d) = d so d is again a neutrosophic multiset idempotent of S.

Let p = {I, I, I, I, 3I, 3I, 3I, 3I} $\in$ S we see l(p $\times$ p) = p is a multiset neutrosophic idempotent.

Suppose q = {I, I, I, I, 3I + 1, 3I + 1, 0, 0, 0, 0, 3I + 1, 3I + 1} $\in$ S we see l(q × q) = {I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1, 0, 0, 0, 0} = q.

q is also a neutrosophic multiset idempotent of S.

Now we test whether p and q are orthogonal multiset neutrosophic idempotents of S.

l(p × q) = l({I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1, 0, 0, 0, 0}) = ({I, I, I, I, 3I, 3I, 3I, 3I, 0, 0, 0, 0}) $\neq$ {0, 0, 0, 0}.

We see l(p × q) = r is a new neutrosophic multiset idempotent different from p and q. However p and q are not orthogonal neutrosophic multiset idempotents of S.

The study of finding the set of all neutrosophic multiset idempotents happens to be a challenging problem. Once even the collection is found, studying the properties enjoyed by them happens to be an interesting one.

Further finding the subcollection of all orthogonal neutrosophic multiset idempotents is interesting and that collection may yield a nice structure or not is only under investigation.

Finally will the neutrosophic multiset idempotents depend on the m of Z_m is an important study.

Now we proceed onto describe the definition in an abstract way.

Definition 3.5. *Let S = {n-M(⟨Z_m ∪ I⟩)} be the collection of all multiset of multiplicity n from ⟨Z_m ∪ I⟩ = {a + bI / a, b ∈ Z_m, I^2 = I; 2 ≤ m < ∞} and 2 ≤ n < ∞. We define this finite collection S as the n-multiplicity neutrosophic multiset of ⟨Z_m ∪ I⟩.*

We have already provided examples of them.

In the first place {S, l(∪)} forms a neutrosophic multiset semilattice of finite order.

Secondly in an analogous way S enjoys the semilattice structure under l(∪) the dual operation of ∪.

Thus {S, ∩} forms a neutrosophic multiset semilattice.

Finally wesee {S, l(∪), ∩} forms a neutrosophic multiset lattice and this lattice is a Smarandache neutrosophic multiset lattice as P(⟨Z_m ∪ I⟩) the power set ⟨Z_m ∪ I⟩ is property contained in S as a subset and this neutrosophic power set P(⟨Zm ∪ I⟩) is infact a neutrosophic Boolean algebra hence {S, l(∪), ∩} enjoys the states of a Smarandache multiset neutrosophic lattice structure.

We will illustrate this situation in case of 2-M(⟨Z_2∪I⟩) = S = {{0}, {1}, {I}, {1 + I}, {0, 0}, {I, 0}, {0, 1}, {0, 1 + I}, …, {0, 0, 1, 1, I, I, 1 + I, 1 + I}}.

Now it is easily see P(⟨Z_2 ∪ I⟩} = {{ϕ}, {0}, {1}, {1, 0}, {0, I}, {I}, {1 + I}, {0, 1, I, 1 + I}, {0, 1 + I}, {1, I}, {I, 1 + I}, {1, 1 + I}, {1, I, 1 + I}, {0, I, 1 + I}, {0, 1 + I, 1}, {0, 1, I}} ⊆ S.

We see P(⟨Z_2 ∪ I⟩) yields a neutrosophic Boolean algebra of order 16 given by the following figure.

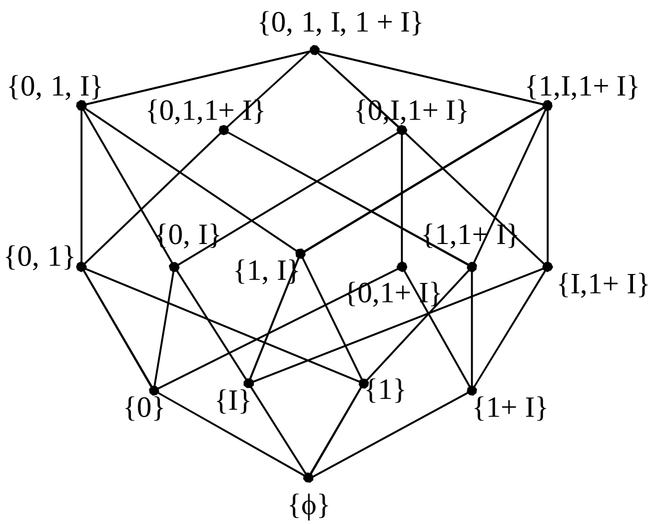

Figure 3.11

Apart from this we have other Boolean Algebra found from the power set {0, 1} given by P({0, 1}) = {ϕ, {0}, {1}, {1, 0}} given by the following diagram which is not a neutrosophic Boolean algebra.

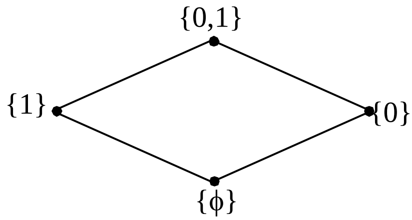

Figure 3.12

We can also speak about the Smarandache neutrosophic multiset idempotents of S.

Consider x = {I, I, I, I, 0, 0, 0, 0, 3I + 1, 3I + 1, 3I + 1, 3I + 1} ∈ S.

Clearly l(x × x) = x so x is a neutrosophic multiset idempotent of S.

Consider y = {I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1} ∈ S.

We see l(y × y) = x and l(x × y) = l({I, I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1} × {0, 0, 0,0, I, I, I, I, 3I + 1, 3I + 1, 3I + 1, 3I + 1}) = x.

Thus l(x × x) = x; l(y × y) = x and l(x × y) = x

Hence x is a Smarandache multiset neutrosophic idempotent of S.

Finding those neutrosophic multiset idempotent which are Smarandache is an interesting problem.

However we wish to keep on record that all neutrosophic multiset idempotents need not in general be Smarandache neutrosophic multiset idempotents and also converse is true; that is all Smarandache neutrosophic multiset idempotents are neutrosophic multiset idempotents. This result follows from the very definition of Smarandache neutrosophic multiset idempotents.

We prove the following result about n-M($\langle Z_m \cup I\rangle$); $2 \leq m < \infty$ and $2 \leq n < \infty$.

Theorem 3.9. *Let S = {n-M($\langle Z_m \cup I\rangle$), l($\cup$), $\cap$} be the neutrosophic lattice on n-multiplicity multiset of $\langle Z_m \cup I\rangle$; ($2 \leq m < \infty$ and $2 \leq n < \infty$). S is a Smarandache neutrosophic multiset lattice and S has atleast two neutrosophic special Boolean algebras of order 2^m and $2^{|\langle Z_m \cup I\rangle|}$.*

Proof: A neutrosophic multiset lattice is a Smarandadche neutrosophic multiset lattice if it has a lattice which is a Boolean algebra. Clearly the power set of $Z_{,m}$, P(Z_m) and power set of $\langle Z_m \cup I\rangle$, P($\langle Z_m \cup I\rangle$) generate Boolean algebras of order 2^m and $2^{|\langle Z_m \cup I\rangle|}$ respectively. Hence the theorem.

Example 3.24. Let S = {3-M($\langle Z_5 \cup I\rangle$, l($\times$)} be the neutrosophic multiset semigroup under level product l($\times$).

We find out whether S can have idempotents, units and zero divisors.

Consider x = {3 + 2I, 3 + 2I, 3 + 2I, 2 + 3I, 4 + I, I + 4} and y = {I, 2I, 3I, 4I} $\in$ S. We see l(x $\times$ y) = {0, 0, 0} so S has neutrosophic multiset zero divisor though Z_5 is a finite field.

Now we work on multiset Smarandache zero divisors in S.

Let x = {3 + 2I, 4 + I, 2 + 3I} and y = {I, 2I} $\in$ S.

We see l(x × y) = {0, 0, 0} so x and y is a neutrosophic multiset zero divisor of S. S also a Smarandache multiset neutrosophic zero divisor of S.

Take a = {I} and b = {2 + 3I} we see l(x × a) = {0, 0, 0} l(b × y) = {0, 0, 0}. Clearly l(a × b) = {0, 0, 0} $\neq$ {0, 0, 0}. Hence the claim. Consider x = {2 + 3I, 0, 3 + 2I} and y = {I, 2I, 3I} $\in$ S.

We see l(x × y) = {0, 0, 0} so x and y is a multiset neutrosophic zero divisor of S.

Consider a = 2I and b = {1 + 4I} $\in$ S.
We see l(x × a) = {0, 0, 0}; l(y × b) = {0, 0, 0}
and l(a × b) = {0} $\neq$ {0, 0, 0}.

Hence x and y are Smarandache neutrosophic multiset zero divisor of S.

Consider another example given in the following.

Example 3.25. Let S = {4-M(⟨Z_{12} $\cup$ I⟩), l(×)} be the 4-multiplicity neutrosophic multiset semigroup under level product l(×).

Let x = {3I, 6I, 9I} and y = {4I, 8} $\in$ S.

We see l(x × y) = {0, 0, 0, 0} is a neutrosophic multiset zero divisor of S.

Consider a = {2I + 10, 6I + 6, 5I + 7, 7I + 5} and b = {4I + 8, 8 + 4I} $\in$ S.

Clearly l(a × b)

l({2I + 10, 6I + 6, 5I + 7, 7I + 5} × {4I + 8, 8 + 4I}

l({8I + 40I + 16I + 80, 24I + 24I + 48I + 48, 20I + 28I + 40I + 56, 28I + 20I + 56I + 40, 16I + 80 + 8I + 40I, 48I + 48, 24I + 24I, 40I + 56 + 20I + 28I, 56I + 40 + 28I + 20I})

= l(4I + 8, 0, 8 + 4I, 8I + 8, 4I + 8, 0, 4I + 8, 8I + 4)

= {0, 0, 4I + 8, 4I + 8, 4I + 8, 8I + 4, 8 + 4I, 8I + 4}

$\neq$ {0, 0, 0, 0}.

Consider l(x × a) = {0, 0, 0, 0} and l(b× y) = {0, 0, 0, 0}.

Hence x, y $\in$ S is a Smarandache multiset neutrosophic zero divisor of S.

We see {n-M($\langle Z_m \cup I \rangle$) l(×)} be the neutrosophic multiset semigroup under the level product l(×); $2 \leq n < \infty$ and $2 \leq m < \infty$. Then we have the following result.

Theorem 3.10. *Let S = {n-M($\langle Z_m \cup I \rangle$) l(×)} be the neutrosophic multiset semigroup under the level product l(×), $2 \leq m < \infty$ and $2 \leq n < \infty$.*

S has Smarandache neutrosophic multiset zero divisor. 2m > n.

Proof. Consider x = {t + sI/t, s $\in Z_m \setminus \{0\}$; t + s = m} and y = {I, 2I, 3I, …, rI} ∈ S, r > n. Clearly $l(x \times y) = \underbrace{\{0,0,...,0\}}_{\text{n-times}}$ is a zero divisor.

Select a = {c + dI}; c + d = m, c, d $\in Z_m \setminus \{0\}$ and b = {gI}, g ≠ {0}.

Clearly $l(x \times b) = \underbrace{\{0,...,0\}}_{\text{n-times}}$ and $l(y \times a) = \underbrace{\{0,...,0\}}_{\text{n-times}}$ but $l(a \times b) = \{0\} \neq \{0, \ldots, 0\}$.

Thus S has Smarandache neutrosophic multiset zero divisors which are nontrivial. Hence the theorem.

Next we proceed onto study idempotents in neutrosophic multiset semigroups and the product l(×).

Example 3.26. Let S = {3-M($\langle Z_{12} \cup I\rangle$), l(×)} be the neutrosophic multiset semigroup under the level product l(×).

We see x = {9, 9, 9, 9I, 9I, 9I, 0, 0, 0} ∈ S is such that l(x × x) = x.

Consider y = {3, 3, 3, 3I, 3I, 3I, 0, 0, 0} ∈ S.

l(y × y) = l({3, 3, 3, 3I, 3I, 3I, 0, 0, 0} × {0, 0, 0, 3, 3, 3, 3I, 3I, 3I)} = {9, 9, 9, 9I, 9I, 9I, 0, 0, 0} = x.

Consider l(x × y) =

l({0, 0, 0, 9, 9, 9, 9I, 9I, 9I} × {0, 0, 0, 3, 3, 3, 3I, 3I, 3I}) = {0, 0, 0, 3, 3, 3, 3I, 3I, 3I} = y.

So x is a Smarandache neutrosophic multiset idempotent of S.

Let x = {4, 4, 4, 4I, 4I, 4I, 0, 0, 0, 0} ∈ S. Clearly l(x × x) = {0, 0, 0, 4, 4, 4, 4I, 4I, 4I} is a neutrosophic multiset idempotent.

Let y = {8, 8, 8, 8I, 8I, 8I, 0, 0, 0} ∈ S. Consider l(y × y) = {4, 4, 4, 4I, 4I, 4I, 0, 0, 0} = x.

Further l(x × y) = l({4, 4, 4, 4I, 4I, 4I, 0, 0, 0} × {8, 8, 8, 8I, 8I, 8I, 0, 0, 0} = {8, 8, 8, 8I, 8I, 8I, 0, 0, 0} = y. Thus x is a Smarandache neutrosophic multiset idempotent of S.

Let x = {2, 2, 2, 0, 0, 0, 4, 4, 4, 8, 8, 8, 2I, 2I, 2I, 4I, 4I, 4I, 1, 1, 1, I, I, I, 8I, 8I, 8I} ∈ S.

l(x × x) = {4, 4, 4, 0, 0, 0, 2I, 2I, 2I, 1, 1, 1, I, I, I, 8I, 8I, 8I, 8, 8, 8, 4I, 4I, 4I, 2, 2, 2} is a neutrosophic multiset idempotent of S.

The reader is left with the task of finding whether x is a Smarandache neutrosophic multiset idempotent of S or not.

Further if S = {n-M($\langle Z_m \cup I\rangle$), l(×)} be the n-multiplicity multiset of neutrosophic elements semigroup under level product, S has nontrivial idempotents whatever be m.

***Theorem 3.11**. S = {n-M($\langle Z_m \cup I\rangle$), l(×)} be the n-multiplicity neutrosophic multiset semigroup under the level product l(×;) $2 \leq n < \infty$ and $2 \leq m < \infty$.*

S has nontrivial neutrosophic multiset idempotents as well as Smarandache neutrosophic multiset idempotents.

Proof. Sicne every, Smarandache multiset idempotents is also an multiset idempotent is enough if we prove S have Smarandache neutrosophic multiset idempotents.

Consider x = $\{\underbrace{I, I, I, I, \ldots, I}_{\text{n-times}}, \underbrace{0, 0, \ldots, 0}_{\text{n-times}}, 3I, 2I, 4I, \ldots, (m-1)I\} \in S$. Now take y = $\{\underbrace{I, I, \ldots, I}_{\text{n-times}}, \underbrace{0, 0, 0, \ldots, 0}_{\text{n-times}}, \underbrace{2I, 2I, \ldots, 2I}_{\text{n-times}}, \underbrace{3I, 3I, \ldots, 3I}_{\text{n-times}}, \underbrace{(m-1)I, \ldots, (m-1)I\}}_{\text{n-times}} \in S$.

We see l(y × y) = y so y is a neutrosophic multiset idempotent of S. Consider l(x × x) = y and l(x × y) = y so x is a Smarandache multiset neutrosophic idempotent of S.

Hence the claim.

Finding neutrosophic multliset nilpotents is a challenging problem.

Example 3.27. Let S = {3-M($\langle Z_{27} \cup I \rangle$), l(×)} be the neutrosophic multiset semigroup under level product l(×).

X = {9I, 9I} ∈ S is a neutrosophic multiset nilpotent element of S.

l(x × x) = l({9I, 9I} × {9I, 9I}) = l({0, 0, 0, 0} = {0, 0, 0}.

Take y = {9, 9I} ∈ S we see l(y × y) = l({9, 9I} × {9, 9I})

= {0, 0, 0} is again a neutrosophic multiset nilpotent of order two.

Infact we can find several neutrosophic multiset nilpotents in S. Thus if Z_m or $\langle Z_m \cup I \rangle$ has no nilpotents then the multisets will not contribute to nilpotents.

In view of all these we have the following theorem.

Theorem 3.12. *Let S = {n-M($\langle Z_m \cup I \rangle$), l(×)} be the neutrosophic multiset semigroup under the level product. S has neutrosophic multiset nilpotents if and only if Z_m and $\langle Z_n \cup I \rangle$ has nontrivial nilpotents and neutrosophic nilpotents respectively.*

Proof is left as an exercise for the reader.

Next we proceed onto show first by examples that the n-neutrosophic multiset semigroup under level product l(×) is a Smarandache neutrosophic multiset semigroup.

Example 3.28. Let S = {5-M(⟨Z_9 ∪ I⟩) l(×)} be the 5-neutrosophic multiset semigroup under the level product l(×).

Consider P = {{1}, {8}} ⊆ 5-M(⟨Z_9 ∪ I⟩).

The table of {P, l(×)} is as follows.

l(×)	{1}	{8}
{1}	{1}	{8}
{8}	{8}	{1}

With the special condition that {1} is the special identity for all multisets in 5-M(⟨Z_9 ∪ I⟩).

So S is a Smarandache multiset neutrosophic semigroup under the operation l(×).

Consider the set B = {{1}, {7}, {2}, {4}, {5}, {8}} ∈ S.

The table for B is as follows.

l(×)	{1}	{2}	{4}	{8}	{7}	{5}
{1}	{1}	{2}	{4}	{8}	{7}	{5}
{2}	{2}	{4}	{8}	{7}	{5}	{1}
{4}	{4}	{8}	{7}	{5}	{1}	{2}
{8}	{8}	{7}	{5}	{1}	{2}	{4}
{7}	{7}	{5}	{1}	{2}	{4}	{8}
{5}	{5}	{1}	{2}	{4}	{8}	{7}

Clearly B is a multiset group under the level product l(×). Hence S is a Smarandache neutrosophic multiset semigroup under level products.

In view of all these we have the following theorem.

Theorem 3.13. *Let S = {n-M(⟨Z_m ∪ I⟩), l(×)} be the neutrosophic multiset semigroup under the level product l(×). S is a Smarandache neutrosophic multiset semigroup.*

Proof. Given S is a neutrosophic multiset semigroup under level product l(×). Clearly Z_m has subgroups under l(×).

Hence S is a Smarandache neutrosophic multiset semigroup.

Now having seen the structure and substructure our next aim would be to find the existence of multiset ideals in S. To this end first we will describe by some examples.

Example 3.29. Let S = {8-M(⟨Z_7 ∪ I⟩), l(×)} be the neutrosophic multiset semigroup under level product.

Consider B = {8-M(Z_7I)} = {all 8-multiplicity multisets from the set Z_7I} ⊆ S = 8-M(⟨Z_7 ∪ I⟩).

We see B is an ideal of S which is a neutrosophic multiset ideal of S.

Example 3.30. Let S = {3-M(⟨Z_7 ∪ I⟩), l(×)} be the neutrosophic multiset semigroup under the level product l(×).

Consider B = {3-M(Z_4I)} = {{0}, {I},{2I}, {3I}, ϕ, {0, I}, {0, 2I}, {0, 3I}, {0, 0, I}, {0, 0, 2I}, {0, 0, 3I}, {0, I, 3I}, {0, 2I, I}, {0, 2I, 3I}, {0, 0, I, 2I}, {0, 0, I, 3I}, {0, 0, 3I, 2I}, …, {0, 0, 0, I, I, I, 2I, 2I, 2I, 3I, 3I, 3I}}.

It is easily verified for every multiset x ∈ S \ B we have for every y ∈ S, l(x ×y) ∈ B.

Further we see {B, l(×)} is a neutrosophic multiset subsemigroup under the level product of S.

Thus B is a neutrosophic multiset ideal of S.

For let

x = {2, 2 + 3I, 0,1 + I, 3I} ∈ S. Take y = {3I, 2I, 2I, 0, I, I, I} ∈ B. We find l(x × y) = l({2 + 3I, 2, 0, 1 + I, 3I} × {3I, 2I, 2I, I, I, I, 0}) = {0, 0, 0, 3I, 3I, 2I, 2I, 2I, I, I, I} $\in$ B. Hence the claim.

Thus we have the following result.

Theorem 3.14. *Let S = {n-M(⟨$Z_m \cup I$⟩), l(×)} be the neutrosophic multiset semigroup under level product l(×). S has a neutrosophic multiset ideal.*

Proof is left as an exercise to the reader.

Multisets using dual numbers ⟨$Z_m \cup g$⟩ = {a + bg/a, b $\in$ Z_m, $g^2 = 0$}. The multiset collection is infinite, however n-multiplicity, multisets are finite $2 \leq n < \infty$; when n = 1 we get the powerset of ⟨$Z_m \cup g$⟩.

We will first illustrate this situation by some examples.

Example 3.31. Let S = {2-M(⟨$Z_2 \cup g$⟩)} = {{0}, {1}, {g}, {1 + g}, {0, 0}, {0, 1}, {0, g}, {0, 1 + g}, {1, 1}, {1, g}, {1, 1 + g}, {g, 1 + g}, {0, 0, 1}, {0, 0, 5}, ..,{1, 1, 0, 0,g, g, 1 + g, 1 + g}}

We can as in case of n-M(Z_m) or n-M(⟨$Z_m \cup g$⟩) we can define on multiset dual numbers the noton of l(+) and l(×) the level sum and product respectively.

We will show by examples how l(+) and l(×) are performed on multiset dual numbers.

Example 3.32. Let S = {4-M(⟨Z_m ∪ g⟩), l(+)} be the dual number multiset semigroup under level addition.

Let x = {1 + g, 2g, 5g, 0, 2g, 1 + g, 8 + 9g} and y = {0, 0, 1, 1, 5g + 5, 4g} ∈ S.

l(x + y) = l({1 + g, 2g, 5g, 0, 2g, 1 + g, 8 + 9g}
+ {0, 0, 1, 1, 5g + 5, 4g})

= l({1 + g, 2g, 5g, 0, 2g, 1 + g, 8 + 9g, 1 + g, 2g, 5g, 0, 2g, 1 + g, 8 + 9g, 2g, 2g + 1, 5g + 1, 1, 2g + 1, 2 + g, 9 + 9g, 2g, 2g + 1, 5g + 1, 1, 2g + 1, 2 + g, 9 + 9g, 6 + 6g, 7g + 5, 5, 5g + 5, 7g + 5,

6 + 6g, 1 + 5g, 6g, 9g, 4g, 6g, 1 +5g, 8 + 3g}) = {1 + g, 2g, 2g, 2g, 2g, 0, 0, 1 + g, 1 + g, 1 + g, 5g, 5g, 2 + g, 2 + g, 2g + 1, 2g+ 1, 2g + 1, 2g + 1, 1 + 5g, 1 + 5g, 5, 5g + 5, 5g + 5, 1 + 5g, 8 + 9g, 8 + 9g, 9 + 9g, 7g + 5, 7g + 5, 9g, 6g, 6g, 8 + 3g, 6 + 6g, 6 + 6g} ∈ S. This is the way the level addition operation is performed on S.

Infact it can be easily proved S = {(n-M(⟨Z_m ∪ g⟩), l(+)} is a multiset dual number semigroup under l(+) and is of finite order.

Now if we want to find an identity S we only arrive at {0}, so we call this {0} as a special level pseudo identity of l(+) on S.

Now we proceed onto give examples of multiset dual number semigroup under l(×), the level product.

Example 3.33. Let S = {(6-M(⟨Z_5 ∪ g⟩), l(×)} be the multiset dual number semigroup under level product l(×).

Consider x = {3g, 4g + 1, 2g + 3, 2, 4 + 3g} and y = {2 + 2g, g, 3g, 3 + g, 4 + 2g} ∈ S.

We find l(x × y) = l({3g, 4g + 1, 2g + 3, 2, 4 + 3g} × {2 + 2g, g, 3g, 3 + g, 4 + 2g})

= l({g, 0, 0, 4g, 2g, 2, g, 3g, 3 + 3g, 3g + 4, 1, 3g, 4g, 4 + 4g, 2, 4 + 4g, 2g, g, 1 + 2g, 3 + 4g, 3 + 4g, 2g, 4g, 2 + 3g, 1})

= {g, 0, 0, 4g, 2g, g, 3g, 3g + 3, 2g, 4g, 3g + 4, 1, 3g, 4g, 4 + 4g, 2, 1, 4 + 4g, 2g, g, 1 + 2g, 3 + 4g, 3 + 4g, 2 + 3g} ∈ S.

This is the way level product operation l(×) is performed on S.

Now we show whatever be Z_m, and n we have nilpotents unlike neutrosophic multiset semigroups.

Consider P = {6-M(Z_5g), l(×)} = {collection of all 6-multiplicity multisets from 6-M(Z_5g), under the level product} = {{0}, {g}, {2g}, {3g}, {4g}, {0, 0}, {g, g}, {2g, 2g}, {3g, 3g}, …, {g g, g, g, g, g}, …, {4g, 4g, 4g, 4g, 4g, 4g},…, {0, 0, 0, 0, 0, 0, g, g, g, g, g, g, 2g, 2g, 2g, 2g, 2g, 2g, 3g, 3g, 3g, 3g, 3g, 3g, 4g, 4g, 4g, 4g, 4g}, l(×)}.

We see for any x, y ∈ P, l(x × y) = {0} or {0, 0} or {0, 0, 0} or {0, 0, 0, 0} or {0, 0, 0, 0, 0} or {0, 0, 0, 0, 0, 0}.

We define P to be a special zero square dual number multiset subsemigroup of S.

Infact for every $x \in S$ and for every $y \in P$ we have $l(x \times y)$ is in P. So P is infact a multiset nil ideal of S.

It is easily verified S has plenty of multiset zero divisors. The reader is left with the task of finding the existence of Smarandache multiset zero divisors in S.

However $\{1\} \in S$ serves as the special multiset unit of S. For every $x \in S$, $xl(x)\{1\} = x$. However $\{1, 1\} \in S$ does not act as an identity for all multisets in S. If $x = \{1, 2, 3, 4, 5\} \in S$ then $l(\{x\}, \times \{1, 1\})$

$= l(\{1, 2, 3, 4, 5\} \times \{1,1\})$

$= \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5\} \neq x$. Hence the claim.

Likewise we see $\{0\}$, $\{0, 0\}$ etc. are not zeros. For if $x = \{2, 1, 3, 3, 4\}$ we see if $x \in \{5\text{-}M(Z_5), l(\times)\} = S$ then $l(x \times \{0\})$ $= l(\{2, 1, 3, 3, 4\} \times \{0\})$

$= \{0, 0, 0, 0, 0\}$.

$\{0, 0, 0, 0, 0\}$ is not a zero of S for if $x = \{4, 4, 4\} \in$ So then $l(x \times \{0, 0, 0, 0, 0\}) = l(\{4, 4, 4\} \times \{0, 0, 0, 0, 0\})$

$= \{0, 0, 0, 0, 0, 0\} \neq \{0, 0, 0, 0, 0\}$.

But it is pertinent to keep on record $l(x \times \{0, 0, 0, 0, 0, 0\})$

$= \{0, 0, 0, 0, 0, 0\}$ for all $x \in S$.

Hence we wish to record that the multiset $P = \{6\text{-}M(Z_g g)\} \subseteq S$ is such that $l(x \times y) = \{0\}$ or $\{0, 0\}$ or $\{0, 0, 0\}$ or $\{0, 0, 0, 0, 0\}$, $\{0, 0, 0, 0\}$ or $\{0, 0, 0, 0, 0, 0\}$.

So we define P to be only a pseudo zero square dual number multiset subsemigroup of S.

Inview of all these we have the following theorem.

Theorem 3.15. *Let $S = \{n\text{-}M(\langle Z_m \cup g\rangle), \langle Z_m \cup g\rangle = \{a + bg / a, b \in Z_m$ with $g^2 = 0; 2 \leq m < \infty$ and $2 \leq n < \infty, l(\times)\}$ be the n-mutliplicity multiset of dual numbers.*

S contains a pseudo zero square multiset dual number subsemigroup P such that $l(P \times P) = \{\{0\}, \{0, 0\}, \{0, 0, 0\}, \ldots, \underbrace{\{0,0,0,\ldots,0\}}_{(n-1)times}, \underbrace{\{0,0,\ldots,0\}}_{n\text{-}times}\}$.

Proof. Consider $S = \{n\text{-}M(\langle Z_m \cup g\rangle), l(\times)\}$ the n-multiplicity multiset of dual modulo integers semigroup under level product $l(\times)$.

Recall a pseudo zero square multiset dual number subsemigroup P of S is such that $l(P \times P) = \{\{0\}, \{0, 0\}, \{0, 0, 0\}, \ldots, \underbrace{\{0,0,0,0,0,0,\ldots,0\}}_{n\text{-times}}$.

Clearly if we take $P = \{n\text{-}M(Z_m g), l(\times)\} \subseteq S$ we see $l(P \times P) = \{\{0\}, \{0, 0\}, \ldots, \underbrace{\{0,0,\ldots,0\}}_{n\text{-times}}\}$. Hence the claim.

A natural question would be does this property hold good only in case of n-multiplicity multiset dual noumbers. The answer is not in general true for we can have multiset semigroups which can have pseudo multiset zero square subsemigroups under the level product l(×).

This is illustrated by the following examples.

Example 3.34. Let S = {3-M(Z_9), l(×)} be the multiset semigroup under level product l(×). Let P = {3,3,3,0}, {3, 0, 0, 0}, {3, 3,0, 0,0}, {0}, {3}, {3, 3}, {3, 0}, {3, 0, 0}, {3, 3, 0}, {3, 3, 0, 0}, {3, 3, 3, 0, 0, 0}, {3, 3, 3, 0, 0}, {0, 0}, {3, 3, 3}, {0, 0, 0}} ⊆ S.

Clearly l(P×P) = {{0}, {0, 0}, {0, 0, 0}}. So P is a pseudo multiset zero square subsemigroup of S under the level product l(×).

Let L = {{0}, {0, 0}, {0, 0, 0}, {3}, {6}, {3, 3}, {6,3}, {6, 6}, {3,3, 6}, {6, 6,3}, {6, 6, 6}, {3,3,3}, {3, 0}, {6, 0}, {3,3, 0}, {3,3, 0, 0}, {3, 3,3, 0, 0, 0}, {3, 3,3, 6, 6}, …, {3, 3, 3, 6, 6, 6, 0, 0, 0}} ⊆ S is such that l(L × L) = {{0}, {0, 0}, {0, 0, 0}}.

Take K = {{0}, {6}. {0, 6}, {6, 6}, {0, 0}, {6, 6, 0}, {0, 0, 0}, {0, 6, 0}, {6, 6, 6}, {6, 6, 0, 0}, {6,6, 6, 0}, {0, 0, 0, 6}, {0, 0, 0, 6, 6}, {6, 6, 6, 0, 0}, {6, 6, 6, 0, 0, 0}} ⊆ S is again a pseudo multiset zero square subsemigroup of S under the level product l(×).

In view of all these we have the following theorem.

Theorem 3.15. *Let S = {n-M(Z_{p^2}), l(×), p a prime} be the n-multiplicity multiset semigroup under level product l(×). S has* *pseudo zero square multiset subsemigroups under the level product l(×).*

Proof. Any subset appropriately choosen from the multiset P = {n-M({p, 2p, 3p,…, (p – 1)p, 0}), l(×)} will give way to pseudo zero square subsemigroups of S.

We cannot claim the above result in case of Z_m, m prime or m not of the form p, p a prime.

Example 3.35. Let S = {4-M(Z_{7^2}), l(×)} be the 4-multiplicity multiset semigroup under the level product l(×).

Consider the P = {4-M({0, 7,14, 21, 28,35,42})}. Any 4-multiset with {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0} using P will yield a pseudo multiset zero square subsemigroup of S.

Q_1 = {{0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {7}, {7, 0}, {7, 7}, {7, 7, 0}, {0, 0, 7}, …, {7, 7, 7, 7, 0, 0, 0, 0}} ⊆ P is a pseudo multiset zero square subsemigroup of S. Several such pseudo multiset zero square subsemigroups can be constructed using the set P by selecting elements appropriately.

Next we proceed onto describe n-multiplicity multiset using the special dual like numbers $\langle Z_m \cup h\rangle = \{a + bh / h^2 = h, a, b \in Z_m\}$.

Study in this direction can be done and we may not get pseudo multiset zero square subsemigroups for all m.

However we will describe this new structure by an example or two.

Example 3.36. Let S = {3-M(⟨Z_4 ∪ h⟩), l(+)} be the multiset special dual like semigroup under the level addition.

We see if x = {3h,2h, 2h, h, 0} and y = {h, 2h, 1+ h, 3+ 2h} ∈ S then l(x + y) = l({3h, 2h, 2h, h,0} + {h, 2h, 1+ h, 3 + 2h})
= {0, 3h, 3h, 2h, h, h, 0, 0, 3h, 2h, 1, 1 + 3yh, 1 + 3h, 1 + 2h, 3 + h, 3, 3, 3 + 3h, 3 + 2h})
= {0, 3h, 3h, 3h, 2h, 2h, 0, 0,1 + 3h, 1 + 3h, h, h, 1, 3 + h, 1+ 2h, 3, 3, 3+ 3h, 3 + 2h} ∈ S.

This is the way level addition operation is performed on S.

It is easily verified S is only a multiset special dual like number semigroup under l(+) and is not a monoid.

Interested reader can study the properties of these structures as it is considered as a matter of routine by authors.

Next we proceed onto give one example of multiset special dual like numbers semigroup under the level product l(×).

Example 3.37. Let S = {4-M(⟨Z_5 ∪ h⟩, l(×)} be the multiset special dual like semigroup under the level product l(×).

We just show how the level product is performed on S.

Let x = {2 + h, h, 2h, 3h, h, 0} and y = {1, 2, 1 + h,1, h, 2h, 3h}$\in$ S. l(x$\times$ y) = l({2 + h, h, 2h, 3h, h, 0} $\times$ {1, 2, 1 + h, 1, h, 2h, 3h}) = l({2 + h, h, 2h, 3h, b, 0, 4 + 2h, 2h, 4h, h,2h, 0, 2 + 4h, 2h, 4h, h, 2h, 0, 2+h, h, 2h, 3h, h, 0, 3h, h, 2h, 3h, h, 0, h, 2h, 4h, h, 2h, 0, 4h, 3h, h, 4h,3h, 0})

= {0, 0, 0, 0, 2h, 2h, 2h, 2h, h, h, h, h, 3h, 3h, 3h, 3h, 4h, 4h, 4h, 4h, 2 + h, 2 + h,2 + 4h,4 + 2h} $\in$ S.

This is the way the level product operation l($\times$) is performed on S.

The reader is left with the task of verifying that S under the level product is a multiset semigroup with {1} as its special identity.

Consider B = {4-M(Z_5), l($\times$)} $\subseteq$ S, clearly B is a multiset special dual like number subsemigroup of S under the level product. Infact B is an multiset ideal of S.

Another interesting feature enjoyed by B is that B^2 = B.

In view of all these we have the following result.

Theorem 3.17. *Let S = {n-M(⟨Z_m ∪ g⟩) (or n-M(⟨Z_m ∪ h⟩)), l(×)} be the multiset dual number semigroup (or multiset of special dual like number semigroup) under the level product l(×).*

S has a multiset ideal given by P = {n-M(Z_mg) (or n-M(Z_mh)) l(×)} ⊆ S

Proof is left as an exercise to the reader.

Next we proceed onto study quasi special dual like numbers multisets by some examples.

Example 3.38. Let S = {2-M(⟨Z_6 ∪ k⟩)} where ⟨Z_6 ∪ k⟩ = {a +bk/a, b ∈ Z_6, k^2 = 5k} be the 2-multiplicity special quasi dual like number multiset collection. We show how the operations l(+) and l(×) are defined on S. As S is a collection of multiset it is easy to verify that usual l(∪) and ∩ can be defined on S.

Let x = {3 + 2k, k, 2k, k, 0, 1, 1} and y = {2 + 4k, 2k, 5k, 5k, 0} ∈ S.

We find the level sum of x and y. l(x + y) = l({3 + 2k, k, 2k, k, 0, 1, 1} + {0, 5k, 5k, 2k, 2 + 4k})

= l({3 + 2k, k, 2k, k, 0, 1, 1, 3 +k, 0,k, 0, 5k, 5k + 1,5k + 1, 3 + k, 0, k, 0, 5k, 5k + 1, 5k + 1, 3 + 4k, 3k, 4k, 3k, 2k + 1, 2k + 1, 5, 5k + 1, 2, 2 + 5k, 2 + 4k, 3 + 4k, 3 + 4k})

= {0, 0, 1, 1, k, k, 3 + 2k, 3 + k, 2k, 3k, 3k, 4k, 5k, 5k+1, 5k+1, 4k, 3 + 4k, 2 + 5k, 2 + 5k, 3 + 4k, 2, 2 + 4k, 3 + 4k, 3 + 4k, 5} I

It is easily verified {S,l(+)} is a multiset special quasi dual number semigroup under level addition.

Consider the level product $l(\times)$ on S for the same $x, y \in S$.

$l(x \times y) = l(\{k, k, 1, 1, 0, 2k, 3 + 2k\} \times \{5k, 5k, 0, 2k, 2 + 4k\}$

$= l(\{k, k, 5k, 5k, 0, 2k, 5k, k, k, 5k, 5k, 0, 2k, 5k, 0, 0, 0, 0, 0, 0, 0, 4k, 4k, 2k, 2k, 0, 2k, 2k, 4k, 4k, 2 + 4k, 2+ 4k, 0, 2k, 2k\})$

$= \{0, 0, k, k, 2k, 2k, 5k, 5k, 4k, 4k, 2 + 4k, 2 + 4k\}$ II

Thus S is closed under the level product $l(\times)$, $\{S, l(\times)\}$ is a multiset special quasi dual number semigroup.

We see I and II are distinct so in general

$l(x \times y) \neq \{x + y)$ for any $x, y \in S$.

Finding multiset zero divisors Smarandache multiset zero divisors multiset nilpotents, multiset idempotents and Smarandache multiset idempotents in case of multiset dual numbers multiset special dual like numbers and multiset special quasi dual numbers.

Now we proceed onto describe the multiset of complex neutrosophic modulo numbers.

Recall $C(\langle Z_m \cup I\rangle) = \{a + bi_F + cI + dIi_F)$ a, b, c, $d \in Z_m$; $I^2 = I$, $i_F^2 = m - 1$, $(i_F I)^2 = (m - 1)I\}$ is the complex neutrosophic modulo integers.

Let n-M(C($\langle Z_m \cup I\rangle$)) be the n-multiplicity multiset of complex neutrosophic modulo integers.

Let x = {I, i, I + 2, $5i_F$ + 3, 4I + 3, I, 1 + i_F + I, $2Ii_F$} and y = {0,I, 2I, 2I, 3I + 1, I, 0} $\in$ 2-M(C($\langle Z_6 \cup I\rangle$)).

Now we show how level sum l(+) and level product l(×) are defined on 2-M(C($\langle Z_6 \cup I\rangle$)).

l(x + y) = l({I, I, 2 + Ii_F, $5i_F$ + 3, 4I + 3, 1 + i_F + I + $2Ii_F$} + {0, 0,I, I, 2I, 2I, 3I + 1}) = l({I, I, 2 + Ii_F, $5i_F$ + 3, 4I + 3, i_F + 1 + I + $2Ii_F$, I, I, 2 + Ii_F, $5i_F$ + 3, 4I + 3, 2I, 2I, 2 + I + Ii_F, 2 + I + i_FI, $5i_F$ + 3 + , 2I, 2I, $5i_F$ + 3 + I, 1 + i_F + 2I + $2Ii_F$, 1 + i_F + 2I + $2Ii_F$, 3I, 3I, 3I, 3I, 2 + 2I + Ii_F, 2 + 2I + $2Ii_F$, $5i_F$ + 3 + 2I, $5i_F$ + 3 + 2I, 3, 3, 1 + i_F + 3I + $2Ii_F$, 1 + i_F + 3I + $2Ii_F$, 4I + 1, 4I + 1, 4I + 1, 4I + 1, 3I + 3 + Ii_F, 4 + $5i_F$ + 3I, I + 4, 4I + 2 + i_F + $2Ii_F$})

= {I, I, 2 + Ii_F, 2 + Ii_F, $5i_F$ + 3, $5i_F$ + 3, 4I + 3, 4I + 3, 2I, 2I, i_F + 1 + I + $2i_F$I, i_F + 1 + I + $2i_F$I, 2 + I + i_FI, 2 + I + i_FI, $5i_F$ + 3 + I, $5i_F$ + 3 + I, 1 + i_F + 2I + $2Ii_F$, 1 + i_F + 2I + $2Ii_F$, 3I, 3I, 2 + 2I + $2Ii_F$, 2 + 2I + $2Ii_F$, 3, 3, $5i_F$ + 3 + 2I, $5i_F$ + 3 + 2I, 1 + i_F + 3I + $2Ii_F$, 1 + i_F + 3I + $2Ii_F$, 4I + 1, 4I + 1, I + 4, 4 + $5i_F$ + 3I, 3I + 3 + Ii_F, 4I + 2 + i_F + $2i_F$I}.

Clearly l(x + y) is again a multiset in that collection.

Infact it can be proved that {2-M(C($\langle Z_6 \cup I\rangle$))} under the level addition is a semigroup of finite order, will be known as the complex-neutrosophic multiset semigroup.

Now we will just show how level product is performed on $2\text{-M}(C(\langle Z_6 \cup I\rangle))$.

Now for the same x, y $\in 2\text{-M}(C(\langle Z_6 \cup I\rangle))$ we find $l(x \times y)$.

$l(x \times y) = l(\{I, I, 2 + Ii_F, 3 + 5i_F, 4I + 3, i_F + 1 + I + 2Ii_F\} \times \{0, 0, I, I, 2I, 2I, 3I + 1\})$

$= l(\{0, 0, 0, 0, 0, 0, 0, 0,0, 0, 0, 0, I, I, 2I + 2Ii_F, 3I + 5Ii_F, I, 3Ii_F + 2I, I, I, 2I + 2Ii_F, I, 3I + 5Ii_F, 3Ii_F + 2I, 2I, 2I, 2I + 2Ii_F, 4i_FI, 2I, 4I, 2I, 2I, 2I + 2Ii_F, 4i_FI, 2I, 4I, 4I, 4I, 2, 2+4i_FI, 3 + 3I + 5i_F + 3Ii_F, I + 3, 4I + 1 + 5Ii_F + i_F\})$

$= \{0, 0, I, I, 2I, 2I, 4I, 4I, 4Ii_F, 4i_FI, 3I + 5Ii_F, 3I + 5Ii_F, 2I + 3Ii_F, 2I + 3Ii_F, 2I + 2Ii_F, 2I + 2Ii_F, I + 3, 2 + 4Ii_F, 3 + 3I + 5Ii_F + 3Ii_F, 4I + 1 + 5Ii_F + i_F\})$.

Clearly $l(x \times y) \in 2\text{-M}(C(\langle Z_6 \cup I\rangle))$.

It is easily verified that $\{2\text{-M}(C(\langle Z_6 \cup I\rangle)), l(\times)\}$ is a complex - neutrosophic mod integers multiset semigroup under the level product ×.

The task of finding mutliset idempotents, Smarandache multiset idempotents, multiset zero divisors, complex - neutrosophic multiset Smarandache idempotents etc. are left as an exercise to the reader.

However we briefly give the abstract definition of multiset complex - neutrosophic modulo integer structures in the following.

Definition 3.6. *S = {n-M(C(⟨$Z_m \cup I$⟩)) is defined as the complex - neutrosophic modulo integers n-multiplicity mutliset where $2 \leq n < \infty$, $2 \leq m < \infty$ and ⟨$Z_m \cup I$⟩ = {a + bI/a, b $\in Z_m$, $I^2 = I$} and C(⟨$Z_m \cup I$⟩) = {a + bi_F+ dI + $ci_F I$ / a, b, c, d $\in Z_m$, $i_F^2 = (m - 1)$, $(Ii_F)^2 = I(m - 1)$}.*

We have already given example of it. But we give yet another example.

Example 3.39. Let S = {4-M(C(⟨$Z_{12} \cup I$⟩)), l(×)} be the complex-neutrosophic modulo integer multiset semigroup under level product l(×).

S has multiset zero divisors and Smarandache multiset zero divisors.

Take B = {aI + $ci_F I$ / a, c $\in Z_{12}$, $I^2 = I$, $i_F^2 = 1$, 1, and $i_F I$ = 11I} and let P = {4-M(B), l(×)} $\subseteq$ S. Clearly P is a complex - neutrosophic modulo integer multiset subsemigroup of S. Infact, P is an ideal of S.

Next we proceed onto define the multiset of complex - dual modulo integers first by some examples.

Let N = $C(\langle Z_m \cup g\rangle) = \{a + bg + ci_F + di_Fg / a, b, c, d, \in Z_m; g^2 = 0, i_F^2 = (m-1), (i_Fg)^2 = 0\}$ denotes the complex - dual modulo integers. n-$M(C(\langle Z_m \cup g\rangle))$ denotes the n-multiplicity multiset of complex-dual modulo integers, $2 \leq n < \infty$ and $2 \leq m < \infty$.

Example 3.40. Let S = {3-$M(C\langle Z_4 \cup g\rangle))$ be the 3-multiplicity multiset of complex - dual numbers.

We now show how the level addition and level product operations are performed on S.

Consider x = $\{3g + i_Fg + 1, 2g, g + i_F, 2i_F, 3g, 2g, 2g\}$ and y = $\{0, 0, g + 2gi_F + i_F + 3, 3, 3, g\} \in S$.

We find $l(x + y) = l(\{2g, 2g, 2g, 3g, 2i_F, g + i_F, 1 + 3g, + 2i_Fg\} + \{0, 0, 3, 3, g, 3 + g + i_F + 2g\})$

$= l(\{2g, 2g, 2g, 3g, 2i_F, g + i_F, 1 + 3g + 2i_Fg, 2g, 2g, 2g, 3g, 2i_F, g + i_F, 1 + 3g + 2i_Fg, 2g + 3, 2g + 3, 2g + 3, 3 + 3g, 2i_F + 3, 3 + g + i_F, 3g + 2i_Fg, 2g + 3, 2g + 3, 2g + 3, 3 + 3g, 2i_F + 3, 3 + g + i_F, 3g + 2i_Fg, 3g, 3g, 3g, 0, 2i_F + g, 2g + i_F, 1 + 2i_Fg, 3 + 3g + i_F, 2gi_F, 3 + 3g + i_F + 2gi_F, 3 + 3g + i_F + 2gi_F, 3 + i_F + 2gi_F, 3 + g + 3i_F + 2gi_F, 2g + 2i_F + 2gi_F + 3, i_F\})$

$=\{2g, 2g, 2g, 3g, 3g, 3g, 2i_F, 2i_F, g + i_F, g + i_F, 1 + 3g + 2i_Fg, 1 + 3g + 2i_Fg, 2g + 3, 2g + 3, 2g + 3, 3 + 3g, 2i_F + 3, 3 + g + i_F, 3+ 3g, 2i_F + 3, 3 + g + i_F, 3g + 2i_Fg, 2i_F + g, 2i_Fg + 1, 2g + i_F\} \in S$.

This is the way the level addition operation l(+) is performed on S.

Infact the reader is left with the task of proving {S, l(+)} is the multiset dual complex number semigroup under level addition.

Consider l(x × y) = l({2g, 2g, 2g, 3g, $2i_F$, 2 + i_F, 1 + 3g + $2i_Fg$} × {0, 0, 3, 3, g, g + $2gi_F$ + i_F + 3}) = l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2g, 2g, 2g, g, $2i_F$, 2 + $3i_F$, 3 + g + $2i_Fg$, 2g, 2g, 2g, g, $2i_F$, 2 + $3i_F$, 3 + g + $2i_Fg$, 0, 0, 0, 0, $2gi_F$, 2g + gi_F, g, 0, 0, 0, 0, $2gi_F$, 2g + i_Fg + $2gi_F$ + 1 + i_F, 3 + i_F + $3i_Fg$})

= {0, 0, 0, 2g, 2g, 2g, g, g, g, $2i_F$, $21i_F$, $2gi_F$, $2gi_F$, 2 + $3i_F$, 3 + g + $2gi_F$, 3 + g + $2i_Fg$, 1 + i_F, 3 + i_F + $3i_Fg$, 2g + $3i_Fg$ + 1 + i_F, 2g + gi_F} ∈ S.

Thus it is left as an exercise for the reader to prove that S under the level product l(×) is a multiset dual complex modulo integer semigroup of finite order.

On similar lines we can define multiset special dual like complex numbers n-M(C(⟨Z_m ∪ h⟩)) where C(⟨Z_m ∪ h⟩) = {a + bh + ci_F + dhi_F / a, b, c, d ∈ Z_m, h^2 = h, i_F^2 = m – 1, (hi_F^2) = h(m – 1)}; 2 ≤ m < ∞ and 2 ≤ n < ∞.

Further we can define on n-M(C(⟨Z_m ∪ h⟩)), the operation l(×) or l(+) or l(∪) or ∩ or both l(∪) and ∩ under which the collection will be a multiset semigroup or a lattice respectively.

Next we proceed onto study special Smarandache properties enjoyed by them.

We set S = {n-M(C(⟨$Z_m \cup h$⟩)), l(×)} under level product has multist zero divisors, idempotents and nilpotents mainly depending on the m; $2 \leq m < \infty$.

Further analogous Smarandache structure can be built. Also S has multiset ideals and multiset subsemigroups which are not multiset ideal.

Study in this direction is a matter of routine so left as exercise to the reader.

However we give some examples which are relevant to this context.

Example 3.41. Let S = {5-M(C(⟨$Z_6 \cup g \cup I \cup k$⟩)); l(×)} be the multiset dual, complex, neutrosophic special quasi dual ring of modulo integers where C(⟨$Z_6 \cup g \cup I \cup k$⟩) = {$a_0 + a_1i_F + a_2g + a_3k + a_4I + a_5i_FI + a_6i_Fg + a_7i_FgI + a_8Ig + a_9Ik + a_{10}Ig + a_{11}kgI + a_{12}kgi_F + a_{13}gIi_F + a_{14}kIi_F + a_{15}gIi_Fk$ / $a_i \in Z_6$; $0 \leq i \leq 15$, $i_F^2 = 5$, $I^2 = I$, $g^2 = 0$, $k^2 = 5k$, $(i_Fk)^2 = k$, $(Ii_F)^2 = 5I$, $(kI)^2 = 5Ik$, $(gI)^2 = 0 = (gi_F)^2 = (gk)^2$, $(gi_Fk)^2 = 0$, $(gIi_F)^2 = 0$ $(gkI)^2 = 0$, $(kIi_F)^2 = Ik$} under the level product l(×) is a multiset semigroup.

This multiset complex - neutrosophic dual number special quasi dual number semigroup has multiset ideals, zero divisors, idempotents nilpotents and their multiset Smarandache analogue. Study of their properties is left as an exercise to the reader.

The properties of n-multisets are different from multisets for in case of multisets even if the set on which multisets are constructed are finite still the collection of multisets are infinite.

However in case of n-multisets if the underlying set is finite then the collection of n-multisets are finite, but in case the underlying set is infinite then the n-multisets is also infinite.

Secondly, we see the special product l(×) and the special addition l(+) has no relevance for in case of multisets addition of product need not be leveled, without leveling the closure axiom is true.

Now one may ask can we study perfect n-multiset in which if an element from the underlying set occurs then it shout occur only n-times, not less than n or not greater than n.

We give examples of them.

Example 3.42. Let $\{3\text{-M}(Z_2)\} = S$ be the perfect 3-multiplicity multiset $S = \{\{0, 0, 0\}, \{1, 1, 1\}, \{0, 0, 0, 1, 1,1\}, \{\phi\}\} \cong \{\phi, \{1\}, \{0\}, \{0, 1\}\} \cong \{\phi, \{1, 1\}, \{0, 0\}, \{0, 0, 1,1\}$ and so on $\cong \{\{0, 0, \ldots, 0\}, \{1, 1, 1, \ldots,1\}, \{0, 0, 0, \ldots, 0, 1, 1, \ldots, 1\}, \{\phi\}$.

Thus we see special n-multiset is isomorphic to the Boolean algebra so it does not contribute any special properties.

One may wonder can we define for any n-multiset under l(+) or l(×) the notion of t-multiset subsemigroup under l(+) (or l(×)) where $1 \leq t < n$. We cannot for closure property will not be true. However under both ∪ and ∩ we see the t-multiset subsemigroup, $t < n$.

Finally we see these n-multisets semigroup under l(×) the level product has both zero divisors and Smarandache zero divisors. They have nilpotents and Smarandache nilpotents depending on the basic set on which n-multisets are built. Study in this direction is interesting and we have found conditions for multisets built using Z_m or $\langle Z_m \cup I\rangle$ or $\langle Z_m \cup g\rangle$ or $\langle Z_m \cup h\rangle$ or $\langle Z_m \cup k\rangle$ or $C(Z_m)$ and so on has only depending on m we can have multiset idempotents and multiset nilpotents and their Smarandache analogue.

We have left open several conjectures in case Smarandache elements in n-multiset $M(Z_m)$, $M(\langle Z_m \cup I\rangle)$, $M(\langle Z_m \cup g\rangle)$, $M(C(Z_m))$, $M(C(\langle Z_m \cup k\rangle))$ and so on.

Finally we keep on record that on a n-multiset the level operations do not distribute over each other, that is

$l(a l(\times) (b l(+) c)) \neq l[l(a \times b)\ l(+)\ l(a \times c)]$ for a, b, c $\in$ n-$M(Z_m)$ (or $M(\langle Z_m \cup I\rangle)$ and so on) in general consequent of this we are not in a position define on the n-multisets semiring or ring structure.

All n-multisets under ∪ and ∩ happens to be a lattice. Infact all these n-multiset lattices are Smarandache lattices.

Interested reader can derive more properties about these structures.

Finally we wish to state that as l(×) and l(+) do not distribute over each other it is difficult to built on the multiset l(×) and l(+) simultaneous, however the only algebraic structure possible is multiset lattice that is $\{n\text{-}M(Z_m), \cup, \cap\}$; $2 \leq m < \infty$ or $2 \leq n < \infty$.

Several applications using multisets can be done. However the scope of this book is only to analyse the algebraic properties and study the special Smarandache elements of these multisets.

Further we are interested in studying the Smarandache structure on these elements. This study is both interesting and innovative.

Finally we suggest a few problems for the reader some of which are open conjectures.

Problems

1. Find the 4-multiplicity multiset of X = {1, 2, 5, 7, 9}.

   i) What is the order of 4-M(X)?

   ii) Obtain the lattice associated with the set 4-M(X).

   iii) How many maximal chains are there in L?

   iv) What is the greatest element of L?

   v) Prove L is a Smarandache lattice?

vi) Is L a modular lattice or a distributive lattice?

vii) Obtain any other striking properties related with 4-M(X).

2. Prove 5-M(X) where X = {7, 9, 18, 2, 3, –1} is only a partially ordered set.

   a) Study questions (i) to (vii) of problem (1) for this 5-M(X).

3. Let X = {a_1, a_2, …, a_m} be a set of cardinality m; $1 \leq m < \infty$. Let n-multiplicity multiset of X be denoted by n-M(X); $1 \leq n < \infty$.

   i) Prove if m = 1 and for all n, 1-M(X) is a chain lattice L=1.

   ii) Find the length of the chain lattice L_1?

   iii) If m = 2 then prove 2-M(X) is not a chain lattice.

   iv) Prove 1-M(X) $\subseteq$ 2-M(X).

   v) How many maximal chain are there in the lattice L_2 associated with 2-M(X).

   vi) What is the cardinality of the set 2-M(X)?

   vii) Is the lattice L_2 modular?

   viii) Prove L_2 is a Smarandache lattice.

   ix) Let m = 3 that is 3-M(X) be the 3-multiplicity mutliset of X. L_3 be the lattice associated with L_3. Prove L_1 and L_2 are sublattices of L_3.

x) Find all other sublattices of L_3.

xi) Find all sublattices lf L_3 which are not sublattices of L_2.

xii) Find all sublattices of L_3 which are not sublattices of L_1.

4. Let X = {a_1, a_2, …, a_9} be a set of order 9. 8M(X) be the 8-multiplicity multiset of X.

i) Find o(8-M(X))

ii) Find the lattice L{8-M(X), $\cup$, $\cap$, $\subseteq$}.

iii) Prove L is a Smarandache lattice. S-lattice.

iv) Proe L has a Boolean algebra of order 2^9 in L.

v) Can L have Boolean algebras of lessor order?

vi) Can L have a sublattice which is modular?

vii) Find the largest modular sublattice in L? (if it exists?)

5. Prove S = {5-M(Z_{12}), l(+)} is a semigroup.

a) Find the order of S.

b) Can S be a Smarandache semigroup?

c) Can S have the identity with respect to l(+)?

d) Can {0} be identity of S with respect to l(+)?

e) Can S have multisets which has inverse with respect to l(+)?

f) Obtain any other special feature enjoyed by {S, l(+)}

6. Let M = {5-M(Z_{23}), l(+)} be the five multiplicity multiset of Z_{23}.

i) Study questions (a) to (f) of problem 5.

ii) Compare this M with S of problem 5.

7. Let B = {n-M(Z_m), l(+)} be the n-multiplicity multiset of Z_m. Study questions (a) to (f) of problem (5) for this B.

8. Let L = {18-M(Z_{99})} be the 18-multiplicity multiset of Z_{99}.

i) Prove {L, l(×)} is a semigroup of finite order.

ii) Find all multiset zero divisors of (L, l(×)).

iii) Does the collection of all multiset zero divisors form a subsemigroup under l(×)?

iv) Find all multiset partial zero divisors of L and prove they are not closed under l(×) the level product.

v) Can {L_1 l(×)} have multiset idempotents which are nontrivial?

vi) Prove there exists a multiset idempotent subsemgiroup of order 7 in L.

vii) Find the collection of strict idempotent multiset of L.

9. Let 8-M(Z_{28}) be the 8-multiplicity multiset of Z_{28}.

i) Find all multiset units of 8-M(Z_{28}).

ii) How many multiset partial units of 8-M(Z_{28}) exist?

iii) Is x = {19, 19, 19, 19, 19, 19, 19, 19} and y = {3, 3, 3, 3, 3, 3, 3, 3, 3} ∈ 8-M(Z_{28}) is such that l(x × y) = {1, 1, 1, 1, 1, 1, 1, 1, 1}?

iv) Find all multiset zero divisors of 8-M(Z_{28}).

v) Find all multiset partial zero divisors of 8-M(Z_{28}).

vi) Can 8-M(Z_{28}) have nontrivial multiset nilpotents?

vii) Find all partial multiset nilpotents of 8-M(Z_{28}).

10. Let S = 9-M(Z_{29}) be the 9-multiplicity multiset of Z_{29}.

i) Prove S cannot have multiset zero divisors.

ii) Can S have partial multiset zero divisors?

iii) Find all multiset units of S.

iv) How many partial multiset unit exists in S?

v) Find all multiset idempotents of S.

vi) Can S contain multiset nilpotents?

vii) Is it possible for S to have partial multiset nilpotents?

11. Let S = {3-M(Z_{29}), l(×)} be the 3-multiplicity multiset of Z_{29}.

i) Find all strict multiset units.

ii) Does the strict multiset units of S form a subsemigroup of S?

iii) Find all strict multiset idempotents (nontrivial) of S and show it forms a subsemigroup of S?

iv) Can S have multiset zero divisors? Justify.

v) Can S have multiset partial zero divisors?

vi) Prove S cannot have multiset nilpotents.

12. Let B = {5-M(Z_{18}), l(×)} be the semigroup 5-multiplicity multiset of Z_{18} under level product.

i) Study questions (i) to (vi) of problem (11) for this B.

ii) Compare this B with S of problem 11.

13. What are the special features enjoyed by {n-M(Z_m), l(+)}?

14. Can {n-M(Z_m)m l(+)} be a Smarandache semigroup?

15. Let S ={9-M(Z_{12}), l(×)} be the 9-multiplicity multiset semigroup under the level product l(×).

i) Define Smarandache multiset strict zero divisors in S.

ii) Can S have Smarandache multiset strict zero divisors?

iii) Define in S Smarandache strict multiset idempotents.

iv) Prove S have Smarandache multiset strict idempotents.

v) Compare S-multiset strict idempotents of S with multiset strict idempotents.

vi) Is every S-multiset strict zero divisor a multiset strict zero divisor?

vii) Can we say if Z_{12} has S-zero divisors then S = {9-M(Z_{12}), l(×)} will have S-strict multiset zero divisors?

viii) Prove or disprove if Z_{12} has S-idempotents then S = {9-M(Z_{12}), l(×)} will have S-strict multiset idempotents and vice versa.

16. Study questions (i) to (viii) problem 15 in case of B = {8-M(Z_{27}), l(×)} and compare it with S of problem 15.

17. Study questions (i) to (viii) of problem 15 in case of D = {11-M(Z_{23}), l(×)} and compare it with S of problem 15 and B of problem 16.

18. Do the collection of all S-strict multi idempotents of S in problem 15 form a subsemigroup?

19. Is M={M(Z_p), l(×)} where p is a prime a S n-multiset semigroup?

20. Will S = {12-M(Z_{10}), l(×)} be a S-n-multiset semigroup under the level product l(×)?

21. Will the collection of S-strict multiset idempotents in problems (15), (16) and (17) from a multiset semigroup?

22. Define for M=(n-M(Z_{p^t}) l(×)} (p a prime t > 1) the notion of Smarandache strict multiset nilpotents?

23. Does the collection of all strict multiset nilpotents of M in problem (22) form a semigroup under level product?

24. Can we say (or prove) the collection of all multiset strict nilpotents of M in problem (22) form a Smarandache semigroup under l(×)?

25. Can we say the collection of all Smarandache strict nilpotents of M in problem (22) form a S-semigroup under l(×)?

26. Obtain any other special feature enjoyed by M given in problem (22).

27. Let S = {9-M(Z_{64})} be a 9-multiset semigroup under the level product l(×).

    i) Find all strict multiset nilpotents of S.

    ii) Find all multiset S-strict nilpotents of S.

    iii) Do the collection of multiset S-strict nilpotents form a subsemigroup under l(×) of S?

    iv) Do the collection of all S-strict multiset idempotents of S form a subsemigroup and l(×) of S?

    v) Prove or disprove the collection of all S-strict multiset zero divisors form a subsemigroup of S under l(×).

28. Let S = {5-M(Z_{10} ∪ I⟩), l(×)} be the 5-multiplicity multiset neutrosophic modulo integer semigroup under level product l(×).

    i) Find all multiset zero divisors of S.
    ii) Prove S has a multiset neutrosophic modulo integer ideal.
    iii) How many multiset neutrosophic modulo subsemigroups of S are not ideals?
    iv) Is B = {5-M(Z_{10}), l(×)} a multiset subsemigroup or an ideal? Justify!
    v) Find all multiset neutrosophic modulo integer zero divisors which are S-zero divisors.
    vi) Enlist all multiset neutrosophic zero divisors which are not S-zero divisors.
    vii) Can S have multiset nilpotents? (if yes enumerate them).
    viii) Can S have multiset neutrosophic idempotents?
    ix) Find all multiset neutrosophic Smarandache idempotents of S.
    x) Find all partial multiset zero divisors of S.

29. Let D = {5-M(⟨Z_{11} ∪ I⟩), l(×)} be the 5-mutliplicity multiset neutrosophic modulo integer semigroups under the level product l(×).

i) Study questions (i) to (x) of problem (28) for this D with appropriate changes.

ii) Compare the results of D and S and prove or disprove that prime Z_p (Z_{11}) has no effect on the multiset structures under the level product l(×).

30. Let W = {3-M(⟨$Z_{12} \cup I$⟩)), l(×)} be the multiset complex neutrosophic semigroup under level product l(×).

i) Study questions (i) to (x) of problem (28) for this W with appropriate changes.

ii) Compare W with S of problem (30) and D of problem (29).

iii) Find the cardinality of W.

iv) If T = {3-M(C(Z_{12})), l(×)} $\subseteq$ W what is the cardinality of T.

v) If R = {3-M(⟨$Z_{12} \cup I$⟩), l(×)} what is the cardinality of R.

vi) Which of the structures T or S enjoy more properties likes S-multiset zero divisor, multiset subsemigroups, multiset ideal?

vii) Let Q = {3-M(Z_{12}), l(×)} $\subseteq$ S, what is the cardinality of Q.

31. Prove or disprove the idempotent multiset subsemigroups of S = {n-M(Z_m), l(×)} can never be a Smarandache idempotent multiset subsemigroup of S.

32. Find all the orthogonal multiset idempotents of S = {5-M(Z_9), l(×)}.

33. Can S = {10-M(Z_{43}), l(×)} have S-multiset idempotent?

34. Let S = {6-M($Z_{3^2 \times 2^5 \times 5^2 \times 7}$), l(×)} be the 6-multiplicity multiset semgroup under level product l(×).

    i) Find all multiset idempotents of S.
    ii) Can S have S-multiset idempotents?
    iii) How many of these multiset idempotents are orthogonal?
    iv) Find the algebraic structure enjoyed by the collection of all orthogonal multiset idempotents.
    v) Find the set of all multiset nilpotents of S.
    vi) Find all multiset zero divisors of S.
    vii) How many of these multiset zero divisors are Smarandache multiset zero divisors?
    viii) Find all multiset partial zero divisors of S.
    ix) Can S have multiset nilpotents?
    x) Enumerate all partial multiset nilpotents of S.
    xi) Obtain all the special features enjoyed by S.

35. Find the Smarandache multiset idempotents of S={5-M(Z_{12}), l(×)}? Let M = {6-M(Z_{210}), l(×)} be the 6-multiplicity multiset of Z_{210}.

    i) Find all mutliset idempotents of M.

    ii) Can M have Smarandache multiset idempotents?

    iii) Do the collection of all Smarandache multiset idempotents form a multiset subsemigroup of M?

    iv) Find all multiset idempotents of M which are not S-multiset idempotents.

    v) Obtain any other interesting result associated with the S-multiset idempotents.

36. Let L = {10-M(Z_{23}), l(×)} be the 10-multiplicity multiset semigroup of Z_{23}.

    i) Study questions (i) to (v) of problem 35 for this L

    ii) Compare M of problem 35 with this L.

    iii) Which of the multisets M or L has more number of S-idempotents?

37. Let W = {12-M(Z_{256}), l(×)} be the multiset semigroup under level product l(×). Study questions of (i) to (v) of problem for this W.

38. Let S = {3-MN(C(Z_6)), l(×)} be the complex 3-multiplicity multiset semigroup on level product using C(Z_6).

    i) Find all complex 3-multiset zero divisors of S={3-M(C(Z_6), l(×)}.

    ii) Find all S-complex multiset zero divisors of S.

    iii) How many complex multiset zero divisors of S are not Smarandache multiset zero divisor?
    iv) Find all complex multiset zero divisors of S.
    v) Find all complex multiset nilpotents and partial nilpotents of S.
    vi) Can S have S-complex multiset idempotents?
    vii) Find all complex multiset idempotents of S which are not S-complex multiset idempotents.

39. Let D = {7-M(C(⟨Z_{13} ∪ I⟩)), l(×)} be the complex neutrosophic multiset semigroup under level product l(×). Study questions (i) to (vii) of problem (38) for this D.

40. Let E = {8-M(C(⟨Z_{15} ∪ I⟩)), l(×)} be the complex neutrosophic multisemigroup under level product l(×).

    i) Study questions (i) to (vii) of problem (38) for this E.

    ii) Compare D of problem (39) with E of this problem.

41. Let M = {99-M($\langle Z_{20} \cup g\rangle$), l(×)} be the multiset modulo dual numbers of multiplicity?

    i) Prove M has a special zero square multiset subsemigroup.

    ii) Find order of M.

    iii) Find the multiset subsemigroups of M which are not multiset ideals.

    iv) Can M have multiset zero divisors which are Smarandache zero divisors?

    v) Obtain any other special feature enjoyed by multiset modulo dual number semigroups undr the level product l(×).

    vi) Can these mutliset dual numbers semigroup have multiset nilpotents of order greater than two? Justify your claim?

    vii) Can this multiset dual numbers semigroup contain Smarandache multiset zero divisors?

    viii) Can this multiset dual number semigorup contain multiset idempotents which are Smarandache multiset idempotnts?

    ix) How many multiset nil ideals does the multiset semigroup M contains?

42. Let B = {5-M(C($\langle Z_{19} \cup g\rangle$)), l(×)} be the multiset complex dual modulo integers semigroup under the level product l(×).

i) Find the cardinality of B.

ii) Show the lattice {B, ∪, ∩} is a Smarandache lattice.

iii) Find all the sublattices which are Boolean algebras of {B, ∪, ∩}.

iv) Can B have multiset complex dual modulo integer ideals? Justify.

v) Find all multiset complex dual modulo integer zero divisors of B.

vi) Find all multiset complex dual modulo integer Smarandache zero divisors of B.

vii) Prove B has a multiset dual complex special zero square subsemigroup under the level product l(×).

viii) How many special multiset zero square subsemirings can B have?

ix) Can B be a Smarandache multiset subsemigroup? Justify your claim.

x) Obtain any other special feature associated with mutliset complex dual semigroups under level product l(×).

43. Let S = {8-M(C(⟨Z_{24} ∪ g⟩)), l(×)} be the multiset complex dual modulo integer semigroup under level product l(×).

i) Study questions (i) to (x) of problem (42) for this S.

ii) Compare B of problem (42) with this S.

iii) Which has more number of S-multiset zero divisors B of problem (42) or this S?

44. Let W = {3-M(C(⟨Z_9 ∪ I⟩)), l(×)} be the multiset complex neutrosophic modulo integers semigroup under the level product l(×).

i) Study questions (i) to (x) of problem (42) for this S.

ii) Which question in problem (42) is irrelevant in this case?

iii) Prove there is a multiset neutrosophic complex modulo integer ideal in W under level product l(×).

45. Let M = {4-M(C(⟨Z_{12} ∪ k⟩)), l(×)} be the multiset complex special quasi dual number semigroup under level product l(×).

i) Study questions (i) to (x) of problem (42) for this M.

ii) Compare this M with B of problem (42).

iii) Obtain all special features enjoyed by M.

46. Let N = {3-M(⟨Z_{10} ∪ g⟩)), l(×)}, where ⟨Z_{10} ∪ g⟩ = {a + bI + ig + dgI/a, b, c, d ∈ Z_{10}, $I^2 = I$, $g^2 = 0$, $(Ig)^2 = 0$} be the multiset neutrosophic dual modulo integer semigroup under level product l(×).

i) Study questions (i) to (x) of problem (42) for this N.

ii) Enumerate all special features enjoyed by N.

iii) Compare this N with B of problem (42).

iv) Compare this N with S of problem (43).

v) Compare this N with W of problem (44).

47. Let D = {5-M($\langle Z_{19} \cup g \cup I\rangle$), l(×)} be the multiset dual neutrosophic modulo integer semigroup under the level product l(×), where $\langle Z_9 \cup g \cup I\rangle$ = {a + bg + cI + dgI /a, b, c, d $\in Z_{19}$, $I^2 = I$, $g^2 = 0$, $(gI)^2 = 0$}

i) Study questions (i) to (x) of problem (42) for this D.

ii) Compare this D with N of problem (46).

48. Let E = {4-M(C$\langle Z_{12} \cup g \cup I\rangle$)), l(×)| be the multiset complex dual neutrosophic modulo integer semigroup under level product l(×) where C($\langle Z_{12} \cup g \cup I\rangle$) = {a + bg + cI + di_F + egi_F + hgI + kIi_F + lIi_Fg/a, b, c, d, e, h, k, l $\in$ Z_{12}, $g^2 = 0$, $i_F^2 = 11$, $I^2 = I$, $(gI)^2 = 0$ $(Ii_F)^2 = 11i_F$, $(gi_F)^2 = 0$, $(i_FgI)^2 = 0$}.

i) Study questions (i) to (x) of problem (42) for this E.

ii) Enumerate all special properties associated with this E.

iii) Compare this E with;

a) D of problem 47,

b) N of problem 46,

c) M of problem 45 and

d) W of problem 44

49. Let F = {8-M(C(⟨Z_9 ∪ I ∪ k ∪ g⟩)), l(×)} be the multiset of complex neutrosophic dual special quasi dual modulo integers semigroup under the level product l(×), where C(⟨Z_9 ∪ I ∪ k ∪ g⟩) = {$a_0 + a_1g + a_2k + a_3I + a_4i_F + a_5gK + a_6gI + a_7gi_F + a_8kI + a_9ki_F + a_{10}Ii_F + a_{11}gIk + a_{12}gIi_F + a_{13}gi_F + a_{14}i_FkI + a_{15}i_FgkI$ / $a_i \in Z_9$; $0 \le i \le 15$, $g^2 = 0$, $I^2 = I$, $k^2 = 8k$, $i_F^2 = 8$, $(gI)^2 = 0 = (gk)^2 = (gi_F)^2$ $(Ii_F)^2 = 8I$, $(Ik)^2 = 8Ik$, $(ki_F)^2 = k$, $(gIk)^2 = 0 = (gIi_F)^2 = (gki_F)^2$, $(Iki_F)^2 = kI$, $(i_FgkI)^2 = 0$}

i) Study questions (i) to (x) of problem 42 for this F.

ii) Enumerate all the special features enjoyed by F.

iii) Find the lattice L, {8-M(C(⟨Z_9 ∪ I ∪ g ∪ k⟩)), ∪, ∩}.

a) Is L a distributive lattice?

b) Is L a Smarandache lattice?

c) How many sublattices of L are Boolean algebras?

iv) Prove F ⊇ {8-M(C((⟨Z_9 ∪ I ∪ k⟩)), l(×)} ⊇ {8-M(C(⟨Z_9 ∪ I⟩)), l(×)} ⊇ {8-M(C(Z_9)), l(×)} ⊇ {8-M(Z_9), l(×)} is chain of only multiset subsemigroup of F and not multiset ideals of F.

Chapter Four

MULTISET MATRICES

In this chapter authors for the first time introduce the new notion of multiset matrices. We call a matrix to be a multiset matrix if its entries are from the multisets M(R or Q or Z or C or Z_n or $\langle Z_n \cup I\rangle$ or $\langle R \cup I\rangle$ or $C(Z_n)$) and so on. All these multiset matrices will be of infinite order. If we need to get finite order multiset matrices then it is mandatory, we define n-multiplicity multisets on finite sets which can be subsets. But in that case also we cannot perform level addition or level multiplication we need to take subsets from Z_m or $C(Z_m)$ or $\langle Z_m \cup I\rangle$ or $\langle Z_m \cup g\rangle$ or so on; $2 \leq m < \infty$ and $2 \leq n < \infty$.

We will first illustrate both the situations by some examples.

Example 4.1. Let S = $\{(a_1, a_2, a_3) / a_i \in M(Z); 1 \leq i \leq 3\}$ be the collection of all 1×3 matrices with entries from multiset M(Z) = {All multisets which takes to entries from Z}.

x = ({3, 3, 3, – 1, – 1, 2, 0}, {1, 1, 1, 1, 2, 9}, {9, 9, 9, 18, 0, – 3, – 3, – 5}) $\in$ S; x is defined as the row multiset matrix or multiset row matrix.

The following observations are vitals

i) o(Z) is infinite so M(Z) the multisets built using Z are infinite; hence S is a row matrix multiset of infinite order.

ii) Clearly on S we can define the operation ∪ or ∩ or both simultaneously.

iii) On S we can define + the usual addition {S, +} is a multiset matrix semigroup of infinite order.

iv) On S we can define ×, the product operation under which {S, ×} is a matrix multiset semigroup of infinite order

First we will illustrate these situations.

Let x = ({3, 3, 0, 1, 9, 2, 5, 0}, {2, 2, – 1, –1, 7, 9, 9, 9}, {9, 2, 0, –7, –9, 8, 2, 0}) and y = ({3, 1, –9, 0}, {2, –9, 9}, {0, 6, 9}) ∈ S. We define

x ∩ y = ((3, 3, 0, 1, 9, 2, 5, 0}, {2, 2, –1, –1, 7, 9, 9, 9}, {9, 2, 0, –7, –9, 8, 0, 2}) ∩ ({3, 1, –9, 0}, {2, –9, 9}, {0, 6, 9}) = ({3, 3, 1, 0, 9, 2, 5, 0} ∩ {3, 1, –9, 0}, {2, 2, –1, –1, 7, 9, 9, 9} ∩ {2, –9, 9}, {9, 2, 0, –7, –9, 8, 0, 2} ∩ {0, 6, 9}) = ({3, 1, 0}, {2, 9}, {0, 9}).

Clearly x ∩ y ∈ S. Thus it is left as an exercise for the reader to prove {S, ∩} is an infinite order multiset matrix semigroup under the intersection operation ∩.

We can now show how the '∪' operation is defined on the matrix multiset x ∪ y = ({3, 3, 0, 1, 9, 2, 5, 0}, {2, 2, –1, –1, 7, 9, 9, 9}, {9, 2, 0, –7, –9, 8, 2, 0}) ∪ ({3, 1, –9, 0}0, {2, –9, 9}, {0, 6, 9}) = ({3, 3, 0, 1, 9, 2, 5, 0} $\cup$ {3, 1, –9, 0}, {2, 2, –1, –1, 7, 9, 9,9} $\cap$ {2, –9, 9}, {9, 2, 0, –7, –9, 8, 2, 0} $\cap$ {0, 6, 9}) = ({3, 3, 0, 1, 9, 2, 5, 0} $\cap$ {3, 1, –9, 0}, {2, 2, –1, –1, 7, 9, 9, 9} $\cap$ {2, –9, 9}, {9, 2, 0, –7, –9, 8, 2, 0} $\cup$ {0, 6, 9}) = ({3, 3, 0, 1, 9, 2, 5, 0, 3, 1, –9, 0}, {2, 2, –1, –1, 7, 9, 9, 9, 2, –9, 9}, {9, 2, 0, –7, –9, 8, 2, 0, 0, 6, 9}) $\in$ S.

It is clear that union operation '$\cup$' increases the cardinality of each and every element in the multiset matrix. The situation it will not remain same even if the multiset matrix X is contained in the multiset matrix Y.

Now is a natural question would be what is the containment relation in multiset matrices. We say a multiset matrix X = ({a}, $\{a_2\}$, $\{a_3\}$) is contained in the multiset matrix Y = ($\{b_1\}$, $\{b_2\}$, $\{b_3\}$) if and only if $\{a_1\} \subseteq \{b_1\}$, $\{a_2\} \subseteq \{b_2\}$ and $\{a_3\} \subseteq \{b_3\}$ or in short if $\{a_i\} \subseteq \{b_i\}$; i = 1, 2, 3 and we denote this by $X \subseteq Y$.

So we can say multiset matrices of same order will satisfy the containment relation. So the collection of all multiset matrices of same order satisfy the partial ordering.

Now based on all these observations we make the following result.

Hence we can prove the following result.

Theorem 4.1. *S = {(a_1, …, a_m) / $a_i \in$ M(Z) (or Q or R or C or Z_n $\langle Z_n \cup I\rangle$ and so on); $1 \leq i \leq m$, $\cup$, $\cap$} is a multiset row matrix lattice of infinite order and S is a special Smarandache lattice.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto show how on these multiset row matrices the notion sum of two row matrices is carried out.

Example 4.2. Let S = {(a_1, a_2, a_3, a_4) / $a_i \in$ M(R); $1 \leq i \leq 4$} be the multiset matrix. We now define the operation of addition on S.

Let x = ({3, 0, –1, 2, 4, 0}, {0, 0, 0, 2, 2}, {–1, –1, 0, –1, –1}, {4, 4, 4, 4, 4, 4}) and y = ({2, 2, 0, 1, 1, 2}, {3, 3, 3, 3, 3}, {4, 2, 0, 1, 1, 1}, {–1, –1, –1, –1, –1}) $\in$ S.

We now find the sum of x + y; x + y = ({3, 0, –1, 2, 4, 0}, {0, 0, 0, 2, 2}, {–1, –1, 0, –1, –1}, {4, 4, 4, 4, 4, 4}) + ({2, 2, 0, 1, 1, 2}, {3, 3, 3, 3, 3}, {4, 2, 0, 1, 1, 1}, {–1, –1, –1, –1, –1}) = ({5, 2, 1, 4, 6, 2, 5, 2, 1, 4, 6, 2, 3, 0, –1, 2, 4, 0, 4, 1, 0, 3, 5, 1, 4, 1, 0, 3, 5, 1, 5, 2, 1, 4, 6, 2}, {3, 3, 3, 5, 5, 3, 3, 3, 5, 5, 3, 3, 3, 5, 5, 3, 3, 3, 5, 5, 3, 3, 3, 5, 5}, {3, 3, 4, 3, 3, 1, 1, 0, 1, 1, –1, –1, 0, –1, –1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1}, {3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3}) $\in$ S.

This is the way + operation is performed on S. It is important to note the following.

i) The cardinality of the multisets of the matrices under consideration increases drastically.

ii) The algebraic structure enjoyed by {S, +} is a semigroup of infinite order. We do not have the additive identity, if we want to have the special additive identity then it is the set {0}; as x + {0} = x for all x however we do not have {x} + {–x} = {0} for any x $\in$ S unless the set {x} is a singleton set.

For if {x} = {0, 2, –1, –1, 2} then {–x} = {0, –2, 1, 1, –2}.

Now {x} + (–x} = {0, 2, –1, –1, 2} + {0, –2, 1, 1, –2}

= {0, 2, –1, –1, 2, 0, –2, –3, –3, 0,

1, –1, 0, 0, –1, 1, –1, 0, 0, –1,

–2, 0, –3, –3, 0} ≠ {0}.

However yet another observation is mandatory.

If x = ({a_1}, {a_2}, …., {a_n}) then we associate with each x the multiset cardinality of x. We know x is a 1 × n row matrix x

The multiset cardinality of x is (|{a_1}|, |{a_2}|, …, |{a_n}|).

Thus |{(a_i}| ≥ 0, and |{a_i}| = 0 if and only if {a_i} = {ϕ};

$1 \leq i \leq n$.

We denote the cardinality of the multiset matrix x by $|x|_C$ = (|{a_1}|, |{a_2}|, …, |{a_n}|).

If $|x|_C$ = (|{a_1}|, |{a_2}|, …, |{a_n}|)

and $|y|_C$ = (|{b_1}|, |{b_2}|, |{b_3}|, …, |{b_n}|)

then $|x + y|_C$ = (|{a_1}} × |{b_1}|, |{a_2}| |{b_2}|,

…, |{a_n}| × |{b_n}|).

It can be easily verified that multiset matrix under + is an abelian semigroup of infinite order. Clearly it is not a monoid.

For ({0}, {0}, {0}, {0}) = ({0}) acts the special identity. The elements of S in which the cardinality of every multiset matrix is only one has inverse and none other multiset matrix has inverse. Though it is pertinent to record that if x = ({a_1},

$\{a_2\}$, $\{a_3\}$, $\{a_4\}$) then –x= ($\{–a_1\}$, $\{–a_2\}$, $\{–a_3\}$, $\{–a_4\}$) ∈ S is such that x + (–x) ≠ ({0}, {0}, {0}, {0}).

Let x = ({3, –3, 2}, {–5, –1, –5, –2, –3, –6},

{2, 1, 4, –3}, {4, 3, –4, 2}) ∈ S

–x = ({–3, 3, –2}, {5, 1, 5, 2, 3, 6}, {–2, –1, –4, 3}

{–4, –3, 4, –2}) {x} + {–x} = ({0, –6, –1, 0, 6, 1, –5, 0}, ({0, 4, 0,3, 2, 1, –4, 0, –4, –1, –2, –5, 0, 4, 0, 3, 2, –1, –3, 1,

–3, 0, –1, –4, –2, 2, –2, 1, 0, –3, 1, 5, 1, 4, 3, 0},

{0, –1, 2, –5, 1, 0, 3, –4, –2, –3, 0, –7, 5, 4, 7, 0},

{0, 0, –1, –8, –2, 1, 0, –7, –1, 8, 7, 0, 6, 2, 1, 4}) ≠ ({0}, {0}, {0}, {0}).

Thus we see for every multiset matrix x we have the multiset matrix –x but x + (–x) is not the zero multiset matrix in general.

So in this case of multisets we see all mathematical operation + and × are different. Multisets under + is only a semigroup. No multiset of cardinality greater than one (or greater than or equal two) can have inverse. So property of inverse element is flouted.

Only {0} is the special additive identity. So {R, +} is the group of infinite order under +.

However M(R) the collection of multisets of R is not a group under + is only a semigroup and further –x; x∈ M(R) has no significance. Thus all the arithmetic theory if x + y = {0} then x = –y or –x + x = 0 is not true in general for any multisets.

So the class of multiset matrices also inherit the deviant behaviours of these multisets. We have just seen multiset row matrices and that collection under $\cup$ or $\cap$ or + are just semigroups only.

Now we proceed onto develop and describe the product operation on row matrix multisets by some examples.

Example 4.3. Let B = {(a_1, a_2, a_3) | a_i ∈M(R) (or Z_p or C(Z_p) or Q or Z or C and so on; 1 ≤ i ≤ 3} be the row matrix multiset.

We now illustrate how product operation is performed on B.

Let x = ({3, 1, 0, 5, 5}, {2, 2, 2, 2}, {7, 0, 0, 0, 3, 4, 3}) and y = ({1, 1, 1, 1, 5, –1}, {0, 2, –1, 3}, {4, 4, 4, 4, 4}) ∈ B

We find out x × y = ({3, 1, 0, 5, 5}, {2, 2, 2, 2}, {7, 0, 0, 0, 3, 4, –3}) × ({1, 1, 1, 1, 5, –1}, {0, 2, –1, 3}, {4, 4, 4, 4, 4}) = ({3, 1, 0, 5, 5, 3, 1, 0, 5, 5, 3, 1, 0, 5, 5, 3, 1, 0, 5, 5, 15, 5, 0, 25, 25, –3, –1, 0, –5, –5}, {0, 0, 0, 0, 4, 4, 4, 4, –2, –2, –2, –2, 6, 6, 6, 6}, {28, 0, 0, 0, 12, 16, –12, 28, 0, 0, 0, 28, 0, 0, 0, 12, 16, –12, 28, 0, 0, 0, 12, 16, –12, 28, 0, 0, 0, 12, 16, –12, 12, –16, –12}) ∈ B.

This is the way × operation is performed on matrix multisets. However {B, ×} is a matrix multiset semigroup under ×.

We see ({1}, {1}, {1}) ∈ B acts as x × ({1}, {1}, {1}) = x but ({1}, {1}, {1}) is defined only as a special row matrix identity.

Further it is pertinent to record that we do not have x, y ∈ B such that x × y = ({1}, {1}, {1}) if the order of even one of the multisets is of order greater than one.

Thus these multiset matrices also behave in an odd way under the product operation.

However we can have multiset matrix special divisors.

If x = ({2, 1, 4}, {0, 0, 0}, {0}) and y = {{0, 0, 0, 0, 0}, {10, 9}, {9, 6}) ∈ B then x × y = ({2, 1, 4}, {0, 0, 0}, {0}) × ({0, 0, 0, 0, 0}, {10, 9}, {9, 6}) = ({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0}) which is a special matrix zero so x and y contribute to special zero divisors.

Multiset matrix semigroups are abundant with special zero divisors. Here it is important to observe that there is no zero which is universal for the multiset zeros of any multiset collection M(R) are {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0} and so on; the number of zeros in multisets can also be infinite varying from $1 \leq n \leq \infty$.

With these properties in mind we see in general a row matrix multiset collection is an infinite order multiset matrix semigroup with infinite number of multiset special zero divisors. Study in this direction is both interesting and innovative with special unit element ({1}, {1}, …, {1}). However if cardinality of any of the multisets in a row matrix multiset is more than one then special inverse does not exist.

So it is very unusual to these semigroups of multiset matrix behave in a very deviant way both under × and +.

Next we proceed onto supply examples of multiset column matrices and define operations on them.

Example 4.4. Let D = $\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ / $a_i \in$ M(Z); $1 \leq i \leq 4\}$ be the multiset column matrix (column multiset matrix). We see $\cup$, $\cap$, + and $\times_n$ can be defined where $\times_n$ is the natural product on matrices.

$$\text{Let } x = \begin{bmatrix} \{3,1,0,5\} \\ \{2,2,2,2\} \\ \{0,0,0,0,0,1\} \\ \{4,2,1,5,1\} \end{bmatrix} \text{ and } y = \begin{bmatrix} \{7,0,1,1,1,2\} \\ \{0,0,1,15\} \\ \{1,2,0\} \\ \{0,1,-1\} \end{bmatrix} \in D.$$

We now define the union operation on D.

$$x \cup y = \begin{bmatrix} \{3,1,0,5\} \\ \{2,2,2,2\} \\ \{0,0,0,0,0,1\} \\ \{4,2,1,5,1\} \end{bmatrix} \cup \begin{bmatrix} \{7,0,1,1,1,2\} \\ \{0,0,1,15\} \\ \{1,2,0\} \\ \{0,1,-1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,1,0,5\} & \cup & \{7,0,1,1,1,2\} \\ \{2,2,2,2\} & \cup & \{0,0,1,1,5\} \\ \{0,0,0,0,0,1\} & \cup & \{1,2,0\} \\ \{4,2,1,5,1\} & \cup & \{0,1,-1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,1,0,5,7,0,1,1,1,2\} \\ \{0,0,1,1,5,2,2,2,2\} \\ \{0,0,0,0,0,2,1,1\} \\ \{4,2,1,5,1,0,1,-1\} \end{bmatrix} \in D.$$

This is the way '$\cup$' operation is performed on D. Infact {D, $\cup$} is a semilattice or a semigroup of column matrix multisets which is not a monoid.

Now we proceed onto define '$\cap$' operation on x, y $\in$ D.

$$x \cap y = \begin{bmatrix} \{3,1,0,5\} & \cap & \{7,0,1,1,1,2\} \\ \{2,2,2,2\} & \cap & \{0,0,1,1,5\} \\ \{0,0,0,0,0,1\} & \cap & \{1,2,0\} \\ \{4,2,1,5,1\} & \cap & \{0,-1,1\} \end{bmatrix}$$

$= \begin{bmatrix} \{0,1\} \\ \{\phi\} \\ \{0\} \\ \{1\} \end{bmatrix} \in$ D. This is the way '$\cap$' operation is performed on D. {D, $\cap$} is a multiset column matrix semigroup of infinite order or infact a semilattice of infinite order.

We further can prove {D, $\cup$, $\cap$} is a column matrix multiset lattice which is infinite order.

All these lattices are special Smarandache lattices. This lattice also has column matrix multisets sublattices which are of finite order.

Study in this direction can help in applying it to crystal theory and other applications where the concept involves multisets. As the scope of this book is only to study the algebraic structure on multiset matrices and find the Smarandache special elements in them like nilpotent multiset matrices, zero divisor multiset matrices, idempotent multiset matrices and their Smarandache analogue on multiset matrix semigroups under the natural product $\times_n$ on them.

Now we illustrate this situation by some examples.

Example 4.5. Let P = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right.$ / $a_i \in$ M (Z (or Z_m or R or Q or C)); $1 \leq i \leq 5\}$ be the multiset column matrix collection.

We show how the addition operation on the multiset column matrix P is performed.

$$\text{Let } x = \begin{bmatrix} \{3,3,3,0,1,2\} \\ \{4,4,0\} \\ \{5,5,1,2\} \\ \{8,9,9,9,9,9\} \\ \{7,0,-1,2,3\} \end{bmatrix} \text{ and } y = \begin{bmatrix} \{8,-3,-3,-1\} \\ \{-4,-4,-4,-4,-9\} \\ \{-1,-2,5,0\} \\ \{-1,0,0,0,-1\} \\ \{2,1,3,0\} \end{bmatrix} \in P.$$

We first find x + y

$$x + y \begin{bmatrix} \{3,3,3,0,1,2\} \\ \{4,4,0\} \\ \{5,5,1,2\} \\ \{8,9,9,9,9\} \\ \{7,0,-1,2,3\} \end{bmatrix} + \begin{bmatrix} \{8,-3,-3,-1\} \\ \{-4,-4,-4,-4,-9\} \\ \{-1,-2,5,0\} \\ \{-1,0,0,0,-1\} \\ \{1,2,3,0\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,3,3,0,1,2\}+\{8,-3,-3,-1\} \\ \{4,4,0\}+\{-4,-4,-4,-9\} \\ \{5,5,1,2\}+\{-1,2,5,0\} \\ \{8,9,9,9,9\}+\{-1,0,0,0,-1\} \\ \{7,0,-1,2,3\}+\{1,2,3,0\} \end{bmatrix}$$

$$= \begin{bmatrix} \{11,11,11,8,9,10,0,0,0,-3,-2,-1, \\ \quad 0,0,0,-3,-2,-1,2,2,2,-1,0,1\} \\ \{0,0,-4,0,0,-4,0,0,-4,0,0,-4,-5,-5,-9\} \\ \{4,4,0,1,7,7,3,4,10,10,6,7,5,5,1,2\} \\ \{7,8,8,8,8,8,9,9,9,9,8,9,9,9,9,8,9,9,9,9, \\ \quad 7,8,8,8,8\} \\ \{8,1,0,3,4,9,2,1,4,5,10,3,2,5,6,7,0,-1,2,3\} \end{bmatrix} \in P.$$

It is easily verified that {P, +} is a multiset column matrix semigroup of infinite order which is commutative.

Clearly {P, +} is a special monoid with a special additive identity $\{0\} = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

We call this as special identity for $x = \begin{bmatrix} \{a_1\} \\ \{a_2\} \\ \{a_3\} \\ \{a_4\} \\ \{a_5\} \end{bmatrix}$ then

$- x = \begin{bmatrix} -\{a_1\} \\ -\{a_2\} \\ -\{a_3\} \\ -\{a_4\} \\ -\{a_5\} \end{bmatrix} \in P$. We see $x + (- x) \neq \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

Recall if $\{a_i\} = \{-5, 6, 2, 3, -1, 0, 0, -1, -5, 6, 2, -1, -1, 1\}$ is in P then $-\{a_i\} = \{-a_i\} = \{5, -6, -2, -3, 1, 0, 0, 1, 5, -6, -2, 1, 1, -1\}$) P; however $\{a_i\} + \{-a_i\} \neq \{0\}$.

Now one can easily prove this situation to be true in case of column multiset matrix also.

We can find column multiset matrix collection which are subsemigroups under the plus operation; '+'.

Now we proceed onto define the order of any x multiset column matrix of M under the operation '+'

$$\text{Let } x = \begin{bmatrix} \{9,-10,2,3,0\} \\ \{1,1,1,1\} \\ \{2,2\} \\ \{5,-5,5\} \\ \{2,5,2,0,1\} \end{bmatrix} \in M. \text{ We first find } x + x;$$

$$x + x = \begin{bmatrix} \{18,-1,11,12,-1,20,-8,-7,-10 \\ 11,-8,4,5,2,12-7,5,6,3,9,-10,2,3,0\} \\ \{2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\} \\ \{4,4,4,4\} \\ \{10,0,10,0,-10,0,10,0,10\} \\ \{4,7,4,2,3,7,10,7,5,6,4,7,4,0,2,3,6,3,0,2\} \end{bmatrix} \in M.$$

We see as we add in m times the cardinality proved of the multisets in x increases exponentially. It can be easily proved as $\underbrace{x + \ldots + x}_{n-\text{times}}$ as $n \to \infty$ makes the cardinality of each multiset in Σx to have infinite number of times as infinite.

$$\text{However } x + (-x) \neq \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}.$$

$$\text{For } x + (-x) = \begin{bmatrix} \{0,-19,-7,-6,-9,19,0,12,13,10,7,-12, \\ 0,1,-2,6,-13,-1,0,-3,9,-10,2,3,0\} \\ \{0,\ 0,\ 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\} \\ \{0,0,0,0\} \\ \{0,-10,0,10,0,10,0,-10,0\} \\ \{0,3,0,-2,-1,-3,0,-3,-5,-4,0,3,0,-2 \\ -1,\ 2,5,2,0,1,1,4,1,-1,0\} \end{bmatrix}$$

$$\neq \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \text{ or any multiset with only zeros.}$$

We call zero multisets as {0}, {0,0}, {0, 0, 0}, {0, 0, 0,0}, …, {0, 0, 0, …,0} and so on.

Now one of the natural question is can we have multiset matrices x such that x + (–x) is a multiset matrix with zero multisets. The answer is we can have so which we will illustrate by some examples.

$$\text{Let } x = \begin{bmatrix} \{5,5,5\} \\ \{6,6,6,6,6\} \\ \{9,9,9,9\} \\ \{-2,-2,-2\} \\ \{-1,-1\} \end{bmatrix} \text{ we find } x + (-x);\ x + (-x)$$

$$= \begin{bmatrix} \{0,0,0,0,0,0,0,0,0\} \\ \{0,0,0,0,0,0,0,0,0,0,0,0,0, \\ 0,0,0,0,0,0,0,0,0,0,0,0\} \\ \{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\} \\ \{0,0,0,0,0,0,0,0,0\} \\ \{0,0,0,0\} \end{bmatrix}$$ is a zero multiset column matrix.

So we can have x + (–x) to be a zero multiset matrix if and only if every multiset in the multiset matrix is just a single element from Z(or Q or C or R or Z_n) which is repeated more than once. If it is not repeated and a multiset has only one element in the multiset matrix then we get x + (–x) is the special zero multiset matrix.

In view of all these we have the following theorem.

Theorem. 4.2. *Let M =* $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \right.$ */* $a_i \in$ *M(Z) (Z or C or Q or R or* Z_n*)* $1 \le i \le n$*, +} be the multiset column matrix semigroup under* + .

i) *For any x* $\in$ *M in general x + (–x) is not a multiset zero matrix.*

ii) *For* $x \in M$ *where* $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $a_i \in M;\ 1 \leq i \leq n$ *is such that* $x + (-x)$ *is a multiset zero matrix only if every* a_i *in* x *is a multiset where only one element from Z(or Q or* Z_m *or R or C) repeats itself.*

iii) *If* $x = \begin{bmatrix} \{x_1\} \\ \{x_2\} \\ \vdots \\ \{0\} \end{bmatrix}$

iv) *If ({0}) is a multiset zero column matrix such that every entry in ({0}) is a zero multiset with cardinality* t_i^2; $1 \leq t_i < \infty$, $1 \leq i \leq n$ *then corresponding to each ({0}) multizero matrix with this property there are infinitely many*

multiset matrix $x = \begin{bmatrix} \{a_1\} \\ \{a_2\} \\ \vdots \\ \{a_n\} \end{bmatrix}$ *such that each* $\{a_i\}$ *ha the same element in the multiset repeated* t_i *times,* $1 \leq i \leq n$.

Proof is left as an exercise to the reader.

However we give one example for the condition (iv) of the theorem with n = 4 and entries from Z so as to enable the reader to understand this notion.

Let ({0}) = $\begin{bmatrix} \{0,0,0,0\} \\ \{0,0,0,0,0,0,0,0,0\} \\ \underbrace{\{0,0,0,0,0,...\}}_{25\,times} \\ \underbrace{\{0,0,...,0\}}_{64\,times} \end{bmatrix}$ $\in$ B = { $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ / $a_i \in$ M(Z); $1 \leq i \leq 4$}.

Now this zero multiset matrix has $t_1^2 = 2^2$, $t_2^2 = 3^2$, $t_3^2 = 5^2$ and $t_4^2 = 8^2$ to number of zeros in the multisets in ({0}) matrix or to be more elaborative first multiset has 4 zeros, second multiset has 9 zeros, third multiset has 25 zeros and the fourth multiset has 64 zero which comprises the zero multiset matrix ({0}).

Now we can have any multiset matrix

$$y = \begin{bmatrix} \{r_1, r_1\} \\ \{r_2, r_2, r_2\} \\ \{r_3, r_3, r_3, r_3, r_3\} \\ \{r_4, r_4, r_4, r_4, r_4, r_4, r_4, r_4\} \end{bmatrix} \in B, \text{ where } r_i \in Z; 1 \leq i \leq 4.$$

It can be verified.

$$(y) + (-y) = \begin{bmatrix} \{0,0,0,0\} \\ \{0,0,0,0,0,0,0,0,0\} \\ \underbrace{\{0,0,0,0,0,...\}}_{25\,times} \\ \underbrace{\{0,0,...,0\}}_{64\,times} \end{bmatrix}.$$

It is left as an exercise to prove that for $r_i \in Z$ are infinitely many, so we can say for a given zero multiset matrix we can have infinitely many multiset matrices (y) of the said from such that (y) + (–y) = ({0}).

Hence the claim.

Now we illustrate by an example the natural product operation $\times_n$ on the multiset column matrices B = { $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ /$a_i \in$ M(Z or C or Q or R or Z_n); $1 \leq i \leq 4$}.

Let x = $\begin{bmatrix} \{3,0,-1,-1,-1,0\} \\ \{2,2,2,2\} \\ \{-1,-1,-1\} \\ \{9,9,9,9,9\} \end{bmatrix}$ and y = $\begin{bmatrix} \{-1,-1,9\} \\ \{-9,-9\} \\ \{-1,0,0\} \\ \{1,1,0,0,0,0\} \end{bmatrix}$ $\in$ B.

$$x \times_n y = \begin{bmatrix} \{-3,0,1,1,1,0,-3,1,1,1,1, \\ 0,+27,0,-27,-27,-27,0\} \\ \{-18,-18,-18,-18,-18,-18,-18,-18\} \\ \{+1,+1,+1,0,0,0,0,0,0\} \\ \{9,9,9,9,9,9,9,9,9,9,0,0,0,0,0,0,0,0,0,0,0,0, \\ 0,0,0,0,0,0,0,0\} \end{bmatrix}.$$

This is the way natural product operation $\times_n$ is performed on multiset column matrices.

It is pertinent to keep on record for any given multiset matrix $x \in$ B. We cannot find the inverse of B. However we call the multiset column matrix ({1}) = $\begin{bmatrix} \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \end{bmatrix}$ to be only a special

unit and other multiset matrices ({x}) = $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ where a_i take only

repeated value 1 as a quasi special multiset units.

However quasi special multiset units cannot act as multiset matrix identity.

For if x is a multiset column matrix and y is a multiset quasi special unit in B then x $\times_n$ y ≠ x; however

$$x \times (\{1\}) = x \times_n \begin{bmatrix} \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \end{bmatrix} = x.$$

We can have the concept of multiset special zero divisors

however ({0}) = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ does not serve as ({0}) multiset matrix

as ({0}) × x ≠ {0}.

All these can be illustrated by examples. Thus B is an abelian multiset column matrix semigroups under the natural product $\times_n$. It is important note B is not a monoid. However only depending on the set under consideration we can show whether B is a Smarandache semigroup or not using only

special multiset matrix unit ({1}) = $\begin{bmatrix} \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \end{bmatrix}$.

Further we wish to state that this multiset matrix semigroup cannot have idempotents. Every element $x \in B$ is torsion free in fact $\underbrace{x \times_n x \times_n \ldots \times_n x}_{m-\text{times}}$ as $m \to \infty$ leads to the resultant being a multiset matrix in which the resultant has the cardinality to be infinite in every multiset of the matrix. However as every multiset matrix is torsion free the question of getting finite multiset matrix subsemigroups does not arise.

Once a multiset in the multiset matrix has cardinality greater than one working with it under + or × only makes the cardinality increase each time.

If $x \in M(Z)$ and if number of elements in x is 20 say then x + x has cardinality 400.

x + x + x has cardinality 20 × 20 × 20 this $\underbrace{x + x + \ldots + x}_{t-\text{times}}$ gives the cardinality of $\left|\underbrace{x + x + \ldots x}_{t-\text{times}}\right| = 20^t$.

Now we just illustrate this situation by an example.

$x = \{2, 0, 2, 2, 3, 3\} \in M(Z)$ $|x| = 6$, we find $x + x$;

$x + x = \{2, 0, 2, 2, 3, 3\} + \{2, 0, 2, 2, 3, 3\} = \{4, 2, 4, 4, 5, 5, 0, 2, 2, 2, 3, 3, 4, 2, 4, 4, 5, 5, 4, 2, 4, 4, 5, 5, 5, 3, 5, 5, 6, 6, 5, 3, 5, 5, 6, 6\} \in M(Z)$ I

Clearly number of elements in $x + x$ is $o(x + x) = |x + x| = 36 = 6^2$.

Hence our claim.

Consider $x = \{2, 2, 2, 0, 3, 3\} \in M(Z)$. $x \times x = x^2 = \{2, 2, 2, 0, 3, 3\} \times \{2, 2, 2, 0, 3, 3\} = \{4, 4, 4, 0, 6, 6, 4, 4, 4, 0, 6, 6, 4, 4, 4, 0, 6, 6, 0, 0, 0, 0, 0, 0, 6, 6, 6, 0, 9, 9\}$ II

$o(x) = 6$ and $0(x^2) = 6^2 = 36$.

Clearly I and II are distinct however $o(x^2) = o(x + x) = 36 = 6^2$.

One can thus say any $x \in M(Z)$ is such that $x + \ldots + x$ as the number of times addition is performed increases the cardinality of the sum reaches exponentially larges hence the sum of infinite terms reaches infinity.

Similar result holds good for product $x \times x \times \ldots \times x = x^t$ as $t \to \infty$; $|x^t| \to \infty$.

Every element in M(Z) under both + and × are torsion free.

Thus in the multiset matrices when we perform infinite addition or infinite natural product.

We get every multiset in that multiset matrix is of infinite cardinality.

Thus all multisets in M(Z or Q or R or C) are torsion free under + and ×. Similarly all multiset matrices with entries from M(Z or Q or R or C) are torsion free under + and $\times_n$.

Thus getting idempotents or nilpotents or zero divisors is not possible.

Further we can have only S = {M(Z), ∪, ∩} to be a lattice as we have partial order defined on multisets M(Z). Finally it is important to record that a × (b + c) ≠ a × b + a × c for a, b, c ∈ M(Z).

Let a = {3, 3, 3, 4}, b = {2, 2, 2, 0} and c = {1, 1,1 } ∈ M(Z); a × (b + c) = a × ({2, 2, 2, 0} + {1, 1, 1})

= a × ({3, 3, 3, 1, 3, 3, 3, 1, 3, 3, 3, 1})

= {3, 3, 3, 4} × {3, 3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1}

= {9, 9, 9, 9, 9, 9, 9, 9, 9, 3, 3, 9, 9, 9, 9, 9, 9,9, 9, 9, 3, 3, 3, 9, 9, 9, 9, 9, 9, 9, 9, 9, 3, 3, 3, 12, 12, 12, 12, 12, 12, 12, 12, 12, 4, 4, 4} I

Consider a × b + a × c = {3, 3, 3, 4} × {2, 2, 2, 0} + {3, 3, 3, 4} × {1, 1, 1} = {6, 6, 6, 0, 6, 6, 6, 0, 6, 6, 6, 0, 8, 8, 8, 0} + {3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4} = {9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 11, 11, 11, 3, 9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 9, 3, 11, 11, 11, 3, 9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 3, 11, 11, 11, 3, 9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 3, 11, 11, 11, 3, 9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 3, 9, 9, 9, 3, 11, 11, 3, … 10, 10, 10, 0, 10, 10, 10, 0, 10, 10, 10, 0, 12, 12, 12, 0} II

Clearly I and II are distinct, hence distributive law is not true in case of mutlisets in general.

Let $x = \begin{bmatrix} \{3,1,1\} & \{0,0,2\} \\ \{1,2,3\} & \{4,5,5,5\} \end{bmatrix}$ and

$y = \begin{bmatrix} \{1,0,0,0\} & \{2,2\} \\ \{5\} & \{0,1\} \end{bmatrix} \in M = \{\begin{bmatrix} \{a_1\} & \{a_2\} \\ \{a_3\} & \{a_4\} \end{bmatrix}$ / $a_i \in M(Z)$;

$1 \leq i \leq 4\}$.

$$x \times y = \begin{bmatrix} \{3,1,1\} & \{0,0,2\} \\ \{1,2,3\} & \{4,5,5,5\} \end{bmatrix} \times \begin{bmatrix} \{1,0,0,0\} & \{2,2\} \\ \{5\} & \{0,1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,1,1\}\times\{1,0,0,0\}+\{0,0,2\}\times\{5\} & \{1,1,1\}\times\{2,2\} \\ \{1,2,3\}\times\{1,0,0,0\}+\{4,5,5,5\}\times\{5\} & \begin{matrix}\{1,2,3\}\times\{2,2\} \\ +\{4,5,5,5\}\times\{0,1\}\end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,0,0,0,1,0,0,0,1,0,0,0\} & \{2,2,2,2,2,2\}+ \\ +\{0,0,10\} & \{0,0,0,0,0,2\} \\ \{1,0,0,0,2,0,0,0,3,0,0,0\} & \{2,4,6,2,4,6\}+ \\ +\{20,25,25,25\} & (0,0,0,0,4,5,5,5\} \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} \{3,0,0,0,1,0,0,0,1,2,0,0\} \\ 3,0,0,0,1,0,0,0,1,0,0,0, \\ 13,10,10,10,11,10, \\ 10,10,11,10,10,10\} \end{matrix} & \begin{matrix} \{2,2,2,2,2,2,2,2,2,2, \\ 2,2,2,2,2,2,2,2, \\ 2,2,2,2,2,2,2,2,2, \\ 2,2,2,4,4,4,4,4,4\} \end{matrix} \\ \begin{matrix} \{21,20,20,20,22,20,20,20, \\ 23,20,20,20,26,25,25,25, \\ 27,25,25,25,28,25,25,25, \\ 26,25,25,25,27,25,25,25, \\ 28,25,25,25,26,25,25,25, \\ 28,25,25,25,27,25,25,25\} \end{matrix} & \begin{matrix} \{2,4,6,2,4,6,2,4,6,2,4,6 \\ 2,4,6,2,4,6,2,4,6,2,4,6, \\ 6,8,10,6,7,8,10,7,9,11, \\ 7,9,11,7,9,11,7,9,11,7, \\ 9,11,7,9,11\} \end{matrix} \end{bmatrix}$$

I

Consider the natural product $\times_n$ of x with y

$$x \times_n y = \left[\begin{bmatrix} \{3,1,1\} & \{0,0,2\} \\ \{1,1,3\} & \{4,5,5,5\} \end{bmatrix} \times_n \begin{bmatrix} \{1,0,0,0\} & \{2,2\} \\ \{5\} & \{0,1\} \end{bmatrix}\right]$$

$$\begin{bmatrix} \{3,1,1,0,0,0,0,0 \\ 0,0,0,0\} & \{0,0,4,0,0,4\} \\ \{5,10,15\} & \{0,0,0,0,4,5,5,5\} \end{bmatrix} \quad \text{II}$$

Clearly I and II are distinct hence we see $x \times y \neq x \times_n y$ in general.

Now we show how the natural product $\times_n$ is performed on $m \times t$ matrices $m \neq t$ by some examples.

Example 4.6. Let $A = \begin{bmatrix} \{3,0,-1,0,0,2,1\} & \{9,9,9,9,9\} \\ \{3,2,1\} & \{1,1,1\} \\ \{4,4,0\} & \{0,0,4,4\} \end{bmatrix}$ and

$B = \begin{bmatrix} \{1,0,0,0\} & \{4,4,0,0\} \\ \{1,1,1,0,1,9\} & \{9,9,9,-9,-9\} \\ [2,-1,-1,0\} & \{-1,-1,-1\} \end{bmatrix}$ be two 3×2 multiset matrices with entries from M(Z).

We now find $A \times_n B$;

$$A \times_n B = \begin{bmatrix} \{3,0,-1,0,0,2,1\} & \{9,9,9,9,9\} \\ \{3,2,1\} & \{1,1,1\} \\ \{4,4,0\} & \{0,0,4,4\} \end{bmatrix} \times_n$$

$$\begin{bmatrix} \{1,0,0,0\} & \{4,4,0,0\} \\ \{1,1,1,10,9\} & \{9,9,9,-9,-9\} \\ \{2,0,-1,-1\} & \{-1,-1,-1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,0,-1,0,0,2,1, & \{36,36,36,36,36,36, \\ 0,0,0,0,0,0,0, & 36,36,36,36,0,0,0,0, \\ 0,0,0,0,0,0,0\} & 0,0,0,0,0,0\} \\ \{3,3,3,3,0,27,2,2,2, & \{9,9,9,-9,-9,9,9,9, \\ 2,0,8,1,1,1,1,0,9\} & -9,-9,9,9,9,-9,-9\} \\ \{8,0,-4,-4,8,0,-4,-4 & \{0,0,-4,-4,0,0,-4,-4, \\ 0,0,0,0\} & 0,0,-4,-4\} \end{bmatrix}$$ is again a

3 × 2 multiset matrix.

This is the way the natural product on multiset matrices are carried out.

Clearly the collection of all multiset m × t matrices under natural product $\times_n$ is a semigroup.

We now give some more examples of them in the following.

Example 4.7. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right.$ / $a_i \in M(Z_6)$; $1 \leq i \leq 12\}$ be the collection of all 3 × 4 multiset matrices. We just illustrate how operations $\cap$, $\cup$, + and $\times_n$ are performed on W.

Consider x, y $\in$ W where x =

$$\begin{bmatrix} \{3,4,0,0\} & \{0,0,0,0,0\} & \{1,1\} & \{4\} \\ \{2,2,2\} & \{3,3\} & \{1,0,1,1\} & \{2,2,2\} \\ \{4,2,1,1\} & \{5\} & \{0,0,0\} & \{0,1,2\} \end{bmatrix}$$ and

$$y = \begin{bmatrix} \{2,2,2\} & \{1,2,3,4\} & \{3,5\} & \{0,0,1\} \\ \{4,1,1\} & \{2,0\} & \{0,1,1\} & \{1,1,5\} \\ \{2\} & \{3,0,1\} & \{0\} & \{1,1,1\} \end{bmatrix} \in W.$$

We first find x ∪ Y;

$$x \cup y = \begin{bmatrix} \begin{matrix}\{3,4,0,0\} \\ \cup\{2,2,2\}\end{matrix} & \begin{matrix}\{0,0,0,0,0\} \\ \cup\{1,2,3,4\}\end{matrix} & \{1,1\} \cup \{3,5\} & \begin{matrix}\{4\} \cup \\ \{0,0,1\}\end{matrix} \\ \begin{matrix}\{2,2,2\} \\ \cup\{4,1,1\}\end{matrix} & \{3,3\} \cup \{2,0\} & \begin{matrix}\{1,1,1,0\} \cup \\ \{0,1,1\}\end{matrix} & \begin{matrix}\{2,2,2\} \cup \\ \{1,1,5\}\end{matrix} \\ \begin{matrix}\{4,2,1,1\} \cup \\ \{2\}\end{matrix} & \{5\} \cup \{3,0,1\} & \begin{matrix}\{0,0,0\} \cup \\ \{0\}\end{matrix} & \begin{matrix}\{0,1,2\} \cup \\ \{1,1,1\}\end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix}\{3,4,0,0, \\ 2,2,2\}\end{matrix} & \begin{matrix}\{0,0,0,0, \\ 1,2,3,4\}\end{matrix} & \{1,1,3,5\} & \{4,0,0,1\} \\ \{2,2,2,4,1,1\} & \{3,3,2,0\} & \{1,1,1,0,0,1,1\} & \{2,2,2,1,1,5\} \\ \{2,2,4,1,1,1\} & \{5,3,0,1\} & \{0,0,0,0\} & \{1,1,1,1,0,2\} \end{bmatrix}.$$

We see x ∪ y ∈ W and infact {W, ∪} is a multiset matrix semi-lattice or semigroup of infinite order.

The repetition of elements in these multisets are only from Z_6. However the elements repeat themselves infinitely. Next for the same pair x, y ∈ W we show the operation of ∩ is performed.

x ∩ y =

$$\begin{bmatrix} \{3,4,0,0\}\cap & \{0,0,0,0,0\} & \{1,1\}\cap & \{4\}\cap \\ 2,2,2\} & \cap 1,2,3,4\} & \{3,5\} & \{0,0,1\} \\ \{2,2,2\}\cap & \{3,3\}\cap & \{1,1,1,0\}\cap & \{2,2,2\}\cap \\ \{4,1,1\} & \{2,0\} & \{0,1,1\} & \{1,1,5\} \\ \{4,2,1,1\}\cap\{2\} & \{5\}\cap & \{0,0,0,0\}\cap & \{0,1,2\}\cap \\ & \{3,0,1\} & \{0\} & \{1,1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \end{bmatrix} \in W.$$

It is easily verified {W, ∩} is a multiset matrix semilattice or semigroup of infinite order. Further it is vital to observe that {W, ∪, ∩} is a lattice multiset matrix of infinite order.

Now for the same pair, x, y we find the sum of x and y.

$$x + y = \begin{bmatrix} \{3,4,0,0\}+ & \{0,0,0,0\} & \{1,1\}+ & \{4\}+ \\ \{2,2,2\} & +\{1,2,3,4\} & \{3,5\} & \{0,0,1\} \\ \{2,2,2\}+ & \{3,3\}+ & \{1,1,1,0\} & \{2,2,2\} \\ \{4,1,1\} & \{2,0\} & +\{0,1,1\} & +\{1,1,5\} \\ \{4,2,1,1\}+ & \{5\}+ & \{0,0,0\} & \{0,1,2\} \\ \{2\} & \{3,0,1\} & +\{0\} & +\{1,1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,0,2,2,5,0, 2,2,5,0,2,2\} & \{1,1,1,1,1, 2,2,2,2,2, 3,3,3,3,4, 4,4,4,4\} & \{4,0,4,0\} & \{4,4,5\} \\ \{0,0,0,3,3, 3,3,3,3\} & \{5,5,3,3\} & \{1,1,1,0, 2,2,2,1,2, 2,2,1\} & (3,3,3,3, 3,3,3, 1,1,1\} \\ \{0,4,3,3\} & \{2,5,0\} & \{0,0,0\} & \{1,2,3,1, 2,31,2,3\} \end{bmatrix}$$

x + y ∈ W.

Infact {W, +} is a multiset matrix semigroup of infinite order. We see {W, +} is not a monoid.

Further we can have elements in W such that x + x = ({0} or {0, 0}, {0, 0, 0}, …, or {0, …, 0}).

We will first illustrate this situation by an example.

$$\text{Let } x = \begin{bmatrix} \{3\} & \{3,3\} & \{3,3,3\} & \{3,3,3,3,3\} \\ \{3,3,3,3\} & \{3\} & \{3,3\} & \{3,3,3,3,3,3,3\} \\ \{3,3,3\} & \{3,3,3,3\} & \{3\} & \{3,3,3,3\} \end{bmatrix}$$

∈ W

$$x + x = \begin{bmatrix} \{0\} & \{0,0,0,0\} & \begin{matrix}\{0,0,0,0,\\ 0,0,0,0,0\}\end{matrix} \\ \begin{matrix}\{0,0,0,0,0,0,\\ 0,0,0,0,0,0,0,\\ 0,0,0\}\end{matrix} & \{0\} & \{0,0,0,0\} \\ \begin{matrix}\{0,0,0,0,0,0,\\ 0,0\}\end{matrix} & \begin{matrix}\{0,0,0,0,0\\ 0,0,0,0,0,0,\\ 0,0,0,0,0\}\end{matrix} & \{0\} \end{bmatrix}$$

$$\begin{matrix} \underbrace{\{0,0,0,\ldots,0\}}_{25\text{ times}} \\ \underbrace{\{0,0,0,0,\ldots,0\}}_{49\text{ times}} \\ \underbrace{\{0,0,0,\ldots,0\}}_{16\text{ times}} \end{matrix}\Bigg] \in W$$ is a multiset mixed zero matrix of W. Thus x + x = ({0}, …, {0, 0, 0, …, 0}) is possible.

Finally we find $x \times_n y$ where $\times_n$ is the natural product of matrices using the given $x, y \in W$.

$$x \times_n y = \begin{bmatrix} \begin{matrix}\{3,4,0,0\}\\ \times\{2,2,2\}\end{matrix} & \begin{matrix}\{0,0,0,0,0\}\\ \times\{1,2,3,4\}\end{matrix} & \begin{matrix}\{1,1\}\times\\ \{3,5\}\end{matrix} & \begin{matrix}\{4\}\times\\ \{0,0,1\}\end{matrix} \\ \begin{matrix}\{2,2,2\}\times\\ \{4,1,1\}\end{matrix} & \begin{matrix}\{3,3\}\times\\ \{2,0\}\end{matrix} & \begin{matrix}\{1,1,1,0\}\\ \times\{0,1,1\}\end{matrix} & \begin{matrix}\{2,2,2\}\times\\ \{1,1,5\}\end{matrix} \\ \begin{matrix}\{4,2,1,1\}\\ \times\{2\}\end{matrix} & \begin{matrix}\{5\}\times\\ \{3,0,1\}\end{matrix} & \begin{matrix}\{0,0,0\}\\ \times\{0\}\end{matrix} & \begin{matrix}\{0,1,2\}\times\\ \{1,1,1\}\end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,2,0,0,0,2, \\ 0,0,0,2,0,0\} & \underbrace{\{0,0,0,\ldots,0\}}_{20\text{ times}} & \{3,5,3,5\} & \{4,0,0\} \\ \{2,2,2,2,2,2, \\ 2,2,2\} & \{0,0,0,0\} & \{1,1,1,0,0,0, \\ 0,0,1,1,1,0\} & \{2,2,2,2,2, \\ 2,4,4,4\} \\ \{2,4,2,2\} & \{3,0,5\} & \{0,0,0\} & \{0,0,0,0, \\ 1,2,3,0,1,2\} \end{bmatrix}$$

$\in$ W. This is the way the natural product operation is performed on W. Clearly $\{W, \times_n\}$ is a multiset matrix semigroup of infinite order. Since Z_6 has zero divisors, and idempotents we see the multiset matrix collection has special quasi zero divisors and idempotents.

We will give some special quasi zero divisors and idempotents in W under the natural product $\times_n$.

Consider $A = \begin{bmatrix} \{3,3,0,3\} & \{3,0\} & \{3,3\} & \{0,0\} \\ \{3,3,3,3,3\} & \{0,3,3\} & \{3\} & \{0\} \\ \{3,3,3,0,0,0\} & \{3\} & \{0\} & \{3,3,3\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{2,0\} & \{2\} & \{2,2,0\} & \{2,2,2\} \\ \{2,2,2,2\} & \{2,2\} & \{2,2,2\} & \{2\} \\ \{2,2,2,2, \\ 0,0,0\} & \{2,2,0\} & \{2\} & \{2,2,0\} \end{bmatrix}$$

$$\text{We see } A \times_n B = \begin{bmatrix} \{0,0,0,0,0 \\ 0,0,0\} & \{0,0\} & \{0,0,0 \\ 0,0,0\} & \{0,0,0,0 \\ 0,0\} \\ \{0,0,0,0,0 \\ 0,0,0,0,0 \\ 0,0,0,0,0 \\ 0,0,0,0,0\} & \{0,0,0, \\ 0,0,0\} & \{0,0,0\} & \{0\} \\ \underbrace{\{0,0,....,0\}}_{42-\text{times}} & \{0,0,0\} & \{0\} & \{0,0,0,0, \\ 0,0,0,0,0\} \end{bmatrix}$$

is a special quasi zero multiset matrix.

Infact W has special quasi multiset matrix zero divisors.

W has no special multiset matrix idempotents if any of the multisets are of cardinality greater than one.

Thus in this case the notion of matrix multiset idempotents or its Smarandache analogous happens to be an impossibility.

Next we proceed onto define the notion of n-multiset matrices where $2 \leq n < \infty$. We have defined n-multisets and described some of the vital properties enjoyed by them.

We first illustrate by examples n-multiset row matrices and define all the four operations on them.

Example 4.8: Let S = $\{(a_1, a_2, a_3, a_4, a_4) / a_i \in 4 - M(Q); 1 \leq i \leq 5\}$ be the collection of all 1×5 row 4 multiset matrices with entries from the multiset 4-M(Q). We define $\cup$, $\cap$, + and $\times$ the operations on S.

Let A = ({3, 0, 6, 4, 2}, {0, 0, 0, 6, 6}, {2, 2, 2, 2, 0}, {9, 9, 9, 9}, {2, 2, 1}) and B = ({2, 0, 6, 6, 6}, {0, 0, 2, 2, 2}, {2, 2, 0,0, 0, 0}, {9, 1, 1, 1, 1}, {3, 3, 5}) ∈ S.

We find A ∪ B, A ∪ B = ({3, 0, 6, 4, 2}, {0, 0, 0, 6, 6}, {2, 2, 2, 2, 0}, {9, 9, 9, 9}, {2, 2, 1}) ∪ ({2, 0, 6, 6, 6}, {0, 0, 2, 2, 2}, {2, 2, 0, 0, 0, 0}, {9, 1, 1, 1, 1}, {3, 3, 5})

= ({3, 0, 6, 4,2} ∪ {2, 0, 6, 6, 6}, {0, 0, 0, 6, 6} ∪ {0, 0, 2, 2, 2}, {2, 2, 2, 0, 2 ∪ {2, 2, 0, 0, 0, 0}, {9, 9, 9, 9} ∪ {9, 1, 1, 1, 1} + {2, 2, 1} ∪ {3, 3, 5}) = ({3, 0, 6, 4, 2, 2, 0, 6, 6, 6}, {0, 0, 0, 6, 6, 0, 0, 2, 2, 2}, {2, 2, 2, 2, 0, 2, 2, 0, 0, 0, 0}, {9, 9, 9, 9, 9, 1, 1, 1, 1}, {2, 2, 1, 3, 3, 5}) ∉S.

So we define leveling of the union operation denoted by l(∪).

l(A ∪ B) = l(({3, 0, 6, 4, 2, 2, 0, 6, 6, 6}, {0, 0, 0, 6, 6, 0, 0, 2, 2, 2}, {2, 2, 2, 2, 0, 2, 2, 0, 0, 0, 0}, {9, 9, 9, 9, 9, 1, 1, 1, 1}, {2, 2, 1, 3, 3, 5}) = (l({3, 0, 6, 4, 2, 2, 0, 6, 6, 6}), l({0, 0, 0, 0, 0, 6, 6, 2, 2, 2}), l({2, 2, 2, 2, 0, 2, 2, 0, 0, 0, 0, 0}) l({9, 9, 9, 9, 9, 1, 1, 1, 1}, l({2, 2, 1, 3, 3, 5})) = ({3, 0, 0, 2, 2, 4, 6, 6, 6}, {0, 0, 0, 0, 6, 6, 6, 2, 2, 2}, {2, 2, 2, 2, 0, 0, 0, 0, 0}, {9, 9, 9, 9, 1, 1, 1, 1}, {1, 3, 3, 5, 2, 2}) ∈ S.

Thus S is not closed under the union operation ∪. Thus we make leveling of '∪' and denote it by l(∪). We see {S, l(∪)} is a 4-multiset row matrix semigroup under l(∪).

Clearly o(S) is infinite so {S, l(∪)} is an infinite commutative 4-multiset row matrix semigroup which is a monoid if {ϕ} the empty set is combined with any x ∈ 4-M(Q) hence if ({ϕ}) = ({ϕ}, {ϕ}, {ϕ}, {ϕ}, {ϕ}) ∈ S then for any A ∈ S, l(A ∪S, l(A ∪ {ϕ}) = A. Hence the claim.

Further as the 4-multiset row matrices can be ordered by the containment relation in the following way we can prove {S, l(∪)} is a leveled semilattice of infinite order.

Let A = ({a_1}, {a_2}, {a_3}, {a_4}, {a_5}) and B = ({b_1}, {b_2}, {b_3}, {b_4}, {b_5}) ∈ S. We say A ⊆ B (A ⊆ B) that is A is contained in B if each [a_i} ⊆ {b_i} i = 1, 2, …, 5.

Infact as 4-multiset M(Q) are partially ordered we can order S also. So {S ⊆} is partially ordered collection of 4-multiset row matrices.

We see (S, l(∪)) is a semilattice of 4-multiset row matrices or equivalent (S, l(∪)) is a 4-multiset row matrix semigroup.

We now describe on S the ∩ operation.

Let A, B ∈ S (given earlier).

A ∩ B = ({3, 0, 6, 4, 2} ∩ {2, 0, 6, 6, 6}, {0, 0, 0, 6, 6} ∩ {0, 0, 2, 2, 2}, {2, 2, 2, 0, 2} ∩ {2, 2, 0, 0, 0, 0, 0}, {9, 9, 9, 9} ∩ {9, 1, 1, 1, 1}, {2, 2, 1} ∩ {3, 3, 5}) = ({2, 0, 6}), {0, 0}, {2, 2, 0}, {9}, {ϕ}) ∈ S.

Clearly in case of intersection we need not use leveling as A ∩ B ∈ S; in particular intersection of any n-multiset is a multiset. Hence {S, ∩} is a n-multiset matrix semilattice of infinite order which is also a n-multiset matrix. It is pertinent to mention that we need not use leveling for ∩ as for any multisets A, B ∈ 4-M(Q), A ∩ B ∈ 4-M(Q).

Now for the same A, B ∈ S we define + operation. A + B = ({3, 0, 6, 4, 2}, {0, 0, 0, 6, 6}, {2, 2, 2, 0, 2}, {9, 9, 9, 9}, {2, 2, 1}) + ({2, 0, 6, 6, 6}, {0, 0, 2, 2, 2}, {2, 2, 0, 0, 0, 0}, {9, 1, 1, 1}, {3, 3, 5}) = ({3, 0, 6, 4, 2} + {2, 0, 6, 6, 6}, {0, 0, 0, 6, 6} + {0, 0, 2, 2, 2}, {2, 2, 2, 2, 0} + {2, 2, 0, 0, 0, 0}, {9, 9, 9, 9} + {9, 1, 1, 1}, {2, 2, 1} + {3, 3, 5}) = ({5, 2, 8, 6, 4, 3, 0, 6,4, 2, 9, 6, 12, 10, 8, 9, 6, 12, 10, 8, 9, 6, 12, 10, 8}, {0, 0, 0, 6, 6, 0, 0, 0, 6, 6, 2, 2, 2, 8, 8, 2, 2, 2, 8, 8, 2, 2, 2, 8, 8}, {4, 4, 4, 4, 2, 4, 4, 4, 4, 2, 2, 2, 2, 2, 0, 2, 2, 2, 2, 0, 2, 2, 2, 2, 0, 2, 2, 2, 2, 0}, {18, 18, 18, 18, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10}, {5, 5, 4, 5, 5, 4, 7, 7, 6}) ∉ S. Hence we now see S not closed under the operation +. So we use only leveling of +, l(+) operation as in case of n-multisets to arrive at the compatability.

Now so l(A + B) = ({5, 2, 2, 3, 4, 4, 6, 6, 6, 6, 0, 8, 8, 8, 8, 9, 9, 9, 12, 12, 12, 10, 10, 10}, {0, 0, 0, 0, 6, 6, 6, 6, 2, 2, 2, 2, 8, 8, 8, 8}, {4, 4, 4, 4, 2, 2, 2, 2, 0, 0, 0, 0}, {18, 18, 18, 18, 10, 10, 10, 10, 10}, {5, 5, 5, 5, 4, 4, 7, 7, 6})∈ S. Hence {S, l(+)} is a 4-multiset matrix monoid of infinite order.

We see ({0}) is the special identity with respect to l(+) as for any with respect to l(+) as for any A = ({a_1}, {a_2}, {a_3}, {a_4}, {a_5}) ∈ S we have A + ({0}) = A where ({0}) = ({0}, {0}, {0}, {0}, {0}).

Next we proceed onto define × on S for the same pair A, B ∈ S.

A × B = ({3, 0, 6, 4, 2} × {2, 0, 6, 6, 6}, {0, 0, 0, 6, 6} × {0, 0, 2, 2, 2}, {2, 2, 2, 2, 0} × {2, 2, 0, 0, 0, 0}, {9, 9, 9, 9} × {9, 1, 1, 1}, {2, 2, 1} × {3, 3, 5}) = ({6, 0, 12, 8, 4, 0, 0, 0, 0, 0, 18, 0, 36, 24, 12, 18, 0, 36, 24, 12, 18, 0, 36, 24, 12}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, 12, 0, 0, 0, 12, 12, 0, 0, 0, 12, 12}, {4, 4, 4, 4, 0, 4, 4, 4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {81, 81, 81, 81, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9}, {6, 6, 3, 6, 6, 3, 10, 10, 5}) ∉ S.

Hence to overcome this problem we define the notion of level product on 4-multiset matrices.

Applying level product on A × B we get

l(A × B) = ({0, 0, 0, 0, 18, 18, 18, 36, 36, 36, 24, 24, 24, 12, 12, 12, 12, 6, 8, 4}, {0, 0, 0, 0, 12, 12, 12, 12}, {4, 4, 4, 4, 0, 0, 0, 0}, {81, 81, 81, 81, 4, 4, 4, 4}, {6, 6, 6, 6, 3, 3, 56, 10, 10}) ∈ S.

Clearly {S, l(×)} is a 4-multiset matrix semigroup.

Clearly {S, l(×)} is not a 4-multiset matrix ({1}) ∈ S acts as the special 4-multiset matrix unit given by ({1}) = ({1}, {1}, {1}, {1}, {1}).

We see for any A ∈ S, A × ({1}) = ({1}) × A = A.

We see (S, l(×)) has idempotent matrices given by

A = ({0, 0, 0, 0}, {1, 1, 1, 1}, {0, 0, 0, 0, 1, 1, 1, 1}, {0, 0, 0, 0}, {1, 1, 1, 1}) ∈ S is such that A l(×)A = A hence A is a 4-multiset matrix idempotent of S.

B = ({1, 1, 1, 1}, {0, 0, 0, 0}, {1, 1, 1, 1}, {0, 0, 0, 0}, {1, 1, 1, 1, 0, 0, 0, 0}) ∈ S is such that B l(×) B = B. B is also a 4-multiset matrix idempotent of S. Though 4-M(Q) has no multiset zero divisors still S has multiset matrix zero divisors given by

A = ({0, 0, 0, 0}, {4, 2, 2, 10}, {0, 0, 0, 0}, {7, 4, 5, 2, 8, 9, 10, 12}, {0, 0, 0, 0}) and B = ({4, 4, 2, 2, 8, 9, 7, 4}, {0, 0, 0, 0}, {3, 8, 9, 7, 4, 2, 1, 7}, {0, 0, 0, 0}, {3, 3, 3}) ∈ S.

We see l(A × B) = ({0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}) is a special 4-multiset matrix zero divisor in S. S has several special 4-multiset zero divisors under the level product. Infact S has infinite number of special 4-multiset zero divisors.

The reader is left with the task of finding the 4-multiset matrix zero divisors in S.

Next we proceed onto describe 3-multiset column matrices with entries from 3-M(Z_{12}).

Example 4.9. Let B = { $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ / $a_i \in$ 3-M(Z_{12}); $1 \leq i \leq 4$} be a 3-multiset column matrix. We indicate the operations l(∪), ∩, l(+) and l($\times_n$) on B and show how B can have zero divisor multiset matrices, idempotent multiset matrices and nilpotent matrix multiset matrices using ({0, 0, 0}) = $\begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}$ is the multiset special zero matrix. However ({0}) = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ is a special zero multiset matrix under which l(A + ({0})) = A for all A ∈ B.

Infact Al(×)({0}) ≠ ({0}) = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ so we cannot call ({0})

as the zero multiset matrix of B.

Further ({0, 0, 0}) = $\begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}$ ∈ B is such that for all A ∈ B,

A × ({0, 0, 0}) = {0, 0, 0} = $\begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}$.

Clearly B is of finite order. The reader is left with the task of finding the order of B.

Now let A = $\begin{bmatrix} \{0,2,2,2,4,4,4,6\} \\ \{1,5,7,8,9,10,11\} \\ \{2,3\} \\ \{6\} \end{bmatrix}$ and

D = $\begin{bmatrix} \{6,6,6\} \\ \{0\} \\ \{6,6,5,4\} \\ \{2,6,10,8,4,3\} \end{bmatrix}$ ∈ B.

We now find l(A ∪ D) = Al(∪)D = $\begin{bmatrix} \{0,2,2,2,4,4,4,6,6,6\} \\ \{1,5,7,8,9,10,11,0\} \\ \{2,3,6,6,5,4\} \\ \{2,6,6,10,8,4,3\} \end{bmatrix}$.

We see only under level union l(∪) we can arrive at the closure operation.

For under '∪' no n-multiset matrix would be closed only multiset matrices will be closed.

So it is mandatory to use leveling of union when n-multiset matrix are analysed. Now in case of intersection leveling is not needed so we can have A ∩ D ∈ B.

Now A ∩ D = $\begin{bmatrix} \{6\} \\ \{\phi\} \\ \{\phi\} \\ \{6\} \end{bmatrix}$ ∈ B.

Thus {B, l(∪)} is a 3-multiset matrix semilattice.

We see in case of 3-multiset matrices {B, l(∪), ∩} is a lattice.

Is {B, l(∪), ∩} a distributive lattice? This is left for the reader to prove as an open problem.

We also leave open the problem of finding whether {B, l(∪), ∩} is a Smarandache lattice of multiset matrices or not?

Now we show how + operation is performed on A and D ∈ B.

$$A + D = \{\begin{bmatrix} \{6,8,8,8,10,10,10,12,6,8,8,8,12, \\ 10,10,10,12,6,8,8,8,10,10,10\} \\ \{1,5,7,8,9,10,11\} \\ \{8,9,8,9,7,8,6,7\} \\ \{8,12,16,14,10,9\} \end{bmatrix} \notin B.$$

Thus we perform the level addition operation on leveling of A + D.

$$l(A + D) = \begin{bmatrix} \{6,6,6,8,8,8,10,10,10,12,12,12\} \\ \{1,5,7,8,9,10,11\} \\ \{8,8,8,9,9,7,7,6\} \\ \{8,9,10,12,14,16\} \end{bmatrix} \text{ is in B.}$$

Hence the claim.

Thus {B, l(+)} is a multiset matrix semigroup of finite order.

Only ({0}) = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ $\in$ B is such that for any A $\in$ B;

A + ({0}) = A. We do not have ({0, 0, 0}) = $\begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}$ to act as additive identity.

For if we take x = $\begin{bmatrix} \{2,2\} \\ \{3,3\} \\ \{4,4\} \\ \{5\} \end{bmatrix}$ ∈ B then l(x + ({0, 0, 0}) = l

$$\left(\begin{bmatrix} \{2,2\} \\ \{3,3\} \\ \{4,4\} \\ \{5\} \end{bmatrix} + \begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}\right) = \begin{bmatrix} \{2,2,2\} \\ \{3,3,3\} \\ \{4,4,4\} \\ \{5,5,5\} \end{bmatrix} \text{ so}$$

l(x + ({0, 0, 0})) ≠ x in general for all x ∈ B. Hence ({0, 0, 0}) cannot serve as the additive identity of B.

We can find 3-multiset matrix subsemigroups of finite order.

Several interesting properties associated with n-multisets can be derived for these 3-multiset matrix.

Next we define natural product $\times_n$ on B. We see B is not closed under the natural product $\times_n$. So we need to define only leveled natural product $\times_n$ on matrices as under $\times_n$ B is not closed.

For the same pair A, D ∈ B we first define

$$A \times_n D = \begin{bmatrix} \{0,0,0,0,0,0,0,6,0,0,0,0,0, \\ 0,0,6,0,0,0,0,0,0,0,6\} \\ \{0,0,0,0,0,0,0\} \\ \{0,0,10,8,6,6,3,0\} \\ \{0,0,0,0,0,6\} \end{bmatrix} \notin B.$$

Thus we see the natural product $\times_n$ is not compatible in the 3-multiset matrix collection B.

Now if we used leveled natural product $\times_n$ instead of $\times$ we get

$$l(A \times_n D) = \begin{bmatrix} 0,0,0,6,6,6\} \\ \{0,0,0\} \\ \{0,0,0,10,8,6,3,6\} \\ \{0,0,0,6\} \end{bmatrix}.$$

Clearly $l(A \times_n D) \in S$. Thus $(S, l(\times_n))$ is a 3-multiset matrix groupoid under the natural product which is leveled.

We can have special zero divisors. It is pertinent to keep on record $(\{0\}) = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ does not serve as the zero element for the zero divisor as any $l(A \times_n (\{0\}) \neq (\{0\})$ in general.

For take A as above we see

$$l(A \times_n(\{0\}) = \begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix} \neq (\{0\}).$$

Infact we have several zero divisors.

Consider $y = \begin{bmatrix} \{8\} \\ \{9,6\} \\ \{1,2,3\} \\ \{4\} \end{bmatrix} \in B$; we see

$$l(y \times_n (\{0, 0, 0\}) = \begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}.$$

So is a special 3-multiset matrix zero of B.

However we see if we take $x = \begin{bmatrix} \{3,4,3,4,5\} \\ \{7,7,7,9\} \\ \{0,0\} \\ \{0,0,0\} \end{bmatrix}$ and

$\begin{bmatrix} \{0\} \\ \{0\} \\ \{8,4,7,8\} \\ \{9\} \end{bmatrix} \in$ B, then

$l(x \times_n y) = \begin{bmatrix} \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \\ \{0,0,0\} \end{bmatrix}$ is a special 3-multiset matrix zero divisors

of $\{S, l(\times_n)\}$.

Consider $x = \begin{bmatrix} \{2,4\} \\ \{8\} \\ \{4\} \\ \{2\} \end{bmatrix}$ and $y = \begin{bmatrix} \{6\} \\ \{6\} \\ \{6,6\} \\ \{6\} \end{bmatrix} \in$ B.

We see $l(x \times_n y) = \begin{bmatrix} \{0,0\} \\ \{0\} \\ \{0,0\} \\ \{0\} \end{bmatrix} \neq (\{0, 0, 0\})$ so x and y are not the 3-multiset matrix special zero divisors of B.

We call these as special pseudo quasi 3-multiset zero divisors of $\{B, l(\times_n)\}$.

Now we proceed onto give some examples of 3-multiset matrix idempotents in $\{B, l(\times_n)\}$.

Let $x = \begin{bmatrix} \{4,4,4\} \\ \{9,9,9\} \\ \{0,0,0\} \\ \{1,1,1\} \end{bmatrix} \in B$.

Clearly $l(x \times_n x) = \begin{bmatrix} \{4,4,4\} \\ \{9,9,9\} \\ \{0,0,0\} \\ \{1,1,1\} \end{bmatrix} = x$. Thus x is a 3-multiset matrix idempotent of $\{B, l(\times_n)\}$.

The reader is left with the task of finding all the 3-multiset matrix idempotents and Smarandache 3-multiset matrix idempotents of $\{B, l(\times_n)\}$.

Next we proceed onto describe the 4-multiset square matrix using 4-M(Z_{10}) by the following example.

Example 4.10. Let $S = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} / a_i \in 4\text{-M}(Z_{10})\ ;\ 1 \leq i \leq 4\}$ be the 4-multiset square matrix collection.

Clearly S is of finite order. Now for this S we can define five different and distinct types of operations on them; viz. l(∪), ∩, l($\times_n$) l(×) and l(+) where l(×) denotes usual product of matrices.

Consider $A = \begin{bmatrix} \{5,5,0,0,2\} & \{4,4,4,4\} \\ \{6,6,6\} & \{2,2,2\} \end{bmatrix}$ and

$B = \begin{bmatrix} \{7,5,2,1,6\} & \{3,2,1\} \\ \{1,2,3,4,5\} & \{4,5,6,7,8,9\} \end{bmatrix} \in S.$

We find out

$$(A\ l(\cup)\ B) = l(A \cup B) = \begin{bmatrix} \{5,5,0,0,2,5 & \{4,4,4,4, \\ 2,1,6,7\} & 3,2,1\} \\ \{6,6,6,1,2, & \{2,2,2,4, \\ 3,4,5\} & 5,6,7,8,9\} \end{bmatrix} \qquad \text{I}$$

Clearly l(A∪ B) ∈ S. Thus {S, l(∪)} is a 4-multiset matrix semilattice (semigroup) of finite order.

We can now find the ∩ operation of the collection S for the same pair. A, B ∈ S.

$$\text{Consider } A \cap B = \begin{bmatrix} \{5,2\} & \{\phi\} \\ \{\phi\} & \{\phi\} \end{bmatrix} \in S \qquad \text{II}$$

Thus (S, ∩} is a 4-multiset matrix semilattice (semigroup) of finite order.

Clearly I and II are distinct. Consider now for the same pair A, B.

$$l(A \times_n B) = \begin{bmatrix} \{5,5,0,0,4,5,5, & \{2,2,2,2,8,8, \\ 0,0,4,2,2\} & 4,4,8,8,4,4\} \\ \{6,2,8,4,0, & \{8,0,2,4,6,8 \\ 6,2,8,4,0 & 8,0,2,4,6,8, \\ 6,2,8,4,0\} & 0,2,4,6\} \end{bmatrix} \quad \text{III}$$

is in S.

We see {S, $l(\times_n)$} is the 4-multiset matrix semigroup of finite order.

This semigroup {S, $l(\times_n)$} has nontrivial 4-multiset matrix zero divisors and 4-multiset matrix idempotents.

Infact ({0, 0, 0, 0}) = $\begin{bmatrix} \{0,0,0,0\} & \{0,0,0,0\} \\ \{0,0,0,0\} & \{0,0,0,0\} \end{bmatrix}$ $\in$ S is the 4-multiset matrix zero of S and for every x $\in$ S.

We have $l(x \times_n \{0, 0, 0, 0\}) = (\{0, 0, 0, 0\})$.

Next we consider the level addition for the same pair A, B in S.

$$l(A + B) = \begin{bmatrix} \{2,2,7,7,9,0,0, & \{7,7,7,7,7, \\ 5,5,7,7,2,2,4,6,6 & 6,6,6,6,5,5, \\ 1,1,3,1,1,6,6,8\} & 5,5\} \\ \{7,8,9,0,1,7,8,9, & \{6,7,8,9,0,1, \\ 0,1,7,8,9,0,1\} & 6,7,8,9,0,1, \\ & 6,7,8,9,0\} \end{bmatrix} \quad \text{IV}$$

Next we find first A × B then l(A × B), the level product of the usual product in matrices.

We show A × B $\notin$ S in general so {S, ×} is not compatible under product.

$$A \times B = \left[\begin{matrix} \{5,5,0,4,5,5, & \{4,8,2,6,0,4,8,2, \\ 0,0,0,0,0,0,0,4, & 6,0,4,8,2,6,0,4,8, \\ 5,5,0,0,2,0,0,0,2\} & 2,6,0\} \\ \{5,5,0,0,6,0,0, & \{6,0,4,8,2,6,6,0, \\ 0,0,4,5,5,0,0,2\} & 4,8,2,6,6,0,4,8,2, \\ & 6,6,0,4,8,2,6\} \end{matrix}\right.$$

$$\left.\begin{matrix} \{2,0,2,6,6,2,0, & + & \{2,4,6,8,0,2,4,6, \\ 2,6,6,2,0,2,6,6\} & & 8,0,2,4,6,8,0\} \\ \{8,2,6,8,2,6, & + & \{8,0,2,4,6,8,8,0,2, \\ 8,2,6\} & & 4,6,8,8,0,2,4,6,8\} \end{matrix}\right] \notin S.$$

Thus we need to level the sum of the set. We just for the sake of understanding whether leveling after sum is the same as summing the level first find the levels of every component given in A × B.

$$\text{Thus } l(A \times B) = \begin{bmatrix} \{5,5,5,5,0,0,0,0 & \{2,2,2,2,6,6,6,6 \\ 2,2,4,4\} + \{4,4,4, & 0,0,0\} + \{2,2,2,4, \\ 4,6,6,6,6,8,8,8,8, & 4,6,6,6,8,8,8,0,0, \\ 0,0,0,0,2,2,2,2\} & 0\} \\ \{5,5,5,5,0,0,0,0, & \{8,8,8,6,6,6,2,2, \\ 2,4,6\} + \{6,6,6,6, & 2\} + \{8,8,8,8,0,0, \\ 0,0,0,0,8,8,8,8, & 0,2,2,2,4,4,4,6, \\ 2,2,2,2\} & 6,6\} \end{bmatrix}$$

$$= \begin{bmatrix} \{9,9,9,9,4,4,4, & \{2,2,2,2,6, \\ 4,6,6,8,8,11,11,11, & 6,6,6,8,8,8, \\ 11,6,6,8,8,10,10,0, & 8,4,4,4,4,0, \\ 0,0,0,7,7,7,7\} & 0,0,0\} \\ \{0,0,0,0,5,5,5,5, & \{8,8,8,8,6,6, \\ 6,6,6,6,2,2,2,2, & 6,6,0,0,0,0, \\ 11,111,11,11,1,1, & 2,2,2,2,4,4, \\ 1,1,4,4,4,4\} & 4,4\} \end{bmatrix} \quad V$$

V is different from I, II, III, IV. Thus {S, l(×)} is a 4-multiset matrix semigroup under the level product of matrices.

We leave it as an exercise for the reader to prove or disprove l(a + b) = l[l(a) + l(b)] where a and b are multiset and l is the level sum (addition of two multisets).

Let x = {3, 3, 3, 4, 4, 4, 3, 4, 5, 3, 4} and y = {9, 9, 9, 9, 9} $\in$ 4-M(Z_{12}).

l (x + y) = l({0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 , 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2})

= {0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2} l[l(x) + l(y)] = l[{3, 3, 3, 3, 4, 4, 4, 4, 5} + {9, 9, 9, 9}0 = l[{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}) = {0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2}.

We see l(l (a) + l(b)) = l(a + b).

Now finding zero divisors idempotents in {S, l(×)} happens to be a challenging problem.

Further {S, l(×)} is a 4-multiset matrix semigroup of finite order which is non commutative.

Clearly $I_{2 \times 2} = \begin{bmatrix} \{1,1,1,1\} & \{0,0,0,0\} \\ \{0,0,0,0\} & \{1,1,1,1\} \end{bmatrix}$ may not serve as the identity with respect to the level product l(×).

Let $A = \begin{bmatrix} \{6,2\} & \{1,5\} \\ \{3\} & \{4,2,1\} \end{bmatrix} \in S.$

$$l(A \times I_{2 \times 2}) = \begin{bmatrix} (\{6,6,6,6,2,2,2,2\} & \{1,1,1,1,5,5,5,5\} \\ +\{0,0,0,0\} & +\{0,0,0,0\} \\ \{0,0,0,0\} & \{4,4,4,4,2,2,2,2 \\ +\{3,3,3,3\} & 1,1,1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{6,6,6,6,2,2,2,2\} & \{1,1,1,1,5,5,5,55\} \\ \{3,3,3,3\} & \{4,4,4,4,2,2,2,2 \\ & 1,1,1,1\} \end{bmatrix} \neq A.$$

Hence the claim.

However if we consider $I = \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix} \in S$ then we see for all A ∈ S, l(A × I) ≠ A so I also cannot be a unit.

Consider $A = \begin{bmatrix} \{1,2,3,4,5\} & \{6,7,8\} \\ \{9,10,11,0\} & \{2,9,9\} \end{bmatrix} \in S;$

$$l(A \times I) = \begin{bmatrix} \{1,2,3,4,5\} & \{6,7,8\} \\ +\{0,0,0,0\} & \{0,0,0\} \\ \{0,0,0,0,0\} & \{0,0,0\} \\ +\{9,10,11,0\} & +\{2,9,9\} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix}\{1,2,3,4,5,1,\\ 2,3,4,5,1,2,3,\\ 4,5,1,2,3,4,5\}\end{matrix} & \begin{matrix}\{6,7,9,6,7,8,\\ 6,7,8\}\end{matrix} \\ \begin{matrix}\{9,10,11,0,9,9,9,\\ 10,10,10,11,11,0,0,0\}\end{matrix} & \begin{matrix}\{2,9,9,2,9,9,\\ 2,9,9\}\end{matrix} \end{bmatrix} \neq A.$$

Hence the claim.

So for {S, l(×)} we do not have the concept of unit hence {S, l(×)} is only a 4-multiset matrix semigroup and is not a 4-multiset matrix monoid.

How if A, B ∈ S we say A and B are zero divisors if Al(×) B is a special zero 4-multiset matrix zero n given by $\begin{bmatrix} \{0,0,0,0\} & \{0,0,0,0\} \\ \{0,0,0,0\} & \{0,0,0,0\} \end{bmatrix}$ = ({0, 0, 0, 0}).

We see A × ({0, 0, 0, 0}) = ({0, 0, 0, 0}) for all A ∈ S.

Now we seen $l(\times_n)$ the leveled natural product for the same pair of 4-multiset matrices A and B.

With respect $l(\times_n)$ also ({0, 0, 0, 0}) serves as the zero for $A \times_n$ ({0, 0, 0, 0}) = {0, 0, 0, 0}) for all A ∈ S.

Finding multiset matrix idempotents and zero divisors are left as an exercise for the reader.

Finally we give yet another example where the usual matrix product ×, cannot be defined.

Example 4.11. Let M = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ / $a_i \in$ 5 - M(Z_{15}); $1 \leq i \leq 8\}$ be the 5-multiset matrices collection.

Clearly order of M; o(M) = |M| is finite.

We can as in case of other n-multiset matrices we can define l(∪), ∩, l(+) and l(×).

We will illustrate only the operation l($\times_n$) on M as only the 5-multiset matrix semigroup {M, l($\times_n$)} has 5-multiset matrix zero divisors and 5 - multiset matrix idempotents.

$$\text{Let } A = \begin{bmatrix} \{5,3,2\} & \{0\} & \{1,1,1,1,1\} & \{5,2\} \\ \{1,1,1\} & \{3\} & \{4,4,4\} & \{3,1,1\} \end{bmatrix}$$

$$B = \begin{bmatrix} \{10,10,10\} & \{4,5,5,1,2\} & \{3,2,4\} & \{2,2,2,2\} \\ \{5,5,5,5\} & \{3,3,3,3\} & \{2,2,2,2\} & \{6,6,9,9\} \end{bmatrix} \in M.$$

We find l(A $\times_n$ B) =

$$\begin{bmatrix} \begin{matrix}\{5,5,5,0,0,\\0,5,5\}\end{matrix} & \{0,0,0,0,0\} & \begin{matrix}\{3,3,3,3,3,2,\\2,2,2,2,4,\\4,4,4,4\}\end{matrix} & \begin{matrix}\{10,10,10,10,\\4,4,4,4\}\end{matrix} \\ \begin{matrix}\{5,5,5,5,5,\\4,4,4,4,4\}\end{matrix} & \{9,9,9,9\} & \begin{matrix}\{8,8,8,8,8,\\2,2,2\}\end{matrix} & \begin{matrix}\{3,6,6,3,6,6,\\12,9,9,12,9,9\}\end{matrix} \end{bmatrix} \in M.$$

This is the way natural product $\times_n$ is performed {M, l($\times_n$)} is infact a finite 5-multiset matrix semigroup.

Now we give an idempotent of the 5-multiset matrix collection.

$$\text{Let } x = \begin{bmatrix} \begin{matrix}\{10,10,10,\\10,10\}\end{matrix} & \begin{matrix}\{0,0,0,\\0,0\}\end{matrix} & \begin{matrix}\{0,0,0,\\0,0\}\end{matrix} & \{6,6,6,6,6\} \\ \{6,6,6,6,6\} & \begin{matrix}\{1,1,1,\\1,1\}\end{matrix} & \begin{matrix}\{1,1,1,\\1,1\}\end{matrix} & \begin{matrix}\{10,10,10,10\}\\10,6,6,6,6,\\6,0,0,0,0\}\end{matrix} \end{bmatrix} \in M.$$

It is easily verified $l(x \times_n x) = x$. Thus we see M has 5-multiset matrix idempotents.

Now we give an illustration of 5-multiset matrix zero divisor in M in the following.

Let A =

$$\begin{bmatrix} \{5,5,5,5,5\} & \{10,10,10,10\} & \{9,9,9\} & \{0,0,0\} \\ \{3,3,3,3,3\} & \{10,5,5\} & \{10,5,10\} & \{5,10,10,10\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{3\} & \{6,6\} & \{10,10,5,5\} & \{3,6,5\} \\ \{10\} & \{6,9\} & \{6,6,9,3\} & \{6,6,9\} \end{bmatrix} \in M.$$

$$A \times_n B = \begin{bmatrix} \{0,0,0,0,0\} & \{0,0,0,0,0\} & \{0,0,0,0,0\} & \{0,0,0,0,0\} \\ \{0,0,0,0,0\} & \{0,0,0,0,0\} & \{0,0,0,0,0\} & \{0,0,0,0,0\} \end{bmatrix}$$

$$= (\{0, 0, 0, 0, 0\}).$$

Thus $A \times_n B$ gives a 5-multiset matrix zero divisor of M.

Interested reader can get all the 5-multiset matrix zero divisors and Smarandache 5-multiset matrix idempotents in any.

Similar study can be made for 5-multiset matrix idempotents of M and then Smarandache analogue.

Finding units for M is a difficult task; however the special unit of the 5-multiset matrix is given by

$$I = \begin{bmatrix} \{1\} & \{1\} & \{1\} & \{1\} \\ \{1\} & \{1\} & \{1\} & \{1\} \end{bmatrix} \in M \text{ is such that for all } A \in M.$$

$A \times_n I = A.$

Infact ({1, 1, 1, 1, 1}) =

$$\begin{bmatrix} \{1,1,1,1,1\} & \{1,1,1,1,1\} & \{1,1,1,1,1\} & \{1,1,1,1,1\} \\ \{1,1,1,1,1\} & \{1,1,1,1,1\} & \{1,1,1,1,1\} & \{1,1,1,1,1\} \end{bmatrix}$$

is not the 5-multiset matrix identity of M for if A =

$\begin{bmatrix} \{2,1,1\} & \{4,4\} & \{5\} & \{6,2\} \\ \{1,3\} & \{9,9\} & \{9\} & \{9,9\} \end{bmatrix} \in$ M then A$\times_n$ ({1, 1, 1, 1, 1}) =

$$\begin{bmatrix} \{2,2,2,2,2, & \{4,4,4, & \{5,5,5, & \{6,6,6,6,6, \\ 1,1,1,1,1\} & 4,4\} & 5,5\} & 2,2,2,2,2\} \\ \{1,1,1,1,1, & \{9,9,9, & \{9,9, & \{9,9, \\ 3,3,3,3,3\} & 9,9\} & 9,9\} & 9,9,9\} \end{bmatrix} \neq A.$$

Thus ({1, 1, 1, 1, 1}) does not act as the 5-multiset matrix identity, which is clear from the above working.

Next we proceed onto prove the n-multiset matrix collection under both l($\cup$) and $\cap$ is a lattice, may be distributive or otherwise. However {M, l(+), l($\times$)} cannot have any algebraic structure as l(+) and l($\times$) does not satisfy distributive law.

With this limitation we can define only special multiset semi vector spaces.

We first proceed onto define that if we have a multiset A and if x $\in$ M(Z) then what is xA. This we will illustrate by an example or two. We need this property if we wish to define the notion of multiset semivector spaces.

Example 4.12. Let S = {M(Z)} be the multiset of Z.

Let x = {3, 4, 0, 0, – 5} and a = 8 $\in$ Z then ax = 8x = {24, 32, 0, 0, –40}.

We see there is no condition on the number of elements in the number of elements in the multisets so multisets in M(Z) (or S) can be even of infinite order.

We see the limitations of the distributive law should not impair the utility of these new algebraic structures.

Consider x = {3, –3, –2, 1, 5, 6, 0} and y = {–3, 2, –1, –5, –6, 0} and a = 9.

Now we find out a(x + y) and ax + ay where x, y ∈ M(Z) and a = 9 ∈ Z = a({3, –3, –2, 1, 5, 6, 0} + {–3, 2, –1, –5, –6, 0}) = 9 × ({0, –6, –5, –2, 2, 3, –3, 5, –1, 0, 3, 7, 8, 2, 2, –4, –3, 0, 4, 5, –1, –2, –8, –7, –4, 0, 1, –5, –3, –9, –8, –5, –1, 0, –6, 3, –3, –2, 1, 5, 6, 0})

= {0, –54, –45, –18, 18, 27, –27, 45, –9, 0, 27 63, 72, 18, 18, –36, –27, 36, 45, –9, –18, –72, –63, –36, 0, 9, –45, –81, –27, –72, –45, –9, 0, –54, 27, –27, –18, 9, 45, 36, 0} I

We find ax + y = [(3, –3, –2, 1, 5, 6, 0} + 9{–3, 2, –1, –5, –6, 0} = {27, –27, –18, 9, 45, 54, 0} + {–27, 18, –9, –45, –54, 0} = {0, –54, –45, –18, 18, 27, –27, 45, –9, 0, 27, 63, 72, 18, 18, –36, –27, 0, 36, +45, –9, –18, –72, –63, –36, 0, 9, –45, –27, –81, –72, –45, –9, 0, –54} II

Clearly I and II are identical.

However if we use n-multisets the above results will not hold good in. To this end we give some examples.

Example 4.13. Let S = {0, 2-M(Z_{12})} be the 2-multiset from using Z_{12}.

Let x = 4 ∈ Z_{12}, a = {0, 0, 4, 4, 6, 6} and b = {3, 3} ∈ M(Z_{12}).

We find out (a + b) × x and ax + bx in the following x(a+b) = 4({0, 0, 4, 4, 6, 6} + {3, 3, 6}) = 4({3, 3, 7, 7,9, 9, 3, 3, 7, 7, 9, 9, 3, 3, 7, 7, 9, 9}) = {0, 0, 4, 4, 0, 0, 0, 0, 4, 4, 0, 0, 0, 0, 4, 4, 0, 0} I

ax + bx = 4{0, 0, 4, 4, 6, 6} + 4{3, 3} = {0, 0, 4, 4, 0, 0} + {0, 0} = {0, 0, 4, 4, 0, 0, 0, 0, 4, 4, 0, 0, 0, 0, 4, 4, 0, 0} II

I and II are identical.

Now consider the set M = (2 - M(Z_{12})} we take x = {3, 9} and y = {9, 3, 2} and a = {1, 1, 3} in M.

Consider l[l(ax) + l(ay)] = l[l({1, 1, 3} × {3, 9})
+ l({1, 1, 3} × {9, 3, 2})]
= l[l({3, 9, 3, 9, 9, 3}) + l({9, 3, 2, 9, 3, 2, 3, 9, 6})]
= l[{3, 3, 9, 9} + {3, 3, 2, 2, 9, 9, 6}]
= l({6, 6, 0, 0, 6, 6, 0, 0, 5, 5, 11, 11, 5, 5, 11, 11, 0, 0, 6, 6, 0, 0, 6, 6, 9, 9, 3, 3})
= {0, 0, 6, 6 5, 5, 11, 11, 9, 9, 3, 3} I

l(a × l(x + y))
= l({1, 1, 3} × l({9, 3} + {9, 3, 2}))
= l({1, 1, 3} × l({6, 0, 0, 6, 11, 5})
= l({1, 1, 3} × {6, 6, 0, 0, 11, 5})
= l({7, 7, 9, 7, 7, 9, 1, 1, 3, 1, 1, 3, 0, 0, 2, 6, 6, 8})
= {7, 7, 9, 9, 1, 1, 3, 3, 0, 0, 2, 8, 6, 6} II

I and II are distinct hence the distributive law is not true for this set a, x, y ∈ 2-M(Z_{12}).

So in general n-M(X) where X is any set and $2 \leq n < \infty$ is such that the distributive law in general is not true for some x, y, z ∈ n-M(X) under the level sum l(+) and level product l(×).

We suggest a few problems in the following:

Problems

1. Prove multiset 3 × 4 matrices with multisets from Z_{16} is of infinite order.

2. Show N = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle/ a_i \in M(Q), \cup \right\}$ is a semilattice of multiset matrices.

3. Prove N in problem (2) cannot have any finite order multiset matrix sublattice of finite order.

4. Let S = $\left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{pmatrix} \middle/ a_i \in M(Z); 1 \le i \le 27, \cap \right\}$

   be the multiset matrix semilattice under $\cap$.

   i) Prove S can have finite order subsemilattices.

   ii) Can every element in S be a trivial subsemilattice?

   iii) Will A = $\left\langle \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{pmatrix} \middle/ a_i \in \{1, 2, 3, \dots, 10\}; 1 \le i \le 27 \right\} \right\rangle$. A generated under the operation $\cap$ be a finite ordered multiset matrix subsemilattice?

   iv) Find the order of A in (iii)

   v) Show if x and y are any two multiset matrices from S such that all the multisets in both the matrices x and y are of finite order then x and y in S under the intersection operation will only generate a multiset matrix subsemilattice T of finite order. Find the order of that T.

vi) Obtain any other special or interesting feature enjoyed by S.

5. Let S = { $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix}$ a_i / $\in M(\langle Z \cup I \rangle)$, $\cup$; $1 \leq i \leq 18$} be the neutrosophic multiset matrix semigroup under $\cup$.

i) Can S contain subsemigroups of finite order?

ii) Can S contain idempotents?

iii) Prove S has subsemigroup of infinite order.

iv) Obtain all special features enjoyed by S.

6. If for the S in problem 5 the operation '$\cup$' is replaced by the natural product $\times_n$ then

i) Prove {S, $\times_n$} cannot have any finite subsemigroups.

ii) Prove {S, $\times_n$} is a torsion free semigroup of infinite order.

iii) Prove {S, $\times_n$} has ideals of infinite order.

iv) Prove {S, $\times_n$} has zero divisors.

v) Can {S, $\times_n$} have Smarandache zero divisors?

vi) Can {S, $\times_n$} have idempotents?

vii) Can {S, $\times_n$} have Smarandache idempotents?

viii) Prove $\{S, \times_n\}$ cannot have nontrivial nilpotents.

ix) Enumerate all special features enjoyed by $\{S, \times_n\}$ and compare it with $\{S, \cup\}$.

7. Let {S, +} be the structure defined on the set S in problem 5.

i) Prove {S, +} is a multiset matrix semigroup of infinite order.

ii) Can {S, +} be a multiset matrix monoid?

iii) Prove if $A \in S$; then $A - A \neq \{0, \ldots, 0\}$; $\{0, \ldots, 0\}$ is the zero multiset in general if $|A| \geq 2$.

iv) Can S have finite order multiset matrix subsemigroups? (Justify)

v) Find all multiset matrix subsemigroups of S and prove all of them are of infinite order.

vi) Can S have multiset matrix ideals?

vii) Prove {S, l (+)} has no additive identity.

8. Let $M = \{M(\langle Q \cup g\rangle)\}$ be a collection of multisets

$$R = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in M(\langle Q \cup g\rangle);\ 1 \leq i \leq 16\right\}$$ be the multiset matrix.

i) Find the structure of $\{M, \cup\}$

ii) Is $\{M, \cap\}$ multiset matrix semilattice?

iii) Can $(M, \times_n\}$ be a multiset matrix monoid? (Justify)

iv) Find multiset matrix idempotents in $\{M, \times_n\}$,

v) Can multiset matrix zero divisors be in $\{M, \times_n\}$?

vi) Prove or disprove all multiset matrix subsemigroups are of infinite order.

vii) What is the algebraic structure enjoyed by $\{M, +\}$?

viii) Can any $A \in M$ be such that $A - A = \{0, \ldots, 0\}$? ($A = \{(a_{ij})) |a_{ij}| \geq 2$ $a_{ij} \in M$?

ix) Prove every multiset matrix P in $\{M, +\}$ is such that $\langle P, +\rangle$, P under addition generates a multiset matrix subsemigroup of infinite order under +.

x) Can $\{M, +\}$ have multiset matrix subsemigroups of infinite order?

xi) Can $\{M, +\}$ be a multiset matrix monoid?

xii) Can $\{M, +\}$ be a multiset matrix Smarandache semigroup?

xiii) Prove $\{M, \cup\}$ is a multiset matrix semilattice.

xiv) Can $\{M, \cup\}$ have multiset matrix subsemigroups subsemilattices of finite order? Justify your claim.

xv) Is $\{M, \cap\}$ a multiset matrix semilattice?

xvi) Prove $\{M, \cup\}$ can have finite order multiset matrix subsemilattice?

xvii) Can $\{M, \cup\}$ have multiset matrix idempotents?

10. Let $B = \{M(\langle Z_{25} \cup I\rangle)\}$ be the multisets collection

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix} \middle/ a_i \in B; 1 \leq i \leq 27 \right\}$$ be the multiset matrix.

i) Prove $\{P, \cap\}$ is a multiset matrix semilattice of infinite order.

ii) Study questions (i) to (xv) of problem 9 for this P.

iii) Prove {P, $\times_n$} has a multiset matrix ideal of infinite order.

11. Let W = {M(⟨C∪g⟩)} be the multiset collection V = { $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{25} & a_{26} & a_{27} \end{bmatrix}$ / $a_i \in M(\langle C \cup g \rangle)$ $1 \leq i \leq 27$} be the multiset matrix of complex.

Study questions (i) to (xv) of problem (9) for this V.

12. Let T = {M(C(⟨Z_{99} ∪ I⟩)} be the multiset.

R = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix}$ / $a^i \in T$; $1 \leq i \leq 20$} be the multiset matrix.

Study questions (i) to (xv) of problem (9) for this R.

13. Let N = {9-M(⟨R ∪ I⟩)} be the 9-multiplicity multiset.

M = { $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ / $a_i \in N$; $1 \leq i \leq 0$} be the 9-multiset matrix collection.

i) Prove {M, ×} is a 4-multiset matrix semigroup of infinite order which is non commutative.

ii) Prove {M, $\times_n$} is a 9-multiset matrix semigroup of infinite order which is commutative.

iii) Is P = P{M, $\cap$} a 9-multiset matrix semigroup (semilattice) of infinite order.

iv) Can P in (iii) have finite order 9-multiset matrix subsemigroups? If so give two or three examples of them.

v) Prove Q = {M, l($\cup$)} is an infinite order 9-multiset matrix semigroup which has no finite order 9-multiset matrix subsemigroups.

vi) Prove L = {M, +} is a 9-multiset matrix semigroup of infinite order.

vii) Can L have finite order 9-multiset matrix subsemigroups?

14. Let W = {3-M($\langle Z_{12} \cup g\rangle$)} be the 3-multiplicity multiset of dual modulo numbers.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \middle/ a_i \in W; 1 \leq i \leq 18 \right\} \text{ be the}$$

3-multiset matrix.

i) Prove |P| is finite

ii) Prove {P, $\times_n$} is a 3-multiset matrix semigroup.

iii) Prove {P, $\times_n$} has 3-multiset matrix zero divisors.

iv) Can {P, $\times_n$} contain 3-multiset matrix Smarandache zero divisors?

v) Prove {P, $\times_n$} has 3-multiset matrix nilpotents.

vi) Prove or disprove {P, $\times_n$} cannot have 3-multiset matrix idempotents or Smarandache 3-multiset matrix idempotents.

vii) Can {P, +} be a 3-mutiset matrix monoid?

viii) Can {P, +} be a Smarandache multiset matrix semigroup?

ix) Can {P, +} have any multiset matrix M = (m_{ij}) with $|M| \geq 2$ have additive inverse?

x) Obtain any other special feature enjoyed by {P, +} and {P, $\times_n$}.

15. Let S = {5-M(C(Z_{24}))} be the 5-multiplicity multiset of finite complex numbers.

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{17} & a_{18} \end{bmatrix} \Big/ a_i \in S, 1 \leq i \leq 18\right\}$$ be the 5-multiplicity multiset matrix. Study questions (i) to (x) of problem (14) for this W.

16. Let T = {9-M(C($\langle Z_{15} \cup I\rangle$))} be the 9-multiplicity finite complex neutrosophic multiset collection.

Let $$X = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{bmatrix} \Big/ a_i \in T; 1 \leq i \leq 27\right\}$$ be 9-matrix multiset {X, $\cup$} is a 9-complex.

i) Prove modulo integer multiset matrix semigroup of finite order.

ii) Prove every A $\in$ {X, $\cup$} in general is not an idempotent.

iii) Prove {P, $\cap$} is a semigroup of finite order.

iv) Find all idempotents of {P, $\cap$}.

v) Is {P, +} a monoid?

vi) Find 3-subsemigroups of finite order in {P, +}.

vii) Prove {P, $\times_n$} is a semigroup.

viii) Find all idempotents of {P, $\times_n$}.

ix) Does {P, $\times_n$} have Smarandache idempotents?

x) Find special zero divisors if any in {P, $\times_n$}.

xi) If (0) = {(0, 0, 0, 0, 0, 0, 0, 0, 0)} is a zero find all x, y ∈ X such that x $\times_n$ y = (0).

xii) Can X have Smarandache zero divisors?

17. Let S = {2-M(⟨Z_5 ∪ I ∪ g ∪ K⟩)} the 2-multiplicity multiset from ⟨Z_5 ∪ g ∪ I ∪ k⟩ = {$a_0 + a_1I + a_2g + a_3K + a_4gK + a_5gI + a_6KI + a_7KIg$ / $a_i \in Z_5 \leq i \leq 7$, $g^2 = 0$, $I^2 = I$, $k^2 = 4k$, $(Ig)^2 = 0$, $(kg)^2 = 0$ $(KI)^2 = 4KI$}.

B = {$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ / $a_i \in S$; $1 \leq i \leq 4$} be the 2-multiset matrix collection.

i) Study question (i) to (xii) of problem (16) for this B.

ii) What is structure enjoyed by {B, $\times_n$} and {B, ×}?

iii) Prove {B, ×} is a non commutative semigroup.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book the authors have introduced the notion of n-multiplicity multisets. They have also defined the notion of multiset semigroups. For these semigroups; special Smarandache elements are defined and developed. By definition, Smarandache special elements are stronger than the classical special elements.

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