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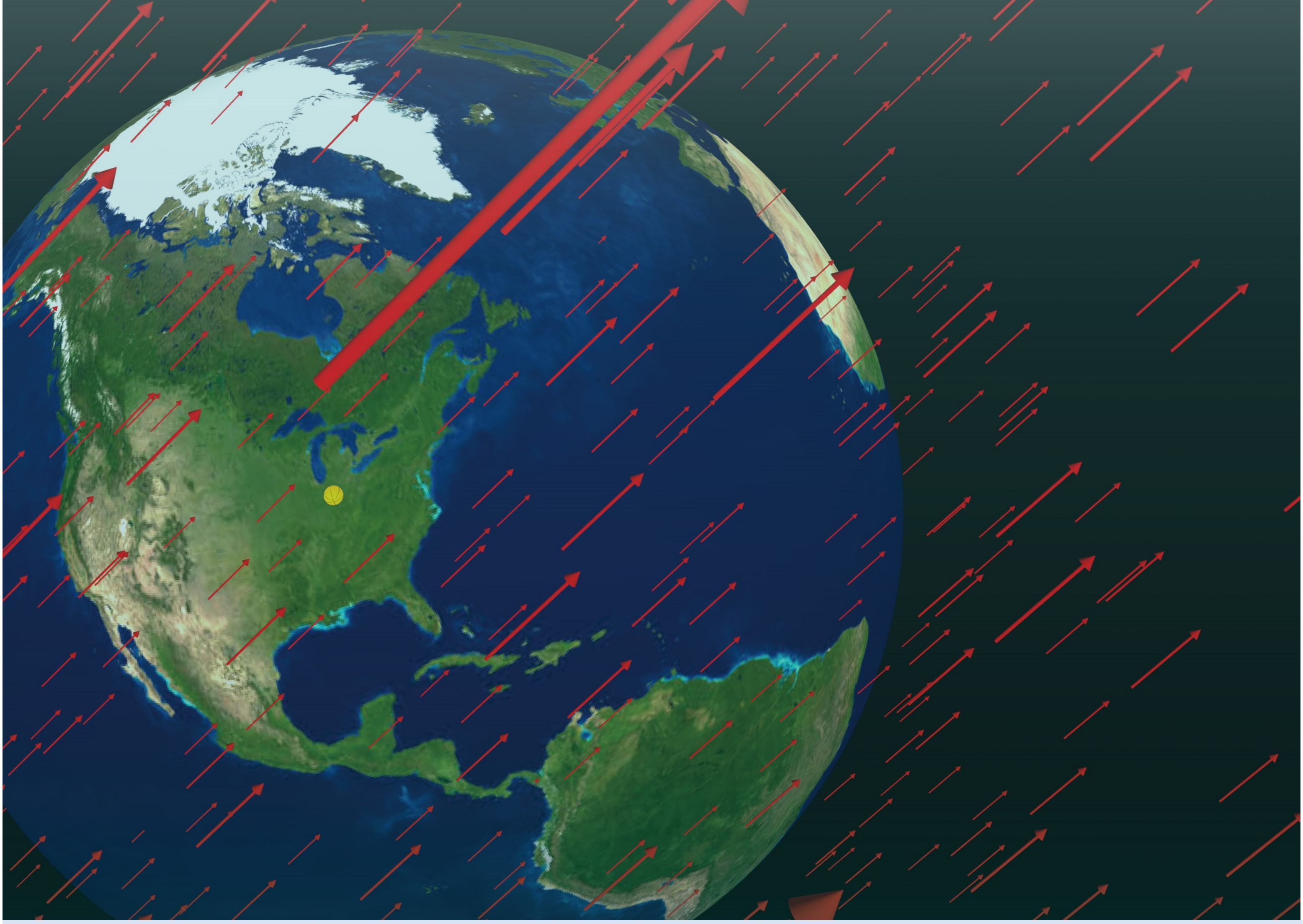

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[The special issue "Neutrosophic Theories Applied in Engineering"]

Neutrosophic sets and logic are generalizations of fuzzy and intuitionistic fuzzy sets and logic.

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Article

A Retinal Vessel Detection Approach Based on Shearlet Transform and Indeterminacy Filtering on Fundus Images

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Abstract: A fundus image is an effective tool for ophthalmologists studying eye diseases. Retinal vessel detection is a significant task in the identification of retinal disease regions. This study presents a retinal vessel detection approach using shearlet transform and indeterminacy filtering. The fundus image's green channel is mapped in the neutrosophic domain via shearlet transform. The neutrosophic domain images are then filtered with an indeterminacy filter to reduce the indeterminacy information. A neural network classifier is employed to identify the pixels whose inputs are the features in neutrosophic images. The proposed approach is tested on two datasets, and a receiver operating characteristic curve and the area under the curve are employed to evaluate experimental results quantitatively. The area under the curve values are 0.9476 and 0.9469 for each dataset respectively, and 0.9439 for both datasets. The comparison with the other algorithms also illustrates that the proposed method yields the highest evaluation measurement value and demonstrates the efficiency and accuracy of the proposed method.

Keywords: retinal vessels detection; shearlet transform; neutrosophic set; indeterminacy filtering; neural network; fundus image

1. Introduction

A fundus image is an important and effective tool for ophthalmologists who diagnose the eyes for determination of various diseases such as cardiovascular, hypertension, arteriosclerosis and diabetes. Recently, diabetic retinopathy (DR) has become a prevalent disease and it is seen as the major cause of permanent vision loss in adults worldwide [1]. Prevention of such adult blindness necessitates the early detection of the DR. DR can be detected early by inspection of the changes in blood vessel structure in fundus images [2,3]. In particular, the detection of the new retinal vessel growth is quite important. Experienced ophthalmologists can apply various clinical methods for the manual diagnosis of DR which require time and steadiness. Hence, automated diagnosis systems for retinal screening are in demand.

Various works have been proposed so far where the authors have claimed to find the retinal vessels automatically on fundus images. Soares et al. proposed two-dimensional Gabor wavelets and supervised classification method to segment retinal vessel [4], which classifies pixels as vessel and non-vessel pixels. Dash et al. presented a morphology-based algorithm to segment retinal vessel [5]. Authors used 2-D Gabor wavelets and the CLAHE method for enhancing retinal images. Segmentation was achieved by geodesic operators. The obtained segmentation result was then refined with post-processing.

Zhao et al. introduced a methodology where level sets and region growing methods were used for retinal vessel segmentation [6]. These authors also used CLAHE and 2D Gabor filters for image enhancement. The enhanced images were further processed by an anisotropic diffusion filter to smooth the retinal images. Finally, the vessels segmentation was achieved by using level sets and region growing method. Levet et al. developed a retinal vessel segmentation method using shearlet transform [7]. The authors introduced a term called ridgeness which was calculated for all pixels at a given scale. Hysteresis thresholding was then applied for extracting the retinal vessels. Another multi-resolution approach was proposed by Bankhead et al. [8], where the authors used wavelets. The authors achieved the vessel segmentation by thresholding the wavelet coefficients. The authors further introduced an alternative approach for center line detection by use of spline fitting. Staal et al. extracted the ridges in images [9]. The extracted ridges were then used to form the line elements which produced a number of image patches. After obtaining the feature vectors, a feature selection mechanism was applied to reduce the number of features. Finally, a K-nearest-neighbors classifier was used for classification. Kande et al. introduced a methodology combining vessel enhancement and the SWFCM method [10]. The vessel enhancement was achieved by matched filtering and the extraction of the vessels was accomplished by the SWFCM method. Chen et al. introduced a hybrid model for automatic retinal vessel extraction [11], which combined the signed pressure force function and the local intensity to construct a robust model for handling the segmentation problem against the low contrast. Wang et al. proposed a supervised approach which segments the vessels in the retinal images hierarchically [12]. It opted to extract features with a trained CNN (convolutional neural network) and used an ensemble random forest to categorize the pixels as a non-vessel or vessel classes. Liskowski et al. utilized a deep learning method to segment the retinal vessels in fundus images [13] using two types of CNN models. One was a standard CNN architecture with nine layers and the other just consisted of convolution layers. Maji et al. introduced an ensemble based methodology for retinal vessels segmentation [14] which considered 12 deep CNN models for constructing the classifier structure. The mean operation was used for the outputs of all networks for the final decision.

In this study, a retinal vessel detection approach is presented using shearlet transform and indeterminacy filtering. Shearlets are capable to capture the anisotropic information which makes it strong in the detection of edges, corners, and blobs where there exists a discontinuity [15–17]. Shearlets are employed to describe the vessel's features and map the image into the neutrosophic domain. An indeterminacy filter is used to remove the uncertain information on the neutrosophic set. A line-like filter is also utilized to enhance the vessel regions. Finally, the vessel is identified via a neural network classifier.

2. Proposed Method

2.1. Shearlet Transform

Shearlet transformation enables image features to be analyzed in more flexible geometric structures with simpler mathematical approaches and is also able to reveal directional and anisotropic information at multi-scales [18]. In the 2-D case, the affine systems are defined as the collection:

$$SH_{\phi} f(a,s,t) = < f,\ \phi_{a,s,t} > \tag{1}$$

$$\phi_{a,s,t}(x) = |\det M_{a,s}|^{-\frac{1}{2}} \phi\left(M_{a,s}^{-1} x - t\right) \tag{2}$$

where $\phi_{a,s,t}$ is the shearlet coefficient. $M_{a,s} = B_s A_a = \begin{pmatrix} a & \sqrt{a}s \\ 0 & \sqrt{a} \end{pmatrix}$, and $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ is parabolic scaling matrix and $B_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ is shear matrix $(a > 0,\ s \in R,\ t \in R^2)$. In this equation a scale parameter is a real number greater than zero and s is a real number. In this case $M_{a,s}$ is the composition of the A_a and B_s.

2.2. Neutrosophic Indeterminacy Filtering

Recently, the neutrosophic theory extended from classical fuzzy theory denotes that neutrosophy has been successfully used in many applications for reducing the uncertainty and indeterminacy [19]. An element g in the neutrosophic set (NS) is defined as g (T, I, F), where T identifies true degree in the set, I identify the indeterminate degree in the set, and F identifies false in the set. T, I and F are the neutrosophic components. The previously reported studies demonstrated that the NS has a vital role in image processing [20–22].

A pixel $P(x,y)$ at the location of (x,y) in an image is described in the NS domain as $P_{NS}(x,y) = \{T(x,y), I(x,y), F(x,y)\}$, where $T(x,y)$, $I(x,y)$ and $F(x,y)$ are the membership values belonging to the bright pixel set, indeterminate set, and non-white set, respectively.

In this study, the fundus image's green channel is mapped into NS domain via shearlet feature values:

$$T(x,y) = \frac{ST_L(x,y) - ST_{Lmin}}{ST_{Lmax} - ST_{Lmin}} \tag{3}$$

$$I(x,y) = \frac{ST_H(x,y) - ST_{Hmin}}{ST_{Hmax} - ST_{Hmin}} \tag{4}$$

where T and I are the true and indeterminate membership values. $ST_L(x,y)$ is the low-frequency component of the shearlet feature at the current pixel $P(x,y)$. In addition, ST_{Lmin} and ST_{Lmax} are the minimum value and maximum value of the low-frequency component of the shearlet feature in the whole image, respectively. $ST_H(x,y)$ is the high-frequency component of the shearlet feature at the current pixel $P(x,y)$. Moreover, ST_{Hmin} and ST_{Hmax} are the minimum value and maximum value of the high-frequency component of the shearlet feature in the whole image, respectively. In the proposed algorithm, we only utilize neutrosophic components T and I for segmentation.

Then an IF (indeterminacy filter) is defined using the indeterminacy membership to reduce the indeterminacy in images. The IF is defined based on the indeterminacy value $I_s(x,y)$ having the kernel function as:

$$O_I(u,v) = \frac{1}{2\pi\sigma_I^2} e^{\left(-\frac{u^2+v^2}{2\sigma_I^2(x,y)}\right)} \tag{5}$$

$$\sigma_I(x,y) = f(I(x,y)) = rI(x,y) + q \tag{6}$$

where $O_I(u,v)$ is the kernel function in the local neighborhood. u and v are coordinator values of local neighborhood in kernel function. σ_I is the standard deviation of the kernel function, which is defined as a linear function associated to the indeterminate degree. r and q are the coefficients in the linear function to control the standard deviation value according to the indeterminacy value. Since the σ_I becomes large with a high indeterminate degree, the IF can create a smooth current pixel by using its neighbors, while with a low indeterminate degree, the value of σ_I is small and the IF performs less smoothing operation.

$$T'(x,y) = T(x,y) \oplus O_I(u,v) = \sum_{v=y-m/2}^{y+m/2} \sum_{u=x-m/2}^{x+m/2} T(x-u, y-v) O_I(u,v) \tag{7}$$

where T' is the indeterminate filtering result.

2.3. Line Structure Enhancement

A multiscale filter is employed on the image to enhance the line-like structure [17]. The local second-order partial derivatives, Hessian matrix, is computed and a line-likeness is defined using its eigenvalues. This measure can describe the vessels region in the fundus images and is shown as follows:

$$En(s) = \begin{cases} 0 & if\ \lambda_2 > 0\ or\ \lambda_3 > 0 \\ \left(1 - e^{-\frac{R_A^2}{2\alpha^2}}\right).e^{-\frac{R_B^2}{2\beta^2}}.\left(1 - e^{-\frac{S^2}{2c^2}}\right) & \text{otherwise} \end{cases} \tag{8}$$

$$S = \sqrt{\sum_{j \le D} \lambda_j^2} \tag{9}$$

$$R_A = R_A = \frac{|\lambda_2|}{|\lambda_3|} \tag{10}$$

$$R_B = R_B = \frac{|\lambda_1|}{\sqrt{|\lambda_2 \lambda_3|}} \tag{11}$$

where λ_k is the eigenvalue with the k-th smallest magnitude of the Hessian matrix. D is the dimension of the image. α, β and c are thresholds to control the sensitivity of the line filter to the measures R_A, R_B and S.

2.4. Algorithm of the Proposed Approach

A retinal vessel detection approach is proposed using shearlet transform and indeterminacy filtering on fundus images. Shearlet transform is employed to describe the vessel's features and map the green channel of the fundus image into the NS domain. An indeterminacy filter is used to remove the indeterminacy information on the neutrosophic set. A multiscale filter is utilized to enhance the vessel regions. Finally, the vessel is detected via a neural network classifier using the neutrosophic image and the enhanced image. The proposed method is summarized as:

1. Take the shearlet transform on green channel Ig;
2. Transform the Ig into neutrosophic set domain using the shearlet transform results, and the neutrosophic components are denoted as T and I;
3. Process indeterminacy filtering on T using I and the result is denoted as T′;
4. Perform the line-like structure enhancement filter on T′ and obtain the En;
5. Obtain the feature vector FV = [T′ I En] for the input of the neural network;
6. Train the neural network as a classifier to identify the vessel pixels;
7. Identify the vessel pixels using the classification results by the neural network.

The whole steps can be summarized using a flowchart in Figure 1.

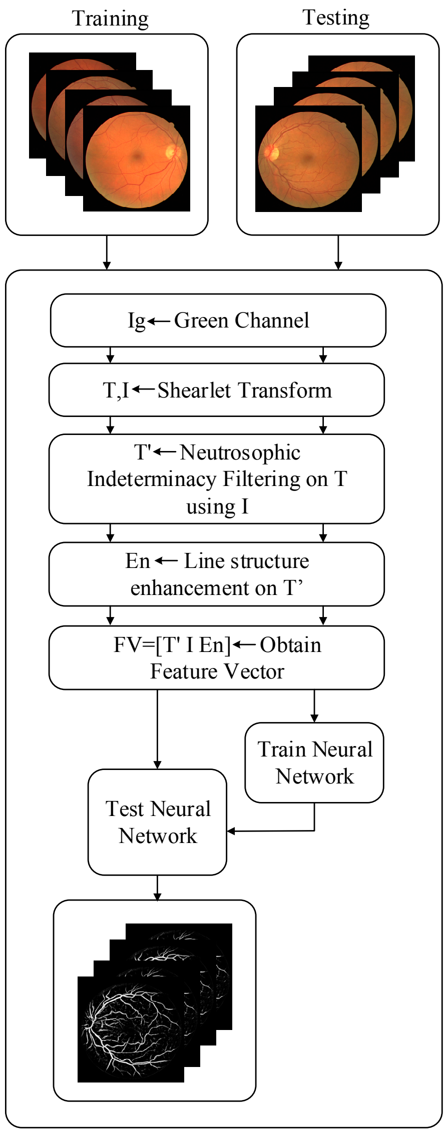


Figure 1. Flowchart for retinal vessel detection.

3. Experimental Results

3.1. Retinal Fundus Image Datasets

In the experimental section, we test the proposed method on two publicly available datasets namely the Digital Retinal Images for Vessel Extraction (DRIVE) and Structured Analysis of the Retina (STARE) datasets.

The DRIVE database was obtained from a diabetic retinopathy screening program in the Netherlands. This database was created to enable comparative studies on the segmentation of blood vessels in retinal images. Researchers are able to test their algorithms and compare their results with other studies in this database [23]. The DRIVE dataset contains 40 total fundus images and has been divided into training and test sets [4]. Training and test sets contain an equal number of images (20). Each image was captured using 8 bits per color plane at 768 by 584 pixels. The field of view (FOV) on each image is circular with a diameter of approximately 540 pixels and all images were cropped using FOV.

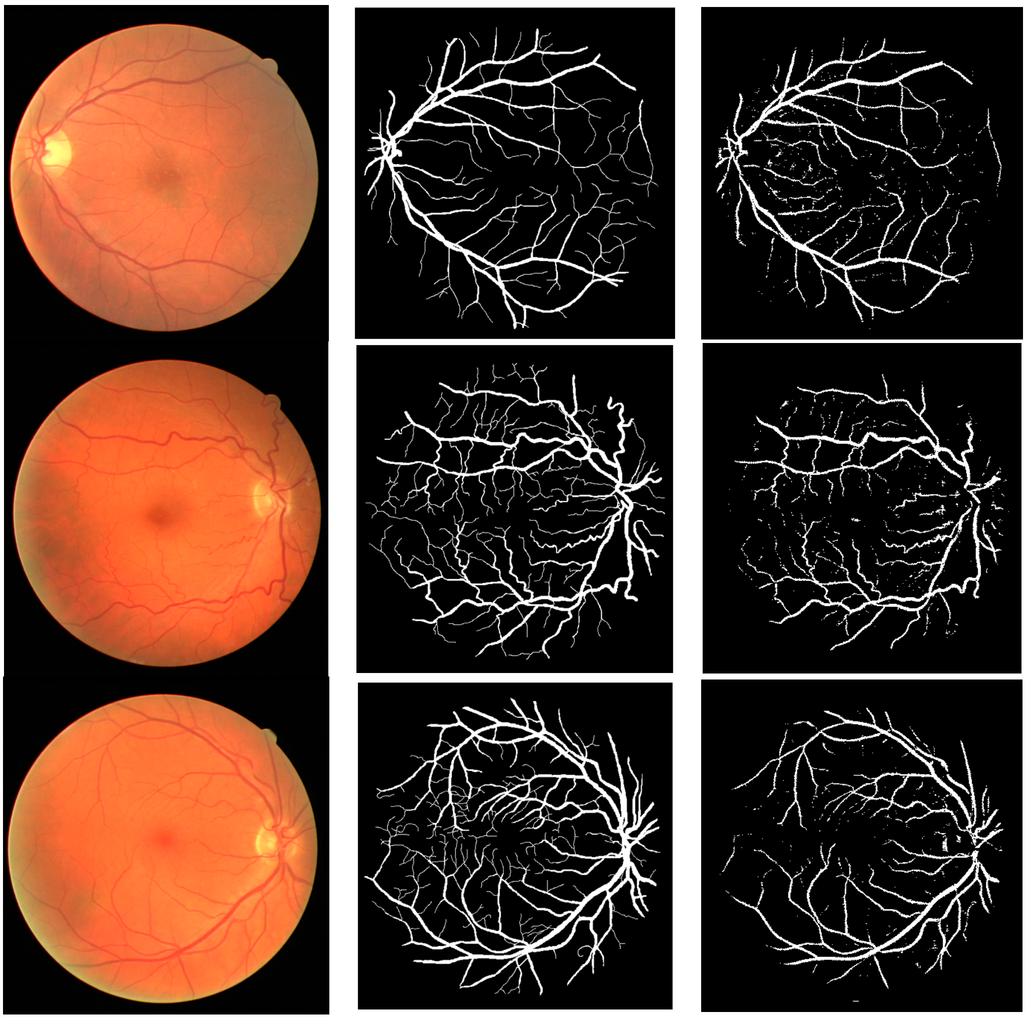
STARE (STructured Analysis of the REtina) Project was designed and initialized in 1975 by Michael Goldbaum, M.D. at the University of California, San Diego. Clinical images were obtained by the

Shiley Eye Center at the University of California, San Diego, and by the Veterans Administration Medical Center in San Diego [24]. The STARE dataset has 400 raw images. Blood vessel segmentation annotations have 20 hand-labeled images [2].

In our experiment, we select 20 images with ground truth results in the training set of the DRIVE dataset as the training samples, and 20 images in the test set from the DRIVE and 20 from the STARE for validation.

3.2. Experiment on Retinal Vessel Detection

The experimental results on DRIVE and STARE were demonstrated in Figures 2 and 3, respectively. The first columns of Figures 2 and 3 show the input retinal images, the second ones in Figures 2 and 3 are the ground-truth segmentation of the retinal vessels and the third ones in Figures 2 and 3 show the obtained results. We used a three-layered neural network classifier. While the input layer of the neural network classifier contained three nodes, hidden and the output layers contained 20 nodes and one node, respectively. The classifier was trained with scaled conjugate gradient backpropagation algorithm. The learning rate was chosen as 0.001, the momentum coefficient was set to 0.01. We used almost 1000 iterations during the training of the network.



(**a**) (**b**) (**c**)

Figure 2. Detection results by our proposed methods on three samples randomly taken from the Digital Retinal Images for Vessel Extraction (DRIVE) dataset: (**a**) Original (**b**) Corresponding ground truth and (**c**) Detection results.

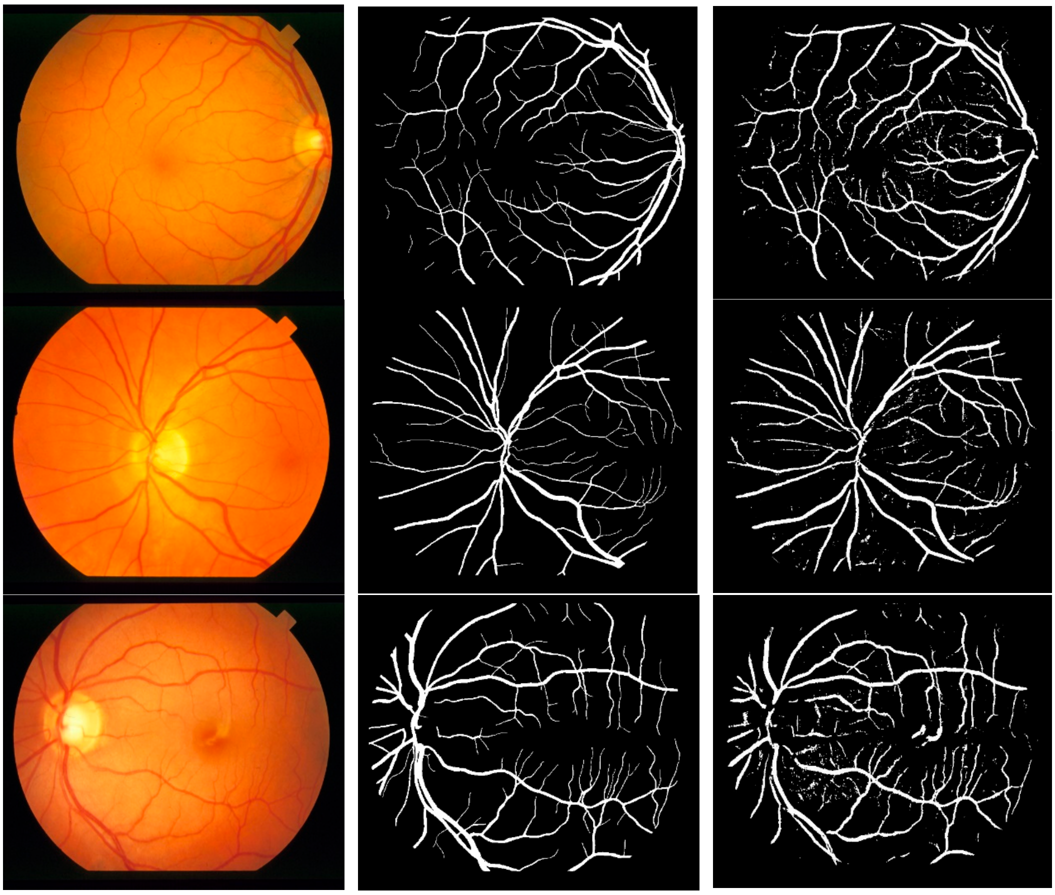


Figure 3. Detection results by our proposed methods on three samples randomly taken in the Structured Analysis of the Retina (STARE) dataset: (**a**) Original (**b**) Corresponding ground truth and (**c**) Detection results.

As seen in Figures 2 and 3, the proposed method obtained reasonable results in which the large vessels were detected perfectly. Only several thin vessel regions were missed. Three ROC curves are drawn to demonstrate the proposed method's performance in DRIVE and STARE datasets.

4. Discussion

The evaluation of the results was carried out using the receiver operating characteristic (ROC) curve, and the area under the ROC curve denoted as AUC. An AUC value tending to 1 demonstrates a successful classifier, AUC equal 0 indicates an unsuccessful classifier.

Figures 4 and 5 illustrate the ROC curves on the test set from DRIVE and STARE datasets, respectively. The AUC values also stressed the successful results of the proposed approach on both datasets. While the calculated the AUC value was 0.9476 for DRIVE data set, and a 0.9469 AUC value was calculated for STARE dataset.

We further compared the obtained results with some early published results. These comparison results in DRIVE and STARE were listed in Tables 1 and 2.

In Table 1, Maji et al. [14] have developed a collective learning method using 12 deep CNN models for vessel segmentation, Fu et al. [25] have proposed an approach combining CNN and CRF (Conditional Random Field) layers, and Niemeijer et al. [26] presented a vessel segmentation algorithm based on pixel classification using a simple feature vector. The proposed method achieved the highest AUC value for the DRIVE dataset. Fu et al. [25] also achieved the second highest AUC value.

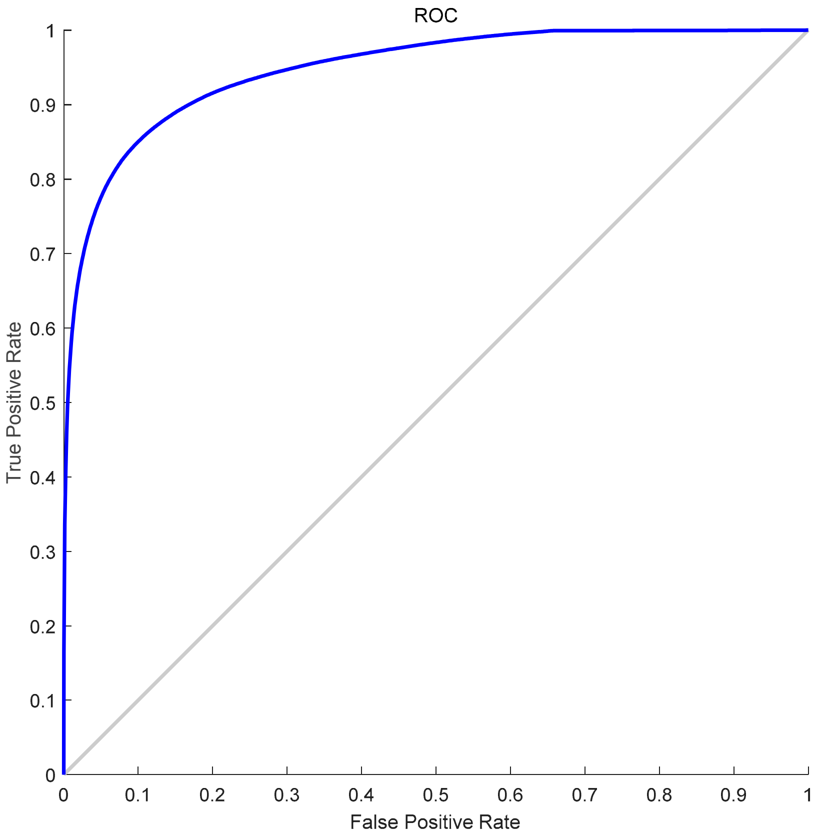


Figure 4. ROC curve in the test set of DRIVE.

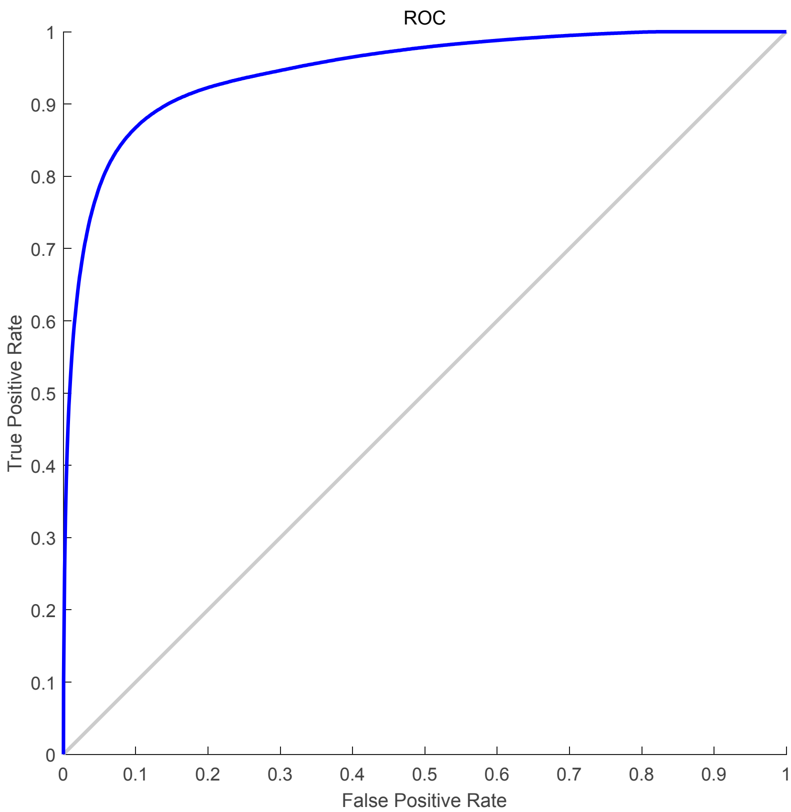


Figure 5. ROC curve in STARE.

In Table 2, Kande et al. [10] have recommended an unsupervised fuzzy based vessel segmentation method, Jiang et al. [2] have proposed an adaptive local thresholding method and Hoover et al. [27] also have combined local and region-based properties to segment blood vessels in retinal images. The highest AUC value was also obtained for STARE dataset with the proposed method.

In the proposed method, the post-processing procedure is not used to deal with the classification results from neural network. In future, we will employ some post-processing methods for improving the quality of the vessel detection.

Table 1. Comparison with the other algorithms on DRIVE dataset.

Method	AUC
Maji et al. [14]	0.9283
Fu et al. [25]	0.9470
Niemeijer et al. [26]	0.9294
Proposed method	0.9476

Table 2. Comparison with the other algorithm on STARE dataset.

Method	AUC
Jiang et al. [2]	0.9298
Hoover et al. [27]	0.7590
Kande et al. [10]	0.9298
Proposed method	0.9469

5. Conclusions

This study proposes a new method for retinal vessel detection. It initially forwards the input retinal fundus images into the neutrosophic domain via shearlet transform. The neutrosophic domain images are then filtered with two neutrosophic filters for noise reduction. Feature extraction and classification steps come after the filtering steps. The presented approach was tested on DRIVE and STARE. The results were evaluated quantitatively. The proposed approach outperformed the others by means of both evaluation methods. The comparison with the existing algorithms also stressed the high accuracy of the proposed approach. In future, we will employ some post-processing methods for improving the quality of the vessel detection.

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Scale Effect and Anisotropy Analyzed for Neutrosophic Numbers of Rock Joint Roughness Coefficient Based on Neutrosophic Statistics

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Abstract: In rock mechanics, the study of shear strength on the structural surface is crucial to evaluating the stability of engineering rock mass. In order to determine the shear strength, a key parameter is the joint roughness coefficient (JRC). To express and analyze JRC values, Ye et al. have proposed JRC neutrosophic numbers (JRC-NNs) and fitting functions of JRC-NNs, which are obtained by the classical statistics and curve fitting in the current method. Although the JRC-NNs and JRC-NN functions contain much more information (partial determinate and partial indeterminate information) than the crisp JRC values and functions in classical methods, the JRC functions and the JRC-NN functions may also lose some useful information in the fitting process and result in the function distortion of JRC values. Sometimes, some complex fitting functions may also result in the difficulty of their expressions and analyses in actual applications. To solve these issues, we can combine the neutrosophic numbers with neutrosophic statistics to realize the neutrosophic statistical analysis of JRC-NNs for easily analyzing the characteristics (scale effect and anisotropy) of JRC values. In this study, by means of the neutrosophic average values and standard deviations of JRC-NNs, rather than fitting functions, we directly analyze the scale effect and anisotropy characteristics of JRC values based on an actual case. The analysis results of the case demonstrate the feasibility and effectiveness of the proposed neutrosophic statistical analysis of JRC-NNs and can overcome the insufficiencies of the classical statistics and fitting functions. The main advantages of this study are that the proposed neutrosophic statistical analysis method not only avoids information loss but also shows its simplicity and effectiveness in the characteristic analysis of JRC.

Keywords: joint roughness coefficient (JRC); neutrosophic number; neutrosophic statistics; scale effect; anisotropy

1. Introduction

The engineering experience shows that rock mass may deform and destroy along the weak structural surfaces. The study of shear strength on the structural surface is crucial to evaluate the stability of engineering rock mass. In order to determine the shear strength in rock mechanics, a key parameter is the joint roughness coefficient (JRC). Since Barton [1] firstly defined the concept of JRC, a lot of methods had been proposed to calculate the JRC value and analyze its anisotropy and scale effect characteristics. Tse et al. [2] gave the linear regression relationship between the JRC value and the root mean square (Z_2). Then, Zhang et al. [3] improved the root mean square (Z_2) by considering the inclination angle, amplitude of asperities, and their directions, and then introduced a new roughness index (λ) by using the modified root mean square (Z_2') to calculate JRC values. To quantify the anisotropic roughness of joint surfaces effectively, a variogram function and a new index were proposed by Chen et al. [4] based on the digital image processing technique, and then they also studied the scale effect by calculating the JRC values of different sample lengths [5]. However, all of these traditional methods do not consider the uncertainties of JRC values in real rock engineering practice.

Recently, Ye et al. [6] not only utilized the crisp average value to express JRC by using traditional statistical methods, but also considered its interval range (indeterminate range) to express the indeterminate information of JRC by means of the neutrosophic function/interval function. They [6] firstly applied the neutrosophic function to calculate JRC values and shear strength, and got the relations between the sampling length and the maximum JRC values and between the sampling length and the minimum JRC values, and then established the neutrosophic functions (thick/interval functions) of JRC and shear strength. However, these thick/interval functions cannot express such an indeterminate function containing the parameters of neutrosophic numbers (NNs) (i.e., indeterminate parameters), where NN is composed of its determinate part a and its indeterminate part bI with indeterminacy I and as denoted by $z = a + bI$ for $a, b \in R$ (R is all real numbers) [7–9]. Obviously, NN is a very useful mathematical tool for the expression of the partial determinate and/or partial indeterminate information in engineering problems. After that, Ye et al. [10] further proposed two NN functions to express the anisotropic ellipse and logarithmic equations of JRC values corresponding to an actual case and to analyze the anisotropy and scale effect of JRC values by the derivative of the two NN functions, and then they further presented a NN function with two-variables so as to express the indeterminate information of JRC values comprehensively in the sample sizes and measurement orientations, and then they analyzed both the anisotropy and scale effect of JRC values simultaneously by the partial derivative of the NN function with two-variables. However, all of these NN functions are obtained by fitting curves of the measured values, where they may still lose some useful information between 3% and 16% in the fitting process and lack a higher fitting accuracy although the fitting degrees of these functions lie in the interval [84%, 97%] in actual applications [10]. Sometimes, some complex fitting functions may also result in the difficulty of their expressions and analyses in actual applications [10]. To overcome these insufficiencies, it is necessary to improve the expression and analysis methods for the JRC values by some new statistical method so that we can retain more vague, incomplete, imprecise, and indeterminate information in the expression and analysis of JRC and avoid the information loss and distortion phenomenon of JRC values. Thus, the neutrosophic interval statistical number (NISN) presented by Ye et al. [11] is composed of both NN and interval probability, and then it only expresses the JRC value with indeterminate information, but they lack the characteristic analysis of JRC values in [11].

However, determinate and/or indeterminacy information is often presented in the real world. Hence, the NNs introduced by Smarandache [7–9] are very suitable for describing determinate and indeterminate information. Then, the neutrosophic statistics presented in [9] is different from classical statistics. The former can deal with indeterminate statistical problems, while the latter cannot do them and can only obtain the crisp values. As mentioned above, since there exist some insufficiencies in the existing analysis methods of JRC, we need a new method to overcome the insufficiencies. For this purpose, we originally propose a neutrosophic statistical method of JRC-NNs to indirectly analyze the scale effect and anisotropy of JRC values by means of the neutrosophic average values and standard deviations of JRC-NNs (JRC values), respectively, to overcome the insufficiencies of existing analysis methods. The main advantages of this study are that the proposed neutrosophic statistical analysis method not only avoid information loss, but also show its simplicity and effectiveness in the characteristic analysis of JRC values.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts of NNs and gives the neutrosophic statistical algorithm to calculate the neutrosophic average value and standard deviation of NNs. Section 3 introduces the source of the JRC data and JRC-NNs in an actual case, where the JRC-NNs of 24 measurement orientations in each sample length and 10 sample lengths in each measurement orientation will be used for neutrosophic statistical analysis of JRC-NNs in the actual case study. In Section 4, the neutrosophic average values and standard deviations of the 24 JRC-NNs of different measurement orientations in each sample length are given based on the proposed neutrosophic statistical algorithm and are used for the scale effect analysis of JRC values. In Section 5, the neutrosophic average values and standard deviations of the 10 JRC-NNs of different sample lengths in each measurement orientations are given based on the proposed neutrosophic statistical algorithm and used for the anisotropic analysis of JRC values. Finally, concluding remarks are given in Section 6.

2. Basic Concepts and Neutrosophic Statistical Algorithm of NNs

NNs and neutrosophic statistics are firstly proposed by Smarandache [7–9]. This section will introduce some basic concepts of NNs and give the neutrosophic statistical algorithm of NNs to calculate the neutrosophic average value and the standard deviation of NNs for the neutrosophic statistical analysis of JRC-NNs in the following study.

A NN $z = a + bI$ consists of a determinate part a and an indeterminate part bI, where a and b are real numbers and $I \in [I^L, I^U]$ is indeterminacy. It is clear that the NN can express the determinate and/or indeterminate information. Here is a numerical example. A NN is $z = 5 + 6I$ for $I \in [0, 0.3]$. Then, the NN is $z \in [5, 6.8]$ for $I \in [0, 0.3]$ and its possible range/interval is $z = [5, 6.8]$, where its determinate part is 5 and its indeterminate part is $6I$. For the numerical example, $z = 5 + 6I$ for $I \in [0, 0.3]$ can be also expressed as another form $z = 5 + 3I$ for $I \in [0, 0.6]$. Therefore, we can specify some suitable interval range $[I^L, I^U]$ for the indeterminacy I according to the different applied demands to adapt the actual representation. In fact, NN is a changeable interval number depending on the indeterminacy $I \in [I^L, I^U]$.

As we know, data in classical statistics are determinate values/crisp values. On the contrary, data in neutrosophic statistics are interval values/indeterminate values/NNs, which contain indeterminacy. If there is no indeterminacy or crisp value in data, neutrosophic statistics is consistent with classical statistics. Let us consider an example of neutrosophic statistics in the following.

Assume that four NNs are $z_1 = 1 + 2I$, $z_2 = 2 + 3I$, $z_3 = 3 + 4I$, and $z_4 = 4 + 5I$ for $I \in [0, 0.2]$, then the average value of these four neutrosophic numbers can be obtained by the following calculational steps:

Firstly, the average value of the four determinate parts is obtained by the following calculation:

$$\bar{a} = (1 + 2 + 3 + 4)/4 = 2.5$$

Secondly, the average value of the four coefficients in the indeterminate parts is yielded by the following calculation:

$$\bar{b} = (2 + 3 + 4 + 5)/4 = 3.5$$

Finally, the neutrosophic average value of the four NNs is given as follows:

$$\bar{z} = 2.5 + 3.5I \text{ for } I \in [0, 0.2]$$

This neutrosophic average value is also called the average NN [9], which still includes its determinate and indeterminate information rather than a crisp value.

However, it is difficult to use the Smarandache's neutrosophic statistics for engineering applications. Thus, Ye et al. [12] presented some new operations of NNs to make them suitable for engineering applications.

Let two NNs be $z_1 = a_1 + b_1I$ and $z_2 = a_2 + b_2I$ for $I \in [I^L, I^U]$. Then, Ye et al. [12] proposed their basic operations:

$$
\begin{aligned}
z_1 + z_2 &= (a_1 + a_2) + (b_1 + b_2)I = [a_1 + a_2 + b_1 I^{\mathrm{L}} + b_2 I^{\mathrm{L}},\ a_1 + a_2 + b_1 I^{\mathrm{U}} + b_2 I^{\mathrm{U}}];\\
z_1 - z_2 &= a_1 - a_2 + (b_1 - b_2)I = [a_1 - a_2 + b_1 I^{\mathrm{L}} - b_2 I^{\mathrm{L}},\ a_1 - a_2 + b_1 I^{\mathrm{U}} - b_2 I^{\mathrm{U}}];\\
z_1 \times z_2 &= a_1 a_2 + (a_1 b_2 + a_2 b_1)I + (b_1 b_2)I^2\\
&= \begin{bmatrix} \min\begin{pmatrix} (a_1 + b_1 I^{\mathrm{L}})(a_2 + b_2 I^{\mathrm{L}}), (a_1 + b_1 I^{\mathrm{L}})(a_2 + b_2 I^{\mathrm{U}}),\\ (a_1 + b_1 I^{\mathrm{U}})(a_2 + b_2 I^{\mathrm{L}}), (a_1 + b_1 I^{\mathrm{U}})(a_2 + b_2 I^{\mathrm{U}}) \end{pmatrix}\\ \max\begin{pmatrix} (a_1 + b_1 I^{\mathrm{L}})(a_2 + b_2 I^{\mathrm{L}}), (a_1 + b_1 I^{\mathrm{L}})(a_2 + b_2 I^{\mathrm{U}}),\\ (a_1 + b_1 I^{\mathrm{U}})(a_2 + b_2 I^{\mathrm{L}}), (a_1 + b_1 I^{\mathrm{U}})(a_2 + b_2 I^{\mathrm{U}}) \end{pmatrix} \end{bmatrix};\\
\frac{z_1}{z_2} &= \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{[a_1 + b_1 I^{\mathrm{L}}, a_1 + b_1 I^{\mathrm{U}}]}{[a_2 + b_2 I^{\mathrm{L}}, a_2 + b_2 I^{\mathrm{U}}]}\\
&= \begin{bmatrix} \min\left(\frac{a_1 + b_1 I^{\mathrm{L}}}{a_2 + b_2 I^{\mathrm{U}}}, \frac{a_1 + b_1 I^{\mathrm{L}}}{a_2 + b_2 I^{\mathrm{L}}}, \frac{a_1 + b_1 I^{\mathrm{U}}}{a_2 + b_2 I^{\mathrm{U}}}, \frac{a_1 + b_1 I^{\mathrm{U}}}{a_2 + b_2 I^{\mathrm{L}}}\right),\\ \max\left(\frac{a_1 + b_1 I^{\mathrm{L}}}{a_2 + b_2 I^{\mathrm{U}}}, \frac{a_1 + b_1 I^{\mathrm{L}}}{a_2 + b_2 I^{\mathrm{L}}}, \frac{a_1 + b_1 I^{\mathrm{U}}}{a_2 + b_2 I^{\mathrm{U}}}, \frac{a_1 + b_1 I^{\mathrm{U}}}{a_2 + b_2 I^{\mathrm{L}}}\right) \end{bmatrix}.
\end{aligned}
\tag{1}
$$

Then, these basic operations are different from the ones introduced in [9], and this makes them suitable for engineering applications.

Based on Equation (1), we can give the neutrosophic statistical algorithm of the neutrosophic average value and standard deviation of NNs.

Let $z_i = a_i + b_i I$ ($i = 1, 2, \ldots, n$) be a group of NNs for $I \in [I^L, I^U]$, then their neutrosophic average value and standard deviation can be calculated by the following neutrosophic statistical algorithm:

Step 1: Calculate the neutrosophic average value of a_i ($i = 1, 2, \ldots, n$):

$$\overline{a} = \frac{1}{n}\sum_{i=1}^{n} a_i \tag{2}$$

Step 2: Calculate the average value of b_i ($i = 1, 2, \ldots, n$):

$$\overline{b} = \frac{1}{n}\sum_{i=1}^{n} b_i \tag{3}$$

Step 3: Obtain the neutrosophic average value:

$$\overline{z} = \overline{a} + \overline{b}I, \quad I \in [I^L, I^U] \tag{4}$$

Step 4: Get the differences between z_i ($i = 1, 2, \ldots, n$) and $\overline{z}$:

$$z_i - \overline{z} = a_i - \overline{a} + (b_i - \overline{b})I, \quad I \in [I^L, I^U] \tag{5}$$

Step 5: Calculate the square of all the differences between z_i ($i = 1, 2, \ldots, n$) and $\overline{z}$:

$$(z_i - \overline{z})^2 = \begin{bmatrix} \min\begin{pmatrix} (a_i + b_i I^{\mathrm{L}})(\overline{a} + \overline{b} I^{\mathrm{L}}), (a_i + b_i I^{\mathrm{L}})(\overline{a} + \overline{b} I^{\mathrm{U}}),\\ (a_i + b_i I^{\mathrm{U}})(\overline{a} + \overline{b} I^{\mathrm{L}}), (a_i + b_i I^{\mathrm{U}})(\overline{a} + \overline{b} I^{\mathrm{U}}) \end{pmatrix}\\ \max\begin{pmatrix} (a_i + b_i I^{\mathrm{L}})(\overline{a} + \overline{b} I^{\mathrm{L}}), (a_i + b_i I^{\mathrm{L}})(\overline{a} + \overline{b} I^{\mathrm{U}}),\\ (a_i + b_i I^{\mathrm{U}})(\overline{a} + \overline{b} I^{\mathrm{L}}), (a_i + b_i I^{\mathrm{U}})(\overline{a} + \overline{b} I^{\mathrm{U}}) \end{pmatrix} \end{bmatrix}, \quad I \in [I^L, I^U] \tag{6}$$

Step 6: Calculate the neutrosophic standard deviation:

$$\sigma_z = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2} \quad (7)$$

In the following sections, we shall apply the proposed neutrosophic statistical algorithm of NNs to the characteristic analysis of JRC data.

3. JRC Values and JRC-NNs in an Actual Case

As an actual case study in this paper, the original roughness profiles were measured by using profilograph and a roughness ruler [13] on a natural rock joint surface in Changshan County of Zhejiang Province, China. In the actual case, based on a classical statistical method we have obtained the average values μ_{ij} and standard deviations σ_{ij} (i = 1, 2, ... , 24; j = 1, 2, ... , 10) of actually measured data in different sample lengths and different measurement orientations, which are shown in Table 1.

Then, we can use NNs $z_{ij} = a_{ij} + b_{ij}I$ (i = 1, 2, ... , 24; j = 1, 2, ... , 10) to express the JRC values in each orientation θ and in each sample length L. Various NNs of the JRC values are indicated by the real numbers of a_{ij} and b_{ij} in z_{ij} (i = 1, 2, ... , 24; j = 1, 2, ... , 10). For convenient neutrosophic statistical analysis, the indeterminacy I is specified as the unified form $I \in [0, 1]$ in all the JRC-NNs. Thus, there is $z_{ij} = a_{ij} + b_{ij}I = \mu_{ij} - \sigma_{ij} + 2\,\sigma_{ij}I$ (i = 1, 2, ... , 24; j = 1, 2, ... , 10), where $a_{ij} = \mu_{ij} - \sigma_{ij}$ is the lower bound of the JRC value and z_{ij} may choose a robust range/confidence interval [$\mu_{ij} - \sigma_{ij}$, $\mu_{ij} + \sigma_{ij}$] for the symmetry about the average value μ_{ij} (see the references [10,11] in detail), and then based on μ_{ij} and σ_{ij} in Table 1 a_{ij} and b_{ij} in z_{ij} (i = 1, 2, ... , 24; j =1, 2, ... , 10) are shown in Table 2. For example, when $\theta = 0°$ and L = 10 cm for i = 1 and j = 1, we can obtain from Table 2 that the JRC-NN is $z_{11} = 8.3040 + 4.4771I$ for $I \in [0, 1]$.

According to the measurement orientation θ and the sample length L in Table 2, the data in the same column consists of a group of the data in each sample length L, and then there are 10 groups in the JRC-NNs. On the other hand, the data in each row are composed of a group of the data in each measurement orientation θ, and then there are 24 groups in the JRC-NNs. In the following, we shall give the neutrosophic statistical analysis of the JRC-NNs based on the proposed neutrosophic statistical algorithm to reflect their scale effect and anisotropy in the actual case.

Table 1. The average values μ_{ij} and standard deviations σ_{ij} of actually measured data in different sample lengths L and different measurement orientations θ.

L	10 cm		20 cm		30 cm		40 cm		50 cm		60 cm		70 cm		80 cm		90 cm		100 cm	
θ	μ_{i1}	σ_{i1}	μ_{i2}	σ_{i2}	μ_{i3}	σ_{i3}	μ_{i4}	σ_{i4}	μ_{i5}	σ_{i5}	μ_{i6}	σ_{i6}	μ_{i7}	σ_{i7}	μ_{i8}	σ_{i8}	μ_{i9}	σ_{i9}	μ_{i10}	σ_{i10}
0°	10.5425	2.2385	9.6532	1.7162	9.2733	1.5227	8.9745	1.7092	8.8222	1.6230	8.8016	1.6069	8.6815	1.6066	8.6009	1.5043	8.5681	1.3465	8.4630	1.2806
15°	10.7111	2.2392	9.9679	1.7379	9.3433	1.5555	9.2708	1.2743	9.2299	1.2850	8.9729	1.3071	8.8332	1.1706	8.5868	0.9413	8.3604	0.7673	8.1404	0.6372
30°	10.5943	2.3528	9.9289	2.0286	9.5715	1.6665	9.1209	1.4207	9.0920	1.4119	8.6006	0.9899	8.7596	1.1489	8.5713	1.0776	8.2927	1.0128	8.1041	0.9664
45°	9.9244	2.3120	9.2005	1.7237	9.0081	1.6464	8.5078	1.1376	8.3336	1.431	8.6237	1.3427	8.3262	1.2184	8.0768	1.2717	7.8458	1.2096	7.5734	1.1294
60°	9.0253	2.4592	8.4047	1.9813	7.8836	1.8199	7.7941	1.8829	7.1873	1.167	8.2678	1.7830	7.3595	1.5956	7.1381	1.4082	6.8722	1.2178	6.7131	0.9627
75°	7.9352	2.1063	7.4604	1.7756	6.7725	1.4153	6.3056	1.0241	6.5446	1.2140	6.4993	1.3108	6.2440	1.1208	6.0933	0.9171	5.9499	0.7311	5.8317	0.5855
90°	7.0467	2.4054	6.6915	1.8482	6.3378	1.4743	5.9993	1.1700	6.1481	1.1920	6.0893	1.1850	5.9543	1.1021	5.8932	0.9630	5.8259	0.9181	5.8219	0.8355
105°	7.7766	2.4105	7.2221	1.7560	6.6770	1.2608	6.2318	0.985	6.4634	1.2288	6.4609	1.5029	6.1670	1.3236	5.9923	1.1016	5.8903	0.9868	5.8359	0.8479
120°	9.1324	2.3250	8.5206	1.8963	8.1998	1.5792	7.9671	1.4094	7.3207	1.0418	7.8245	1.1807	7.2472	1.0637	7.0649	0.9507	6.8537	0.8122	6.6909	0.7715
135°	9.2258	1.9104	8.5670	1.5412	8.0898	1.3452	7.8194	0.9910	7.3735	0.9848	7.6660	1.2845	7.3846	1.1608	7.0872	1.1589	6.9154	1.0345	6.7586	0.9157
150°	10.4673	2.4365	9.5650	1.9065	8.9102	1.6863	8.9059	1.4562	8.3930	1.1855	8.8162	1.5870	8.2064	1.3432	8.0153	1.1287	7.6556	1.0101	7.4443	0.9080
165°	10.6035	2.2090	9.9647	1.6606	9.5320	1.5695	8.8760	1.5994	8.6121	1.4899	8.6463	1.5942	8.3931	1.3637	8.1107	1.2203	7.9051	1.0893	7.7175	1.0050
180°	9.8501	2.1439	9.0984	1.8556	8.7574	1.7300	8.6002	1.6753	8.2973	1.5862	8.1266	1.6278	7.9647	1.4864	7.8981	1.3395	7.8338	1.1935	7.8291	1.0616
195°	9.9383	2.2254	9.2299	1.8331	8.6781	1.6791	8.7993	1.4556	8.5308	1.5551	8.1016	1.5598	7.9219	1.2559	7.6562	0.9674	7.4610	0.8060	7.3131	0.7402
210°	9.5903	1.9444	8.9414	1.5298	8.6532	1.6227	8.2601	1.5626	8.2065	1.5438	7.3828	1.2507	7.7527	1.2989	7.5050	1.1484	7.2495	1.0876	7.0479	0.9558
225°	8.9167	1.9764	8.2550	1.4256	8.1330	1.4751	7.7012	1.2124	7.6798	1.4502	7.4365	1.1748	7.3183	1.2086	7.1309	1.2749	6.8652	1.2190	6.6742	1.1571
240°	7.8582	1.8456	7.3032	1.4385	6.8241	1.1626	6.7427	1.2022	6.3250	0.8971	6.8181	1.1123	6.3526	1.0430	6.1521	0.9953	5.9138	0.8906	5.7515	0.7329
255°	7.2166	1.9341	6.8638	1.3901	6.3349	1.2705	6.1050	1.0350	6.0333	0.9671	6.0693	1.1394	5.8924	0.9417	5.7122	0.8153	5.7803	0.8598	5.3946	0.5627
270°	6.8025	2.1165	6.3123	1.6374	6.0061	1.3786	5.8815	1.3700	5.7871	1.1783	5.9707	1.2858	5.8530	1.2711	5.7376	1.1886	5.8259	0.9181	5.5856	1.0273
285°	7.0061	1.5474	6.4941	1.1183	6.1107	0.9586	5.8455	0.9821	5.7563	0.9033	6.0606	1.3603	5.8403	1.2714	5.6386	1.1359	5.4716	1.0374	5.3629	0.9501
300°	8.4720	1.7448	7.8124	1.3531	7.5303	1.2127	7.2813	1.0247	6.9533	1.1089	7.0673	0.8880	6.8002	0.9202	6.6414	0.8727	6.4460	0.8434	6.3104	0.7904
315°	10.1428	2.4790	9.4554	2.1149	8.9644	1.7308	8.5698	1.4949	8.1224	1.4089	8.6863	1.5162	8.3659	1.5934	7.6582	1.3811	7.4641	1.1563	7.3537	1.0960
330°	9.8295	2.2844	9.0011	1.6139	8.3261	1.6005	8.3290	1.3232	7.8712	1.2376	8.0526	1.2755	7.9134	1.1209	7.6498	1.0157	7.3466	0.9740	7.0927	0.9342
345°	9.6831	2.0192	9.1761	1.6305	8.7732	1.1686	8.4741	1.1887	7.8597	1.1436	7.8485	1.0332	7.7270	1.0174	7.4667	0.9254	7.1781	0.821	7.0038	0.7346

Table 2. The values of a_{ij} and b_{ij} in JRC neutrosophic numbers (JRC-NNs) z_{ij} (i = 1, 2, . . . , 24; j =1, 2, . . . , 10) for each orientation θ and each sample length L.

L	**10 cm**		**20 cm**		**30 cm**		**40 cm**		**50 cm**		**60 cm**		**70 cm**		**80 cm**		**90 cm**		**100 cm**	
θ	a_{i1}	b_{i1}	a_{i2}	b_{i2}	a_{i3}	b_{i3}	a_{i4}	b_{i4}	a_{i5}	b_{i5}	a_{i6}	b_{i6}	a_{i7}	b_{i7}	a_{i8}	b_{i8}	a_{i9}	b_{i9}	a_{i10}	b_{i10}
0°	8.3040	4.4771	7.9370	3.4325	7.7506	3.0454	7.2653	3.4184	7.1992	3.2459	7.1947	3.2138	7.0750	3.2132	7.0966	3.0085	7.2216	2.6930	7.1824	2.5612
15°	8.4719	4.4784	8.2300	3.4759	7.7878	3.1110	7.9964	2.5487	7.9449	2.5700	7.6657	2.6142	7.6627	2.3412	7.6456	1.8825	7.5931	1.5347	7.5032	1.2745
30°	8.2415	4.7057	7.9003	4.0572	7.9051	3.3330	7.7002	2.8414	7.6801	2.8239	7.6107	1.9798	7.6107	2.2977	7.4938	2.1552	7.2799	2.0256	7.1377	1.9328
45°	7.6124	4.6240	7.4768	3.4474	7.3616	3.2929	7.3701	2.2753	6.9018	2.8636	7.2810	2.6853	7.1078	2.4369	6.8051	2.5434	6.6362	2.4192	6.4440	2.2589
60°	6.5660	4.9185	6.4234	3.9627	6.0638	3.6397	5.9112	3.7658	6.0203	2.3341	6.4848	3.5660	5.7639	3.1912	5.7299	2.8163	5.6544	2.4355	5.7504	1.9253
75°	5.8289	4.2126	5.6847	3.5513	5.3573	2.8306	5.2815	2.0483	5.3307	2.4279	5.1885	2.6216	5.1232	2.2416	5.1762	1.8342	5.2188	1.4622	5.2462	1.1710
90°	4.6413	4.8108	4.8432	3.6965	4.8635	2.9486	4.8293	2.3399	4.9561	2.3841	4.9043	2.3701	4.8522	2.2043	4.9302	1.9260	4.9078	1.8362	4.9865	1.6709
105°	5.3661	4.821	5.4661	3.5119	5.4162	2.5216	5.2460	1.9717	5.2346	2.4576	4.9580	3.0058	3.0054	2.6472	4.8907	2.2031	4.9034	1.9737	4.9881	1.6957
120°	6.8074	4.6500	6.6243	3.7926	6.6206	3.1584	6.5577	2.8188	6.2789	2.0837	6.6438	2.3614	6.1834	2.1274	6.1142	1.9014	6.0415	1.6243	5.9194	1.5430
135°	7.3153	3.8208	7.0258	3.0824	6.7446	2.6904	6.8283	1.9821	6.3887	1.9696	6.3815	2.5690	6.2238	2.3216	5.9283	2.3178	5.8810	2.0689	5.8429	1.8314
150°	8.0308	4.8731	7.6585	3.8130	7.2240	3.3725	7.4497	2.9125	7.2075	2.3710	7.2292	7.2291	6.8633	2.6863	6.8866	2.2573	6.6454	2.0203	6.5363	1.8161
165°	8.3945	4.4180	8.3040	3.3213	7.9625	3.1391	7.2766	3.1988	7.1222	2.9799	7.0521	3.1884	7.0294	2.7274	6.8904	2.4406	6.8158	2.1787	6.7124	2.0101
180°	7.7062	4.2877	7.2427	3.7113	7.0273	3.4601	6.9249	3.3506	6.7111	3.1724	6.4988	3.2556	6.4782	2.9729	6.5586	2.6790	6.6403	2.3871	6.7675	2.1232
195°	7.7130	4.4507	7.3968	3.6661	6.9990	3.3583	7.3437	2.9113	6.9757	3.1102	6.5419	3.1195	6.6660	2.5119	6.6888	1.9348	6.6550	1.6120	6.5729	1.4803
210°	7.6459	3.8887	7.4116	3.0596	7.0305	3.2453	6.6975	3.1252	6.6628	3.0875	6.1321	2.5014	6.4538	2.5977	6.3566	2.2967	6.1619	2.1752	6.0921	1.9116
225°	6.9402	3.9529	6.8294	2.8512	6.6580	2.9502	6.4888	2.4248	6.2296	2.9004	6.2617	2.3495	6.1097	2.4172	5.8560	2.5498	5.6462	2.4379	5.5170	2.3143
240°	6.0125	3.6913	5.8648	2.8769	5.6615	2.3252	5.5405	2.4044	5.4280	1.7941	5.7058	2.2246	5.3096	2.0861	5.1568	1.9906	5.0231	1.7812	5.0186	1.4658
255°	5.2825	3.8683	5.4738	2.7801	5.0644	2.5410	5.0700	2.0701	5.0662	1.9343	4.9300	2.2788	4.9507	1.8834	4.8968	1.6307	4.9204	1.7197	4.8319	1.1253
270°	4.6859	4.2330	4.6748	3.2749	4.6275	2.7571	4.5115	2.7401	4.6088	2.3565	4.6849	2.5716	4.5820	2.5422	4.5490	2.3772	4.9078	1.8362	4.5584	2.0545
285°	5.4587	3.0948	5.3757	2.2367	5.1521	1.9172	4.8634	1.9642	4.8530	1.8066	4.7003	2.7205	4.5688	2.5429	4.5027	2.2719	4.4341	2.0749	4.4128	1.9002
300°	6.7272	3.4897	6.4594	2.7061	6.3176	2.4254	6.2566	2.0494	5.8444	2.2178	6.1793	1.7760	5.8800	1.8404	5.7687	1.7453	5.6025	1.6869	5.5200	1.5808
315°	7.6638	4.9579	7.3405	4.2297	7.2336	3.4616	7.0749	2.9898	6.7135	2.8178	7.1701	3.0324	6.7725	3.1868	6.2771	2.7622	6.3079	2.3125	6.2577	2.1921
330°	7.5450	4.5689	7.3872	3.2277	6.7256	3.2009	7.0058	2.6464	6.6335	2.4751	6.7770	2.5510	6.7925	2.2418	6.6340	2.0314	6.3726	1.9480	6.1586	1.8684
345°	7.6639	4.0383	7.5456	3.2610	7.6046	2.3372	7.2854	2.3774	6.7161	2.2872	6.8153	2.0664	6.7096	2.0348	6.5413	1.8508	6.3570	1.6421	6.2692	1.4692

4. Scale Effect Analysis in Different Sample Lengths Based on the Neutrosophic Statistical Algorithm

In this section, we give the neutrosophic statistical analysis of JRC-NNs of each column in Table 2 based on the neutrosophic statistical algorithm to reflect the scale effect of JRC-NNs in different sample lengths.

In the neutrosophic statistical analysis of JRC-NNs, we need to calculate the neutrosophic average value and the standard deviation of each group of JRC-NNs in each column by using Equations (2)–(7). To show their calculational procedures in detail, we give the following example.

For example, the neutrosophic average value and standard deviation of the JRC-NNs in L = 10 cm is calculated. Then, we give the following calculational steps based on the neutrosophic statistical algorithm.

Step 1: By Equation (2), calculate the average value of the determinate parts a_{i1} (i = 1, 2, ... , 24) in the JRC-NNs corresponding to the first column as follows:

$$\begin{aligned}\bar{a}_1 &= \frac{1}{24}\sum_{i=1}^{24} a_{i1} = (8.304 + 8.4719 + 8.2415 + 7.6124 + 6.566 + 5.8289 + 4.6413 + 5.3661 \\ &\quad +6.8074 + 7.3153 + 8.0308 + 8.3945 + 7.7062 + 7.713 + 7.6459 + 6.9402 + \\ &\quad 6.0125 + 5.2825 + 4.6859 + 5.4587 + 6.7272 + 7.6638 + 7.545 + 7.6639)/24 \\ &= 6.9427.\end{aligned}$$

Step 2: By Equation (3), calculate the average value of the indeterminate coefficients b_{i1} (i = 1, 2, ... , 24) in the JRC-NNs:

$$\begin{aligned}\bar{b}_1 &= \frac{1}{24}\sum_{i=1}^{24} b_{i1} = (4.4771 + 4.4784 + 4.7057 + 4.624 + 4.9185 + 4.2126 + 4.8108 + 4.821 \\ &\quad +4.65 + 3.8208 + 4.8731 + 4.418 + 4.2877 + 4.4507 + 3.8887 + 3.9529 + \\ &\quad 3.6913 + 3.8683 + 4.233 + 3.0948 + 3.4897 + 4.9579 + 4.5689 + 4.0383)/24 \\ &= 4.3055.\end{aligned}$$

Step 3: By Equation (4), obtain the neutrosophic average value of the JRC-NNs in the first column:

$$\bar{z}_1 = \bar{a}_1 + \bar{b}_1 I = 6.9427 + 4.3055I, \quad I \in [0,1].$$

Step 4: By Equation (5), calculate the differences between z_{i1} (i = 1, 2, ... , 24) and $\bar{z}_1$:

$$\begin{aligned}z_{11} - \bar{z}_1 &= (a_{11} - \bar{a}_1) + (b_{11} - \bar{b}_1)I = 1.3613 + 0.1716I, \ldots, \\ z_{241} - \bar{z}_1 &= (a_{241} - \bar{a}_1) + (b_{241} - \bar{b}_1)I = 0.7212 - 0.2672I, \quad I \in [0,1].\end{aligned}$$

Step 5: By Equation (6), calculate the square of all the differences:

$$\begin{aligned}(z_{11} - \bar{z}_1)^2 &= [\min((a_{11} - \bar{a}_1)^2, (a_{11} - \bar{a}_1)((a_{11} - \bar{a}_1) + 1 \times (b_{11} - \bar{b}_1)), ((a_{11} - \bar{a}_1) + 1 \times (b_{11} - \bar{b}_1))^2), \\ &\quad \max((a_{11} - \bar{a}_1)^2, (a_{11} - \bar{a}_1)((a_{11} - \bar{a}_1) + 1 \times (b_{11} - \bar{b}_1)), ((a_{11} - \bar{a}_1) + 1 \times (b_{11} - \bar{b}_1))^2)] \\ &= [(a_{11} - \bar{a}_1)^2, ((a_{11} - \bar{a}_1) + 1 \times (b_{11} - \bar{b}_1))^2] = [1.8530, 2.3495], \ldots, \\ (z_{241} - \bar{z}_1)^2 &= [0.2061, 0.5202].\end{aligned}$$

Step 6: By Equation (7), calculate the neutrosophic standard deviation:

$$\begin{aligned}\sigma_{z1} &= \sqrt{\frac{1}{24}\sum_{i=1}^{24}(z_{i1} - \bar{z}_1)^2} = \left[\sqrt{\frac{1}{24}(1.8530 + \ldots + 0.2061)}, \sqrt{\frac{1}{24}(2.3495 + \ldots + 0.5202}\right] \\ &= [1.0866, 1.4375].\end{aligned}$$

Thus, the neutrosophic average value and the standard deviation of the JRC-NNs in the first column are obtained by the above calculational steps. By the similar calculational steps, the neutrosophic average values and standard deviations of JRC-NNs in other columns can be also obtained and all of the results are shown in Table 3. Then, the neutrosophic average values and standard deviations of JRC-NNs in different sample lengths are depicted in Figures 1 and 2.

Table 3. The neutrosophic average values and standard deviations of JRC-NNs in different sample lengths.

Sample Length L	Average Value			Standard Deviation σ_{zj}
	$\bar{a}_j$	$\bar{b}_j$	$\bar{z}_j$ ($I \in [0, 1]$)	
10 cm	6.9427	4.3055	[6.9427, 11.2482]	[1.0866, 1.4375]
20 cm	6.7740	3.3761	[6.7740, 10.1501]	[0.9894, 1.3176]
30 cm	6.5483	2.9609	[6.5483, 9.5092]	[0.9878, 1.3073]
40 cm	6.4490	2.6322	[6.4490, 9.0812]	[0.9607, 1.3257]
50 cm	6.2795	2.5196	[6.2795, 8.7991]	[0.8988, 1.2243]
60 cm	6.2913	2.6582	[6.2913, 8.9495]	[0.8594, 1.1493]
70 cm	6.1505	2.4706	[6.1505, 8.6211]	[0.8711, 1.1260]
80 cm	6.0573	2.2253	[6.0573, 8.2826]	[0.8352, 1.0883]
90 cm	5.9928	1.9952	[5.9928, 7.9880]	[0.7960, 1.0300]
100 cm	5.9261	1.7990	[5.9261, 7.7251]	[0.7644, 1.0553]

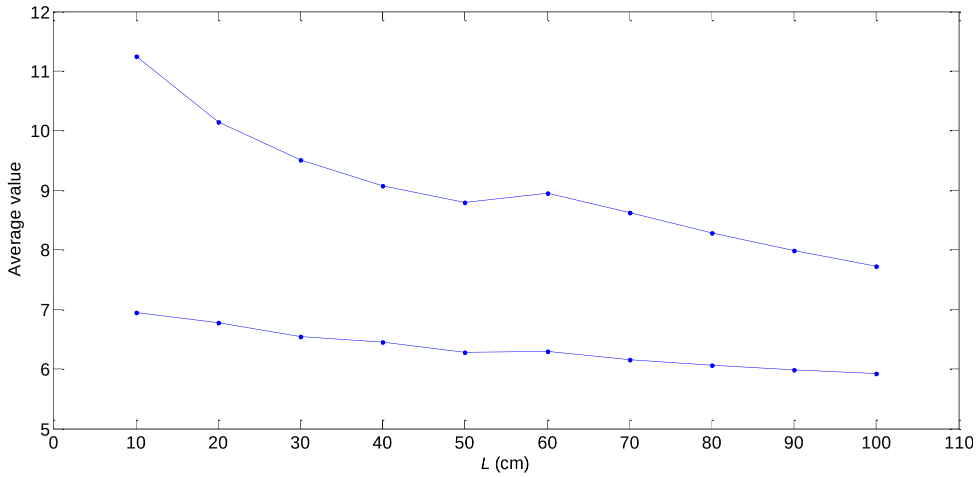

Figure 1. The neutrosophic average values of JRC-NNs in different sample lengths L.

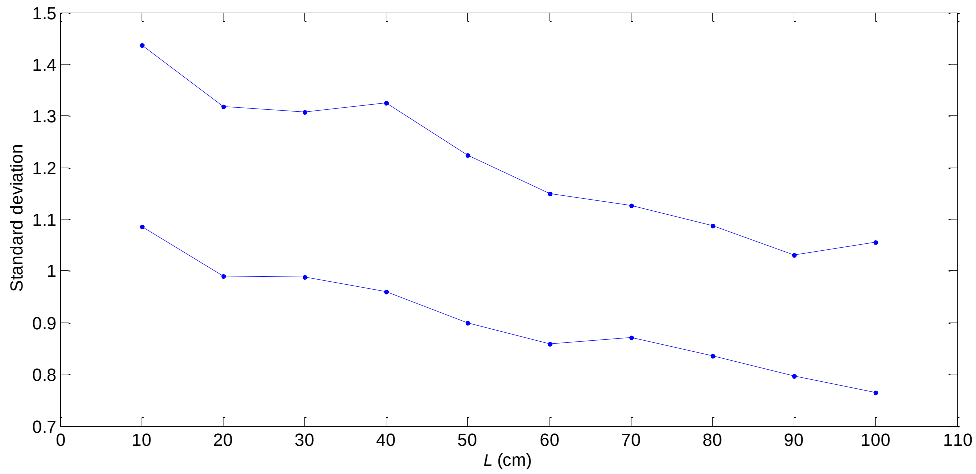

Figure 2. The neutrosophic standard deviations of JRC-NNs in different sample lengths L.

Figure 1 shows that the neutrosophic average values (ranges) of JRC-NNs decrease with the sample length increases. It is obvious that they can reflect the scale effect in different lengths. In other words, the larger the length L is, the smaller the average value (range) of JRC-NNs is. Thus, the scale effect in different lengths is consistent with that of the literature [10].

In Figure 2, we can see that the neutrosophic standard deviations of JRC-NNs decrease with the sample length increases. Since the standard deviation is used to indicate the dispersion degree of data, the neutrosophic standard deviation in some length L means the dispersion degree of the JRC-NNs. The larger the standard deviation is, the more discrete the JRC-NNs is. Under some sample lengths, its standard deviation means the dispersion degree of the JRC-NNs in different orientations. The larger the neutrosophic standard deviation is, the more obvious the anisotropy of the JRC-NNs under this length is. Hence, the neutrosophic standard deviations of JRC-NNs can also indicate the scale effect of the anisotropy of JRC-NNs. What's more, when the sample length is large enough, the anisotropy of the JRC values may decrease to some stable tendency. This situation is consistent with the tendency in [10].

Obviously, both neutrosophic average values and neutrosophic standard deviations of JRC-NNs can reflect the scale effect of JRC-NNs. Then, the neutrosophic average values reflect the scale effect of JRC values, while the neutrosophic standard deviations reflect the scale effect of the anisotropy of JRC values.

5. Anisotropic Analysis in Different Measurement Orientations Based on the Neutrosophic Statistical Algorithm

In this section, we give the neutrosophic statistical analysis of JRC-NNs of each row in Table 2 based on the neutrosophic statistical algorithm to reflect the anisotropy of JRC-NNs in different measurement orientations.

To indicate the neutrosophic statistical process, we take the orientation of $\theta = 0°$ for $i = 1$ as an example to show the detailed calculational steps of the neutrosophic average value and the standard deviation of the JRC-NNs in the orientation based on the neutrosophic statistical algorithm.

Step 1: By Equation (2), calculate the average value of the determinate parts a_{1j} ($j = 1, 2, \ldots, 10$) of the JRC-NNs in the first row ($i = 1$) as follows:

$$\bar{a}_1 = \frac{1}{10}\sum_{j=1}^{10} a_{1j} = (8.304 + 7.9370 + 7.7506 + 7.2653 + 7.1992 + 7.1947 + 7.0750 + 7.0966$$
$$+7.2216 + 7.1824)/10 = 7.4226.$$

Step 2: By Equation (3), calculate the average value of b_{1j} ($j = 1, 2, \ldots, 10$) in the indeterminate parts of the JRC-NNs:

$$\bar{b}_1 = \frac{1}{10}\sum_{j=1}^{10} b_{1j} = (4.4771 + 3.4325 + 3.0454 + 3.4184 + 3.2459 + 3.2138 + 3.2132 + 3.0085$$
$$+2.6930 + 2.5612)/10 = 3.2309.$$

Step 3: By Equation (4), get the neutrosophic average value of the JRC-NNs in the first row:

$$\bar{z}_1 = \bar{a}_1 + \bar{b}_1 I = 7.4226 + 3.2309I, \quad I \in [0, 1].$$

Step 4: By Equation (5), calculate the differences between z_{1j} ($j = 1, 2, \ldots, 10$) and $\bar{z}_1$:

$$z_{11} - \bar{z}_1 = (a_{11} - \bar{a}_1) + (b_{11} - \bar{b}_1)I = 0.8814 + 1.2462I, \ldots,$$
$$z_{110} - \bar{z}_1 = (a_{110} - \bar{a}_1) + (b_{110} - \bar{b}_1)I = 0.2402 - 0.6697I, \quad I \in [0, 1].$$

Step 5: By Equation (6), calculate the square of these differences:

$$\begin{aligned}(z_{11}-\bar{z}_1)^2 &= [\min((a_{11}-\bar{a}_1)^2,(a_{11}-\bar{a}_1)((a_{11}-\bar{a}_1)+1\times(b_{11}-\bar{b}_1)),((a_{11}-\bar{a}_1)+1\times(b_{11}-\bar{b}_1))^2),\\ &\quad \max((a_{11}-\bar{a}_1)^2,(a_{11}-\bar{a}_1)((a_{11}-\bar{a}_1)+1\times(b_{11}-\bar{b}_1)),((a_{11}-\bar{a}_1)+1\times(b_{11}-\bar{b}_1))^2)]\\ &= [(a_{11}-\bar{a}_1)^2,((a_{11}-\bar{a})+1\times(b_{11}-\bar{b}_1))^2]=[0.7767,4.5263],\ldots,\\ (z_{110}-\bar{z}_1)^2 &= [0.0577,0.8279].\end{aligned}$$

Step 6: By Equation (7), calculate the neutrosophic standard deviation:

$$\begin{aligned}\sigma_{z1} &= \sqrt{\tfrac{1}{10}\textstyle\sum_{j=1}^{10}(z_{1j}-\bar{z}_1)^2}=\left[\sqrt{\tfrac{1}{10}(0.7767+\ldots+0.0577)},\sqrt{\tfrac{1}{10}(4.5263+\ldots+0.8279}\right]\\ &= [0.3844,0.8420].\end{aligned}$$

By the similar calculational steps, the neutrosophic average values and standard deviations of JRC-NNs in other rows can be also obtained and all the results are shown in Table 4. Then, the neutrosophic average values and standard deviations of JRC-NNs in different orientations are depicted in Figures 3 and 4.

Table 4. The average values and standard deviations in each orientation θ.

Orientation θ	Average Value			Standard Deviation σ_{zi}
	$\bar{a}_i$	$\bar{b}_i$	$\bar{z}_i$ ($I \in [0, 1]$)	
0°	7.4226	3.2309	[7.4226, 10.6535]	[0.3844, 0.8420]
15°	7.8501	2.5831	[7.8501, 10.4332]	[0.2843, 1.1698]
30°	7.6560	2.8152	[7.6560, 10.4712]	[0.3013, 1.1842]
45°	7.0997	2.8847	[7.0997, 9.9844]	[0.3385, 0.9850]
60°	6.0368	3.2555	[6.0368, 9.2923]	[0.3130, 1.1182]
75°	5.3436	2.4401	[5.3436, 7.7837]	[0.2130, 1.0704]
90°	4.8714	2.6187	[4.8714, 7.4901]	[0.0907, 0.8406]
105°	5.1312	2.6809	[5.1312, 7.8121]	[0.1902, 1.0122]
120°	6.3791	2.6061	[6.3791, 8.9852]	[0.2789, 1.2189]
135°	6.4560	2.4654	[6.4560, 8.9214]	[0.4636, 1.0067]
150°	7.1731	2.9296	[7.1731, 10.1027]	[0.4368, 1.2946]
165°	7.3560	2.9602	[7.3560, 10.3162]	[0.5843, 1.1961]
180°	6.8556	3.1400	[6.8556, 9.9956]	[0.3554, 0.9170]
195°	6.9553	2.8155	[6.9553, 9.7708]	[0.3640, 1.2298]
210°	6.6645	2.7889	[6.6645, 9.4534]	[0.5157, 1.0531]
225°	6.2537	2.7148	[6.2537, 8.9685]	[0.4522, 0.8612]
240°	5.4721	2.2640	[5.4721, 7.7361]	[0.3255, 0.9058]
255°	5.0487	2.1831	[5.0487, 7.2318]	[0.1818, 0.8701]
270°	4.6391	2.6743	[4.6391, 7.3134]	[0.1003, 0.6426]
285°	4.8322	2.2530	[4.8322, 7.0852]	[0.3335, 0.6340]
300°	6.0556	2.1518	[6.0556, 8.2074]	[0.3653, 0.9042]
315°	6.8812	3.1943	[6.8812, 10.0755]	[0.4565, 1.2396]
330°	6.8032	2.6760	[6.8032, 9.4792]	[0.3983, 1.1377]
345°	6.9508	2.3364	[6.9508, 9.2872]	[0.5018, 1.1878]

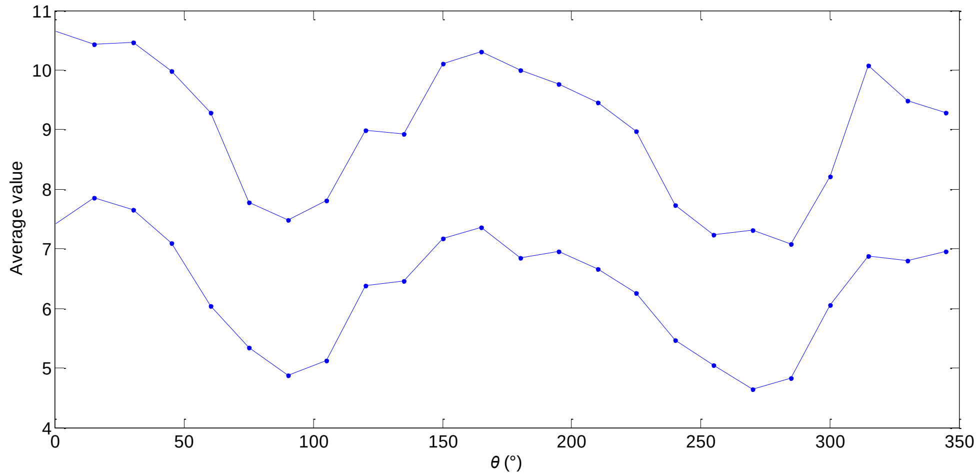

Figure 3. The neutrosophic average values of JRC-NNs in different orientations θ.

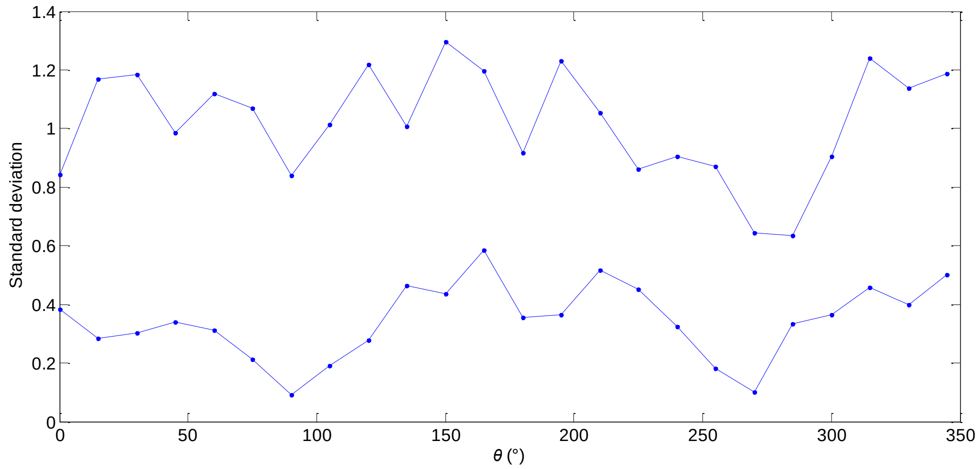

Figure 4. The neutrosophic standard deviations of JRC-NNs in different orientations θ.

Figure 3 shows that the neutrosophic average values (ranges) of JRC-NNs are very different in every orientation. Their changing curves look somewhat like trigonometric functions, which show the anisotropy of JRC-NNs.

In Figure 4, the neutrosophic standard deviations indicate the dispersion degrees of JRC-NNs under different sample lengths in some orientation. The larger the neutrosophic standard deviation is, the more discrete the JRC-NNs is. This case indicates that the scale effect of JRC-NNs is more obvious in the orientation. Although the changing curves in Figure 4 are irregular, it is clear that the dispersion degree of each orientation is very different. For example, the neutrosophic standard deviation of $\theta = 270°$ is obviously smaller than that of other orientations. Especially, if the JRC-NNs of all rows have the same neutrosophic standard deviations in such a special case, then the two curve area in Figure 4 will be reduced to the area between two parallel lines without the anisotropy in each sample scale.

From the above analysis, it is obvious that the neutrosophic average values and standard deviations of JRC-NNs (JRC values) also imply the anisotropy in different orientations. Thus, the neutrosophic average values reflect the anisotropy of JRC values, while the neutrosophic standard deviations reflect the anisotropy of the scale effect. Obviously, this neutrosophic statistical analysis method is more detailed and more effective than existing methods and avoids the difficulty of the curve fitting and analysis in some complex cases.

6. Conclusion Remarks

According to the JRC data obtained in an actual case and the expressions and operations of JRC-NNs, we provided a new neutrosophic statistical analysis method based on the neutrosophic statistical algorithm of the neutrosophic average values and the standard deviations of JRC-NNs in different columns (different sample lengths) and different rows (different measurement orientations). It is obvious that the two characteristic analyses (scale effect and anisotropy) of JRC values were indicated in this study. For the first characteristic, we analyzed the scale effect of JRC-NNs in different sample lengths, where the neutrosophic average values reflect the scale effect of JRC-NNs, while the neutrosophic standard deviations reflect the scale effect of the anisotropy of JRC-NNs. For the second characteristic, we analyzed the anisotropy of JRC values in different measurement orientations, where the neutrosophic average values reflect the anisotropy of JRC-NNs, while the neutrosophic standard deviations reflect the anisotropy of the scale effect. Therefore, the neutrosophic statistical analysis of the actual case demonstrates that the neutrosophic average values and neutrosophic standard deviations of JRC-NNs can reflect the scale effect and anisotropic characteristics of JRC values reasonably and effectively.

However, the obtained analysis results and the performance benefits of the presented neutrosophic statistical algorithm in this study are summarized as follows:

(1) The neutrosophic statistical analysis method without fitting functions is more feasible and more reasonable than the existing method [10].
(2) The neutrosophic statistical analysis method based on the neutrosophic average values and neutrosophic standard deviations of JRC-NNs can retain much more information and reflect the scale effect and anisotropic characteristics of JRC values in detail.
(3) The presented neutrosophic statistical algorithm can analyze the scale effect and the anisotropy of JRC-NNs (JRC values) directly and effectively so as to reduce the information distortion.
(4) The presented neutrosophic statistical algorithm based on the neutrosophic statistical averages and standard deviations of JRC-NNs is more convenient and simpler than the existing curve fitting and derivative analysis of JRC-NN functions in [10].
(5) The presented neutrosophic statistical algorithm can overcome the insufficiencies of the existing method in the fitting and analysis process [10].
(6) This study can extend the existing related methods with JRC-NNs [10,11] and show its easy analysis advantage in complex cases.

From what has been discussed above, the proposed neutrosophic statistical analysis method of JRC-NNs provides a more convenient and feasible new way for the scale effect and anisotropic characteristic analysis of JRC values in rock mechanics.

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Forecasting Model Based on Neutrosophic Logical Relationship and Jaccard Similarity

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Abstract: The daily fluctuation trends of a stock market are illustrated by three statuses: up, equal, and down. These can be represented by a neutrosophic set which consists of three functions—truth-membership, indeterminacy-membership, and falsity-membership. In this paper, we propose a novel forecasting model based on neutrosophic set theory and the fuzzy logical relationships between the status of historical and current values. Firstly, the original time series of the stock market is converted to a fluctuation time series by comparing each piece of data with that of the previous day. The fluctuation time series is then fuzzified into a fuzzy-fluctuation time series in terms of the pre-defined up, equal, and down intervals. Next, the fuzzy logical relationships can be expressed by two neutrosophic sets according to the probabilities of different statuses for each current value and a certain range of corresponding histories. Finally, based on the neutrosophic logical relationships and the status of history, a Jaccard similarity measure is employed to find the most proper logical rule to forecast its future. The authentic Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) time series datasets are used as an example to illustrate the forecasting procedure and performance comparisons. The experimental results show that the proposed method can successfully forecast the stock market and other similar kinds of time series. We also apply the proposed method to forecast the Shanghai Stock Exchange Composite Index (SHSECI) to verify its effectiveness and universality.

Keywords: fuzzy time series; forecasting; fuzzy logical relationship; neutrosophic set; Jaccard similarity

1. Introduction

It is well known that there is a statistical long-range dependency between current values and historical values at different times in certain time series [1]. Therefore, many researchers have developed various models to predict the future of such time series based on historical data sets, for example the regression analysis model [2], the autoregressive moving average (ARIMA) model [3], the autoregressive conditional heteroscedasticity (ARCH) model [4], the generalized ARCH (GARCH) model [5], and so on. However, crisp data used in those models are sometimes unavailable as such time series contain many uncertainties. In fact, models that satisfy the constraints precisely can miss the true optimal design within the confines of practical and realistic approximations. Therefore, Song and Chissom proposed the fuzzy time series (FTS) forecasting model [6–8] to predict the future of such nonlinear and complicated problems. In a financial context, FTS approaches have been widely applied to stock index forecasting [9–13]. In order to improve the accuracy of forecasts for stock market indices, some researchers combine fuzzy and non-fuzzy time series with heuristic optimization methods in their forecasting strategies [14]. Other approaches even introduce neural networks and machine learning procedures in order to find forecasting rules from historical time series [15–17].

The major points in FTS models are related to the fuzzifying of original time series, the establishment of fuzzy logical relationships from historical training datasets, and the forecasting and defuzzification of the outputs. Various proposals have been considered to determine the basic steps of the fuzzifying method, such as the effective length of intervals—e.g., determining the optimal interval length based on averages and distribution methods [18], using statistical theory [18–23], the unequal interval length method based on ratios of data [24], or the length determination method based on particle swarm optimization (PSO) techniques [10], etc. To state appropriate fuzzy logical relationships, Yu [25] proposed a weight assignation model, based on the recurrent fuzzy relationships, for each individual relationship. Aladag et al. [26] considered artificial neural networks to be a basic high-order method for the establishment of logical relationships. Fuzzy auto regressive (AR) models and fuzzy auto regressive and moving average (ARMA) models are also widely used to reflect the recurrence and weights of different fuzzy logical relationships [9,10,27–35]. These obtained logical relationships will be used as rules during the forecasting process. However, the proportions of the lagged variables in AR or ARMA models only represent the general best fitness for certain training datasets, without taking into account the differences between individual relationships. Although the weight assignation model considers the differences between individual relationships, it has to deal with special relationships that appear in the testing dataset but never happen in the training dataset. These FTS methods look for point forecasts without taking into account the implicit uncertainty in the ex post forecasts.

For a financial system, if anything, future fluctuation is more important than the indicated number itself. Therefore, the crucial ingredients for financial forecasting are the fluctuation orientations (including up, equal, and down) and to what extent the trends would be realized. Inspired by this, we first changed the original time series into a fluctuation time series for further rule generation. Meanwhile, comparing the three statuses with the concept of the neutrosophic set, the trends and weights of the relationships between historical and current statuses can be represented by the different dimensions of the neutrosophic sets, respectively. The concept of the neutrosophic set was originally proposed from a philosophical point of view by Smarandache [36]. A neutrosophic set is characterized independently by a truth-membership function, an indeterminacy-membership function and a falsity-membership function. Its similarity measure plays a key role in decision-making in uncertain environments. Researchers have proposed various similarity measures and mainly applied them to decision-making—e.g., Jaccard, Dice and Cosine similarity measures [37], distance-based similarity measures [38], entropy measures [39], etc. Although neutrosophic sets have been successfully applied to decision-making [37–42], they have rarely been applied to forecasting problems.

In this paper, we introduce neutrosophic sets to stock market forecasting. We propose a novel forecasting model based on neutrosophic set theory and the fuzzy logical relationships between current and historical statuses. Firstly, the original time series of the stock market is converted to a fluctuation time series by comparing each piece of data with that of the previous day. The fluctuation time series is then fuzzified into a fuzzy-fluctuation time series in terms of the pre-defined up, equal, and down intervals. Next, the fuzzy logical relationships can be expressed by two neutrosophic sets according to the probabilities for different statuses of each current value and a certain range of corresponding histories. Finally, based on the neutrosophic logical relationships and statuses of recent history, the Jaccard similarity measure is employed to find the most proper logical rule with which to forecast its future.

The remaining content of this paper is organized as follows: Section 2 introduces some preliminaries of fuzzy-fluctuation time series and concepts, and the similarity measures of neutrosophic sets. Section 3 describes a novel approach for forecasting based on fuzzy-fluctuation trends and logical relationships. In Section 4, the proposed model is used to forecast the stock market using Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) datasets from 1997 to 2005 and Shanghai Stock Exchange Composite Index (SHSECI) from 2007 to 2015. Conclusions and potential issues for future research are summarized in Section 5.

2. Preliminaries

2.1. Definition of Fuzzy-Fluctuation Time Series (FFTS)

Song and Chissom [6–8] combined fuzzy set theory with time series and defined fuzzy time series. In this section, we extend fuzzy time series to fuzzy-fluctuation time series (FFTS) and propose the related concepts.

Definition 1. *Let $L = \{l_1, l_2, \ldots, l_g\}$ be a fuzzy set in the universe of discourse U; it can be defined by its membership function, $\mu_L : U \to [0,1]$, where $\mu_L(u_i)$ denotes the grade of membership of u_i, $U = \{u_1, u_2, \ldots u_i, \ldots, u_l\}$.*

The fluctuation trends of a stock market can be expressed by a linguistic set $L = \{l_1, l_2, l_3\}$ = {*down, equal, up*}. The element l_i and its subscript i are strictly monotonically increasing [43], so the function can be defined as follows: $f : l_i = f(i)$.

Definition 2. *Let $F(t)(t = 1, 2, \ldots, T)$ be a time series of real numbers, where T is the number of the time series. $G(t)$ is defined as a fluctuation time series, where $G(t) = F(t) - F(t-1), (t = 2, 3, \ldots, T)$. Each element of $G(t)$ can be represented by a fuzzy set $S(t)(t = 2, 3, \ldots, T)$ as defined in Definition 1. Then we call the time series $G(t)$, which is to be fuzzified into a fuzzy-fluctuation time series (FFTS), $S(t)$.*

Definition 3. *Let $S(t)$ $(t = n+1, n+2, \ldots, T, n \geq 1)$ be a FFTS. If $S(t)$ is determined by $S(t-1), S(t-2), \ldots, S(t-n)$, then the fuzzy-fluctuation logical relationship is represented by:*

$$S(t-1), S(t-2), \ldots, S(t-n) \to S(t) \tag{1}$$

and it is called the nth-order fuzzy-fluctuation logical relationship (FFLR) of the fuzzy-fluctuation time series, where $S(t-n), \ldots, S(t-2)S(t-1)$ is called the left-hand side (LHS) and $S(t)$ is called the right-hand side (RHS) of the FFLR, and $S(k)(k = t, t-1, t-2, \ldots, t-n) \in L$.

2.2. Basic Concept of Neutrosophic Logical Relationship (NLR)

Smarandache [36] originally presented the neutrosophic set theory. Based on neutrosophic set theory, we propose the concept of the fuzzy-neutrosophic logical relationship, which employs the three terms of a neutrosophic set to reflect the fuzzy-fluctuation trends and weights of an *n*th-order FFLR.

Definition 4. *Let $P^i_{A(t)}$ be the probabilities of each element $l_i (l_i \in L)$ in the LHS of an nth-order FFLR $S(t-1), S(t-2), \ldots, S(t-n) \to S(t)$, and it can be generated by:*

$$P^i_{A(t)} = \frac{\sum_{j=1}^{n} w_{i,j}}{n} \qquad i = 1, 2, 3 \tag{2}$$

where $w_{i,j} = 1$ if $S(t-j) = i$ and 0 otherwise. Let X be a universal set, and the left-hand side of a neutrosophic logical relationship is defined by:

$$A(t) = \left\{ \left\langle x, P^1_{A(t)}, P^2_{A(t)}, P^3_{A(t)} \right\rangle | x \in X \right\} \tag{3}$$

Definition 5. *For S(t)(t = n + 1, n + 2, ... , T) is a FFTS and A(t) is the LHS of a neutrosophic logical relationship. The FFLRs with the same A(t) can be grouped into a FFLRG by putting all their RHSs together as on the RHS of the FFLRG. The RHSs of the FFRLG for A(t) can be represented by a neutrosophic set as described by Definition 4:*

$$B_{A(t)} = \left\{ \left\langle x, P^1_{B_{A(t)}}, P^2_{B_{A(t)}}, P^3_{B_{A(t)}} \right\rangle | x \in X \right\} \tag{4}$$

where $P^i_{B_{A(t)}}$ $(i = 1, 2, 3)$ *represent the down, equal or up probability of the RHSs of the FFRLG for* $A(t)$*.* $P^i_{B_{A(t)}}$ $(i = 1, 2, 3)$ *is called the righ-hand side of a neutrosophic logical relationship.*

In this way, the FFLR $S(t-1), S(t-2), \ldots, S(t-n) \rightarrow S(t)$ can be represented by a neutrosophic logical relationship (NLR) $A(t) \rightarrow B_{A(t)}$.

Definition 6 [37]. *Let* $A(t_1)$ *and* $A(t_2)$ *be two neutrosophic sets. The Jaccard similarity measure between* $A(t_1)$ *and* $A(t_2)$ *in vector space is defined as follows:*

$$J(A(t_1), A(t_2)) = \frac{\sum_{i=1}^{3} P^i_{A(t_1)} P^i_{A(t_2)}}{\sum_{i=1}^{3} (P^i_{A(t_1)})^2 + \sum_{i=1}^{3} (P^i_{A(t_2)})^2 - \sum_{i=1}^{3} P^i_{A(t_1)} P^i_{A(t_2)}} \quad (5)$$

3. A Novel Forecasting Model Based on Neutrosophic Logical Relationships

In this paper, we propose a novel forecasting model based on high-order neutrosophic logical relationships and Jaccard similarity measures. In order to compare the forecasting results with other researchers' work [9,17,23,25,44–48], the authentic TAIEX (Taiwan Stock Exchange Capitalization Weighted Stock Index) is employed to illustrate the forecasting process. The data from January 1999 to October 1999 are used as the training time series and the data from November 1999 to December 1999 are used as the testing dataset. The basic steps of the proposed model are shown in Figure 1.

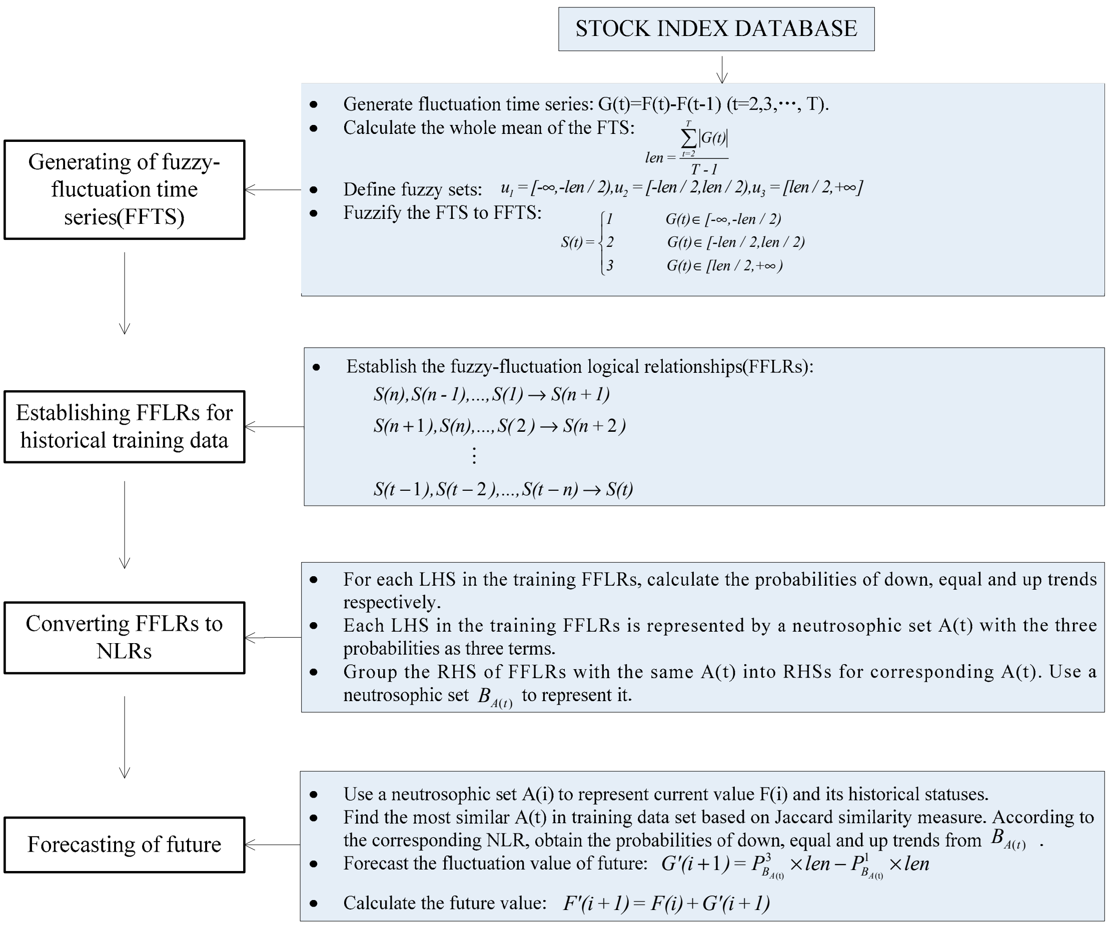

Figure 1. Flowchart of our proposed forecasting model.

Step 1. Construct FFTS for the historical training data

For each element $F(t)(t = 1, 2, \ldots, T)$ in the historical training time series, its fluctuation trend is determined by $G(t) = F(t) - F(t-1), (t = 2, 3, \ldots, T)$. According to the range and orientation of the fluctuations, $G(t)(t = 2, 3, \ldots, T)$ can be fuzzified into a linguistic set {down, equal, up}. Let *len* be the whole mean of all elements in the fluctuation time series $G(t)(t = 2, 3, \ldots, T)$, define $u_1 = [-\infty, -len/2]$, $u_2 = [-len/2, len/2]$, $u_3 = [len/2, +\infty]$, and then $G(t)(t = 2, 3, \ldots, T)$ can be fuzzified into a fuzzy-fluctuation time series $S(t)(t = 2, 3, \ldots, T)$.

Step 2. Establish *n*th-order FFLRs for the training data set

According to Definition 3, each $S(t)(t > n)$ in the historical training data set can be represented by its previous *n* days' fuzzy-fluctuation numbers to establish the training FFLRs.

Step 3. Convert the FFLRs to NLRs

According to Definition 4, the LHS of each FFLR can be expressed by a neutrosophic set $A(t)$. Then, we can generate the RHSs $B_{A(t)}$ for different LHSs respectively, as described in Definition 5. Thus, the FFLRs for the historical training dataset are converted into NLRs.

Step 4. Forecast test time series

For each observed point $F(i)$ in the test time series, we can use a neutrosophic set $A(i)$ to represent its *n*th-order fuzzy-fluctuation trends. Then, for each $A(t)$ obtained in step 3, compare $A(i)$ with $A(t)$ respectively, and find the most similar one based on the Jaccard similarity measure method described in Definition 6. Next, use the corresponding $B_{A(t)}$ as the forecasting rule to predict the fluctuation value $G'(i+1)$ of the next point. Finally, obtain the forecasting value by $F'(i+1) = F(i) + G'(i+1)$.

4. Empirical Analysis

4.1. Forecasting Taiwan Stock Exchange Capitalization Weighted Stock Index

Many studies use TAIEX1999 as an example to illustrate their proposed forecasting methods [9,17,25,34,44–48]. In order to compare the accuracy with their models, we also use TAIEX1999 to illustrate the proposed method.

Step 1: Calculate the fluctuation trend for each element in the historical training dataset of TAIEX1999. Then, we use the whole mean of the fluctuation numbers of the training dataset to fuzzify the fluctuation trends into FFTS. For example, the whole mean of the historical dataset of TAIEX1999 from January to October is 85. That is to say, *len* = 85. For $F(1) = 6152.43$ and $F(2) = 6199.91$, $G(2) = 47.48$, $S(2) = 3$. In this way, the historical training dataset can be represented by a fuzzified fluctuation dataset as shown in Table A1.

Step 2: Based on the FFTS from 5 January to 30 October 1999—shown in Table A1—the *n*th-order FFLRs for the forecasting model are established as shown in Table A2. The subscript *i* is used to represent element l_i in the FFLRs for convenience.

Step 3: In order to convert the FFLRs to NLRs, first of all the LHSs of the FFLRs in Table A2 are represented by a neutrosophic set, respectively (shown in Table A2). Then, the RHSs of the FFLRs are grouped with the same LHS neutrosophic set value into the RHSs group. A neutrosophic set is used to represent the RHSs group. For example, the LHS of FFLR 2,3,1,1,1,2,2,3,3→1 can be represented by the neutrosophic set (0.33,0.33,0.33). The detailed grouping and converting processes are shown in Figure 2.

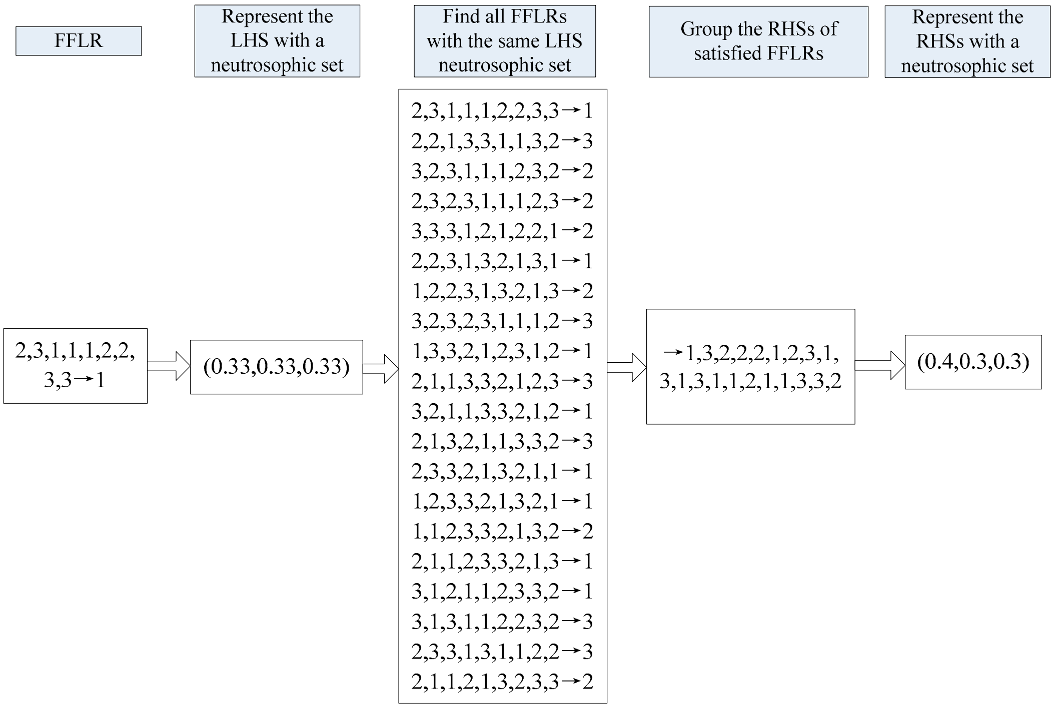

Figure 2. Group and converting processes for FFLR 2,3,1,1,1,2,2,3,3→2.

In this way, the FFLR 2,3,1,1,1,2,2,3,3→1 and other members of the same group are converted into an NLR (0.33,0.33,0.33)→(0.4,0.3,0.3). Therefore, the FFLRs in Table A2 can be converted into NLRs as shown in Table 1.

Table 1. Neutrosophic logical relationships (NLRs) for the historical training data of TAIEX1999.

NLRs	NLRs	NLRs
(0.33,0.33,0.33)→(0.4,0.3,0.3)	(0.22,0.33,0.44)→(0,0.6,0.4)	(0.22,0.78,0)→(0.5,0.5,0)
(0.44,0.33,0.22)→(0.23,0.46,0.31)	(0.22,0.44,0.33)→(0.33,0.33,0.33)	(0.33,0.67,0)→(0,0,1)
(0.44,0.44,0.11)→(0.4,0.33,0.27)	(0.11,0.56,0.33)→(0.17,0.5,0.33)	(0.11,0.11,0.78)→(0,1,0)
(0.33,0.44,0.22)→(0.54,0.23,0.23)	(0.11,0.67,0.22)→(0.17,0.33,0.5)	(0,0.22,0.78)→(0,1,0)
(0.33,0.56,0.11)→(0.25,0.5,0.25)	(0.22,0.56,0.22)→(0.25,0.5,0.25)	(0,0.33,0.67)→(0,1,0)
(0.56,0.33,0.11)→(0.36,0.27,0.36)	(0.11,0.44,0.44)→(0,0.38,0.63)	(0.56,0.22,0.22)→(0.25,0.25,0.5)
(0.67,0.22,0.11)→(0,1,0)	(0.11,0.33,0.56)→(0.33,0.17,0.5)	(0.44,0.11,0.44)→(0.5,0.5,0)
(0.56,0.44,0)→(0,0,1)	(0.11,0.22,0.67)→(0.43,0.43,0.14)	(0.56,0.11,0.33)→(1,0,0)
(0.44,0.22,0.33)→(0.29,0.43,0.29)	(0,0.56,0.44)→(0.33,0,0.67)	(0.67,0,0.33)→(0,1,0)
(0.33,0.22,0.44)→(0.31,0.38,0.31)	(0,0.44,0.56)→(0.14,0.43,0.43)	(0.67,0.11,0.22)→(0.5,0.25,0.25)
(0.22,0.22,0.56)→(0.25,0.25,0.5)	(0.11,0.78,0.11)→(0,0.8,0.2)	(0.22,0.67,0.11)→(0,0,1)
(0.33,0.11,0.56)→(0,0.5,0.5)	(0,0.89,0.11)→(0.25,0.75,0)	
(0.22,0.11,0.67)→(0.29,0.29,0.43)	(0.11,0.89,0)→(0.5,0.5,0)	

Step 4: Use the NLRs obtained from historical training data to forecast the test dataset from 1 November to 30 December 1999. For example, the forecasting value of the TAIEX on 1 November 1999 is calculated as follows:

First, the ninth-order historical fuzzy-fluctuation trends 3,2,2,2,2,3,1,2,2 on 1 November 1999 can be represented by a neutrosophic set (0.11,0.67,0.22). Then, we use the Jaccard similarity measure method as described by Definition 6 to choose the most optimal NLR from the NLRs listed in Table 1. The NLR (0.11,0.67,0.22)→(0.17,0.33,0.5) is evidently the best rule for further forecasting. Therefore, the forecasted fuzzy-fluctuation number is:

$$S'(i+1) = (-0.17) + 0.5 = 0.33$$

The forecasted fluctuation from the current value to the next value can be obtained by defuzzifying the fluctuation fuzzy number:

$$G'(i+1) = S'(i+1) \times len = 0.33 \times 85 = 28.05$$

Finally, the forecasted value can be obtained by the current value and the fluctuation value:

$$F'(i+1) = F(i) + G'(i+1) = 7854.85 + 28.05 = 7882.9$$

The other forecasting results are shown in Table 2 and Figure 3.

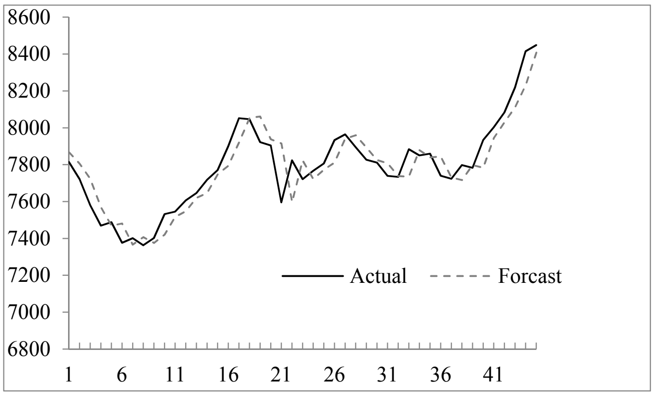

Figure 3. Forecasting results from 1 November 1999 to 30 December 1999.

Table 2. Forecasting results from 1 November 1999 to 30 December 1999.

Date (MM/DD/YYYY)	Actual	Forecast	(Forecast − Actual)^2	Date (MM/DD/YYYY)	Actual	Forecast	(Forecast − Actual)^2
11/1/1999	7814.89	7882.90	4625.36	12/1/1999	7766.20	7720.87	2054.81
11/2/1999	7721.59	7842.94	14,725.82	12/2/1999	7806.26	7766.20	1604.80
11/3/1999	7580.09	7721.59	20,022.25	12/3/1999	7933.17	7797.76	18,335.87
11/4/1999	7469.23	7580.09	12,289.94	12/4/1999	7964.49	7924.67	1585.63
11/5/1999	7488.26	7469.23	362.14	12/6/1999	7894.46	7955.99	3785.94
11/6/1999	7376.56	7488.26	12,476.89	12/7/1999	7827.05	7885.96	3470.39
11/8/1999	7401.49	7365.51	1294.56	12/8/1999	7811.02	7827.05	256.96
11/9/1999	7362.69	7390.44	770.06	12/9/1999	7738.84	7802.52	4055.14
11/10/1999	7401.81	7351.64	2517.03	12/10/1999	7733.77	7745.64	140.90
11/11/1999	7532.22	7486.82	2061.16	12/13/1999	7883.61	7707.42	31,042.92
11/15/1999	7545.03	7521.17	569.30	12/14/1999	7850.14	7857.26	50.69
11/16/1999	7606.20	7545.03	3741.77	12/15/1999	7859.89	7823.79	1303.21
11/17/1999	7645.78	7606.20	1566.58	12/16/1999	7739.76	7859.89	14,431.22
11/18/1999	7718.06	7673.83	1956.29	12/17/1999	7723.22	7728.71	30.14
11/19/1999	7770.81	7731.66	1532.72	12/18/1999	7797.87	7723.22	5572.62
11/20/1999	7900.34	7799.71	10,126.40	12/20/1999	7782.94	7797.87	222.90
11/22/1999	8052.31	7924.99	16,210.38	12/21/1999	7934.26	7782.94	22,897.74
11/23/1999	8046.19	8052.31	37.45	12/22/1999	8002.76	7947.86	3014.01
11/24/1999	7921.85	8046.19	15,460.44	12/23/1999	8083.49	8056.32	738.21
11/25/1999	7904.53	7936.30	1009.33	12/24/1999	8219.45	8137.05	6789.76
11/26/1999	7595.44	7918.98	104,678.13	12/27/1999	8415.07	8233.90	32,822.57
11/29/1999	7823.90	7629.44	37,814.69	12/28/1999	8448.84	8390.42	3412.90
11/30/1999	7720.87	7845.15	15,445.52	Root Mean Square Error(RMSE)			98.76

The forecasting performance can be assessed by comparing the difference between the forecasted values and the actual values. The widely used indicators in time series model comparisons are the mean squared error (MSE), the root of the mean squared error (RMSE), the mean absolute error (MAE), and the mean percentage error (MPE), etc. To compare the performance of different forecasting methods, the Diebold-Mariano test statistic (S) is also widely used [49]. These indicators are defined by Equations (6)–(10):

$$MSE = \frac{\sum_{t=1}^{n} (forecast(t) - actual(t))^2}{n} \tag{6}$$

$$RMSE = \sqrt{\frac{\sum_{t=1}^{n} (forecast(t) - actual(t))^2}{n}} \tag{7}$$

$$MAE = \frac{\sum_{t=1}^{n} |(forecast(t) - actual(t))|}{n} \tag{8}$$

$$MPE = \frac{\sum_{t=1}^{n} |(forecast(t) - actual(t))|/actual(t)}{n} \tag{9}$$

$$S = \frac{\bar{d}}{(Variance(\bar{d}))^{1/2}},\ \bar{d} = \frac{\sum_{t=1}^{n} (error\quad of\quad forecast1)_t^2 - \sum_{t=1}^{n} (error\quad of\quad forecast2)_t^2}{n} \tag{10}$$

where n denotes the number of values forecasted, *forecast*(t) and *actual*(t) denote the predicted value and actual value at time t, respectively. S is a test statistic of the Diebold method that is used to compare the predictive accuracy of two forecasts obtained by different methods. *Forecast*1 represents the dataset obtained by method 1, and *Forecast*2 represents another dataset from method 2. If $S > 0$ and $|S| > Z = 1.64$ at the 0.05 significance level, then *Forecast*2 has better predictive accuracy than *Forecast*1. With respect to the proposed method for the ninth order, the MSE, RMSE, MAE, and MPE are 9753.63, 98.76, 76.32, and 0.01, respectively.

Let the order number n vary from two to 10; the RMSEs for different nth-order forecasting models are listed in Table 3. The item "Average" refers to the RMSE for the average forecasting results of these different nth-order (n = 2, 3, ..., 10) models.

Table 3. Comparison of forecasting errors for different nth orders.

	n									Average
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
RMSE	100.22	100.9	100.66	99.81	102.83	103.48	100.36	98.76	108.99	99.03

In practical forecasting, the average of the results of different nth-order (n = 2, 3, ..., 9) forecasting models is adopted to avoid the uncertainty. The proposed method is employed to forecast the TAIEX from 1997 to 2005. The forecasting results and errors are shown in Figure 4 and Table 4.

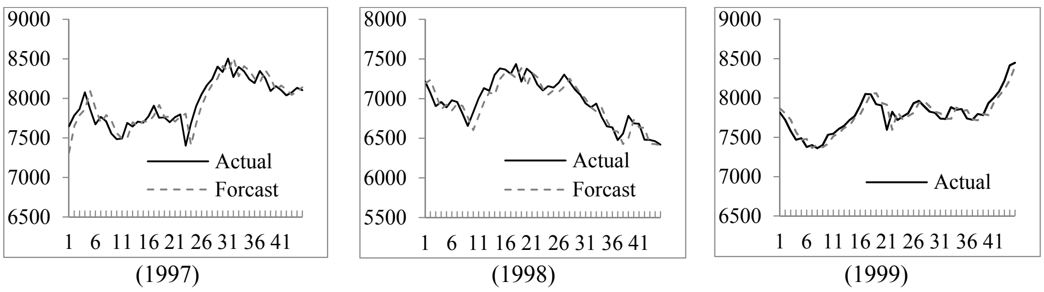

Figure 4. *Cont.*

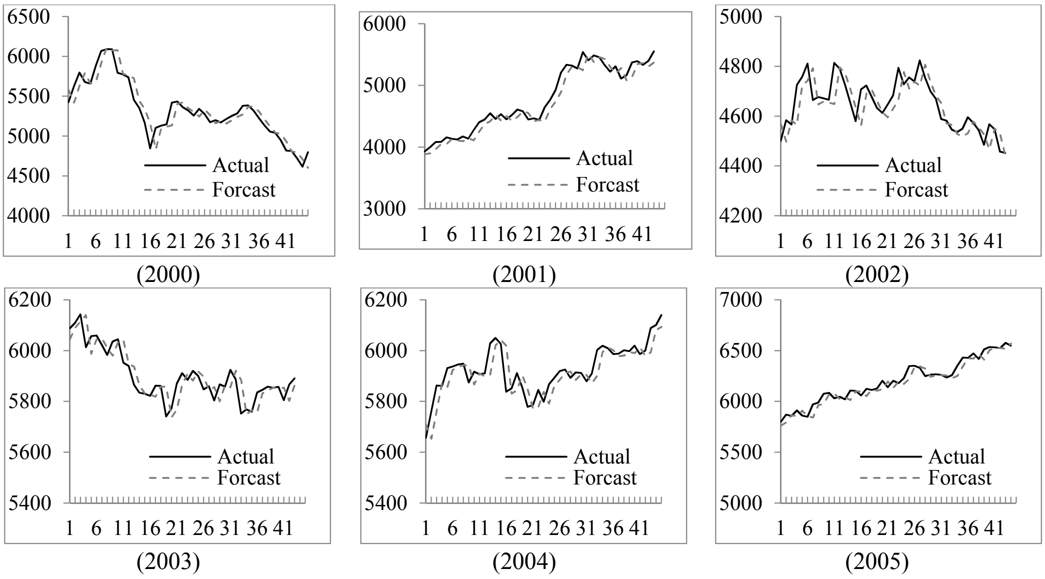

Figure 4. The stock market fluctuation for the TAIEX test dataset (1997–2005).

Table 4. RMSEs of forecast errors for TAIEX 1997 to 2005.

	Year								
	1997	**1998**	**1999**	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**
RMSE	141.89	119.85	99.03	128.62	125.64	66.29	53.2	56.11	55.83

Table 5 shows a comparison between the RMSEs of different methods for forecasting the TAIEX1999. From this table, we can see that the performance of the proposed method is acceptable. The greatest advantage of the proposed method is that it does not need to determine the boundary of discourse or the intervals for number fuzzifying. Meanwhile, the introduction of neutrosophic sets into the expression of logical relationships makes it possible to employ a similar comparison method to locate the most appropriate rules for further forecasting. Therefore, the proposed method, to some extent, is more rigorous than other methods that just use meaningless values in the case of missing rules in the training data. Though the RMSEs of some of the other methods outperform the proposed method, they often need to determine complex discretization partitioning rules or use adaptive expectation models to justify the final forecasting results. The method proposed in this paper is simpler and more easily realized by a computer program.

Table 5. A comparison of RMSEs for different methods for forecasting the TAIEX1999.

Methods	**RMSE**	**S**
Yu's Method (2005) [25]	145	1.82 **
Hsieh et al.'s Method (2011) [48]	94	−0.42
Chang et al.'s Method (2011) [45]	100	0.21
Cheng et al.'s Method (2013) [47]	103	0.42
Chen et al.'s Method (2013) [46]	102.11	0.39
Chen and Chen's Method (2015) [9]	103.9	0.29
Chen and Chen's Method (2015) [44]	92	−0.51
Zhao et al.'s Method (2016) [23]	110.85	1.16
Jia et al.'s Method (2017) [17]	99.31	0.11
The Proposed Method	99.03	-

** The proposed method has better predictive accuracy than the method at the 5% significance level.

4.2. Forecasting Shanghai Stock Exchange Composite Index

The SHSECI is the most famous stock market index in China. In the following, we apply the proposed method to forecast the SHSECI from 2007 to 2015. For each year, the authentic datasets of the historical daily SHSECI closing prices between January and October are used as the training data, and the datasets from November to December are used as the testing data. The RMSEs of forecast errors are shown in Table 6.

From Table 6, we can see that the proposed method can successfully predict the SHSECI stock market.

Table 6. RMSEs of forecast errors for SHSECI from 2007 to 2015.

	Year								
	2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**
RMSE	113.47	71.6	49.14	45.35	27.74	25.83	19.95	41.42	64.6

5. Conclusions

In this paper, a novel forecasting model is proposed based on neutrosophic logical relationships, the Jaccard similarity measure, and on fluctuations of the time series. The high-order fuzzy-fluctuation logical relationships are represented by neutrosophic logical relationships. Therefore, we can use the Jaccard similarity measure method to find the optimal forecasting rules. The biggest advantage of this method is that it can deal with the problem of lack of rules. Considering the fact that future fluctuation is more important than the indicated number itself, this method focuses on the forecasting of fluctuation orientations in terms of the extent of the fluctuation rather than on the real numbers. Meanwhile, utilizing NLRs instead of FLRs makes it possible to select the most appropriate rules for further forecasting. Therefore, the proposed method is more rigorous and interpretable. Experiments show that the parameters generated by the training dataset can be successfully used for future datasets as well. In order to compare the performance with that of other methods, we took the TAIEX 1999 as an example. We also forecasted TAIEX 1997–2005 and SHSECI 2007–2015 to verify its effectiveness and universality. In the future, we will consider other factors that might affect the fluctuation of the stock market, such as the trade volume, the beginning value, the end value, etc. We will also consider the influence of other stock markets, such as the Dow Jones, the National Association of Securities Dealers Automated Quotations (NASDAQ), the M1b, and so on.

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Author Contributions: Hongjun Guan designed the experiments and wrote the paper; Shuang Guan performed the experiments and analyzed the data; and Aiwu Zhao conceived the main idea of the method.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. Historical training data and fuzzified fluctuation data of TAIEX 1999.

Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified	Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified	Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified
1/5/1999	6152.43	-	-	4/17/1999	7581.5	114.68	3	7/26/1999	7595.71	−128.81	1
1/6/1999	6199.91	47.48	3	4/19/1999	7623.18	41.68	2	7/27/1999	7367.97	−227.74	1
1/7/1999	6404.31	204.4	3	4/20/1999	7627.74	4.56	2	7/28/1999	7484.5	116.53	3
1/8/1999	6421.75	17.44	2	4/21/1999	7474.16	−153.58	1	7/29/1999	7359.37	−125.13	1
1/11/1999	6406.99	−14.76	2	4/22/1999	7494.6	20.44	2	7/30/1999	7413.11	53.74	3
1/12/1999	6363.89	−43.1	1	4/23/1999	7612.8	118.2	3	7/31/1999	7326.75	−86.36	1
1/13/1999	6319.34	−44.55	1	4/26/1999	7629.09	16.29	2	8/2/1999	7195.94	−130.81	1
1/14/1999	6241.32	−78.02	1	4/27/1999	7550.13	−78.96	1	8/3/1999	7175.19	−20.75	2
1/15/1999	6454.6	213.28	3	4/28/1999	7496.61	−53.52	1	8/4/1999	7110.8	−64.39	1
1/16/1999	6483.3	28.7	2	4/29/1999	7289.62	−206.99	1	8/5/1999	6959.73	−151.07	1
1/18/1999	6377.25	−106.05	1	4/30/1999	7371.17	81.55	3	8/6/1999	6823.52	−136.21	1
1/19/1999	6343.36	−33.89	2	5/3/1999	7383.26	12.09	2	8/7/1999	7049.74	226.22	3
1/20/1999	6310.71	−32.65	2	5/4/1999	7588.04	204.78	3	8/9/1999	7028.01	−21.73	2
1/21/1999	6332.2	21.49	2	5/5/1999	7572.16	−15.88	2	8/10/1999	7269.6	241.59	3
1/22/1999	6228.95	−103.25	1	5/6/1999	7560.05	−12.11	2	8/11/1999	7228.68	−40.92	2
1/25/1999	6033.21	−195.74	1	5/7/1999	7469.33	−90.72	1	8/12/1999	7330.24	101.56	3
1/26/1999	6115.64	82.43	3	5/10/1999	7484.37	15.04	2	8/13/1999	7626.05	295.81	3
1/27/1999	6138.87	23.23	2	5/11/1999	7474.45	−9.92	2	8/16/1999	8018.47	392.42	3
1/28/1999	6063.41	−75.46	1	5/12/1999	7448.41	−26.04	2	8/17/1999	8083.43	64.96	3
1/29/1999	5984	−79.41	1	5/13/1999	7416.2	−32.21	2	8/18/1999	7993.71	−89.72	1
1/30/1999	5998.32	14.32	2	5/14/1999	7592.53	176.33	3	8/19/1999	7964.67	−29.04	2
2/1/1999	5862.79	−135.53	1	5/15/1999	7576.64	−15.89	2	8/20/1999	8117.42	152.75	3
2/2/1999	5749.64	−113.15	1	5/17/1999	7599.76	23.12	2	8/21/1999	8153.57	36.15	2
2/3/1999	5743.86	−5.78	2	5/18/1999	7585.51	−14.25	2	8/23/1999	8119.98	−33.59	2
2/4/1999	5514.89	−228.97	1	5/19/1999	7614.6	29.09	2	8/24/1999	7984.39	−135.59	1
2/5/1999	5474.79	−40.1	2	5/20/1999	7608.88	−5.72	2	8/25/1999	8127.09	142.7	3
2/6/1999	5710.18	235.39	3	5/21/1999	7606.69	−2.19	2	8/26/1999	8097.57	−29.52	2
2/8/1999	5822.98	112.8	3	5/24/1999	7588.23	−18.46	2	8/27/1999	8053.97	−43.6	1
2/9/1999	5723.73	−99.25	1	5/25/1999	7417.03	−171.2	1	8/30/1999	8071.36	17.39	2
2/10/1999	5798	74.27	3	5/26/1999	7426.63	9.6	2	8/31/1999	8157.73	86.37	3
2/20/1999	6072.33	274.33	3	5/27/1999	7469.01	42.38	2	9/1/1999	8273.33	115.6	3
2/22/1999	6313.63	241.3	3	5/28/1999	7387.37	−81.64	1	9/2/1999	8226.15	−47.18	1
2/23/1999	6180.94	−132.69	1	5/29/1999	7419.7	32.33	2	9/3/1999	8073.97	−152.18	1
2/24/1999	6238.87	57.93	3	5/31/1999	7316.57	−103.13	1	9/4/1999	8065.11	−8.86	2
2/25/1999	6275.53	36.66	2	6/1/1999	7397.62	81.05	3	9/6/1999	8130.28	65.17	3
2/26/1999	6318.52	42.99	3	6/2/1999	7488.03	90.41	3	9/7/1999	7945.76	−184.52	1
3/1/1999	6312.25	−6.27	2	6/3/1999	7572.91	84.88	3	9/8/1999	7973.3	27.54	2

Table A1. *Cont.*

Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified	Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified	Date (MM/DD/YYYY)	TAIEX	Fluctuation	Fuzzified
3/2/1999	6263.54	−48.71	1	6/4/1999	7590.44	17.53	2	9/9/1999	8025.02	51.72	3
3/3/1999	6403.14	139.6	3	6/5/1999	7639.3	48.86	3	9/10/1999	8161.46	136.44	3
3/4/1999	6393.74	−9.4	2	6/7/1999	7802.69	163.39	3	9/13/1999	8178.69	17.23	2
3/5/1999	6383.09	−10.65	2	6/8/1999	7892.13	89.44	3	9/14/1999	8092.02	−86.67	1
3/6/1999	6421.73	38.64	2	6/9/1999	7957.71	65.58	3	9/15/1999	7971.04	−120.98	1
3/8/1999	6431.96	10.23	2	6/10/1999	7996.76	39.05	2	9/16/1999	7968.9	−2.14	2
3/9/1999	6493.43	61.47	3	6/11/1999	7979.4	−17.36	2	9/17/1999	7916.92	−51.98	1
3/10/1999	6486.61	−6.82	2	6/14/1999	7973.58	−5.82	2	9/18/1999	8016.93	100.01	3
3/11/1999	6436.8	−49.81	1	6/15/1999	7960	−13.58	2	9/20/1999	7972.14	−44.79	1
3/12/1999	6462.73	25.93	2	6/16/1999	8059.02	99.02	3	9/27/1999	7759.93	−212.21	1
3/15/1999	6598.32	135.59	3	6/17/1999	8274.36	215.34	3	9/28/1999	7577.85	−182.08	1
3/16/1999	6672.23	73.91	3	6/21/1999	8413.48	139.12	3	9/29/1999	7615.45	37.6	2
3/17/1999	6757.07	84.84	3	6/22/1999	8608.91	195.43	3	9/30/1999	7598.79	−16.66	2
3/18/1999	6895.01	137.94	3	6/23/1999	8492.32	−116.59	1	10/1/1999	7694.99	96.2	3
3/19/1999	6997.29	102.28	3	6/24/1999	8589.31	96.99	3	10/2/1999	7659.55	−35.44	2
3/20/1999	6993.38	−3.91	2	6/25/1999	8265.96	−323.35	1	10/4/1999	7685.48	25.93	2
3/22/1999	7043.23	49.85	3	6/28/1999	8281.45	15.49	2	10/5/1999	7557.01	−128.47	1
3/23/1999	6945.48	−97.75	1	6/29/1999	8514.27	232.82	3	10/6/1999	7501.63	−55.38	1
3/24/1999	6889.42	−56.06	1	6/30/1999	8467.37	−46.9	1	10/7/1999	7612	110.37	3
3/25/1999	6941.38	51.96	3	7/2/1999	8572.09	104.72	3	10/8/1999	7552.98	−59.02	1
3/26/1999	7033.25	91.87	3	7/3/1999	8563.55	−8.54	2	10/11/1999	7607.11	54.13	3
3/29/1999	6901.68	−131.57	1	7/5/1999	8593.35	29.8	2	10/12/1999	7835.37	228.26	3
3/30/1999	6898.66	−3.02	2	7/6/1999	8454.49	−138.86	1	10/13/1999	7836.94	1.57	2
3/31/1999	6881.72	−16.94	2	7/7/1999	8470.07	15.58	2	10/14/1999	7879.91	42.97	3
4/1/1999	7018.68	136.96	3	7/8/1999	8592.43	122.36	3	10/15/1999	7819.09	−60.82	1
4/2/1999	7232.51	213.83	3	7/9/1999	8550.27	−42.16	2	10/16/1999	7829.39	10.3	2
4/3/1999	7182.2	−50.31	1	7/12/1999	8463.9	−86.37	1	10/18/1999	7745.26	−84.13	1
4/6/1999	7163.99	−18.21	2	7/13/1999	8204.5	−259.4	1	10/19/1999	7692.96	−52.3	1
4/7/1999	7135.89	−28.1	2	7/14/1999	7888.66	−315.84	1	10/20/1999	7666.64	−26.32	2
4/8/1999	7273.41	137.52	3	7/15/1999	7918.04	29.38	2	10/21/1999	7654.9	−11.74	2
4/9/1999	7265.7	−7.71	2	7/16/1999	7411.58	−506.46	1	10/22/1999	7559.63	−95.27	1
4/12/1999	7242.4	−23.3	2	7/17/1999	7366.23	−45.35	1	10/25/1999	7680.87	121.24	3
4/13/1999	7337.85	95.45	3	7/19/1999	7386.89	20.66	2	10/26/1999	7700.29	19.42	2
4/14/1999	7398.65	60.8	3	7/20/1999	7806.85	419.96	3	10/27/1999	7701.22	0.93	2
4/15/1999	7498.17	99.52	3	7/21/1999	7786.65	−20.2	2	10/28/1999	7681.85	−19.37	2
4/16/1999	7466.82	−31.35	2	7/22/1999	7678.67	−107.98	1	10/29/1999	7706.67	24.82	2
4/17/1999	7581.5	114.68	3	7/23/1999	7724.52	45.85	3	10/30/1999	7854.85	148.18	3

Table A2. The FFLRs and the converted left hand side of NLRs for historical training data of TAIEX1999.

Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR
1/18/1999	2,3,1,1,1,2,2,3,3→2	(0.33,0.33,0.33)	4/3/1999	3,3,2,2,1,3,3,1,1→3	(0.33,0.22,0.44)	6/11/1999	2,3,3,3,3,2,3,3,3→2	(0,0.22,0.78)	8/21/1999	3,2,1,3,3,3,3,2,3→3	(0.11,0.22,0.67)
1/19/1999	1,2,3,1,1,1,2,2,3→1	(0.44,0.33,0.22)	4/6/1999	1,3,3,2,2,1,3,3,1→1	(0.33,0.22,0.44)	6/14/1999	2,2,3,3,3,3,2,3,3→2	(0,0.33,0.67)	8/23/1999	2,3,2,1,3,3,3,3,2→2	(0.11,0.33,0.56)
1/20/1999	2,1,2,3,1,1,1,2,2→2	(0.44,0.44,0.11)	4/7/1999	2,1,3,3,2,2,1,3,3→2	(0.22,0.33,0.44)	6/15/1999	2,2,2,3,3,3,3,2,3→2	(0,0.44,0.56)	8/24/1999	2,2,3,2,1,3,3,3,3→2	(0.11,0.33,0.56)
1/21/1999	2,2,1,2,3,1,1,1,2→2	(0.44,0.44,0.11)	4/8/1999	2,2,1,3,3,2,2,1,3→2	(0.22,0.44,0.33)	6/16/1999	2,2,2,2,3,3,3,3,2→2	(0,0.56,0.44)	8/25/1999	1,2,2,3,2,1,3,3,3→1	(0.22,0.33,0.44)
1/22/1999	2,2,2,1,2,3,1,1,1→2	(0.44,0.44,0.11)	4/9/1999	3,2,2,1,3,3,2,2,1→3	(0.22,0.44,0.33)	6/17/1999	3,2,2,2,2,3,3,3,3→3	(0,0.44,0.56)	8/26/1999	3,1,2,2,3,2,1,3,3→3	(0.22,0.33,0.44)
1/25/1999	1,2,2,2,1,2,3,1,1→1	(0.44,0.44,0.11)	4/12/1999	2,3,2,2,1,3,3,2,2→2	(0.11,0.56,0.33)	6/21/1999	3,3,2,2,2,2,3,3,3→3	(0,0.44,0.56)	8/27/1999	2,3,1,2,2,3,2,1,3→2	(0.22,0.44,0.33)
1/26/1999	1,1,2,2,2,1,2,3,1→1	(0.44,0.44,0.11)	4/13/1999	2,2,3,2,2,1,3,3,2→2	(0.11,0.56,0.33)	6/22/1999	3,3,3,2,2,2,2,3,3→3	(0,0.44,0.56)	8/30/1999	1,2,3,1,2,2,3,2,1→1	(0.33,0.44,0.22)
1/27/1999	3,1,1,2,2,2,1,2,3→3	(0.33,0.44,0.22)	4/14/1999	3,2,2,3,2,2,1,3,3→3	(0.11,0.44,0.44)	6/23/1999	3,3,3,3,2,2,2,2,3→3	(0,0.44,0.56)	8/31/1999	2,1,2,3,1,2,2,3,2→2	(0.22,0.56,0.22)
1/28/1999	2,3,1,1,2,2,2,1,2→2	(0.33,0.56,0.11)	4/15/1999	3,3,2,2,3,2,2,1,3→3	(0.11,0.44,0.44)	6/24/1999	1,3,3,3,3,2,2,2,2→1	(0.11,0.44,0.44)	9/1/1999	3,2,1,2,3,1,2,2,3→3	(0.22,0.44,0.33)
1/29/1999	1,2,3,1,1,2,2,2,1→1	(0.44,0.44,0.11)	4/16/1999	3,3,3,2,2,3,2,2,1→3	(0.11,0.44,0.44)	6/25/1999	3,1,3,3,3,3,2,2,2→3	(0.11,0.33,0.56)	9/2/1999	3,3,2,1,2,3,1,2,2→3	(0.22,0.44,0.33)
1/30/1999	1,1,2,3,1,1,2,2,2→1	(0.44,0.44,0.11)	4/17/1999	2,3,3,3,2,2,3,2,2→2	(0,0.56,0.44)	6/28/1999	1,3,1,3,3,3,3,2,2→1	(0.22,0.22,0.56)	9/3/1999	1,3,3,2,1,2,3,1,2→1	(0.33,0.33,0.33)
2/1/1999	2,1,1,2,3,1,1,2,2→2	(0.44,0.44,0.11)	4/19/1999	3,2,3,3,3,2,2,3,2→3	(0,0.44,0.56)	6/29/1999	2,1,3,1,3,3,3,3,2→2	(0.22,0.22,0.56)	9/4/1999	1,1,3,3,2,1,2,3,1→1	(0.44,0.22,0.33)
2/2/1999	1,2,1,1,2,3,1,1,2→1	(0.56,0.33,0.11)	4/20/1999	2,3,2,3,3,3,2,2,3→2	(0,0.44,0.56)	6/30/1999	3,2,1,3,1,3,3,3,3→3	(0.22,0.11,0.67)	9/6/1999	2,1,1,3,3,2,1,2,3→2	(0.33,0.33,0.33)
2/3/1999	1,1,2,1,1,2,3,1,1→1	(0.67,0.22,0.11)	4/21/1999	2,2,3,2,3,3,3,2,2→2	(0,0.56,0.44)	7/2/1999	1,3,2,1,3,1,3,3,3→1	(0.33,0.11,0.56)	9/7/1999	3,2,1,1,3,3,2,1,2→3	(0.33,0.33,0.33)
2/4/1999	2,1,1,2,1,1,2,3,1→2	(0.56,0.33,0.11)	4/22/1999	1,2,2,3,2,3,3,3,2→1	(0.11,0.44,0.44)	7/3/1999	3,1,3,2,1,3,1,3,3→3	(0.33,0.11,0.56)	9/8/1999	1,3,2,1,1,3,3,2,1→1	(0.44,0.22,0.33)
2/5/1999	1,2,1,1,2,1,1,2,3→1	(0.56,0.33,0.11)	4/23/1999	2,1,2,2,3,2,3,3,3→2	(0.11,0.44,0.44)	7/5/1999	2,3,1,3,2,1,3,1,3→2	(0.33,0.22,0.44)	9/9/1999	2,1,3,2,1,1,3,3,2→2	(0.33,0.33,0.33)
2/6/1999	2,1,2,1,1,2,1,1,2→2	(0.56,0.44,0)	4/26/1999	3,2,1,2,2,3,2,3,3→3	(0.11,0.44,0.44)	7/6/1999	2,2,3,1,3,2,1,3,1→2	(0.33,0.33,0.33)	9/10/1999	3,2,1,3,2,1,1,3,3→3	(0.33,0.22,0.44)
2/8/1999	3,2,1,2,1,1,2,1,1→3	(0.56,0.33,0.11)	4/27/1999	2,3,2,1,2,2,3,2,3→2	(0.11,0.56,0.33)	7/7/1999	1,2,2,3,1,3,2,1,3→1	(0.33,0.33,0.33)	9/13/1999	3,3,2,1,3,2,1,1,3→3	(0.33,0.22,0.44)
2/9/1999	3,3,2,1,2,1,1,2,1→3	(0.44,0.33,0.22)	4/28/1999	1,2,3,2,1,2,2,3,2→1	(0.22,0.56,0.22)	7/8/1999	2,1,2,2,3,1,3,2,1→2	(0.33,0.44,0.22)	9/14/1999	2,3,3,2,1,3,2,1,1→2	(0.33,0.33,0.33)
2/10/1999	1,3,3,2,1,2,1,1,2→1	(0.44,0.33,0.22)	4/29/1999	1,1,2,3,2,1,2,2,3→1	(0.33,0.44,0.22)	7/9/1999	3,2,1,2,2,3,1,3,2→3	(0.22,0.44,0.33)	9/15/1999	1,2,3,3,2,1,3,2,1→1	(0.33,0.33,0.33)
2/20/1999	3,1,3,3,2,1,2,1,1→3	(0.44,0.22,0.33)	4/30/1999	1,1,1,2,3,2,1,2,2→1	(0.44,0.44,0.11)	7/12/1999	2,3,2,1,2,2,3,1,3→2	(0.22,0.44,0.33)	9/16/1999	1,1,2,3,3,2,1,3,2→1	(0.33,0.33,0.33)
2/22/1999	3,3,1,3,3,2,1,2,1→3	(0.33,0.22,0.44)	5/3/1999	3,1,1,1,2,3,2,1,2→3	(0.44,0.33,0.22)	7/13/1999	1,2,3,2,1,2,2,3,1→1	(0.33,0.44,0.22)	9/17/1999	2,1,1,2,3,3,2,1,3→2	(0.33,0.33,0.33)
2/23/1999	3,3,3,1,3,3,2,1,2→3	(0.22,0.22,0.56)	5/4/1999	2,3,1,1,1,2,3,2,1→2	(0.44,0.33,0.22)	7/14/1999	1,1,2,3,2,1,2,2,3→1	(0.33,0.44,0.22)	9/18/1999	1,2,1,1,2,3,3,2,1→1	(0.44,0.33,0.22)
2/24/1999	1,3,3,3,1,3,3,2,1→1	(0.33,0.11,0.56)	5/5/1999	3,2,3,1,1,1,2,3,2→3	(0.33,0.33,0.33)	7/15/1999	1,1,1,2,3,2,1,2,2→1	(0.44,0.44,0.11)	9/20/1999	3,1,2,1,1,2,3,3,2→3	(0.33,0.33,0.33)
2/25/1999	3,1,3,3,3,1,3,3,2→3	(0.22,0.11,0.67)	5/6/1999	2,3,2,3,1,1,1,2,3→2	(0.33,0.33,0.33)	7/16/1999	2,1,1,1,2,3,2,1,2→2	(0.44,0.44,0.11)	9/27/1999	1,3,1,2,1,1,2,3,3→1	(0.44,0.22,0.33)
2/26/1999	2,3,1,3,3,3,1,3,3→2	(0.22,0.11,0.67)	5/7/1999	2,2,3,2,3,1,1,1,2→2	(0.33,0.44,0.22)	7/17/1999	1,2,1,1,1,2,3,2,1→1	(0.56,0.33,0.11)	9/28/1999	1,1,3,1,2,1,1,2,3→1	(0.56,0.22,0.22)
3/1/1999	3,2,3,1,3,3,3,1,3→3	(0.22,0.11,0.67)	5/10/1999	1,2,2,3,2,3,1,1,1→1	(0.44,0.33,0.22)	7/19/1999	1,1,2,1,1,1,2,3,2→1	(0.56,0.33,0.11)	9/29/1999	1,1,1,3,1,2,1,1,2→1	(0.67,0.22,0.11)
3/2/1999	2,3,2,3,1,3,3,3,1→2	(0.22,0.22,0.56)	5/11/1999	2,1,2,2,3,2,3,1,1→2	(0.33,0.44,0.22)	7/20/1999	2,1,1,2,1,1,1,2,3→2	(0.56,0.33,0.11)	9/30/1999	2,1,1,1,3,1,2,1,1→2	(0.67,0.22,0.11)
3/3/1999	1,2,3,2,3,1,3,3,3→1	(0.22,0.22,0.56)	5/12/1999	2,2,1,2,2,3,2,3,1→2	(0.22,0.56,0.22)	7/21/1999	3,2,1,1,2,1,1,1,2→3	(0.56,0.33,0.11)	10/1/1999	2,2,1,1,1,3,1,2,1→2	(0.56,0.33,0.11)
3/4/1999	3,1,2,3,2,3,1,3,3→3	(0.22,0.22,0.56)	5/13/1999	2,2,2,1,2,2,3,2,3→2	(0.11,0.67,0.22)	7/22/1999	2,3,2,1,1,2,1,1,1→2	(0.56,0.33,0.11)	10/2/1999	3,2,2,1,1,1,3,1,2→3	(0.44,0.33,0.22)
3/5/1999	2,3,1,2,3,2,3,1,3→2	(0.22,0.33,0.44)	5/14/1999	2,2,2,2,1,2,2,3,2→2	(0.11,0.78,0.11)	7/23/1999	1,2,3,2,1,1,2,1,1→1	(0.56,0.33,0.11)	10/4/1999	2,3,2,2,1,1,1,3,1→2	(0.44,0.33,0.22)
3/6/1999	2,2,3,1,2,3,2,3,1→2	(0.22,0.44,0.33)	5/15/1999	3,2,2,2,2,1,2,2,3→3	(0.11,0.67,0.22)	7/26/1999	3,1,2,3,2,1,1,2,1→3	(0.44,0.33,0.22)	10/5/1999	2,2,3,2,2,1,1,1,3→2	(0.33,0.44,0.22)
3/8/1999	2,2,2,3,1,2,3,2,3→2	(0.11,0.56,0.33)	5/17/1999	2,3,2,2,2,2,1,2,2→2	(0.11,0.78,0.11)	7/27/1999	1,3,1,2,3,2,1,1,2→1	(0.44,0.33,0.22)	10/6/1999	1,2,2,3,2,2,1,1,1→1	(0.44,0.44,0.11)
3/9/1999	2,2,2,2,3,1,2,3,2→2	(0.11,0.67,0.22)	5/18/1999	2,2,3,2,2,2,2,1,2→2	(0.11,0.78,0.11)	7/28/1999	1,1,3,1,2,3,2,1,1→1	(0.56,0.22,0.22)	10/7/1999	1,1,2,2,3,2,2,1,1→1	(0.44,0.44,0.11)
3/10/1999	3,2,2,2,2,3,1,2,3→3	(0.11,0.56,0.33)	5/19/1999	2,2,2,3,2,2,2,2,1→2	(0.11,0.78,0.11)	7/29/1999	3,1,1,3,1,2,3,2,1→3	(0.44,0.22,0.33)	10/8/1999	3,1,1,2,2,3,2,2,1→3	(0.33,0.44,0.22)
3/11/1999	2,3,2,2,2,2,3,1,2→2	(0.11,0.67,0.22)	5/20/1999	2,2,2,2,3,2,2,2,2→2	(0,0.89,0.11)	7/30/1999	1,3,1,1,3,1,2,3,2→1	(0.44,0.22,0.33)	10/11/1999	1,3,1,1,2,2,3,2,2→1	(0.33,0.44,0.22)
3/12/1999	1,2,3,2,2,2,2,3,1→1	(0.22,0.56,0.22)	5/21/1999	2,2,2,2,2,3,2,2,2→2	(0,0.89,0.11)	7/31/1999	3,1,3,1,1,3,1,2,3→3	(0.44,0.11,0.44)	10/12/1999	3,1,3,1,1,2,2,3,2→3	(0.33,0.33,0.33)
3/15/1999	2,1,2,3,2,2,2,2,3→2	(0.11,0.67,0.22)	5/24/1999	2,2,2,2,2,2,3,2,2→2	(0,0.89,0.11)	8/2/1999	1,3,1,3,1,1,3,1,2→1	(0.56,0.11,0.33)	10/13/1999	3,3,1,3,1,1,2,2,3→3	(0.33,0.22,0.44)
3/16/1999	3,2,1,2,3,2,2,2,2→3	(0.11,0.67,0.22)	5/25/1999	2,2,2,2,2,2,2,3,2→2	(0,0.89,0.11)	8/3/1999	1,1,3,1,3,1,1,3,1→1	(0.67,0,0.33)	10/14/1999	2,3,3,1,3,1,1,2,2→2	(0.33,0.33,0.33)
3/17/1999	3,3,2,1,2,3,2,2,2→3	(0.11,0.56,0.33)	5/26/1999	1,2,2,2,2,2,2,2,3→1	(0.11,0.78,0.11)	8/4/1999	2,1,1,3,1,3,1,1,3→2	(0.56,0.11,0.33)	10/15/1999	3,2,3,3,1,3,1,1,2→3	(0.33,0.22,0.44)
3/18/1999	3,3,3,2,1,2,3,2,2→3	(0.11,0.44,0.44)	5/27/1999	2,1,2,2,2,2,2,2,2→2	(0.11,0.89,0)	8/5/1999	1,2,1,1,3,1,3,1,1→1	(0.67,0.11,0.22)	10/16/1999	1,3,2,3,3,1,3,1,1→1	(0.44,0.11,0.44)
3/19/1999	3,3,3,3,2,1,2,3,2→3	(0.11,0.33,0.56)	5/28/1999	2,2,1,2,2,2,2,2,2→2	(0.11,0.89,0)	8/6/1999	1,1,2,1,1,3,1,3,1→1	(0.67,0.11,0.22)	10/18/1999	2,1,3,2,3,3,1,3,1→2	(0.33,0.22,0.44)
3/20/1999	3,3,3,3,3,2,1,2,3→3	(0.11,0.22,0.67)	5/29/1999	1,2,2,1,2,2,2,2,2→1	(0.22,0.78,0)	8/7/1999	1,1,1,2,1,1,3,1,3→1	(0.67,0.11,0.22)	10/19/1999	1,2,1,3,2,3,3,1,3→1	(0.33,0.22,0.44)

Table A2. *Cont.*

Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR	Date (MM/DD/YYYY)	FFLR	LHS of NLR
3/22/1999	2,3,3,3,3,3,2,1,2→2	(0.11,0.33,0.56)	5/31/1999	2,1,2,2,1,2,2,2,2→2	(0.22,0.78,0)	8/9/1999	3,1,1,1,2,1,1,3,1→3	(0.67,0.11,0.22)	10/20/1999	1,1,2,1,3,2,3,3,1→1	(0.44,0.22,0.33)
3/23/1999	3,2,3,3,3,3,3,2,1→3	(0.11,0.22,0.67)	6/1/1999	1,2,1,2,2,1,2,2,2→1	(0.33,0.67,0)	8/10/1999	2,3,1,1,1,2,1,1,3→2	(0.56,0.22,0.22)	10/21/1999	2,1,1,2,1,3,2,3,3→2	(0.33,0.33,0.33)
3/24/1999	1,3,2,3,3,3,3,3,2→1	(0.11,0.22,0.67)	6/2/1999	3,1,2,1,2,2,1,2,2→3	(0.33,0.56,0.11)	8/11/1999	3,2,3,1,1,1,2,1,1→3	(0.56,0.22,0.22)	10/22/1999	2,2,1,1,2,1,3,2,3→2	(0.33,0.44,0.22)
3/25/1999	1,1,3,2,3,3,3,3,3→1	(0.22,0.11,0.67)	6/3/1999	3,3,1,2,1,2,2,1,2→3	(0.33,0.44,0.22)	8/12/1999	2,3,2,3,1,1,1,2,1→2	(0.44,0.33,0.22)	10/25/1999	1,2,2,1,1,2,1,3,2→1	(0.44,0.44,0.11)
3/26/1999	3,1,1,3,2,3,3,3,3→3	(0.22,0.11,0.67)	6/4/1999	3,3,3,1,2,1,2,2,1→3	(0.33,0.33,0.33)	8/13/1999	3,2,3,2,3,1,1,1,2→3	(0.33,0.33,0.33)	10/26/1999	3,1,2,2,1,1,2,1,3→3	(0.44,0.33,0.22)
3/29/1999	3,3,1,1,3,2,3,3,3→3	(0.22,0.11,0.67)	6/5/1999	2,3,3,3,1,2,1,2,2→2	(0.22,0.44,0.33)	8/16/1999	3,3,2,3,2,3,1,1,1→3	(0.33,0.22,0.44)	10/27/1999	2,3,1,2,2,1,1,2,1→2	(0.44,0.44,0.11)
3/30/1999	1,3,3,1,1,3,2,3,3→1	(0.33,0.11,0.56)	6/7/1999	3,2,3,3,3,1,2,1,2→3	(0.22,0.33,0.44)	8/17/1999	3,3,3,2,3,2,3,1,1→3	(0.22,0.22,0.56)	10/28/1999	2,2,3,1,2,2,1,1,2→2	(0.33,0.56,0.11)
3/31/1999	2,1,3,3,1,1,3,2,3→2	(0.33,0.22,0.44)	6/8/1999	3,3,2,3,3,3,1,2,1→3	(0.22,0.22,0.56)	8/18/1999	3,3,3,3,2,3,2,3,1→3	(0.11,0.22,0.67)	10/29/1999	2,2,2,3,1,2,2,1,1→2	(0.33,0.56,0.11)
4/1/1999	2,2,1,3,3,1,1,3,2→2	(0.33,0.33,0.33)	6/9/1999	3,3,3,2,3,3,3,1,2→3	(0.11,0.22,0.67)	8/19/1999	1,3,3,3,3,2,3,2,3→1	(0.11,0.22,0.67)	10/30/1999	2,2,2,2,3,1,2,2,1→2	(0.22,0.67,0.11)
4/2/1999	3,2,2,1,3,3,1,1,3→3	(0.33,0.22,0.44)	6/10/1999	3,3,3,3,2,3,3,3,1→3	(0.11,0.11,0.78)	8/20/1999	2,1,3,3,3,3,2,3,2→2	(0.11,0.33,0.56)			

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An Efficient Image Segmentation Algorithm Using Neutrosophic Graph Cut

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Abstract: Segmentation is considered as an important step in image processing and computer vision applications, which divides an input image into various non-overlapping homogenous regions and helps to interpret the image more conveniently. This paper presents an efficient image segmentation algorithm using neutrosophic graph cut (NGC). An image is presented in neutrosophic set, and an indeterminacy filter is constructed using the indeterminacy value of the input image, which is defined by combining the spatial information and intensity information. The indeterminacy filter reduces the indeterminacy of the spatial and intensity information. A graph is defined on the image and the weight for each pixel is represented using the value after indeterminacy filtering. The segmentation results are obtained using a maximum-flow algorithm on the graph. Numerous experiments have been taken to test its performance, and it is compared with a neutrosophic similarity clustering (NSC) segmentation algorithm and a graph-cut-based algorithm. The results indicate that the proposed NGC approach obtains better performances, both quantitatively and qualitatively.

Keywords: image segmentation; neutrosophic set; graph cut; indeterminate filtering

1. Introduction

With a classical definition, image segmentation refers to dividing an input image into several sub-images according to a pre-defined criterion where the sub-images are disjointed, homogenous and meaningful. Image segmentation is also known as an important and crucial step in many computer vision and pattern-recognition applications. Many researchers have been working on image segmentation, and works have been done [1].

Among the published works, graph-based segmentation algorithms constitute an important image segmentation category [2]. A graph G can be denoted as $G = (V, E)$ where V and E are a set of vertices and edges. On an image, vertices can be either pixels or regions, and edges connect the neighboring vertices [3]. A weight is a non-negative measure of dissimilarity which is associated with each edge using some property of the pixels.

In this paper, using the advantages of neutrosophic to interpret the indeterminacy on the image, we combine neutrosophic set into the graph cut for image segmentation. Neutrosophic set (NS) was an extension of the fuzzy set [4]. In NS theory, a member of a set has degrees to the truth, falsity, and indeterminacy, respectively [5]. Therefore, it has an ability to deal with the indeterminacy information and has attracted much attention in almost all engineering communities and subsequently a great number of works have been studied, such as NS-based color and texture segmentation [6–14], NS-based clustering [15–17], NS-based similarity for image thresholding [18], NS-based edge detection [19] and NS-based level set [20].

Firstly, the image is interpreted using neutrosophic set and indeterminacy degree is calculated accordingly. Then an indeterminacy filter is constructed using the indeterminacy value on the image which is defined by combining the spatial and intensity information. The indeterminacy filter reduces the indeterminacy in the intensity and spatial information respectively. A graph is defined on the image and the weight for each pixel is represented using the value after indeterminacy filtering, and the energy function is also redefined using the neutrosophic value. A maximum-flow algorithm on the graph is employed to obtain the final segmentation results.

The proposed method has the following new contributions: (1) an indeterminate filter is proposed to reduce the uncertain information in the image; and (2) a new energy function in graph model is defined in neutrosophic domain and used to segment the image with better performance.

The rest of the paper is structured: Section 2 briefly reviews the previous works. Section 3 describes the proposed method based on neutrosophic graph cut. Section 4 provides the experimental results. Conclusions are drawn in Section 5.

2. Previous Works

As mentioned in the Introduction Section, graph based image segmentation has gained much attention from the domain researchers with many published papers. A systematic survey work on graph-based image segmentation was conducted by Peng et al. [21]. In this survey, authors categorized the graph-based image segmentation methods into five groups. The first category is minimal spanning tree (MST)-based method. The MST is a popular concept in graph theory with numerous works. In [22], a hierarchical image segmentation method was proposed based on MST [22]. This method segmented the input image iteratively. At each iteration, one sub-graph was produced and, in the final segmentation, there were a given number of sub-graphs. In [23], a region merging procedure was adopted to produce a MST-based image segmentation algorithm using the differences between two sub graphs and inside graphs.

Cost-function-based graph cut methods constitute the second category. The most popular graph-based segmentation methods are in this category. Wu et al. [3] applied the graph theory to image segmentation and proposed the popular minimal cut method to minimize a cost function. A graph-based image segmentation approach namely normalized cut (Ncut) was presented [24]. It alleviates shortcomings of the minimal cut method by introducing an eigen system. Wang et al. [25] presented a graph-based method and a cost function and defined it as the ratio of the sum of different weights of edges along the cut boundary. Ding et al. [26] presented a cost function to alleviate the weakness of the minimal cut method, in which the similarity between two subgraphs was minimized, and the similarity within each subgraph was maximized. Another efficient graph-based image segmentation method was proposed in [27], and considered both the interior and boundary information. It minimized the ratio between the exterior boundary and interior region. The Mean-Cut incorporates the edge weight function [25] to minimize the mean edge weight on the cut boundary.

Methods based on Markov random fields (MRF) are in the third class, and the shortest-path-based methods are classified in the fourth class. Generally, MRF-based graph cut methods form a graph structure with a cost function and try to minimize that cost function to solve the segmentation problem. The shortest path based methods searched the shortest path between two vertices [21], and the boundaries of segments were achieved by employing the shortest path. The shortest-path-based segmentation methods need interaction from users.

The other graph-based methods are categorized into the fifth class. The random walker (RW) method by Grady [28] used a weighted graph to obtain labels of pixels and then these weights were considered as the likelihood that RW went across the edge. Finally, a pixel label was assigned by the seed point where the RW reached first.

3. Proposed Method

3.1. Neutrosophic Image

An element in *NS* is defined as: let $A = \{A_1, A_2, , \ldots, A_m\}$ as a set of alternatives in neutrosophic set. The alternative A_i is $\{T(A_i),\ I(A_i), F(A_i)\}/A_i$, where $T(A_i)$, $I(A_i)$ and $F(A_i)$ are the membership values to the true, indeterminate and false set.

An image I_m in *NS* is called neutrosophic image, denoted as I_{NS} which is interpreted using *Ts*, *Is* and *Fs*. Given a pixel *P(x,y)* in I_{NS}, it is interpreted as $P_{NS}(x,y) = \{Ts(x,y),\ Is(x,y), Fs(x,y)\}$. $Ts(x,y)$, $Is(x,y)$ and $Fs(x,y)$ represent the memberships belonging to foreground, indeterminate set and background, respectively.

Based on the intensity value and local spatial information, the true and indeterminacy memberships are used to describe the indeterminacy among local neighborhood as:

$$Ts(x,y) = \frac{g(x,y) - g_{\min}}{g_{\max} - g_{\min}} \tag{1}$$

$$Is(x,y) = \frac{Gd(x,y) - Gd_{\min}}{Gd_{\max} - Gd_{\min}} \tag{2}$$

where *g(x,y)* and *Gd(x,y)* are the intensity and gradient magnitude at the pixel of *(x,y)* on the image.

We also compute the neutrosophic membership values based on the global intensity distribution which considers the indeterminacy on intensity between different groups. The neutrosophic c-means clustering (NCM) overcomes the disadvantages on handling indeterminate points in other algorithms [16]. Here, we use NCM to obtain the indeterminacy values between different groups on intensity to be segmented.

Using NCM, the truth and indeterminacy memberships are defined as:

$$K = \left[\frac{1}{\varpi_1}\sum_{j=1}^{C}(x_i - c_j)^{-\frac{2}{m-1}} + \frac{1}{\varpi_2}(x_i - \bar{c}_{i\max})^{-\frac{2}{m-1}} + \frac{1}{\varpi_3}\delta^{-\frac{2}{m-1}}\right]^{-1} \tag{3}$$

$$Tn_{ij} = \frac{K}{\varpi_1}(x_i - c_j)^{-\frac{2}{m-1}} \tag{4}$$

$$In_i = \frac{K}{\varpi_2}(x_i - \bar{c}_{i\max})^{-\frac{2}{m-1}} \tag{5}$$

where $T_{n_{ij}}$ and I_{n_i} are the true and indeterminacy membership value of point i, and the cluster centers is c_j. $\bar{c}_{i\max}$ is obtained using to indexes of the largest and second largest value of T_{ij}. They are updated at each iteration until $\left|T_{n_{ij}}^{(k+1)} - T_{n_{ij}}^{(k)}\right| < \varepsilon$, where ε is a termination criterion.

3.2. Indeterminacy Filtering

A filter is newly defined based on the indeterminacy and used to remove the effect of indeterminacy information for segmentation, in which the kernel function is defined using a Gaussian function as follows:

$$G_I(u,v) = \frac{1}{2\pi\sigma_I^2}\exp\left(-\frac{u^2+v^2}{2\sigma_I^2}\right) \tag{6}$$

$$\sigma_I(x,y) = f(I(x,y)) = aI(x,y) + b \tag{7}$$

where σ_I is the standard deviation value where is defined as a function $f(\cdot)$ associated to the indeterminacy degree. When the indeterminacy level is high, σ_I is large and the filtering can make the current local neighborhood more smooth. When the indeterminacy level is low, σ_I is small and the filtering takes a less smooth operation on the local neighborhood. The reason to use Gaussian function is that it can map the indeterminate degree to a filter weight more smooth.

An indeterminate filtering is taken on $T_s(x, y)$, and it becomes more homogeneous.

$$T_s'(x,y) = Ts(x,y) \oplus G_{Is}(u,v) = \sum_{v=y-m/2}^{y+m/2} \sum_{u=x-m/2}^{x+m/2} Ts(x-u, y-v) G_{Is}(u,v) \tag{8}$$

$$G_{Is}(u,v) = \frac{1}{2\pi\sigma_{Is}^2} \exp\left(-\frac{u^2+v^2}{2\sigma_{Is}^2}\right) \tag{9}$$

$$\sigma_{Is}(x,y) = f(Is(x,y)) = aIs(x,y) + b \tag{10}$$

where T_s' is the indeterminate filtering result. a and b are the parameters in the linear function to transform the indeterminacy level to parameter value.

The filtering is also used on $T_{n_{ij}}(x, y)$ after NCM. The input of NCM is the local spatial neutrosophic value after indeterminacy filtering.

$$Tn_{ij}'(x,y) = Tn_{ij}(x,y) \oplus G_{In}(u,v) = \sum_{v=y-m/2}^{y+m/2} \sum_{u=x-m/2}^{x+m/2} Tn_{ij}(x-u, y-v) G_{In}(u,v) \tag{11}$$

$$G_{In}(u,v) = \frac{1}{2\pi\sigma_{In}^2} \exp\left(-\frac{u^2+v^2}{2\sigma_{In}^2}\right) \tag{12}$$

$$\sigma_{In}(x,y) = f(In(x,y)) = cIn(x,y) + d \tag{13}$$

where Tn_{ij}' is the indeterminate filtering result on T_s and m is the size of the filter kernel. Tn_{ij}' is employed to construct a graph, and a maximum-flow algorithm is used to segment the image.

3.3. Neutrosophic Graph Cut

A cut $C = (S,T)$ partitions a graph $G = (V,E)$ into two subsets: S and T. The cut set of a cut $C = (S,T)$ is the set $\{(u, v) \in E | u \in S, v \in T\}$ of edges that have one endpoint in S and the other endpoint in T. Graph cuts can efficiently solve image segmentation problems by formulating in terms of energy minimization, which is transformed into the maximum flow problem in a graph or a minimal cut of the graph.

The energy function often includes two components: data constrict E_{data} and smooth constrict E_{smooth} as:

$$E(f) = E_{data}(f) + E_{smooth}(f) \tag{14}$$

where f is a map which assigns pixels to different groups. E_{data} measures the disagreement between f and the assigned region, which can be represented as a t-link, while E_{smooth} evaluates the extent of how f is piecewise smooth and can be represented as an n-link in a graph.

Different models have different forms in the implementation of the energy function. The function based on Potts model is defined as:

$$E(f) = \sum_{p \in P} D_p(f_p) + \sum_{\{p,q\} \in N} V_{\{p,q\}}(f_p, f_q) \tag{15}$$

where p and q are pixels, and N is the neighborhood of p. D_p evaluates how appropriate a segmentation is for the pixel p.

In the proposed neutrosophic graph cut (NGC) algorithm, the data function D_p and smooth function $V_{\{p,q\}}$ are defined as:

$$D_{ij}(p) = \left| Tn_{ij}'(p) - C_j \right| \tag{16}$$

$$V_{\{p,q\}}(f_p, f_q) = u\delta(f_p \neq f_q) \quad (17)$$

$$\delta(f_p \neq f_q) = \begin{cases} 1 & \text{if } f_p \neq f_q \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

where u is a constant number in [0, 1] and used for a penalty of the disagree of labeling of pixel p and q.

After the energy function is redefined in the neutrosophic set domain, a maximum flow algorithm in graph cut theory is used to segment the objects from the background.

All steps can be summarized as:

Step 1: Compute the local neutrosophic value T_s and I_s.
Step 2: Take indeterminate filtering on T_s using I_s.
Step 3: Use NCM algorithm on the filtered T_s subset to obtain T_n and I_n.
Step 4: Filter T_n using indeterminate filter based on I_n.
Step 5: Define the energy function based on the $T_n{}'$ value.
Step 6: Partition the image using the maximum flow algorithm.

The flowchart of the proposed approach is shown in Figure 1 as:

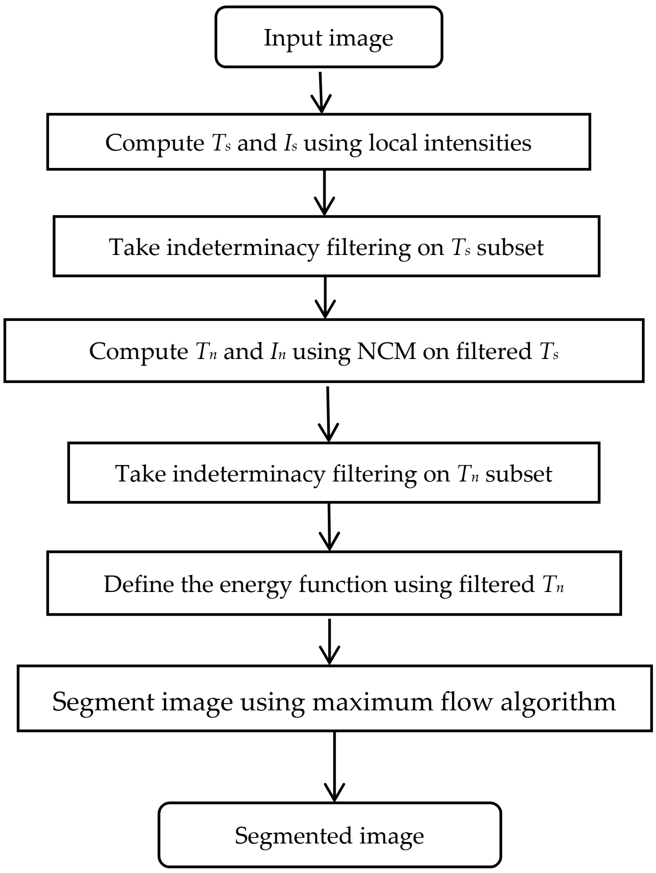


Figure 1. The flowchart of the proposed method.

To show the steps of the whole algorithm, some intermediate results are demonstrated using an example image in Figure 2.

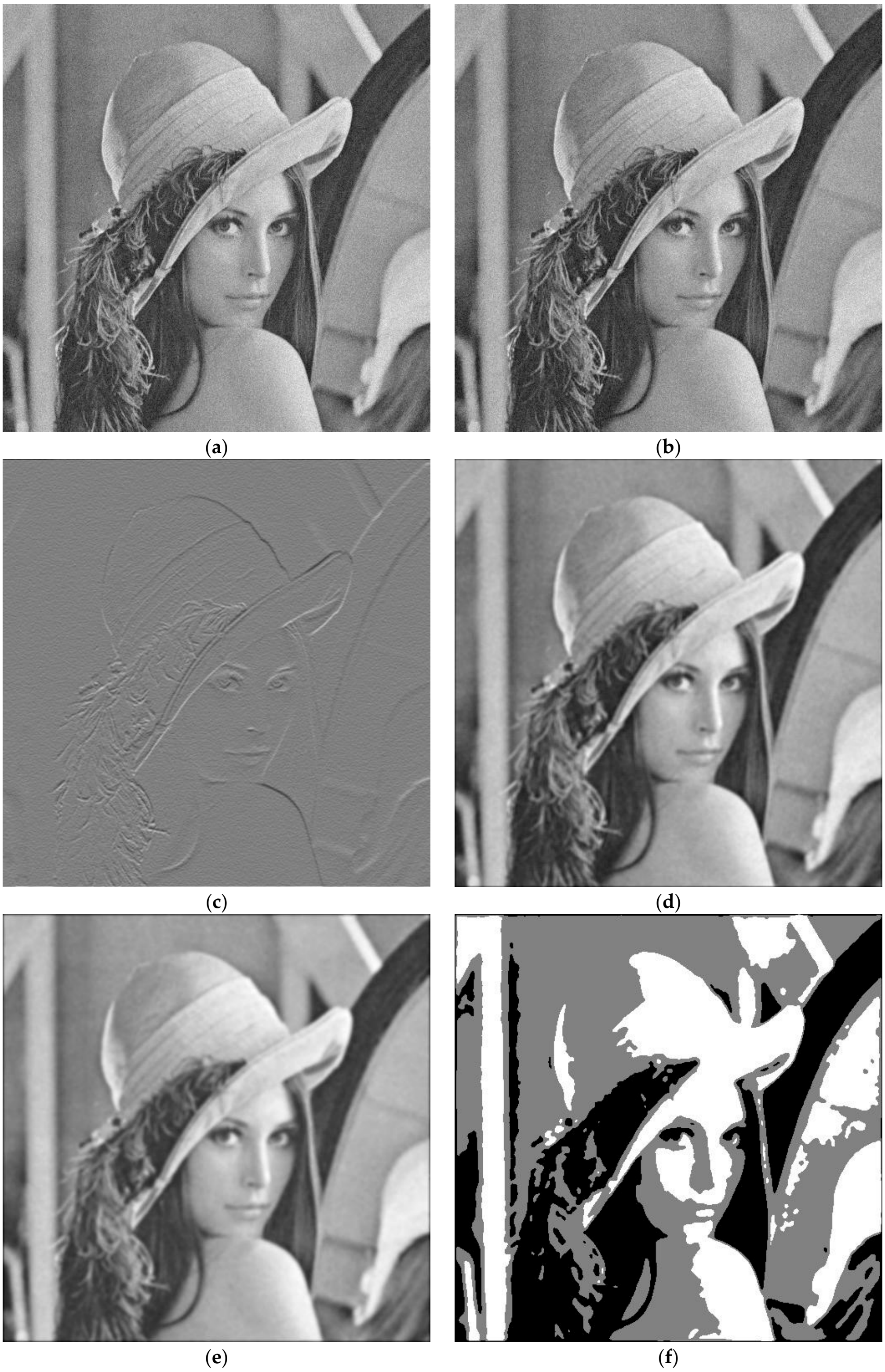


Figure 2. Intermediate results for "Lena" image: (a) Original image; (b) Result of Ts; (c) Result of Is; (d) Filtered result of Ts; (e) Filter result of Tn; (f) Final result.

4. Experimental Results

It is challenging to segment images having uncertain information such as noise. Different algorithms have been developed to solve this problem. To validate the performance of the NGC approach on image segmentation, we test it on many images and compare its performance with a newly published neutrosophic similarity clustering (NSC) method [12] which performed better than previous methods [6], and a newly developed graph cut (GC) method [29].

All experiments are taken using the same parameters: a = 10; b = 0.25; c = 10; d = 0.25; and u = 0.5.

4.1. Quantitatively Evaluation

Simulated noisy images are employed to compare the NGC with NSC and GC methods visually, and then their performances are tested quantitatively by using two metrics. In the NSC method [12], simulated noisy images were employed to evaluate its performance. To make the comparison fair and consistent, we use the same images and noise and test three algorithms on them.

A simulated image having intensities of 64, 128, and 192 is added with Gaussian noises and used to evaluate the performance of NGC, NSC, and GC algorithms. Figure 3a shows the original noisy images with noise mean values are 0 and variance values: 80, 100, and 120, respectively. Figure 3b–d lists results by the NSC, GC, and NGC methods, respectively. The results in Figure 3 also show the NGC performs visually better than NSC and GC methods on the simulated images with low contrast and noises. Pixels in Figure 3b,c that are segmented into wrong groups are assigned into the right groups by NGC method in Figure 3d. Boundary pixels, which are challenging to label, are also segmented into right categories by NGC.

Misclassification error (ME) is used to evaluate the segmentation performances [30–32]. The ME measures the percentage of background wrongly categorized into foreground, and vice versa.

$$ME = 1 - \frac{|B_o \cap B_T| + |F_o \cap F_T|}{|B_o| + |F_o|} \tag{19}$$

where F_o, B_o, F_T, and B_T are the object and background pixels on the ground truth image and the resulting image, respectively.

In addition, FOM [31] is used to evaluate the difference between the segmented results with the ground truth:

$$FOM = \frac{1}{\max(N_I, N_A)} \sum_{k=1}^{N_A} \frac{1}{1 + \beta d^2(k)} \tag{20}$$

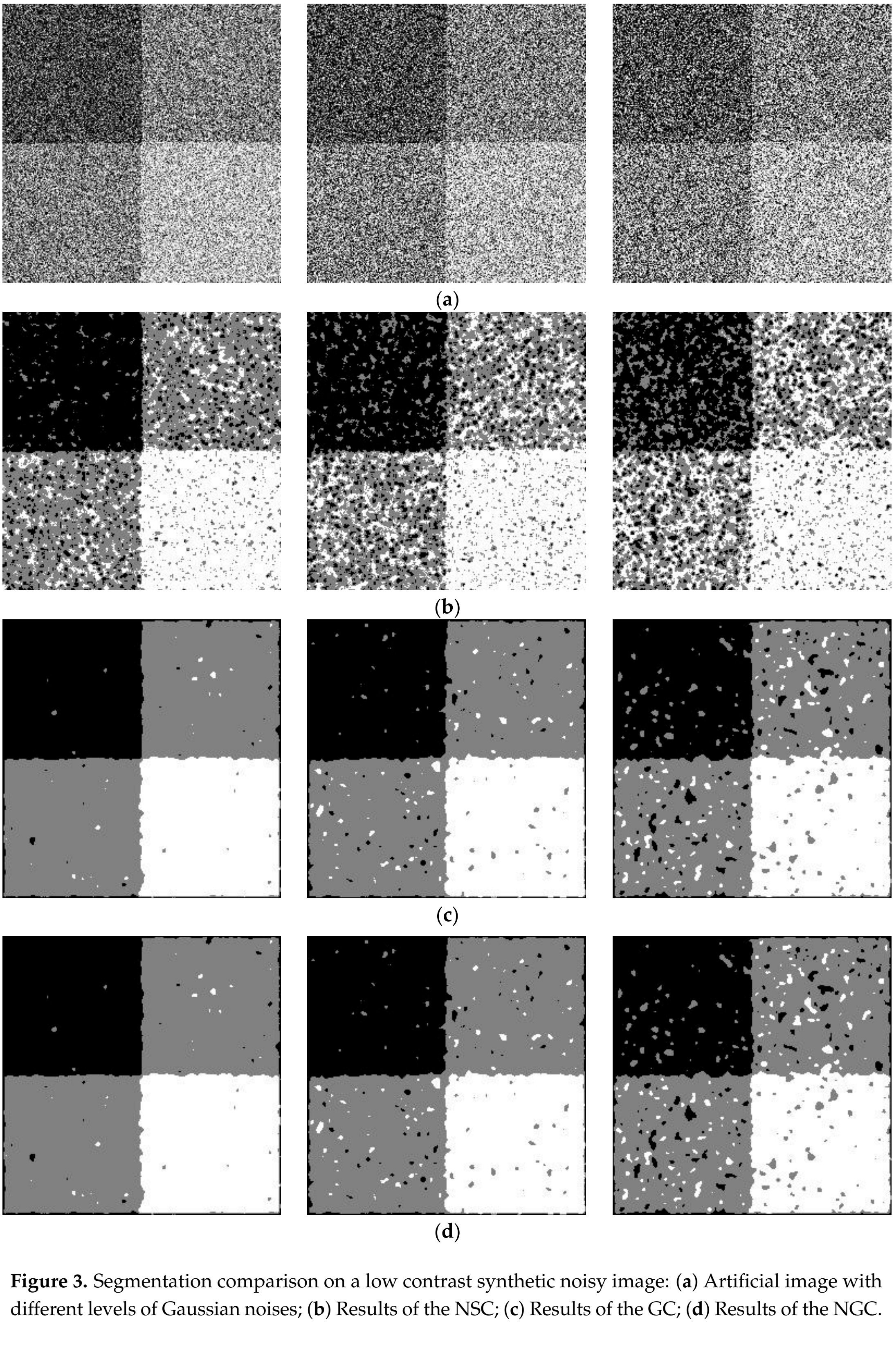
where N_I and N_A are the numbers of the segment object and the true object pixels. $d(k)$ is the distance from the k_{th} actual pixel to the nearest segmented result pixel. β is a constant and set as $1/9$ in [31].

The quality of the noisy image is measured via a signal to noise ratio (SNR):

$$SNR = 10 \log \left[\frac{\sum_{r=1}^{H-1} \sum_{c=1}^{W-1} I^2(r,c)}{\sum_{r=1}^{H-1} \sum_{c=1}^{W-1} (I(r,c) - I_n(r,c))^2} \right] \tag{21}$$

where $I_n(r,c)$and $I(r,c)$ are the intensities of point (r,c) in the noisy and original images, respectively.

The results of ME and FOM are drawn in Figures 4 and 5, where $*$ denotes NSC method, o denotes GC method, and + is NGC method. NGC method has the lowest ME values. All ME by NGC are smaller than 0.043, and all values from NSC and GC methods are larger than those from NGC method. The NGC obtains the best performance with ME = 0.0068 when SNR is 5.89 dB, while NSC has the lowest value ME = 0.1614 and GC ME = 0.0327. NGC also has bigger FOM than NSC and GC, especially at the low SNR. The comparison results are listed in Table 1. The mean and standard deviation of the ME and FOM are 0.247 ± 0.058 and 0.771 ± 0.025 using NSC method, 0.062 ± 0.025 and 0.897 ± 0.027 using GC method, 0.015 ± 0.011 and 0.987 ± 0.012 using NGC method, respectively. The NGC method achieves better performance with lesser values of ME and FOM than the NSC and GC methods.



(a) (b) (c) (d)

Figure 3. Segmentation comparison on a low contrast synthetic noisy image: (**a**) Artificial image with different levels of Gaussian noises; (**b**) Results of the NSC; (**c**) Results of the GC; (**d**) Results of the NGC.

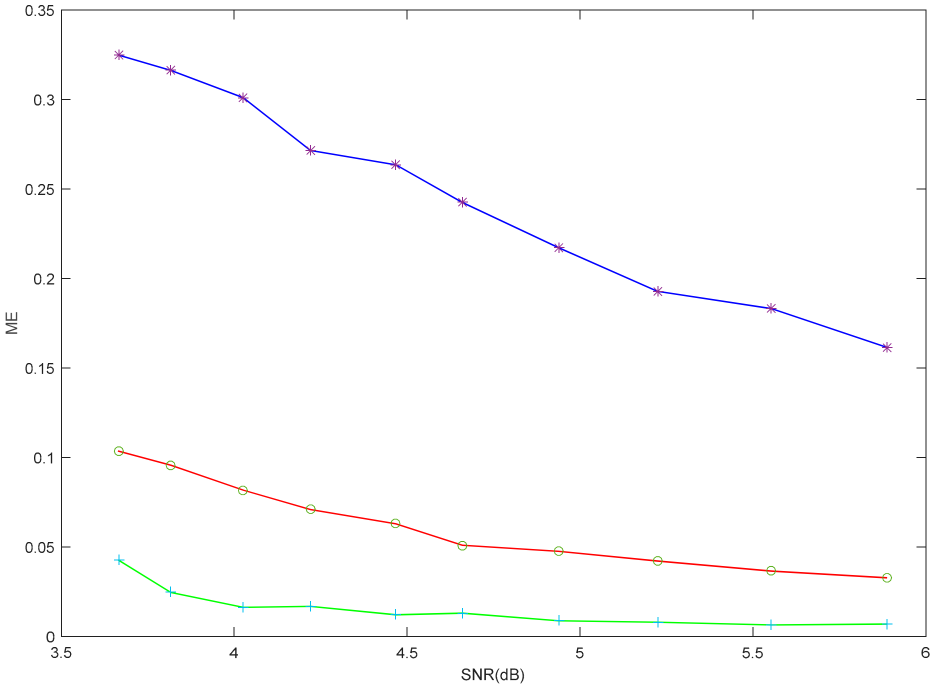


Figure 4. Plot of ME: ∗, NSC method; o, GC method; +, NGC method.

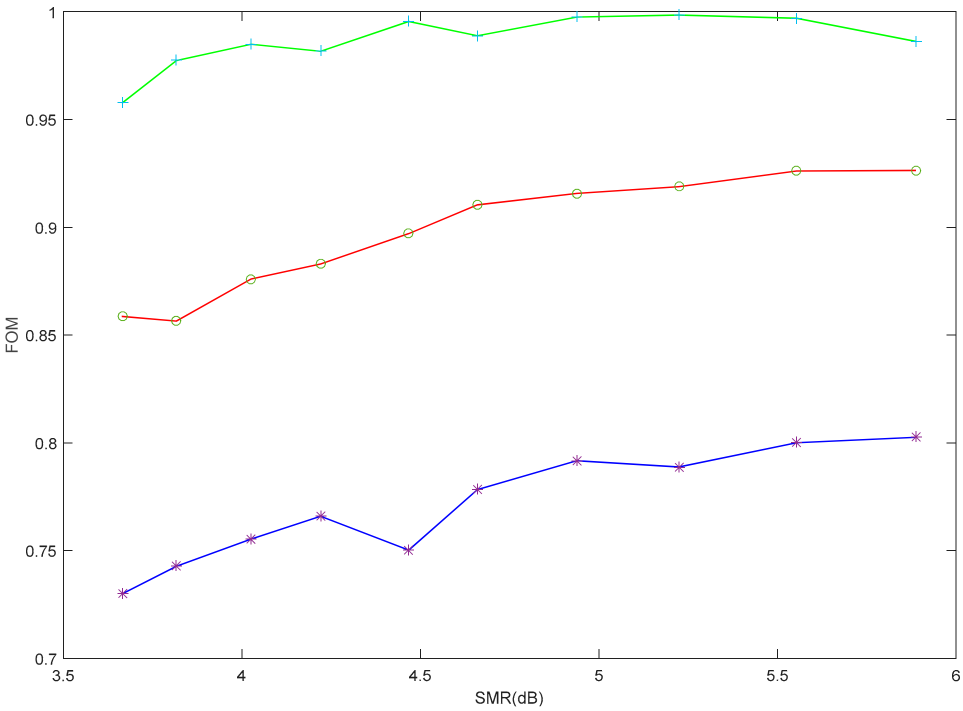


Figure 5. Plot of FOM: ∗, NSC method; o, GC method; +, NGC method.

Table 1. Performance comparisons on evaluation metrics.

Metrics	NSC	GC	NGC
ME	0.247 ± 0.058	0.062 ± 0.025	0.015 ± 0.011
FOM	0.771 ± 0.025	0.897 ± 0.027	0.987 ± 0.012

4.2. Performance on Natural Images

Many images are employed to validate the NGC's performance. We also compare the results with a newly developed image segmentation algorithm based on an improved kernel graph cut (KGC) algorithm [33]. Here, five images are randomly selected to show the NGC method's segmentation performance. The first row in Figures 6–10 shows the original images and segmentation results of NSC, GC, KGC, and NGC, respectively. The other rows demonstrate the results on the noisy images. The results by NGC have better quality than those of NSC, GC, and KGC visually. On the original images, the NGC and GC obtain similarly accurate results, while the KGC obtains under-segmented results. When the noise is increased, the NSC and GC are deeply affected and have a lot of over-segmentation, and the KGC results are under-segmentation and lose some details. However, NGC is not affected by noise and most pixels are categorized into the right groups, and the details on the boundary are well segmented.

Figure 6 shows the segmentation results on the "Lena" image. The results in the fourth columns are better than in the second and third columns. Regions of face, nose, mouth, and eyes are segmented correctly by NGC. The noisy regions as hair region and the area above the hat are also segmented correctly. However, the NSC and GC methods obtain wrong segmentations, especially in the region above the hat. The KGC results lose some detail information on face and eyes. In the observation, the NGC algorithm is better than NSC.

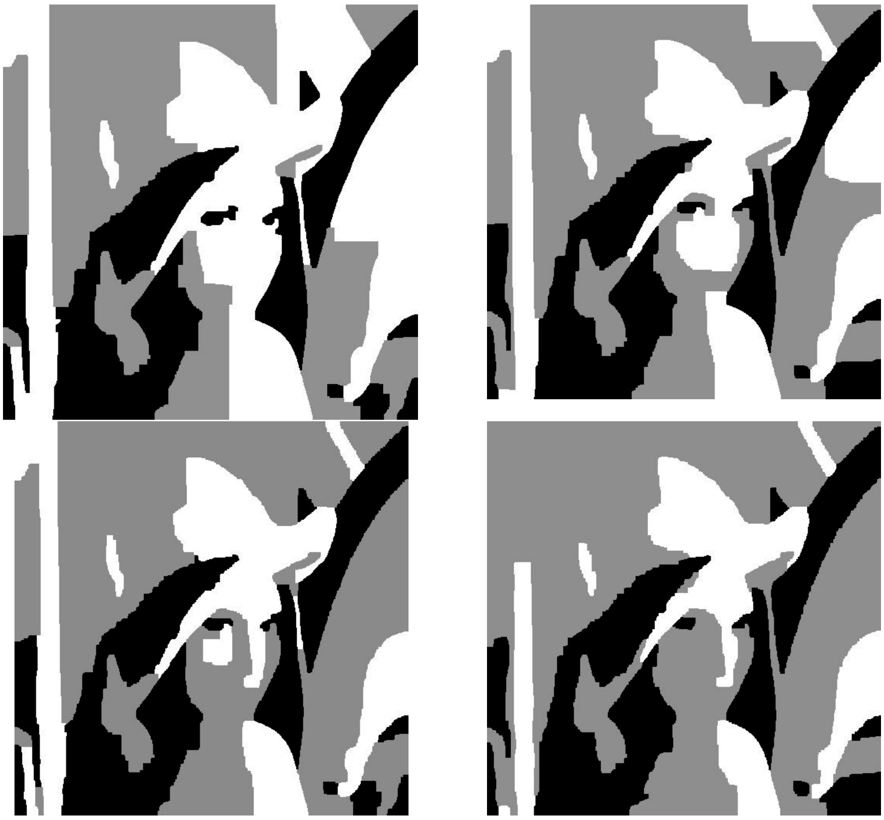
(**a**)

Figure 6. *Cont.*

(b)

(c)

Figure 6. *Cont.*



(**d**)

(**e**)

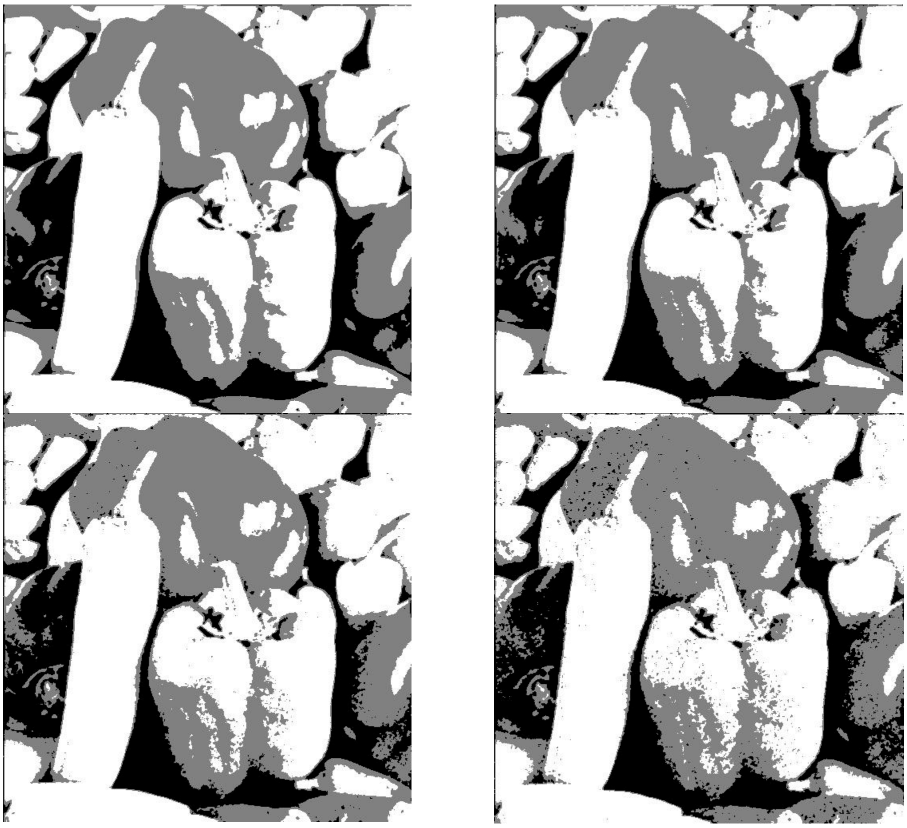
Figure 6. Comparison results on "Lena" image: (**a**) "Lena" image with different Gaussian noise level: variance: 0, 10, 20, 30; (**b**) Segmentation results of NSC; (**c**) Segmentation results of GC; (**d**) Segmentation results of KGC; (**e**) Segmentation results of NGC.

We also compared the performances of all methods on the "Peppers" image, as shown in Figure 7. As mentioned earlier for other comparisons, for zero noise level, GC, NGC, and KGC produced similar segmentations. GC, KGC and NGC methods produced better segmentation results than NSC in all noise levels. When the noise level increased, the efficiency of the proposed NGC method became more obvious. There were some wrong segmentation regions (black regions in gray pepper regions) in the GC results. Some of the background regions were also wrongly segmented by the GC method. More proper segmentations were obtained with the proposed NGC method. Especially, for noise levels 20 and 30, the NGC method's segmentation achievement was visually better than the others, with less wrongly segmented regions produced. On this image, the KGC achieves similar performance as NGC on the segmentation results.

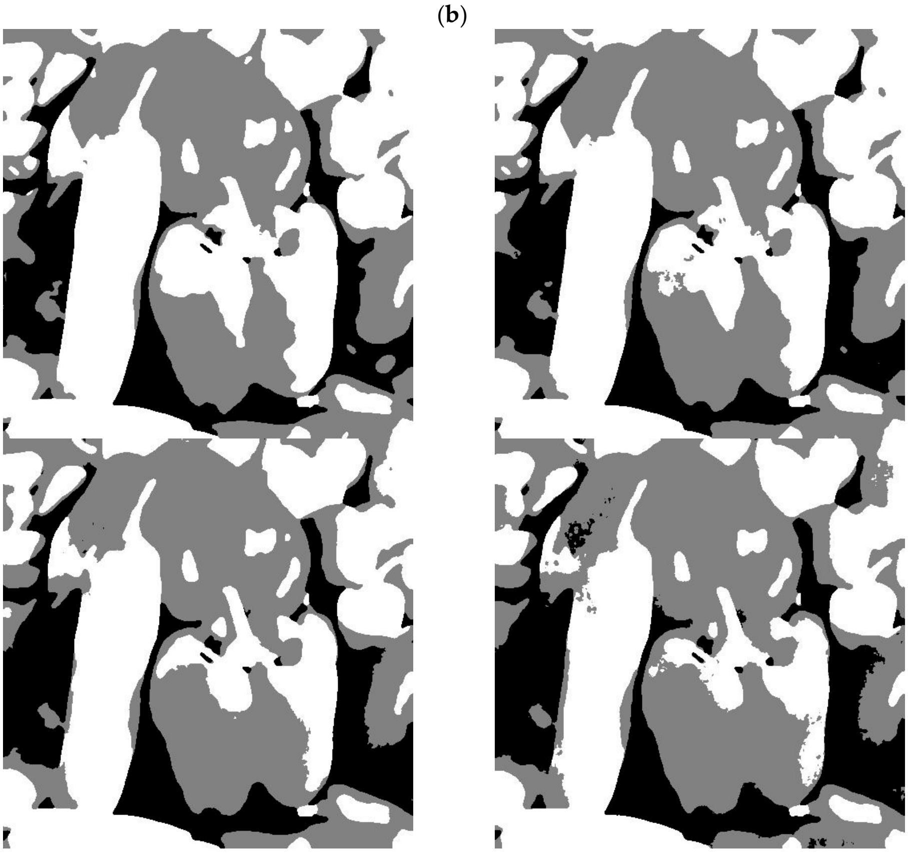
The comparison results on the "Woman" image are given in Figure 8. It is obvious that the NSC method produced worse segmentations when the noise level increased. The GC and KGC methods produced better results when compared to the NSC method, with more homogeneous regions produced. It is also worth mentioning that the GC, KGC and NGC methods produced the same segmentation results for the noiseless case. However, when the noise level increased, the face of the woman became more complicated. On the other hand, the proposed NGC method produced more distinctive regions when compared to other methods. On the results of KGC, the boundary of eyes and nose cannot be recognized. In addition, the edges of the produced regions by NGC were smoother than for the others.

(**a**)

Figure 7. *Cont.*

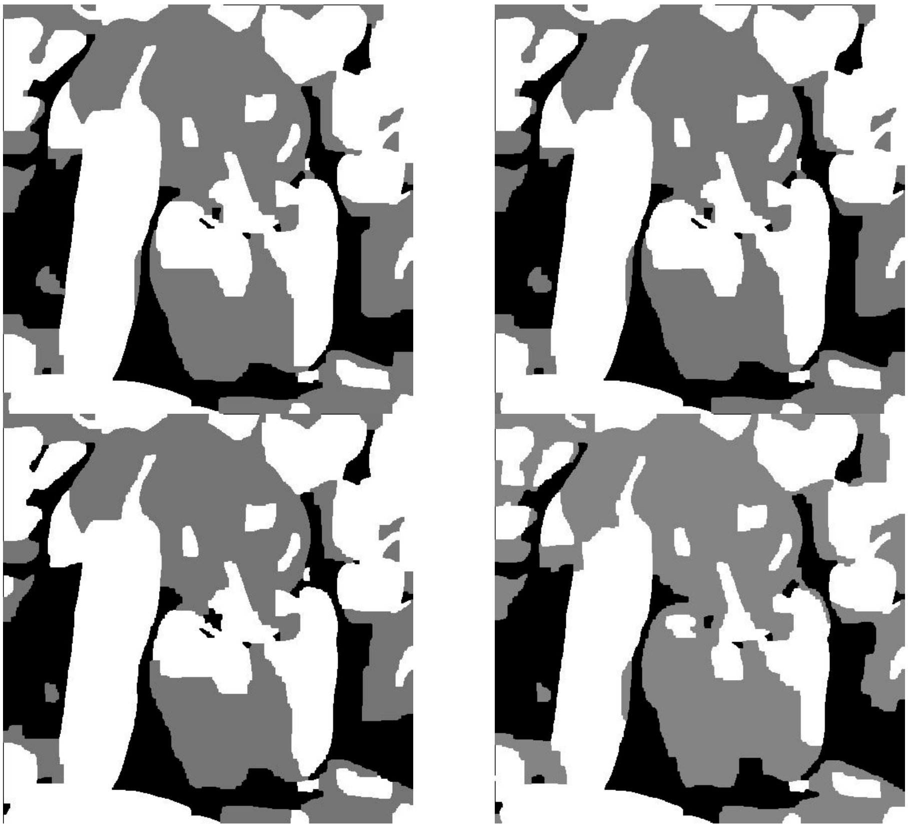


(**b**)

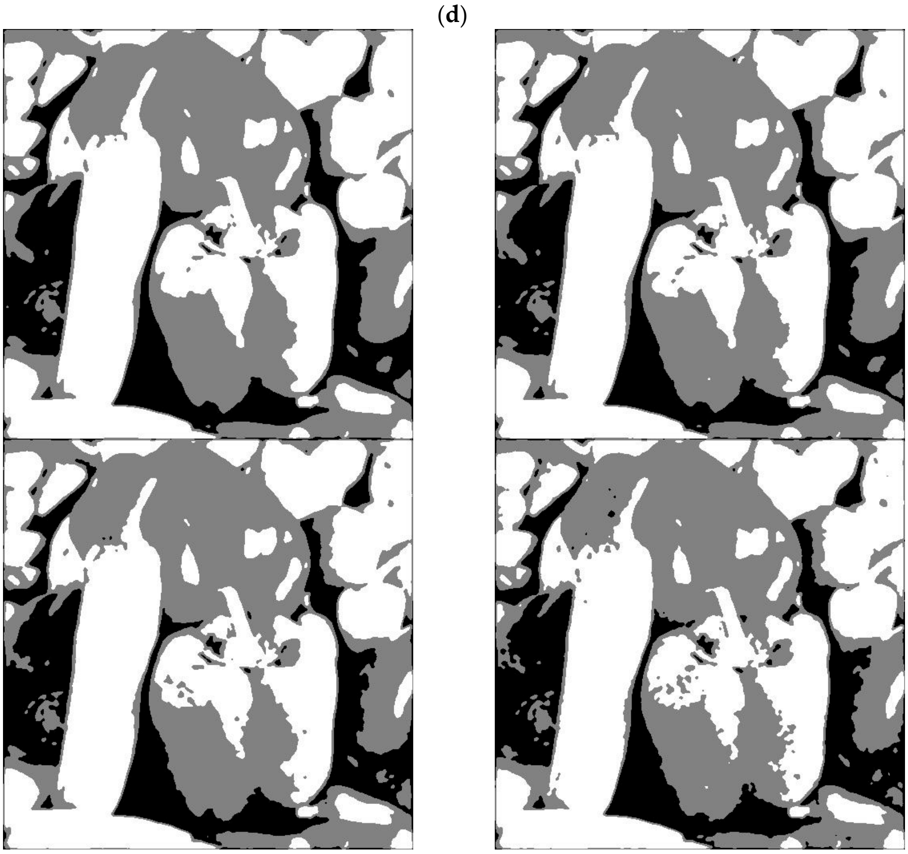


(**c**)

Figure 7. *Cont.*



(**d**)



(**e**)

Figure 7. Comparison results on "Peppers" image: (**a**) "Peppers" image with different Gaussian noise level: variance: 0, 10, 20, 30; (**b**) Segmentation results of NSC; (**c**) Segmentation results of GC; (**d**) Segmentation results of KGC; (**e**) Segmentation results of NGC.

(a)

(b)

Figure 8. *Cont.*

(**c**)

(**d**)

Figure 8. *Cont.*

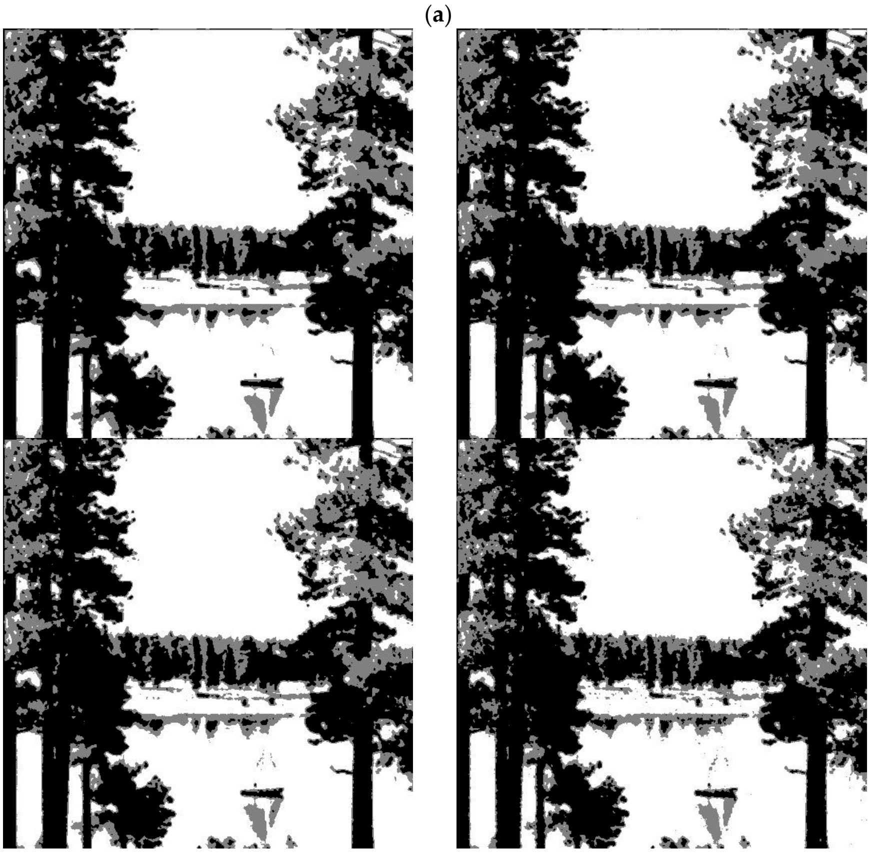
(e)

Figure 8. Comparison results on "Woman" image: (**a**) "Woman" image with different Gaussian noise level: variance: 0, 10, 20, 30; (**b**) Segmentation results of NSC; (**c**) Segmentation results of GC; (**d**) Segmentation results of KGC; (**e**) Segmentation results of NGC.

We also compared these methods on the "Lake" image, as shown in Figure 9. In the comparisons, it is seen that GC, KGC and NGC methods produced better results than for the NSC method. The results are especially better at high noise levels. It should be specified that GC and KGC methods produced more homogeneous regions, but, in that case, the boundary information was lost. This is an important disadvantage of the GC method. On the other hand, the proposed NGC method also produced comparable homogeneous regions, while preserving the edge information. The proposed method especially yielded better results at high noise levels.

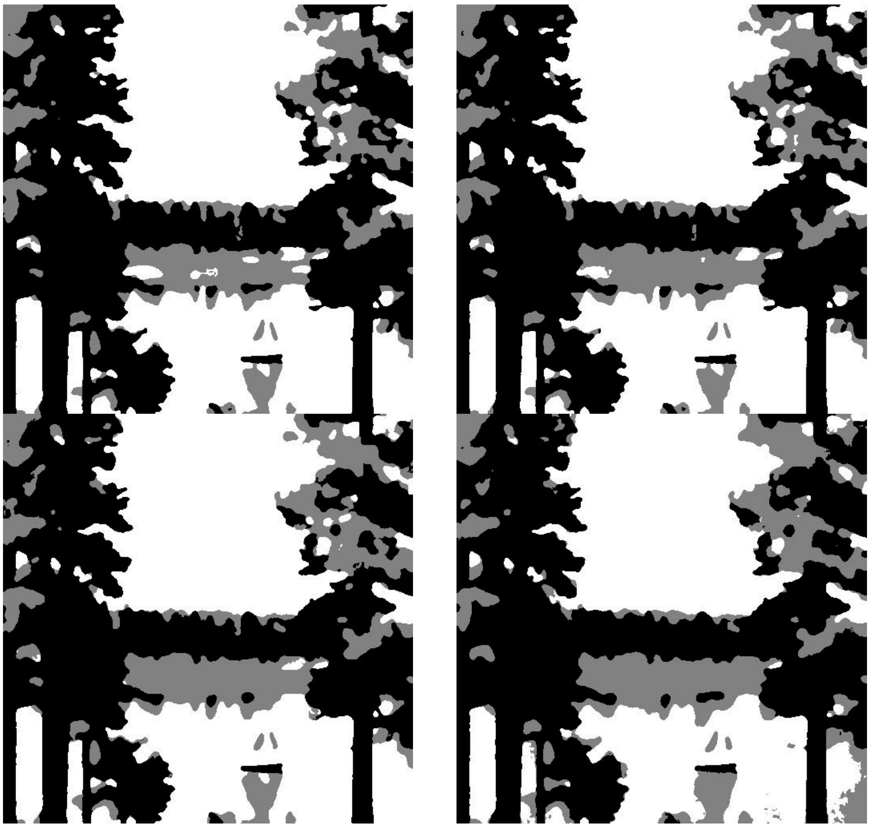
In Figure 10, a more convenient image was used for comparison purposes. While the blood cells can be considered as objects, the rest of the image can be considered as background. In the "Blood" image, the NSC and NGC methods produced similar segmentation results. The KGC has some wrong segmentation on the background region. The NGC has better results on the noisy blood images where the blood cells are extracted accurately and completely. The superiority of the NGC algorithm can also be observed in this image.

(**a**)

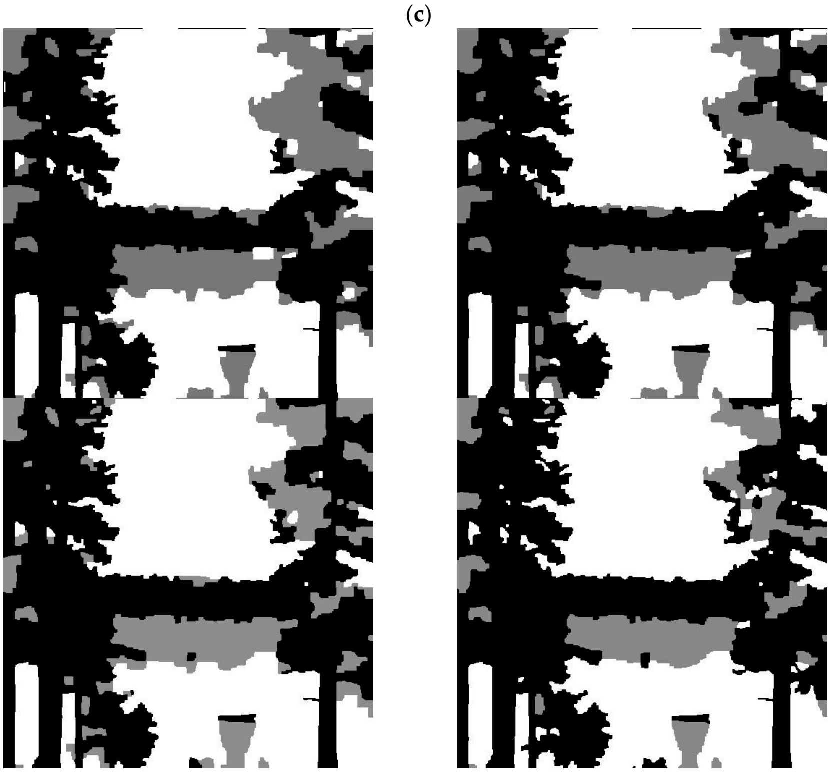


(**b**)

Figure 9. *Cont.*

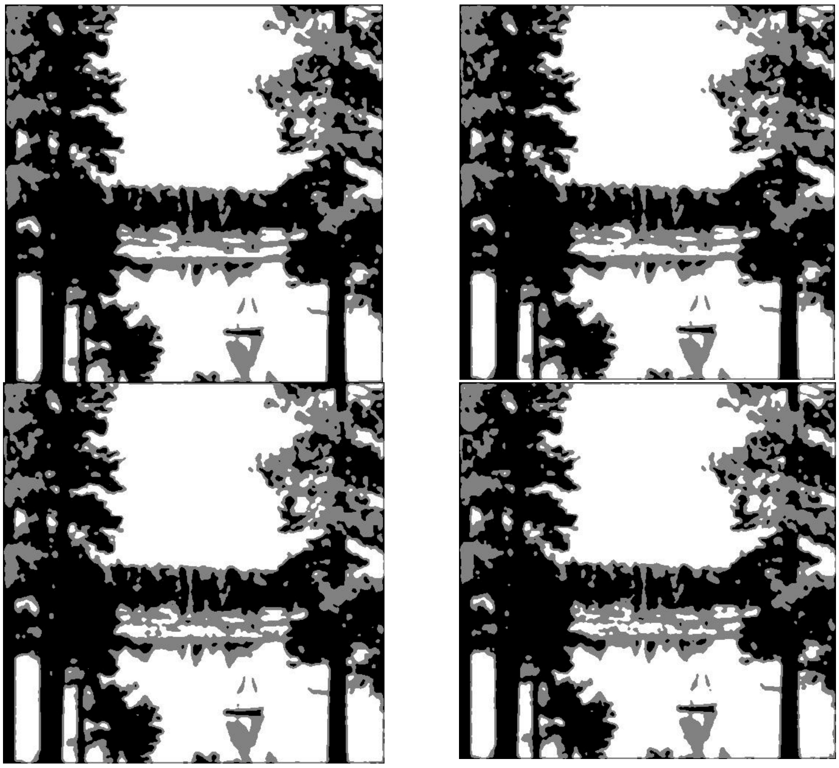


(**c**)



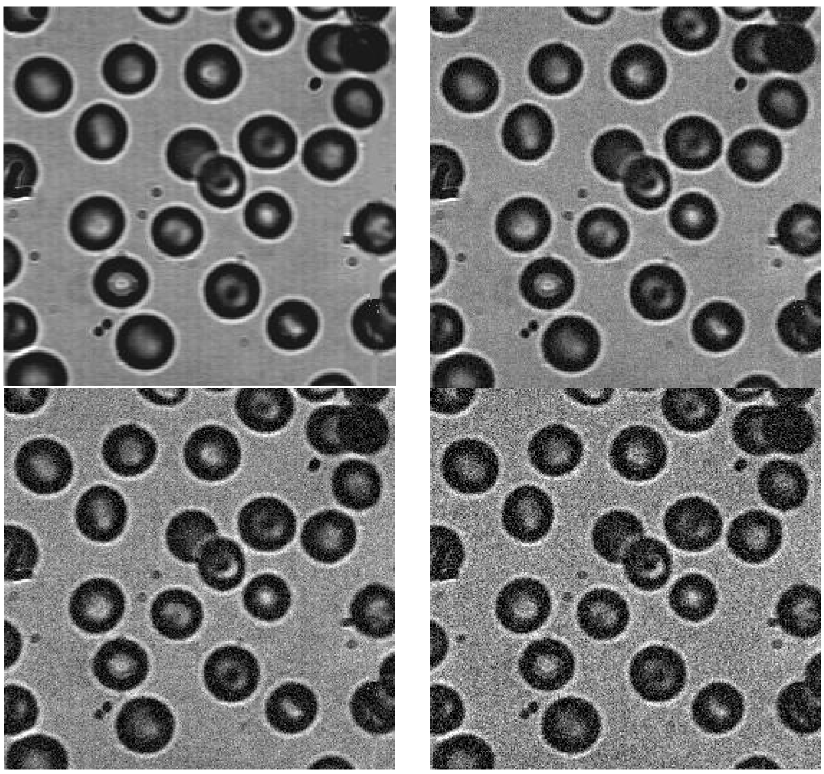
(**d**)

Figure 9. *Cont.*



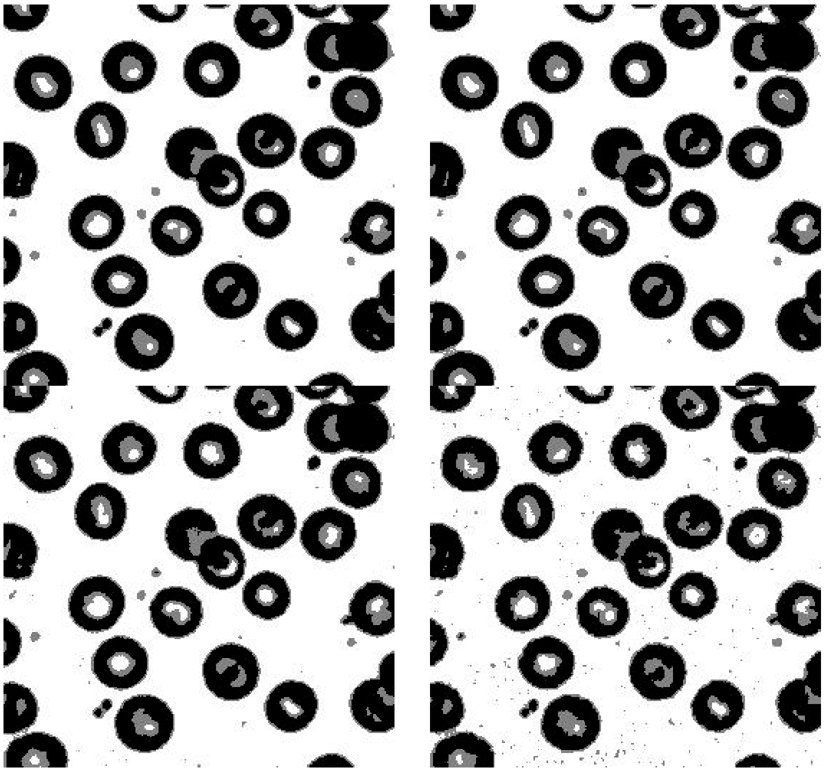
(e)

Figure 9. Comparison results on "Lake" image: (**a**) "Lake" image with different Gaussian noise level: variance: 0, 10, 20, 30; (**b**) Segmentation results of NSC; (**c**) Segmentation results of GC; (**d**) Segmentation results of KGC; (**e**) Segmentation results of NGC.

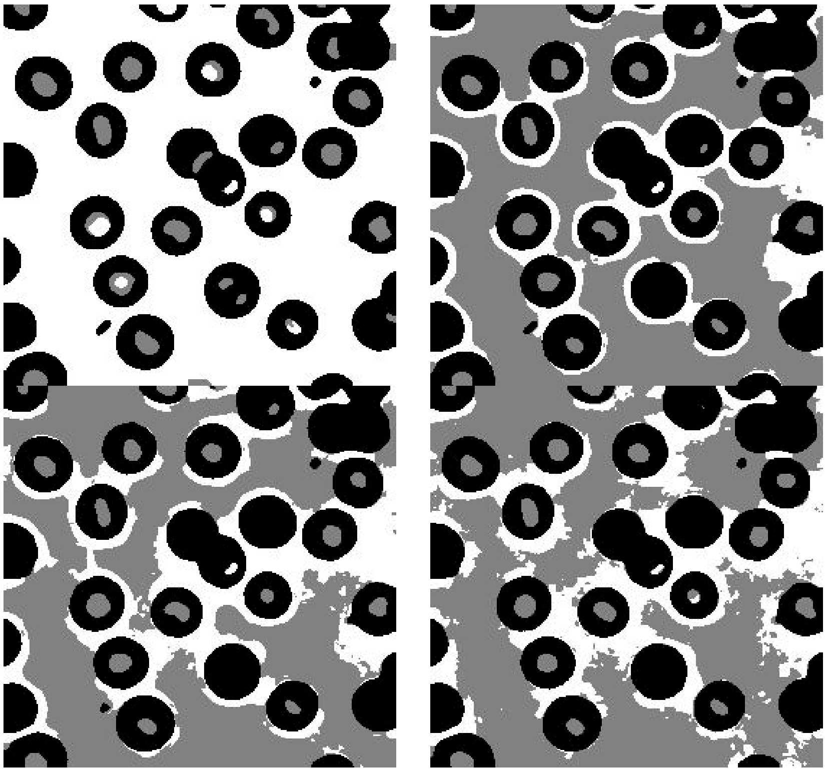


(a)

Figure 10. *Cont.*

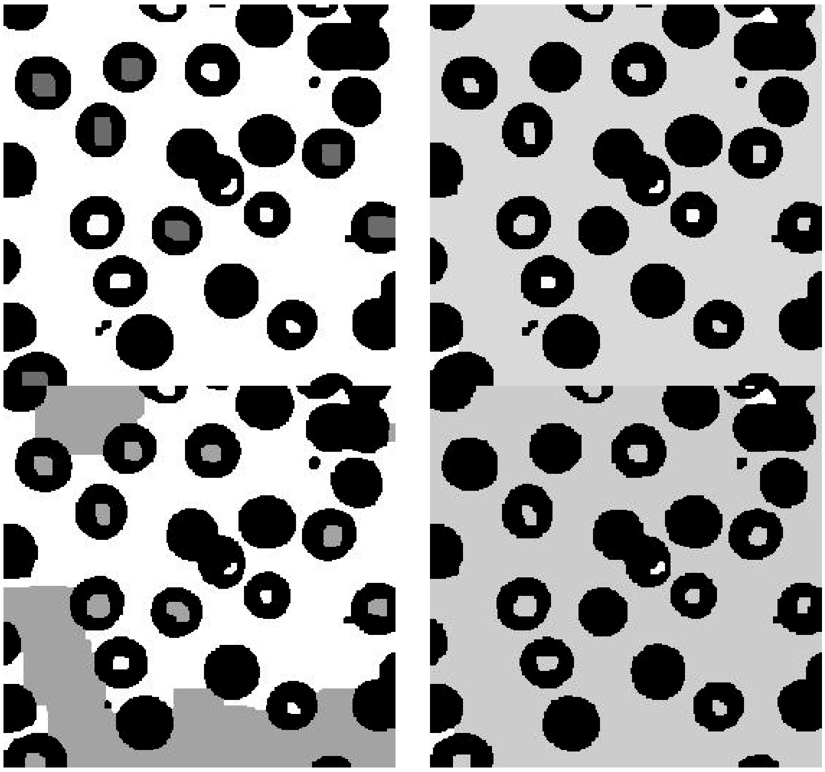


(**b**)

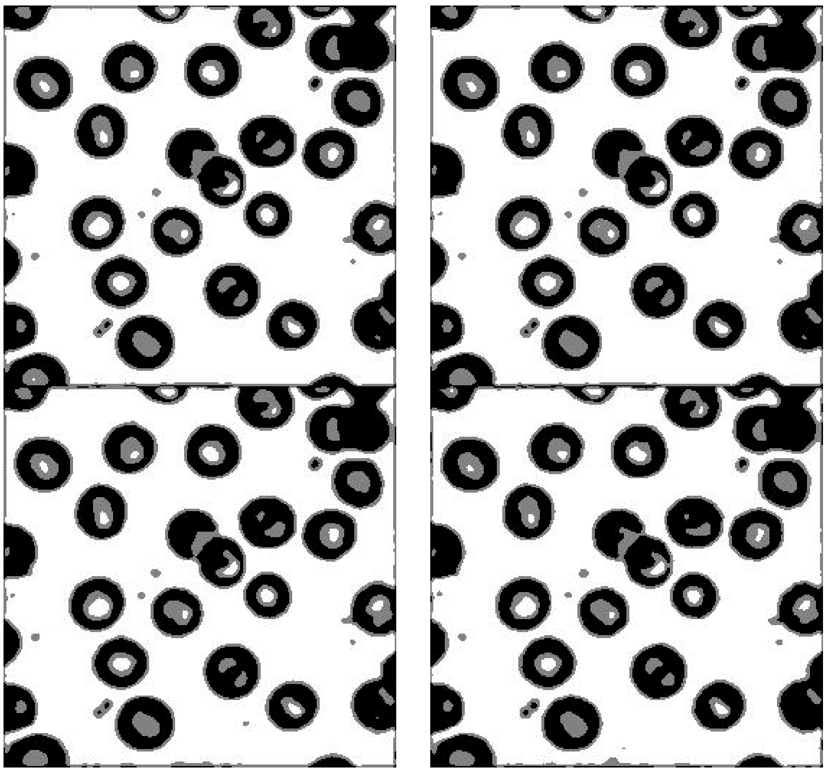


(**c**)

Figure 10. *Cont.*



(**d**)



(**e**)

Figure 10. Comparison results on "Blood" image: (**a**) "Blood" image with different Gaussian noise level: variance: 0, 10, 20, 30, 40; (**b**) Segmentation results of NSC; (**c**) Segmentation results of GC; (**d**) Segmentation results of KGC; (**e**) Segmentation results of NGC.

5. Conclusions

This study aims to develop an efficient method to segment images having uncertain information such as noise. To overcome this challenge, a novel image segmentation method is proposed based on neutrosophic graph cut in this paper. An image is mapped into the neutrosophic set domain and filtered using a newly defined indeterminacy filter. Then, a new energy function is designed according to the neutrosophic values after indeterminacy filtering. The indeterminacy filtering operation removes the indeterminacy in the global intensity and local spatial information. The segmentation results are obtained by maximum flow algorithm. Comparison results demonstrate the better performance of the proposed method than existing methods, in both quantitative and qualitative terms. It also shows that the presented method can segment the images properly and effectively, on both clean images and noisy images, because the indeterminacy information in the image has been handled well in the proposed approach.

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NS-*k*-NN: Neutrosophic Set-Based *k*-Nearest Neighbors Classifier

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Abstract: *k*-nearest neighbors (*k*-NN), which is known to be a simple and efficient approach, is a non-parametric supervised classifier. It aims to determine the class label of an unknown sample by its *k*-nearest neighbors that are stored in a training set. The *k*-nearest neighbors are determined based on some distance functions. Although *k*-NN produces successful results, there have been some extensions for improving its precision. The neutrosophic set (NS) defines three memberships namely T, I and F. T, I, and F shows the truth membership degree, the false membership degree, and the indeterminacy membership degree, respectively. In this paper, the NS memberships are adopted to improve the classification performance of the *k*-NN classifier. A new straightforward *k*-NN approach is proposed based on NS theory. It calculates the NS memberships based on a supervised neutrosophic *c*-means (NCM) algorithm. A final belonging membership U is calculated from the NS triples as $U = T + I - F$. A similar final voting scheme as given in fuzzy *k*-NN is considered for class label determination. Extensive experiments are conducted to evaluate the proposed method's performance. To this end, several toy and real-world datasets are used. We further compare the proposed method with *k*-NN, fuzzy *k*-NN, and two weighted *k*-NN schemes. The results are encouraging and the improvement is obvious.

Keywords: *k*-NN; Fuzzy *k*-NN; neutrosophic sets; data classification

1. Introduction

The *k*-nearest neighbors (*k*-NN), which is known to be the oldest and simplest approach, is a non-parametric supervised classifier [1,2]. It aims to determine the class label of an unknown sample by its *k*-nearest neighbors that are stored in a training set. The *k*-nearest neighbors are determined based on some distance functions. As it is simplest and oldest approach, there have been so many data mining and pattern recognition applications, such as ventricular arrhythmia detection [3], bankruptcy prediction [4], diagnosis of diabetes diseases [5], human action recognition [6], text categorization [7], and many other successful ones.

Although *k*-NN produces successful results, there have been some extensions for improving its precision. Fuzzy theory-based *k*-NN (Fuzzy *k*-NN) has been among the most successful ones. As *k*-NN produces crisp memberships for training data samples, fuzzy *k*-NN replaces the crisp memberships with a continuous range of memberships which enhances the class label determination. Keller et al. [8] was the one who incorporated the fuzzy theory in the *k*-NN approach. Authors proposed three different methods for assigning fuzzy memberships to the labeled samples. After determination of the fuzzy memberships, some distance function was used to weight the fuzzy memberships for final class label determination of the test sample. The membership assignment by the conventional fuzzy *k*-NN algorithm has a disadvantage in that it depends on the choice of some distance function. To alleviate this drawback, Pham et al. [9] proposed an optimally-weighted fuzzy *k*-NN approach. Author introduced a computational scheme for determining optimal weights which were used to improve the efficiency of the fuzzy *k*-NN approach. Denœux et al. [10] proposed a *k*-NN method where Dempster-Shafer theory was used to calculate the memberships of the training data samples. Author assumed that each neighbor of a sample to be classified was considered as an item of evidence and the degree of support was defined as a function of the distance. The final class label assignment was handled by Dempster's rule of combination. Another evidential theory-based *k*-NN approach, denoted by E*k*-NN, has been proposed by Zouhal et al. [11]. In addition to the belonging degree, the authors introduced the ignorant class to model the uncertainty. Then, Zouhal et al. [12] proposed the generalized E*k*-NN approach, denoted by FEk-NN. Authors adopted fuzzy theory for improving the E*k*-NN classification performance. The motivation for the FE*k*-NN was arisen from the fact that each training sample was considered having some degree of membership to each class. In addition, Liu et al. [13] proposed an evidential reasoning based fuzzy-belief *k*-nearest neighbor (FBK-NN) classifier. In FBK-NN, each labeled sample was assigned with a fuzzy membership to each class according to its neighborhood and the test sample's class label was determined by the K basic belief assignments which were determined from the distances between the object and its K nearest neighbors. A belief theory based *k*-NN, denoted by the BK-NN classifier was introduced by Liu et al. [14]. The author aimed to deal with uncertain data using the meta-class. Although, the proposed method produced successful results, the computation complexity and the sensitivity to *k* makes the approach inconvenient for many classification application. Derrac et al. [15] proposed an evolutionary fuzzy *k*-NN approach where interval-valued fuzzy sets were used. The authors not only defined a new membership function, but also a new voting scheme was proposed. Dudani et al. [16] proposed a weighted voting method for *k*-NN which was called the distance-weighted *k*-NN (WKNN). Authors presumed that the closer neighbors were weighted more heavily than the farther ones, using the distance-weighted function. Gou et al. [17] proposed a distance-weighted *k*-NN (DWKNN) method where a dual distance-weighted function was introduced. The proposed method has improved the traditional *k*-NN's performance by using a new method for selection of the *k* value.

In [18–21], Smarandache proposed neutrosophic theories. Neutrosophy was introduced as a new branch of philosophy which deals with the origin, nature, and scope of neutralities, and their interactions with different ideational spectra [19]. Neutrosophy is the base for the neutrosophic set (NS), neutrosophic logic, neutrosophic probability, neutrosophic statistics, and so on. In NS theory, every event has not only a certain degree of truth, but also a falsity degree and an indeterminacy degree that have to be considered independently from each other [20]. Thus, an event, or entity, {A} is considered with its opposite {Anti-A} and the neutrality {Neut-A}. NS provides a powerful tool to deal with the indeterminacy. In this paper, a new straightforward *k*-NN approach was developed which is based on NS theory. We adopted the NS memberships to improve the classification performance of the *k*-NN classifier. To do so, the neutrosophic c-means (NCM) algorithm was considered in a supervised manner, where labeled training data was used to obtain the centers of clusters. A final belonging membership degree U was calculated from the NS triples as $U = T + I - F$. A similar final voting scheme as given in fuzzy *k*-NN was employed for class label determination.

The paper is organized as follows: In the next section, we briefly reviewed the theories of *k*-NN and fuzzy *k*-NN. In Section 3, the proposed method was introduced and the algorithm of the proposed method was tabulated in Table 1. The experimental results and related comparisons were given in Section 4. The paper was concluded in Section 5.

2. Related works

2.1. k-Nearest Neighbor (k-NN) Classifier

As it was mentioned earlier, k-NN is the simplest, popular, supervised, and non-parametric classification method which was proposed in 1951 [1]. It is a distance based classifier which needs to measure the similarity of the test data to the data samples stored in the training set. Then, the test data is labelled by a majority vote of its k-nearest neighbors in the training set.

Let $X = \{x_1, x_2, \ldots, x_N\}$ denote the training set where $x_i \in R^n$ is a training data point in the n-dimensional feature space and let $Y = \{y_1, y_2, \ldots, y_N\}$ denotes the corresponding class labels. Given a test data point $\acute{x}$ whose class label is unknown, it can be determined as follows:

- Calculate the similarity measures between test sample and training samples by using a distance function (e.g., Euclidean distance)
- Find the test sample's k nearest neighbors in training data samples according to the similarity measure and determine the class label by the majority voting of its nearest neighbors.

2.2. Fuzzy k-Nearest Neighbor (k-NN) Classifier

In k-NN, a training data sample x is assumed to belong to one of the given classes so the membership U of that training sample to each class of C is given by an array of values in {0, 1}. If training data sample x belongs to class c_1 then $U_{c_1}(x) = 1$ and $U_{c_2}(x) = 0$ where $C = \{c_1, c_2\}$.

However, in fuzzy k-NN, instead of using crisp memberships, continuous range of memberships is used due to the nature of fuzzy theory [8]. So, the membership of training data sample can be calculated as:

$$U_{c_1}(x) = \begin{cases} 0.51 + 0.49\frac{k_{c_1}}{K} \; if \; c = c_1 \\ 0.49\frac{k_{c_1}}{K} \text{ otherwise} \end{cases} \tag{1}$$

where k_{c_1} shows the number of instances belonging to class c_1 found among the k neighbors of $\acute{x}$ and k is an integer value between [3,9].

After fuzzy membership calculation, a test sample's class label can be determined as following. Determine the k nearest neighbors of the test sample via Euclidean distance and produce a final vote for each class and neighbor using the Euclidean norm and the memberships:

$$V(k_j, c) = \frac{\frac{U_c(k_j)}{(\|\acute{x} - k_j\|)^{\frac{2}{m-1}}}}{\sum_{i=1}^{k} \frac{1}{(\|\acute{x} - k_i\|)^{\frac{2}{m-1}}}} \tag{2}$$

where k_j is the jth nearest neighbor and m = 2 is a parameter. The votes of each neighbor are then added to obtain the final classification.

3. Proposed Neutrosophic-*k*-NN Classifier

As traditional k-NN suffers from assigning equal weights to class labels in the training dataset, neutrosophic memberships are adopted in this work to overcome this limitation. Neutrosophic memberships reflect the data point's significance in its class and these memberships can be used as a new procedure for k-NN approach.

Neutrosophic set can determine a sample's memberships belonging to truth, false, and indeterminacy. An unsupervised neutrosophic clustering algorithm (NCM) is used in a supervised manner [22,23]. Crisp clustering methods assumed that every data points should belong to a cluster according to their nearness to the center of clusters. Fuzzy clustering methods assigned fuzzy memberships to each data point according to their nearness to the center of cluster. Neutrosophic clustering assigned memberships (T, I, and F) to each data point not only according to its nearness to a cluster center, but also according to the nearness to the center mean of the two clusters. Readers may refer to [22] for detailed information about the NCM clustering. As the labels of a training dataset samples are known in a supervised learning, the centers of the clusters can be calculated accordingly. Then, the related memberships of true (T), false (F), and indeterminacy (I) can be calculated as follows:

$$T_{ij} = \frac{(x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \bar{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}} \tag{3}$$

$$F_i = \frac{(\delta)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \bar{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}} \tag{4}$$

$$I_i = \frac{(x_i - \bar{c}_{imax})^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \bar{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}} \tag{5}$$

where m is a constant, δ is a regularization parameter and c_j shows the center of cluster j. For each point i, the $\bar{c}_{imax}$ is the mean of two cluster centers where the true membership values are greater than the others. T_{ij} shows the true membership value of point i for class j. F_i shows the falsity membership of point i and I_i determines the indeterminacy membership value for point i. Larger T_{ij} means that the point i is near a cluster and less likely to be a noise. Larger I_i means that the point i is between any two clusters and larger F_i indicates that point i is likely to be a noise. A final membership value for point i can be calculated by adding indeterminacy membership value to true membership value and subtracting the falsity membership value as shown in Equation (6).

After determining the neutrosophic membership triples, the membership for an unknown sample x_u to class label j, can be calculated as [9]:

$$\mu_{ju} = \frac{\sum_{i=1}^{k} d_i (T_{ij} + I_i - F_i)}{\sum_{i=1}^{k} d_i} \tag{6}$$

$$d_i = \frac{1}{\|x_u - x_i\|^{\frac{2}{q-1}}} \tag{7}$$

where d_i is the distance function to measure the distance between x_i and x_u, k shows the number of k-nearest neighbors and q is an integer. After the assignment of the neutrosophic membership grades of an unknown sample x_u to all class labels, the neutrosophic k-NN assigns x_u to the class whose neutrosophic membership is maximum. The following steps are used for construction of the proposed NS-k-NN method:

Step 1: Initialize the cluster centers according to the labelled dataset and employ Equations (3)–(5) to calculate the T, I, and F values for each data training data point.

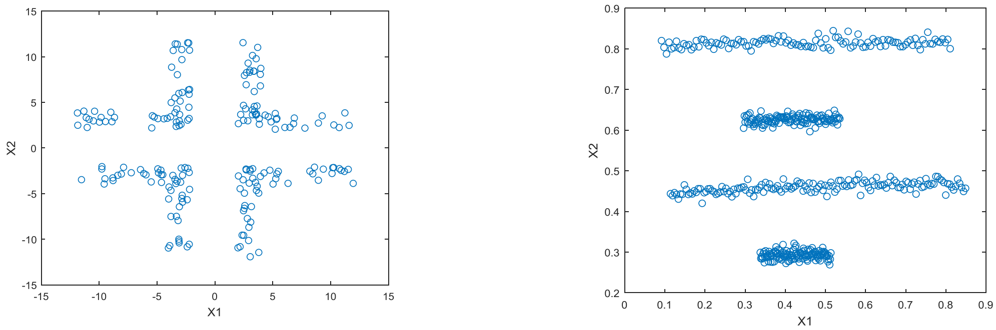
Step 2: Compute membership grades of test data samples according to the Equations (6) and (7).

Step 3: Assign class labels of the unknown test data points to the class whose neutrosophic membership is maximum.

4. Experimental Works

The efficiency of the proposed method was evaluated with several toy and real datasets. Two toy datasets were used to test the proposed method and investigate the effect of the parameters change on classification accuracy. On the other hand, several real datasets were used to compare the proposed method with traditional k-NN and fuzzy k-NN methods. We further compare the proposed method with several weighted k-NN methods such as weighted k-NN (WKNN) and distance-weighted k-nearest neighbor (DWKNN).

The toy dataset that were used in the experiments were shown in Figure 1a,b respectively. Both toy datasets contain two dimensional data with four classes. Randomly selected half of the toy datasets were used for training and the other half were used for testing. The k value was chosen to be 5, 10, and 15 and the δ parameter was chosen to be 0.01, 0.1, and 1, respectively. The obtained results were shown in Figure 2, respectively. As seen in the first row of Figure 2, the proposed method obtained 100% classification accuracy with $k = 10$ and $\delta = 0.01$ values for both toy datasets. However, 100% correct classification did not obtained for the other parameters as shown in the second and the third rows of Figure 2. This situation shows that the proposed method needs a parameter tuning mechanism in the k vs. δ space. So, k was set to an integer value between [2, 15] and δ parameter was also searched on $\{2^{-10}, 2^{-8}, \ldots, 2^{8}, 2^{10}\}$.



(**a**) (**b**)

Figure 1. Toy datasets. (**a**) Corner data; (**b**) line data.

We conducted further experiments on 39 real-world datasets which were downloaded from KEEL dataset repository [24]. Each dataset was already partitioned according to the cross validation procedure (five-folds or 10-folds). Table 1 shows several characteristics of the each dataset such as number of samples, number of features, and number of classes. All feature values were normalized to $[-1, 1]$ and a five-folds cross validation procedure was adopted in all experiments. The accuracies were calculated as the ratio of the number of correctly classified samples to the total number of samples.

Table 1. Data sets and their properties.

Data Sets	Instance (#)	Attribute (#)	Class (#)	Data Sets	Instance (#)	Attribute (#)	Class (#)
Appendicitis	106	7	2	Penbased	10,992	16	10
Balance	625	4	3	Phoneme	5404	5	2
Banana	5300	2	2	Pima	768	8	2
Bands	365	19	2	Ring	7400	20	2
Bupa	345	6	2	Satimage	6435	36	7
Cleveland	297	13	5	Segment	2310	19	7
Dermatology	358	34	6	Sonar	208	60	2
Ecoli	336	7	8	Spectfheart	267	44	2
Glass	214	9	7	Tae	151	5	3
Haberman	306	3	2	Texture	5500	40	11
Hayes-roth	160	4	3	Thyroid	7200	21	3
Heart	270	13	2	Twonorm	7400	20	2
Hepatitis	80	19	2	Vehicle	846	18	4
Ionosphere	351	33	2	Vowel	990	13	11
Iris	150	4	3	Wdbc	569	30	2
Mammographic	830	5	2	Wine	178	13	3
Monk-2	432	6	2	Winequality-red	1599	11	11
Movement	360	90	15	Winequality-white	4898	11	11
New thyroid	215	5	3	Yeast	1484	8	10
Page-blocks	5472	10	5	-	-	-	-

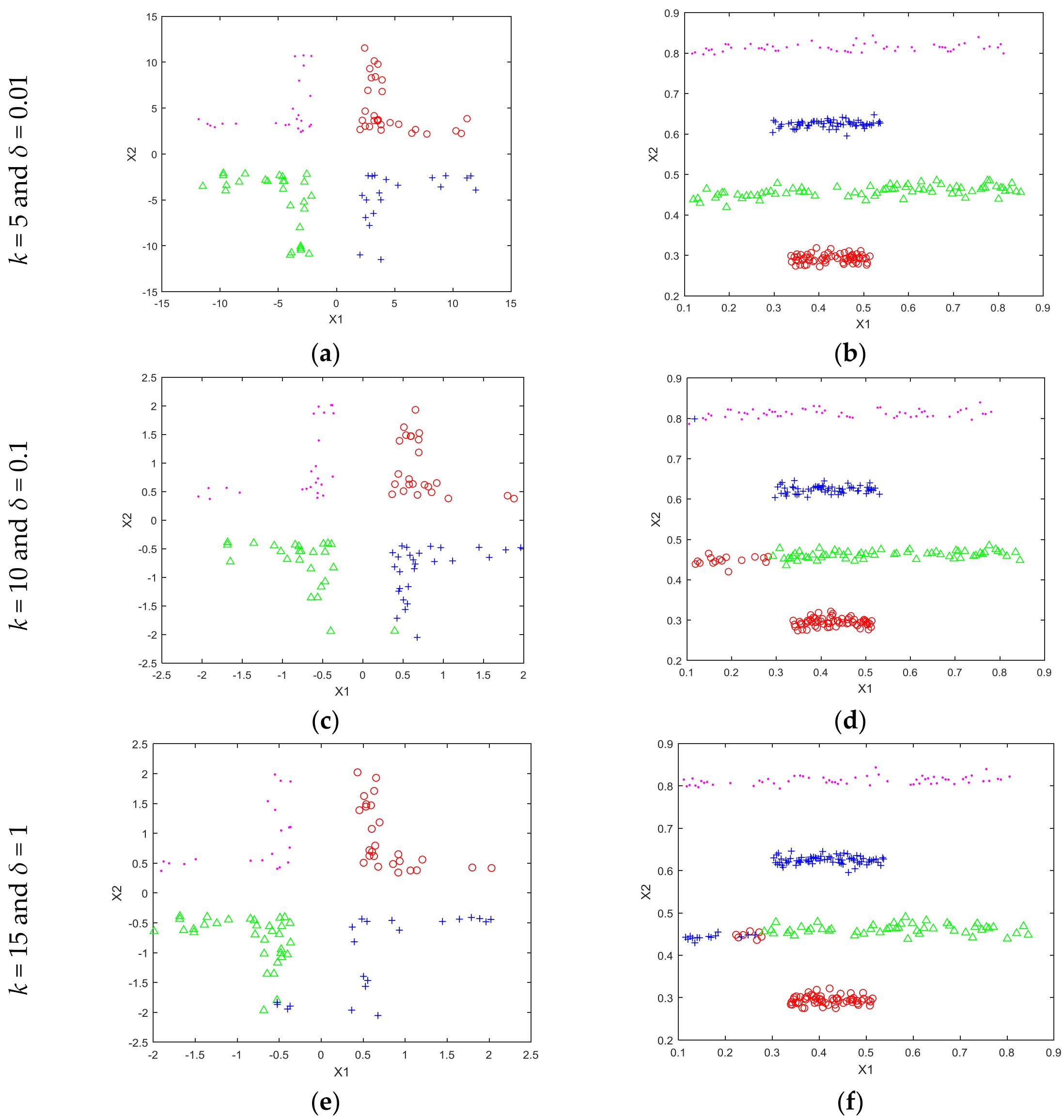


Figure 2. Some classification results for various k and δ parameters. (**a**) classification result 1 of corner data with various parameters; (**b**) classification result 1 of line data with various parameters (**c**) classification result 2 of corner data with various parameters (**d**) classification result 2 of line data with various parameters (**e**) classification result 3 of corner data with various parameters; (**f**) classification result 3 of line data with various parameters.

In addition to our results, we also compared our results with k-NN and fuzzy k-NN results on the same datasets. The obtained results were tabulated in Table 2 where the best results were indicated with bold-face. As seen in Table 2, the proposed method performed better than the other methods in 27 of 39 datasets. In addition, k-NN and fuzzy k-NN performed better on six and seven datasets out of 39 datasets, respectively. Our proposal obtained 100% accuracy for two datasets (new thyroid and wine). Moreover, for 13 datasets, the proposed method obtained accuracy values higher than 90%. On the other hand, the worse result was recorded for "Wine quality-white" dataset where the accuracy was 33.33%. Moreover, there were a total of three datasets where the accuracy was lower than 50%. We further conducted experiments on several datasets from UCI-data repository [25]. Totally, 11 datasets were considered in these experiments and compared results with two weighted k-NN approaches, namely WKNN and DWKNN. The characteristics of the each dataset from UCI-data repository were shown in Table 3, and the obtained all results were tabulated in Table 4. The boldface in Table 4 shows the higher accuracy values for each dataset.

Table 2. Experimental results of *k*-NN and fuzzy *k*-NN vs. the proposed method.

Data Sets	*k*-NN	Fuzzy *k*-NN	Proposed Method	Data Sets	*k*-NN	Fuzzy *k*-NN	Proposed Method
Appendicitis	87.91	**97.91**	90.00	Penbased	99.32	**99.34**	86.90
Balance	89.44	88.96	**93.55**	Phoneme	88.49	**89.64**	79.44
Banana	**89.89**	89.42	60.57	Pima	73.19	73.45	**81.58**
Bands	71.46	70.99	**75.00**	Ring	71.82	63.07	**72.03**
Bupa	62.53	66.06	**70.59**	Satimage	90.94	90.61	**92.53**
Cleveland	56.92	56.95	**72.41**	Segment	95.41	96.36	**97.40**
Dermatology	96.90	96.62	**97.14**	Sonar	83.10	83.55	**85.00**
Ecoli	82.45	83.34	**84.85**	Spectfheart	77.58	78.69	**80.77**
Glass	70.11	72.83	**76.19**	Tae	45.79	67.67	**86.67**
Haberman	71.55	68.97	**80.00**	Texture	**98.75**	**98.75**	80.73
Hayes-roth	30.00	65.63	**68.75**	Thyroid	**94.00**	93.92	74.86
Heart	80.74	80.74	**88.89**	Twonorm	97.11	97.14	**98.11**
Hepatitis	**89.19**	85.08	87.50	Vehicle	**72.34**	71.40	54.76
Ionosphere	96.00	96.00	**97.14**	Vowel	97.78	**98.38**	49.49
Iris	85.18	84.61	**93.33**	Wdbc	97.18	97.01	**98.21**
Mammographic	81.71	80.37	**86.75**	Wine	96.63	97.19	**100.00**
Monk-2	96.29	89.69	**97.67**	Winequality-red	55.60	**68.10**	46.84
Movement	**78.61**	36.11	50.00	Winequality-white	51.04	**68.27**	33.33
New thyroid	95.37	96.32	**100.00**	Yeast	57.62	59.98	**60.81**
Page-blocks	95.91	95.96	**96.34**	-	-	-	-

Table 3. Several datasets and their properties from UCI dataset.

Data set	Features	Samples	Classes	Training Samples	Testing Samples
Glass	10	214	7	140	74
Wine	13	178	3	100	78
Sonar	60	208	2	120	88
Parkinson	22	195	2	120	75
Iono	34	351	2	200	151
Musk	166	476	2	276	200
Vehicle	18	846	4	500	346
Image	19	2310	7	1310	1000
Cardio	21	2126	10	1126	1000
Landsat	36	6435	7	3435	3000
Letter	16	20,000	26	10,000	10,000

Table 4. The accuracy values for DWKNN vs. NSKNN.

Data set	WKNN (%)	DWKNN (%)	Proposed Method (%)
Glass	69.86	**70.14**	60.81
Wine	71.47	71.99	**79.49**
Sonar	81.59	82.05	**85.23**
Parkinson	83.53	83.93	**90.67**
Iono	84.27	84.44	**85.14**
Musk	84.77	85.10	**86.50**
Vehicle	63.96	64.34	**71.43**
Image	95.19	95.21	**95.60**
Cardio	70.12	**70.30**	66.90
Landsat	90.63	90.65	**91.67**
Letter	94.89	**94.93**	63.50

As seen in Table 4, the proposed method performed better than the other methods in eight of 11 datasets and DWKNN performed better in the rest datasets. For three datasets (Parkinson, Image and Landsat), the proposed method yielded accuracy value higher than 90% and the worse result was found for the 'Glass' dataset where the accuracy was 60.81%. DWKNN and the WKNN produced almost same accuracy values and performed significantly better than the proposed method on 'Letter and Glass' datasets. We further compared the running times of each method on each KEEL dataset and the obtained running times were tabulated in Table 5. We used MATLAB 2014b (The MathWorks Inc., Natick, MA, USA) on a computer having an Intel Core i7-4810 CPU and 32 GB memory. As seen in Table 5, for some datasets, the *k*-NN and fuzzy *k*-NN methods achieved lower running times than our proposal's achievement. However, when the average running times took into consideration, the proposed method achieved the lowest running time with 0.69 s. The *k*-NN method also obtained the second lowest running time with 1.41 s. The fuzzy *k*-NN approach obtained the average slowest running time when compared with the other methods. The fuzzy *k*-NN method's achievement was 3.17 s.

Table 5. Comparison of running times for each method.

Data Sets	*k*-NN	Fuzzy *k*-NN	Proposed Method	Data Sets	*k*-NN	Fuzzy *k*-NN	Proposed Method
Appendicitis	0.11	0.16	0.15	Penbased	10.21	18.20	3.58
Balance	0.15	0.19	0.18	Phoneme	0.95	1.88	0.71
Banana	1.03	1.42	0.57	Pima	0.45	0.58	0.20
Bands	0.42	0.47	0.19	Ring	6.18	10.30	2.55
Bupa	0.14	0.28	0.16	Satimage	8.29	15.25	1.96
Cleveland	0.14	0.18	0.19	Segment	1.09	1.76	0.63
Dermatology	0.33	0.31	0.22	Sonar	0.15	0.21	0.23
Ecoli	0.12	0.26	0.17	Spectfheart	0.14	0.25	0.22
Glass	0.10	0.18	0.18	Tae	0.13	0.12	0.16
Haberman	0.13	0.24	0.16	Texture	6.72	12.78	4.30
Hayes-roth	0.07	0.11	0.16	Thyroid	5.86	9.71	2.14
Heart	0.22	0.33	0.17	Twonorm	5.89	10.27	2.69
Hepatitis	0.06	0.06	0.16	Vehicle	0.17	0.31	0.27
Ionosphere	0.13	030	0.25	Vowel	0.47	0.62	0.31
Iris	0.23	0.13	0.16	Wdbc	0.39	0.46	0.26
Mammographic	0.21	0.22	0.20	Wine	0.08	0.14	0.17
Monk-2	0.27	0.33	0.17	Winequality-red	0.28	0.46	0.34
Movement	0.16	0.34	0.35	Winequality-white	1.38	1.95	0.91
New thyroid	0.14	0.18	0.17	Yeast	0.44	0.78	0.30
Page-blocks	1.75	2.20	0.93	Average	1.41	3.17	0.69

Generally speaking, the proposed NS-*k*-NN method can be announced successful when the accuracy values which were tabulated in Tables 3–5, were considered. The NS-*k*-NN method obtained these high accuracies because it incorporated the NS theory with the distance learning for constructing an efficient supervised classifier. The running time evaluation was also proved that the NS-*k*-NN was quite an efficient classifier than the compared other related classifiers.

5. Conclusions

In this paper, we propose a novel supervised classification method based on NS theory called neutrosophic *k*-NN. The proposed method assigns the memberships to training samples based on the supervised NCM clustering algorithm, and classifies the samples based on their neutrosophic memberships. This approach can be seen as an extension of the previously-proposed fuzzy *k*-NN method by incorporating the falsity and indeterminacy sets. The efficiency of the proposed method was demonstrated with extensive experimental results. The results were also compared with other improved *k*-NN methods. According to the obtained results, the proposed method can be used in various classification applications. In the future works, we plan to apply the proposed NS-*k*-NN on imbalanced dataset problems. We would like to analyze the experimental results with some non-parametric statistical methods, such as the Freidman test and Wilcoxon signed-ranks test. In addition, some other evaluation metrics such as AUC will be used for comparison purposes. We will also explore the *k*-NN method where Dezert-Smarandache theory will be used to calculate the data samples' memberships, replacing Dempster's rule by Proportional Conflict Redistribution Rule #5 (PCR5), which is more performative in order to handle the assignments of the final class.

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Vector Similarity Measures between Refined Simplified Neutrosophic Sets and Their Multiple Attribute Decision-Making Method

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Abstract: A refined single-valued/interval neutrosophic set is very suitable for the expression and application of decision-making problems with both attributes and sub-attributes since it is described by its refined truth, indeterminacy, and falsity degrees. However, existing refined single-valued/interval neutrosophic similarity measures and their decision-making methods are scarcely studied in existing literature and cannot deal with this decision-making problem with the weights of both attributes and sub-attributes in a refined interval and/or single-valued neutrosophic setting. To solve the issue, this paper firstly introduces a refined simplified neutrosophic set (RSNS), which contains the refined single-valued neutrosophic set (RSVNS) and refined interval neutrosophic set (RINS), and then proposes vector similarity measures of RSNSs based on the Jaccard, Dice, and cosine measures of simplified neutrosophic sets in vector space, and the weighted Jaccard, Dice, and cosine measures of RSNSs by considering weights of both basic elements and sub-elements in RSNS. Further, a decision-making method with the weights of both attributes and sub-attributes is developed based on the weighted Jaccard, Dice, and cosine measures of RSNSs under RSNS (RINS and/or RSVNS) environments. The ranking order of all the alternatives and the best one can be determined by one of weighted vector similarity measures between each alternative and the ideal solution (ideal alternative). Finally, an actual example on the selecting problem of construction projects illustrates the application and effectiveness of the proposed method.

Keywords: refined simplified neutrosophic set; refined single-valued neutrosophic set; refined interval neutrosophic set; vector similarity measure; decision-making

1. Introduction

Since fuzzy set theory was introduced by Zadeh [1] in 1965, it has been successfully applied to decision-making areas, and fuzzy decision-making has become a research focal point since then. With the increasing complexity of decision-making problems in actual applications, the fuzzy set is not suitable for fuzzy expression, which involves the membership degree and non-membership degree. Hence, an intuitionistic fuzzy set (IFS) [2] and an interval-valued IFS [3] were introduced as the generalization of fuzzy set and applied to decision-making problems. However, the incomplete, indeterminate, and inconsistent problems in real life cannot be explained by means of the IFS and interval-valued IFS. Therefore, Smarandache [4] proposed the concept of a neutrosophic set from a philosophical point of view, which consists of the truth, indeterminacy, and falsity memership functions, denoted by T, I, F, to represent incomplete, indeterminate, and inconsistent information in the real world. Since the truth, indeterminacy, and falsity membership degrees of T, I, F

in the neutrosophic set lie in the real standard/nonstandard interval $]^{-}0, 1^{+}[$, Smarandache [4], Wang et al. [5,6], and Ye [7,8] constrained the three membership degrees in the neutrosophic set to the single-valued membership degrees and the interval membership degrees. These become a single-valued neutrosophic set (SVNS), an interval neutrosophic set (INS), and a simplified neutrosophic set (SNS) (including SVNS and INS), respectively. Obviously, they are subclasses of the neutrosophic set for convenient applications in science and engineering fields, such as decision-making [7–13] and fault diagnosis [14]. However, because there are both arguments and sub-arguments/refined arguments in the truth, indeterminacy, and falsity membership degrees of T, I, F in the neutrosophic set to express complex problems of the real world in detail, one needs to refine truth, indeterminacy, and falsity information. Hence, Smarandache [15] further extended the neutrosophic logic to n-valued refined neutrosophic logic, where he refined/split the truth, indeterminacy, and falsity functions T, I, F into T_1, T_2, ..., T_r, I_1, I_2, ..., I_s, and F_1, F_2, ..., F_t, respectively, and constructed them as a n-valued refined neutrosophic set. Moreover, some researchers extended the neutrosophic set to multi-valued neutrosophic set/neutrosophic multiset/neutrosophic refined sets and applied them to medical diagnoses [16–18] and decision-making [19–21]. In fact, the multi-valued neutrosophic sets/neutrosophic refined sets are neutrosophic multisets in their expressed forms [22,23]. Hence, these multi-valued neutrosophic sets/neutrosophic refined sets, that is, neutrosophic multisets, and their decision-making methods cannot express and deal with decision-making problems with both attributes and sub-attributes. To solve the issue, Ye and Smarandache [22] proposed a refined single-valued neutrosophic set (RSVNS), where the neutrosophic set {T, I, F} was refined into the RSVNS {(T_1, T_2, ..., $\mathrm{T_r}$), (I_1, I_2, ..., I_r), (F_1, F_2, ..., F_r)}, and proposed the similarity measures based on union and intersection operations of RSVNSs to solve decision-making problems with both attributes and sub-attributes. Then, Fan and Ye [23] further presented the cosine measures of RSVNSs and refined interval neutrosophic sets (RINSs) based the distance and cosine function and applied them to the decision-making problems with both attributes and sub-attributes under refined single-value/interval neutrosophic environments. However, these cosine measures cannot handle such a decision-making problem with the weights of both attributes and sub-attributes.

In fact, RINSs and/or RSVNSs are scarcely studied and applied in science and engineering fields. Therefore, it is necessary to develop new similarity measures and their decision-making method in refined interval and/or single-value neutrosophic environments. However, in existing literature [22,23], the similarity measures of RSVNSs and RINSs and their decision-making methods only took into account the basic element (single-valued/interval neutrosophic number in RSVNS/RINS)/attribute weights rather than sub-element/sub-attribute weights (weights of refined elements/refined attributes) in the measures of RSVNSs and RINSs and their decision-making methods. To overcome these drawbacks, this paper firstly introduces a refined simplified neutrosophic set (RSNS), which includes the concepts of RSVNS and RINS, and proposes the vector similarity measures of RSNSs based on the Jaccard, Dice, and cosine measures between SNSs in vector space [8]. Further, a decision-making method is established based on the Jaccard/Dice/cosine measures between RSNSs to solve multiple attribute decision-making problems with both attribute weights and sub-attribute weights under refined simplified (interval and/or single-value) neutrosophic environments. The main advantages of the proposed approach are that it can solve decision-making problems with the weights of both attributes and sub-attributes and extend existing similarity measures and decision-making methods in [22,23], because the existing similarity measures and decision-making methods cannot deal with such a decision-making problem with the weights of both attributes and sub-attributes under RSNS (RINS and/or RSVNS) environments.

The rest of the paper is structured as follows. Section 2 reviews basic concepts of SNSs and vector similarity measures of SNSs. In Section 3, we introduces a RSNS concept, including RSVNS and RINS. Section 4 proposes the Jaccard, Dice, and cosine similarity measures (three vector similarity measures) between RSNSs by considering weights of elements and sub-elements/refined elements in RSNSs. Section 5 develops a multiple attribute decision-making method with both attribute weights and sub-attribute weights based on one of three vector similarity measures under refined simplified (interval and/or single-value) neutrosophic environments. In Section 6, an actual example on the selection problem of construction projects is provided as the multiple attribute decision-making problem with both attribute weights and sub-attribute weights to illustrate the application and effectiveness of the proposed method. Finally, conclusions and future research are contained in Section 7.

2. Basic Concepts of SNSs and Vector Similarity Measures of SNSs

In 1995, Smarandache [4] proposed a concept of neutrosophic sets from a philosophical point of view, which is a part of neutrosophy and extends the concepts of fuzzy sets, interval valued fuzzy sets, IFSs, and interval valued IFSs. A neutrosophic set is characterized independently by the truth, indeterminacy and falsity membership functions, which lie in a real standard interval [0, 1] or a nonstandard interval $]^-0, 1^+[$. For convenient science and engineering applications, we need to constrain them in the real standard interval [0, 1] from a science and engineering point of view. Thus, Ye [7,8] introduced the concept of SNS as a simplified form/subclass of the neutrosophic set.

A SNS A in a universe of discourse X is characterized by its truth, indeterminacy, and falsity membership functions $T_A(x)$, $I_A(x)$, and $F_A(x)$, which is denoted as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton subintervals/subsets in the real standard [0, 1], such that $T_A(x)$: $X \to [0, 1]$, $I_S(x)$: $X \to [0, 1]$, and $F_S(x)$: $X \to [0, 1]$. Then, the SNS A contains SVNS for $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ and INS for $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0, 1]$.

For convenient expression, a basic element $\langle x, T_A(x), I_A(x), F_A(x)\rangle$ in A is simply denoted as a simplified neutrosophic number (SNN) $a = <T_a, I_a, F_a>$, where a contains a single-value neutrosophic number (SVNN) for T_a, I_a, $F_a \in [0, 1]$ and an interval neutrosophic number (INN) for T_a, I_a, $F_a \subseteq [0, 1]$.

Assume that two SNSs are $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $a_j = <T_{aj}, I_{aj}, F_{aj}>$ and $b_j = <T_{bj}, I_{bj}, F_{bj}>$ for $j = 1, 2, \ldots, n$ are two collections of SNNs. Based on the Jaccard, Dice, and cosine measures between two vectors, Ye [8] presented the their similarity measures between SNSs (SVNSs and INSs) A and B in vector space, respectively, as follows:

(1) Three vector similarity measures between A and B for SVNSs:

$$M_J(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}}{\left[\left(T_{aj}^2 + I_{aj}^2 + F_{aj}^2\right) + \left(T_{bj}^2 + I_{bj}^2 + F_{bj}^2\right) - \left(T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}\right)\right]} \quad (1)$$

$$M_D(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{aj}\right)}{\left(T_{aj}^2 + I_{aj}^2 + F_{aj}^2\right) + \left(T_{bj}^2 + I_{bj}^2 + F_{bj}^2\right)} \quad (2)$$

$$M_C(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}}{\sqrt{T_{aj}^2 + I_{aj}^2 + F_{aj}^2}\sqrt{T_{bj}^2 + I_{bj}^2 + F_{bj}^2}} \quad (3)$$

(2) Three vector similarity measures between A and B for INSs:

$$M_J(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\inf T_{aj}\inf T_{bj} + \sup T_{aj}\sup T_{bj} + \inf I_{aj}\inf I_{bj} + \sup I_{aj}\sup I_{bj} + \inf F_{aj}\inf F_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{l}(\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2 \\ +(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2 \\ -(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj}) \\ -(\sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj})\end{array}\right)} \quad (4)$$

$$M_D(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj} + \sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{l}(\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2 \\ +(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2\end{array}\right)} \quad (5)$$

$$M_C(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj} + \sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{c}\sqrt{(\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2} \\ \sqrt{(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2}\end{array}\right)} \quad (6)$$

Clearly, Equations (1)–(3) are special cases of Equations (4)–(6) when the upper and lower limits of the interval numbers for T_{aj} = [inf T_{aj}, sup T_{aj}] I_{aj} = [inf I_{aj}, sup I_{aj}], F_{aj} = [inf F_{aj}, sup F_{aj}], T_{bj} = [inf T_{bj}, sup T_{bj}], I_{bj} = [inf I_{bj}, sup I_{bj}], and F_{bj} = [inf F_{bj}, sup F_{bj}] are equal.

Then, the Jaccard, Dice, and cosine measures $M_k(A, B)$ (k = J, D, C) contains the following properties [8]:

(P1) $0 \le M_k(A, B) \le 1$;
(P2) $M_k(A, B) = M_k(B, A)$;
(P3) $M_k(A, B) = 1$ if $A = B$, i.e., $T_{aj}, = T_{bj}$, $I_{aj} = I_{bj}$, and $F_{aj} = F_{bj}$ for $j = 1, 2, \ldots, n$.

3. Refined Simplified Neutrosophic Sets

As the concept of SNS [7,8], a SNS A in a universe of discourse X is denoted as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where the values of its truth, indeterminacy, and falsity membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ for $x \in X$ are single-value and/or interval values in [0, 1]. Then, SNS contain INS and/or SVNS.

If the components $T_A(x)$, $I_A(x)$, $F_A(x)$ in SNS are refined (split) into $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$, respectively, for $x \in X$, $x = \{x_1, x_2, ..., x_r\}$, and a positive integer r, then they can be constructed as RSNS by the refinement of SNS, which is defined below.

Definition 1. *Let X be a universe of discourse, then a RSNS A in X can be defined as*

$$A = \{\langle x, (T_A(x_1), T_A(x_2), \ldots, T_A(x_r)), (I_A(x_1), I_A(x_2), \ldots, I_A(x_r)), (F_A(x_1), F_A(x_2), \ldots, F_A(x_r))\rangle | x \in X, x_j \in x\},$$

where $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$, $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$ *for* $x \in X$, $x_j \in x = \{x_1, x_2, ..., x_r\}$ ($j = 1, 2, \ldots, r$), *and a positive integer r are subintervals/subsets in the real standard interval [0, 1], such that* $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$: $X \to [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$: $X \to [0, 1]$, *and* $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$: $X \to [0, 1]$.

Then, the RSNS A contains the following two concepts:

(1) If $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r) \in [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r) \in [0, 1]$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r) \in [0, 1]$ in A for $x \in X$ and $x_j \in x$ (j =1, 2, ... , r) are considered as single/exact values in [0, 1], then A reduces to RSVNS [22], which satisfies the condition $0 \le T_A(x_j) + I_A(x_j) + F_A(x_j) \le 3$ for $j = 1, 2, \ldots, r$;

(2) If $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r) \subseteq [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r) \subseteq [0, 1]$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r) \subseteq [0, 1]$ in A for $x \in X$ and $x_j \in x$ (j =1, 2, ... , r) are considered as interval values in [0, 1], then A reduces to RINS [23], which satisfies the condition $0 \le \sup T_A(x_j) + \sup I_A(x_j) + \sup F_A(x_j) \le 3$ for $j = 1, 2, \ldots, r$.

Particularly when the lower and upper limits of $T_A(x_j)$ = [inf $T_A(x_j)$, sup $T_A(x_j)$], $I_A(x_j)$ = [inf $I_A(x_j)$, sup $I_A(x_j)$] and $F_A(x_j)$ = [inf $F_A(x_j)$, sup $F_A(x_j)$] in A for $x \in X$ and $x_j \in x$ ($j = 1, 2, \ldots, r$) are equal, the RINS A reduces to the RSVNS A. Clearly, RSVNS is a special case of RINS. If some lower and upper limits of $T_A(x_j)$ = [inf $T_A(x_j)$, sup $T_A(x_j)$]/$I_A(x_j)$ = [inf $I_A(x_j)$, sup $I_A(x_j)$]/$F_A(x_j)$ = [inf $F_A(x_j)$, sup $F_A(x_j)$] in RINS are equal, then it can be denoted as a special interval (equal interval of the lower and upper limits) $T_A(x_j)$ = [$T_A(x_j)$, $T_A(x_j)$]/$I_A(x_j)$ = [$I_A(x_j)$, $I_A(x_j)$]/$F_A(x_j)$ = [$F_A(x_j)$, $F_A(x_j)$]. Hence, RINS can contain RINS and/or SVNS information (hybrid information of both).

For convenient expression, a basic element <x, $(T_A(x_1), T_A(x_2), ..., T_A(x_r))$, $(I_A(x_1), I_A(x_2), ..., I_A(x_r))$, $(F_A(x_1), F_A(x_2), ..., F_A(x_r))$> in A is simply denoted as a = <$(T_{a1}, T_{a2}, \ldots, T_{ar})$, $(I_{a1}, I_{a2}, \ldots, I_{ar})$, $(F_{a1}, F_{a2}, \ldots, F_{ar})$>, which is called a refined simplified neutrosophic number (RSNN).

Let two RSNNs be a = <$(T_{a1}, T_{a2}, \ldots, T_{ar})$, $(I_{a1}, I_{a2}, \ldots, I_{ar})$, $(F_{a1}, F_{a2}, \ldots, F_{ar})$> and b = <$(T_{b1}, T_{b2}, \ldots, T_{br})$, $(I_{b1}, I_{b2}, \ldots, I_{br})$, $(F_{b1}, F_{b2}, \ldots, F_{br})$> for $T_{aj}, T_{bj}, I_{aj}, I_{bj}, F_{aj}, F_{bj} \in [0, 1]$ ($j = 1, 2, \ldots, r$). Then, there are the following relations between a and b:

(1) Containment: $a \subseteq b$, if and only if $T_{aj} \leq T_{bj}$, $I_{aj} \geq I_{bj}$, $F_{aj} \geq F_{bj}$ for $j = 1, 2, \ldots, r$;
(2) Equality: $a = b$, if and only if $a \subseteq b$ *and* $b \subseteq a$, i.e., $T_{aj} = T_{bj}$, $I_{aj} = I_{bj}$, $F_{aj} = F_{bj}$ for $j = 1, 2, \ldots, r$;
(3) Union:

$$a \cup b = \langle (T_{a1} \vee T_{b1}, T_{a2} \vee T_{b2}, \ldots, T_{ar} \vee T_{br}), (I_{a1} \wedge I_{b1}, I_{a2} \wedge I_{b2}, \ldots, I_{ar} \wedge I_{br}), (F_{a1} \wedge F_{b1}, F_{a2} \wedge F_{b2}, \ldots, F_{ar} \wedge F_{br}) \rangle;$$

(4) Intersection:

$$a \cap b = \langle (T_{a1} \wedge T_{b1}, T_{a2} \wedge T_{b2}, \ldots, T_{ar} \wedge T_{br}), (I_{a1} \vee I_{b1}, I_{a2} \vee I_{b2}, \ldots, I_{ar} \vee I_{br}), (F_{a1} \vee F_{b1}, F_{a2} \vee F_{b2}, \ldots, F_{ar} \vee F_{br}) \rangle.$$

Let two RSNNs be $a = \langle (T_{a1}, T_{a2}, \ldots, T_{ar}), (I_{a1}, I_{a2}, \ldots, I_{ar}), (F_{a1}, F_{a2}, \ldots, F_{ar}) \rangle$ and $b = \langle (T_{b1}, T_{b2}, \ldots, T_{br}), (I_{b1}, I_{b2}, \ldots, I_{br}), (F_{b1}, F_{b2}, \ldots, F_{br}) \rangle$ for $T_{aj}, T_{bj}, I_{aj}, I_{bj}, F_{aj}, F_{bj} \subseteq [0, 1]$ ($j = 1, 2, \ldots, r$). Then, there are the following relations of a and b:

(1) Containment: $a \subseteq b$, if and only if inf $T_{aj} \leq$ inf T_{bj}, sup $T_{aj} \leq$ sup T_{bj}, inf $I_{aj} \geq$ inf I_{bj}, sup $I_{aj} \geq$ sup I_{bj}, inf $F_{aj} \geq$ inf F_{bj}, and sup $F_{aj} \geq$ sup F_{bj} for $j = 1, 2, \ldots, r$;
(2) Equality: $a = b$, if and only if $a \subseteq b$ and $b \subseteq a$, i.e., inf T_{aj} = inf T_{bj}, sup T_{aj} = sup T_{bj}, inf I_{aj} = inf I_{bj}, sup I_{aj} = sup I_{bj}, inf F_{aj} = inf F_{bj}, and sup F_{aj} = sup F_{bj} for $j = 1, 2, \ldots, r$;
(3) Union:

$$a \cup b = \left\langle \begin{array}{l} ([\inf T_{a1} \vee \inf T_{b1}, \sup T_{a1} \vee \sup T_{b1}], [\inf T_{a2} \vee \inf T_{b2}, \sup T_{a2} \vee \sup T_{b2}], \ldots, [\inf T_{ar} \vee \inf T_{br}, \sup T_{ar} \vee \sup T_{br}]), \\ ([\inf I_{a1} \wedge \inf I_{b1}, \sup I_{a1} \wedge \sup I_{b1}], [\inf I_{a2} \wedge \inf I_{b2}, \sup I_{a2} \wedge \sup I_{b2}], \ldots, [\inf I_{ar} \wedge \inf I_{br}, \sup I_{ar} \wedge \sup I_{br}]), \\ ([\inf F_{a1} \wedge \inf F_{b1}, \sup F_{a1} \wedge \sup F_{b1}], [\inf F_{a2} \wedge \inf F_{b2}, \sup F_{a2} \wedge \sup F_{b2}], \ldots, [\inf F_{ar} \wedge \inf F_{br}, \sup F_{ar} \wedge \sup F_{br}]) \end{array} \right\rangle$$

(4) Intersection:

$$a \cap b = \left\langle \begin{array}{l} ([\inf T_{a1} \wedge \inf T_{b1}, \sup T_{a1} \wedge \sup T_{b1}], [\inf T_{a2} \wedge \inf T_{b2}, \sup T_{a2} \wedge \sup T_{b2}], \ldots, [\inf T_{ar} \wedge \inf T_{br}, \sup T_{ar} \wedge \sup T_{br}]), \\ ([\inf I_{a1} \vee \inf I_{b1}, \sup I_{a1} \vee \sup I_{b1}], [\inf I_{a2} \vee \inf I_{b2}, \sup I_{a2} \vee \sup I_{b2}], \ldots, [\inf I_{ar} \vee \inf I_{br}, \sup I_{ar} \vee \sup I_{br}]), \\ ([\inf F_{a1} \vee \inf F_{b1}, \sup F_{a1} \vee \sup F_{b1}], [\inf F_{a2} \vee \inf F_{b2}, \sup F_{a2} \vee \sup F_{b2}], \ldots, [\inf F_{ar} \vee \inf F_{br}, \sup F_{ar} \vee \sup F_{br}]) \end{array} \right\rangle$$

4. Vector Similarity Measures of RSNSs

Based on the Jaccard, Dice, and cosine measures between SNSs in vector space [8], this section proposes the three vector similarity measures between RSNSs.

Definition 2. *Let two RSNSs be A* = {a_1, a_2, ..., a_n} *and B* = {b_1, b_2, ..., b_n}, *where* $a_j = \left\langle (T_{a_j1}, T_{a_j2}, \ldots, T_{a_jr_j}), (I_{a_j1}, I_{a_j2}, \ldots, I_{a_jr_j}), (F_{a_j1}, F_{a_j2}, \ldots, F_{a_jr_j}) \right\rangle$ *and* $b_j = \left\langle (T_{b_j1}, T_{b_j2}, \ldots, T_{b_jr_j}), (I_{b_j1}, I_{b_j2}, \ldots, I_{b_jr_j}), (F_{b_j1}, F_{b_j2}, \ldots, F_{b_jr_j}) \right\rangle$ *for j = 1, 2, ..., n are two collections of RSNNs for* $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \in [0, 1]$ *or* $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \subseteq [0, 1]$ *(j = 1, 2, ..., n; k = 1, 2, ...,* r_j*). Then, the Jaccard, Dice, and cosine measures between A and B are defined, respectively, as follows:*

(1) *Three vector similarity measures between A and B for RSVNSs:*

$$R_J(A, B) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{T_{a_jk}T_{b_jk} + I_{a_jk}I_{b_jk} + F_{a_jk}F_{b_jk}}{\left[\left(T_{a_jk}^2 + I_{a_jk}^2 + F_{a_jk}^2\right) + \left(T_{b_jk}^2 + I_{b_jk}^2 + F_{b_jk}^2\right) - \left(T_{a_jk}T_{b_jk} + I_{a_jk}I_{b_jk} + F_{a_jk}F_{b_jk}\right)\right]} \tag{7}$$

$$R_D(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{2\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)}{\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)} \quad (8)$$

$$R_C(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\sqrt{T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2}\sqrt{T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2}} \quad (9)$$

(2) *Three vector similarity measures between A and B for RINSs:*

$$R_J(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\sup T_{a_jk}\sup T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}\\+\sup I_{a_jk}\sup I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\\-(\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk})\\-(\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk})\end{pmatrix}}, \quad (10)$$

$$R_D(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{2\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\end{pmatrix}}, \quad (11)$$

$$R_C(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}\sqrt{(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2}\\\sqrt{(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2}\end{pmatrix}} \quad (12)$$

Clearly, Equations (7)–(9) are special cases of Equations (10)–(12) when the upper and lower limits of the interval numbers for $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \subseteq [0, 1]$ (j = 1, 2, … , n; k = *1, 2,* … , r_j) are equal. Especially when k = 1, Equations (7)–(12) are reduced to Equations (1)–(6).

Based on the properties of the Jaccard, Dice, and cosine measures of SNSs [8], it is obvious that the Jaccard, Dice, and cosine measures of RSNSs for $R_s(A, B)$ (s = J, D, C) also contain the following properties (P1)–(P3):

(P1) $0 \le R_s(A, B) \le 1$;
(P2) $R_s(A, B) = R_s(B, A)$;
(P3) $R_s(A, B) = 1$ if $A = B$, i.e., $T_{a_jk} = T_{b_jk}$, $I_{a_jk} = I_{b_jk}$, $F_{a_jk} = F_{b_jk}$ for j = 1, 2, … , n and k = 1, 2, ..., r_j.

When we consider the weights of different elements and sub-elements in RSNS, the weight of elements a_j and b_j (j = 1, 2, … , n) in the RSNSs A and B is given as $w_j \in [0, 1]$ with $\sum_{j=1}^{n} w_j = 1$ and the weight of the refined components (sub-elements) $T_{a_jk}, I_{a_jk}, F_{a_jk}$ and $T_{b_jk}, I_{b_jk}, F_{b_jk}$ (k = 1, 2, … , r_j) in $a_j = \left\langle (T_{a_j1}, T_{a_j2}, \ldots, T_{a_jr_j}), (I_{a_j1}, I_{a_j2}, \ldots, I_{a_jr_j}), (F_{a_j1}, F_{a_j2}, \ldots, F_{a_jr_j}) \right\rangle$ and $b_j = \left\langle (T_{b_j1}, T_{b_j2}, \ldots, T_{b_jr_j}), (I_{b_j1}, I_{b_j2}, \ldots, I_{b_jr_j}), (F_{b_j1}, F_{b_j2}, \ldots, F_{b_jr_j}) \right\rangle$ (j = 1, 2, … , n) is considered as $\omega_k \in [0, 1]$ with $\sum_{k=1}^{r_j} \omega_k = 1$, the weighted Jaccard, Dice, and cosine measures between A and B are presented, respectively, as follows:

(1) Three weighted vector similarity measures between A and B for RSVNSs:

$$R_{WJ}(A,B)=\sum_{j=1}^{n} w_j \sum_{k=1}^{r_j}\omega_k\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\left[\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)-\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)\right]} \quad (13)$$

$$R_{WD}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{2\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)}{\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)} \quad (14)$$

$$R_{WC}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\sqrt{T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2}\sqrt{T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2}} \quad (15)$$

(2) Three weighted vector similarity measures between A and B for RINSs:

$$R_{WJ}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\sup T_{a_jk}\sup T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}\\+\sup I_{a_jk}\sup I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\\-(\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk})\\-(\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk})\end{pmatrix}}, \quad (16)$$

$$R_{WD}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{2\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\end{pmatrix}}, \quad (17)$$

$$R_{WC}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}\sqrt{(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2}\\\sqrt{(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2}\end{pmatrix}}. \quad (18)$$

Clearly, Equations (13)–(15) are special cases of Equations (16)–(18) when the upper and lower limits of the interval numbers for $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \subseteq [0, 1]$ (j = 1, 2, ... , n; k = 1, 2, ... , r_j) are equal. Especially when each $w_j = 1/n$ and $\omega_k = 1/r_j$ (j = 1, 2, ... , n; k = 1, 2, ... , r_j), Equations (13)–(18) are reduced to Equations (7)–(12).

Obviously, the weighted Jaccard, Dice, and cosine measures of RSNSs for $R_{Ws}(A, B)$ (s = J, D, C) also satisfies the following properties (P1)–(P3):

(P1) $0 \le R_{Ws}(A, B) \le 1$;
(P2) $R_{Ws}(A, B) = R_{Ws}(B, A)$;
(P3) $R_{Ws}(A, B) = 1$ if $A = B$, i.e., $T_{a_jk} = T_{b_jk}, I_{a_jk} = I_{b_jk}, F_{a_jk} = F_{b_jk}$ for j = 1, 2, ... , n and k = 1, 2, ..., r_j.

5. Decision-Making Method Using the Vector Similarity Measures

In a decision-making problem with multiple attributes and sub-attributes, assume that $A = \{A_1, A_2, \ldots, A_m\}$ is a set of m alternatives, which needs to satisfies a set of n attributes $B = \{b_1, b_2, \ldots, b_n\}$, where b_j (j = 1, 2, ... , n) may be refined/split into a set of r_j sub-attributes $b_j = \left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ (j = 1, 2, ... , n). If the decision-maker provides the suitability evaluation values of attributes $b_j = \left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ (j =1, 2, ..., n) on the alternative A_i (i = 1, 2, ... , m) by using RSNS: $A_i = \left\{\left\langle b_j, (T_{A_i}(b_{j1}), T_{A_i}(b_{j2}), \ldots, T_{A_i}(b_{jr_j})), (I_{A_i}(b_{j1}), I_{A_i}(b_{j2}), \ldots, I_{A_i}(b_{jr_j})), (F_{A_i}(b_{j1}), F_{A_i}(b_{j2}), \ldots, F_{A_i}(b_{jr_j}))\right\rangle \middle| b_j \in B, b_{jk} \in b_j\right\}$.

For convenient expression, each basic element in the RSNS A_i is represented by RSNN: $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j})\right\rangle$ for i = 1, 2, ... , m and j = 1, 2, ... , n.

Hence, we can construct the refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$, as shown in Table 1.

Table 1. Refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$.

	b_1	b_2	...	b_n
	$\{b_{11}, b_{12}, \ldots, b_{1r_1}\}$	$\{b_{21}, b_{22}, \ldots, b_{2r_2}\}$		$\{b_{n1}, b_{n2}, \ldots, b_{nr_n}\}$
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

When the weights of each attribute b_j ($j = 1, 2, \ldots, n$) and its sub-attributes are considered as having different importance, the weight vector of the attributes is given by $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and the weight vector for each sub-attribute set $\left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ is given as $\boldsymbol{\omega}_j = \left\{\omega_{j1}, \omega_{j2}, \ldots, \omega_{jr_j}\right\}$ ($j = 1, 2, \ldots, n$) with $\omega_{jk} \in [0, 1]$ and $\sum_{k=1}^{r_j} \omega_{jk} = 1$. Thus, the decision steps are described as follows:

Step 1: We determine the ideal solution (ideal RSNN) from the refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$ as follows:

$$a_j^* = \left\langle \begin{array}{c} (\max_i(T_{j1}), \max_i(T_{j2}), \ldots, \max_i(T_{jr_j})), (\min_i(I_{j1}), \min_i(I_{j2}), \\ \ldots, \min_i(I_{jr_j})), (\min_i(F_{j1}), \min_i(F_{j2}), \ldots, \min_i(F_{jr_j})) \end{array} \right\rangle \text{ for RSVNN} \quad (19)$$

$$\text{or } a_j^* = \left\langle \begin{array}{c} ([\max_i(\inf T_{j1}), \max_i(\sup T_{j1})], [\max_i(\inf T_{j2}), \max_i(\sup T_{j2})], \ldots, [\max_i(\inf T_{jr_j}), \max_i(\sup T_{jr_j})]), \\ ([\min_i(\inf I_{j1}), \min_i(\sup I_{j1})], [\min_i(\inf I_{j2}), \min_i(\sup I_{j2})], \ldots, [\min_i(\inf I_{jr_j}), \min_i(\sup I_{jr_j})]), \\ ([\min_i(\inf F_{j1}), \min_i(\sup F_{j1})], [\min_i(\inf F_{j2}), \min_i(\sup F_{j2})], \ldots, [\min_i(\inf F_{jr_j}), \min_i(\sup F_{jr_j})]) \end{array} \right\rangle \text{ for RINN,} \quad (20)$$

which is constructed as the ideal alternative $A^* = \left\{a_1^*, a_2^*, \ldots, a_n^*\right\}$.

Step 2: The similarity measure between each alternative A_i ($i = 1, 2, \ldots, m$) and the ideal alternative A^* can be calculated by using one of Equations (13)–(15) or Equations (16)–(18), and obtained as the values of $R_{Ws}(A_i, A^*)$ for $i = 1, 2, \ldots, m$ and s = J or D or C.

Step 3: According to the values of $R_{Ws}(A_i, A^*)$ for $i = 1, 2, \ldots, m$ and s = J or D or C, the alternatives are ranked in a descending order. The greater value of $R_{Ws}(A_i, A^*)$ means the best alternative.

Step 4: End.

6. Illustrative Example on the Selection of Construction Projects

In this section, we apply the proposed decision-making method to the selection of construction projects adapted from [23].

Some construction company wants to select one of potential construction projects. The company provides four potential construction projects as their set $A = \{A_1, A_2, A_3, A_4\}$. To select the best one of them, experts or decision-makers need to make a decision of these construction projects corresponding to three attributes and their seven sub-attributes, which are described as follows:

(1) Financial state (b_1) contains two sub-attributes: budget control (b_{11}) and risk/return ratio (b_{12});
(2) Environmental protection (b_2) contains three sub-attributes: public relation (b_{21}), geographical location (b_{22}), and health and safety (b_{23});
(3) Technology (b_3) contains tow sub-attributes: technical know-how (b_{31}), technological capability (b_{32}).

Then, the weight vector of the three attributes is given by $\boldsymbol{W}$ = (0.4, 0.3, 0.3) and the weight vectors of the three sub-attribute sets $\{b_{11}, b_{12}\}$, $\{b_{21}, b_{22}, b_{23}\}$, and $\{b_{31}, b_{32}\}$ are given, respectively, by $\boldsymbol{\omega}_1$ = (0.6, 0.4), $\boldsymbol{\omega}_2$ = (0.25, 0.4, 0.35), and $\boldsymbol{\omega}_3$ = (0.45, 0.55).

In the following, we use the proposed decision-making method for solving the decision-making problem of construction projects under RSVNN and/or RINN environments to show the applications and effectiveness of the proposed decision-making method.

Under RSVNN environment, experts or decision-makers are required to evaluate the four possible alternatives under the above three attributes including seven sub-attributes by suitability judgments, which are described by RSVNN $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j}) \right\rangle$ for $T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j} \in [0, 1]$, $I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j} \in [0, 1]$, and $F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j} \in [0, 1]$ (i = 1, 2, 3, 4; j = 1, 2, 3; r_1 = 2, r_2 = 3, r_3 = 2). Thus, we can construct the following refined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$, which is shown in Table 2.

Table 2. Defined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$ under refined single-valued neutrosophic set (RSVNS) environment.

	b_1	b_2	b_3
	$\{b_{11}, b_{12}\}$	$\{b_{21}, b_{22}, b_{23}\}$	$\{b_{31}, b_{32}\}$
A_1	<(0.6, 0.7), (0.2, 0.1), (0.2, 0.3)>	<(0.9, 0.7, 0.8), (0.1, 0.3, 0.2), (0.2, 0.2, 0.1)>	<(0.6, 0.8), (0.3, 0.2), (0.3, 0.4)>
A_2	<(0.8, 0.7), (0.1, 0.2), (0.3, 0.2)>	<(0.7, 0.8, 0.7), (0.2, 0.4, 0.3), (0.1, 0.2, 0.1)>	<(0.8, 0.8), (0.1, 0.2), (0.1, 0.2)>
A_3	<(0.6, 0.8), (0.1, 0.3), (0.3, 0.4)>	<(0.8, 0.6, 0.7), (0.3, 0.1, 0.1), (0.2, 0.1, 0.2)>	<(0.8, 0.7), (0.4, 0.3), (0.2, 0.1)>
A_4	<(0.7, 0.6), (0.1, 0.2), (0.2, 0.3)>	<(0.7, 0.8, 0.7), (0.2, 0.2, 0.1), (0.1, 0.2, 0.2)>	<(0.7, 0.7), (0.2, 0.3), (0.2, 0.3)>

Under RSVNS environment, the proposed decision-making method is applied to the selection problem of the construction projects. The decision steps are described as follows:

Step 1: By Equation (19), the ideal solution (ideal RSVNS) can be determined as the following ideal alternative: A^* = {<(0.8, 0.8), (0.1, 0.1), (0.2, 0.2)>, <(0.9, 0.8, 0.8), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1)>, <(0.8, 0.8), (0.1, 0.2), (0.1, 0.1)>}.

Step 2: According to one of Equations (13)–(15), the weighted similarity measure values between each alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* can be obtained and all the results are shown in Table 3.

Step 3: In Table 3, since all the measure values are $R_{Ws}(A_2, A^*) > R_{Ws}(A_4, A^*) > R_{Ws}(A_3, A^*) > R_{Ws}(A_1, A^*)$ for s = J, D, C, all the ranking orders of the four alternatives are $A_2 \succ A_4 \succ A_3 \succ A_1$. Hence, the alternative A_2 is the best choice among all the construction projects.

Table 3. All the measure values between A_i (i = 1, 2, 3, 4) and A^* for RSVNSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$W(A_i, A^*)$ [23]	$W(A_1, A^*) = 0.9848$, $W(A_2, A^*) = 0.9938$, $W(A_3, A^*) = 0.9858$, $W(A_4, A^*) = 0.9879$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WJ}(A_i, A^*)$	$R_{WJ}(A_1, A^*) = 0.9187$, $R_{WJ}(A_2, A^*) = 0.9610$, $R_{WJ}(A_3, A^*) = 0.9249$, $R_{WJ}(A_4, A^*) = 0.9320$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WD}(A_i, A^*)$	$R_{WD}(A_1, A^*) = 0.9568$, $R_{WD}(A_2, A^*) = 0.9797$, $R_{WD}(A_3, A^*) = 0.9607$, $R_{WD}(A_4, A^*) = 0.9646$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WC}(A_i, A^{*'})$	$R_{WC}(A_1, A^*) = 0.9646$, $R_{WC}(A_2, A^*) = 0.9832$, $R_{WC}(A_3, A^*) = 0.9731$, $R_{WC}(A_4, A^*) = 0.9780$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

Under RINS environment, on the other hand, experts or decision-makers are required to evaluate the four possible alternatives under the above three attributes including seven sub-attributes by suitability judgments, which are described by RINN $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j}) \right\rangle$ for $T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j} \subseteq [0, 1]$, $I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j} \subseteq [0, 1]$, and $F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j} \subseteq [0, 1]$ (i = 1, 2, 3, 4; j = 1, 2, 3; r_1 = 2, r_2 = 3,

r_3 = 2). Thus, we can construct the following refined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$, which is shown in Table 4.

Table 4. Defined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$ under refined interval neutrosophic set (RINS) environment.

	b_1	b_2	b_3
	$\{b_{11}, b_{12}\}$	$\{b_{21}, b_{22}, b_{23}\}$	$\{b_{31}, b_{32}\}$
A_1	<([0.6,0.7],[0.7,0.8]),([0.2,0.3],[0.1,0.2]),([0.2,0.3],[0.3,0.4])>	<([0.8,0.9],[0.7,0.8],[0.8,0.9]),([0.1,0.2],[0.3,0.4],[0.2,0.3]),([0.2,0.3],[0.2,0.3],[0.1,0.2])>	<([0.6,0.7],[0.8,0.9]),([0.3,0.4],[0.2,0.3]),([0.3,0.4],[0.4,0.5])>
A_2	<([0.8,0.9],[0.7,0.8]),([0.1,0.2],[0.2,0.3]),([0.3,0.4],[0.2,0.3])>	<([0.7,0.8],[0.8,0.9],[0.7,0.8]),([0.2,0.3],[0.4,0.5],[0.3,0.4]),([0.1,0.2],[0.2,0.3],[0.1,0.2])>	<([0.8,0.9],[0.8,0.9]),([0.1,0.2],[0.2,0.3]),([0.1,0.2],[0.2,0.3])>
A_3	<([0.6,0.7],[0.8,0.9]),([0.1,0.2],[0.3,0.4]),([0.3,0.4],[0.4,0.5])>	<([0.8,0.9],[0.6,0.7],[0.7,0.8]),([0.3,0.4],[0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.1,0.2],[0.2,0.3])>	<([0.8,0.9],[0.7,0.8]),([0.4,0.5],[0.3,0.4]),([0.2,0.3],[0.1,0.2])>
A_4	<([0.7,0.8],[0.6,0.7]),([0.1,0.2],[0.2,0.3]),([0.2,0.3],[0.3,0.4])>	<([0.7,0.8],[0.8,0.9],[0.7,0.8]),([0.2,0.3],[0.2,0.3],[0.1,0.2]),([0.1,0.2],[0.2,0.3],[0.2,0.3])>	<([0.7,0.8],[0.7,0.8]),([0.2,0.3],[0.3,0.4]),([0.2,0.3],[0.3,0.4])>

Under RINS environment, the proposed decision-making method is applied to the selection problem of the construction projects. The decision steps are described as follows:

Step 1: By Equation (20), the ideal solution (ideal RINS) can be determined as the following ideal alternative:

A^* = {<([0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.1, 0.2]), ([0.2, 0.3], [0.2, 0.3])>, <([0.8, 0.9], [0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.1, 0.2], [0.1, 0.2]), ([0.1, 0.2], [0.1, 0.2], [0.1, 0.2])>, <([0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.2, 0.3]), ([0.1, 0.2], [0.1, 0.2])>}.

Step 2: By using one of Equations (16)–(18), the weighted similarity measure values between each alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* can be calculated, and then all the results are shown in Table 5.

Step 3: In Table 5, since all the measure values are $R_{Ws}(A_2, A^*) > R_{Ws}(A_4, A^*) > R_{Ws}(A_3, A^*) > R_{Ws}(A_1, A^*)$ for s = J, D, C, all the ranking orders of the four alternatives are $A_2 \succ A_4 \succ A_3 \succ A_1$. Hence, the alternative A_2 is the best choice among all the construction projects.

Table 5. All the measure values between A_i (i = 1, 2, 3, 4) and A^* for RINSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$W(A_i, A^*)$ [23]	$W(A_1, A^*) = 0.9848$, $W(A_2, A^*) = 0.9932$, $W(A_3, A^*) = 0.9868$, $W(A_4, A^*) = 0.9886$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WJ}(A_i, A^*)$	$R_{WJ}(A_1, A^*) = 0.9314$, $R_{WJ}(A_2, A^*) = 0.9693$, $R_{WJ}(A_3, A^*) = 0.9369$, $R_{WJ}(A_4, A^*) = 0.9430$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WD}(A_i, A^*)$	$R_{WD}(A_1, A^*) = 0.9639$, $R_{WD}(A_2, A^*) = 0.9841$, $R_{WD}(A_3, A^*) = 0.9672$, $R_{WD}(A_4, A^*) = 0.9705$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WC}(A_i, A^*)$	$R_{WC}(A_1, A^*) = 0.9697$, $R_{WC}(A_2, A^*) = 0.9860$, $R_{WC}(A_3, A^*) = 0.9775$, $R_{WC}(A_4, A^*) = 0.9805$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

For convenient comparison with existing related method [23], the decision results based on the cosine function without considering sub-attribute weights in the literature [23] are also indicated in Tables 3 and 5. Obviously, all the ranking orders are identical, which indicate the feasibility and effectiveness of the developed decision-making method based on the proposed measures R_{Ws} for

s = J, D, C. However, the existing related decision-making methods with RSVNSs and RINSs [22,23] cannot deal with such a decision-making problem with both attribute weights and sub-attribute weights in this paper. Although the same computational complexity in decision-making algorithms is shown by comparison of the method of this study with the related methods introduced in [22,23], the developed method in this study extends the methods in [22,23] and is more feasible and more general than the existing related decision-making methods [22,23]. It is obvious that the new developed decision-making method in a RSNS (RINS and/or SVNS) setting is superior to the existing related methods in a RINS or SVNS setting [22,23].

Compared with traditional decision-making approaches without sub-attributes [7–13,19–21], the decision-making approach proposed in this study can deal with decision-making problems with both attributes and sub-attributes; while traditional decision-making approaches [7–13,19–21] cannot deal with such a decision-making problem with both attributes and sub-attributes. Hence, the proposed decision-making approach is superior to traditional ones [7–13,19–21].

However, the study in this paper provides new three vector measures and their decision-making method as the main contributions due to no study of existing literature on the vector similarity measures and decision-making methods with RSNSs (RSVNSs and/or RINSs). Clearly, the main advantages of this study are that it can solve decision-making problems with the weights of both attributes and sub-attributes, which all existing methods cannot deal with, and extend existing similarity measures and decision-making methods.

To analyze the sensitivities of the proposed approach, let us change the RINS of the alternative A_4 into the RSNS $A_{4'}$ = {<([0.7,0.7], [0.6,0.6]), ([0.2,0.2], [0.2,0.2]), ([0.3,0.3], [0.3,0.3])>, <([0.7,0.8], [0.8,0.9], [0.7,0.8]), ([0.2,0.3], [0.2,0.3], [0.1,0.2]), ([0.1,0.2], [0.2,0.3], [0.2,0.3])>, <([0.7,0.8], [0.7,0.8]), ([0.2,0.3], [0.3,0.4]), ([0.2,0.3], [0.3,0.4])> with hybrid information of both RSVNNs and RINNs. Then, by above similar computation steps, we can obtain all the measure values, which are shown in Table 6.

Table 6. All the measure values between A_i (i = 1, 2, 3, 4′) and A^* for RSNSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$R_{WJ}(A_i, A_*)$	$R_{WJ}(A_1, A_*)$ = 0.9314, $R_{WJ}(A_2, A_*)$ = 0.9693, $R_{WJ}(A_3, A_*)$ = 0.9369, $R_{WJ}(A_4, A_*)$ = 0.9356	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
$R_{WD}(A_i, A_*)$	$R_{WD}(A_1, A_*)$ = 0.9639, $R_{WD}(A_2, A_*)$ = 0.9841, $R_{WD}(A_3, A_*)$ = 0.9672, $R_{WD}(A_4, A_*)$ = 0.9665	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
$R_{WC}(A_i, A_*)$	$R_{WC}(A_1, A_*)$ = 0.9697, $R_{WC}(A_2, A_*)$ = 0.9860, $R_{WC}(A_3, A_*)$ = 0.9775, $R_{WC}(A_4, A_*)$ = 0.9780	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

The results of Table 6 demonstrate the ranking orders based on $R_{WJ}(A_i, A^*)$ and $R_{WD}(A_i, A^*)$ are the same, but their decision-making method can change the previous ranking orders and show some difference between two alternatives A_3 and A_4; while the best one is still A_2. Clearly, the decision-making approach based on the Jaccard and Dice measures shows some sensitivity in this case. However, the ranking order based on $R_{WC}(A_i, A^*)$ still keeps the previous ranking order, and then the decision-making approach based on the cosine measure shows some robustness/insensitivity in this case.

In actual decision-making problems, decision-makers can select one of three vector measures of RSNSs to apply it to multiple attribute decision-making problems with weights of attributes and sub-attributes according to their preference and actual requirements.

7. Conclusions

This paper introduced RSNSs, including the concepts of RSVNSs and RINSs, and proposed the vector similarity measures of RSNSs, including the Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) in vector space. It then presented the weighted Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) by considering the weights of basic elements in

RSNSs and the weights of sub-elements (the refined weights) in each RSNN. Further, we established a decision-making method based on the weighted Jaccard/Dice/cosine measures of RSNSs (RSVNSs and RINSs) to deal with multiple attribute decision-making problems with both attribute weights and sub-attribute weights under RSNS (RINS and/or RSVNS) environments. In the decision-making process, through the Jaccard/Dice/cosine measures between each alternative and the ideal alternative, the ranking order of all alternatives and the best one can be determined based on the measure values. Finally, an actual example on the decision-making problem of construction projects with RSNS (RSVNS and/or RINS) information is provided to demonstrate the application and effectiveness of the proposed method. The proposed approach is very suitable for actual applications in decision-making problems with weights of both attributes and sub-attributes under RSNS (RINS and/or RSVNS) environments, and provides a new decision-making method. In the future, we shall further extend the proposed method to group decision-making, clustering analysis, medical diagnosis, fault diagnosis, and so forth.

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Author Contributions: Jun Ye proposed the vector similarity measures of RSNSs, including the weighted Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) in vector space; Jiqian Chen and Jun Ye established a decision-making method based on the weighted Jaccard/Dice/cosine measures of RSNSs (RSVNSs and RINSs) to deal with multiple attribute decision-making problems with both attribute weights and sub-attribute weights under RSNS (RINS and/or RSVNS) environments; Jiqian Chen and Shigui Du gave an actual example on the decision-making problem of construction projects with RSNS (RSVNS and/or RINS) information and its calculation and comparative analysis; we wrote the paper together.

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Evaluating Investment Risks of Metallic Mines Using an Extended TOPSIS Method with Linguistic Neutrosophic Numbers

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Abstract: The investment in and development of mineral resources play an important role in the national economy. A good mining project investment can improve economic efficiency and increase social wealth. Faced with the complexity and uncertainty of a mine's circumstances, there is great significance in evaluating investment risk scientifically. In order to solve practical engineering problems, this paper presents an extended TOPSIS method combined with linguistic neutrosophic numbers (LNNs). Firstly, considering that there are several qualitative risk factors of mining investment projects, the paper describes evaluation information by means of LNNs. The advantage of LNNs is that major original information is reserved with linguistic truth, indeterminacy, and false membership degrees. After that, a number of distance measures are defined. Furthermore, a common status is that the decision makers can't determine the importance degrees of every risk factor directly for a few reasons. With respect to this situation, the paper offers a weight model based on maximizing deviation to obtain the criteria weight vector objectively. Subsequently, a decision-making approach through improving classical TOPSIS with LNNs comes into being. Next, a case study of the proposed method applied in metallic mining projects investment is given. Some comparison analysis is also submitted. At last, the discussions and conclusions are finished.

Keywords: metallic mine project; investment risks evaluation; linguistic neutrosophic numbers; maximum deviation; extended TOPSIS

1. Introduction

The assessment of investment risk has always attracted the attention of many researchers in different fields [1]. For example, Wu et al. [2] proposed an improved Analytical Hierarchy Process (AHP) approach to select an optimal financial investment strategy. An extended TOPSIS method was provided by Hatami-Marbini and Kangi [3], and applied in the Tehran stock exchange. Yazdani-Chamzini et al. [4] constructed a model on the basis of AHP, decision-making trial and evaluation, and TOPSIS to evaluate investment risk in the private sector of Iran. A VIKOR-DANP method was presented by Shen et al. [5] and a case study of Taiwan's semiconductor industry was also given to demonstrate the effectiveness of the approach. Dincer and Hacioglu [6] discussed the relationships of financial stress and conflict risk in emerging capital markets with a fuzzy AHP-TOPSIS and VIKOR method. In high-tech fields, such as nanotechnology, Hashemkhani, Zolfani, and Bahrami [7] provided a SWARA-COPRAS decision-making method. Unlike other general industries, investment in the mining industry usually has a long cycle and large uncertainty [8]. There are a lot of risk factors in the process of mining investment. Consequently, identifying and assessing the investment venture of a mine accurately and properly is vital for any project.

The widely-used risk evaluation methods of mining investment can be divided into two main categories [9]. Traditional methods include fault tree analysis, Monte Carlo Simulation, breakeven analysis, the decision tree method, and so on. Another kind contains Analytic Hierarchy process (AHP), fuzzy comprehensive evaluation, and so on. Many researchers have paid particular attention to the latter method, which is based on fuzzy mathematics. Chen et al. [10] sorted and summarized the risk elements of metallic mines, and then presented a method based on a fuzzy set and a neural network. Wang et al. [11] constructed the fuzzy comprehensive appraisal model through creating a risk estimation indicator system. San et al. [12] focused on Tongxin mine, and established an investment risk assessment model with a fuzzy analytic hierarchy process. These methods take the ambiguity of the assessment process into consideration.

However, the fuzzy numbers, such as interval numbers [13], triangular fuzzy numbers [14,15], and trapezoidal fuzzy numbers [16], used in most approaches have some limitations. On the one hand, they only described limited consistent information, while the hesitant and inconsistent values are not indicated. Furthermore, qualitative information is also not expressed. Smarandache [17] firstly put forward the concept of neutrosophic sets (NSs) to deal with consistent, hesitant, and inconsistent information simultaneously. After that, many extensions based on NSs have been presented [18–20]. Related decision-making methods include TOPSIS [21], VIKOR [22], TODIM [23], COPRAS [24,25], WASPAS [26], MULTIMOORA [27], ELECTRE [28,29], QUALIFLEX [30], and other approaches [31,32]. Among them, TOPSIS is widely used. The basic idea of this method is that the distance of the optimal alternative with the positive ideal solution is nearest, and the negative-position ideal solution is farthest [21]. It is easy to understand and operate for decision makers.

In order to qualitatively evaluate risk, like social environment risk and management risk in a mining project, linguistic variables may be a good description [33,34]. Much literature has focused on risk assessment with linguistic information. A venture analysis method on the basis of Dempster–Shafer theory under linguistic environment was presented in the literature [35]. Liu et al. [36] established a risk linguistic decision matrix and discussed the situation when weight informationis unknown. An unbalanced linguistic weighted geometric average operator was proposed to deal with fuzzy risk evaluation problems in [37]. Peiris et al. [38] built three linguistic models to assess alien plants' invasion risks.

For the sake of keeping as much linguistic evaluation information as possible, multiple extensions about language were suggested. For example, the notion of 2-dimensional uncertain linguistic variables occurred some researchers [39–41]. The idea of single-valued neutrosophic linguistic numbers occurred to Ye [42]. Other extensive forms are intuitionistic linguistic sets [43], hesitant fuzzy linguistic term sets [44,45], probabilistic linguistic term sets [46,47], and so on [48,49]. It is worth noting that Chen et al. [50] proposed a group decision-making method in the light of linguistic intuitionistic fuzzy numbers (LIFNs). They connected linguistic values with intuitionistic fuzzy numbers [51]. Then, the linguistic intuitionistic Frank Heronian average operator [52] and some improved linguistic intuitionistic fuzzy operators [53] were proposed.

However, there are only linguistic membership degrees and linguistic non-membership degrees reflected in LIFNs. To overcome this shortcoming, Fang and Ye [54] came up with the concept of linguistic neutrosophic numbers (LNNs). They are based on linguistic terms and simplified neutrosophic numbers [55]. The truth-membership, indeterminacy-membership, and false-membership in a linguistic neutrosophic number (LNN) are found using linguistic information. The difference of LNNs with neutrosophic linguistic numbers (NLNs) [56] is that there is only a linguistic value in NLNs, and the truth-membership, indeterminacy-membership, and false-membership are crisp numbers. For instance, (s_1, s_2, s_3) is a LNN, while $(s_1, < 0.1, 0.2, 0.3 >)$ is a neutrosophic linguistic number (NLN). Of course, they are independent of each other as well. In addition, Fang and Ye [54] defined the operations and comparison rules of LNNs, and then decision-making methods based on of several weighted mean operators were raised.

From this we can see, considering the complicacy of mine environment and the ambiguity of the human mind, assessing the ventures of mining projects on the basis of LNNs may be feasible and advisable. As a result, this paper considers metallic mine investment risk under a linguistic neutrosophic situation with incomplete weight information. A new and reasonable way to evaluate risk degrees by means of LNNs is proposed. In summary, the fundamental innovations of this article are conveyed as follows:

(1) Present a number of distance measures between two LNNs, such as the Hamming distance, the Euclidean distance, and the Hausdorff distance. Equally important, prove relevant properties of these formulas;
(2) Use the thought of maximum deviation for our reference, build a model with respect to linguistic neutrosophic environment to obtain the values of mine risk evaluation criteria weight;
(3) Come up with the extended TOPSIS model with LNNs. Importantly, utilize this method to cope with investment decision-making matter of metallic mine projects;
(4) Compare with other methods, in order to demonstrate the significance and superiority.

We methodize the rest of this article as follows. In Section 2, basic background and knowledge related to risk factors, linguistic information, and LNNs are presented. The extended TOPSIS method with LNNs is depicted after defining the distance measures of LNNs and constructing the weight model in Section 3. Section 4 studies a case of metallic mining investment, and the proposed approach is applied in it. In Section 5, we make comparison with several current literatures. And then, conclusions are made in the last section.

2. Background

In this section, some preliminaries about mining investment risk factors, linguistic term sets, linguistic scale functions, and LNNs are presented.

2.1. Risk Factors of Mining Project Investment

The economic factors of mines and the risk influence factors of metallic mines are introduced in this subsection.

According to the situation of mining investment in China and the research results of the World Bank's investment preference, Pan [57] divided the economic venture of mining investment into five types. They are financial risk, production risk, market risk, personnel risk, and environmental risk, respectively. More details can be seen in Table 1.

Table 1. The economic risk factors of mines.

Risk Factors	Explanations
Financial risk	Caused by the unexpected changes in the mine's balance of payments. It largely consists of financial balance, exchange rate, interest rate, and other factors.
Production risk	Caused by accident, which makes it impossible to produce the production plan according to the predetermined cost. Mainly including production cost, technical conditions, selection scheme, and so on.
Market risk	Caused by the unexpected changes in the market, which makes the mine unable to sell its products according to the original plan. It chiefly contains demand forecasting, substitution products, peer competition, and other factors.
Personnel risk	Caused by accident or change of the important personnel in the mine, which causes a significant impact on the production and operation of the mine. The main factors include accidental casualties, confidential leaks, and personnel changes.
Environmental risk	Caused by the changes of the external environment of the mining industry, which primarily comprises the national policies, geological conditions, and pollution control.

In 2011, Chen et al. [10] summarized and classified the influence factors of the metallic mining investment process based on strategic angle of investment implementation. The presented risk system (see Table 2.) includes two levels of indicators, which are five primary indicators and sixty secondary indicators. The secondary index corresponds to the attributes of the primary one.

Table 2. Investment risk evaluation system of metallic mining.

Assessment Indicators	
Primary indicators	Secondary indicators
Production risk	Mining type, production equipment level, and mining technology
Geological risk	Geological grade, mine reserves, hydrogeology, and surrounding rock conditions
Social environment	Marco economy, national industrial policy, and international environment
Market risk	Marketing ability, product market price, and potential competition
Management risk	Rationality of enterprise organization, scientific decision, and management personnel

2.2. Linguistic Term Sets and Linguistic Scale Function

Xu [58] first put forward the concept of linguistic term sets. For a certain linguistic term set, there are a group of linguistic values s_i $(i = 0, 1, \ldots, 2g)$. Consequently, the linguistic term set can be denoted as $\overline{S} = \{s_i | i = 0, 1, \ldots, 2g\}$.

While the linguistic values in the above-mentioned linguistic term set are discrete, they may not work on aggregated linguistic information. Accordingly, Xu [58] redefined the linguistic term set with $S = \{s_i | i \in [0, 2u]\}$ $(u > g)$, where the elements are continuous. Moreover, we can compare arbitrary linguistic terms in accordance with their subscripts. Namely, when $i > j$, $s_i > s_j$ is established. The operational rules of any two linguistic values $s_i, s_j \in S$ are indicated: (1) the addition operator $s_i \oplus s_j = s_{i+j}$; (2) the scalar multiplication $\tau s_i = s_{\tau i}$, $0 \le \tau \le 1$; (3) the negation operator $ne(s_i) = s_{-i}$.

Definition 1. *[59] The linguistic scale function is regarded as a mapping from linguistic values s_i $(i = 0, 1, \cdots, 2g)$ to a corresponding crisp number $cn_i \in [0, 1]$. Furthermore, it should meet the requirement of monotonically increasing, that is to say, $0 \le cn_0 < cn_1 < \cdots < cn_{2g} \le 1$.*

As the continuous linguistic term sets are defined, we use $f(s_i) = cn_i = \frac{i}{2u}$ $(i \in [0, 2u])$ as the linguistic scale function in this essay. The inverse function can be described as $f^{-1}(cn_i) = 2u \cdot cn_i$ $(i \in [0, 2u])$.

2.3. Linguistic Neutrosophic Numbers

Definition 2. *[54] Given the linguistic term set $S = \{s_i | i \in [0, 2u]\}$, if $s_T, s_I, s_F \in S$, then $\eta = (s_T, s_I, s_F)$ can be regarded as a LNN, where s_T, s_I, and s_F are independent, and describe the linguistic truth-membership degree, the linguistic indeterminacy-membership degree, and the linguistic falsity-membership degree in turn.*

Definition 3. *[54] Assume $\eta_1 = (s_{T_1}, s_{I_1}, s_{F_1})$ and $\eta_2 = (s_{T_2}, s_{I_2}, s_{F_2})$ are two LNNs, then the operations of them are represented as follows:*

(1) $\eta_1 \oplus \eta_2 = (s_{T_1}, s_{I_1}, s_{F_1}) \oplus (s_{T_2}, s_{I_2}, s_{F_2}) = (s_{T_1 + T_2 - \frac{T_1 T_2}{2u}}, s_{\frac{I_1 I_2}{2u}}, s_{\frac{F_1 F_2}{2u}})$;

(2) $\eta_1 \otimes \eta_2 = (s_{T_1}, s_{I_1}, s_{F_1}) \oplus (s_{T_2}, s_{I_2}, s_{F_2}) = (s_{\frac{T_1 T_2}{2u}}, s_{I_1 + I_2 - \frac{I_1 I_2}{2u}}, s_{F_1 + F_2 - \frac{F_1 F_2}{2u}})$;

(3) $q\eta_1 = q(s_{T_1}, s_{I_1}, s_{F_1}) = (s_{2u - 2u(1 - \frac{T_1}{2u})^q}, s_{2u(\frac{I_1}{2u})^q}, s_{2u(\frac{F_1}{2u})^q})$, $q > 0$;

(4) $\eta_1{}^q = (s_{T_1}, s_{I_1}, s_{F_1})^q = (s_{2u(\frac{T_1}{2u})^q}, s_{2u - 2u(1 - \frac{I_1}{2u})^q}, s_{2u - 2u(1 - \frac{F_1}{2u})^q})$, $q > 0$.

Definition 4. [54] *Suppose $\eta = (s_T, s_I, s_F)$ is an optional LLN, the following are the score function and the accuracy function, respectively:*

$$SC(\eta) = (4u + T - I - F)/(6u), \tag{1}$$

$$AC(\eta) = (T - F)/(2u). \tag{2}$$

Definition 5. [54] *If $\eta_1 = (s_{T_1}, s_{I_1}, s_{F_1})$ and $\eta_2 = (s_{T_2}, s_{I_2}, s_{F_2})$ are two LNNs, then the comparison rule is:*

(1) $\eta_1 > \eta_2$ if $SC(\eta_1) > SC(\eta_2)$;
(2) $\eta_1 > \eta_2$ if $SC(\eta_1) = SC(\eta_2)$ and $AC(\eta_1) > AC(\eta_2)$;
(3) $\eta_1 = \eta_2$ if $SC(\eta_1) = SC(\eta_2)$ and $AC(\eta_1) = AC(\eta_2)$.

Definition 6. [54] *Assume there are a group of LNNs $\eta_i = (s_{T_i}, s_{I_i}, s_{F_i})$ $(i = 1, 2, \ldots, n)$, the linguistic neutrosophic weight arithmetic mean (LNWAM) operator is:*

$$LNWAM(\eta_1, \eta_2, \ldots, \eta_n) = \sum_{i=1}^{n} \gamma_i \eta_i = (s_{2u-2u\prod_{i=1}^{n}(1-\frac{T_i}{2u})^{\gamma_i}}, s_{2u\prod_{i=1}^{n}(\frac{I_i}{2u})^{\gamma_i}}, s_{2u\prod_{i=1}^{n}(\frac{F_i}{2u})^{\gamma_i}}), \tag{3}$$

where γ_i is the corresponding weight value of η_i, $0 \le \gamma_i \le 1$ and $\sum_{i=1}^{n} \gamma_i = 1$.

Definition 7. [54] *Assume $\eta_i = (s_{T_i}, s_{I_i}, s_{F_i})$ $(i = 1, 2, \ldots, n)$ are a set of LNNs, the linguistic neutrosophic weight geometric mean (LNWGM) operator is:*

$$LNWGM(\eta_1, \eta_2, \ldots, \eta_n) = \sum_{i=1}^{n} \eta_i^{\gamma_i} = (s_{2u\prod_{i=1}^{n}(\frac{T_i}{2u})^{\gamma_i}}, s_{2u-2u\prod_{i=1}^{n}(1-\frac{I_i}{2u})^{\gamma_i}}, s_{2u-2u\prod_{i=1}^{n}(1-\frac{F_i}{2u})^{\gamma_i}}), \tag{4}$$

where γ_i is the related weight value of η_i, $0 \le \gamma_i \le 1$ and $\sum_{i=1}^{n} \gamma_i = 1$.

3. Extended TOPSIS Method with Incomplete Weight Information

In this section, we present the idea of an extended TOPSIS method with LNNs, and discuss the situation in which weight information is completely unknown.

3.1. Descriptions

With respect to the multi-criteria decision-making problems under linguistic neutrosophic situations, k decision makers evaluate a set of options $X = \{x_1, x_2, \ldots, x_n\}$ under some attributes $A = \{a_1, a_2, \ldots, a_m\}$. ω_j is the corresponding weight of a_j, which is completely unknown, but satisfies $\omega_j \in [0, 1]$ and $\sum_{i=1}^{m} \omega_j = 1$. There are k decision makers $\{b_1, b_2, \ldots, b_k\}$ with the related weight $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$, $0 \le \gamma_l \le 1$ $(l = 1, 2, \ldots, k)$ and $\sum_{l=1}^{k} \gamma_l = 1$. $S = \{s_i | i \in [0, 2u]\}$ is the predefined linguistic term set. In order to rank the objects or pick out the optimal one(s), each decision-maker $(b_l(l = 1, 2, \ldots, k))$ makes evaluations and then constructs the corresponding decision-making matrix, that is:

$$N^{(l)} = (\eta_{ij}^{(l)})_{n\times m} = \begin{bmatrix} \eta_{11}^{(l)} & \cdots & \eta_{1m}^{(l)} \\ \eta_{21}^{(l)} & \cdots & \eta_{2m}^{(l)} \\ \vdots & \cdots & \vdots \\ \eta_{n1}^{(l)} & \cdots & \eta_{nm}^{(l)} \end{bmatrix} = \begin{bmatrix} (s_{T_{11}}^{(l)}, s_{T_{11}}^{(l)}, s_{T_{11}}^{(l)}) & \cdots & (s_{T_{1m}}^{(l)}, s_{T_{1m}}^{(l)}, s_{T_{1m}}^{(l)}) \\ (s_{T_{21}}^{(l)}, s_{T_{21}}^{(l)}, s_{T_{21}}^{(l)}) & \cdots & (s_{T_{2m}}^{(l)}, s_{T_{2m}}^{(l)}, s_{T_{2m}}^{(l)}) \\ \vdots & \cdots & \vdots \\ (s_{T_{n1}}^{(l)}, s_{T_{n1}}^{(l)}, s_{T_{n1}}^{(l)}) & \cdots & (s_{T_{nm}}^{(l)}, s_{T_{nm}}^{(l)}, s_{T_{nm}}^{(l)}) \end{bmatrix}, (l = 1, 2, \ldots, k).$$

The basic elements of the matrix $N^{(l)}$ are by means of LNNs, where $\eta_{ij}^{(l)} = (s_{T_{ij}}^{(l)}, s_{T_{ij}}^{(l)}, s_{T_{ij}}^{(l)})(s_{T_{ij}}^{(l)}, s_{T_{ij}}^{(l)}, s_{T_{ij}}^{(l)} \in S)$ means the assessment information of b_l about x_i related to criteria a_j.

3.2. Distance Measures of LNNs

In this subsection, we intend to introduce several distance formulas of LNNs, so that the discussion behind these can be smoothly advanced.

Definition 8. *Let $\eta_1 = (s_{T_1}, s_{I_1}, s_{F_1})$ and $\eta_2 = (s_{T_2}, s_{I_2}, s_{F_2})$ be two haphazard LNNs. $S = \{s_i | i \in [0, 2u]\}$ is the linguistic term set, and $f(s_i) = \frac{i}{2u}$ is the linguistic scale function. Then, the distance between η_1 and η_2 are denoted as follows:*

$$d(\eta_1, \eta_2) = (\tfrac{1}{3}(|f(s_{T_1}) - f(s_{T_2})|^\lambda + |f(s_{2u-I_1}) - f(s_{2u-I_2})|^\lambda + |f(s_{2u-F_1}) - f(s_{2u-F_2})|^\lambda))^{\frac{1}{\lambda}}, \lambda > 0. \quad (5)$$

Remarkably:

(1) when $\lambda = 1$, the Hamming distance

$$d_{Hm}(\eta_1, \eta_2) = \frac{1}{3}(|f(s_{T_1}) - f(s_{T_2})| + |f(s_{2u-I_1}) - f(s_{2u-I_2})| + |f(s_{2u-F_1}) - f(s_{2u-F_2})|); \quad (6)$$

(2) when $\lambda = 2$, the Euclidean distance

$$d_{Ed}(\eta_1, \eta_2) = \sqrt{\frac{1}{3}(|f(s_{T_1}) - f(s_{T_2})|^2 + |f(s_{2t-I_1}) - f(s_{2t-I_2})|^2 + |f(s_{2t-F_1}) - f(s_{2t-F_2})|^2)}; \quad (7)$$

(3) the Hausdorff distance

$$d_{Hd}(\eta_1, \eta_2) = \max\{|f(s_{T_1}) - f(s_{T_2})|, |f(s_{2t-I_1}) - f(s_{2t-I_2})|, |f(s_{2t-F_1}) - f(s_{2t-F_2})|\}. \quad (8)$$

Property 1. *Given three arbitrary LNNs $\eta_1 = (s_{T_1}, s_{I_1}, s_{F_1})$, $\eta_2 = (s_{T_2}, s_{I_2}, s_{F_2})$ and $\eta_3 = (s_{T_3}, s_{I_3}, s_{F_3})$. The linguistic term set is $S = \{s_i | i \in [0, 2u]\}$, and the universal set of LNNs is Ω . For any $\eta_1, \eta_2, \eta_3 \in \Omega$, the following properties are met:*

(1) $0 \le d(\eta_1, \eta_2) \le 1$;
(2) $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$;
(3) $d(\eta_1, \eta_2) = 0$ if $\eta_1 = \eta_2$;
(4) $d(\eta_1, \eta_3) \le d(\eta_1, \eta_2) + d(\eta_2, \eta_3)$.

Proof.

(1) Because $f(s_i) = \frac{i}{2u} \in [0, 1] \Rightarrow |f(s_{T_1}) - f(s_{T_2})| \in [0, 1]$, $|f(s_{2u-I_1}) - f(s_{2u-I_2})|$ and $|f(s_{2u-F_1}) - f(s_{2u-F_2})|$, as $\lambda > 0$, then $0 \le d(\eta_1, \eta_2) \le 1$.
(2) This proof is obvious.

(3) Since $\eta_1 = \eta_2$, then $SC(\eta_1) = SC(\eta_2)$ and $AC(\eta_1) = AC(\eta_2)$
$\Rightarrow$ $(4u + T_1 - I_1 - F_1)/(6u) = (4u + T_2 - I_2 - F_2)/(6u)$ and $(T_1 - F_1)/(2u) = (T_2 - F_2)/(2u)$
$\Rightarrow$ $T_1 - I_1 - F_1 = T_2 - I_2 - F_2$ and $T_1 - F_1 = T_2 - F_2$ $\Rightarrow$ $I_1 = I_2$ and $T_1 - F_1 = T_2 - F_2$.

$$\begin{aligned}
\text{Thus, } d(\eta_1,\eta_2) &= \left(\tfrac{1}{3}\left(|f(s_{T_1}) - f(s_{T_2})|^\lambda + |f(s_{2u-I_1}) - f(s_{2u-I_2})|^\lambda + |f(s_{2u-F_1}) - f(s_{2u-F_2})|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= \left(\tfrac{1}{3}\left(|\tfrac{T_1-T_2}{2u}|^\lambda + |\tfrac{I_2-I_1}{2u}|^\lambda + |\tfrac{F_2-F_1}{2u}|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= \left(\tfrac{1}{3}\left(|\tfrac{T_1-F_1+F_1-T_2}{2u}|^\lambda + |\tfrac{I_2-I_1}{2u}|^\lambda + |\tfrac{F_2-T_2+T_2-F_1}{2u}|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= \left(\tfrac{1}{3}\left(|\tfrac{T_2-F_2+F_1-T_2}{2u}|^\lambda + |\tfrac{I_2-I_1}{2u}|^\lambda + |\tfrac{F_1-T_1+T_2-F_1}{2u}|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= \left(\tfrac{1}{3}\left(|\tfrac{F_1-F_2}{2u}|^\lambda + |\tfrac{I_1-I_2}{2u}|^\lambda + |\tfrac{T_2-T_1}{2u}|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= \left(\tfrac{1}{3}\left(|f(s_{T_2}) - f(s_{T_1})|^\lambda + |f(s_{2u-I_2}) - f(s_{2u-I_1})|^\lambda + |f(s_{2u-F_2}) - f(s_{2u-F_1})|^\lambda\right)\right)^{\frac{1}{\lambda}} \\
&= d(\eta_2,\eta_1)
\end{aligned}$$

(4) As $|f(s_{T_1}) - f(s_{T_2})| = |f(s_{T_1}) - f(s_{T_2}) + f(s_{T_2}) - f(s_{T_3})|$
$\le |f(s_{T_1}) - f(s_{T_2})| + |f(s_{T_2}) - f(s_{T_3})|$,
$|f(s_{2u-I_1}) - f(s_{2u-I_2})| = |f(s_{2u-I_1}) - f(s_{2u-I_2}) + f(s_{2u-I_2}) - f(s_{2u-I_3})|$
$\le |f(s_{2u-I_1}) - f(s_{2u-I_2})| + |f(s_{2u-I_2}) - f(s_{2u-I_3})|$,
and $|f(s_{2u-F_1}) - f(s_{2u-F_2})| = |f(s_{2u-F_1}) - f(s_{2u-F_2}) + f(s_{2u-F_2}) - f(s_{2u-F_3})|$
$\le |f(s_{2u-F_1}) - f(s_{2u-F_2})| + |f(s_{2u-F_2}) - f(s_{2u-F_3})|$,
hence, $d(\eta_1,\eta_3) \le d(\eta_1,\eta_2) + d(\eta_2,\eta_3)$.

Example 1. *If $u = 4$, two LNNs $\eta_1 = (s_1, s_2, s_4)$ and $\eta_2 = (s_5, s_3, s_6)$, the Hamming distance is $d_{Hm}(\eta_1,\eta_2) \approx 0.292$, the Euclidean distance is $d_{Ed}(\eta_1,\eta_2) \approx 0.331$, and the Hausdorff distance is $d_{Hd}(\eta_1,\eta_2) = 0.500$.*

3.3. Weight Model Based on Maximum Deviation

Because the weight information is completely unknown, we use the maximum deviation approach to determine the weight vector of criteria in this subsection.

The basic idea of the maximum deviation method is that [60]:

(1) If there is a tiny difference of evaluation values η_{ij} among all objects under criteria $a_j (j = 1, 2, \ldots, m)$, it indicates that the criteria a_j has little effect on the sorting results. Accordingly, it is appropriate to allocate a small value of the related weight ω_j.
(2) Conversely, if there is a significant variance of assessment information η_{ij} among all alternatives under criteria $a_j (j = 1, 2, \ldots, m)$, then the criteria a_j may be very important to the ranking orders. In this case, giving a large weight value ω_j is reasonable.
(3) Notably, if η_{ij} are the same values among all options under criteria a_j $((j = 1, 2, \ldots, m))$, it means that the criteria a_j doesn't affect the ranking results. Therefore, we can make the corresponding weight $\omega_j = 0$.

For the sake of obtaining the difference values, we define the deviation degree of a certain object $x_i (i = 1, 2, \ldots, n)$ to all objects for a certain criteria $a_j (j = 1, 2, \ldots, m)$ as follows:

$$D_{ij}(\omega_j) = \sum_{e=1}^{n} d(\eta_{ij}, \eta_{ej})\omega_j, \tag{9}$$

where $d(\eta_{ij}, \eta_{ej})$ is the distance measure between η_{ij} and η_{ej}.

Subsequently, the deviation degrees of all options under the criteria $a_j(j = 1, 2, \ldots, m)$ can be denoted as:

$$D_j(\omega_j) = \sum_{i=1}^{n} D_{ij}(\omega_j) = \sum_{i=1}^{n}\sum_{e=1}^{n} d(\eta_{ij}, \eta_{ej})\omega_j. \tag{10}$$

Thus, the total deviation of all alternatives with all criteria is proposed in the following:

$$D(\omega) = \sum_{j=1}^{m} D_j(\omega_j) = \sum_{j=1}^{m}\sum_{i=1}^{n} D_{ij}(\omega_j) = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{e=1}^{n} d(\eta_{ij}, \eta_{ej})\omega_j. \tag{11}$$

As a result, we can build the weight model based on maximum deviation as follows:

$$\begin{aligned} \max D(\omega) &= \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{e=1}^{n} d(\eta_{ij}, \eta_{ej})\omega_j \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{e=1}^{n} \left(\tfrac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - f(s_{2u-I_{ej}})|^{\lambda} + |f(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda})\right)^{\frac{1}{\lambda}}\omega_j \\ s.t & \begin{cases} \sum_{j=1}^{m} \omega_j^2 = 1 \\ 0 \le \omega_j \le 1,\ j = 1, 2, \ldots, m \end{cases} \end{aligned} \tag{12}$$

In order to get the solution, we can construct the Lagrange function as that:

$$\begin{aligned} L(\omega, p) = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{e=1}^{n} (\tfrac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - & f(s_{2u-I_{ej}})|^{\lambda} + \\ & |f(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda}))^{\frac{1}{\lambda}}\omega_j + \tfrac{p}{2}(\sum_{j=1}^{m} \omega_j^2 - 1) \end{aligned} \tag{13}$$

Taking the partial deviation of this function, we have:

$$\begin{aligned} & \begin{cases} \frac{\partial L(\omega,p)}{\partial \omega} = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{e=1}^{n} (\frac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - f(s_{2u-I_{ej}})|^{\lambda} + |f(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda}))^{\frac{1}{\lambda}} + p\omega_j = 0 \\ \frac{\partial L(\omega,p)}{\partial p} = \sum_{j=1}^{m} \omega_j^2 - 1 = 0 \end{cases} \\ & \Rightarrow p = \sqrt{\sum_{j=1}^{m} (\sum_{i=1}^{n}\sum_{e=1}^{n} (\tfrac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - f(s_{2u-I_{ej}})|^{\lambda} + |(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda}))^{\frac{1}{\lambda}})^2} \text{ and} \\ & \omega_j = \frac{\sum_{i=1}^{n}\sum_{e=1}^{n} (\frac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - f(s_{2u-I_{ej}})|^{\lambda} + |(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda}))^{\frac{1}{\lambda}}}{\sqrt{\sum_{j=1}^{m} (\sum_{i=1}^{n}\sum_{e=1}^{n} (\frac{1}{3}(|f(s_{T_{ij}}) - f(s_{T_{ej}})|^{\lambda} + |f(s_{2u-I_{ij}}) - f(s_{2u-I_{ej}})|^{\lambda} + |(s_{2u-F_{ij}}) - f(s_{2u-F_{ej}})|^{\lambda}))^{\frac{1}{\lambda}})^2}}. \end{aligned} \tag{14}$$

In the end, we can use the following formula to normalize the criteria weights:

$$\omega_j^* = \frac{\omega_j}{\sum_{j=1}^{m} \omega_j}, j = 1, 2, \ldots, m. \tag{15}$$

3.4. The Extended TOPSIS Method with LNNs

In this subsection, an extended TOPSIS approach under a linguistic neutrosophic environment is proposed.

The detailed steps are described as follows:

Step 1: Obtain the normalized decision-making matrix $N^{\bullet(l)} = (\eta_{ij}^{\bullet(l)})_{n\times m} = (s_{T_{ij}}^{\bullet(l)}, s_{I_{ij}}^{\bullet(l)}, s_{F_{ij}}^{\bullet(l)})_{n\times m}$. If the criteria belong to cost type, let $s_{T_{ij}}^{\bullet(l)} = s_{2u-T_{ij}}^{(l)}$, $s_{I_{ij}}^{\bullet(l)} = s_{2u-I_{ij}}^{(l)}$ and $s_{F_{ij}}^{\bullet(l)} = s_{2u-F_{ij}}^{(l)}$. If the criteria belong to benefit type, then the matrix remains, that is to say $s_{T_{ij}}^{\bullet(l)} = s_{T_{ij}}^{(l)}$, $s_{I_{ij}}^{\bullet(l)} = s_{I_{ij}}^{(l)}$ and $s_{F_{ij}}^{\bullet(l)} = s_{F_{ij}}^{(l)}$.

Step 2: Get the comprehensive decision-making matrix $N^{\bullet} = (\eta_{ij}^{\bullet})_{n\times m} = (s_{T_{ij}}^{\bullet}, s_{I_{ij}}^{\bullet}, s_{F_{ij}}^{\bullet})_{n\times m}$ using the LNWAM operator or LNWGM operator on the basis of Formula (3) or Formula (4).

Step 3: Use the weight model to calculate the weight values ω_j $(j = 1,2,\ldots,m)$ based on Formula (9), and then normalize the weight information in line with Formula (10), denoted as $\omega_j^\bullet$ $(j = 1,2,\ldots,m)$.

Step 4: Establish the weight standardized decision-making matrix $N^* = (\eta_{ij}^*)_{n\times m} = (s_{T_{ij}}^*, s_{I_{ij}}^*, s_{F_{ij}}^*)_{n\times m}$ through multiplying the normalized matrix with weight vector, where $s_{T_{ij}}^* = \omega_j s_{T_{ij}}^\bullet$, $s_{I_{ij}}^* = \omega_j s_{I_{ij}}^\bullet$ and $s_{F_{ij}}^* = \omega_j s_{F_{ij}}^\bullet$.

Step 5: Distinguish the positive ideal solution η^+ and the negative ideal solution η^-, respectively, then:

$$\eta^+ = (\eta_1^+, \eta_1^+, \ldots, \eta_m^+), \eta_j^+ = \max_i(\eta_{ij}^*),\ (j = 1,2,\ldots,m) \tag{16}$$

And

$$\eta^- = (\eta_1^-, \eta_1^-, \ldots, \eta_m^-), \eta_j^- = \min_i(\eta_{ij}^*),\ (j = 1,2,\ldots,m) \tag{17}$$

Step 6: Based on Formula (5), calculate the distance measures of the positive ideal solution to all options, and the distance measures of the negative ideal solution to all options in proper sequence. The computation formulas are:

$$d^+ = (d_1^+, d_1^+, \ldots, d_n^+), d_i^+ = \sum_{j=1}^{m} d(\eta_j^+, \eta_{ij}^*),\ (i = 1,2,\ldots,n) \tag{18}$$

And

$$d^- = (d_1^-, d_1^-, \ldots, d_n^-), d_i^- = \sum_{j=1}^{m} d(\eta_j^-, \eta_{ij}^*),\ (i = 1,2,\ldots,n). \tag{19}$$

Step 7: For each option x_i $(i = 1,2,\ldots,n)$, compute the values of correlation coefficient D_i with the following equation:

$$D_i = \frac{d_i^-}{d_i^+ + d_i^-}. \tag{20}$$

Step 8: Achieve the ranking orders according to the values of D_i $(i = 1,2,\ldots,n)$. The bigger the value of D_i, the better the alternative x_i is.

4. Case Study

In this section, we study a case of evaluating investment risks of a gold mine using the proposed approach.

Recently, a construction investment company in Hunan province, called JK MINING Co., Ltd., had a plan for investing in a domestic metal mine. After an initial investigation and screening, four famous metal mines, described as $\{x_1, x_2, x_3, x_4\}$, have been under consideration. The enterprise establishes a team of three experts to conduct field explorations and surveys in depth, so that the optimal mine can be selected. The specialists need to evaluate the investment risk in line with their findings, professional knowledge, and experience. Assume the importance of each professional is equal, that is to say $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$. After heated discussions, five attributions are recognized as the evaluation criteria. They are geological risk (a_1), production risk (a_2), market risk (a_3), management risk (a_4), and social environment risk (a_5), separately. Then, the experts defined the linguistic term set, $S = \{s_i | i \in [0,8]\}$, where $s = \{s_0 = \textit{exceedingly low},\ \ s_1 = \textit{pretty low},\ \ s_2 = \textit{low},\ \ s_3 = \textit{slightly low},\ \ s_4 = \textit{medium},\ \ s_5 = \textit{slightly high},\ \ s_2 = \textit{high},\ \ s_3 = \textit{pretty high},\ \ s_4 = \textit{exceedingly high}\}$. Afterwards, they can give scores (or score ranges) or linguistic information directly of options under each attribute. The corresponding relationships between grade and linguistic term can been seen in Table 3.

Table 3. Reference of investment risk evaluation.

Grade	**0~19**	**20~29**	**30~39**	**40~49**	**50~59**	**60~69**	**70~79**	**80~89**	**90~100**
Evaluation	exceedingly low	pretty low	low	slightly low	medium	slightly high	high	pretty high	exceedingly high
Linguistic term	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

In order to describe the ambiguity and uncertainty of risks, their evaluation information is represented by LNNs. Subsequently, these assessment matrices are formed as Tables 4–6:

Table 4. Decision-making matrix $N^{(1)}$.

$N^{(1)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_1, s_2, s_1)	(s_2, s_3, s_2)	(s_4, s_4, s_3)	(s_1, s_5, s_1)	(s_3, s_3, s_2)
x_2	(s_2, s_6, s_2)	(s_3, s_8, s_2)	(s_2, s_4, s_1)	(s_3, s_1, s_2)	(s_1, s_2, s_1)
x_3	(s_2, s_3, s_1)	(s_3, s_2, s_3)	(s_1, s_4, s_1)	(s_3, s_5, s_1)	(s_5, s_2, s_4)
x_4	(s_3, s_1, s_2)	(s_1, s_7, s_1)	(s_4, s_6, s_3)	(s_2, s_5, s_1)	(s_4, s_6, s_4)

Table 5. Decision-making matrix $N^{(2)}$.

$N^{(2)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_1, s_6, s_1)	(s_4, s_3, s_4)	(s_2, s_6, s_2)	(s_3, s_5, s_2)	(s_5, s_2, s_4)
x_2	(s_1, s_4, s_1)	(s_3, s_2, s_1)	(s_2, s_3, s_4)	(s_4, s_0, s_5)	(s_2, s_6, s_4)
x_3	(s_3, s_5, s_2)	(s_2, s_4, s_3)	(s_1, s_6, s_5)	(s_3, s_5, s_3)	(s_2, s_6, s_1)
x_4	(s_2, s_7, s_2)	(s_4, s_6, s_1)	(s_3, s_7, s_2)	(s_4, s_4, s_2)	(s_3, s_8, s_4)

Table 6. Decision-making matrix $N^{(3)}$.

$N^{(3)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_2, s_4, s_1)	(s_3, s_5, s_2)	(s_5, s_1, s_4)	(s_2, s_6, s_1)	(s_3, s_3, s_2)
x_2	(s_1, s_2, s_1)	(s_2, s_4, s_2)	(s_1, s_5, s_3)	(s_4, s_2, s_0)	(s_0, s_5, s_6)
x_3	(s_2, s_3, s_3)	(s_1, s_5, s_2)	(s_2, s_4, s_5)	(s_0, s_4, s_6)	(s_3, s_2, s_4)
x_4	(s_2, s_3, s_2)	(s_4, s_2, s_1)	(s_1, s_4, s_3)	(s_3, s_4, s_5)	(s_0, s_4, s_5)

Next, the extended TOPSIS approach presented in Section 3.4 is employed to identify the optimal metal mine. A concrete calculation process is delivered as follows:

Step 1: Obtain the normalized decision matrix. As all the criteria are risk element, regarded as a part of cost, then normalizing evaluation values with function $s^{\bullet(l)}_{T_{ij}} = s^{(l)}_{2u-T_{ij}}$, $s^{\bullet(l)}_{I_{ij}} = s^{(l)}_{2u-I_{ij}}$ and $s^{\bullet(l)}_{F_{ij}} = s^{(l)}_{2u-F_{ij}}$. The followings (Tables 7–9) are the normalized decision-making matrix of each expert.

Table 7. Normalized decision-making matrix $N^{\bullet(1)}$.

$N^{\bullet(1)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_7, s_6, s_7)	(s_6, s_5, s_6)	(s_4, s_4, s_5)	(s_7, s_3, s_7)	(s_5, s_5, s_6)
x_2	(s_6, s_2, s_6)	(s_5, s_0, s_6)	(s_6, s_4, s_7)	(s_6, s_7, s_6)	(s_7, s_6, s_7)
x_3	(s_6, s_5, s_7)	(s_5, s_6, s_5)	(s_7, s_4, s_7)	(s_5, s_3, s_7)	(s_3, s_6, s_4)
x_4	(s_5, s_7, s_6)	(s_7, s_1, s_7)	(s_4, s_2, s_5)	(s_6, s_3, s_7)	(s_4, s_2, s_4)

Table 8. Normalized decision-making matrix $N^{\bullet(2)}$.

$N^{\bullet(2)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_7, s_2, s_7)	(s_4, s_5, s_4)	(s_6, s_2, s_6)	(s_5, s_3, s_6)	(s_3, s_6, s_4)
x_2	(s_7, s_4, s_7)	(s_5, s_6, s_7)	(s_6, s_5, s_4)	(s_4, s_8, s_3)	(s_6, s_2, s_4)
x_3	(s_5, s_3, s_6)	(s_6, s_4, s_5)	(s_7, s_2, s_3)	(s_5, s_3, s_5)	(s_6, s_2, s_7)
x_4	(s_6, s_1, s_6)	(s_4, s_3, s_7)	(s_5, s_1, s_6)	(s_4, s_4, s_6)	(s_5, s_0, s_4)

Table 9. Normalized decision-making matrix $N^{\bullet(3)}$.

$N^{\bullet(3)}$	a_1	a_2	a_3	a_4	a_5
x_1	(s_6, s_4, s_7)	(s_5, s_3, s_6)	(s_3, s_7, s_4)	(s_6, s_2, s_7)	(s_5, s_5, s_6)
x_2	(s_7, s_6, s_7)	(s_6, s_4, s_6)	(s_7, s_3, s_5)	(s_4, s_6, s_8)	(s_8, s_3, s_2)
x_3	(s_6, s_5, s_5)	(s_7, s_3, s_6)	(s_6, s_4, s_3)	(s_8, s_4, s_2)	(s_5, s_6, s_4)
x_4	(s_6, s_5, s_6)	(s_4, s_6, s_7)	(s_7, s_4, s_5)	(s_5, s_4, s_3)	(s_8, s_4, s_3)

Step 2: Using the LNWAM operator in line with Formula (3) to get the comprehensive decision matrix as Table 10:

Table 10. Comprehensive decision-making matrix $N^{\bullet}$.

$N^{\bullet}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_{6.74}, s_{3.63}, s_7)$	$(s_{5.12}, s_{4.22}, s_{5.24})$	$(s_{4.58}, s_{3.83}, s_{4.93})$	$(s_{6.18}, s_{2.62}, s_{6.65})$	$(s_{4.44}, s_{5.31}, s_{5.24})$
x_2	$(s_{6.74}, s_{3.63}, s_{6.65})$	$(s_{5.38}, s_0, s_{6.32})$	$(s_{6.18}, s_{3.91}, s_{5.19})$	$(s_{4.83}, s_{6.95}, s_{5.24})$	$(s_8, s_{3.3}, s_{3.83})$
x_3	$(s_{5.71}, s_{4.22}, s_{5.94})$	$(s_{6.18}, s_{4.16}, s_{5.31})$	$(s_{6.74}, s_{3.17}, s_{3.98})$	$(s_8, s_{3.3}, s_{4.12})$	$(s_{4.89}, s_{4.16}, s_{4.82})$
x_4	$(s_{5.71}, s_{3.27}, s_6)$	$(s_{5.48}, s_{2.62}, s_7)$	$(s_{5.71}, s_2, s_{5.31})$	$(s_{5.11}, s_{3.63}, s_{5.01})$	$(s_8, s_0, s_{3.63})$

Step 3: Calculate the values of the criteria weight ω_j (suppose $\lambda = 1$) on the basis of Formula (9) as follows: $\omega_1 \approx 0.17$, $\omega_2 \approx 0.42$, $\omega_3 \approx 0.31$, $\omega_4 \approx 0.55$ and $\omega_1 \approx 0.63$. Normalize them based on Formula (10): $\omega_1^{\bullet} = \frac{\omega_1}{\omega_1+\omega_2+\omega_3+\omega_4+\omega_5} \approx 0.08$, $\omega_2^{\bullet} \approx 0.20$, $\omega_3^{\bullet} \approx 0.15$, $\omega_4^{\bullet} \approx 0.27$ and $\omega_5^{\bullet} \approx 0.30$.

Step 4: Establish the weight standardized decision-making matrix as Table 11.

Table 11. Weight standardized decision-making matrix N^w.

N^w	a_1	a_2	a_3	a_4	a_5
x_1	$(s_{1.1}, s_{7.51}, s_{7.91})$	$(s_{1.48}, s_{7.04}, s_{7.35})$	$(s_{0.96}, s_{7.16}, s_{7.44})$	$(s_{2.64}, s_{5.92}, s_{7.61})$	$(s_{1.73}, s_{7.07}, s_{7.05})$
x_2	$(s_{1.1}, s_{7.51}, s_{7.88})$	$(s_{1.6}, s_0, s_{7.63})$	$(s_{1.59}, s_{7.19}, s_{7.5})$	$(s_{1.77}, s_{7.7}, s_{7.14})$	$(s_8, s_{6.13}, s_{6.41})$
x_3	$(s_{0.76}, s_{7.6}, s_{7.81})$	$(s_{2.05}, s_{7.02}, s_{7.37})$	$(s_{1.94}, s_{6.96}, s_{7.2})$	$(s_8, s_{6.3}, s_{6.69})$	$(s_{1.97}, s_{6.57}, s_{6.87})$
x_4	$(s_{0.76}, s_{7.45}, s_{7.82})$	$(s_{1.65}, s_{6.4}, s_{7.79})$	$(s_{1.37}, s_{6.5}, s_{7.53})$	$(s_{1.92}, s_{6.46}, s_{7.05})$	$(s_8, s_0, s_{6.31})$

Step 5: Identify the positive ideal solution and the negative ideal solution, respectively. See Table 12.

Table 12. Positive ideal solution and negative ideal solution.

η_1^+	η_2^+	η_3^+	η_4^+	η_5^+
$(s_{1.1}, s_{7.51}, s_{7.88})$	$(s_{1.6}, s_0, s_{7.63})$	$(s_{1.94}, s_{6.96}, s_{7.2})$	$(s_8, s_{6.3}, s_{6.69})$	$(s_8, s_0, s_{6.31})$
η_1^-	η_2^-	η_3^-	η_4^-	η_5^-
$(s_{0.76}, s_{7.6}, s_{7.81})$	$(s_{1.48}, s_{7.04}, s_{7.35})$	$(s_{0.96}, s_{7.16}, s_{7.44})$	$(s_{1.77}, s_{7.7}, s_{7.14})$	$(s_{1.73}, s_{7.07}, s_{7.05})$

Step 6: In line with Formula (5), the distances are measured as follows (assume $\lambda = 1$): $d_1^+ \approx 9.88$, $d_2^+ \approx 5.06$, $d_3^+ = 7.13$, $d_4^+ = 5.01$, $d_1^- \approx 1.22$, $d_2^- \approx 5.50$, $d_3^- \approx 3.68$ and $d_4^- \approx 6.04$.

Step 7: Compute the values of correlation coefficient: $D_1 \approx 0.11$, $D_2 \approx 0.52$, $D_3 \approx 0.34$ and $D_4 \approx 0.55$.

Step 8: Since $D_4 > D_2 > D_3 > D_1$, then the ranking order is $x_4 > x_2 > x_3 > x_1$, and the best metal mine is x_4.

5. Comparison Analysis

In this section, several related studies are compared through solving the same problem of gold mine venture assessment.

The comparison results can be seen in Table 13, and the particularized discussions and analysis are depicted in the following:

Table 13. Ranking orders using different approaches.

Approaches	Ranking Orders	Optimal Alternatives	Worst Alternatives
Approach with the LNWAM operator [54]	$x_4 > x_2 > x_3 > x_1$	x_4	x_1
Approach with the LNWGM operator [54]	$x_4 > x_3 > x_2 > x_1$	x_4	x_1
Approach with $u_{ij} = \frac{1}{3}s_{T_{ij}}$ [50]	$x_4 > x_3 > x_2 > x_1$	x_4	x_1
Approach with $u_{ij} = \frac{1}{3}s_{T_{ij}} + \frac{1}{6}s_{I_{ij}}$ [50]	$x_1 > x_3 > x_2 > x_4$	x_1	x_4
Approach with $u_{ij} = \frac{1}{3}s_{T_{ij}} + \frac{1}{3}s_{F_{ij}}$ [50]	$x_2 > x_1 > x_3 > x_4$	x_2	x_4
Approach with SVNLN-TOPSIS [42]	$x_4 > x_2 > x_3 > x_1$	x_4	x_1
The presented approach	$x_4 > x_2 > x_3 > x_1$	x_4	x_1

(1) The information in Reference [54] is LNNs. The multi-criteria group decision-making methods based on the LNWM operator or LNGM operator are presented. If we use the LNWAM operator to deal with the same problem in this paper, we have the comprehensive evaluations of each alternative as follows: $c_1 = (s_{5.4}, s_{3.87}, s_{5.67})$, $c_2 = (s_8, s_0, s_{5.04})$ $c_3 = (s_8, s_{3.76}, s_{4.65})$ $c_4 = (s_8, s_0, s_{4.98})$. Since the score function $SC(c_1) \approx 0.494$, $SC(c_2) = 0.790$, $SC(c_3) \approx 0.650$, $SC(c_4) \approx 0.793$, then $SC(c_4) > SC(c_2) > SC(c_3) > SC(c_1) \Rightarrow x_4 > x_2 > x_3 > x_1$. If the LNWGM operator is used, then $c_1^{\#} = (s_{5.19}, s_{4.15}, s_{5.87})$, $c_2^{\#} = (s_{6.12}, s_{4.6}, s_{5.32})$, $c_3^{\#} = (s_{6.22}, s_{3.81}, s_{4.75})$, $c_4^{\#} = (s_{6.08}, s_{2.24}, s_{5.43})$. As $SC(c_1^{\#}) \approx 0.465$, $SC(c_2^{\#}) \approx 0.508$, $SC(c_3^{\#}) \approx 0.569$, $SC(c_4^{\#}) \approx 0.600$, we have $SC(c_4^{\#}) > SC(c_3^{\#}) > SC(c_2^{\#}) > SC(c_1^{\#}) \Rightarrow x_4 > x_3 > x_2 > x_1$.

(2) The information in Reference [50] is linguistic intuitionistic fuzzy numbers (LIFNs). In the first place, it is necessary to translate LNNs into LIFNs. However, there is no existing universal conversion method. In this case, we have three ideas. The first idea is that all the linguistic indeterminacy-membership degrees in LNNs are allocated to linguistic non-membership degrees in LIFNs. In other words, $u_{ij} = \frac{1}{3}s_{T_{ij}}$ and $v_{ij} = \frac{1}{3}s_{I_{ij}} + \frac{1}{3}s_{F_{ij}}$. For example, a LNN (s_3, s_6, s_6) can be changed into a linguistic intuitionistic fuzzy number (LIFN) (s_1, s_4). The second opinion is that linguistic indeterminacy-membership degrees in LNNs are assigned to linguistic membership degrees and linguistic non-membership degrees in LIFNs on average. That is to say, $u_{ij} = \frac{1}{3}s_{T_{ij}} + \frac{1}{6}s_{I_{ij}}$ and $v_{ij} = \frac{1}{6}s_{I_{ij}} + \frac{1}{3}s_{F_{ij}}$. For instance, the LIFN (s_2, s_3) may take the place of a LNN (s_3, s_6, s_6). On the contrary, the last attitude is that all the linguistic indeterminacy-membership degrees in LNNs are allotted to linguistic membership degrees in LIFNs. So to speak, $u_{ij} = \frac{1}{3}s_{T_{ij}} + \frac{1}{3}s_{F_{ij}}$ and $v_{ij} = \frac{1}{3}s_{F_{ij}}$. As an example, a LNN (s_3, s_6, s_6) may be replaced by a LIFN (s_3, s_2).

Owing to the limited space, we take the first idea as an example in the following. The converted decision-making matrices of each expert are shown as Tables 14–16:

Table 14. Converted decision-making matrix $N^{co(1)}$.

$N^{co(1)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_{7/3}, s_{13/3})$	$(s_2, s_{11/3})$	$(s_{4/3}, s_3)$	$(s_{7/3}, s_{10/3})$	$(s_{5/3}, s_{11/3})$
x_2	$(s_2, s_{8/3})$	$(s_{5/3}, s_2)$	$(s_2, s_{11/3})$	$(s_2, s_{13/3})$	$(s_{7/3}, s_{13/3})$
x_3	(s_2, s_4)	$(s_{5/3}, s_{11/3})$	$(s_{7/3}, s_{11/3})$	$(s_{5/3}, s_{10/3})$	$(s_1, s_{10/3})$
x_4	$(s_{5/3}, s_{13/3})$	$(s_{7/3}, s_{8/3})$	$(s_{4/3}, s_{7/3})$	$(s_2, s_{10/3})$	$(s_{4/3}, s_2)$

Table 15. Converted decision-making matrix $N^{co(2)}$.

$N^{co(2)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_{7/3}, s_3)$	$(s_{4/3}, s_3)$	$(s_2, s_{8/3})$	$(s_{5/3}, s_3)$	$(s_1, s_{10/3})$
x_2	$(s_{7/3}, s_{11/3})$	$(s_{5/3}, s_{13/3})$	(s_2, s_3)	$(s_{4/3}, s_{11/3})$	(s_2, s_2)
x_3	$(s_{5/3}, s_3)$	(s_2, s_3)	$(s_{7/3}, s_{5/3})$	$(s_{5/3}, s_{8/3})$	(s_2, s_3)
x_4	$(s_2, s_{7/3})$	$(s_{4/3}, s_{10/3})$	$(s_{5/3}, s_{7/3})$	$(s_{4/3}, s_{10/3})$	$(s_{5/3}, s_{4/3})$

Table 16. Converted decision-making matrix $N^{co(3)}$.

$N^{co(3)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_2, s_{11/3})$	$(s_{5/3}, s_3)$	$(s_1, s_{11/3})$	(s_2, s_3)	$(s_{5/3}, s_{11/3})$
x_2	$(s_{7/3}, s_{13/3})$	$(s_2, s_{10/3})$	$(s_{7/3}, s_{8/3})$	$(s_{4/3}, s_{14/3})$	$(s_{8/3}, s_{5/3})$
x_3	$(s_2, s_{10/3})$	$(s_{7/3}, s_3)$	$(s_2, s_{7/3})$	$(s_{8/3}, s_2)$	$(s_{5/3}, s_{10/3})$
x_4	$(s_2, s_{11/3})$	$(s_{4/3}, s_{13/3})$	$(s_{7/3}, s_3)$	$(s_{5/3}, s_{7/3})$	$(s_{8/3}, s_{7/3})$

Then, using the method in Reference [50], the collective evaluations of each option are $x_1 = (s_{83/48}, s_{173/52})$, $x_2 = (s_{103/52}, s_{68/21})$, $x_3 = (s_{133/73}, s_{257/84})$ and $x_4 = (s_{173/96}, s_{29/11})$ (let the position weight $w = (0.2, 0.3, 0.5)^T$). Then the score functions are $L(x_1) \approx -1.60$, $L(x_2) \approx -1.26$, $L(x_3) \approx -1.24$ and $L(x_4) \approx -0.83$. Because $L(x_4) > L(x_3) > L(x_2) > L(x_1)$, the ranking result is $x_4 > x_3 > x_2 > x_1$.

Likewise, we use the approach in Reference [50] with the second and third thought to deal with the same problem, successively. Afterwards, we get the corresponding ranking orders are $x_1 > x_3 > x_2 > x_4$ and $x_2 > x_1 > x_3 > x_4$, respectively (suppose the position weight is constant and that $w = (0.2, 0.3, 0.5)^T$).

(3) The information in Reference [42] consists of single valued neutrosophic linguistic numbers (SVNLNs). The first step is to change the LNNs into SVNLNs. For a certain LNN $\eta = (s_T, s_I, s_F)$, if $g = \max(T, I, F)$, we can make the linguistic value in a single valued neutrosophic linguistic number (SVNLN) equal to s_g, then the truth-membership, indeterminacy-membership, and false-membership degrees in a SVNLN are described as T/g, I/g and F/g in proper order. So to say, a LNN $\eta = (s_T, s_I, s_F)$ may be converted into a SVNLN $(s_g, < T/g, I/g, F/g >)$. For example, a LNN (s_3, s_3, s_6) and a SVNLN $(s_6, < 0.5, 0.5, 1 >)$ are equivalent in manner.

The transformed decision-making matrices of each specialist are listed as Tables 17–19:

Table 17. Transformed decision-making matrix $N^{tr(1)}$.

$N^{tr(1)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_7, < 1, 6/7, 1 >)$	$(s_6, < 1, 5/6, 1 >)$	$(s_5, < 4/5, 4/5, 1 >)$	$(s_7, < 1, 3/7, 1 >)$	$(s_6, < 5/6, 5/6, 1 >)$
x_2	$(s_6, < 1, 1/3, 1 >)$	$(s_6, < 5/6, 0, 1 >)$	$(s_7, < 6/7, 4/7, 1 >)$	$(s_7, < 6/7, 1, 6/7 >)$	$(s_7, < 1, 6/7, 1 >)$
x_3	$(s_7, < 6/7, 5/7, 1 >)$	$(s_6, < 5/6, 1, 5/6 >)$	$(s_7, < 1, 4/7, 1 >)$	$(s_7, < 5/7, 3/7, 1 >)$	$(s_6, < 1/2, 1, 2/3 >)$
x_4	$(s_7, < 5/7, 1, 6/7 >)$	$(s_7, < 1, 1/7, 1 >)$	$(s_5, < 4/5, 2/5, 1 >)$	$(s_7, < 6/7, 3/7, 1 >)$	$(s_4, < 1, 1/2, 1 >)$

Table 18. Transformed decision-making matrix $N^{tr(2)}$.

$N^{tr(2)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_7, < 1, 2/7, 1 >)$	$(s_5, < 4/5, 1, 4/5 >)$	$(s_6, < 1, 1/3, 1 >)$	$(s_6, < 5/6, 1/2, 1 >)$	$(s_6, < 1/2, 1, 2/3 >)$
x_2	$(s_7, < 1, 4/7, 1 >)$	$(s_7, < 5/7, 6/7, 1 >)$	$(s_6, < 1, 5/6, 2/3 >)$	$(s_8, < 1/2, 1, 3/8 >)$	$(s_6, < 1, 1/3, 2/3 >)$
x_3	$(s_6, < 5/6, 1/2, 1 >)$	$(s_6, < 1, 2/3, 5/6 >)$	$(s_7, < 1, 2/7, 3/7 >)$	$(s_5, < 1, 3/5, 1 >)$	$(s_7, < 6/7, 2/7, 1 >)$
x_4	$(s_6, < 1, 1/6, 1 >)$	$(s_7, < 4/7, 3/7, 1 >)$	$(s_6, < 5/6, 1/6, 1 >)$	$(s_6, < 2/3, 2/3, 1 >)$	$(s_5, < 1, 0, 4/5 >)$

Table 19. Transformed decision-making matrix $N^{tr(3)}$.

$N^{tr(3)}$	a_1	a_2	a_3	a_4	a_5
x_1	$(s_7, < 6/7, 4/7, 1 >)$	$(s_6, < 5/6, 1/2, 1 >)$	$(s_7, < 3/7, 1, 4/7 >)$	$(s_7, < 6/7, 2/7, 1 >)$	$(s_6, < 5/6, 5/6, 1 >)$
x_2	$(s_7, < 1, 6/7, 1 >)$	$(s_6, < 1, 2/3, 1 >)$	$(s_7, < 1, 3/7, 5/7 >)$	$(s_8, < 1/2, 3/4, 1 >)$	$(s_8, < 1, 3/8, 1/4 >)$
x_3	$(s_6, < 1, 5/6, 5/6 >$	$(s_7, < 1, 3/7, 6/7 >)$	$(s_6, < 1, 2/3, 1/2 >)$	$(s_8, < 1, 1/2, 1/4 >)$	$(s_6, < 5/6, 1, 2/3 >)$
x_4	$(s_6, < 1, 5/6, 1 >)$	$(s_7, < 4/7, 6/7, 1 >)$	$(s_7, < 1, 4/7, 5/7 >)$	$(s_5, < 1, 4/5, 3/5 >)$	$(s_8, < 1, 1/2, 3/8 >)$

After that, the extended SVNLN-TOPSIS approach in literature [42] is employed to assess the metal mine's investment venture. The relative closeness coefficients of each mine are calculated as follows: $rc_1 = 21/25$, $rc_2 = 54/67$, $rc_3 = 5/6$ and $rc_4 = 29/36$. Because $rc_4 < rc_2 < rc_3 < rc_1$, we have $x_4 > x_2 > x_3 > x_1$.

From Table 13, we can see that there are diverse ranking results with distinct methods. In order to attain the ideal ranking order, we can assign grades for alternatives in these seven rankings successively. The better the option is, the higher the score is. That is to say, the optimal alternative in a ranking may be distributed with 4, the second is 3, the third is 2, and the worst is 1. As an illustration, according to the ranking $x_4 > x_2 > x_3 > x_1$ in literature [54] with the LNWAM operator, we have $G_1(x_4) = 4$, $G_1(x_2) = 3$, $G_1(x_3) = 2$ and $G_1(x_1) = 1$. Similarly, grades in other ranking methods can be determined. In the end, the overall grades of all alternatives can be earned through summation as follows: $G(x_1) = 12$, $G(x_2) = 19$, $G(x_3) = 17$ and $G(x_4) = 22$. Because $G(x_4) > G(x_2) > G(x_3) > G(x_1)$, the ideal ranking result may be regarded as $x_4 > x_2 > x_3 > x_1$. It is obvious that the result is the same with the proposed method in this paper. The feasibility and availability of the presented approach are indicated.

Besides, the best and worst objects are identical in the literature [42,54] and our approach. The reasons for the differences between literature [54] with our method may be the decision-making thought. Our measure is based on distance, while the literature [54] is based on aggregation operators. Some initial information may be missing in the process of aggregating. Moreover, diverse conclusions may occur with different aggregation operators, which has been demonstrated in the second and third line in Table 13. Both the method in Reference [42] and ours are in line with TOPSIS, and the same orders are received. However, there may be some limitations in [42]. Because the attribute weight vector is given directly, the positive and negative ideal solutions are absolute. In addition, the rankings in literature [50] are all different from the presented method. The reason for the distinction may be that the indeterminacy-membership information in LNNs is unavoidably distorted in LIFNs to some extent.

From the analysis above, the advantages of the proposed method can be summarized as follows:

(1) Evaluating the risk degree of mining projects under qualitative criteria by means of LNNs is a good choice. As all the consistent, hesitant, and inconsistent linguistic information are taken into account.

(2) The flexibility has increased because various distance measures, aggregation operators, and linguistic scale functions can be chosen according to the savants' experience or reality.

(3) A common situation, in which the criteria weight information is unknown, is under consideration. There are many complex risk factors in the process of metallic mining investment. Thus, it is difficult or unrealistic for decision makers to give the weight vector directly. The weight

model based on the thought of maximum deviation may be a simple and suitable way to resolve this problem.

(4) Instead of using absolute ideal points, the extended TOPSIS method defined the relative ideal solutions. The strength of it is that different ideal solutions are calculated corresponding with the different original information of different mining projects. This may be more in line with reality.

6. Discussion and Conclusions

To evaluate risk is the beginning of a metallic mining project investment. Proper risk assessments have great significance on the success of investments. Owing to the uncertainty and complexity in mine surroundings, this paper advised an extended TOPSIS method with LNNs to rise to this challenge. LNNs were suggested to manifest the indeterminate and inconsistent linguistic values, so that the evaluation information can be retained as much as possible. Then, generalized distance formulas were presented to calculate the difference degrees of two LNNs. As it is not easy for the mining investment decision makers to directly determine criteria weight values, a weight model based on maximum deviation was recommended. Afterwards, the method of ranking mines was shown by a case study. Furthermore, the effectiveness and highlights of the presented approach can be reflected in the comparison analysis.

Even though the extended TOPSIS with LNNs method is a good solution, there are still some limitations. For example, the determination of the criteria weight values does not take the subjective elements into consideration. Hence, a more reasonable weight determination method should be further proposed. Besides, the sub-attribute risk factors may be considered in the future. The presented method with LNNs for evaluating the investment risks may be extended to interval linguistic neutrosophic numbers.

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A Novel Neutrosophic Weighted Extreme Learning Machine for Imbalanced Data Set

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Abstract: Extreme learning machine (ELM) is known as a kind of single-hidden layer feedforward network (SLFN), and has obtained considerable attention within the machine learning community and achieved various real-world applications. It has advantages such as good generalization performance, fast learning speed, and low computational cost. However, the ELM might have problems in the classification of imbalanced data sets. In this paper, we present a novel weighted ELM scheme based on neutrosophic set theory, denoted as neutrosophic weighted extreme learning machine (NWELM), in which neutrosophic *c*-means (NCM) clustering algorithm is used for the approximation of the output weights of the ELM. We also investigate and compare NWELM with several weighted algorithms. The proposed method demonstrates advantages to compare with the previous studies on benchmarks.

Keywords: extreme learning machine (ELM); weight; neutrosophic *c*-means (NCM); imbalanced data set

1. Introduction

Extreme learning machine (ELM) was put forward in 2006 by Huang et al. [1] as a single-hidden layer feedforward network (SLFN). The hidden layer parameters of ELM are arbitrarily initialized and output weights are determined by utilizing the least squares algorithm. Due to this characteristic, ELM has fast learning speed, better performance and efficient computation cost [1–4], and has, as a result, been applied in different areas.

However, ELM suffers from the presence of irrelevant variables in the large and high dimensional real data set [2,5]. The unbalanced data set problem occurs in real applications such as text categorization, fault detection, fraud detection, oil-spills detection in satellite images, toxicology, cultural modeling, and medical diagnosis [6]. Many challenging real problems are characterized by imbalanced training data in which at least one class is under-represented relative to others.

The problem of imbalanced data is often associated with asymmetric costs of misclassifying elements of different classes. In addition, the distribution of the test data set might differ from that of the training samples. Class imbalance happens when the number of samples in one class is much more than that of the other [7]. The methods aiming to tackle the problem of imbalance can be classified into four groups such as algorithmic based methods, data based methods, cost-sensitive methods and ensembles of classifiers based methods [8]. In algorithmic based approaches, the minority class classification accuracy is improved by adjusting the weights for each class [9]. Re-sampling methods can be viewed in the data based approaches where these methods did not improve the classifiers [10]. The cost-sensitive approaches assign various cost values to training samples of the majority class and the minority class, respectively [11]. Recently, ensembles based methods have been widely used in classification of imbalanced data sets [12]. Bagging and boosting methods are the two popular ensemble methods.

The problem of class imbalance has received much attention in the literature [13]. Synthetic minority over–sampling technique (SMOTE) [9] is known as the most popular re-sampling method that uses pre-processing for obtaining minority class instances artificially. For each minority class sample, SMOTE creates a new sample on the line joining it to the nearest minority class neighbor. Borderline SMOTE [14], SMOTE-Boost [15], and modified SMOTE [14] are some of the improved variants of the SMOTE algorithm. In addition, an oversampling method was proposed that identifies some minority class samples that are hard to classify [16]. Another oversampling method was presented that uses bagging with oversampling [17]. In [18], authors opted to use double ensemble classifier by combining bagging and boosting. In [19], authors combined sampling and ensemble techniques to improve the classification performance for skewed data. Another method, namely random under sampling (RUS), was proposed that removes the majority class samples randomly until the training set becomes balanced [19]. In [20], authors proposed an ensemble of an support vector machine (SVM) structure with boosting (Boosting-SVM), where the minority class classification accuracy was increased compared to pure SVM. In [21], a cost sensitive approach was proposed where k-nearest neighbors (k-NN) classifier was adopted. In addition, in [22], an SVM based cost sensitive approach was proposed for class imbalanced data classification. Decision trees [23] and logistic regression [24] based methods were also proposed in order to handle with the imbalanced data classification.

An ELM classifier trained with an imbalanced data set can be biased towards the majority class and obtain a high accuracy on the majority class by compromising minority class accuracy. Weighted ELM (WELM) was employed to alleviate the ELM's classification deficiency on imbalanced data sets, and which can be seen as one of the cost-proportionate weighted sampling methods [25]. ELM assigns the same misclassification cost value to all data points such as positive and negative samples in a two-class problem. When the number of negative samples is much larger than that of the number of positive samples or vice versa, assigning the same misclassification cost value to all samples can be seen one of the drawbacks of traditional ELM. A straightforward solution is to obtain misclassification cost values adaptively according to the class distribution, in the form of a weight scheme inversely proportional to the number of samples in the class.

In [7], the authors proposed a weighted online sequential extreme learning machine (WOS-ELM) algorithm for alleviating the imbalance problem in chunk-by-chunk and one-by-one learning. A weight setting was selected in a computationally efficient way. Weighted Tanimoto extreme learning machine (T-WELM) was used to predict chemical compound biological activity and other data with discrete, binary representation [26]. In [27], the authors presented a weight learning machine for a SLFN to recognize handwritten digits. Input and output weights were globally optimized with the batch learning type of least squares. Features were assigned into the prescribed positions. Another weighted ELM algorithm, namely ESOS-ELM, was proposed by Mirza et al. [28], which was inspired from WOS-ELM. ESOS-ELM aims to handle class imbalance learning (CIL) from a concept-drifting data stream. Another ensemble-based weighted ELM method was proposed by Zhang et al. [29], where the weight of each base learner in the ensemble is optimized by differential evolution algorithm. In [30], the authors further improved the re-sampling strategy inside Over-sampling based online bagging (OOB) and Under-sampling based online bagging (UOB) in order to learn class imbalance.

Although much awareness of the imbalance has been raised, many of the key issues remain unresolved and encountered more frequently in massive data sets. How to determine the weight values is key to designing WELM. Different situations such as noises and outlier data should be considered.

The noises and outlier data in a data set can be treated as a kind of indeterminacy. Neutrosophic set (NS) has been successfully applied for indeterminate information processing, and demonstrates advantages to deal with the indeterminacy information of data and is still a technique promoted for data analysis and classification application. NS provides an efficient and accurate way to define imbalance information according to the attributes of the data.

In this study, we present a new weighted ELM scheme using neutrosophic *c*-means (NCM) clustering to overcome the ELM's drawbacks in highly imbalanced data sets. A novel clustering algorithm NCM was proposed for data clustering [31,32]. NCM is employed to determine a sample's belonging, noise, and indeterminacy memberships, and is then used to compute a weight value for that sample [31–33]. A weighted ELM is designed using the weights from NCM and utilized for imbalanced data set classification.

The rest of the paper is structured as follows. In Section 2, a brief history of the theory of ELM and weighted ELM is introduced. In addition, Section 2 introduces the proposed method. Section 3 discusses the experiments and comparisons, and conclusions are drawn in Section 4.

2. Proposed Method

2.1. Extreme Learning Machine

Backpropagation, which is known as gradient-based learning method, suffers from slow convergence speed. In addition, stuck in the local minimum can be seen as another disadvantage of a gradient-based learning algorithm. ELM was proposed by Huang et al. [1] as an alternative method that overcomes the shortcomings of gradient-based learning methods. The ELM was designed as an SLFN, where the input weights and hidden biases are selected randomly. These weights do not need to be adjusted during the training process. The output weights are determined analytically with Moore–Penrose generalized inverse of the hidden-layer output matrix.

Mathematically speaking, the output of the ELM with L hidden nodes and activation function g(·) can be written as:

$$o_i = \sum_{j=1}^{L} \beta_j g(a_j, b_j, x_j), \quad i = 1, 2, \ldots, N, \tag{1}$$

where x_j is the jth input data, $a_j = [a_{j1}, a_{j2}, \ldots, a_{jn}]^T$ is the weight vector, $\beta_j = [\beta_{j1}, \beta_{j2}, \ldots, \beta_{jn}]^T$ is the output weight vector, b_j is the bias of the jth hidden node and o_i is the ith output node and N shows the number of samples. If ELM learns these N samples with 0 error, then Equation (1) can be updated as follows:

$$t_i = \sum_{j=1}^{L} \beta_j g(a_j, b_j, x_j), \quad i = 1, 2, \ldots, N, \tag{2}$$

where t_i shows the actual output vector. Equation (2) can be written compactly as shown in Equation (3):

$$H\beta = T, \tag{3}$$

where $H = \{h_{ij}\} = g(a_j, b_j, x_j)$ is the hidden-layer output matrix. Thus, the output weight vector can be calculated analytically with Moore–Penrose generalized inverse of the hidden-layer output matrix as shown in Equation (4):

$$\hat{\beta} = H^{+}T, \tag{4}$$

where H^{+} is the Moore–Penrose generalized inverse of matrix H.

2.2. Weighted Extreme Learning Machine

Let us consider a training data set $[x_i, t_i], i = 1, \ldots, N$ belonging to two classes, where $x_i \in R^n$and t_i are the class labels. In binary classification, t_i is either -1 or $+1$. Then, a $N \times N$ diagonal matrix W_{ii} is considered, where each of them is associated with a training sample x_i. The weighting procedure generally assigns larger W_{ii} to x_i, which comes from the minority class.

An optimization problem is employed to maximize the marginal distance and to minimize the weighted cumulative error as:

$$Minimize: \quad \|H\beta - T\|^2 \ \ and \ \ \|\beta\|. \tag{5}$$

Furthermore:

$$Minimize: \quad L_{ELM} = \frac{1}{2}\|\beta\|^2 + CW\frac{1}{2}\sum_{i=1}^{N}\|\xi_i\|^2, \tag{6}$$

$$Subjected\ to: \ \ h(x_i)\beta = t_i^T - \xi_i^T, \ \ i = 1, 2, \ldots, N, \tag{7}$$

where $T = [t_1, \ldots, t_N]$, ξ_i is the error vector and $h(x_i)$ is the feature mapping vector in the hidden layer with respect to x_i, and β. By using the Lagrage multiplier and Karush–Kuhn–Tucker theorem, the dual optimization problem can be solved. Thus, hidden layer's output weight vector β becomes can be derived from Equation (7) regarding left pseudo-inverse or right pseudo-inverse. When presented data with small size, right pseudo-inverse is recommended because it involves the inverse of an $N \times N$ matrix. Otherwise, left pseudo-inverse is more suitable since it is much easier to compute matrix inversion of size $L \times L$ when L is much smaller than N:

$$When\ N\ is\ small: \ \ \beta = H^T(\frac{I}{C} + WHH^T)^{-1}WT, \tag{8}$$

$$When\ N\ is\ large: \ \ \beta = H^T(\frac{I}{C} + H^TWT)^{-1}H^TWT. \tag{9}$$

In the weighted ELM, the authors adopted two different weighting schemes. In the first one, the weights for the minority and majority classes are calculated as:

$$W_{minority} = \frac{1}{\#(t_i^+)} \ \ and \ \ W_{majority} = \frac{1}{\#(t_i^-)}, \tag{10}$$

and, for the second one, the related weights are calculated as:

$$W_{minority} = \frac{0.618}{\#(t_i^+)} \ \ and \ \ W_{majority} = \frac{1}{\#(t_i^-)}. \tag{11}$$

The readers may refer to [25] for detail information about determination of the weights.

2.3. Neutrosophic Weighted Extreme Learning Machine

Weighted ELM assigns the same weight value to all samples in the minority class and another same weight value to all samples in the majority class. Although this procedure works quite well in some imbalanced data sets, assigning the same weight value to all samples in a class may not be a good choice for data sets that have noise and outlier samples. In other words, to deal with noise and outlier data samples in an imbalanced data set, different weight values are needed for each sample in each class that reflects the data point's significance in its class. Therefore, we present a novel method to determine the significance of each sample in its class. NCM clustering can determine a sample's belonging, noise and indeterminacy memberships, which can then be used in order to compute a weight value for that sample.

Guo and Sengur [31] proposed the NCM clustering algorithms based on the neutrosophic set theorem [34–37]. In NCM, a new cost function was developed to overcome the weakness of the Fuzzy *c*-Means (FCM) method on noise and outlier data points. In the NCM algorithm, two new types of rejection were developed for both noise and outlier rejections. The objective function in NCM is given as follows:

$$J_{NCM}(T, I, F, C) = \sum_{i=1}^{N}\sum_{j=1}^{C}(\overline{w}_1 T_{ij})^m \|x_i - c_j\|^2 + \sum_{i=1}^{N}(\overline{w}_2 I_i)^m \|x_i - \overline{c}_{imax}\|^2 + \delta^2 \sum_{i=1}^{N}(\overline{w}_3 F_i)^m, \quad (12)$$

where m is a constant. For each point i, the $\overline{c}_{imax}$ is the mean of two centers. T_{ij}, I_i and F_i are the membership values belonging to the determinate clusters, boundary regions and noisy data set. $\theta < T_{ij}, I_i, F_i < 1$:

$$\sum_{j=1}^{c} T_{ij} + I_i + F_i = 1. \quad (13)$$

Thus, the related membership functions are calculated as follows:

$$T_{ij} = \frac{\overline{w}_2 \overline{w}_3 (x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \overline{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \quad (14)$$

$$I_i = \frac{\overline{w}_1 \overline{w}_3 (x_i - c_{imax})^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \overline{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \quad (15)$$

$$F_i = \frac{\overline{w}_1 \overline{w}_2 (\delta)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - \overline{c}_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \quad (16)$$

$$C_j = \frac{\sum_{i=1}^{N}(\overline{w}_1 T_{ij})^m x_i}{\sum_{i=1}^{N}(\overline{w}_1 T_{ij})^m}, \quad (17)$$

where c_j shows the center of cluster j, $\overline{w}_1$, $\overline{w}_2$, and $\overline{w}_3$ are the weight factors and δ is a regularization factor which is data dependent [31]. Under the above definitions, every input sample in each minority and majority class is associated with a triple T_{ij}, I_i, F_i. While the larger T_{ij} means that the sample belongs to the labeled class with a higher probability, the larger I_i means that the sample is indeterminate with a higher probability. Finally, the larger F_i means that the sample is highly probable to be a noise or outlier data.

After clustering procedure is applied in NCM, the weights for each sample of minority and majority classes are obtained as follows:

$$W_{ii}^{minority} = \frac{C_r}{T_{ij} + I_i - F_i} \;\; and \;\; W_{ii}^{majority} = \frac{1}{T_{ij} + I_i - F_i}, \quad (18)$$

$$C_r = \frac{\#(t_i^-)}{\#(t_i^+)}, \quad (19)$$

where C_r is the ratio of the number of samples in the majority class to the number of the samples in the minority class.

The algorithm of the neutrosophic weighted extreme learning machine (NWELM) is composed of four steps. The first step necessitates applying the NCM algorithm based on the pre-calculated cluster centers, according to the class labels of the input samples. Thus, the T, I and F membership values are determined for the next step. The related weights are calculated from the determined T, I and F membership values in the second step of the algorithm.

In Step 3, the ELM parameters are tuned and samples and weights are fed into the ELM in order to calculate the H matrix. The hidden layer weight vector β is calculated according to the H, W and class labels. Finally, the determination of the labels of the test data set is accomplished in the final step of the algorithm (Step 4).

The neutrosophic weighted extreme learning machine (NWELM) algorithm is given as following:

Input: Labelled training data set.

Output: Predicted class labels.
Step 1: Initialize the cluster centers according to the labelled data set and run NCM algorithm in order to obtain the T, I and F value for each data point.
Step 2: Compute $W_{ii}^{minority}$ and $W_{ii}^{majority}$ according to Equations (18) and (19).
Step 3: Adapt the ELM parameters and run NWELM. Compute H matrix and obtain β according to Equation (8) or Equation (9).
Step 4: Calculate the labels of test data set based on β.

3. Experimental Results

The geometric mean (G_{mean}) is used to evaluate the performance of the proposed NWELM method. The G_{mean} is computed as follows:

$$G_{mean} = \sqrt{R\frac{TN}{TN+FP}}, \tag{20}$$

$$R = \frac{TP}{TP+FN}, \tag{21}$$

where R denotes the recall rate and TN, FP denotes true-negative and false-positive detections, respectively. G_{mean} values are in the range of [0–1] and it represents the square root of positive class accuracy and negative class accuracy. The performance evaluation of NWELM classifier is tested on both toy data sets and real data sets, respectively. The five-fold cross-validation method is adopted in the experiments. In the hidden node of the NWELM, the radial basis function (RBF) kernel is considered. A grid search of the trade-off constant C on $\{2^{-18}, 2^{-16}, \ldots, 2^{48}, 2^{50}\}$ and the number of hidden nodes L on $\{10, 20, \ldots, 990, 2000\}$ was conducted in seeking the optimal result using five-fold cross-validation. For real data sets, a normalization of the input attributes into $[-1, 1]$ is considered. In addition, for NCM, the following parameters are chosen such as $\varepsilon = 10^{-5}$, $\overline{w}_1 = 0.75$, $\overline{w}_2 = 0.125$, $\overline{w}_3 = 0.125$ respectively, which were obtained by means of trial and error. The δ parameter of NCM method is also searched on $\{2^{-10}, 2^{-8}, \ldots, 2^{8}, 2^{10}\}$.

3.1. Experiments on Artificial Data Sets

Four two-class artificial imbalance data sets were used to evaluate the classification performance of the proposed NWELM scheme. The illustration of the data sets is shown in Figure 1 [38]. The decision boundary between classes is complicated. In Figure 1a, we illustrate the first artificial data set that follows a uniform distribution. As can be seen, the red circles of Figure 1a belong to the minority class, with the rest of the data samples shown by blue crosses as the majority class. The second imbalance data set, namely Gaussian-1, is obtained using two Gaussian distributions with a 1:9 ratio of samples as shown in Figure 1b. While the red circles illustrate the minority class, the blue cross samples show the majority class.

Another Gaussian distribution-based imbalance data set, namely Gaussian-2, is given in Figure 1c. This data set consists of nine Gaussian distributions with the same number of samples arranged in a 3×3 grid. The red circle samples located in the middle belong to the minority class while the blue cross samples belong to the majority class. Finally, Figure 1d shows the last artificial imbalance data set. It is known as a complex data set because it has a 1:9 ratio of samples for the minority and majority classes.

Table 1 shows the G_{mean} achieved by the two methods on these four data sets in ten independent runs. For Gaussian-1, Gaussian-2 and the Uniform artificial data sets, the proposed NWELM method yields better results when compared to the weighted ELM scheme; however, for the Complex artificial data sets, the weighted ELM method achieves better results. The better resulting cases are shown in bold text. It is worth mentioning that, for the Gaussian-2 data set, NWELM achieves a higher G_{mean} across all trials.

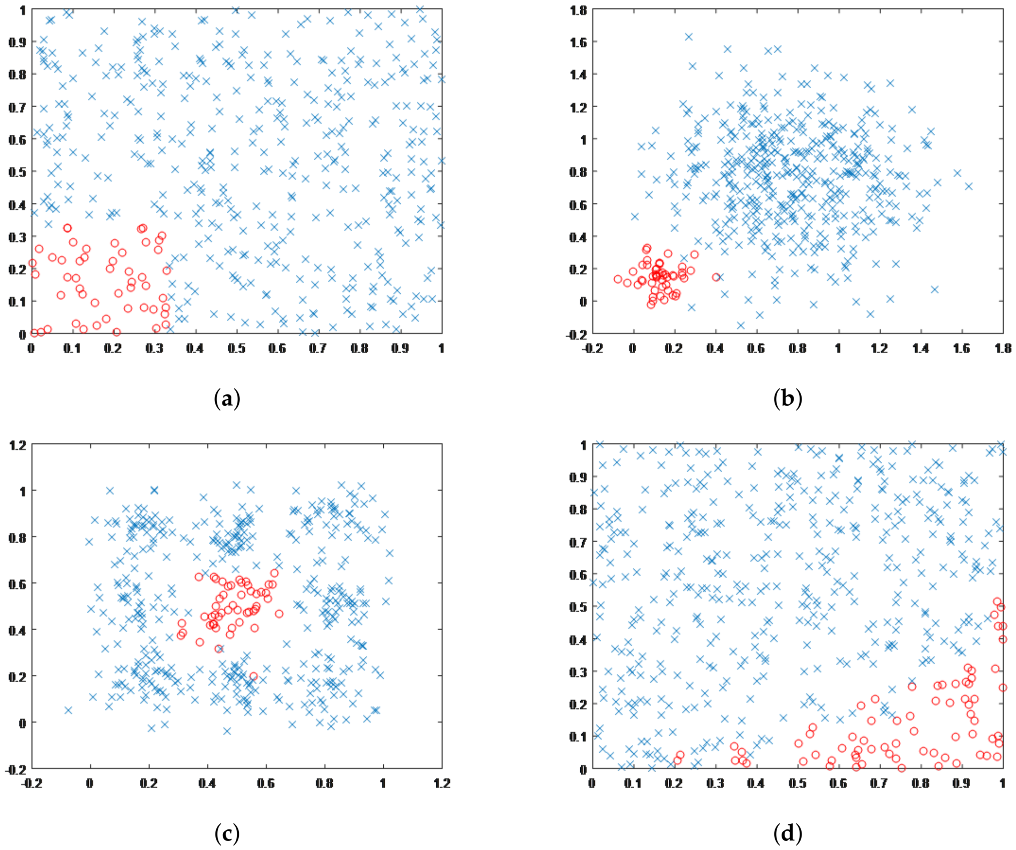

Figure 1. Four 2-dimensional artificial imbalance data sets (X_1, X_2): (**a**) uniform; (**b**) gaussian-1; (**c**) gaussian-2; and (**d**) complex.

Table 1. Comparison of weighted extreme learning machine (ELM) vs. NWELM on artificial data sets.

Data Sets	Weighted ELM	NWELM	Data Sets	Weighted ELM	NWELM
	G_{mean}	G_{mean}		G_{mean}	G_{mean}
Gaussian-1-1	0.9811	**0.9822**	Gaussian-2-1	0.9629	**0.9734**
Gaussian-1-2	0.9843	**0.9855**	Gaussian-2-2	0.9551	**0.9734**
Gaussian-1-3	0.9944	**0.9955**	Gaussian-2-3	0.9670	**0.9747**
Gaussian-1-4	0.9866	**0.9967**	Gaussian-2-4	0.9494	**0.9649**
Gaussian-1-5	**0.9866**	0.9833	Gaussian-2-5	0.9467	**0.9724**
Gaussian-1-6	**0.9899**	0.9685	Gaussian-2-6	0.9563	**0.9720**
Gaussian-1-7	**0.9833**	0.9685	Gaussian-2-7	0.9512	**0.9629**
Gaussian-1-8	0.9967	**0.9978**	Gaussian-2-8	0.9644	**0.9785**
Gaussian-1-9	**0.9944**	0.9798	Gaussian-2-9	0.9441	**0.9559**
Gaussian-1-10	0.9846	**0.9898**	Gaussian-2-10	0.9402	**0.9623**
Uniform-1	0.9836	**0.9874**	Complex-1	**0.9587**	0.9481
Uniform-2	**0.9798**	0.9750	Complex-2	**0.9529**	0.9466
Uniform-3	0.9760	**0.9823**	Complex-3	0.9587	**0.9608**
Uniform-4	0.9811	**0.9836**	Complex-4	**0.9482**	0.9061
Uniform-5	0.9811	**0.9823**	Complex-5	**0.9587**	0.9297
Uniform-6	0.9772	0.9772	Complex-6	0.9409	**0.9599**
Uniform-7	**0.9734**	0.9403	Complex-7	**0.9644**	0.9563
Uniform-8	0.9785	**0.9812**	Complex-8	**0.9575**	0.9553
Uniform-9	**0.9836**	0.9762	Complex-9	**0.9551**	0.9446
Uniform-10	0.9695	**0.9734**	Complex-10	0.9351	**0.9470**

3.2. Experiments on Real Data Set

In this section, we test the achievement of the proposed NWELM method on real data sets [39]. A total of 21 data sets with different numbers of features, training and test samples, and imbalance ratios are shown in Table 2. The selected data sets can be categorized into two classes according to their imbalance ratios. The first class has the imbalance ratio range of 0 to 0.2 and contains yeast-1-2-8-9_vs_7, abalone9_18, glass-0-1-6_vs_2, vowel0, yeast-0-5-6-7-9_vs_4, page-blocks0, yeast3, ecoli2, new-thyroid1 and the new-thyroid2 data sets.

Table 2. Real data sets and their attributes.

Data Sets	Features (#)	Training Data (#)	Test Data (#)	Imbalance Ratio
yeast-1-2-8-9_vs_7	8	757	188	0.0327
abalone9_18	8	584	147	0.0600
glass-0-1-6_vs_2	9	153	39	0.0929
vowel0	13	790	198	0.1002
yeast-0-5-6-7-9_vs_4	8	422	106	0.1047
page-blocks0	10	4377	1095	0.1137
yeast3	8	1187	297	0.1230
ecoli2	7	268	68	0.1806
new-thyroid1	5	172	43	0.1944
new-thyroid2	5	172	43	0.1944
ecoli1	7	268	68	0.2947
glass-0-1-2-3_vs_4-5-6	9	171	43	0.3053
vehicle0	18	676	170	0.3075
vehicle1	18	676	170	0.3439
haberman	3	244	62	0.3556
yeast1	8	1187	297	0.4064
glass0	9	173	43	0.4786
iris0	4	120	30	0.5000
pima	8	614	154	0.5350
wisconsin	9	546	137	0.5380
glass1	9	173	43	0.5405

On the other hand, second class contains the data sets, such as ecoli1, glass-0-1-2-3_vs_4-5-6, vehicle0, vehicle1, haberman, yeast, glass0, iris0, pima, wisconsin and glass1, that have imbalance ratio rates between 0.2 and 1.

The comparison results of the proposed NWELM with the weighted ELM, unweighted ELM and SVM are given in Table 3. As the weighted ELM method used a different weighting scheme (W_1, W_2), in our comparisons, we used the higher G_{mean} value. As can be seen in Table 3, the NWELM method yields higher G_{mean} values for 17 of the imbalanced data sets. For three of the data sets, both methods yield the same G_{mean}. Just for the page-blocks0 data set, the weighted ELM method yielded better results. It is worth mentioning that the NWELM method achieves 100% G_{mean} values for four data sets (vowel0, new-thyroid1, new-thyroid2, iris0). In addition, NWELM produced higher G_{mean} values than SVM for all data sets.

The obtained results were further evaluated by area under curve (AUC) values [40]. In addition, we compared the proposed method with unweighted ELM, weighted ELM and SVM based on the achieved AUC values as tabulated in Table 4. As seen in Table 4, for all examined data sets, our proposal's AUC values were higher than the compared other methods. For further comparisons of the proposed method with unweighted ELM, weighted ELM and SVM methods appropriately, statistical tests on AUC results were considered. The paired t-test was chosen [41]. The paired t-test results between each compared method and the proposed method for AUC was tabulated in Table 5 in terms of p-value. In Table 5, the results showing a significant advantage to the proposed method were shown in bold–face where p-values are equal or smaller than 0.05. Therefore, the proposed method performed better than the other methods in 39 tests out of 63 tests when each data set and pairs of methods are considered separately.

Table 3. Experimental results of binary data sets in terms of the G_{mean}. The best results on each data set are emphasized in bold-face.

	G_{mean}	Data (Imbalance Ratio)						
		Gaussian Kernel				Radial Base Kernel		
		Unweighted ELM		Weighted ELM max (W_1,W_2)		SVM	Neutrosophic Weighted ELM	
		C	G_{mean}(%)	C	G_{mean}(%)	G_{mean}(%)	C	G_{mean}(%)
imbalance ratio: 0, 0.2	yeast-1-2-8-9_vs_7 (0.0327)	2^{48}	60.97	2^{4}	71.41	47.88	2^{-7}	**77.57**
	abalone9_18 (0.0600)	2^{18}	72.71	2^{28}	89.76	51.50	2^{23}	**94.53**
	glass-0-1-6_vs_2 (0.0929)	2^{50}	63.20	2^{32}	83.59	51.26	2^{7}	**91.86**
	vowel0 (0.1002)	2^{-18}	100.00	2^{-18}	100.00	99.44	2^{7}	100.00
	yeast-0-5-6-7-9_vs_4 (0.1047)	2^{-6}	68.68	2^{4}	82.21	62.32	2^{-10}	**85.29**
	page-blocks0 (0.1137)	2^{4}	89.62	2^{16}	**93.61**	87.72	2^{20}	93.25
	yeast3 (0.1230)	2^{44}	84.13	2^{48}	93.11	84.71	2^{3}	**93.20**
	ecoli2 (0.1806)	2^{-18}	94.31	2^{8}	94.43	92.27	2^{10}	**95.16**
	new-thyroid1 (0.1944)	2^{0}	99.16	2^{14}	99.72	96.75	2^{7}	**100.00**
	new-thyroid2 (0.1944)	2^{2}	99.44	2^{12}	99.72	98.24	2^{7}	**100.00**
imbalance ratio: 0.2, 1	ecoli1 (0.2947)	2^{0}	88.75	2^{10}	91.04	87.73	2^{20}	**92.10**
	glass-0-1-2-3_vs_4-5-6 (0.3053)	2^{10}	93.26	2^{-18}	95.41	91.84	2^{7}	**95.68**
	vehicle0 (0.3075)	2^{8}	99.36	2^{20}	99.36	96.03	2^{10}	99.36
	vehicle1 (0.3439)	2^{18}	80.60	2^{24}	86.74	66.04	2^{10}	**88.06**
	haberman (0.3556)	2^{42}	57.23	2^{14}	66.26	37.35	2^{7}	**67.34**
	yeast1 (0.4064)	2^{0}	65.45	2^{10}	73.17	61.05	2^{10}	**73.19**
	glass0 (0.4786)	2^{0}	85.35	2^{0}	85.65	79.10	2^{13}	**85.92**
	iris0 (0.5000)	2^{-18}	100.00	2^{-18}	100.00	98.97	2^{10}	100.00
	pima (0.5350)	2^{0}	71.16	2^{8}	75.58	70.17	2^{10}	**76.35**
	wisconsin (0.5380)	2^{-2}	97.18	2^{8}	97.70	95.67	2^{7}	**98.22**
	glass1 (0.5405)	2^{-18}	77.48	2^{2}	80.35	69.64	2^{17}	**81.77**

Table 4. Experimental result of binary data sets in terms of the average area under curve (AUC). The best results on each data set are emphasized in bold-face.

	AUC	Data (Imbalance Ratio)						
		Gaussian Kernel				Radial Base Kernel		
		Unweighted ELM		Weighted ELM max (W_1,W_2)		SVM	Neutrosophic Weighted ELM	
		C	AUC (%)	C	AUC (%)	AUC (%)	C	AUC (%)
imbalance ratio: 0, 0.2	yeast-1-2-8-9_vs_7 (0.0327)	2^{48}	61.48	2^{4}	65.53	56.67	2^{-7}	74.48
	abalone9_18 (0.0600)	2^{18}	73.05	2^{28}	89.28	56.60	2^{23}	95.25
	glass-0-1-6_vs_2 (0.0929)	2^{50}	67.50	2^{32}	61.14	53.05	2^{7}	93.43
	vowel0 (0.1002)	2^{-18}	93.43	2^{-18}	99.22	99.44	2^{7}	99.94
	yeast-0-5-6-7-9_vs_4 (0.1047)	2^{-6}	66.35	2^{4}	80.09	69.88	2^{-10}	82.11
	page-blocks0 (0.1137)	2^{4}	67.42	2^{16}	71.55	88.38	2^{20}	91.49
	yeast3 (0.1230)	2^{44}	69.28	2^{48}	90.92	83.92	2^{3}	93.15
	ecoli2 (0.1806)	2^{-18}	71.15	2^{8}	94.34	92.49	2^{10}	94.98
	new-thyroid1 (0.1944)	2^{0}	90.87	2^{14}	98.02	96.87	2^{7}	100.00
	new-thyroid2 (0.1944)	2^{2}	84.29	2^{12}	96.63	98.29	2^{7}	100.00
imbalance ratio: 0.2, 1	ecoli1 (0.2947)	2^{0}	66.65	2^{10}	90.28	88.16	2^{20}	92.18
	glass-0-1-2-3_vs_4-5-6 (0.3053)	2^{10}	88.36	2^{-18}	93.94	92.02	2^{7}	95.86
	vehicle0 (0.3075)	2^{8}	71.44	2^{20}	62.41	96.11	2^{10}	98.69
	vehicle1 (0.3439)	2^{18}	58.43	2^{24}	51.80	69.10	2^{10}	88.63
	haberman (0.3556)	2^{42}	68.11	2^{14}	55.44	54.05	2^{7}	72.19
	yeast1 (0.4064)	2^{0}	56.06	2^{10}	70.03	66.01	2^{10}	73.66
	glass0 (0.4786)	2^{0}	74.22	2^{0}	75.99	79.81	2^{13}	81.41
	iris0 (0.5000)	2^{-18}	100.00	2^{-18}	100.00	99.00	2^{10}	100.00
	pima (0.5350)	2^{0}	59.65	2^{8}	50.01	71.81	2^{10}	75.21
	wisconsin (0.5380)	2^{-2}	83.87	2^{8}	80.94	95.68	2^{7}	98.01
	glass1 (0.5405)	2^{-18}	75.25	2^{2}	80.46	72.32	2^{17}	81.09

Table 5. Paired *t*-test results between each method and the proposed method for AUC results.

	Data Sets	Unweighted ELM	Weighted ELM	SVM
imbalance ratio: 0, 0.2	yeast-1-2-8-9_vs_7 (0.0327)	**0.0254**	0.0561	**0.0018**
	abalone9_18 (0.0600)	**0.0225**	0.0832	**0.0014**
	glass-0-1-6_vs_2 (0.0929)	**0.0119**	**0.0103**	**0.0006**
	vowel0 (0.1002)	**0.0010**	0.2450	0.4318
	yeast-0-5-6-7-9_vs_4 (0.1047)	**0.0218**	0.5834	0.0568
	page-blocks0 (0.1137)	**0.0000**	**0.0000**	**0.0195**
	yeast3 (0.1230)	**0.0008**	**0.0333**	**0.0001**
	ecoli2 (0.1806)	**0.0006**	0.0839	0.0806
	new-thyroid1 (0.1944)	**0.0326**	0.2089	0.1312
	new-thyroid2 (0.1944)	**0.0029**	0.0962	0.2855
imbalance ratio: 0.2, 1	ecoli1 (0.2947)	**0.0021**	0.1962	0.0744
	glass-0-1-2-3_vs_4-5-6 (0.3053)	0.0702	0.4319	**0.0424**
	vehicle0 (0.3075)	**0.0000**	**0.0001**	0.0875
	vehicle1 (0.3439)	**0.0000**	**0.0000**	**0.0001**
	haberman (0.3556)	0.1567	**0.0165**	**0.0007**
	yeast1 (0.4064)	**0.0001**	0.0621	**0.0003**
	glass0 (0.4786)	**0.0127**	0.1688	0.7072
	iris0 (0.5000)	NaN	NaN	0.3739
	pima (0.5350)	**0.0058**	**0.0000**	**0.0320**
	wisconsin (0.5380)	**0.0000**	**0.0002**	**0.0071**
	glass1 (0.5405)	**0.0485**	0.8608	**0.0293**

Another statistical test, namely the Friedman aligned ranks test, has been applied to compare the obtained results based on AUC values [42]. This test is a non-parametric test and the Holm method was chosen as the post hoc control method. The significance level was assigned 0.05. The statistics were obtained with the STAC tool [43] and recorded in Table 6. According to these results, the highest rank value was obtained by the proposed NWELM method and SVM and WELM rank values were greater than the ELM. In addition, the comparison's statistics, adjusted *p*-values and hypothesis results were given in Table 6.

Table 6. Friedman Aligned Ranks test (significance level of 0.05).

Statistic	*p*-Value	Result
29.6052	0.0000	H0 is rejected

Ranking

Algorithm	Rank
ELM	21.7619
WELM	38.9047
SVM	41.5238
NWELM	67.8095

Comparison	Statistic	Adjusted *p*-Value	Result
NWELM vs. ELM	6.1171	0.0000	H0 is rejected
NWELM vs. WELM	3.8398	0.0003	H0 is rejected
NWELM vs. SVM	3.4919	0.0005	H0 is rejected

We further compared the proposed NWELM method with two ensemble-based weighted ELM methods on 12 data sets [29]. The obtained results and the average classification G_{mean} values are recorded in Table 7. The best classification result for each data set is shown in bold text. A global view on the average classification performance shows that the NWELM yielded the highest average G_{mean} value against both the ensemble-based weighted ELM methods. In addition, the proposed NWELM method evidently outperforms the other two compared algorithms in terms of G_{mean} in 10 out of 12 data sets, with the only exceptions being the yeast3 and glass2 data sets.

As can be seen through careful observation, the NWELM method has not significantly improved the performance in terms of the glass1, haberman, yeast1_7 and abalone9_18 data sets, but slightly outperforms both ensemble-based weighted ELM methods.

Table 7. Comparison of the proposed method with two ensemble-based weighted ELM methods.

	Vote-Based Ensemble		DE-Based Ensemble		NWELM	
	C	G_{mean}(%)	C	G_{mean}(%)	C	G_{mean}(%)
glass1	2^{30}	74.32	2^{18}	77.72	2^{17}	**81.77**
haberman	2^{12}	63.10	2^{28}	62.68	2^{7}	**67.34**
ecoli1	2^{40}	89.72	2^{0}	91.39	2^{20}	**92.10**
new-thyroid2	2^{10}	99.47	2^{32}	99.24	2^{7}	**100.00**
yeast3	2^{4}	94.25	2^{2}	**94.57**	2^{3}	93.20
ecoli3	2^{10}	88.68	2^{18}	89.50	2^{17}	**92.16**
glass2	2^{8}	86.45	2^{16}	**87.51**	2^{7}	85.58
yeast1_7	2^{20}	78.95	2^{38}	78.94	2^{-6}	**84.66**
ecoli4	2^{8}	96.33	2^{14}	96.77	2^{10}	**98.85**
abalone9_18	2^{4}	89.24	2^{16}	90.13	2^{23}	**94.53**
glass5	2^{18}	94.55	2^{12}	94.55	2^{7}	**95.02**
yeast5	2^{12}	94.51	2^{28}	94.59	2^{17}	**98.13**
Average		87.46		88.13		**90.53**

A box plots illustration of the compared methods is shown in Figure 2. The box generated by the NWELM is shorter than the boxes generated by the compared vote-based ensemble and differential evolution (DE)- based ensemble methods. The dispersion degree of NWELM method is relatively low. It is worth noting that the box plots of all methods consider the G_{mean} of the haberman data set as an exception. Finally, the box plot determines the proposed NWELM method to be more robust when compared to the ensemble-based weighted ELM methods.

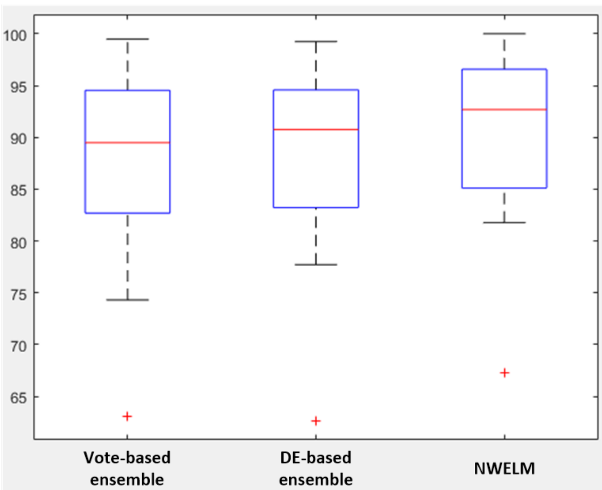

Figure 2. Box plots illustration of the compared methods.

4. Conclusions

In this paper, we propose a new weighted ELM model based on neutrosophic clustering. This new weighting scheme introduces true, indeterminacy and falsity memberships of each data point into ELM. Thus, we can remove the effect of noises and outliers in the classification stage and yield better classification results. Moreover, the proposed NWELM scheme can handle the problem of class imbalance more effectively. In the evaluation experiments, we compare the performance of the NWELM method with weighted ELM, unweighted ELM, and two ensemble-based weighted ELM methods. The experimental results demonstrate the NEWLM to be more effective than the compared methods for both artificial and real binary imbalance data sets. In the future, we are planning to extend our study to multiclass imbalance learning.

Author Contributions: Abdulkadir Sengur provided the idea of the proposed method. Yanhui Guo and Florentin Smarandache proved the theorems. Yaman Akbulut analyzed the model's application. Abdulkadir Sengur, Yanhui Guo, Yaman Akbulut, and Florentin Smarandache wrote the paper.

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Another Note on Paraconsistent Neutrosophic Sets

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Abstract: In an earlier paper, we proved that Smarandache's definition of neutrosophic paraconsistent topology is neither a generalization of Çoker's intuitionistic fuzzy topology nor a generalization of Smarandache's neutrosophic topology. Recently, Salama and Alblowi proposed a new definition of neutrosophic topology, that generalizes Çoker's intuitionistic fuzzy topology. Here, we study this new definition and its relation to Smarandache's paraconsistent neutrosophic sets.

Keywords: logic; set-theory; topology; Atanassov's intuitionistic fuzzy sets

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1. Introduction

In various papers, Smarandache [1,2] has generalized Atanassov's intuitionistic fuzzy sets [3] to neutrosophic sets.

Çoker [4] defined and studied intuitionistic fuzzy topological spaces.

On the other hand, various authors including Priest et al. [5] worked on paraconsistent logic, that is, logic where some contradiction is admissible. We refer the reader to studies of References [6–8] as well as the work on paraconsistent fuzzy logic conducted in Reference [9].

Smarandache [2] also defined neutrosophic paraconsistent sets, and proposed a natural definition of neutrosophic paraconsistent topology.

In an earlier paper [10], we proved that this Smarandache's definition of neutrosophic paraconsistent topology is neither a generalization of Çoker's intuitionistic fuzzy topology nor of Smarandache's general neutrosophic topology.

Recently, Salama and Alblowi [11] proposed a new definition of neutrosophic topology that generalizes Çoker's intuitionistic fuzzy topology.

In this paper, we study this new definition and its relation to Smarandache's paraconsistent neutrosophic sets.

The interest of neutrosophic paraconsistent topology was previously shown by us [12] (Section 4).

2. Materials and Methods

First, we present some basic definitions:

Robinson [13] developed the non-standard analysis, a formalization of analysis and a branch of mathematical logic, which rigorously defines infinitesimals. Formally, a number x is said to be infinitesimal if for all positive integers n, one has $|x| < 1/n$. Let $\varepsilon \geq 0$ be such an infinitesimal number. The hyper-real number set is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers. Let us consider the non-standard finite numbers $(1+) = 1 + \varepsilon$, where "1" is its standard part and "ε" its non-standard part, and $(-0) = 0 - \varepsilon$, where "0" is its standard part and "ε" its non-standard part. Then, we denote $]-0, 1+[$ to indicate a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely smaller but less than 0 or infinitely smaller but greater than 1, belong to the non-standard unit interval. It can be proven that S is a standard finite set if and only if every element of S is standard (See Reference [14]).

Definition 1. *In Reference [2], let T,I,F be real standard or non-standard subsets of the non-standard unit interval]−0,1+[, with*

$sup\ T = t_{sup}\ inf\ T = t_{inf},$
$sup\ I = i_{sup},\ inf\ I = i_{inf};$
$sup\ F = f_{sup}\ ,\ inf\ F = f_{inf}$ *and*
$n_{sup} = t_{sup} + i_{sup} + f_{sup}\ ,\ n_{inf} = t_{inf} + i_{inf} + f_{inf}.$

T, I, F are called neutrosophic components. Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T,I,F) and belongs to M in the following way: it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F. The set M is called a neutrosophic set (NS).

Definition 2. *In Reference [2], a neutrosophic set x (T,I,F) is called paraconsistent, if inf (T) + inf (I) + inf (F) > 1.*

Definition 3. *In Reference [11], the NSs 0_N and 1_N are defined as follows:*

0_N may be defined as:

(0_1) $0_N = x\ (0, 0, 1)$
(0_2) $0_N = x\ (0, 1, 1)$
(0_3) $0_N = x\ (0, 1, 0)$
(0_4) $0_N = x\ (0, 0, 0)$

1_N may be defined as:

(1_1) $1_N = x\ (1, 0, 0)$
(1_2) $1_N = x\ (1, 0, 1)$
(1_3) $1_N = x\ (1, 1, 0)$
(1_4) $1_N = x\ (1, 1, 1)$

Definition 4. *In Reference [11], let X be a non-empty set and $A = x\ (T_A, I_A, F_A)$, $B = x\ (T_B, I_B, F_B)$ be NSs. Then:*
A∩B may be defined as:

(I_1) $A \cap B = x\ (T_A \cdot T_B, I_A \cdot I_B, F_A \cdot F_B)$
(I_2) $A \cap B = x\ (T_A \wedge T_B, I_A \wedge I_B, F_A \vee F_B)$
(I_3) $A \cap B = x\ (T_A \wedge T_B, I_A \vee I_B, F_A \vee F_B)$

A∪B may be defined as:

(U_1) $A \cup B = x\ (T_A \vee T_B, I_A \vee I_B, F_A \wedge F_B)$
(U_2) $A \cup B = x\ (T_A \vee T_B, I_A \wedge I_B, F_A \wedge F_B)$

Definition 5. *In Reference [11], let $\{A_j | j \in J\}$ be an arbitrary family of NSs in X, then:*

(1) *$\cap A_j$ may be defined as:*

(i) $\cap A_j = x\ (\wedge, \wedge, \vee)$
(ii) $\cap A_j = x\ (\wedge, \vee, \vee)$

(2) *$\cup A_j$ may be defined as:*

(i) $\cup A_j = x\ (\vee, \vee, \wedge)$
(ii) $\cup A_j = x\ (\vee, \wedge, \wedge)$

Definition 6. *In Reference [11], a neutrosophic topology on a non-empty set X is a family τ of NSs in X satisfying the following properties:*

(1) 0_N *and* $1_N \epsilon \tau_\tau$*;*
(2) $G_1 \cap G_2 \in \tau$*, for any* $G_1, G_2 \in \tau$*;*
(3) $\cup G_j \in \tau$ *or any subfamily* $\{G_j\}_{j \in J}$ *of* τ*.*

In this case, the pair (**X**,$\boldsymbol{\tau}$) is called a neutrosophic topological space.

3. Results

Proposition 1. *The set of paraconsistent NSs with the definitions above is not a bounded lattice.*

Proof.

(1) It is necessary to omit a definition of $\cap$, because we will need $\cap$ of paraconsistent NSs to be paraconsistent. Indeed, let **A** = x (1/2, 1/2, 1/2) and B = x (1/2, 1/3, 1/3) (both are paraconsistent NSs), but 1/4 + 1/6 + 1/6 is not > 1. Then, the case with product ((I_1), in Definition 4) must be deleted for paraconsistent NSs.
(2) The definitions of 0_N and 1_N also have problems for paraconsistent NSs:
    (a) Only (0_2) and (1_2), (1_3), (1_4) are paraconsistent;
    (b) If we want all NSs: $0_N \cup 0_N$, $0_N \cup 1_N$, $1_N \cup 1_N$, $0_N \cap 0_N$, and $0_N \cap 1_N$ to be paraconsistent NSs, it is necessary to delete 1_2 in Definition 3, because with this definition,

$0_N \cap 1_N$ is equal either to x (0, 0, 1) which is not paraconsistent, or to x (0,1,1) = 0_N.

The other cases have no problems: $0_N \cup 0_N$ = x (0, 1, 1) = 0_N,

$0_N \cup 1_N$ is equal either to x (1,0,1), or to x (1,1,0), or x (1,1,1), i.e equal to 1_N,
$1_N \cup 1_N$ is equal either to x (1,0,1), or to x (1,1,0), or x (1,1,1), i.e equal to 1_N,
$0_N \cap 0_N$ = x (0,1,1) = 0_N,
$1_N \cap 1_N$ is equal either to x (1,0,1), or to x (1,1,0), or x (1,1,1), i.e equal to 1_N.

Then, after these changes in Definitions 3 and 4, Definition 6 is suitable for Smarandache's paraconsistent NSs, and one can work on paraconsistent neutrosophic topological spaces. □

Definition 7. *Let X be a non-empty set. A family τ of neutrosophic paraconsistent sets in X will be called a paraconsistent neutrosophic topology if:*

(1) 0_N *= x (0,1,1), and* 1_N *= x (1,1,0) or x (1,1,1), are in* τ*;*
(2) $G_1 \cap G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$ *(where* $\cap$ *is defined by (*I_2*) or (*I_3*));*
(3) $\cup G_j \in \tau$ *for any subfamily* $\{G_j\}_{j \in J}$ *of* τ *(where* $\cup$ *is defined by Definition 5).*

In this case, the pair (X,τ) is called a paraconsistent neutrosophic topological space.

Remark. *The above notion of paraconsistent neutrosophic topology generalizes Çoker's intuitionistic fuzzy topology when all sets are paraconsistent.*

4. Discussion

Definition 7 is suitable for the work on paraconsistent neutrosophic topological spaces. In fact:

Proposition 2. *The set of paraconsistent NSs with the following definitions,*

(a) 0_N *= x (0,1,1), and* 1_N *= x (1,1,0) or x (1,1,1)*
(b) $\cap$ *defined by (*I_2*) or (*I_3*)*
(c) $\cup$ *defined by Definition 5 is a bounded lattice.*

Proof. Obvious from proof of Proposition 1. □

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Merger and Acquisition Target Selection Based on Interval Neutrosophic Multigranulation Rough Sets over Two Universes

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Abstract: As a significant business activity, merger and acquisition (M&A) generally means transactions in which the ownership of companies, other business organizations or their operating units are transferred or combined. In a typical M&A procedure, M&A target selection is an important issue that tends to exert an increasingly significant impact on different business areas. Although some research works based on fuzzy methods have been explored on this issue, they can only deal with incomplete and uncertain information, but not inconsistent and indeterminate information that exists universally in the decision making process. Additionally, it is advantageous to solve M&A problems under the group decision making context. In order to handle these difficulties in M&A target selection background, we introduce a novel rough set model by combining interval neutrosophic sets (INSs) with multigranulation rough sets over two universes, called an interval neutrosophic (IN) multigranulation rough set over two universes. Then, we discuss the definition and some fundamental properties of the proposed model. Finally, we establish decision making rules and computing approaches for the proposed model in M&A target selection background, and the effectiveness of the decision making approach is demonstrated by an illustrative case analysis.

Keywords: merger and acquisition (M&A) target selection; group decision making; interval neutrosophic sets; multigranulation rough sets over two universes

1. Introduction

In the era of business intelligence, the development of computational intelligence approaches has far-reaching effects on business organizations' daily activities, including project management, human resource allocation optimization, merger and acquisition (M&A), and so on. As a key business activity in many organizations, M&A requires business administrators to make effective decisions by analyzing massive business data. Additionally, in M&A, one of the most important elements that determines the success ratio of business organizations is M&A target selection [1]. To deal with this issue, some efforts have been made through combining fuzzy approaches with the classical M&A research [2–5]. On the basis of the fuzzy set (FS) theory [6], fuzzy approaches have been widely used in realistic decision making problems. However, there is much uncertain information induced from various vague sources, and this often leads to some limitations when analyzing information systems through using FSs. Consequently, many new concepts of high-order FSs were established over the past decades. Among them, as a typical representative of generalized FSs, intuitionistic fuzzy sets (IFSs) [7] are characterized by a membership degree and a non-membership degree that describe whether one element belongs to a certain set or not, which provide a flexible framework to handle imprecise data, both complete and incomplete in nature. However, IFSs cannot cope with all kinds of uncertainties perfectly, such as problems including inconsistent and indeterminate information. Hence, it is necessary to develop some new theories.

Smarandache [8,9] presented the concept of neutrosophic logic and neutrosophic sets (NSs) from philosophical standpoints. Additionally, an NS is characterized by each element of the universe owning a degree of truth, indeterminacy and falsity, respectively. However, NSs can only be applied in philosophical problems. In order to utilize NSs easily in real-world situations, Wang et al. [10] constructed the definition and some operational laws of interval neutrosophic sets (INSs). Ever since the establishment of INSs, many scholars have studied INSs from different viewpoints and obtained an increasing number of academic achievements [11–21]. In light of the above, M&A target selection using IN information could handle uncertain situations and indeterminate information well and provide corporate acquirers with more exemplary and flexible access to convey their understandings about the M&A knowledge base.

As a typical model in the granular computing paradigm [22], the multigranulation rough set model [23,24] aims to analyze complicated problems from multiple views and levels, and it is seen as an efficient way to integrate and analyze information in group decision making procedures. Specifically, the advantages of utilizing multigranulation rough sets to solve group decision making problems can be summed up as follows:

1. In group decision making procedures, according to actual requirements of realistic problems, multigranulation rough set model-based computations consider multiple binary relations at the same time, which could increase the efficiency of the whole knowledge discovery process for multi-source information systems.
2. In light of different risk attitudes, the multigranulation rough set model can be divided into two parts, i.e., optimistic multigranulation rough sets [23] based on the "seeking common ground while reserving differences" strategy and pessimistic multigranulation rough sets [24] based on the "seeking common ground while eliminating differences" strategy. Thus, the multigranulation rough set model is suitable for solving risk-based group decision making problems.

Additionally, the classical rough set [25] is usually expressed and computed based on a single universe; this may lead to a limitation when describing group decision making information that is made up of multiple aspects. Through extending a single universe to two universes, it is beneficial to express a complicated group decision making knowledge base. Hence, Pei and Xu [26] studied the concept of rough sets over two universes systematically. Since then, several scholars have researched rough sets over two universes widely according to numerous practical requirements [27–33]. Recently, in order to expand the application scopes of rough sets over two universes from the ideas of granular computing, Sun and Ma [34] introduced multigranulation rough sets over two universes, which could not only describe real-life decision making information effectively and reasonably through different universes of discourse, but also integrate each expert's opinion to form an ultimate conclusion by aggregating multiple binary relations. Therefore, multigranulation rough sets over two universes constitute another approach to aid group decision making [35–40].

In this article, in order to handle the problems of IN data analysis and group decision making, it is necessary to introduce IN multigranulation rough sets over two universes through fusing multigranulation rough sets over two universes with INSs; both the general definition and some main properties of the proposed model are discussed. Then, we construct a new decision making method for M&A target selection problems by utilizing the proposed rough set model. Moreover, we give an illustrative case to interpret fundamental steps and a practical application to M&A target selection.

The rest of the article is structured as follows. In Section 2, we briefly introduce some concepts such as NSs, INSs and IN rough sets over two universes. Section 3 introduces IN multigranulation rough sets over two universes and some related properties. In Section 4, we establish decision making rules and an algorithm for M&A target selection problems. In Section 5, we give the steps of the proposed decision making approach by a case study, and conclusions and future research directions are illustrated in Section 6.

2. Preliminaries

In this section, we first review some fundamental concepts such as NSs, INSs and their properties. Next, we develop the definition of IN rough sets over two universes.

2.1. Neutrosophic Sets

NSs were defined by Smarandache [8] from philosophical standpoints. According to [8], NSs derive their origin from neutrosophy. In what follows, we review the concept of NSs.

Definition 1. *Let U be the universe of discourse, then an NS A can be expressed as the form $A = \{\langle x, \mu_A(x), \nu_A(x), \omega_A(x)\rangle | x \in U\}$, where the functions $\mu_A(x)$, $\nu_A(x)$ and $\omega_A(x)$ denote the degree of membership, the degree of indeterminacy and the degree of non-membership of the element $x \in U$ to the set A. Additionally, the functions $\mu_A(x)$, $\nu_A(x)$ and $\omega_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is $\mu_A(x) : U \to]0^-, 1^+[$, $\nu_A(x) : U \to]0^-, 1^+[$ and $\omega_A(x) : U \to]0^-, 1^+[$. There is no restriction on the sum of $\mu_A(x)$, $\nu_A(x)$ and $\omega_A(x)$. Thus, $0^- \le \sup \mu_A(x) + \sup \nu_A(x) + \sup \omega_A(x) \le 3^+$ [8].*

2.2. Interval Neutrosophic Sets

Since it is hard to utilize NSs in various practical situations, Wang et al. [10] developed the concept of INSs, which can be regarded as a subclass of NSs and a powerful structure in reflecting an expert's inconsistent and indeterminate preferences in real-life decision making procedures. In what follows, we present the concept of INSs.

Definition 2. *[10] Let U be the universe of discourse; an INS A is characterized by a truth-membership function $\mu_A(x)$, an indeterminacy-membership function $\nu_A(x)$ and a falsity-membership function $\omega_A(x)$. Then, an INS A can be denoted as the following mathematical symbol:*

$$A = \left\{\left\langle x, \left[\mu_A^L(x), \mu_A^U(x)\right], \left[\nu_A^L(x), \nu_A^U(x)\right], \left[\omega_A^L(x), \omega_A^U(x)\right]\right\rangle | x \in U\right\},$$

where $[\mu_A^L(x), \mu_A^U(x)]$, $[\nu_A^L(x), \nu_A^U(x)]$, $[\omega_A^L(x), \omega_A^U(x)] \subseteq [0,1]$ for all $x \in U$ to the set A. Thus, the sum of $\mu_A^U(x)$, $\nu_A^U(x)$ and $\omega_A^U(x)$ satisfies the condition: $0 \le \mu_A^U(x) + \nu_A^U(x) + \omega_A^U(x) \le 3$.

Suppose that U is the universe of discourse, then the set of all INSs on U is represented by $IN(U)$. Moreover, $\forall A \in IN(U)$. Based on the above definition, Wang et al. [10] defined the following operational laws on INSs.

Definition 3. *Let U be the universe of discourse, $\forall A, B \in IN(U)$, then [10]:*

1. *the complement of A is denoted by A^c such that $\forall x \in U$,* $A^c = \{\langle x, [\omega_A^L(x), \omega_A^U(x)], [1 - \nu_A^U(x), 1 - \nu_A^L(x)], [\mu_A^L(x), \mu_A^U(x)]\rangle | x \in U\}$;
2. *the intersection of A and B is denoted by $A \cap B$ such that $\forall x \in U$,* $A \cap B = \{\langle x, [\min(\mu_A^L(x), \mu_B^L(x)), \min(\mu_A^U(x), \mu_B^U(x))], [\max(\nu_A^L(x), \nu_B^L(x)), \max(\nu_A^U(x), \nu_B^U(x))], [\max(\omega_A^L(x), \omega_B^L(x)), \max(\omega_A^U(x), \omega_B^U(x))]\rangle | x \in U\}$;
3. *the union of A and B is denoted by $A \cup B$ such that $\forall x \in U$,* $A \cup B = \{\langle x, [\max(\mu_A^L(x), \mu_B^L(x)), \max(\mu_A^U(x), \mu_B^U(x))], [\min(\nu_A^L(x), \nu_B^L(x)), \min(\nu_A^U(x), \nu_B^U(x))], [\min(\omega_A^L(x), \omega_B^L(x)), \min(\omega_A^U(x), \omega_B^U(x))]\rangle | x \in U\}$;
4. *inclusion: $A \subseteq B$ if and only if $\mu_A^L(x) \le \mu_B^L(x)$, $\mu_A^U(x) \le \mu_B^U(x)$, $\nu_A^L(x) \ge \nu_B^L(x)$, $\nu_A^U(x) \ge \nu_B^U(x)$, $\omega_A^L(x) \ge \omega_B^L(x)$ and $\omega_A^U(x) \ge \omega_B^U(x)$ for any x in U;*

5. *equality:* $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

Next, we introduce some fundamental properties of the above operations, which state some important algebraic properties of the operations defined on INSs.

Theorem 1. *Let U be the universe of discourse; suppose that A, B and C are three INSs. Then, the followings are true [10]:*

1. *Double negation law:* $(A^c)^c = A$;
2. *De Morgan's laws:* $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$;
3. *Commutativity:* $A \cup B = B \cup A$, $A \cap B = B \cap A$;
4. *Associativity:* $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$;
5. *Distributivity:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

To compare the magnitude of different interval neutrosophic numbers (INNs), Zhang et al. [15] introduced the following comparison laws.

Definition 4. *Let* $x = \langle [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)], [\omega_A^L(x), \omega_A^U(x)] \rangle$ *be an INN; the score function of x is defined as follows [15]:*

$$s(x) = \left[\mu_A^L(x) + 1 - \nu_A^U(x) + 1 - \omega_A^U(x), \mu_A^U(x) + 1 - \nu_A^L(x) + 1 - \omega_A^L(x)\right].$$

It is noted that the score value is a significant index in ranking INNs. For an INN x, the bigger truth-membership $\mu_A(x)$ is, the less indeterminacy-membership $\nu_A(x)$ is, and the less falsity-membership $\omega_A(x)$ is, the greater INN x is.

2.3. Interval Neutrosophic Rough Sets over Two Universes

In this subsection, we introduce IN relations over two universes firstly.

Definition 5. *Let U, V be two non-empty and finite universes of discourse; an IN relation R from U to V can be denoted as the following form:*

$$R = \left\{ \left\langle (x,y), \left[\mu_R^L(x,y), \mu_R^U(x,y)\right], \left[\nu_R^L(x,y), \nu_R^U(x,y)\right], \left[\omega_R^L(x,y), \omega_R^U(x,y)\right] \right\rangle | (x,y) \in U \times V \right\},$$

where $[\mu_A^L(x), \mu_A^U(x)]$, $[\nu_A^L(x), \nu_A^U(x)]$, $[\omega_A^L(x), \omega_A^U(x)] \subseteq [0,1]$, *denoting the truth-membership, the indeterminacy-membership and the falsity-membership for all* $(x,y) \in U \times V$, *respectively. Furthermore, the family of all IN relations on* $U \times V$ *is represented by* $INR(U \times V)$.

According to IN relations over two universes, we present the definition of IN rough sets over two universes below.

Definition 6. *Let U, V be two non-empty and finite universes of discourse, and* $R \in INR(U \times V)$, *the pair* (U, V, R) *is called an IN approximation space over two universes. For any* $A \in IN(V)$, *the IN rough lower and upper approximations of A with respect to* (U, V, R), *denoted by* $\underline{R}(A)$ *and* $\overline{R}(A)$, *are defined as follows:*

$$\underline{R}(A) = \left\{ \left\langle x, \left[\mu_{\underline{R}(A)}^L(x), \mu_{\underline{R}(A)}^U(x)\right], \left[\nu_{\underline{R}(A)}^L(x), \nu_{\underline{R}(A)}^U(x)\right], \left[\omega_{\underline{R}(A)}^L(x), \omega_{\underline{R}(A)}^U(x)\right] \right\rangle | x \in U \right\},$$

$$\overline{R}(A) = \left\{ \left\langle x, \left[\mu_{\overline{R}(A)}^L(x), \mu_{\overline{R}(A)}^U(x)\right], \left[\nu_{\overline{R}(A)}^L(x), \nu_{\overline{R}(A)}^U(x)\right], \left[\omega_{\overline{R}(A)}^L(x), \omega_{\overline{R}(A)}^U(x)\right] \right\rangle | x \in U \right\},$$

where

$$\mu^L_{\underline{R}(A)}(x) = \wedge_{y\in V}\left\{\omega^L_R(x,y) \vee \mu^L_A(y)\right\}, \mu^U_{\underline{R}(A)}(x) = \wedge_{y\in V}\left\{\omega^U_R(x,y) \vee \mu^U_A(y)\right\},$$
$$\nu^L_{\underline{R}(A)}(x) = \vee_{y\in V}\left\{\left(1-\nu^U_R(x,y)\right) \wedge \nu^L_A(y)\right\}, \nu^U_{\underline{R}(A)}(x) = \vee_{y\in V}\left\{\left(1-\nu^L_R(x,y)\right) \wedge \nu^U_A(y)\right\},$$
$$\omega^L_{\underline{R}(A)}(x) = \vee_{y\in V}\left\{\mu^L_R(x,y) \wedge \omega^L_A(y)\right\}, \omega^U_{\underline{R}(A)}(x) = \vee_{y\in V}\left\{\mu^U_R(x,y) \wedge \omega^U_A(y)\right\},$$
$$\mu^L_{\overline{R}(A)}(x) = \vee_{y\in V}\left\{\mu^L_R(x,y) \wedge \mu^L_A(y)\right\}, \mu^U_{\overline{R}(A)}(x) = \vee_{y\in V}\left\{\mu^U_R(x,y) \wedge \mu^U_A(y)\right\},$$
$$\nu^L_{\overline{R}(A)}(x) = \wedge_{y\in V}\left\{\nu^L_R(x,y) \vee \nu^L_A(y)\right\}, \nu^U_{\overline{R}(A)}(x) = \wedge_{y\in V}\left\{\nu^U_R(x,y) \vee \nu^U_A(y)\right\},$$
$$\omega^L_{\overline{R}(A)}(x) = \wedge_{y\in V}\left\{\omega^L_R(x,y) \vee \omega^L_A(y)\right\}, \omega^U_{\overline{R}(A)}(x) = \wedge_{y\in V}\left\{\omega^U_R(x,y) \vee \omega^U_A(y)\right\}.$$

The pair $\left(\underline{R}(A), \overline{R}(A)\right)$ is called an IN rough set over two universes of A.

3. Interval Neutrosophic Multigranulation Rough Sets over Two Universes

In this section, we present IN relations over two universes from a single granulation to multiple granulations, both the definition and properties of IN multigranulation rough sets over two universes will be elaborated on.

3.1. Optimistic in Multigranulation Rough Sets over Two Universes

Definition 7. *Let U, V be two non-empty and finite universes of discourse and $R_i \in INR(U \times V)\,(i = 1, 2, \ldots, m)$ be m IN relations over $U \times V$; the pair (U, V, R_i) is called an IN multigranulation approximation space over two universes. For any $A \in IN(V)$, the optimistic IN multigranulation rough lower and upper approximations of A with respect to (U, V, R_i), denoted by $\underline{\sum_{i=1}^{m} R_i}^{O}(A)$ and $\overline{\sum_{i=1}^{m} R_i}^{O}(A)$, are defined as follows:*

$$\underline{\sum_{i=1}^{m} R_i}^{O}(A) = \left\{\left\langle x, \left[\mu^L_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \mu^U_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right], \left[\nu^L_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \nu^U_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right], \left[\omega^L_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \omega^U_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right]\right\rangle | x \in U\right\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{O}(A) = \left\{\left\langle x, \left[\mu^L_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \mu^U_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right], \left[\nu^L_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \nu^U_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right], \left[\omega^L_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x), \omega^U_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x)\right]\right\rangle | x \in U\right\},$$

where:

$$\mu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \omega^{L}_{R_i}(x,y) \vee \mu^{L}_{A}(y) \right\}, \mu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \omega^{U}_{R_i}(x,y) \vee \mu^{U}_{A}(y) \right\},$$

$$\nu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \left(1 - \nu^{U}_{R_i}(x,y)\right) \wedge \nu^{L}_{A}(y) \right\},$$

$$\nu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \left(1 - \nu^{L}_{R_i}(x,y)\right) \wedge \nu^{U}_{A}(y) \right\},$$

$$\omega^{L}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \mu^{L}_{R_i}(x,y) \wedge \omega^{L}_{A}(y) \right\}, \omega^{U}_{\underline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \mu^{U}_{R_i}(x,y) \wedge \omega^{U}_{A}(y) \right\},$$

$$\mu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \mu^{L}_{R_i}(x,y) \wedge \mu^{L}_{A}(y) \right\}, \mu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V} \left\{ \mu^{U}_{R_i}(x,y) \wedge \mu^{U}_{A}(y) \right\},$$

$$\nu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \nu^{L}_{R_i}(x,y) \vee \nu^{L}_{A}(y) \right\}, \nu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \nu^{U}_{R_i}(x,y) \vee \nu^{U}_{A}(y) \right\},$$

$$\omega^{L}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \omega^{L}_{R_i}(x,y) \vee \omega^{L}_{A}(y) \right\}, \omega^{U}_{\overline{\sum_{i=1}^{m} R_i}^{O}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V} \left\{ \omega^{U}_{R_i}(x,y) \vee \omega^{U}_{A}(y) \right\}.$$

The pair $\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A), \overline{\sum_{i=1}^{m} R_i}^{O}(A) \right)$ *is called an optimistic IN multigranulation rough set over two universes of A.*

Theorem 2. *Let U, V be two non-empty and finite universes of discourse and $R_i \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be m IN relations over $U \times V$. Then, for any $A, A' \in IN(V)$, the following properties hold:*

1. $\underline{\sum_{i=1}^{m} R_i}^{O}(A^c) = \left(\overline{\sum_{i=1}^{m} R_i}^{O}(A) \right)^c,$
   $\overline{\sum_{i=1}^{m} R_i}^{O}(A^c) = \left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \right)^c;$
2. $A \subseteq A' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{O}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A'),$
   $A \subseteq A' \Rightarrow \overline{\sum_{i=1}^{m} R_i}^{O}(A) \subseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A');$
3. $\underline{\sum_{i=1}^{m} R_i}^{O}(A \cap A') = \underline{\sum_{i=1}^{m} R_i}^{O}(A) \cap \underline{\sum_{i=1}^{m} R_i}^{O}(A'),$
   $\overline{\sum_{i=1}^{m} R_i}^{O}(A \cup A') = \overline{\sum_{i=1}^{m} R_i}^{O}(A) \cup \overline{\sum_{i=1}^{m} R_i}^{O}(A');$
4. $\underline{\sum_{i=1}^{m} R_i}^{O}(A \cup A') \supseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A) \cup \underline{\sum_{i=1}^{m} R_i}^{O}(A'),$
   $\overline{\sum_{i=1}^{m} R_i}^{O}(A \cap A') \subseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A) \cap \overline{\sum_{i=1}^{m} R_i}^{O}(A').$

Proof.

1. For all $x \in U$, we have:

$$\underline{\sum_{i=1}^{m} R_i}^{O}(A^c) =$$
$$\left\{\left\langle x, \left[\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{L}_{R_i}(x,y)\vee\mu^{L}_{A^c}(y)\right\}, \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{U}_{R_i}(x,y)\vee\mu^{U}_{A^c}(y)\right\}\right],\right.\right.$$
$$\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{U}_{R_i}(x,y)\right)\wedge\nu^{L}_{A^c}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{L}_{R_i}(x,y)\right)\wedge\nu^{U}_{A^c}(y)\right\}\right],$$
$$\left.\left.\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{L}_{R_i}(x,y)\wedge\omega^{L}_{A^c}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{U}_{R_i}(x,y)\wedge\omega^{U}_{A^c}(y)\right\}\right]\right\rangle | x\in U\right\}$$
$$= \left\{\left\langle x, \left[\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left(\sim\left\{\mu^{L}_{R_i}(x,y)\wedge\mu^{L}_{A}(y)\right\}\right), \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left(\sim\left\{\mu^{U}_{R_i}(x,y)\wedge\mu^{U}_{A}(y)\right\}\right)\right],\right.\right.$$
$$\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left(\sim\left\{\nu^{L}_{R_i}(x,y)\vee\nu^{L}_{A}(y)\right\}\right), \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left(\sim\left\{\nu^{U}_{R_i}(x,y)\vee\nu^{U}_{A}(y)\right\}\right)\right],$$
$$\left.\left.\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left(\sim\left\{\omega^{L}_{R_i}(x,y)\vee\omega^{L}_{A}(y)\right\}\right), \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left(\sim\left\{\omega^{U}_{R_i}(x,y)\vee\omega^{U}_{A}(y)\right\}\right)\right]\right\rangle | x\in U\right\}$$
$$= \left\{\left\langle x, \left[\sim\left(\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{L}_{R_i}(x,y)\wedge\mu^{L}_{A}(y)\right\}\right), \sim\left(\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{U}_{R_i}(x,y)\wedge\mu^{U}_{A}(y)\right\}\right)\right],\right.\right.$$
$$\left[\sim\left(\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\nu^{L}_{R_i}(x,y)\vee\nu^{L}_{A}(y)\right\}\right), \sim\left(\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\nu^{U}_{R_i}(x,y)\vee\nu^{U}_{A}(y)\right\}\right)\right],$$
$$\left.\left.\left[\sim\left(\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{L}_{R_i}(x,y)\vee\omega^{L}_{A}(y)\right\}\right), \sim\left(\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{U}_{R_i}(x,y)\vee\omega^{U}_{A}(y)\right\}\right)\right]\right\rangle | x\in U\right\}$$
$$= \left(\overline{\sum_{i=1}^{m} R_i}^{O}(A)\right)^{c}.$$

Thus, we obtain $\underline{\sum_{i=1}^{m} R_i}^{O}(A^c) = \left(\overline{\sum_{i=1}^{m} R_i}^{O}(A)\right)^{c}$. Then, $\overline{\sum_{i=1}^{m} R_i}^{O}(A^c) = \left(\underline{\sum_{i=1}^{m} R_i}^{O}(A)\right)^{c}$ is obtained in an identical fashion.

2. Since $A \subseteq A'$, we have $\mu^{L}_{A}(y) \leq \mu^{L}_{A'}(y)$, $\mu^{U}_{A}(y) \leq \mu^{U}_{A'}(y)$, $\nu^{L}_{A}(y) \geq \nu^{L}_{A'}(y)$, $\nu^{U}_{A}(y) \geq \nu^{U}_{A'}(y)$, $\omega^{L}_{A}(y) \geq \omega^{L}_{A'}(y)$ and $\omega^{U}_{A}(y) \geq \omega^{U}_{A'}(y)$.
So it follows that:

$$\left\{\left\langle x, \left[\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{L}_{R_i}(x,y)\vee\mu^{L}_{A}(y)\right\}, \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{U}_{R_i}(x,y)\vee\mu^{U}_{A}(y)\right\}\right],\right.\right.$$
$$\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{U}_{R_i}(x,y)\right)\wedge\nu^{L}_{A}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{L}_{R_i}(x,y)\right)\wedge\nu^{U}_{A}(y)\right\}\right],$$
$$\left.\left.\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{L}_{R_i}(x,y)\wedge\omega^{L}_{A}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{U}_{R_i}(x,y)\wedge\omega^{U}_{A}(y)\right\}\right]\right\rangle | x\in U\right\} \leq$$
$$\left\{\left\langle x, \left[\overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{L}_{R_i}(x,y)\vee\mu^{L}_{A'}(y)\right\}, \overset{m}{\underset{i=1}{\vee}} \wedge_{y\in V}\left\{\omega^{U}_{R_i}(x,y)\vee\mu^{U}_{A'}(y)\right\}\right],\right.\right.$$
$$\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{U}_{R_i}(x,y)\right)\wedge\nu^{L}_{A'}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\left(1-\nu^{L}_{R_i}(x,y)\right)\wedge\nu^{U}_{A'}(y)\right\}\right],$$
$$\left.\left.\left[\overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{L}_{R_i}(x,y)\wedge\omega^{L}_{A'}(y)\right\}, \overset{m}{\underset{i=1}{\wedge}} \vee_{y\in V}\left\{\mu^{U}_{R_i}(x,y)\wedge\omega^{U}_{A'}(y)\right\}\right]\right\rangle | x\in U\right\}.$$

Therefore, we have $A \subseteq A' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{O}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A')$. Then, $A \subseteq A' \Rightarrow \overline{\sum_{i=1}^{m} R_i}^{O}(A) \subseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A')$ is obtained in a similar manner.

3. In light of the previous results, it is easy to obtain:
$\underline{\sum_{i=1}^{m} R_i}^{O}(A \cap A') = \underline{\sum_{i=1}^{m} R_i}^{O}(A) \cap \underline{\sum_{i=1}^{m} R_i}^{O}(A'),$
$\overline{\sum_{i=1}^{m} R_i}^{O}(A \cup A') = \overline{\sum_{i=1}^{m} R_i}^{O}(A) \cup \overline{\sum_{i=1}^{m} R_i}^{O}(A').$
4. In light of the previous results, it is easy to obtain:
$\underline{\sum_{i=1}^{m} R_i}^{O}(A \cup A') \supseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A) \cup \underline{\sum_{i=1}^{m} R_i}^{O}(A'),$
$\overline{\sum_{i=1}^{m} R_i}^{O}(A \cap A') \subseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A) \cap \overline{\sum_{i=1}^{m} R_i}^{O}(A').$

□

Theorem 3. *Let U, V be two non-empty and finite universes of discourse and $R_i, R_i' \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be two IN relations over $U \times V$. If $R_i \subseteq R_i'$, for any $A \in IN(V)$, the following properties hold:*

1. $\underline{\sum_{i=1}^{m} R_i'}^{O}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A)$, *for all $A \in IN(V)$;*
2. $\overline{\sum_{i=1}^{m} R_i'}^{O}(A) \supseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A)$, *for all $A \in IN(V)$.*

Proof. Since $R_i \subseteq R_i'$, we have $\mu^L_{R_i}(x,y) \le \mu^L_{R_i'}(x,y)$, $\mu^U_{R_i}(x,y) \le \mu^U_{R_i'}(x,y)$, $\nu^L_{R_i}(x,y) \ge \nu^L_{R_i'}(x,y)$, $\nu^U_{R_i}(x,y) \ge \nu^U_{R_i'}(x,y)$, $\omega^L_{R_i}(x,y) \ge \omega^L_{R_i'}(x,y)$ and $\omega^U_{R_i}(x,y) \ge \omega^U_{R_i'}(x,y)$.

Therefore, it follows that:

$$\begin{aligned}
&\Big\{\Big\langle x, \Big[\bigvee_{i=1}^{m} \wedge_{y\in V}\left\{\omega^L_{R_i}(x,y) \vee \mu^L_A(y)\right\}, \bigvee_{i=1}^{m} \wedge_{y\in V}\left\{\omega^U_{R_i}(x,y) \vee \mu^U_A(y)\right\}\Big],\\
&\Big[\bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\left(1-\nu^U_{R_i}(x,y)\right) \wedge \nu^L_A(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\left(1-\nu^L_{R_i}(x,y)\right) \wedge \nu^U_A(y)\right\}\Big],\\
&\Big[\bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\mu^L_{R_i}(x,y) \wedge \omega^L_A(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\mu^U_{R_i}(x,y) \wedge \omega^U_A(y)\right\}\Big]\Big\rangle \mid x \in U\Big\} \ge\\
&\Big\{\Big\langle x, \Big[\bigvee_{i=1}^{m} \wedge_{y\in V}\left\{\omega^L_{R_i'}(x,y) \vee \mu^L_A(y)\right\}, \bigvee_{i=1}^{m} \wedge_{y\in V}\left\{\omega^U_{R_i'}(x,y) \vee \mu^U_A(y)\right\}\Big],\\
&\Big[\bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\left(1-\nu^U_{R_i'}(x,y)\right) \wedge \nu^L_A(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\left(1-\nu^L_{R_i'}(x,y)\right) \wedge \nu^U_A(y)\right\}\Big],\\
&\Big[\bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\mu^L_{R_i'}(x,y) \wedge \omega^L_A(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y\in V}\left\{\mu^U_{R_i'}(x,y) \wedge \omega^U_A(y)\right\}\Big]\Big\rangle \mid x \in U\Big\}.
\end{aligned}$$

Thus, we obtain $\underline{\sum_{i=1}^{m} R_i'}^{O}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A)$. Then, $\overline{\sum_{i=1}^{m} R_i'}^{O}(A) \supseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A)$ is obtained in an identical fashion. □

3.2. Pessimistic in Multigranulation Rough Sets over Two Universes

Definition 8. *Let U, V be two non-empty and finite universes of discourse and $R_i \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be m IN relations over $U \times V$; the pair (U, V, R_i) is called an IN multigranulation approximation space over two universes. For any $A \in IN(V)$, the pessimistic IN* *multigranulation rough lower and upper approximations of A with respect to* (U, V, R_i)*, denoted by* $\underline{\sum_{i=1}^{m} R_i}^{P}(A)$ *and* $\overline{\sum_{i=1}^{m} R_i}^{P}(A)$*, are defined as follows:*

$$\underline{\sum_{i=1}^{m} R_i}^{P}(A) = \left\{ \left\langle x, \left[\mu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \mu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right], \left[\nu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \nu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right], \left[\omega^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \omega^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right] \right\rangle | x \in U \right\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{P}(A) = \left\{ \left\langle x, \left[\mu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \mu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right], \left[\nu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \nu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right], \left[\omega^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x), \omega^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x)\right] \right\rangle | x \in U \right\},$$

where:

$$\mu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\omega^{L}_{R_i}(x,y) \vee \mu^{L}_{A}(y)\right\}, \mu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\omega^{U}_{R_i}(x,y) \vee \mu^{U}_{A}(y)\right\},$$

$$\nu^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\left(1 - \nu^{U}_{R_i}(x,y)\right) \wedge \nu^{L}_{A}(y)\right\},$$

$$\nu^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\left(1 - \nu^{L}_{R_i}(x,y)\right) \wedge \nu^{U}_{A}(y)\right\},$$

$$\omega^{L}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\mu^{L}_{R_i}(x,y) \wedge \omega^{L}_{A}(y)\right\}, \omega^{U}_{\underline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\mu^{U}_{R_i}(x,y) \wedge \omega^{U}_{A}(y)\right\},$$

$$\mu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\mu^{L}_{R_i}(x,y) \wedge \mu^{L}_{A}(y)\right\}, \mu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigvee_{i=1}^{m} \vee_{y \in V} \left\{\mu^{U}_{R_i}(x,y) \wedge \mu^{U}_{A}(y)\right\},$$

$$\nu^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\nu^{L}_{R_i}(x,y) \vee \nu^{L}_{A}(y)\right\}, \nu^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\nu^{U}_{R_i}(x,y) \vee \nu^{U}_{A}(y)\right\},$$

$$\omega^{L}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\omega^{L}_{R_i}(x,y) \vee \omega^{L}_{A}(y)\right\}, \omega^{U}_{\overline{\sum_{i=1}^{m} R_i}^{P}(A)}(x) = \bigwedge_{i=1}^{m} \wedge_{y \in V} \left\{\omega^{U}_{R_i}(x,y) \vee \omega^{U}_{A}(y)\right\}.$$

The pair $\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A), \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$ *is called a pessimistic IN multigranulation rough set over two universes of A.*

Theorem 4. *Let U, V be two non-empty and finite universes of discourse and* $R_i \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ *be m IN relations over* $U \times V$*. Then, for any* $A, A' \in IN(V)$*, the following properties hold:*

1. $\underline{\sum_{i=1}^{m} R_i}^{P}(A^c) = \left(\overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)^c$,
$\overline{\sum_{i=1}^{m} R_i}^{P}(A^c) = \left(\underline{\sum_{i=1}^{m} R_i}^{P}(A)\right)^c$;
2. $A \subseteq A' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{P}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{P}(A')$,
$A \subseteq A' \Rightarrow \overline{\sum_{i=1}^{m} R_i}^{P}(A) \subseteq \overline{\sum_{i=1}^{m} R_i}^{P}(A')$;
3. $\underline{\sum_{i=1}^{m} R_i}^{P}(A \cap A') = \underline{\sum_{i=1}^{m} R_i}^{P}(A) \cap \underline{\sum_{i=1}^{m} R_i}^{P}(A')$,
$\overline{\sum_{i=1}^{m} R_i}^{P}(A \cup A') = \overline{\sum_{i=1}^{m} R_i}^{P}(A) \cup \overline{\sum_{i=1}^{m} R_i}^{P}(A')$;
4. $\underline{\sum_{i=1}^{m} R_i}^{P}(A \cup A') \supseteq \underline{\sum_{i=1}^{m} R_i}^{P}(A) \cup \underline{\sum_{i=1}^{m} R_i}^{P}(A')$,
$\overline{\sum_{i=1}^{m} R_i}^{P}(A \cap A') \subseteq \overline{\sum_{i=1}^{m} R_i}^{P}(A) \cap \overline{\sum_{i=1}^{m} R_i}^{P}(A')$.

Theorem 5. *Let U, V be two non-empty and finite universes of discourse and $R_i, R_i' \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be two IN relations over $U \times V$. If $R_i \subseteq R_i'$, for any $A \in IN(V)$, the following properties hold:*

1. $\underline{\sum_{i=1}^{m} R_i'}^{P}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{P}(A)$, *for all* $A \in IN(V)$;
2. $\overline{\sum_{i=1}^{m} R_i'}^{P}(A) \supseteq \overline{\sum_{i=1}^{m} R_i}^{P}(A)$, *for all* $A \in IN(V)$.

3.3. Relationships between Optimistic and Pessimistic in Multigranulation Rough Sets over Two Universes

Theorem 6. *Let U, V be two non-empty and finite universes of discourse and $R_i \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be m IN relations over $U \times V$. Then, for any $A \in IN(V)$, the following properties hold:*

1. $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A)$;
2. $\overline{\sum_{i=1}^{m} R_i}^{P}(A) \supseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A)$.

Proof. For any $x \in U$, we have:

$$\underline{\sum_{i=1}^{m} R_i}^{O}(A) = \left\{\left\langle x, \left[\bigvee_{i=1}^{m} \wedge_{y \in V}\left\{\omega_{R_i}^{L}(x,y) \vee \mu_A^{L}(y)\right\}, \bigvee_{i=1}^{m} \wedge_{y \in V}\left\{\omega_{R_i}^{U}(x,y) \vee \mu_A^{U}(y)\right\}\right],\right.\right.$$
$$\left[\bigwedge_{i=1}^{m} \vee_{y \in V}\left\{\left(1-\nu_{R_i}^{U}(x,y)\right) \wedge \nu_A^{L}(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y \in V}\left\{\left(1-\nu_{R_i}^{L}(x,y)\right) \wedge \nu_A^{U}(y)\right\}\right],$$
$$\left.\left.\left[\bigwedge_{i=1}^{m} \vee_{y \in V}\left\{\mu_{R_i}^{L}(x,y) \wedge \omega_A^{L}(y)\right\}, \bigwedge_{i=1}^{m} \vee_{y \in V}\left\{\mu_{R_i}^{U}(x,y) \wedge \omega_A^{U}(y)\right\}\right]\right\rangle \mid x \in U\right\}$$
$$\geq \left\{\left\langle x, \left[\bigwedge_{i=1}^{m} \wedge_{y \in V}\left\{\omega_{R_i}^{L}(x,y) \vee \mu_A^{L}(y)\right\}, \bigwedge_{i=1}^{m} \wedge_{y \in V}\left\{\omega_{R_i}^{U}(x,y) \vee \mu_A^{U}(y)\right\}\right],\right.\right.$$
$$\left[\bigvee_{i=1}^{m} \vee_{y \in V}\left\{\left(1-\nu_{R_i}^{U}(x,y)\right) \wedge \nu_A^{L}(y)\right\}, \bigvee_{i=1}^{m} \vee_{y \in V}\left\{\left(1-\nu_{R_i}^{L}(x,y)\right) \wedge \nu_A^{U}(y)\right\}\right],$$
$$\left.\left.\left[\bigvee_{i=1}^{m} \vee_{y \in V}\left\{\mu_{R_i}^{L}(x,y) \wedge \omega_A^{L}(y)\right\}, \bigvee_{i=1}^{m} \vee_{y \in V}\left\{\mu_{R_i}^{U}(x,y) \wedge \omega_A^{U}(y)\right\}\right]\right\rangle \mid x \in U\right\} = \underline{\sum_{i=1}^{m} R_i}^{P}(A).$$

Therefore, we have $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O}(A)$. Then, $\overline{\sum_{i=1}^{m} R_i}^{P}(A) \supseteq \overline{\sum_{i=1}^{m} R_i}^{O}(A)$ is obtained in an identical fashion. □

4. The Model and Approach of Merger and Acquisition Target Selection

In this section, in order to explore M&A target selection, we develop a new method by utilizing IN multigranulation rough sets over two universes. Specifically, some main points of the established decision making method can be summarized in the following subsections.

4.1. The Application Model

Suppose that $U = \{x_1, x_2, \ldots, x_j\}$ is a set of optional M&A targets and $V = \{y_1, y_2, \ldots, y_k\}$ is a set of evaluation factors. Let $R_i \in INR(U \times V)$ $(i = 1, 2, \ldots, m)$ be m IN relations, which reflect the related degree between some optional M&A targets and evaluation factors provided by m decision makers. In what follows, suppose that $A \in IN(V)$ is a standard set, representing some requirements of corporate acquirers. Thus, we construct an IN information system that is denoted as (U, V, R_i, A). On the basis of the above descriptions, we calculate the optimistic and pessimistic IN multigranulation rough lower and upper approximations of A according to Definitions 7 and Definition 8, respectively. Then, with reference to operational laws developed in [15]: $\forall A, B \in IN(U)$, we have $A \oplus B = \{\langle x, [\mu_A^L(x) + \mu_B^L(x) - \mu_A^L(x)\mu_B^L(x), \mu_A^U(x) + \mu_B^U(x) - \mu_A^U(x)\mu_B^U(x)], [\nu_A^L(x)\nu_B^L(x), \nu_A^U(x)\nu_B^U(x)], [\omega_A^L(x)\omega_B^L(x), \omega_A^U(x)\omega_B^U(x)]\rangle \mid x \in U\}$, $\lambda A = \left\{\left\langle x, \left[1 - (1 - \mu_A^L(x))^{\lambda}, 1 - (1 - \mu_A^U(x))^{\lambda}\right], \left[(\nu_A^L(x))^{\lambda}, (\nu_A^U(x))^{\lambda}\right], \left[(\omega_A^L(x))^{\lambda}, (\omega_A^U(x))^{\lambda}\right]\right\rangle \mid x \in U\right\}$.

By virtue of the above operational laws, we further calculate the sets $\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)$ and $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)$. Since optimistic multigranulation rough sets are based on the "seeking common ground while reserving differences" strategy, which implies that one reserves both common decisions and inconsistent decisions at the same time, thus, the set $\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)$ can be seen as a risk-seeking decision strategy. Similarly, pessimistic multigranulation rough sets are based on the "seeking common ground while eliminating differences" strategy, which implies that one reserves common decisions while deleting inconsistent decisions. Hence, the set $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)$ can be seen as a risk-averse decision strategy.

In realistic M&A target selection procedures, the enterprise's person in charge can choose the optimal M&A target through referring to the above mentioned risk-seeking and risk-averse decision strategies. In what follows, in order to make better use of those decision strategies, we present a compromise decision strategy with a risk coefficient of decision makers, and the risk coefficient is denoted as $\lambda\,(\lambda \in [0,1])$. Based on this, the compromise decision strategy can be described as $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$. Additionally, it is noted that the compromise decision strategy reduces to the risk-seeking decision strategy and the risk-averse decision strategy when $\lambda = 1$ and $\lambda = 0$, respectively. Moreover, the larger λ is, business administrators are more prone to seek the maximum of risk, while the smaller λ is, business administrators are more prone to seek the minimum of risk. Finally, by virtue of the risk coefficient, the optimal decision result can be obtained by selecting the M&A target with the largest score value in $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$.

4.2. Algorithm for Merger and Acquisition Target Selection Used IN Multigranulation Rough Sets over Two Universes

In what follows, we present an algorithm for M&A target selection based on IN multigranulation rough sets over two universes.

Algorithm 1 M&A target selection based on IN multigranulation rough sets over two universes

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Require: The relations between the universes U and V provided by multiple experts (U, V, R_i) and a standard set A.

Ensure: The determined M&A target.

Step 1. Calculate the following sets: $\underline{\sum_{i=1}^{m} R_i}^{O}(A)$, $\overline{\sum_{i=1}^{m} R_i}^{O}(A)$, $\underline{\sum_{i=1}^{m} R_i}^{P}(A)$ and $\overline{\sum_{i=1}^{m} R_i}^{P}(A)$, respectively;

Step 2. Calculate the following sets: $\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)$, $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)$ and $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$, respectively;

Step 3. Determine the score values for optional M&A targets in compromise decision strategy $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$;

Step 4. The optimal solution is the M&A target with the largest score value in $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$.

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5. Numerical Example

In this section, by utilizing an illustrative case analysis that was previously modeled in [2], we show fundamental steps of the proposed decision making method in M&A target selection background.

5.1. Case Description

Suppose there is a steel corporate acquirer who aims to evaluate which target organization is suitable for the acquiring firm. In order to reflect the fairness of the procedure, the acquiring firm invites three specialists to establish M&A target selection information systems, denoted as (U, V, R_i, A). Suppose that $U = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of optional M&A targets and another universe $V = \{y_1, y_2, y_3, y_4, y_5\}$ is a set of evaluation factors, where the evaluation factor represents mineral output, mining difficulty, proved reserves, reserve-production ratio, science and technology contribution rate, respectively. Based on the above, the M&A knowledge base can be composed of two kinds of objections related to M&A target selection, i.e., the optional M&A targets set and the evaluation factors set. In order to solve this group decision making problem, each decision maker provides his or her own thought about the M&A knowledge base that is shown as the following Tables 1–3.

Moreover, assume that decision makers provide a standard set that is represented by the following INS:

$$\begin{aligned} A = \; & \{\langle y_1, \langle [0.7, 0.8], [0.4, 0.6], [0.2, 0.3] \rangle\rangle, \langle y_2, \langle [0.5, 0.6], [0.3, 0.4], [0.6, 0.7] \rangle\rangle, \\ & \langle y_3, \langle [0.7, 0.8], [0.2, 0.3], [0.4, 0.5] \rangle\rangle, \langle y_4, \langle [0.2, 0.3], [0.5, 0.7], [0.6, 0.7] \rangle\rangle, \\ & \langle y_5, \langle [0.4, 0.5], [0.2, 0.3], [0.8, 0.9] \rangle\rangle\}. \end{aligned}$$

5.2. Decision Making Process

According to Algorithm 1, we aim to seek the determined M&A target by utilizing the proposed model. At first, by virtue of Definitions 7 and 8, we calculate the optimistic and pessimistic IN multigranulation rough lower and upper approximations of A:

$$\underline{\sum_{i=1}^{3} R_i}^{O}(A) = \{\langle x_1, \langle [0.4, 0.5], [0.5, 0.6], [0.6, 0.7]\rangle\rangle, \langle x_2, \langle [0.4, 0.5], [0.5, 0.7], [0.7, 0.8]\rangle\rangle, \langle x_3, \langle [0.5, 0.6], [0.4, 0.6], [0.6, 0.7]\rangle\rangle, \langle x_4, \langle [0.4, 0.5], [0.5, 0.6], [0.6, 0.7]\rangle\rangle, \langle x_5, \langle [0.2, 0.3], [0.5, 0.6], [0.7, 0.8]\rangle\rangle\};$$

$$\overline{\sum_{i=1}^{3} R_i}^{O}(A) = \{\langle x_1, \langle [0.7, 0.8], [0.3, 0.4], [0.4, 0.5]\rangle\rangle, \langle x_2, \langle [0.5, 0.6], [0.3, 0.5], [0.6, 0.7]\rangle\rangle, \langle x_3, \langle [0.7, 0.8], [0.2, 0.3], [0.5, 0.6]\rangle\rangle, \langle x_4, \langle [0.7, 0.8], [0.2, 0.3], [0.5, 0.6]\rangle\rangle, \langle x_5, \langle [0.7, 0.8], [0.3, 0.4], [0.2, 0.3]\rangle\rangle\};$$

$$\underline{\sum_{i=1}^{3} R_i}^{P}(A) = \{\langle x_1, \langle [0.4, 0.5], [0.5, 0.7], [0.6, 0.7]\rangle\rangle, \langle x_2, \langle [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]\rangle\rangle, \langle x_3, \langle [0.5, 0.6], [0.4, 0.6], [0.6, 0.7]\rangle\rangle, \langle x_4, \langle [0.3, 0.4], [0.5, 0.7], [0.7, 0.8]\rangle\rangle, \langle x_5, \langle [0.2, 0.3], [0.5, 0.7], [0.8, 0.9]\rangle\rangle\};$$

$$\overline{\sum_{i=1}^{3} R_i}^{P}(A) = \{\langle x_1, \langle [0.7, 0.8], [0.2, 0.3], [0.4, 0.5]\rangle\rangle, \langle x_2, \langle [0.5, 0.7], [0.3, 0.4], [0.5, 0.6]\rangle\rangle, \langle x_3, \langle [0.7, 0.8], [0.2, 0.3], [0.4, 0.5]\rangle\rangle, \langle x_4, \langle [0.7, 0.8], [0.2, 0.3], [0.4, 0.5]\rangle\rangle, \langle x_5, \langle [0.7, 0.8], [0.3, 0.4], [0.2, 0.3]\rangle\rangle\}.$$

By virtue of the above results, we further calculated the sets $\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)$ and $\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)$.

$$\underline{\sum_{i=1}^{3} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(A) = \{\langle x_1, \langle [0.82, 0.90], [0.15, 0.24], [0.24, 0.35]\rangle\rangle, \langle x_2, \langle [0.70, 0.80], [0.15, 0.35], [0.42, 0.56]\rangle\rangle, \langle x_3, \langle [0.85, 0.92], [0.08, 0.18], [0.30, 0.42]\rangle\rangle, \langle x_4, \langle [0.82, 0.90], [0.10, 0.18], [0.30, 0.42]\rangle\rangle, \langle x_5, \langle [0.76, 0.86], [0.15, 0.24], [0.14, 0.24]\rangle\rangle\},$$

$$\underline{\sum_{i=1}^{3} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(A) = \{\langle x_1, \langle [0.82, 0.90], [0.10, 0.21], [0.24, 0.35]\rangle\rangle, \langle x_2, \langle [0.70, 0.85], [0.15, 0.28], [0.40, 0.54]\rangle\rangle, \langle x_3, \langle [0.85, 0.92], [0.08, 0.18], [0.24, 0.35]\rangle\rangle, \langle x_4, \langle [0.79, 0.88], [0.10, 0.21], [0.28, 0.40]\rangle\rangle, \langle x_5, \langle [0.76, 0.86], [0.15, 0.28], [0.16, 0.27]\rangle\rangle\}.$$

Next, suppose business managers take $\lambda = 0.6$, then $\lambda\left(\underline{\sum_{i=1}^{m} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{O}(A)\right) \oplus (1-\lambda)\left(\underline{\sum_{i=1}^{m} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{m} R_i}^{P}(A)\right)$ can be obtained as follows:

$$0.6\left(\underline{\sum_{i=1}^{3} R_i}^{O}(A) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(A)\right) \oplus (1-0.6)\left(\underline{\sum_{i=1}^{3} R_i}^{P}(A) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(A)\right) = \{\langle x_1, \langle [0.82, 0.90], [0.13, 0.23], [0.24, 0.35]\rangle\rangle, \langle x_2, \langle [0.70, 0.82], [0.15, 0.32], [0.41, 0.55]\rangle\rangle, \langle x_3, \langle [0.85, 0.92], [0.08, 0.18], [0.28, 0.39]\rangle\rangle, \langle x_4, \langle [0.81, 0.89], [0.10, 0.19], [0.29, 0.41]\rangle\rangle, \langle x_5, \langle [0.76, 0.86], [0.15, 0.25], [0.15, 0.25]\rangle\rangle\}.$$

Table 1. The knowledge of merger and acquisition (M&A) target selection given by Expert 1.

R_1	y_1	y_2	y_3	y_4	y_5
x_1	$\langle[0.8,0.9],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.3,0.4],[0.3,0.4],[0.8,0.9]\rangle$	$\langle[0.6,0.7],[0.5,0.6],[0.3,0.4]\rangle$	$\langle[0.6,0.7],[0.4,0.5],[0.4,0.5]\rangle$	$\langle[0.2,0.3],[0.2,0.3],[0.8,0.9]\rangle$
x_2	$\langle[0.4,0.5],[0.3,0.4],[0.7,0.8]\rangle$	$\langle[0.3,0.4],[0.4,0.5],[0.7,0.8]\rangle$	$\langle[0.5,0.6],[0.4,0.5],[0.6,0.7]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.5,0.6]\rangle$	$\langle[0.7,0.8],[0.3,0.5],[0.3,0.4]\rangle$
x_3	$\langle[0.7,0.8],[0.4,0.5],[0.5,0.6]\rangle$	$\langle[0.7,0.9],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.3,0.4],[0.5,0.6],[0.6,0.7]\rangle$	$\langle[0.2,0.3],[0.6,0.7],[0.6,0.7]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.7,0.8]\rangle$
x_4	$\langle[0.8,0.9],[0.2,0.4],[0.4,0.6]\rangle$	$\langle[0.5,0.6],[0.3,0.4],[0.6,0.7]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.7,0.8]\rangle$	$\langle[0.6,0.7],[0.4,0.5],[0.3,0.5]\rangle$	$\langle[0.6,0.7],[0.2,0.3],[0.7,0.8]\rangle$
x_5	$\langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle$	$\langle[0.4,0.5],[0.1,0.2],[0.8,0.9]\rangle$	$\langle[0.6,0.7],[0.5,0.6],[0.2,0.3]\rangle$	$\langle[0.7,0.8],[0.4,0.5],[0.1,0.2]\rangle$	$\langle[0.7,0.9],[0.5,0.6],[0.1,0.2]\rangle$

Table 2. The knowledge of M&A target selection given by Expert 2.

R_2	y_1	y_2	y_3	y_4	y_5
x_1	$\langle[0.7,0.8],[0.2,0.3],[0.4,0.6]\rangle$	$\langle[0.3,0.4],[0.2,0.3],[0.7,0.8]\rangle$	$\langle[0.6,0.8],[0.4,0.6],[0.3,0.4]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.2,0.4],[0.3,0.4],[0.7,0.9]\rangle$
x_2	$\langle[0.4,0.6],[0.2,0.3],[0.7,0.8]\rangle$	$\langle[0.3,0.5],[0.4,0.5],[0.6,0.7]\rangle$	$\langle[0.5,0.6],[0.3,0.4],[0.5,0.6]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.5,0.6]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.3,0.4]\rangle$
x_3	$\langle[0.7,0.8],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.8,0.9],[0.3,0.4],[0.3,0.4]\rangle$	$\langle[0.3,0.5],[0.4,0.5],[0.5,0.6]\rangle$	$\langle[0.2,0.4],[0.5,0.6],[0.5,0.7]\rangle$	$\langle[0.4,0.6],[0.2,0.3],[0.7,0.8]\rangle$
x_4	$\langle[0.7,0.9],[0.2,0.3],[0.5,0.6]\rangle$	$\langle[0.5,0.7],[0.2,0.3],[0.5,0.6]\rangle$	$\langle[0.4,0.5],[0.3,0.4],[0.7,0.8]\rangle$	$\langle[0.5,0.7],[0.2,0.4],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.2,0.3],[0.6,0.7]\rangle$
x_5	$\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle$	$\langle[0.4,0.6],[0.1,0.2],[0.7,0.8]\rangle$	$\langle[0.6,0.7],[0.4,0.6],[0.3,0.4]\rangle$	$\langle[0.7,0.8],[0.3,0.4],[0.1,0.2]\rangle$	$\langle[0.7,0.8],[0.3,0.6],[0.1,0.2]\rangle$

Table 3. The knowledge of M&A target selection given by Expert 3.

R_3	y_1	y_2	y_3	y_4	y_5
x_1	$\langle[0.7,0.8],[0.3,0.4],[0.5,0.6]\rangle$	$\langle[0.3,0.5],[0.2,0.4],[0.7,0.8]\rangle$	$\langle[0.7,0.8],[0.4,0.5],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.3,0.5],[0.4,0.5]\rangle$	$\langle[0.3,0.4],[0.2,0.4],[0.7,0.8]\rangle$
x_2	$\langle[0.5,0.6],[0.2,0.3],[0.8,0.9]\rangle$	$\langle[0.4,0.5],[0.4,0.6],[0.6,0.7]\rangle$	$\langle[0.5,0.7],[0.3,0.5],[0.5,0.7]\rangle$	$\langle[0.5,0.7],[0.2,0.4],[0.5,0.6]\rangle$	$\langle[0.7,0.8],[0.3,0.4],[0.3,0.4]\rangle$
x_3	$\langle[0.7,0.8],[0.3,0.5],[0.4,0.5]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.3,0.5]\rangle$	$\langle[0.4,0.5],[0.4,0.5],[0.5,0.7]\rangle$	$\langle[0.3,0.4],[0.5,0.7],[0.5,0.6]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.8,0.9]\rangle$
x_4	$\langle[0.7,0.8],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.6,0.7],[0.2,0.4],[0.5,0.6]\rangle$	$\langle[0.4,0.5],[0.2,0.4],[0.7,0.9]\rangle$	$\langle[0.5,0.6],[0.3,0.4],[0.3,0.4]\rangle$	$\langle[0.6,0.7],[0.2,0.3],[0.6,0.8]\rangle$
x_5	$\langle[0.7,0.8],[0.1,0.2],[0.1,0.3]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.7,0.8]\rangle$	$\langle[0.6,0.7],[0.4,0.5],[0.3,0.4]\rangle$	$\langle[0.7,0.8],[0.3,0.5],[0.1,0.2]\rangle$	$\langle[0.8,0.9],[0.3,0.5],[0.1,0.2]\rangle$

At last, we determine the score values for five optional target organizations in $0.6\left(\underline{\sum_{i=1}^{3}R_i}^{O}(A)\oplus\overline{\sum_{i=1}^{3}R_i}^{O}(A)\right)\oplus(1-0.6)\left(\underline{\sum_{i=1}^{3}R_i}^{P}(A)\oplus\overline{\sum_{i=1}^{3}R_i}^{P}(A)\right)$; it is not difficult to obtain the ordering result for M&A targets in the compromise decision strategy: $x_3 \succ x_5 \succ x_1 \succ x_4 \succ x_2$. Thus, we can see that the optimal selection is the third target enterprise.

5.3. Comparative Analysis

The above subsection shows basic steps and procedures of the proposed algorithm based on IN multigranulation rough sets over two universes in the M&A target selection background. In order to illustrate the effectiveness of the proposed decision making approach, we compare the newly-proposed decision making rules with the decision making method based on IN aggregation operators in this subsection. In the literature [15], Zhang et al. developed several common IN aggregation operators and applied them to a multiple criteria decision making problem. In what follows, a comparison analysis is conducted by utilizing the proposed M&A target selection approach with the interval neutrosophic number weighted averaging (INNWA) operator and the interval neutrosophic number weighted geometric (INNWG) operator presented in [15]. Prior to the specific comparison analysis, we review the above-mentioned IN aggregation operators. Let $A_j = \left\langle \left[T_{A_j}^L, T_{A_j}^U\right], \left[I_{A_j}^L, I_{A_j}^U\right], \left[F_{A_j}^L, F_{A_j}^U\right] \right\rangle (j = 1, 2, \ldots, n)$ be a collection of INNs and $w = (1/n, 1/n, \ldots, 1/n)^T$ be the weight of A_j with equal weight, then the INNWA operator and the INNWG operator are presented below.

1. The interval neutrosophic number weighted averaging (INNWA) operator:

$$INNWA_w(A_1, A_2, \ldots, A_n) = \sum_{j=1}^{n}\left(\frac{1}{n}A_j\right) =$$
$$\left\langle \begin{array}{l} \left[1-\prod_{j=1}^{n}\left(1-T_{A_j}^L\right)^{1/n}, 1-\prod_{j=1}^{n}\left(1-T_{A_j}^U\right)^{1/n}\right], \\ \left[\prod_{j=1}^{n}\left(I_{A_j}^L\right)^{1/n}, \prod_{j=1}^{n}\left(I_{A_j}^U\right)^{1/n}\right], \left[\prod_{j=1}^{n}\left(F_{A_j}^L\right)^{1/n}, \prod_{j=1}^{n}\left(F_{A_j}^U\right)^{1/n}\right] \end{array} \right\rangle,$$

2. The interval neutrosophic number weighted geometric (INNWG) operator:

$$INNWG_w(A_1, A_2, \ldots, A_n) = \prod_{j=1}^{n}\left(A_j\right)^{1/n} =$$
$$\left\langle \begin{array}{l} \left[\prod_{j=1}^{n}\left(T_{A_j}^L\right)^{1/n}, \prod_{j=1}^{n}\left(T_{A_j}^U\right)^{1/n}\right], \left[1-\prod_{j=1}^{n}\left(1-I_{A_j}^L\right)^{1/n}, 1-\prod_{j=1}^{n}\left(1-I_{A_j}^U\right)^{1/n}\right], \\ \left[1-\prod_{j=1}^{n}\left(1-F_{A_j}^L\right)^{1/n}, 1-\prod_{j=1}^{n}\left(1-F_{A_j}^U\right)^{1/n}\right] \end{array} \right\rangle.$$

On the basis of the INNWA and INNWG operators introduced above, the knowledge of M&A target selection given by Expert 1, Expert 2 and Expert 3 could be aggregated by using INNWA and INNWG operators, i.e., we aggregate IN relations over two universes R_1, R_2 and R_3 to a single IN relation over two universes, represented by R_{INNWA} and R_{INNWG}, which is shown as the following Tables 4 and 5.

Next, by virtue of IN rough sets over two universes presented in Definition 6, we calculate the IN rough lower and upper approximations of A with respect to (U, V, R_{INNWA}) and (U, V, R_{INNWG}), respectively.

$$\begin{aligned}
\underline{R_{INNWA}}(A) = \{ & \langle x_1, \langle [0.40, 0.50], [0.50, 0.67], [0.60, 0.70] \rangle \rangle, \langle x_2, \langle [0.40, 0.50], [0.50, 0.70], [0.74, 0.85] \rangle \rangle, \\
& \langle x_3, \langle [0.50, 0.60], [0.40, 0.60], [0.60, 0.70] \rangle \rangle, \langle x_4, \langle [0.33, 0.46], [0.50, 0.70], [0.63, 0.74] \rangle \rangle, \\
& \langle x_5, \langle [0.20, 0.30], [0.50, 0.67], [0.74, 0.88] \rangle \rangle \}; \\
\overline{R_{INNWA}}(A) = \{ & \langle x_1, \langle [0.70, 0.80], [0.23, 0.37], [0.33, 0.46] \rangle \rangle, \langle x_2, \langle [0.52, 0.65], [0.33, 0.47], [0.60, 0.70] \rangle \rangle, \\
& \langle x_3, \langle [0.70, 0.80], [0.20, 0.30], [0.43, 0.52] \rangle \rangle, \langle x_4, \langle [0.70, 0.80], [0.20, 0.30], [0.43, 0.56] \rangle \rangle, \\
& \langle x_5, \langle [0.70, 0.80], [0.30, 0.40], [0.20, 0.30] \rangle \rangle \}; \\
\underline{R_{INNWG}}(A) = \{ & \langle x_1, \langle [0.40, 0.50], [0.50, 0.67], [0.60, 0.70] \rangle \rangle, \langle x_2, \langle [0.40, 0.50], [0.50, 0.70], [0.72, 0.84] \rangle \rangle, \\
& \langle x_3, \langle [0.50, 0.60], [0.40, 0.60], [0.60, 0.70] \rangle \rangle, \langle x_4, \langle [0.33, 0.48], [0.50, 0.70], [0.62, 0.72] \rangle \rangle, \\
& \langle x_5, \langle [0.20, 0.30], [0.50, 0.67], [0.72, 0.88] \rangle \rangle \}; \\
\overline{R_{INNWG}}(A) = \{ & \langle x_1, \langle [0.70, 0.80], [0.23, 0.37], [0.40, 0.50] \rangle \rangle, \langle x_2, \langle [0.50, 0.62], [0.33, 0.48], [0.54, 0.67] \rangle \rangle, \\
& \langle x_3, \langle [0.70, 0.80], [0.20, 0.30], [0.44, 0.54] \rangle \rangle, \langle x_4, \langle [0.70, 0.80], [0.20, 0.30], [0.44, 0.57] \rangle \rangle, \\
& \langle x_5, \langle [0.70, 0.80], [0.30, 0.40], [0.20, 0.30] \rangle \rangle \}.
\end{aligned}$$

Then, in a single granulation context, we further calculate the sets $\lambda \left(\underline{R_{INNWA}}(A) \right) \oplus (1 - \lambda) \left(\overline{R_{INNWA}}(A) \right)$ and $\lambda \left(\underline{R_{INNWG}}(A) \right) \oplus (1 - \lambda) \left(\overline{R_{INNWG}}(A) \right)$ when $\lambda = 0.6$:

$$\begin{aligned}
& 0.6 \left(\underline{R_{INNWA}}(A) \right) \oplus (1 - 0.6) \left(\overline{R_{INNWA}}(A) \right) = \{ \langle x_1, \langle [0.54, 0.65], [0.37, 0.53], [0.51, 0.62] \rangle \rangle, \\
& \quad \langle x_2, \langle [0.44, 0.56], [0.42, 0.59], [0.64, 0.76] \rangle \rangle, \langle x_3, \langle [0.59, 0.69], [0.31, 0.46], [0.53, 0.62] \rangle \rangle, \\
& \quad \langle x_4, \langle [0.51, 0.63], [0.35, 0.50], [0.54, 0.66] \rangle \rangle, \langle x_5, \langle [0.46, 0.57], [0.41, 0.55], [0.44, 0.58] \rangle \rangle \}, \\
& 0.6 \left(\underline{R_{INNWG}}(A) \right) \oplus (1 - 0.6) \left(\overline{R_{INNWG}}(A) \right) = \{ \langle x_1, \langle [0.54, 0.65], [0.37, 0.53], [0.51, 0.62] \rangle \rangle, \\
& \quad \langle x_2, \langle [0.44, 0.55], [0.42, 0.61], [0.64, 0.77] \rangle \rangle, \langle x_3, \langle [0.59, 0.69], [0.31, 0.46], [0.53, 0.63] \rangle \rangle, \\
& \quad \langle x_4, \langle [0.51, 0.64], [0.35, 0.50], [0.54, 0.66] \rangle \rangle, \langle x_5, \langle [0.46, 0.57], [0.41, 0.55], [0.43, 0.58] \rangle \rangle \}.
\end{aligned}$$

Table 4. The aggregated knowledge of M&A target selection by using the interval neutrosophic number weighted averaging (INNWA) operator.

R_{INNWA}	y_1	y_2	y_3	y_4	y_5
x_1	$\langle[0.74, 0.85], [0.23, 0.33], [0.43, 0.56]\rangle$	$\langle[0.30, 0.44], [0.23, 0.37], [0.72, 0.84]\rangle$	$\langle[0.63, 0.77], [0.43, 0.56], [0.33, 0.43]\rangle$	$\langle[0.63, 0.74], [0.33, 0.46], [0.40, 0.50]\rangle$	$\langle[0.23, 0.37], [0.23, 0.37], [0.72, 0.88]\rangle$
x_2	$\langle[0.44, 0.57], [0.23, 0.33], [0.72, 0.84]\rangle$	$\langle[0.33, 0.48], [0.40, 0.52], [0.62, 0.72]\rangle$	$\langle[0.50, 0.63], [0.33, 0.46], [0.52, 0.65]\rangle$	$\langle[0.54, 0.67], [0.23, 0.37], [0.50, 0.60]\rangle$	$\langle[0.74, 0.85], [0.33, 0.46], [0.30, 0.40]\rangle$
x_3	$\langle[0.70, 0.80], [0.33, 0.46], [0.43, 0.52]\rangle$	$\langle[0.77, 0.90], [0.33, 0.43], [0.33, 0.46]\rangle$	$\langle[0.33, 0.48], [0.43, 0.52], [0.52, 0.65]\rangle$	$\langle[0.23, 0.37], [0.52, 0.65], [0.52, 0.65]\rangle$	$\langle[0.44, 0.57], [0.20, 0.30], [0.72, 0.84]\rangle$
x_4	$\langle[0.74, 0.88], [0.20, 0.33], [0.43, 0.56]\rangle$	$\langle[0.54, 0.67], [0.23, 0.37], [0.52, 0.62]\rangle$	$\langle[0.40, 0.50], [0.23, 0.37], [0.70, 0.84]\rangle$	$\langle[0.53, 0.67], [0.29, 0.43], [0.33, 0.46]\rangle$	$\langle[0.63, 0.74], [0.20, 0.30], [0.62, 0.76]\rangle$
x_5	$\langle[0.74, 0.88], [0.18, 0.29], [0.12, 0.26]\rangle$	$\langle[0.44, 0.57], [0.12, 0.23], [0.72, 0.84]\rangle$	$\langle[0.60, 0.70], [0.43, 0.56], [0.26, 0.37]\rangle$	$\langle[0.70, 0.80], [0.33, 0.46], [0.10, 0.20]\rangle$	$\langle[0.74, 0.88], [0.35, 0.56], [0.10, 0.20]\rangle$

Table 5. The aggregated knowledge of M&A target selection by using the interval neutrosophic number weighted geometric (INNWG) operator.

R_{INNWG}	y_1	y_2	y_3	y_4	y_5
x_1	$\langle[0.72, 0.84], [0.23, 0.33], [0.44, 0.57]\rangle$	$\langle[0.30, 0.43], [0.23, 0.37], [0.74, 0.85]\rangle$	$\langle[0.62, 0.76], [0.44, 0.57], [0.33, 0.44]\rangle$	$\langle[0.62, 0.72], [0.33, 0.48], [0.40, 0.50]\rangle$	$\langle[0.23, 0.37], [0.23, 0.37], [0.74, 0.88]\rangle$
x_2	$\langle[0.43, 0.56], [0.23, 0.33], [0.74, 0.85]\rangle$	$\langle[0.33, 0.46], [0.40, 0.54], [0.63, 0.74]\rangle$	$\langle[0.50, 0.62], [0.33, 0.48], [0.54, 0.67]\rangle$	$\langle[0.52, 0.65], [0.23, 0.37], [0.50, 0.60]\rangle$	$\langle[0.72, 0.84], [0.33, 0.48], [0.30, 0.40]\rangle$
x_3	$\langle[0.70, 0.80], [0.33, 0.48], [0.44, 0.54]\rangle$	$\langle[0.76, 0.90], [0.33, 0.44], [0.33, 0.48]\rangle$	$\langle[0.33, 0.46], [0.44, 0.54], [0.54, 0.67]\rangle$	$\langle[0.23, 0.37], [0.54, 0.67], [0.54, 0.67]\rangle$	$\langle[0.43, 0.56], [0.20, 0.30], [0.74, 0.85]\rangle$
x_4	$\langle[0.72, 0.88], [0.20, 0.33], [0.44, 0.57]\rangle$	$\langle[0.52, 0.65], [0.23, 0.37], [0.54, 0.63]\rangle$	$\langle[0.40, 0.50], [0.23, 0.37], [0.70, 0.85]\rangle$	$\langle[0.53, 0.65], [0.30, 0.44], [0.33, 0.48]\rangle$	$\langle[0.62, 0.72], [0.20, 0.30], [0.63, 0.77]\rangle$
x_5	$\langle[0.72, 0.88], [0.20, 0.30], [0.12, 0.26]\rangle$	$\langle[0.43, 0.56], [0.12, 0.23], [0.74, 0.85]\rangle$	$\langle[0.60, 0.70], [0.44, 0.57], [0.26, 0.37]\rangle$	$\langle[0.70, 0.80], [0.33, 0.48], [0.10, 0.20]\rangle$	$\langle[0.72, 0.88], [0.37, 0.57], [0.10, 0.20]\rangle$

Finally, we compute the score values for five optional target organizations in $0.6\left(\underline{R_{INNWA}}(A)\right) \oplus (1-0.6)\left(\overline{R_{INNWA}}(A)\right)$ and $0.6\left(\underline{R_{INNWG}}(A)\right) \oplus (1-0.6)\left(\overline{R_{INNWG}}(A)\right)$ and further determine the M&A target with the largest score value. Then, the ranking results by utilizing INNWA and INNWG operators show that the best alternative is x_3, namely the third target enterprise, which is consistent with the decision result determined by Algorithm 1.

5.4. Result Analysis and Discussion

From the above ranking results, we can see that the proposed model takes full advantage of INSs and multigranulation rough sets over two universes in M&A target selection procedures, and the superiorities of utilizing INSs and multigranulation rough sets over two universes to deal with group decision making problems can be summarized as follows:

1. In the process of describing decision making information, INSs provide experts with more exemplary and flexible access to convey their understandings about the M&A knowledge base. Specifically, it is worth noting that a variety of decision making information systems based on INSs outperform some common extended forms of classical FSs, such as IFSs, interval-valued fuzzy sets (IVFSs), interval-valued intuitionistic fuzzy sets (IVIFSs) and single-valued neutrosophic sets (SVNSs). In the concept of INSs, by introducing the degree of indeterminacy and the degree of non-membership of an element to a set, decision makers could express their incomplete, indeterminate and inconsistent preferences more precisely, as well as avoid the loss of decision making information through considering the truth, indeterminacy and falsity membership functions in various decision making processes. Additionally, the expression of the degree of membership, the degree of indeterminacy and the degree of non-membership using an interval number enables decision makers to better model insufficiency in available information. Thus, INSs could effectively deal with the more uncertain information in M&A target selection.
2. In group decision making procedures, multigranulation rough sets over two universes can be seen as an efficient information fusion approach that could integrate each expert's preference to form an ultimate conclusion by considering optimistic and pessimistic decision making strategies. Moreover, compared with classical IN group decision making approaches based on INNWA and INNWG operators, it is noted that INNWA and INNWG operators could only offer a one-fold information fusion strategy, i.e., the information fusion strategy based on averaging operators or geometric operators without considering the risk appetite of experts, which may cause the loss of risk-based information and further preclude the solution of risk-based group decision making problems. However, the proposed decision making approach based on multigranulation rough sets over two universes could not only provide risk-seeking and risk-averse decision strategies simultaneously, but also provide a compromise decision strategy that considers the risk preference of decision makers. Hence, multigranulation rough sets over two universes can be regarded as a multiple information fusion strategy that is suitable for solving risk-based group decision making problems.

As discussed previously, the advantages of the proposed model could provide a reasonable way to express some complicated decision making information, utilize risk-seeking and risk-averse decision strategies through considering the risk preference of decision makers and also increase the efficiency of M&A target selection.

6. Conclusions

In this article, in order to conduct group decision making from the granular computing paradigm, by combining multigranulation rough sets over two universes with INSs, a novel rough set model named IN multigranulation rough sets over two universes is developed. The definition and some properties of optimistic and pessimistic IN multigranulation rough sets over two universes are studied systematically. Then, by virtue of IN multigranulation rough sets over two universes, we further construct decision making rules and computing approaches for M&A target selection problems. At last, we illustrate the newly-proposed decision making approach on the basis of a practical M&A case study. It is desirable to study attribute reduction algorithms and uncertainty measures based on IN multigranulation rough sets over two universes in the future. Another future research direction is to apply the proposed decision making approach to other business intelligence issues.

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Cosine Measures of Neutrosophic Cubic Sets for Multiple Attribute Decision-Making

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Abstract: The neutrosophic cubic set can contain much more information to express its interval neutrosophic numbers and single-valued neutrosophic numbers simultaneously in indeterminate environments. Hence, it is a usual tool for expressing much more information in complex decision-making problems. Unfortunately, there has been no research on similarity measures of neutrosophic cubic sets so far. Since the similarity measure is an important mathematical tool in decision-making problems, this paper proposes three cosine measures between neutrosophic cubic sets based on the included angle cosine of two vectors, distance, and cosine functions, and investigates their properties. Then, we develop a cosine measures-based multiple attribute decision-making method under a neutrosophic cubic environment in which, from the cosine measure between each alternative (each evaluated neutrosophic cubic set) and the ideal alternative (the ideal neutrosophic cubic set), the ranking order of alternatives and the best option can be obtained, corresponding to the cosine measure values in the decision-making process. Finally, an illustrative example about the selection problem of investment alternatives is provided to illustrate the application and feasibility of the developed decision-making method.

Keywords: neutrosophic cubic set; decision-making; similarity measure; cosine measure; interval neutrosophic set; single-valued neutrosophic set

1. Introduction

The classic fuzzy set, as presented by Zadeh [1], is only described by the membership degree in the unit interval [0, 1]. In the real world, it is often difficult to express the value of a membership function by an exact value in a fuzzy set. In such cases, it may be easier to describe vagueness and uncertainty in the real world using both an interval value and an exact value, rather than unique interval/exact values. Thus, the hybrid form of an interval value and an exact value may be a very useful expression for a person to describe certainty and uncertainty due to his/her hesitant judgment in complex decision-making problems. For this purpose, Jun et al. [2] introduced the concept of (fuzzy) cubic sets, including internal cubic sets and external cubic sets, by the combination of both an interval-valued fuzzy number (IVFN) and a fuzzy value, and defined some logic operations of cubic sets, such as the P-union, P-intersection, R-union, and R-intersection of cubic sets. Also, Jun and Lee [3] and Jun et al. [4–6] applied the concept of cubic sets to BCK/BCI-algebras and introduced the concepts of cubic subalgebras/ideals, cubic o-subalgebras and closed cubic ideals in BCK/BCI-algebras.

However, the cubic set is described by two parts simultaneously, where one represents the membership degree range by the interval value and the other represents the membership degree by a fuzzy value. Hence, a cubic set is the hybrid set combined by both an IVFN and a fuzzy value. Obviously, the advantage of the cubic set is that it can contain much more information to express the IVFN and fuzzy value simultaneously.

As the generalization of fuzzy sets [1], interval-valued fuzzy sets (IVFSs) [7], intuitionistic fuzzy sets (IFSs) [8], and interval-valued intuitionistic fuzzy sets (IVIFSs) [9], Smarandache [10] initially introduced a concept of neutrosophic sets to express incomplete, indeterminate, and inconsistent information. As simplified forms of neutrosophic sets, Smarandache [10], Wang et al. [11,12] and Ye [13] introduced single-valued neutrosophic sets (SVNSs) and interval neutrosophic sets (INSs), and simplified neutrosophic sets (SNSs) as subclasses of neutrosophic sets for easy engineering applications. Since then, SVNSs, INSs, and SNSs have been widely applied to various areas, such as image processing [14–16], decision-making [17–32], clustering analyses [33,34], medical diagnoses [35,36], and fault diagnoses [37]. Recently, Ali et al. [38] and Jun et al. [39] have extended cubic sets to the neutrosophic sets and proposed the concepts of neutrosophic cubic sets (NCSs), including internal NCSs and external NCSs, subsequently introducing some logic operations of NCSs, such as the P-union, P-intersection, R-union, and R-intersection of NCSs. Furthermore, Ali et al. [38] introduced a distance measure between NCSs and applied it to pattern recognition. Subsequently, Banerjee et al. [40] further presented a multiple attribute decision-making (MADM) method with NCSs based on grey relational analysis, in which they introduced the Hamming distances of NCSs for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and then gave the relative closeness coefficients in order to rank the alternatives.

From the above review, we can see that the existing literature mainly focus on the theoretical studies of cubic sets and NCSs, rather than the studies on their similarity measures and their applications. On the other hand, the NCS contains much more information than the general neutrosophic set (INS/SVNS) because the NCS is expressed by the combined information of both INS and SVNS. Hence, NCSs used for attribute evaluation in decision making may show its rationality and affectivity since general neutrosophic decision-making methods with INSs/SVNSs may lose some useful evaluation information (either INSs or SVNSs) of attributes, which may affect decision results, resulting in the distortion phenomenon. Moreover, the similarity measure is an important mathematical tool in decision-making problems. Currently, since there is no study on similarity measures of cubic sets and NCSs under a neutrosophic cubic environment, we need to develop new similarity measures for NCSs for MADM problems with neutrosophic cubic information, since the cubic set is a special case of the NCS. For these reasons, this paper aims to propose three cosine measures between NCSs based on the included angle cosine of two vectors, distance, and cosine function, and their MADM method in a neutrosophic cubic environment.

The remainder of the article is organized as follows. Section 2 briefly describes some concepts of cubic sets and NCSs. Section 3 presents three cosine measures of NCSs and discusses their properties. In Section 4, we develop an MADM approach based on the cosine measures of NCSs under a neutrosophic cubic environment. In Section 5, an illustrative example about the selection problem of investment alternatives is provided to illustrate the application and feasibility of the developed method. Section 6 contains conclusions and future research.

2. Some Basic Concepts of Cubic Sets and NCSs

By the combination of a fuzzy value and an IVFN, Jun et al. [2] defined a (fuzzy) cubic set.

A cubic set S in a universe of discourse X is constructed as follows [2]:

$$S = \{x, T(x), \mu(x) \mid x \in X\},$$

where $T(x) = [T^-(x), T^+(x)]$ is an IVFN for $x \in X$ and μ is a fuzzy value for $x \in X$. Then, we call

(i) $S = \{x, T(x), \mu(x) \mid x \in X\}$ an internal cubic set if $T^-(x) \le \mu(x) \le T^+(x)$ for $x \in X$;

(ii) $S = \{x, T(x), \mu(x) \mid x \in X\}$ an external cubic set if $\mu(x) \notin \left(T^-(x), T^+(x)\right)$ for $x \in X$.

Then, Ali et al. [38] and Jun et al. [39] proposed a NCS based on the combination of an interval neutrosophic number (INN) and a single-valued neutrosophic number (SVNN) as the extension of the (fuzzy) cubic set.

A NCS S in X is constructed as the following form [38,39]:

$$P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\},$$

where <$T(x)$, $U(x)$, $F(x)$> is an INN, and $T(x) = [T^-(x), T^+(x)] \subseteq [0, 1]$, $U(x) = [U^-(x), U^+(x)] \subseteq [0, 1]$, and $F(x) = [F^-(x), F^+(x)] \subseteq [0, 1]$ for $x \in X$ are the truth-interval, indeterminacy-interval, and falsity-interval, respectively; then <$t(x)$, $u(x)$, $f(x)$> is a SVNN, and $t(x)$, $u(x)$, $f(x) \in [0, 1]$ for $x \in X$ are the truth, indeterminacy, and falsity degrees, respectively.

An NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ is said to be [38,39]:

(i) An internal NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ if $T^-(x) \le t(x) \le T^+(x)$, $U^-(x) \le u(x) \le U^+(x)$, and $F^-(x) \le f(x) \le F^+(x)$ for $x \in X$;

(ii) An external NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ if $t(x) \notin \left(T^-(x), T^+(x)\right)$, $u(x) \notin \left(U^-(x), U^+(x)\right)$, and $f(x) \notin \left(F^-(x), F^+(x)\right)$ for $x \in X$.

For convenience, a basic element $(x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) >)$ in an NCS P is simply denoted by p = (<T, U, F>, <t, u, f>), which is called a neutrosophic cubic number (NCN), where T, U, $F \subseteq [0, 1]$ and t, u, $f \in [0, 1]$, satisfying $0 \le T^+(x) + U^+(x) + F^+(x) \le 3$ and $0 \le t + u + f \le 3$.

Let p_1 = (<T_1, U_1, F_1>, <t_1, u_1, f_1>) and p_2 = (<T_2, U_2, F_2>, <t_2, u_2, f_2>) be two NCNs. Then, there are the following relations [38,39]:

(1) $p_1^c = \left(\left\langle \left[F_1^-, F_1^+\right], \left[1-U_1^+, 1-U_1^-\right], \left[T_1^-, T_1^+\right]\right\rangle, \left\langle f_1, 1-u_1, t_1\right\rangle\right)$ (complement of p_1);

(2) $p_1 \subseteq p_2$ if and only if $T_1 \subseteq T_2$, $U_1 \supseteq U_2$, $F_1 \supseteq F_2$, $t_1 \le t_2$, $u_1 \ge u_2$, and $f_1 \ge f_2$ (P-order);

(3) $p_1 = p_2$ if and only if $p_2 \subseteq p_1$ and $p_1 \subseteq p_2$, i.e., <T_1, U_1, F_1> = <T_2, U_2, F_2> and <t_1, u_1, f_1> = <t_2, u_2, f_2>.

3. Cosine Measures of NCSs

In this section, we propose three cosine measures between NCSs.

Definition 1. *Let X ={x_1, x_2, …, x_n} be a finite set and two NCSs be P ={p_1, p_2, …, p_n} and Q ={q_1, q_2, …, q_n}, where p_j = (<T_{pj}, U_{pj}, F_{pj}>, <t_{pj}, u_{pj}, f_{pj}>) and q_j = (<T_{qj}, U_{qj}, F_{qj}>, <t_{qj}, u_{qj}, f_{qj}>) for j = 1, 2, …, n are two collections of NCNs. Then, three cosine measures of P and Q are proposed based on the included angle cosine of two vectors, distance, and cosine function, respectively, as follows:*

(1) Cosine measure based on the included angle cosine of two vectors

$$S_1(P,Q) = \frac{1}{2n}\left\{ \sum_{j=1}^{n} \frac{T_{pj}^- T_{qj}^- + T_{pj}^+ T_{qj}^+ + U_{pj}^- U_{qj}^- + U_{pj}^+ U_{qj}^+ + F_{pj}^- F_{qj}^- + F_{pj}^+ F_{qj}^+}{\left\{ \begin{array}{l} \sqrt{(T_{pj}^-)^2 + (T_{pj}^+)^2 + (U_{pj}^-)^2 + (U_{pj}^+)^2 + (F_{pj}^-)^2 + (F_{pj}^+)^2} \\ \times \sqrt{(T_{qj}^-)^2 + (T_{qj}^+)^2 + (U_{qj}^-)^2 + (U_{qj}^+)^2 + (F_{qj}^-)^2 + (F_{qj}^+)^2} \end{array} \right\}} + \sum_{j=1}^{n} \frac{t_{pj} t_{qj} + u_{pj} u_{qj} + f_{pj} f_{qj}}{\left\{ \sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2} \right\}} \right\} \quad (1)$$

(2) Cosine measure based on distance

$$S_2(P,Q) = \frac{1}{2n} \sum_{j=1}^{n} \left\{ \begin{array}{l} \cos\left(\dfrac{\left|T_{pj}^- - T_{qj}^-\right| + \left|T_{pj}^+ - T_{qj}^+\right| + \left|U_{pj}^- - U_{qj}^-\right| + \left|U_{pj}^+ - U_{qj}^+\right| + \left|F_{pj}^- - F_{qj}^-\right| + \left|F_{pj}^+ - F_{qj}^+\right|}{12} \pi \right) \\ + \cos\left(\dfrac{\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|}{6} \pi \right) \end{array} \right\} \quad (2)$$

(3) Cosine measure based on cosine function

$$S_3(P,Q) = \frac{1}{2n}\left\{ \frac{1}{3(\sqrt{2}-1)} \sum_{j=1}^{n} \left\{ \begin{array}{l} \left[\sqrt{2}\cos\left(\frac{T_{pj}^- + T_{pj}^+ - T_{qj}^- - T_{qj}^+}{8}\pi \right) - 1 \right] \\ + \left[\sqrt{2}\cos\left(\frac{U_{pj}^- + U_{pj}^+ - U_{qj}^- - U_{qj}^+}{8}\pi \right) - 1 \right] \\ + \left[\sqrt{2}\cos\left(\frac{F_{pj}^- + F_{pj}^+ - F_{qj}^- - F_{qj}^+}{8}\pi \right) - 1 \right] \end{array} \right\} + \frac{1}{3(\sqrt{2}-1)} \sum_{j=1}^{n} \left\{ \begin{array}{l} \left[\sqrt{2}\cos\left(\frac{t_{pj} - t_{qj}}{4}\pi \right) - 1 \right] \\ + \left[\sqrt{2}\cos\left(\frac{u_{pj} - u_{qj}}{4}\pi \right) - 1 \right] \\ + \left[\sqrt{2}\cos\left(\frac{f_{pj} - f_{qj}}{4}\pi \right) - 1 \right] \end{array} \right\} \right\} \quad (3)$$

Obviously, the three cosine measures $S_k(P, Q)$ (k = 1, 2, 3) satisfy the following properties (S_1)–(S_3):

(S_1) $0 \le S_k(P, Q) \le 1$;

(S_2) $S_k(P, Q) = S_k(Q, P)$;

(S_3) $S_k(P, Q) = 1$ *if* $P = Q$, *i.e.*, $<T_{pj}, U_{pj}, F_{pj}>, = <T_{qj}, U_{qj}, F_{qj}>$ *and* $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$.

Proof.

Firstly, we prove the properties (S_1)–(S_3) of $S_1(P, Q)$.

(S_1) The inequality $S_1(P, Q) \ge 0$ is obvious. Then, we only prove $S_1(P, Q) \le 1$.

Based on the Cauchy–Schwarz inequality:

$$\left(x_1y_1 + x_2y_2 + \cdots + x_ny_n\right)^2 \le \left(x_1^2 + x_2^2 + \cdots x_n^2\right) \times \left(y_1^2 + y_2^2 + \cdots y_n^2\right),$$

where $(x_1, x_2, \ldots, x_n) \in R^n$ and $(y_1, y_2, \ldots, y_n) \in R^n$, we can give the following inequality:

$$\left(x_1y_1 + x_2y_2 + \cdots + x_ny_n\right) \le \sqrt{\left(x_1^2 + x_2^2 + \cdots x_n^2\right)} \times \sqrt{\left(y_1^2 + y_2^2 + \cdots y_n^2\right)}.$$

According to the above inequality, we have the following inequality:

$$T_{pj}^- T_{qj}^- + T_{pj}^+ T_{qj}^+ + U_{pj}^- U_{qj}^- + U_{pj}^+ U_{qj}^+ + F_{pj}^- F_{qj}^- + F_{pj}^+ F_{qj}^+ \le$$

$$\sqrt{(T_{pj}^-)^2 + (T_{pj}^+)^2 + (U_{pj}^-)^2 + (U_{pj}^+)^2 + (F_{pj}^-)^2 + (F_{pj}^+)^2} \times \sqrt{(T_{qj}^-)^2 + (T_{qj}^+)^2 + (U_{qj}^-)^2 + (U_{qj}^+)^2 + (F_{qj}^-)^2 + (F_{qj}^+)^2},$$

$$t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj} \le \sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2}.$$

Hence, there is the following result:

$$\frac{1}{n}\sum_{j=1}^{n} \frac{T_{pj}^- T_{qj}^- + T_{pj}^+ T_{qj}^+ + U_{pj}^- U_{qj}^- + U_{pj}^+ U_{qj}^+ + F_{pj}^- F_{qj}^- + F_{pj}^+ F_{qj}^+}{\left\{ \begin{array}{l} \sqrt{(T_{pj}^-)^2 + (T_{pj}^+)^2 + (U_{pj}^-)^2 + (U_{pj}^+)^2 + (F_{pj}^-)^2 + (F_{pj}^+)^2} \\ \times \sqrt{(T_{qj}^-)^2 + (T_{qj}^+)^2 + (U_{qj}^-)^2 + (U_{qj}^+)^2 + (F_{qj}^-)^2 + (F_{qj}^+)^2} \end{array} \right\}} \le 1,$$

$$\frac{1}{n}\sum_{j=1}^{n} \frac{t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj}}{\left\{ \sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2} \right\}} \le 1.$$

Based on Equation (1), we have $S_1(P, Q) \le 1$. Hence, $0 \le S_1(P, Q) \le 1$ holds.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $<T_{pj}, U_{pj}, F_{pj}> = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for j = 1, 2, …, n. Hence $S_1(P, Q) = 1$ holds.

Secondly, we prove the properties (S_1)–(S_3) of $S_2(P, Q)$.

(S_1) Let $x_1 = \left(\left|T_{pj}^- - T_{qj}^-\right| + \left|T_{pj}^+ - T_{qj}^+\right| + \left|U_{pj}^- - U_{qj}^-\right| + \left|U_{pj}^+ - U_{qj}^+\right| + \left|F_{pj}^- - F_{qj}^-\right| + \left|F_{pj}^+ - F_{qj}^+\right|\right)/6$ and $x_2 = \left(\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|\right)/3$. It is obvious that there exist $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$. Thus, there are $0 \le \cos(x_1\pi/2) \le 1$ and $0 \le \cos(x_2\pi/2) \le 1$. Hence, $0 \le S_2(P, Q) \le 1$ holds.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $\langle T_{pj}, U_{pj}, F_{pj}\rangle = \langle T_{qj}, U_{qj}, F_{qj}\rangle$ and $\langle t_{pj}, u_{pj}, f_{pj}\rangle = \langle t_{qj}, u_{qj}, f_{qj}\rangle$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for $j = 1, 2, \ldots, n$. Hence, $S_2(P, Q) = 1$ holds.

Thirdly, we prove the properties (S_1)–(S_3) of $S_3(P, Q)$.

(S_1) Let $y_1 = (T_{pj}^- + T_{pj}^+ - T_{qj}^- - T_{qj}^+)/2$, $y_2 = (U_{pj}^- + U_{pj}^+ - U_{qj}^- - U_{qj}^+)/2$, $y_3 = (F_{pj}^- + F_{pj}^+ - F_{qj}^- - F_{qj}^+)/2$, $y_4 = t_{pj} - t_{qj}$, $y_5 = u_{pj} - u_{qj}$, and $y_6 = f_{pj} - f_{qj}$. Obviously, there exists $-1 \le y_k \le +1$ for $k = 1, 2, \ldots, 6$. Thus, $\sqrt{2}/2 \le \cos(y_k\pi/4) \le 1$, and then there exists $0 \le S_3(P, Q) \le 1$.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $\langle T_{pj}, U_{pj}, F_{pj}\rangle = \langle T_{qj}, U_{qj}, F_{qj}\rangle$ and $\langle t_{pj}, u_{pj}, f_{pj}\rangle = \langle t_{qj}, u_{qj}, f_{qj}\rangle$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for $j = 1, 2, \ldots, n$. Hence, $S_3(P, Q) = 1$ holds. □

When the weight of the elements p_j and q_j ($j = 1, 2, \ldots, n$) is taken into account, $\boldsymbol{w} = \{w_1, w_2, \ldots, w_n\}$ is given as the weight vector of the elements p_j and q_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then, we have the following three weighted cosine measures between P and Q, respectively:

$$S_{w1}(P,Q) = \frac{1}{2}\left\{\begin{array}{l} \sum_{j=1}^{n} w_j \dfrac{T_{pj}^- T_{qj}^- + T_{pj}^+ T_{qj}^+ + U_{pj}^- U_{qj}^- + U_{pj}^+ U_{qj}^+ + F_{pj}^- F_{qj}^- + F_{pj}^+ F_{qj}^+}{\left\{\begin{array}{l}\sqrt{(T_{pj}^-)^2 + (T_{pj}^+)^2 + (U_{pj}^-)^2 + (U_{pj}^+)^2 + (F_{pj}^-)^2 + (F_{pj}^+)^2} \\ \times\sqrt{(T_{qj}^-)^2 + (T_{qj}^+)^2 + (U_{qj}^-)^2 + (U_{qj}^+)^2 + (F_{qj}^-)^2 + (F_{qj}^+)^2}\end{array}\right\}} \\ + \sum_{j=1}^{n} w_j \dfrac{t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj}}{\left\{\sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2}\right\}} \end{array}\right\}, \tag{4}$$

$$S_{w2}(P,Q) = \frac{1}{2}\sum_{j=1}^{n} w_j \left\{\begin{array}{l} \cos\left(\dfrac{\left|T_{pj}^- - T_{qj}^-\right| + \left|T_{pj}^+ - T_{qj}^+\right| + \left|U_{pj}^- - U_{qj}^-\right| + \left|U_{pj}^+ - U_{qj}^+\right| + \left|F_{pj}^- - F_{qj}^-\right| + \left|F_{pj}^+ - F_{qj}^+\right|}{12}\pi\right) \\ + \cos\left(\dfrac{\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|}{6}\pi\right) \end{array}\right\}, \tag{5}$$

$$S_{w3}(P,Q) = \frac{1}{2}\left\{\frac{1}{3(\sqrt{2}-1)}\sum_{j=1}^{n} w_j \left\{\left(\begin{array}{l} \left[\sqrt{2}\cos\left(\dfrac{T_{pj}^- + T_{pj}^+ - T_{qj}^- - T_{qj}^+}{8}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\dfrac{U_{pj}^- + U_{pj}^+ - U_{qj}^- - U_{qj}^+}{8}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\dfrac{F_{pj}^- + F_{pj}^+ - F_{qj}^- - F_{qj}^+}{8}\pi\right) - 1\right] \end{array}\right) + \left(\begin{array}{l} \left[\sqrt{2}\cos\left(\dfrac{t_{pj} - t_{qj}}{4}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\dfrac{u_{pj} - u_{qj}}{4}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\dfrac{f_{pj} - f_{qj}}{4}\pi\right) - 1\right] \end{array}\right)\right\}\right\}. \tag{6}$$

It is obvious that the three cosine measures $S_{wk}(P, Q)$ (k=1, 2, 3) also satisfy the following properties (S_1)-(S_3):

(S_1) $0 \le S_{wk}(P, Q) \le 1$;

(S_2) $S_{wk}(P, Q) = S_{wk}(Q, P)$;

(S_3) $S_{wk}(P, Q) = 1$ if $P = Q$, i.e., $\langle T_{pj}, U_{pj}, F_{pj}\rangle = \langle T_{qj}, U_{qj}, F_{qj}\rangle$ and $\langle t_{pj}, u_{pj}, f_{pj}\rangle = \langle t_{qj}, u_{qj}, f_{qj}\rangle$.

By similar proof ways, we can prove the properties (S_1)–(S_3) for $S_{wk}(P, Q)$ (k = 1, 2, 3). Their proofs are omitted here.

4. Decision-Making Method Using Cosine Measures

In this section, we propose an MADM method by using one of three cosine measures to solve decision-making problems with neutrosophic cubic information.

In an MADM problem, let $P = \{P_1, P_2, \ldots, P_m\}$ be a set of m alternatives and $R = \{R_1, R_2, \ldots, R_n\}$ be a set of n attributes. The evaluation value of an attribute R_j ($j = 1, 2, \ldots, n$) with respect to an alternative P_i ($i = 1, 2, \ldots, m$) is expressed by a NCN $p_{ij} = (<T_{ij}, U_{ij}, F_{ij}>, <t_{ij}, u_j, f_{ij}>)$ ($j = 1, 2, \ldots, n$; $i = 1, 2, \ldots, m$), where $T_{ij}, U_{ij}, F_{ij} \subseteq [0,1]$ and $t_{ij}, u_{ij}, f_{ij} \in [0,1]$. Therefore, all the evaluation values expressed by NCNs can be constructed as the neutrosophic cubic decision matrix $P = (p_{ij})_{m\times n}$. Then, the weight vector of the attributes R_j ($j = 1, 2, \ldots, n$) is considered as $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$, satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, the proposed decision steps are described as follows:

Step 1: Establish an ideal solution (ideal alternative) $P^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$ by the ideal NCN

$$p_j^* = \left(\left\langle \left[\max_i(T_{ij}^-), \max_i(T_{ij}^+)\right], \left[\min_i(U_{ij}^-), \min_i(U_{ij}^+)\right], \left[\min_i(F_{ij}^-), \min_i(F_{ij}^+)\right]\right\rangle, \left\langle \max_i(t_{ij}), \min_i(u_{ij}), \min_i(f_{ij})\right\rangle\right)$$

corresponding to the benefit type of attributes and

$$p_j^* = \left(\left\langle \left[\min_i(T_{ij}^-), \min_i(T_{ij}^+)\right], \left[\max_i(U_{ij}^-), \max_i(U_{ij}^+)\right], \left[\max_i(F_{ij}^-), \max_i(F_{ij}^+)\right]\right\rangle, \left\langle \min_i(t_{ij}), \max_i(u_{ij}), \max_i(f_{ij})\right\rangle\right)$$

corresponding to the cost type of attributes.

Step 2: Calculate the weighted cosine measure values between an alternative P_i ($i = 1, 2, \ldots, m$) and the ideal solution P^* by using Equation (4) or Equation (5) or Equation (6) and get the values of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$ ($i = 1, 2, \ldots, m$).

Step 3: Rank the alternatives in descending order corresponding to the weighted cosine measure values and select the best one(s) according to the bigger value of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$.

Step 4: End.

5. Illustrative Example and Comparison Analysis

In this section, an illustrative example of the selection problem of investment alternatives is provided in order to demonstrate the application of the proposed MADM method with neutrosophic cubic information.

5.1. Illustrative Example

An investment company wants to invest a sum of money for one of four potential alternatives: (a) P_1 is a textile company; (b) P_2 is an automobile company; (c) P_3 is a computer company; (d) P_4 is a software company. The evaluation requirements of the four alternatives are on the basis of three attributes: (a) R_1 is the risk; (b) R_2 is the growth; (c) R_3 is the environmental impact; where the attributes R_1 and R_2 are benefit types, and the attribute R_3 is a cost type. The weight vector of the three attributes is $\boldsymbol{w} = (0.32, 0.38, 0.3)$. When the expert or decision maker is requested to evaluate the four potential alternatives on the basis of the above three attributes using the form of NCNs. Thus, we can construct the following neutrosophic cubic decision matrix:

$$P = \begin{bmatrix} (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.6,0.8],[0.2,0.3],[0.1,0.2]\rangle,\langle 0.7,0.2,0.1\rangle) \\ (\langle[0.6,0.8],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.6,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.1,0.2]\rangle,\langle 0.7,0.4,0.1\rangle) \\ (\langle[0.4,0.6],[0.2,0.3],[0.1,0.3]\rangle,\langle 0.6,0.2,0.2\rangle) & (\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.3,0.4\rangle) & (\langle[0.5,0.7],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) \\ (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.8,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle,\langle 0.7,0.3,0.2\rangle) \end{bmatrix}.$$

Hence, the proposed MADM method can be applied to this decision-making problem with NCSs by the following steps:

Firstly, corresponding to the benefit attributes R_1, R_2, and the cost attribute R_3, we establish an ideal solution (ideal alternative):

$$P^* = \{p_1^*, p_2^*, \ldots, p_n^*\} = \left\{ \begin{array}{l} \left(\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle, \langle 0.8,0.1,0.2\rangle\right), \\ \left(\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle, \langle 0.7,0.1,0.2\rangle\right), \\ \left(\langle[0.5,0.7],[0.3,0.4],[0.3,0.4]\rangle, \langle 0.6,0.4,0.3\rangle\right) \end{array} \right\}.$$

Then, we calculate the weighted cosine measure values between an alternative P_i (i = 1, 2, 3, 4) and the ideal solution P^* by using Equation (4) or Equation (5) or Equation (6), get the values of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$ (i = 1, 2, 3, 4), and rank the four alternatives, which are shown in Table 1.

Table 1. All the cosine measure values between P_i and P^* and ranking orders of the four alternatives.

$S_{wk}(P_i, P^*)$	Cosine Measure Value	Ranking Order	The Best Alternative
$S_{w1}(P_i, P^*)$	0.9564, 0.9855, 0.9596, 0.9945	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w2}(P_i, P^*)$	0.9769, 0.9944, 0.9795, 0.9972	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w3}(P_i, P^*)$	0.9892, 0.9959, 0.9897, 0.9989	$P_4 > P_2 > P_3 > P_1$	P_4

From the results of Table 1, we can see that all the ranking orders of the four alternatives and best choice return the same results corresponding to the three cosine measures in the decision-making problem with neutrosophic cubic information. It is obvious that P_4 is the best one.

5.2. Related Comparison

For relative comparison, we compare our decision-making method with the only existing related decision-making method based on the grey relational analysis under neutrosophic cubic environment [40]. Because the decision-making problem/method with CNS weights in [40] is different from ours, which has exact/crisp weights, we cannot compare them under different decision-making conditions. However, we only gave the comparison of decision-making complexity to show our simple method.

The proposed decision-making method based on the cosine measures of NCSs directly uses the cosine measures between an alternative P_i (i = 1, 2, …, m) and the ideal alternative (ideal solution) P^* to rank all the alternatives; while the existing decision-making method with NCSs introduced in [40] firstly determines the Hamming distances of NCSs for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and then derives the relative closeness coefficients in order to rank the alternatives. It is obvious that our decision-making method is simpler and easier than the existing decision-making method with NCSs introduced in [40]. But, our decision-making method can only deal with decision-making problems with exact/crisp weights, rather than NCS weights [40].

Compared with existing related decision-making methods with general neutrosophic sets (INSs or SVNSs) [17–39], the proposed decision-making method with NCSs contains much more evaluation information of attributes, which consists of both INSs and SVNSs; while the existing decision-making methods [17–39] contain either INS or SVNS information, which may lose some useful evaluation information of attributes in the decision-making process and affect the decision results, resulting in the distortion phenomenon. Furthermore, the existing decision-making methods [17–39] cannot deal with the decision-making problem with NCSs.

5.3. Sensitive Analysis

To show the sensitivities of these cosine measures on the decision results, we can only change the internal NCS of the alternative P_4 into the external NCS and reconstruct the following neutrosophic cubic decision matrix:

$$P' = \begin{bmatrix} (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.6,0.8],[0.2,0.3],[0.1,0.2]\rangle,\langle 0.7,0.2,0.1\rangle) \\ (\langle[0.6,0.8],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.6,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.1,0.2]\rangle,\langle 0.7,0.4,0.1\rangle) \\ (\langle[0.4,0.6],[0.2,0.3],[0.1,0.3]\rangle,\langle 0.6,0.2,0.2\rangle) & (\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.3,0.4\rangle) & (\langle[0.5,0.7],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) \\ (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.9,0.3,0.3\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.8,0.3,0.4\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle,\langle 0.8,0.5,0.4\rangle) \end{bmatrix}.$$

Then, the corresponding ideal solution (ideal alternative) is changed into the following form:

$$P^{*'} = \{p_1^{*'}, p_2^{*'}, ..., p_n^{*'}\} = \begin{Bmatrix} (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.9,0.1,0.2\rangle), \\ (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.8,0.1,0.2\rangle), \\ (\langle[0.5,0.7],[0.3,0.4],[0.3,0.4]\rangle,\langle 0.6,0.5,0.4\rangle) \end{Bmatrix}.$$

According to the results of Table 2, both the cosine measure based on the included angle cosine of two vectors S_{w1} and the cosine measure based on cosine function S_{w3} still hold the same ranking orders; while the cosine measure based on distance S_{w2} shows another ranking form. In this case, S_{w2} is sensitive to the change of the evaluation values, since its ranking order changes with the change of the evaluation values for the alternative P_4.

Table 2. All the cosine measure values between P_i' and $P^{*'}$and ranking orders of the four alternatives.

$S_{wk}(P_i', P^{*'})$	Cosine Measure Value	Ranking Order	The Best Alternative
$S_{w1}(P_i', P^{*'})$	0.9451, 0.9794, 0.9524, 0.9846	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w2}(P_i', P^{*'})$	0.9700, 0.9906, 0.9732, 0.9877	$P_2 > P_4 > P_3 > P_1$	P_2
$S_{w3}(P_i', P^{*'})$	0.9867, 0.9942, 0.9877, 0.9968	$P_4 > P_2 > P_3 > P_1$	P_4

Nevertheless, this study provides a new and effective method for decision makers, due to the limited study on similarity measures and decision-making methods with NCSs in the existing literature. In this study, decision makers can select one of three cosine measures of NCSs to apply to MADM problems, according to their preferences and actual requirements.

6. Conclusions

This paper proposed three cosine measures of NCSs based on the included angle cosine of two vectors, distance, and cosine function, and discussed their properties. Then, we developed an MADM method with neutrosophic cubic information by using one of three cosine measures of NCSs. An illustrative example about the selection problem of investment alternatives was provided to demonstrate the applications of the proposed MADM method with neutrosophic cubic information.

The cosine measures-based MADM method developed in this paper is simpler and easier than the existing decision-making method with neutrosophic cubic information based on the grey related analysis, and shows the main advantage of its simple and easy decision-making process. However, this study can only deal with decision-making problems with exact/crisp weights, rather than NCS weights [40], which is its chief limitation. Therefore, the three cosine measures of NCSs that were developed, and their decision-making method are the main contributions of this paper. The developed MADM method provides a new and effective method for decision makers under neutrosophic cubic environments. In future work, we will further propose some new similarity measures of NCSs and their applications in other fields, such as image processing, medical diagnosis, and fault diagnosis.

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A Novel Rough Set Model in Generalized Single Valued Neutrosophic Approximation Spaces and Its Application

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Abstract: In this paper, we extend the rough set model on two different universes in intuitionistic fuzzy approximation spaces to a single-valued neutrosophic environment. Firstly, based on the (α, β, γ)-cut relation $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$, we propose a rough set model in generalized single-valued neutrosophic approximation spaces. Then, some properties of the new rough set model are discussed. Furthermore, we obtain two extended models of the new rough set model—the degree rough set model and the variable precision rough set model—and study some of their properties. Finally, we explore an example to illustrate the validity of the new rough set model.

Keywords: neutrosophic sets; single-valued neutrosophic sets; generalized single-valued neutrosophic approximation spaces; rough sets; single-valued neutrosophic relations

1. Introduction

Smarandache [1,2] introduced the concept of the neutrosophic set (NS), which consists of three membership functions (truth membership function, indeterminacy membership function and falsity membership function), where each function value is a real standard or nonstandard subset of the nonstandard unit interval $[0^-, 1^+]$. The neutrosophic set generalizes the concepts of the classical set, fuzzy set [3], interval-valued fuzzy set [4], intuitionistic fuzzy set [5] and interval-valued intuitionistic fuzzy set [6]. The neutrosophic set model is an important tool for dealing with real scientific and engineering applications because it can handle not only incomplete information, but also the inconsistent information and indeterminate information that exist commonly in real situations.

For easily applying NSs in the real world, Smarandache [1] and Wang et al. [7] proposed single-valued neutrosophic sets (SVNSs) by simplifying NSs. SVNSs can also be seen as an extension of intuitionistic fuzzy sets [5], in which three membership functions are unrelated and their function values belong to the unit closed interval. SVNSs has been a hot research issue. Ye [8,9] proposed decision making based on correlation coefficients and weighted correlation coefficients of SVNSs and illustrated the application of the proposed methods. Baušys et al. [10] applied SVNSs to multi-criteria decision making and proposed a new extension of the crisp complex proportional assessment (COPRAS) method named COPRAS-SVNS. Zavadskas et al. [11] applied SVNSs to the weighted aggregated sum product assessment (WASPAS) method, named WASPAS-SVNS, and used the new method to solve sustainable assessment of alternative sites for the construction of a waste incineration plant. Zavadskas et al. [12] also applied WASPAS-SVNS to the selection of a lead-zinc flotation circuit design. Zavadskas et al. [13] proposed a single-valued neutrosophic multi-attribute market value assessment method and applied this method to the sustainable market valuation of

Croydon University Hospital. Li et al. [14] applied the Heronian mean to the neutrosophic set, proposed some Heronian mean operators and illustrated their application in multiple attribute group decision making. Baušys and Juodagalvienė [15] demonstrated garage location selection for a residential house. In [16], Ye proposed similarity measures between interval neutrosophic sets and applied them to multi-criteria decision making problems under the interval neutrosophic environment. Ye [17] proposed three vector similarity measures of simplified neutrosophic sets and applied them to a multi-criteria decision making problem with simplified neutrosophic information. Majumdar and Samanta [18] studied the distance, similarity and entropy of SVNSs from a theoretical aspect. Peng et al. [19] developed a new outranking approach for multi-criteria decision making problems in the context of a simplified neutrosophic environment. Liu and Wang [20] introduced an interval neutrosophic prioritized ordered weighted aggregation operator w.r.t. interval neutrosophic numbers and discussed its application in multiple attribute decision making. To deal with difficulties in steam turbine fault diagnosis, Zhang et al. [21] investigated a single-valued neutrosophic multi-granulation rough set over two universes. Şahin [22] proposed two kinds of interval neutrosophic cross-entropies based on the extension of fuzzy cross-entropy and single-valued neutrosophic cross-entropy and developed two multi-criteria decision making methods using the interval neutrosophic cross-entropy. Ye [23] proposed similarity measures between SVNSs based on the tangent function and a multi-period medical diagnosis method based on the similarity measure and the weighted aggregation of multi-period information to solve multi-period medical diagnosis problems with single-valued neutrosophic information. Yang et al. [24] proposed SVNRs and studied some kinds of kernels and closures of SVNRs. Ye [25] presented a simplified neutrosophic harmonic averaging projection measure and its multiple attribute decision making method with simplified neutrosophic information. Stanujkic et al. [26] proposed a new extension of the multi-objective optimization (MULTIMOORA) method adapted for usage with a neutrosophic set.

Rough set theory, initiated by Pawlak [27,28], is a mathematical tool for the study of intelligent systems characterized by insufficient and incomplete information. The theory has been successfully applied to many fields, such as machine learning, knowledge acquisition, decision analysis, etc. To extend the application domain of rough set theory, more and more researchers have made some efforts toward the study of rough set models based on two different universes [29–39].

In recent years, many researchers have paid attention to combining neutrosophic sets with rough sets. Salama and Broumi [40] investigated the roughness of neutrosophic sets. Broumi and Smarandache put forward rough neutrosophic sets [41,42], as well as interval neutrosophic rough sets [43]. Yang et al. [44] proposed single-valued neutrosophic rough sets, which comprise a hybrid model of single-valued neutrosophic sets and rough sets. Along this line, this paper attempts to do some work regarding the fusion of single-valued neutrosophic sets and rough sets again. Concretely, we will extend the rough set model in [29] to a single-valued neutrosophic environment. Furthermore, we will apply the new model to a multi-attribute decision making problem.

The rest of this paper is organized as follows. In Section 2, we recall some basic notions related to Pawlak rough sets, SVNSs and single-valued neutrosophic rough sets. In Section 3, we propose a rough set model in generalized single-valued neutrosophic approximation spaces. Section 4 gives two extended models and studies some related properties. Section 5 explores an example to illustrate the new rough set model's application in multi-attribute decision making. The last section summarizes the conclusions.

2. Preliminaries

In this section, we recall some basic notions of Pawlak rough sets, SVNSs and single-valued neutrosophic rough sets.

2.1. Pawlak Rough Sets

Definition 1. ([27,28]) *Let U be a nonempty finite universe and R be an equivalence relation in U. (U, R) is called a Pawlak approximation space. $\forall X \subseteq U$, the lower and upper approximations of X, denoted by $\underline{R}(X)$ and $\overline{R}(X)$, are defined as follows, respectively:*

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

where $[x]_R = \{y \in U \mid (x, y) \in R\}$. $\underline{R}$ and $\overline{R}$ are called the lower and upper approximation operators, respectively. The pair $(\underline{R}(X), \overline{R}(X))$ is called a Pawlak rough set.

Furthermore, the positive region, boundary region and negative region of the subset X are defined by

$$\mathrm{pos}(X) = \underline{R}(X), \quad \mathrm{neg}(X) = U - \overline{R}(X), \quad \mathrm{bn}(X) = \overline{R}(X) - \underline{R}(X).$$

2.2. Single-Valued Neutrosophic Sets and Single-Valued Neutrosophic Rough Sets

Definition 2. ([7]) *Let U be a space of points (objects), with a generic element in U denoted by x. A SVNS $\widetilde{A}$ in U is characterized by three membership functions, a truth membership function $T_{\widetilde{A}}$, an indeterminacy membership function $I_{\widetilde{A}}$ and a falsity membership function $F_{\widetilde{A}}$, where $\forall x \in U$, $T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x) \in [0, 1]$.*

The SVNS $\widetilde{A}$ can be denoted by $\widetilde{A} = \{\langle x, T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x)\rangle \mid x \in U\}$ or $\widetilde{A} = (T_{\widetilde{A}}, I_{\widetilde{A}}, F_{\widetilde{A}})$. $\forall x \in U$, $\widetilde{A}(x) = (T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x))$, and $(T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x))$ is called a single-valued neutrosophic number.

Definition 3. ([44]) *An SVNS $\widetilde{R}$ in $U \times U$ is called a single-valued neutrosophic relation (SVNR) in U, denoted by $\widetilde{R} = \{\langle (x, y), T_{\widetilde{R}}(x, y), I_{\widetilde{R}}(x, y), F_{\widetilde{R}}(x, y)\rangle \mid (x, y) \in U \times U\}$, where $T_{\widetilde{R}} : U \times U \longrightarrow [0, 1]$, $I_{\widetilde{R}} : U \times U \longrightarrow [0, 1]$ and $F_{\widetilde{R}} : U \times U \longrightarrow [0, 1]$ denote the truth membership function, indeterminacy membership function and falsity membership function of $\widetilde{R}$, respectively.*

Definition 4. ([45]) *Let $\widetilde{R}, \widetilde{S}$ be two SVNRs in U. If $\forall x, y \in U$, $T_{\widetilde{R}}(x, y) \leq T_{\widetilde{S}}(x, y)$, $I_{\widetilde{R}}(x, y) \geq I_{\widetilde{S}}(x, y)$ and $F_{\widetilde{R}}(x, y) \geq F_{\widetilde{S}}(x, y)$, then we say $\widetilde{R}$ is contained in $\widetilde{S}$, denoted by $\widetilde{R} \widetilde{\subseteq} \widetilde{S}$. In other words, we say $\widetilde{S}$ contains $\widetilde{R}$, denoted by $\widetilde{S} \widetilde{\supseteq} \widetilde{R}$.*

Definition 5. ([24]) *Let $\widetilde{R}$ be an SVNR in U. If $\forall x \in U$, $T_{\widetilde{R}}(x, x) = 1$ and $I_{\widetilde{R}}(x, x) = F_{\widetilde{R}}(x, x) = 0$, then $\widetilde{R}$ is called a reflexive SVNR. If $\forall x, y \in U$, $T_{\widetilde{R}}(x, y) = T_{\widetilde{R}}(y, x)$, $I_{\widetilde{R}}(x, y) = I_{\widetilde{R}}(y, x)$ and $F_{\widetilde{R}}(x, y) = F_{\widetilde{R}}(y, x)$, then $\widetilde{R}$ is called a symmetric SVNR. If $\forall x \in U$, $\bigvee_{y \in U} T_{\widetilde{R}}(x, y) = 1$ and $\bigwedge_{y \in U} I_{\widetilde{R}}(x, y) = \bigwedge_{y \in U} F_{\widetilde{R}}(x, y) = 0$, then $\widetilde{R}$ is called a serial SVNR. If $\forall x, y, z \in U$, $\bigvee_{y \in U}(T_{\widetilde{R}}(x, y) \wedge T_{\widetilde{R}}(y, z)) \leq T_{\widetilde{R}}(x, z)$, $\bigwedge_{y \in U}(I_{\widetilde{R}}(x, y) \vee I_{\widetilde{R}}(y, z)) \geq I_{\widetilde{R}}(x, z)$ and $\bigwedge_{y \in U}(F_{\widetilde{R}}(x, y) \vee F_{\widetilde{R}}(y, z)) \geq F_{\widetilde{R}}(x, z)$, then $\widetilde{R}$ is called a transitive SVNR, where "$\vee$" and "$\wedge$" denote maximum and minimum, respectively.*

Definition 6. ([24]) *Let $\widetilde{R}$ be an SVNR in U; the tuple $(U, \widetilde{R})$ is called a single-valued neutrosophic approximation space. $\forall \widetilde{A} \in SVNS(U)$, the lower and upper approximations of $\widetilde{A}$ w.r.t. $(U, \widetilde{R})$, denoted by $\underline{\widetilde{R}}(\widetilde{A})$ and $\overline{\widetilde{R}}(\widetilde{A})$, are two SVNSs whose membership functions are defined as: $\forall x \in U$,*

$T_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y \in U}(F_{\widetilde{R}}(x, y) \vee T_{\widetilde{A}}(y))$,
$I_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y \in U}((1 - I_{\widetilde{R}}(x, y)) \wedge I_{\widetilde{A}}(y))$,
$F_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y \in U}(T_{\widetilde{R}}(x, y) \wedge F_{\widetilde{A}}(y))$;
$T_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y \in U}(T_{\widetilde{R}}(x, y) \wedge T_{\widetilde{A}}(y))$,
$I_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y \in U}(I_{\widetilde{R}}(x, y) \vee I_{\widetilde{A}}(y))$,
$F_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y \in U}(F_{\widetilde{R}}(x, y) \vee F_{\widetilde{A}}(y))$.

The pair $(\underline{\widetilde{R}}(\widetilde{A}), \overline{\widetilde{R}}(\widetilde{A}))$ is called the single-valued neutrosophic rough set of $\widetilde{A}$ w.r.t. $(U, \widetilde{R})$. $\underline{\widetilde{R}}$ and $\overline{\widetilde{R}}$ are referred to as the single-valued neutrosophic lower and upper approximation operators, respectively.

3. Rough Set Model in Generalized Single-Valued Neutrosophic Approximation Spaces

Guo et al. [29] studied the rough set model on two different universes in intuitionistic fuzzy approximation space. In this section, we will extend the rough set model in [29] to a single-valued neutrosophic environment.

Yang et al. [24] proposed the notions of single-valued neutrosophic relations from U to V and generalized single-valued neutrosophic approximation spaces as follows.

Definition 7. ([24]) *Let U and V be two nonempty finite universes. The relation $\widetilde{R}$ in $U \times V$ is called a single-valued neutrosophic relation from U to V, denoted by $\widetilde{R} = \{\langle (x,y), T_{\widetilde{R}}(x,y), I_{\widetilde{R}}(x,y), F_{\widetilde{R}}(x,y)\rangle | (x,y) \in U \times V\}$, where $T_{\widetilde{R}} : U \times V \longrightarrow [0,1]$, $I_{\widetilde{R}} : U \times V \longrightarrow [0,1]$ and $F_{\widetilde{R}} : U \times V \longrightarrow [0,1]$ denote the truth membership function, indeterminacy membership function and falsity membership function of $\widetilde{R}$, respectively.*

The triple $(U, V, \widetilde{R})$ is called a generalized single-valued neutrosophic approximation space on two different universes.

Remark 1. *If $U = V$, then we call $\widetilde{R}$ a single-valued neutrosophic relation in U.*

Definition 8. *Let $\widetilde{R}$ be an SVNR from U to V. If $\forall x \in U, y \in V$, $T_{\widetilde{R}}(x,y) = T_{\widetilde{R}}(y,x)$, $I_{\widetilde{R}}(x,y) = I_{\widetilde{R}}(y,x)$ and $F_{\widetilde{R}}(x,y) = F_{\widetilde{R}}(y,x)$, then $\widetilde{R}$ is called a symmetric SVNR. If $\forall x \in U$, $\bigvee_{y \in V} T_{\widetilde{R}}(x,y) = 1$ and $\bigwedge_{y \in V} I_{\widetilde{R}}(x,y) = \bigwedge_{y \in V} F_{\widetilde{R}}(x,y) = 0$, then $\widetilde{R}$ is called a serial SVNR.*

The union, intersection and containmentof two SVNRs from U to V are defined as follows, respectively.

Definition 9. *Let R, S be two SVNRs from U to V.*

(1) The union $R \widetilde{\cup} S$ of R and S is defined by $R \widetilde{\cup} S = \{\langle (x,y), \max\{T_R(x,y), T_S(x,y)\}, \min\{I_R(x,y), I_S(x,y)\}, \min\{F_R(x,y), F_S(x,y)\}\rangle | (x,y) \in U \times V\}$.

(2) The intersection $R \widetilde{\cap} S$ of R and S is defined by $R \widetilde{\cap} S = \{\langle (x,y), \min\{T_R(x,y), T_S(x,y)\}, \max\{I_R(x,y), I_S(x,y)\}, \max\{F_R(x,y), F_S(x,y)\}\rangle | (x,y) \in U \times V\}$.

(3) If $\forall (x,y) \in U \times V$, $T_R(x,y) \le T_S(x,y)$, $I_R(x,y) \ge I_S(x,y)$ and $F_R(x,y) \ge F_S(x,y)$, then we say R is contained in S, denoted by $R \widetilde{\subseteq} S$.

Next, we give the notion of (α, β, γ)-cut relation $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ of a single-valued neutrosophic relation $\widetilde{R}$ from U to V.

Definition 10. *Let U, V be two nonempty finite universes and $\widetilde{R}$ be a single-valued neutrosophic relation from U to V. For any $\alpha, \beta, \gamma \in (0,1]$, we define the (α, β, γ)-cut relation $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ of $\widetilde{R}$ as follows:*

$$\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} = \{(x,y) \in U \times V | T_{\widetilde{R}}(x,y) \ge \alpha, I_{\widetilde{R}}(x,y) \le \beta, F_{\widetilde{R}}(x,y) \le \gamma\}.$$

According to Definition 10, if $(x,y) \in \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$, it indicates that the truth membership degree of the relationships of x and y w.r.t. SVNR $\widetilde{R}$ is not less than α, and the indeterminacy membership degree and falsity membership degree of the relationships of x and y w.r.t. SVNR $\widetilde{R}$ are not more than β and γ, respectively.

Definition 11. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is the (α, β, γ)-cut relation defined in Definition 8. For any $x \in U, y \in V$, we define*

$$\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha, I_{\widetilde{R}}(x,y) \leq \beta, F_{\widetilde{R}}(x,y) \leq \gamma\},$$
$$\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}(y) = \{x \in U | T_{\widetilde{R}}(x,y) \geq \alpha, I_{\widetilde{R}}(x,y) \leq \beta, F_{\widetilde{R}}(x,y) \leq \gamma\}.$$

The following Definition 12 gives a rough set model on two universes based on the (α, β, γ)-cut relation $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ induced by a single-valued neutrosophic relation $\widetilde{R}$ from U to V.

Definition 12. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. Suppose $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is the (α, β, γ)-cut relation given in Definition 10 from U to V. For any set $Y \subseteq V$, the lower approximation and upper approximation of Y on two universes w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ) are defined by*

$$\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\},$$
$$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}.$$

The pair $(\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y), \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y))$ is called the rough set of Y w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ). If $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, then Y is called the definable set w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ). If $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \neq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, then Y is called the undefinable set w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ).

Next, we define the positive region $\text{pos}_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, negative region $\text{neg}_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ and boundary region $\text{bn}_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ of Y, respectively:

$$pos_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y),$$
$$neg_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \overline{U - \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y),$$
$$bn_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) - \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

Remark 2. *If $\widetilde{R}$ is a series single-valued neutrosophic relation from U to V, i.e., $\forall x \in U, \bigvee_{y \in U} T_{\widetilde{R}}(x,y) = 1$ and $\bigwedge_{y \in U} I_{\widetilde{R}}(x,y) = \bigwedge_{y \in U} F_{\widetilde{R}}(x,y) = 0$, then there exists $y \in V$ such that $T_{\widetilde{R}}(x,y) = 1$ and $I_{\widetilde{R}}(x,y) = 0, F_{\widetilde{R}}(x,y) = 0$ for all $x \in U$ since V is finite. Therefore, for any $\alpha, \beta, \gamma \in (0,1]$, we have $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset$. Therefore, we have*

$$\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\},$$
$$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\},$$
$$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}.$$
$$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\}.$$

In the following, we discuss some properties of the lower approximation and the upper approximation given in Definition 12.

Theorem 1. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. Suppose $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is the (α, β, γ)-cut relation given in Definition 10 from U to V. For any $Y, Y_1, Y_2 \subseteq V$, the following properties hold:*

(1) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;

(2) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\emptyset) = \emptyset, \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(V) = U$;

(3) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) = \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2) = \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$;

(4) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$;

(5) *If* $Y_1 \subseteq Y_2$, *then* $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$ *and* $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$;

(6) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \sim \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\sim Y)$, $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \sim \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\sim Y)$.

Proof. We only prove (3) and (6).

(3) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y_1 \cap Y_2 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y_1 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y_2 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y_1 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \cap \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y_2 \text{ and }$
$\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$;
$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (Y_1 \cup Y_2) \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}$
$= \{x \in U | (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y_1) \cup (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y_2) \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y_1 \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \cup \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y_2 \neq \emptyset$
$\text{or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}$
$= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$.

(6) $\sim \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\sim Y)$
$= \sim \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (\sim Y) \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (\sim Y) = \emptyset \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;
$\sim \underline{\widetilde{R}_{\{}(\alpha,\beta,\gamma)\}}(\sim Y)$
$= \sim \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq (\sim Y) \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \sim \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y = \emptyset \text{ and } x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}$
$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}$
$= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)}}(Y)$. □

Remark 3. *In general,*

(1) $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\emptyset) \neq \emptyset$, $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(V) \neq U$;

(2) $\underline{\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}}(Y_1 \cup Y_2) \neq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$ *and* $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) \neq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}}(Y_2)$,

as shown in the following example.

Example 1. *Let* $U = \{x_1, x_2, x_3\}$, $V = \{y_1, y_2, y_3, y_4\}$. $Y_1 = \{y_1\}$ *and* $Y_2 = \{y_3\}$. *The single-valued neutrosophic relation* $\widetilde{R}$ *from* U *to* V *is given in Table 1.*

Table 1. The single-valued neutrosophic relation $\widetilde{R}$ from U to V.

$\widetilde{R}$	y_1	y_2	y_3	y_4
x_1	(0.7,0,1,0.2)	(0.8,0.3,0.2)	(0.7,0.2,0.1)	(1,0,0)
x_2	(1,0.3,0.1)	(0,0,1)	(0.7,0.3,0.2)	(0,0,1)
x_3	(0.2,0.1,0.8)	(0.1,0.2,0.7)	(0.8,0.2,0.1)	(0,0.3,1)

(1) *Take* $\alpha = 0.9$, $\beta = 0.3$ *and* $\gamma = 0.2$; *we have*

$\widetilde{R}_{\{(0.9,0.3,0.2)\}}(x_1) = \{y_4\}$, $\widetilde{R}_{\{(0.9,0.3,0.2)\}}(x_2) = \{y_1\}$, $\widetilde{R}_{\{(0.9,0.3,0.2)\}}(x_3) = \{y_3\}$.

By Definition 12, we have $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(\emptyset) = \{x_3\} \neq \emptyset$ *and* $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(V) = \{x_1, x_2\} \neq U$.

(2) *Take* $\alpha = 0.5$, $\beta = 0.3$ *and* $\gamma = 0.2$; *we have*

$\widetilde{R}_{\{(0.5,0.3,0.2)\}}(x_1) = \{y_1, y_2, y_3, y_4\}$, $\widetilde{R}_{\{(0.5,0.3,0.2)\}}(x_2) = \{y_1, y_3\}$, $\widetilde{R}_{\{(0.5,0.3,0.2)\}}(x_3) = \emptyset$.

By Definition 12, we have

$$I\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) = \emptyset, \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2) = \{x_3\}, \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2) = \{x_2, x_3\}.$$
$$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) = \{x_1, x_2\}, \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2) = \{x_1, x_2, x_3\}, \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) = \emptyset.$$

Obviously, $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2) \neq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$ *and* $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) \neq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)$.

Theorem 2. *Let* $(U, V, \widetilde{R})$ *be a generalized single-valued neutrosophic approximation space.* $\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}$ *and* $\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}$ *are two relations defined in Definition 10. If* $\widetilde{R}$ *is a series,* $\alpha_1 \leq \alpha_2$, $\beta_1 \geq \beta_2$ *and* $\gamma_1 \geq \gamma_2$, *then*

(1) $\underline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$;
(2) $\overline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$.

Proof. (1) Since $\alpha_1 \leq \alpha_2$, $\beta_1 \geq \beta_2$ and $\gamma_1 \geq \gamma_2$, for any $x \in U$, we have

$$\begin{aligned}\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) &= \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha_1, I_{\widetilde{R}}(x,y) \leq \beta_1, F_{\widetilde{R}}(x,y) \leq \gamma_1\} \\ &\supseteq \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha_2, I_{\widetilde{R}}(x,y) \leq \beta_2, F_{\widetilde{R}}(x,y) \leq \gamma_2\} \\ &= \widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}(x).\end{aligned}$$

By Definition 12, for any $x \in \underline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y)$, we have $\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) \subseteq Y$. Thus $\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}(x) \subseteq \widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) \subseteq Y$, which implies that $x \in \underline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$. Hence, $\underline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$.

(2) By (1), for any $x \in U$, we have $\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) \supseteq \widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}(x)$. So

$$\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) \cap Y \supseteq \widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}(x) \cap Y \textit{ for any } x \in U.$$

By Definition 12, for any $x \in \overline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$, we have $\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}(x) \cap Y \neq \emptyset$. Thus, $\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}(x) \cap Y \neq \emptyset$, which implies that $x \in \overline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y)$. Hence, $\overline{\widetilde{R}_{\{(\alpha_1,\beta_1,\gamma_1)\}}}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha_2,\beta_2,\gamma_2)\}}}(Y)$. □

Theorem 3. *Let* $\widetilde{R}, \widetilde{S}$ *be two series single-valued neutrosophic relations from* U *to* V. *If* $\widetilde{R} \widetilde{\subseteq} \widetilde{S}$, *then* $\forall Y \subseteq V$, *we have:*

(1) $\underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;

(2) $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

Proof. (1) Since $\widetilde{R}\widetilde{\subseteq}\widetilde{S}$, we have

$$T_{\widetilde{S}}(x,y) \geq T_{\widetilde{R}}(x,y), I_{\widetilde{S}}(x,y) \leq I_{\widetilde{R}}(x,y) \text{ and } F_{\widetilde{S}}(x,y) \leq F_{\widetilde{R}}(x,y) \text{ for any } (x,y) \in U \times V.$$
$$\text{Then}, \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) = \{y \in V | T_{\widetilde{S}}(x,y) \geq \alpha, I_{\widetilde{S}}(x,y) \leq \beta, F_{\widetilde{S}}(x,y) \leq \gamma\}$$
$$\supseteq \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha, I_{\widetilde{R}}(x,y) \leq \beta, F_{\widetilde{R}}(x,y) \leq \gamma\}$$
$$= \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x).$$

By Definition 12, for any $x \in \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, we have $\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y$. Thus, $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y$, which implies that $x \in \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$. Hence, $\underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

(2) By (1), for any $x \in U$, we have $\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \supseteq \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x)$. Thus, $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \subseteq \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y$ for any $x \in U$. By Definition 12, for any $x \in \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, we have $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset$. Thus, $\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset$, which implies that $x \in \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$. Hence, $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$. □

Lemma 1. *Let $\widetilde{R}, \widetilde{S}$ be two single-valued neutrosophic relations from U to V. For any $x \in U$ and $\alpha, \beta, \gamma \in (0,1]$, we have:*

(1) $(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) \supseteq \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cup \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)$;
(2) $(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) = \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)$.

Proof. (1) For any $x \in U$, we have:

$$(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) = \{y \in V | \max(T_{\widetilde{R}}(x,y), T_{\widetilde{S}}(x,y)) \geq \alpha, \min(I_{\widetilde{R}}(x,y), I_{\widetilde{S}}(x,y)) \leq \beta,$$
$$\min(F_{\widetilde{R}}(x,y), F_{\widetilde{S}}(x,y)) \leq \gamma\}$$
$$\supseteq \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha, I_{\widetilde{R}}(x,y) \leq \beta, F_{\widetilde{R}}(x,y) \leq \gamma\}$$
$$\cup \{y \in V | T_{\widetilde{S}}(x,y) \geq \alpha, I_{\widetilde{S}}(x,y) \leq \beta, F_{\widetilde{S}}(x,y) \leq \gamma\}$$
$$= \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cup \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)$$

(2) For any $x \in U$, we have:

$$(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) = \{y \in V | \min(T_{\widetilde{R}}(x,y), T_{\widetilde{S}}(x,y)) \geq \alpha, \max(I_{\widetilde{R}}(x,y), I_{\widetilde{S}}(x,y)) \leq \beta,$$
$$\max(F_{\widetilde{R}}(x,y), F_{\widetilde{S}}(x,y)) \leq \gamma\}$$
$$= \{y \in V | T_{\widetilde{R}}(x,y) \geq \alpha, I_{\widetilde{R}}(x,y) \leq \beta, F_{\widetilde{R}}(x,y) \leq \gamma\}$$
$$\cap \{y \in V | T_{\widetilde{S}}(x,y) \geq \alpha, I_{\widetilde{S}}(x,y) \leq \beta, F_{\widetilde{S}}(x,y) \leq \gamma\}$$
$$= \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)$$

□

Theorem 4. *Let $\widetilde{R}, \widetilde{S}$ be two series single-valued neutrosophic relations from U to V. For any $Y \subseteq V$ and $\alpha, \beta, \gamma \in (0,1]$, we have:*

(1) $\underline{(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;
(2) $\overline{(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \supseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

Proof. (1) By Lemma 1 (1), we have

$$\underline{(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | (\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\}$$

$$\subseteq \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cup \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\}$$
$$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\} \cap \{x \in U | \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\}$$
$$= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

So

$$\underline{(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

(2) By Lemma 1 (1), we have

$$\overline{(\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | (\widetilde{R}\widetilde{\cup}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\}$$
$$\supseteq \{x \in U | (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cup \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)) \cap Y \neq \emptyset\}$$
$$= \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\} \cup \{x \in U | \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\}$$
$$= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

□

Theorem 5. *Let $\widetilde{R}, \widetilde{S}$ be two series single-valued neutrosophic relations from U to V. For any $Y \subseteq V$ and $\alpha, \beta, \gamma \in (0, 1]$, we have:*

(1) $\underline{(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;

(2) $\overline{(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

Proof. (1) By Lemma 1 (2), we have

$$\underline{(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | (\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\}$$
$$= \{x \in U | (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x)) \subseteq Y\}$$
$$\supseteq \{x \in U | (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\} \cup \{x \in U | (\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y\}$$
$$= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

Therefore,

$$\underline{(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cup \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \underline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

(2) By Lemma 1 (2), we have

$$\overline{(\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}}(Y) = \{x \in U | (\widetilde{R}\widetilde{\cap}\widetilde{S})_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\}$$
$$= \{x \in U | (\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} \cap \widetilde{S}_{\{(\alpha,\beta,\gamma)\}})(x) \cap Y \neq \emptyset\}$$
$$\subseteq \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\} \cap \{x \in U | \widetilde{S}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset\}$$
$$= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \cap \overline{\widetilde{S}_{\{(\alpha,\beta,\gamma)\}}}(Y).$$

□

Next, we define the inverse lower approximation and upper approximation on two universes w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ) as follows:

Definition 13. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. For any $\alpha, \beta, \gamma \in (0, 1]$, $X \subseteq U$, the inverse lower approximation and upper approximation of X on two universes w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ) are defined as:*

$$\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X) = \{y \in V | \widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}(y) \subseteq X \text{ and } \widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}(y) \neq \emptyset\},$$
$$\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X) = \{y \in V | \widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}(y) \cap X \neq \emptyset \text{ or } \widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}(y) = \emptyset\}.$$

The pair $(\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X), \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X))$ is called the inverse rough set of X w.r.t. $(U, V, \widetilde{R})$ and (α, β, γ).

Theorem 6. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is the (α, β, γ)-cut relation given in Definition 10 from U to V, where $\alpha, \beta, \gamma \in (0, 1]$. For any $X, X_1, X_2 \subseteq U$, we have:*

(1) $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X) \subseteq \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X)$;

(2) $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(\emptyset) = \emptyset$, $\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(U) = V$;

(3) $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1 \cap X_2) = \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \cap \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$,

$\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1 \cup X_2) = \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \cup \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$;

(4) $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1 \cup X_2) \supseteq \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \cup \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$,

$\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1 \cap X_2) \subseteq \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \cap \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$;

(5) *If $X_1 \subseteq X_2$, then $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \subseteq \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$ and $\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_1) \subseteq \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X_2)$;*

(6) $\underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X) = \sim \overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(\sim X)$, $\overline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(X) = \sim \underline{\widetilde{R}^{-1}_{\{(\alpha,\beta,\gamma)\}}}(\sim X)$.

Proof. The proof is similar to that of Theorem 1. □

Definition 14. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is a (α, β, γ)-cut relation defined in Definition 10. For any $Y \subseteq V$, the approximate precision $\rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ of Y w.r.t. $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is defined as follows:*

$$\rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)|},$$

where $|Y|$ represents the cardinality of the set Y.

Let $\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = 1 - \rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, and $\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ is called the rough degree of with regard to $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$. It is obviously that $0 \leq \rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \leq 1$ and $0 \leq \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \leq 1$.

The following Theorem 7 discusses the properties of approximation precision and rough degree.

Theorem 7. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is a (α, β, γ)-cut relation defined in Definition 10. For any $Y_1, Y_2 \subseteq V(Y_1 \neq \emptyset, Y_2 \neq \emptyset)$, then the rough degree and the approximate precision of the set $Y_1, Y_2, Y_1 \cup Y_2$, and $Y_1 \cap Y_2$ satisfy the following properties:*

(1) $\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \leq \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$;

(2) $\rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \geq \rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + \rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - \rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(A \cap Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$.

Proof. According to the definition of the rough degree, we have

$$\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2) = 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|}$$
$$= 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}$$
$$\leq 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}.$$

Then, we have

$$\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$$
$$\leq |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|.$$

Similarly, we have

$$\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2) = 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|}$$
$$= 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|}$$
$$\leq 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}.$$

Hence, $\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \leq |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$.

Furthermore, we know $|A \cup B| = |A| + |B| - |A \cap B|$ holds for any sets A and B. Then,

$$\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$$
$$\leq |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$$
$$= |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|$$
$$-|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| + |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|$$
$$= |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - (|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)$$
$$\cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|)$$
$$\leq |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)$$
$$|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|.$$

Furthermore, by

$$\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) = 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|} \textit{ and } \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2) = 1 - \frac{|\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}{|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|}, \textit{we have}$$
$$|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| = \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| \textit{ and}$$
$$|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| = \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|.$$

Therefore,

$$
\begin{aligned}
&\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cup Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \\
&\leq |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + |\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| - |\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \\
&\quad -\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| \\
&= \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1)| + \mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)| \\
&\quad -\mu_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1 \cap Y_2)|\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y_2)|.
\end{aligned}
$$

□

4. Two Extended Models

In this section, we give two extended rough set models of the model in Section 3, i.e., the degree rough set model and the variable precision rough set model.

4.1. The Degree Rough Set Model on Two Different Universes

Definition 15. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space and $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ be a (α,β,γ)-cut relation defined in Definition 10. For any $Y \subseteq V$, we define the degree lower and upper approximations of Y w.r.t. the degree k, $(U, V, \widetilde{R})$ and (α,β,γ) as follows:*

$$
\begin{aligned}
\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) &= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x)| - |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\}; \\
\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) &= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y| > k \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\}.
\end{aligned}
$$

where k is a finite nonnegative integer and $|Y|$ denotes the cardinality of the set Y.

The pair $(\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y), \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y))$ is called the degree rough set of Y w.r.t. the degree k, $(U, V, \widetilde{R})$ and (α,β,γ).

We also define the positive region $\text{pos}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}}(Y)$, negative region $\text{neg}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}}(Y)$ and boundary region $\text{bn}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}(Y)}$ of Y as follows:

$$
\begin{aligned}
\text{pos}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}}(Y) &= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y), \\
\text{neg}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}}(Y) &= U - \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y), \\
\text{bn}_{\widetilde{R}^k_{\{(\alpha,\beta,\gamma)\}}}(Y) &= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) - \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y).
\end{aligned}
$$

Remark 4. *In Definition 15, if $k = 0$, then*

$$
\begin{aligned}
\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y) &= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq 0 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| = 0 \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \{x \in U \,|\, \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \subseteq Y \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y); \\
\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y) &= \{x \in U \,|\, |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y| > 0 \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \\
&= \{x \in U \,|\, \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y \neq \emptyset \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \\
&= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y),
\end{aligned}
$$

which implies that the lower and upper approximation operators in Definition 12 are special cases of the degree lower and upper approximation operators in Definition 15, respectively.

In following Theorem 8, we discuss some properties of the degree lower and upper approximation operators.

Theorem 8. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space, and $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ is the (α, β, γ)-cut relation defined in Definition 10. For any $Y, Y_1, Y_2 \subseteq V$, we have:*

(1) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_1 \cap Y_2) = \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_2)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_1 \cup Y_2) = \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_1 \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_2;$

(2) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cap Y_2) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cup Y_2) \supseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \cup \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$;

(3) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cup A_2) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cap Y_2) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \cap \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$;

(4) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) =\sim \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(\sim Y)$,

$\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) =\sim \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(\sim Y)$;

(5) *If $Y_1 \subseteq Y_2$, then $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$ and $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$;*

(6) *If $k \geq l$, then $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{l}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y)$ and $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{l}(Y)$, where k is a finite positive integer.*

Proof. We only prove (4) and (6).

(4) Notice that $Y_1 - Y_2 = Y_1 \cap (\sim Y_2)$ and $\sim (\sim Y) = Y$ for any set $Y \in V$; we have

$$\begin{aligned}
\sim \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(\sim Y) &=\sim \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (\sim Y)| > k \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \\
&=\sim \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| > k \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \\
&= \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (\sim Y)| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y), \\
\sim \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(\sim Y) &=\sim \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - (\sim Y)| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&=\sim \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (Y)| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (Y)| > k \text{ or } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\} \\
&= \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y).
\end{aligned}$$

(6) Since $k \geq l$, for any $x \in U$ and $Y \subseteq V$, we have

$$\begin{aligned}
\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{l}(Y) &= \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq l \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&\subseteq \{x \in U || \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq k \text{ and } \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \\
&= \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y).
\end{aligned}$$

For any $x \in \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{l}(Y)$, we have $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq l$ and $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset$, then we have $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) - Y| \leq k$ and $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset$, so $x \in \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y)$. Hence, $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{l}(Y) \subseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y)$. □

Remark 5. *In general, $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cap Y_2) \supseteq \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \cap \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$ does not hold, where k is a finite positive integer. The reason is that if $k > 0$, then $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} - Y_1| \leq k$ and $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} - Y_2| \leq k \not\Longrightarrow |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} - Y_1 \cap Y_2| \leq k$. Besides, $\overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1 \cup Y_2) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_1) \cup \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{k}(Y_2)$*

also does not hold in general, where k is a finite positive integer. The reason is that if $k > 0$, then $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap (Y_1 \cup Y_2)| > k \not\Longrightarrow |\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} \cap Y_1| > k$ or $|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}} \cap Y_2| > k$.

4.2. The Variable Precision Rough Set Model on Two Different Universes

Definition 16. *Let $(U, V, \widetilde{R})$ be a generalized single valued neutrosophic approximation space. For any $Y \subseteq V, \alpha, \beta, \gamma \in (0, 1]$, we define the variable precision lower and upper approximation of Y w.r.t. the control parameter ν, $(U, V, \widetilde{R})$ and (α, β, γ) as follows, respectively:*

$$\underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y) = \{x \in U | \frac{|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y|}{|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x)|} \geq 1 - \nu, \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\},$$

$$\overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y) = \{x \in U | \frac{|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \cap Y|}{|\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x)|} > \nu, \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) \neq \emptyset\} \cup \{x \in U | \widetilde{R}_{\{(\alpha,\beta,\gamma)\}}(x) = \emptyset\},$$

where $\nu \in [0, 0.5)$, $|Y|$ denotes the cardinality of the set Y.

The pair $(\underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}, \overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu})$ is called the variable precision single-valued neutrosophic rough set of A with regard to the control parameter ν, $(U, V, \widetilde{R})$ and (α, β, γ).

We also define the positive region $\text{pos}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, negative region $\text{neg}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ and boundary region $\text{bn}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y)$ of Y about $\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}(Y)$ as follows:

$$\text{pos}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y),$$

$$\text{neg}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y) = U - \overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y),$$

$$\text{bn}_{V\widetilde{R}^{\nu}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y) - \underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y).$$

The following Theorems 9 and 10 can be easily obtained by Definition 16.

Theorem 9. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. For any $Y \subseteq V$, $\alpha, \beta, \gamma \in [0, 1]$, $\nu = [0, 0.5)$, then:*

(1) $\underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) \subseteq \underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y)$;

(2) $\overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y) \subseteq \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

Proof. The proof is straightforward from Definition 16. □

Remark 6. *In Theorem 9, if $\nu = 0$, then:*

(1) $\underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y) = \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$;

(2) $\overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0}(Y) = \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$.

Theorem 10. *Let $(U, V, \widetilde{R})$ be a generalized single-valued neutrosophic approximation space. For any $Y \subseteq V$, $\alpha, \beta, \gamma \in [0, 1]$. Then:*

(1) $\underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0.5}(Y) = \bigcup_{\nu \in [0,0.5)} \underline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y)$;

(2) $\overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{0.5}(Y) = \bigcap_{\nu \in [0,0.5)} \overline{V\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}^{\nu}(Y)$.

Proof. The proof is straightforward from Definition 16. □

5. An Application in Multi-Attribute Decision Making

In this section, we illustrate the application of the rough set model in generalized single-valued neutrosophic spaces proposed in Section 3.

We present the medical decision procedure based on the proposed rough set model in a single-valued neutrosophic environment as follows.

Assume that the universe $U = \{x_1, x_2, x_3, \cdots, x_m\}$ denotes a set of patients and the universe $V = \{x_1, x_2, x_3, \cdots, x_n\}$ denotes a set of symptoms. Let $\widetilde{R}$ be an SVNR from U to V. For any $(x_i, y_j) \in U \times V$, $T_{\widetilde{R}}(x_i, y_j), I_{\widetilde{R}}(x_i, y_j), F_{\widetilde{R}}(x_i, y_j)$ represent the truth membership degree, indeterminacy membership degree and falsity membership degree of patient x_i with symptoms y_j, respectively. Given a patient x_i, the doctor can present the relationship degree decreased by a single-valued neutrosophic number, i.e., $\widetilde{R}(x_i, y_j)$ between the patient x_i and the symptom y_j. The (α, β, γ) is given in advance by another doctor and represents the doctors' lowest requirements on the membership degree. For any $Y \subseteq V$, Y denotes a certain disease and has the basic symptoms $y_j \in Y$. For a patient x_i, if $x_i \in \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, then we can be sure that the patient x_i is suffering from the disease Y and must receive treatment immediately; if $x_i \in \overline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) - \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, we cannot be sure whether the patient x_i is suffering from the disease Y or not. Therefore, the doctor needs to carry out a second choice to decide whether the patient x_i is suffering from the disease Y or not; if $x_i \in U - \underline{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y)$, then we can be sure that the patient x_i is not suffering from the disease Y and does not need to receive treatment.

Next, we give an example to illustrate the method above.

Example 2. *Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be five patients, $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ be seven symptoms, where y_j $(j = 1, 2, 3, 4, 5, 6, 7)$ stand for "tired", "a stuffed-up nose", "headache", "chill","stomach pain", "dry cough" and "chest-pain". $\widetilde{R}$ is the medical knowledge statistic data of the relationship of the patient x_i $(x_i \in U)$ and the symptom y_j $(y_j \in V)$, and $\widetilde{R}$ is an SVNR from U to V (given in Table 2). For any $(x_i, y_j) \in U \times V$, $T_{\widetilde{R}}(x_i, y_j), I_{\widetilde{R}}(x_i, y_j), F_{\widetilde{R}}(x_i, y_j)$ represent the truth membership degree, indeterminacy membership degree and falsity membership degree of patient x_i with symptoms y_j, respectively. For example, $\widetilde{R}(x_1, y_1) = (0.2, 0.1, 0.8)$ indicates that the truth membership, indeterminacy membership and falsity membership of patient x_1 with symptoms y_1 is $0.2, 0.1, 0.8$, respectively.*

Table 2. The single-valued neutrosophic relation $\widetilde{R}$ of the symptoms and patients.

$\widetilde{R}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	(0.6,0,1,0.4)	(1,0,0)	(0.6,0.2,0.2)	(0.8,0.3,0.2)	(0,0,1)	(0.9,0.1,0.2)	(0.1,0.1,0.9)
x_2	(1,0.2,0)	(0,0,1)	(0.8,0.1,0.1)	(0.1,0.1,1.7)	(0,0,1)	(0.8,0.2,0.1)	(0.2,0.1,0.6)
x_3	(0.8,0.1,0.5)	(0,0.3,1)	(0.2,0.2,0.8)	(0.2,0.1,0.8)	(0.7,0.1,0.2)	(0.1,0.2,0.8)	(1,0,0)
x_4	(1,0.3,0.1)	(0,0,1)	(0.3,0.1,0.7)	(0,0,1)	(0,0,1)	(0.7,0.3,0.2)	(0,0,1)
x_5	(0.1,0.2,0.7)	(0,0,1)	(0,0.2,0.9)	(0.2,0.1,0.7)	(1,0,0)	(0,0,1)	(0.7,0.3,0.2)

Let $Y = \{y_1, y_2, y_3, y_6\} \subseteq V$ denote a certain disease showing four basic symptoms in the clinic.

Case 1. Take $(\alpha, \beta, \gamma) = (0.5, 0.5, 0.5)$; by Definition 10, we can get the cut relation $\widetilde{R}_{\{(0.5,0.5,0.5)\}}$ (given in Table 3).

Table 3. The cut relation $\widetilde{R}_{\{(0.5,0.5,0.5)\}}$.

$\widetilde{R}_{\{(0.5,0.5,0.5)\}}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	1	1	1	1	0	1	0
x_2	1	0	1	0	0	1	0
x_3	1	0	0	0	1	0	1
x_4	1	0	0	0	0	1	0
x_5	0	0	0	0	1	0	1

According to Definition 11, we can get

$\widetilde{R}_{\{(0.5,0.5,0.5)\}}(x_1) = \{y_1, y_2, y_3, y_4, y_6\}$,

$\widetilde{R}_{\{(0.5,0.5,0.5)\}}(x_2) = \{y_1, y_3, y_6\}$,
$\widetilde{R}_{\{(0.5,0.5,0.5)\}}(x_3) = \{y_1, y_5, y_7\}$,
$\widetilde{R}_{\{(0.5,0.5,0.5)\}}(x_4) = \{y_1, y_6\}$,
$\widetilde{R}_{\{(0.5,0.5,0.5)\}}(x_5) = \{y_5, y_7\}$.

Then, by Definition 12, we can calculate the lower approximation, the upper approximation, the boundary region and the negative region of Y as follows, respectively.

$\underline{\widetilde{R}_{\{(0.5,0.5,0.5)\}}}(Y) = \{x_2, x_4\}$,
$\overline{\widetilde{R}_{\{(0.5,0.5,0.5)\}}}(Y) = \{x_1, x_2, x_3, x_4\}$,
$bn_{\widetilde{R}_{\{(0.5,0.5,0.5)\}}}(Y) = \{x_1, x_3\}$,
$neg_{\widetilde{R}_{\{(0.5,0.5,0.5)\}}}(Y) = \{x_5\}$.

By Definition 14, we also can calculate the approximate precision of the set Y ($Y \subseteq V$) as follows:

$\rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \frac{1}{2}$.

Thus, we can obtain the following conclusions:

(1) *Patients x_2 and x_4 are suffering from the disease Y and must receive treatment immediately.*
(2) *We cannot determine whether patients x_1 and x_3 are suffering from the disease Y or not.*
(3) *The patient x_5 is not suffering from the disease Y and does not need to receive treatment.*

Case 2. Take $(\alpha, \beta, \gamma) = (0.7, 0.4, 0.3)$. We can obtain the cut relation $\widetilde{R}_{\{(0.7,0.4,0.3)\}}$ (given in Table 4).

Table 4. The cut relation $\widetilde{R}_{\{(0.7,0.4,0.3)\}}$.

$\widetilde{R}_{\{(0.7,0.4,0.3)\}}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	0	1	1	0	0	1	0
x_2	1	0	1	0	0	1	0
x_3	0	0	0	0	1	0	1
x_4	1	0	0	0	0	1	0
x_5	0	0	0	0	1	0	1

According to Definition 11, we can get

$\widetilde{R}_{\{(0.7,0.4,0.3)\}}(x_1) = \{y_2, y_3, y_4, y_6\}$,
$\widetilde{R}_{\{(0.7,0.4,0.3)\}}(x_2) = \{y_1, y_3, y_6\}$,
$\widetilde{R}_{\{(0.7,0.4,0.3)\}}(x_3) = \{y_5, y_7\}$,
$\widetilde{R}_{\{(0.7,0.4,0.3)\}}(x_4) = \{y_1, y_6\}$,
$\widetilde{R}_{\{(0.7,0.4,0.3)\}}(x_5) = \{y_5, y_7\}$.

Then, by Definition 12, we can calculate the lower approximation, the upper approximation, the boundary region and the negative region of Y as follows, respectively.

$\underline{\widetilde{R}_{\{(0.7,0.4,0.3)\}}}(Y) = \{x_2, x_4\}$,
$\overline{\widetilde{R}_{\{(0.7,0.4,0.3)\}}}(Y) = \{x_1, x_2, x_4\}$,
$bn_{\widetilde{R}_{\{(0.7,0.4,0.3)\}}}(Y) = \{x_1\}$,
$neg_{\widetilde{R}_{\{(0.7,0.4,0.3)\}}}(Y) = \{x_3, x_5\}$.

By Definition 14, we also can calculate the approximate precision of the set Y ($Y \subseteq V$) as follows:

$\rho_{\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}}(Y) = \frac{2}{3}$.

Thus, we can obtain the following conclusions:

(1) *Patients x_2 and x_4 are suffering from the disease Y and must receive treatment immediately.*
(2) *We cannot determine whether patient x_1 is suffering from the disease Y or not.*
(3) *Patients x_3 and x_5 are not suffering from the disease Y and do not need to receive treatment.*

Based on the above analysis, the proposed model and method could help decision makers make a scientific and precise decision as they face the similar cases in practice. Besides, the model presented in this paper also permits controlling the risk of misdiagnosis in practice.

To explore the effectiveness of the method proposed in this paper, we compare it with the method proposed in [29]. The method given in [29] only deals with the decision making problems with intuitionistic fuzzy information, but not the decision making problems with the single-valued neutrosophic information; while the model proposed in the present paper can handle the decision making problems not only with intuitionistic fuzzy information (since intuitionistic fuzzy sets can be regarded as a special case of SVNSs), but also with single-valued neutrosophic information. Thus, the proposed method is more general, and its application domain is wider than that of the method in [29].

The proposed model is based on the level cut relation of single-valued neutrosophic relations. There are two advantages. One advantage is that the level parameter in the model can control the risk of the misdiagnosis. Another advantage is that the model can approximate the crisp concept by converting a single-valued neutrosophic fuzzy relation into a crisp binary relation.

The rough set method does not depend on any other extra knowledge besides the given dataset. Rough set theory can be applied as a component of hybrid solutions in machine learning and data mining. They have been found to be particularly useful for rule induction and feature selection. Decision makers can control the size of the loss of information by changing the level parameter.

6. Conclusions

In the present paper, based on the (α, β, γ)-cut relation $\widetilde{R}_{\{(\alpha,\beta,\gamma)\}}$ $(\alpha, \beta, \gamma \in (0, 1])$, we propose a new rough set model in generalized single-valued neutrosophic approximation spaces and obtain two extended models of the model. Some properties are studied. Finally, we use an example to illustrate the proposed rough set model's application in a multi-attribute decision making problem.

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Multiple Attribute Group Decision-Making Method Based on Linguistic Neutrosophic Numbers

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Abstract: Existing intuitionistic linguistic variables can describe the linguistic information of both the truth/membership and falsity/non-membership degrees, but it cannot represent the indeterminate and inconsistent linguistic information. To deal with the issue, this paper originally proposes the concept of a linguistic neutrosophic number (LNN), which is characterized independently by the truth, indeterminacy, and falsity linguistic variables. Then, we define the basic operational laws of LNNs and the score and accuracy functions of LNN for comparing LNNs. Next, we develop an LNN-weighted arithmetic averaging (LNNWAA) operator and an LNN-weighted geometric averaging (LNNWGA) operator to aggregate LNN information and investigate their properties. Further, a multiple attribute group decision-making method based on the proposed LNNWAA or LNNWGA operator is established under LNN environment. Finally, an illustrative example about selecting problems of investment alternatives is presented to demonstrate the application and effectiveness of the developed approach.

Keywords: linguistic neutrosophic number; score function; accuracy function; linguistic neutrosophic number weighted arithmetic averaging (LNNWAA) operator; linguistic neutrosophic number weighted geometric averaging (LNNWGA) operator; multiple attribute group decision-making

1. Introduction

In complex decision-making problems, human judgments, including preference information, may be difficultly stated in numerical values due to the ambiguity of human thinking about the complex objective things in the real world, and then may be easily expressed in linguistic terms, especially for some qualitative attributes. Thus, decision-making problems under linguistic environments are interesting research topics, which have received more and more attentions from researchers in past decades. Zadeh [1] firstly introduced the concept of linguistic variables and the application in fuzzy reasoning. Later, Herrera et al. [2] and Herrera and Herrera-Viedma [3] presented linguistic decision analyses to deal with decision-making problems with linguistic information. Next, Xu [4] put forward a linguistic hybrid arithmetic averaging operator for multiple attribute group decision-making (MAGDM) problems with linguistic information. Further, Xu [5] developed goal programming models for multiple attribute decision-making (MADM) problems with linguistic information. Some scholars [6–8] also proposed two-dimension uncertain linguistic operations and aggregation operators and applied them to decision-making. By combining intuitionistic fuzzy numbers (IFNs) (basic elements in intuitionistic fuzzy sets) introduced in [9] and linguistic variables introduced in [1], Chen et al. [10] proposed the linguistic intuitionistic fuzzy number (LIFN) denoted by the form of $s = (l_p, l_q)$, where l_p and l_q stand for the linguistic variables of the truth/membership and falsity/non-membership degrees, respectively, and developed a MAGDM method with LIFNs. Then, Liu and Wang [11] presented some improved LIFN aggregation operators for MADM. It is obvious that the LIFN consists of two linguistic variables l_p and l_q and describes the linguistic information of both the truth/membership and falsity/non-membership degrees, which are expressed by linguistic values rather than exact values like IFNs. However, LIFNs cannot describe indeterminate and inconsistent linguistic information. Then, a single-valued neutrosophic number (SVNN), which is a basic element in a single-valued neutrosophic set (SVNS) [12,13], can only express the truth, indeterminacy, and falsity degrees independently, and describe the incomplete, indeterminate, and inconsistent information in SVNN rather than linguistic information; then, it cannot express linguistic information in linguistic decision-making problems, while linguistic variables can represent the qualitative information for attributes in complex MADM problems. Hence, Ye [13] proposed the single-valued neutrosophic linguistic number (SVNLN), which is composed of a linguistic variable and an SVNN, where the linguistic variable is represented as the decision-maker's judgment to an evaluated object and the SVNN is expressed as the reliability of the given linguistic variable, and developed an extended TOPSIS method for MAGDM problems with SVNLNs. However, SVNLN cannot also describe the truth, indeterminacy, and falsity linguistic information according to a linguistic term set. Tian et al. [14] put forward a simplified neutrosophic linguistic MAGDM approach for green product development. Liu and Tang [15] presented an interval neutrosophic uncertain linguistic Choquet integral method for MAGDM. Liu and Shi [16] introduced some neutrosophic uncertain linguistic number Heronian mean operators for MAGDM. However, all existing linguistic decision-making methods cannot express and deal with decision-making problems with indeterminate and inconsistent linguistic information.

To overcome the aforementioned insufficiency for SVNNs, LIFNs, and SVNLNs, a feasible solution is to represent the truth, indeterminacy, and falsity degrees independently by three linguistic variables to an evaluated object. On the other hand, human judgments under a linguistic decision-making environment should also contain the linguistic information of truth/determinacy, indeterminacy, and falsity degrees since SVNN contains the information of the truth/determinacy, indeterminacy, and falsity degrees. Based on this idea, it is necessary to propose the concept of a linguistic neutrosophic number (LNN) by combining SVNN and linguistic variables, where its truth, indeterminacy, and falsity degrees can be described by three linguistic variables rather than three exact values, like an SVNN, or both a linguistic value and an SVNN, like an SVNLN. For example, a company wants to select a supplier. Suppose that a decision-maker evaluates it based on a linguistic term set $L = \{l_0$ = extremely low, l_1 = very low, l_2 = low, l_3 = slightly low, l_4 = medium, l_5 = slightly high, l_6 = high, l_7 = very high, l_8 = extremely high$\}$. If the evaluation of a supplier with respect to its service performance is given as l_6 for the truth/membership degree, l_2 for the indeterminacy degree, and l_3 for the falsity/non-membership degree, respectively, by the decision-maker corresponding to the linguistic term set L then, for the concept of an LNN, it can be expressed as the form of an LNN $e = <l_6, l_2, l_3>$. Obviously, LIFN and SVNLN cannot express such kinds of linguistic evaluation values; while LNN can easily describe them in a linguistic setting by the extension of SVNN and LIFN to LNN. Therefore, it is necessary to introduce LNN for expressing indeterminate and inconsistent linguistic information corresponding to human fuzzy thinking about complex problems, especially for some qualitative evaluations for attributes, and solving linguistic decision-making problems with indeterminate and inconsistent linguistic information. However, LNNs are very suitable for describing more complex linguistic information of human judgments under linguistic decision-making environment since LNNs contain the advantages of both SVNNs and linguistic variables, which imply the truth, falsity, and indeterminate linguistic information. To aggregate LNN information in MAGDM problems, we have to develop some weighted aggregation operators, including an LNN-weighted arithmetic averaging (LNNWAA) operator and an LNN-weighted geometric averaging (LNNWGA) operator, which are usually used for MADM/MAGDM problems, score, and accuracy functions for the comparison of LNNs, and their decision-making method. Thus, the purposes of this paper are (1) to propose LNNs and their basic operational laws; (2) to introduce the score and accuracy functions of the LNN for comparing LNNs; (3) to present the LNNWAA and LNNWGA operators, their properties, and special cases; (4) to develop a MAGDM method based on the LNNWAA or LNNWGA operator under an LNN environment; and (5) to explain the advantages of the proposed method.

The rest of this paper is organized as follows: Section 2 briefly reviews the basic concept of LIFNs, the basic operational laws of LIFNs, and the score and accuracy functions of LIFN for the comparison of LIFNs. In Section 3, LNNs and their basic operational laws are presented as the extension of LIFNs, and then the score and accuracy functions for an LNN are defined to compare LNNs. Section 4 develops the LNNWAA and LNNWGA operators for aggregating LNNs and discusses their properties and some special cases. In Section 5, a MAGDM method is developed by using the LNNWAA or LNNWGA operator under LNN environment. In Section 6, an illustrative example about selecting problem of investment alternatives demonstrates the application of the presented method. Section 7 gives conclusions and future research directions.

2. Linguistic Intuitionistic Fuzzy Numbers

Under a linguistic intuitionistic fuzzy environment, Chen et al. [10] introduced the concept of LIFNs and gave the following definition:

Definition 1. *[10] Assume that $L = \{l_0, l_1, \ldots, l_t\}$ is a linguistic term set with odd cardinality $t + 1$, where l_j ($j = 0, 1, \ldots, t$) is a possible value for a linguistic variable. If there is $s = (l_p, l_q)$ for $l_p, l_q \in L$ and $p, q \in [0, t]$, then s is called LIFN.*

Definition 2. *[10] Let $s = (l_p, l_q)$, $s_1 = (l_{p_1}, l_{q_1})$, and $s_2 = (l_{p_2}, l_{q_2})$ be three LIFNs in L and $\rho > 0$, then there are the following operational laws of the LIFNs:*

$$s_1 \oplus s_2 = (l_{p_1}, l_{q_1}) \oplus (l_{p_2}, l_{q_2}) = \left(l_{p_1+p_2-\frac{p_1p_2}{t}}, l_{\frac{q_1q_2}{t}}\right); \tag{1}$$

$$s_1 \otimes s_2 = (l_{p_1}, l_{q_1}) \otimes (l_{p_2}, l_{q_2}) = \left(l_{\frac{p_1p_2}{t}}, l_{q_1+q_2-\frac{q_1q_2}{t}}\right); \tag{2}$$

$$\rho s = \rho(l_p, l_q) = \left(l_{t-t(1-\frac{p}{t})^\rho}, l_{t(\frac{q}{t})^\rho}\right); \tag{3}$$

$$s^\rho = (l_p, l_q)^\rho = \left(l_{t(\frac{p}{t})^\rho}, l_{t-t(1-\frac{q}{t})^\rho}\right). \tag{4}$$

Then, Chen et al. [10] defined the score and accuracy functions for the comparison of LIFNs.

Definition 3. *[10] Let $s = (l_p, l_q)$ be a LIFN in L, then the score and accuracy functions are defined as follows:*

$$S(s) = p - q; \tag{5}$$

$$H(s) = p + q. \tag{6}$$

Definition 4. *[10] Let $s_1 = (l_{p_1}, l_{q_1})$ and $s_2 = (l_{p_2}, l_{q_2})$ be two LIFNs in L, then there are the following comparative relations:*

(1) If $S(s_1) < S(s_2)$, then $s_1 \prec s_2$;
(2) If $S(s_1) > S(s_2)$, then $s_1 \succ s_2$;
(3) If $S(s_1) = S(s_2)$ and $H(s_1) < H(s_2)$, then $s_1 \prec s_2$;
(4) If $S(s_1) = S(s_2)$ and $H(s_1) > H(s_2)$, then $s_1 \succ s_2$;
(5) If $S(s_1) = S(s_2)$ and $H(s_1) = H(s_2)$, then $s_1 = s_2$.

3. Linguistic Neutrosophic Numbers

An SVNS is described independently by the truth, indeterminacy, and falsity membership functions, which is a subclass of a neutrosophic set [12]. Then, an SVNN (a basic element in an SVNS) consists of the truth T, indeterminacy I, and falsity F, which is denoted by $N = <T, I, F>$ for T, I, $F \in [0, 1]$ and $0 \leq T + I + F \leq 3$. Then in some complex decision situations (especially for some qualitative arguments), it is difficult for decision-makers to give the truth, indeterminacy, and falsity degrees with crisp numbers. A feasible solution is to express them by linguistic arguments. Based on this idea, we can introduce a linguistic neutrosophic concept to express incomplete, indeterminate, inconsistent linguistic information. In this section, we propose an LNN, which consists of the truth, indeterminacy, and falsity linguistic variables. Intuitively, LNNs can more easily deal with fuzzy linguistic information because the three linguistic variables in an LNN can be expressed independently by three linguistic values rather than exact values, like a SVNN.

Definition 5. *Assume that $L = \{l_0, l_1, \ldots, l_t\}$ is a linguistic term set with odd cardinality $t + 1$. If $e = <l_p, l_q, l_r>$ is defined for $l_p, l_q, l_r \in L$ and $p, q, r \in [0, t]$, where l_p, l_q, and l_r express independently the truth degree, indeterminacy degree, and falsity degree by linguistic terms, respectively, then e is called an LNN.*

Definition 6. *Let $e = \langle l_p, l_q, l_r \rangle$, $e_1 = \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle$, and $e_2 = \langle l_{p_2}, l_{q_2}, , l_{r_2} \rangle$ be three LNNs in L and $\rho > 0$, then there are the following operational laws of the LNNs:*

$$e_1 \oplus e_2 = \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle \oplus \langle l_{p_2}, l_{q_2}, l_{r_2} \rangle = \left(l_{p_1 + p_2 - \frac{p_1 p_2}{t}}, l_{\frac{q_1 q_2}{t}}, l_{\frac{r_1 r_2}{t}} \right); \tag{7}$$

$$e_1 \otimes e_2 = \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle \otimes \langle l_{p_2}, l_{q_2}, l_{r_2} \rangle = \left\langle l_{\frac{p_1 p_2}{t}}, l_{q_1 + q_2 - \frac{q_1 q_2}{t}}, l_{r_1 + r_2 - \frac{r_1 r_2}{t}} \right\rangle; \tag{8}$$

$$\rho e = \rho \langle l_p, l_q, l_r \rangle = \left\langle l_{t - t(1 - \frac{p}{t})^\rho}, l_{t(\frac{q}{t})^\rho}, l_{t(\frac{r}{t})^\rho} \right\rangle; \tag{9}$$

$$e^\rho = \langle l_p, l_q, l_r \rangle^\rho = \left\langle l_{t(\frac{p}{t})^\rho}, l_{t - t(1 - \frac{q}{t})^\rho}, l_{t - t(1 - \frac{r}{t})^\rho} \right\rangle. \tag{10}$$

It is obvious that the above operational results are still LNNs.

Example 1. *Assume that $e_1 = <l_6, l_2, l_3>$ and $e_2 = <l_5, l_1, l_2>$ be two LNNs in L and $\rho = 0.5$, then there are the following operational results:*

(1)
$$\begin{aligned} e_1 \oplus e_2 &= \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle \oplus \langle l_{p_2}, l_{q_2}, l_{r_2} \rangle = \left\langle l_{p_1 + p_2 - \frac{p_1 p_2}{t}}, l_{\frac{q_1 q_2}{t}}, l_{\frac{r_1 r_2}{t}} \right\rangle \\ &= \langle l_{6+5-6\times5/8}, l_{2\times1/8}, l_{3\times2/8} \rangle = \langle l_{7.25}, l_{0.25}, l_{0.75} \rangle, \end{aligned}$$

(2)
$$\begin{aligned} e_1 \otimes e_2 &= \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle \otimes \langle l_{p_2}, l_{q_2}, l_{r_2} \rangle = \left\langle l_{\frac{p_1 p_2}{t}}, l_{q_1 + q_2 - \frac{q_1 q_2}{t}}, l_{r_1 + r_2 - \frac{r_1 r_2}{t}} \right\rangle \\ &= \left\langle l_{\frac{6\times5}{8}}, l_{2+1-\frac{2\times1}{8}}, l_{3+2-\frac{3\times2}{8}} \right\rangle = \langle l_{3.75}, l_{2.75}, l_{4.25} \rangle, \end{aligned}$$

(3)
$$\begin{aligned} \rho e_1 &= \rho \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle = \left\langle l_{t - t(1 - \frac{p_1}{t})^\rho}, l_{t(\frac{q_1}{t})^\rho}, l_{t(\frac{r_1}{t})^\rho} \right\rangle = \left\langle l_{8 - 8(1 - \frac{6}{8})^{0.5}}, l_{8(\frac{2}{8})^{0.5}}, l_{8(\frac{3}{8})^{0.5}} \right\rangle \\ &= \langle l_4, l_4, l_{4.899} \rangle, \end{aligned}$$

(4) $e_1^\rho = \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle^\rho = \left\langle l_{t(\frac{p_1}{t})^\rho}, l_{t-t(1-\frac{q_1}{t})^\rho}, l_{t-t(1-\frac{r_1}{t})^\rho} \right\rangle = \left\langle l_{8(\frac{6}{8})^{0.5}}, l_{8-8(1-\frac{2}{8})^{0.5}}, l_{8-8(1-\frac{3}{8})^{0.5}} \right\rangle$
$= \langle l_{6.9282}, l_{1.0718}, l_{1.6754} \rangle$.

Then, we can define the score function and accuracy function for the comparison of LNNs.

Definition 7. *Let e = <l_p, l_q, l_r> be an LNN in L. Then the score and accuracy functions of e are defined as follows:*

$$Q(e) = (2t + p - q - r)/(3t) \text{ for } Q(e) \in [0, 1]; \tag{11}$$

$$T(e) = (p - r)/t \text{ for } T(e) \in [-1, 1]. \tag{12}$$

Definition 8. *Let* $e_1 = \langle l_{p_1}, l_{q_1}, l_{r_1} \rangle$ *and* $e_2 = \langle l_{p_2}, l_{q_2}, , l_{r_2} \rangle$ *be two LNNs in L, then their comparative relations are as follows:*

(1) *If $Q(e_1) < Q(e_2)$, then $e_1 \prec e_2$;*
(2) *If $Q(e_1) > Q(e_2)$, then $e_1 \succ e_2$;*
(3) *If $Q(e_1) = Q(e_2)$ and $T(e_1) < T(e_2)$, then $e_1 \prec e_2$;*
(4) *If $Q(e_1) = Q(e_2)$ and $T(e_1) > T(e_2)$, then $e_1 \succ e_2$;*
(5) *If $Q(e_1) = Q(e_2)$ and $T(e_1) = T(e_2)$, then $e_1 = e_2$.*

Example 2. *Assume that e_1 = <l_6, l_3, l_4>, e_2 = <l_5, l_1, l_3>, and e_3 = <l_6, l_4, l_3> be three LNNs in L, then the values of their score and accuracy functions are as follows:*

$Q(e_1) = (2 \times 8 + 6 - 3 - 4)/24 = 0.625$, $Q(e_2) = (2 \times 8 + 5 - 1 - 3)/24 = 0.7083$, and $Q(e_3) = (2 \times 8 + 6 - 4 - 3)/24 = 0.625$;
$T(e_1) = (6 - 4)/8 = 0.25$ and $T(e_3) = (6 - 3)/8 = 0.375$.
According to Definition 8, their ranking order is $e_2 \succ e_3 \succ e_1$.

4. Weighted Aggregation Operators of LNNs

4.1. LNNWAA Operator

Definition 9. *Let* $e_j = \left\langle l_{p_j}, l_{q_j}, l_{r_j} \right\rangle$ *(j =1, 2, ... , n) be a collection of LNNs in L, then we can define LNNWAA operator as follows:*

$$LNNWAA(e_1, e_2, ..., e_n) = \sum_{j=1}^{n} w_j e_j, \tag{13}$$

where $w_j \in [0, 1]$ is the weight of e_j (j = 1, 2, ... , n), satisfying $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 6 and 9, we can present the following theorem:

Theorem 1. *Let* $e_j = \left\langle l_{p_j}, l_{q_j}, l_{r_j} \right\rangle$ *(j = 1, 2, ... , n) be a collection of LNNs in L, then the aggregation result obtained by Equation (13) is still an LNN, and has the following aggregation formula:*

$$LNNWAA(e_1, e_2, ..., e_n) = \sum_{j=1}^{n} w_j e_j = \left\langle l_{t - t\prod_{j=1}^{n}(1-\frac{p_j}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{q_j}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{r_j}{t})^{w_j}} \right\rangle, \tag{14}$$

where $w_j \in [0, 1]$ is the weight of e_j (j = 1, 2, ... , n), satisfying $\sum_{j=1}^{n} w_j = 1$.

Theorem 1 can be proved by mathematical induction.

Proof.

(1) When $n = 2$, by Equation (9), we obtain:

$$w_1 e_1 = \left\langle l_{t-t(1-\frac{p_1}{t})^{w_1}}, l_{t(\frac{q_1}{t})^{w_1}}, l_{t(\frac{r_1}{t})^{w_1}} \right\rangle,$$

$$w_2 e_2 = \left\langle l_{t-t(1-\frac{p_2}{t})^{w_2}}, l_{t(\frac{q_2}{t})^{w_2}}, l_{t(\frac{r_2}{t})^{w_2}} \right\rangle.$$

By Equation (7), there is the following result:

$$\begin{aligned} LNNWAA(e_1, e_2) = w_1 e_1 \oplus w_2 e_2 &= \left\langle l_{t-t(1-\frac{p_1}{t})^{w_1}+t-t(1-\frac{p_2}{t})^{w_2}-\frac{(t-t(1-\frac{p_1}{t})^{w_1})(t-t(1-\frac{p_2}{t})^{w_2})}{t}}, l_{t(\frac{q_1}{t})^{w_1}(\frac{q_2}{t})^{w_2}}, l_{t(\frac{r_1}{t})^{w_1}(\frac{r_2}{t})^{w_2}} \right\rangle \\ &= \left\langle l_{t-t(1-\frac{p_1}{t})^{w_1}+t-t(1-\frac{p_2}{t})^{w_2}-(t-t(1-\frac{p_1}{t})^{w_1}-t(1-\frac{p_2}{t})^{w_2}+t(1-\frac{p_1}{t})^{w_1}(1-\frac{p_2}{t})^{w_2})}, l_{t(\frac{q_1}{t})^{w_1}(\frac{q_2}{t})^{w_2}}, l_{t(\frac{r_1}{t})^{w_1}(\frac{r_2}{t})^{w_2}} \right\rangle \\ &= \left\langle l_{t-t(1-\frac{p_1}{t})^{w_1}(1-\frac{p_2}{t})^{w_2}}, l_{t(\frac{q_1}{t})^{w_1}(\frac{q_2}{t})^{w_2}}, l_{t(\frac{r_1}{t})^{w_1}(\frac{r_2}{t})^{w_2}} \right\rangle = \left\langle l_{t-t\prod_{j=1}^{2}(1-\frac{p_j}{t})^{w_j}}, l_{t\prod_{j=1}^{2}(\frac{q_j}{t})^{w_j}}, l_{t\prod_{j=1}^{2}(\frac{r_j}{t})^{w_j}} \right\rangle. \end{aligned} \tag{15}$$

(2) When $n = k$, by applying Equation (14), we obtain:

$$LNNWAA(e_1, e_2, \ldots, e_k) = \sum_{j=1}^{k} w_j e_j = \left\langle l_{t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}}, l_{t\prod_{j=1}^{k}(\frac{q_j}{t})^{w_j}}, l_{t\prod_{j=1}^{k}(\frac{r_j}{t})^{w_j}} \right\rangle, \tag{16}$$

(3) When $n = k + 1$, by applying Equations (15) and (16), which yields:

$$\begin{aligned} LNNWAA(e_1, e_2, \ldots, e_{k+1}) &= \sum_{j=1}^{k+1} w_j e_j \\ &= \left\langle l_{t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}+t-t(1-\frac{p_{k+1}}{t})^{w_{k+1}}-\frac{(t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j})(t-t(1-\frac{p_{k+1}}{t})^{w_{k+1}})}{t}}, l_{t\prod_{j=1}^{k}(\frac{q_j}{t})^{w_j}(\frac{q_{k+1}}{t})^{w_{k+1}}}, l_{t\prod_{j=1}^{k}(\frac{r_j}{t})^{w_j}(\frac{r_{k+1}}{t})^{w_{k+1}}} \right\rangle \\ &= \left\langle l_{t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}+t-t(1-\frac{p_{k+1}}{t})^{w_{k+1}}-(t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}-t(1-\frac{p_{k+1}}{t})^{w_{k+1}}+t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}(1-\frac{p_{k+1}}{t})^{w_{k+1}})}, l_{t\prod_{j=1}^{k}(\frac{q_j}{t})^{w_j}(\frac{q_{k+1}}{t})^{w_{k+1}}}, l_{t\prod_{j=1}^{k}(\frac{r_j}{t})^{w_j}(\frac{r_{k+1}}{t})^{w_{k+1}}} \right\rangle \\ &= \left\langle l_{t-t\prod_{j=1}^{k}(1-\frac{p_j}{t})^{w_j}(1-\frac{p_{k+1}}{t})^{w_{k+1}}}, l_{t\prod_{j=1}^{k}(\frac{q_j}{t})^{w_j}(\frac{q_{k+1}}{t})^{w_{k+1}}}, l_{t\prod_{j=1}^{k}(\frac{r_j}{t})^{w_j}(\frac{r_{k+1}}{t})^{w_{k+1}}} \right\rangle = \left\langle l_{t-t\prod_{j=1}^{k+1}(1-\frac{p_j}{t})^{w_j}}, l_{t\prod_{j=1}^{k+1}(\frac{q_j}{t})^{w_j}}, l_{t\prod_{j=1}^{k+1}(\frac{r_j}{t})^{w_j}} \right\rangle. \end{aligned}$$

Corresponding to the above results, we have Equation (14) for any n. This finishes the proof. □

It is obvious that the LNNWAA operator satisfies the following properties:

(1) Idempotency: Let e_j ($j = 1, 2, \ldots, n$) be a collection of LNNs in L. If e_j ($j = 1, 2, \ldots, n$) is equal, i.e., $e_j = e$ for $j = 1, 2, \ldots, n$, then $LNNWAA(e_1, e_2, \cdots, e_n) = e$.

(2) Boundedness: Let e_j ($j = 1, 2, \ldots, n$) be a collection of LNNs in L and let $e^- = \left\langle \min_j(l_{p_j}), \max_j(l_{q_j}), \max_j(l_{r_j}) \right\rangle$ and $e^+ = \left\langle \max_j(l_{p_j}), \min_j(l_{q_j}), \min_j(l_{r_j}) \right\rangle$. Then $e^- \leq LNNWAA(e_1, e_2, \cdots, e_n) \leq e^+$.

(3) Monotonicity: Let e_j ($j = 1, 2, \ldots, n$) be a collection of LNNs in L. If $e_j \leq e_j^*$ for $j = 1, 2, \ldots, n$, then $LNNWAA(e_1, e_2, \cdots, e_n) \leq LNNWAA(e_1^*, e_2^*, \cdots, e_n^*)$.

Proof.

(1) Since $e_j = e$, i.e., $p_j = p$; $q_j = q$; $t_j = r$ for $j = 1, 2, \ldots, n$, we have:

$$LNNWAA(e_1, e_2, \ldots, e_n) = \sum_{j=1}^{n} w_j e_j = \left\langle l_{t-t\prod_{j=1}^{n}(1-\frac{p_j}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{q_j}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{r_j}{t})^{w_j}} \right\rangle$$
$$= \left\langle l_{t-t(1-\frac{p}{t})^{\sum_{j=1}^{n} w_j}}, l_{t(\frac{q}{t})^{\sum_{j=1}^{n} w_j}}, l_{t(\frac{r}{t})^{\sum_{j=1}^{n} w_j}} \right\rangle = \left\langle l_{t-t(1-\frac{p}{t})}, l_{t(\frac{q}{t})}, l_{t(\frac{r}{t})} \right\rangle$$
$$= \langle l_p, l_q, l_r \rangle = e.$$

(2) Since the minimum LNN is e^- and the maximum LNN is e^+, $e^- \leq e_j \leq e^+$. Thus, $\sum_{j=1}^{n} w_j e^- \leq \sum_{j=1}^{n} w_j e_j \leq \sum_{j=1}^{n} w_j e^+$. According to the above property (1), $e^- \leq \sum_{j=1}^{n} w_j e_j \leq e^+$, i.e., $e^- \leq LNNWAA(e_1, e_2, \cdots, e_n) \leq e^+$.

(3) Since $e_j \leq e_j^*$ for $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} w_j e_j \leq \sum_{j=1}^{n} w_j e_j^*$, i.e., $LNNWAA(e_1, e_2, \cdots, e_n) \leq LNNWAA(e_1^*, e_2^*, \cdots, e_n^*)$.

Thus, the proofs of these properties are completed. □

Especially when $w_j = 1/n$ for $j = 1, 2, \ldots, n$, the LNNWAA operator is reduced to the LNN arithmetic averaging operator.

4.2. LNNWGA Operator

Definition 10. *Let* $e_j = \left\langle l_{p_j}, l_{q_j}, l_{r_j} \right\rangle$ *(j = 1, 2, … , n) be a collection of LNNs in L, then we can define LNNWGA operator as follows:*

$$LNNWGA(e_1, e_2, \cdots, e_n) = \prod_{j=1}^{n} e_j^{w_j}, \tag{17}$$

where $w_j \in [0, 1]$ *is the weight of* e_j *(j = 1, 2, … , n), satisfying* $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 6 and 10, we can present the following theorem:

Theorem 2. *Let* $e_j = \left\langle l_{p_j}, l_{q_j}, l_{r_j} \right\rangle$ *(j = 1,2, … , n) be a collection of LNNs in L, then the aggregation result obtained by Equation (17) is still an LNN, and has the following aggregation formula:*

$$LNNWGA(e_1, e_2, \ldots, e_n) = \prod_{j=1}^{n} e_j^{w_j} = \left\langle l_{t\prod_{j=1}^{n}(\frac{p_j}{t})^{w_j}}, l_{t-t\prod_{j=1}^{n}(1-\frac{q_j}{t})^{w_j}}, l_{t-t\prod_{j=1}^{n}(1-\frac{r_j}{t})^{w_j}} \right\rangle, \tag{18}$$

where $w_j \in [0, 1]$ *is the weight of* e_j *(j =1, 2, … , n), satisfying* $\sum_{j=1}^{n} w_j = 1$. *Especially when* $w_j = 1/n$ *for j = 1, 2, … , n, the LNNWGA operator is reduced to the LNN geometric averaging operator.*

Since the proof manner of Theorem 2 is similar to that of Theorem 1, it is not repeated here.
It is obvious that the LNNWGA operator implies the following properties:

(1) Idempotency: Let e_j ($j = 1, 2, \ldots, n$) be a collection of LNNs in L. If e_j ($j = 1, 2, \ldots, n$) is equal, i.e., $e_j = e$ for $j = 1, 2, \ldots, n$, then $LNNWGA(e_1, e_2, \cdots, e_n) = e$.

(2) Boundedness: Let e_j (j = 1, 2, ..., n) be a collection of LNNs in L and let $e^- = \left\langle \min_j(l_{p_j}), \max_j(l_{q_j}), \max_j(l_{r_j}) \right\rangle$ and $e^+ = \left\langle \max_j(l_{p_j}), \min_j(l_{q_j}), \min_j(l_{r_j}) \right\rangle$. Then $e^- \leq LNNWGA(e_1, e_2, \cdots, e_n) \leq e^+$.

(3) Monotonicity: Let e_j (j = 1, 2, ..., n) be a collection of LNNs in L. If $e_j \leq e_j^*$ for j = 1, 2, ..., n, then $LNNWGA(e_1, e_2, \cdots, e_n) \leq LNNWGA(e_1^*, e_2^*, \cdots, e_n^*)$.

Due to the similar proof manner of the properties of the LNNWAA operator we can prove these properties, which are omitted here.

5. MAGDM Method Based on the LNNWAA or LNNWGA Operator

In this section, the LNNWAA and LNNWGA operators and the score and accuracy functions are applied to MAGDM problems with LNN information.

In a MAGDM problem, let $Y = \{Y_1, Y_2, \ldots, Y_m\}$ be a set of alternatives and $Z = \{Z_1, Z_2, \ldots, Z_n\}$ be a set of attributes. The weigh vector of the attributes Z_j (j = 1, 2, ..., n) is $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$. Then, a group of decision-makers $D = \{D_1, D_2, \ldots, D_d\}$ can be assigned with a corresponding weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_d)^{\mathrm{T}}$ to evaluate the alternatives Y_i (i = 1, 2, ..., m) on the attributes Z_j (j = 1, 2, ..., n) by LNNs from the linguistic term set L = {l_0 = extremely low, l_1 = very low, l_2 = low, l_3 = slightly low, l_4 = medium, l_5 = slightly high, l_6 = high, l_7 = very high, l_8 = extremely high}. In the evaluation process, the decision-makers can assign the three linguistic values of the truth, falsity, and indeterminacy degrees, composed of an LNN, to each attribute Z_j on an alternative Y_i according to the linguistic terms. Thus, the LNN evaluation information of the attributes Z_j (j = 1, 2, ..., n) on the alternatives Y_i (i = 1, 2, ..., m) provided by each decision maker D_k (k = 1, 2, ..., d) can be established as an LNN decision matrix $M^k = (e_{ij}^k)_{m\times n}$, where $e_{ij}^k = \left\langle l_{p_{ij}^k}, l_{q_{ij}^k}, l_{r_{ij}^k} \right\rangle$ (k = 1, 2, ..., d; i = 1,2, ..., m; j = 1, 2, ..., n) is an LNN.

Then, we apply the LNNWAA or LNNWGA operator and the score function (accuracy function if necessary) to the MAGDM problem with LNN information to rank the alternatives and to select the best one. The decision-making steps are introduced as follows:

Step 1: Obtain the integrated matrix $R = (e_{ij})_{m\times n}$, where $e_{ij} = \left\langle l_{p_{ij}}, l_{q_{ij}}, l_{r_{ij}} \right\rangle$ (i = 1, 2, ..., m; j = 1, 2, ..., n) is an integrated LNN, by using the following LNNWAA operator:

$$e_{ij} = LNNWAA(e_{ij}^1, e_{ij}^2, \ldots, e_{ij}^d) = \sum_{k=1}^{d} \omega_k e_{ij}^k = \left\langle l_{t - t\prod_{k=1}^{d}(1-\frac{p_{ij}^k}{t})^{\omega_k}}, l_{t\prod_{k=1}^{d}(\frac{q_{ij}^k}{t})^{\omega_k}}, l_{t\prod_{k=1}^{d}(\frac{r_{ij}^k}{t})^{\omega_k}} \right\rangle. \quad (19)$$

Step 2: Obtain the collective overall LNN e_i for Y_i (i = 1, 2, ..., m) by using the following LNNWAA operator or LNNWGA operator:

$$e_i = LNNWAA(e_{i1}, e_{i2}, \ldots, e_{in}) = \sum_{j=1}^{n} w_j e_{ij} = \left\langle l_{t - t\prod_{j=1}^{n}(1-\frac{p_{ij}}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{q_{ij}}{t})^{w_j}}, l_{t\prod_{j=1}^{n}(\frac{r_{ij}}{t})^{w_j}} \right\rangle, \quad (20)$$

or

$$e_i = LNNWGA(e_{i1}, e_{i2}, \ldots, e_{in}) = \prod_{j=1}^{n} e_{ij}^{w_j} = \left\langle l_{t\prod_{j=1}^{n}(\frac{p_{ij}}{t})^{w_j}}, l_{t - t\prod_{j=1}^{n}(1-\frac{q_{ij}}{t})^{w_j}}, l_{t - t\prod_{j=1}^{n}(1-\frac{r_{ij}}{t})^{w_j}} \right\rangle. \quad (21)$$

Step 3: Calculate the score function $Q(e_i)$ (accuracy function $T(e_i)$ if necessary) (i = 1, 2, ..., m) of the collective overall LNN e_i (i = 1, 2, ..., m) by Equation (11) (Equation (12) if necessary).

Step 4: Rank the alternatives corresponding to the score (accuracy if necessary) values, and then select the best one.

Step 5: End.

6. An Illustrative Example

This section considers the selection problem of investment alternatives in an investment company as an illustrative example, which is adapted from [13], in order to demonstrate the application of the proposed method.

Some investment company needs to invest a sum of money to an industry. A panel provides a set of four possible investment alternatives $Y = \{Y_1, Y_2, Y_3, Y_4\}$, where Y_1 is a car company; Y_2 is a food company; Y_3 is a computer company; Y_4 is an arms company. The evaluation of the four alternatives must satisfy the requirements of three attributes: (1) Z_1 is the risk; (2) Z_2 is the growth; (3) Z_3 is the environmental impact. The importance of the three attributes is provided by the weigh vector $\boldsymbol{W} = (0.35, 0.25, 0.4)^{\mathrm{T}}$. Then, three decision-makers are invited and denoted as a set of the decision-makers $D = \{D_1, D_2, D_3\}$ and the importance of the three decision-makers is given as a weight vector $\boldsymbol{\omega} = (0.37, 0.33, 0.3)^{\mathrm{T}}$. The three decision-makers are required to give the suitability evaluation of the four possible alternatives Y_i (i = 1, 2, 3, 4) with respect to the three attributes Z_j (j = 1, 2, 3) by the expression of the linguistic values of LNNs from the linguistic term set $L = \{l_0$ = extremely low, l_1 = very low, l_2 = low, l_3 = slightly low, l_4 = medium, l_5 = slightly high, l_6 = high, l_7 = very high, l_8 = extremely high$\}$ with the odd cardinality $t + 1 = 9$. Thus, the linguistic evaluation information given by each decision-maker D_k (k = 1, 2, 3) can be established as the following the LNN decision matrix M^k:

$$M^1 = \begin{bmatrix} \langle l_6, l_1, l_2 \rangle & \langle l_7, l_2, l_1 \rangle & \langle l_6, l_2, l_2 \rangle \\ \langle l_7, l_1, l_1 \rangle & \langle l_7, l_3, l_2 \rangle & \langle l_7, l_2, l_1 \rangle \\ \langle l_6, l_2, l_2 \rangle & \langle l_7, l_1, l_1 \rangle & \langle l_6, l_2, l_2 \rangle \\ \langle l_7, l_1, l_2 \rangle & \langle l_7, l_2, l_3 \rangle & \langle l_7, l_2, l_1 \rangle \end{bmatrix},$$

$$M^2 = \begin{bmatrix} \langle l_6, l_1, l_2 \rangle & \langle l_6, l_1, l_1 \rangle & \langle l_4, l_2, l_3 \rangle \\ \langle l_7, l_2, l_3 \rangle & \langle l_6, l_1, l_1 \rangle & \langle l_4, l_2, l_3 \rangle \\ \langle l_5, l_1, l_2 \rangle & \langle l_5, l_1, l_2 \rangle & \langle l_5, l_4, l_2 \rangle \\ \langle l_6, l_1, l_1 \rangle & \langle l_5, l_1, l_1 \rangle & \langle l_5, l_2, l_3 \rangle \end{bmatrix},$$

$$M^3 = \begin{bmatrix} \langle l_7, l_3, l_4 \rangle & \langle l_7, l_3, l_3 \rangle & \langle l_5, l_2, l_5 \rangle \\ \langle l_6, l_3, l_4 \rangle & \langle l_5, l_1, l_2 \rangle & \langle l_6, l_2, l_3 \rangle \\ \langle l_7, l_2, l_4 \rangle & \langle l_6, l_1, l_2 \rangle & \langle l_7, l_2, l_4 \rangle \\ \langle l_7, l_2, l_3 \rangle & \langle l_5, l_2, l_1 \rangle & \langle l_6, l_1, l_1 \rangle \end{bmatrix}.$$

Hence, the proposed method can be applied to this decision-making problem and the computational procedures are given as follows:

Step 1: Get the following integrated matrix $R = (e_{ij})_{m \times n}$ by using Equation (19):

$$R = \begin{bmatrix} \langle l_{6.3755}, l_{1.3904}, l_{2.4623} \rangle & \langle l_{6.7430}, l_{1.7969}, l_{1.3904} \rangle & \langle l_{5.1608}, l_{2.0000}, l_{3.0097} \rangle \\ \langle l_{6.7689}, l_{1.7477}, l_{2.1781} \rangle & \langle l_{6.2523}, l_{1.5015}, l_{1.5911} \rangle & \langle l_{6.0547}, l_{2.0000}, l_{1.9980} \rangle \\ \langle l_{6.1429}, l_{1.5911}, l_{2.4623} \rangle & \langle l_{6.2309}, l_{1.0000}, l_{1.5476} \rangle & \langle l_{6.1429}, l_{2.5140}, l_{2.4623} \rangle \\ \langle l_{6.7430}, l_{1.2311}, l_{1.7969} \rangle & \langle l_{6.0020}, l_{1.5911}, l_{1.5015} \rangle & \langle l_{6.2309}, l_{1.6245}, l_{1.4370} \rangle \end{bmatrix}.$$

Step 2: By using Equation (20), the collective overall LNNs of e_i for Y_i (i = 1, 2, 3, 4) can be obtained as follows:

$$e_1 = \langle l_{6.0951}, l_{1.7145}, l_{2.3129} \rangle,\ e_2 = \langle l_{6.3863}, l_{1.7759}, l_{1.9453} \rangle,\ e_3 = \langle l_{6.1653}, l_{1.7011}, l_{2.1924} \rangle, \text{and } e_4 = \langle l_{6.3818}, l_{1.4666}, l_{1.5711} \rangle.$$

Step 3: Calculate the score values of $Q(e_i)$ (i = 1, 2, 3, 4) of the collective overall LNNs of e_i (i = 1, 2, 3, 4) by Equation (11):

$$Q(e_1) = 0.7528, Q(e_2) = 0.7777, Q(e_3) = 0.7613, \text{ and } Q(e_4) = 0.8060.$$

Step 4: Ranking order of the four alternatives is $Y_4 \succ Y_2 \succ Y_3 \succ Y_1$ corresponding to the score values. Thus, the alternative Y_4 is the best choice among the four alternatives.

Or by using Equation (21), the computational procedures are given as follows:

Step 1': The same as Step 1.

Step 2': By using Equation (21), the collective overall LNNs of e_i for Y_i (i = 1, 2, 3, 4) are obtained as follows:

$$e_1 = \langle l_{5.9413}, l_{1.7414}, l_{2.4479} \rangle,\ e_2 = \langle l_{6.3464}, l_{1.7902}, l_{1.9634} \rangle, e_3 = \langle l_{6.1648}, l_{1.8433}, l_{2.2465} \rangle, \text{ and } e_4 = \langle l_{6.3459}, l_{1.4810}, l_{1.5811} \rangle.$$

Step 3': By using Equation (11), we calculate the score values of $Q(e_i)$ (i = 1, 2, 3, 4) of the collective overall LNNs of e_i (i = 1, 2, 3, 4) as follows:

$$Q(e_1) = 0.7397, Q(e_2) = 0.7747, Q(e_3) = 0.7531, \text{ and } Q(e_4) = 0.8035.$$

Step 4': The ranking order of the four alternatives is $Y_4 \succ Y_2 \succ Y_3 \succ Y_1$. Thus, the alternative Y_4 is still the best choice among the four alternatives.

Clearly, the above two ranking orders and the best alternative based on the LNNWAA and LNNWGA operators are the same, which are in agreement with Ye's results [13].

Compared with the relevant papers [10,11] which proposed the decision-making approaches with LIFNs, the decision information used in [10,11] is LIFNs, whereas the decision information in this paper are LNNs. As mentioned above, the LNN is a further generalization of the LIFN and contains more information than the LIFN. Thus, the decision-making method proposed in this paper is more typical and more general in application since the decision-making method proposed in [10,11] cannot handle indeterminate and inconsistent linguistic information and the MAGDM problem with LNN information in this paper. Furthermore, compared with the relevant papers [6–8,13–16], the decision-making approach proposed in this study can be used to solve decision-making problems with LNN information, while the MADM/MAGDM methods with various linguistic information presented in [6–8,13–16] are not suitable for handling the decision-making problems with LNN information in this paper since existing various linguistic numbers in [6–8,13–16] cannot express indeterminate and inconsistent linguistic information.

In fact, all decision-making methods based on various linguistic variables in existing literature not only cannot express indeterminate and inconsistent linguistic information, but also lose the useful information in linguistic evaluation process, and then they cannot also deal with decision-making problems with indeterminate and inconsistent linguistic information; while the linguistic method proposed in the study is a generalization of existing linguistic methods and can represent and handle linguistic decision-making problems with LNN information. Obviously, the main contribution in this study is that our new method can express indeterminate and inconsistent linguistic information corresponding to human fuzzy thinking about complex problems, especially for some qualitative evaluations of attributes, and solve linguistic decision-making problems with indeterminate and inconsistent linguistic information.

From above comparative analyses with relevant papers, one can see that main advantages of the developed new method are summarized as follows:

(1) The developed new method is more suitable for expressing and handling indeterminate and inconsistent linguistic information in linguistic decision-making problems to overcome the insufficiency of various linguistic decision-making methods in the existing literature.

(2) The developed new method contains much more information (the three linguistic variables of truth, indeterminate, and falsity degrees contained in an LNN) than the existing method in [10,11] (the two linguistic variables of truth and falsity degrees contained in a LIFN) and can better describe people's linguistic expression to objective things evaluated in detail.

(3) The developed new method enriches the neutrosophic theory and decision-making method under a linguistic environment and provides a new way for solving linguistic MAGDM problems with indeterminate and inconsistent linguistic information.

7. Conclusions

This paper originally presented LNNs, the operational laws of LNNs, and the score and accuracy functions of LNNs. Then, we proposed the LNNWAA and LNNWGA operators to aggregate LNNs and investigated their properties and special cases. Further, we developed a MAGDM method based on the LNNWAA or LNNWGA operator and the score and accuracy functions to solve MAGDM problems with LNN information. Finally, an illustrative example was provided to demonstrate the application of the developed MAGDM method under LNN environment. The developed MAGDM method with LNNs enriches fuzzy decision-making theory and provides a new way for decision-makers under LNN environment. In the future research directions, we shall further develop new aggregation operators of LNNs and apply them to decision-making, pattern recognition, medical diagnosis, and so on.

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Solving Solar-Wind Power Station Location Problem Using an Extended Weighted Aggregated Sum Product Assessment (WASPAS) Technique with Interval Neutrosophic Sets

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Abstract: As one of the promising renewable energy resources, solar-wind energy has increasingly become a regional engine in leading the economy and raising competitiveness. Selecting a solar-wind power station location can contribute to efficient utilization of resource and instruct long-term development of socio-economy. Since the selection procedure consists of several location alternatives and many influential criteria factors, the selection can be recognized as a multiple criteria decision-making (MCDM) problem. To better express multiple uncertainty information during the selection procedure, fuzzy set theory is introduced to manage that issue. Interval neutrosophic sets (INSs), which are characterized by truth-membership, indeterminacy-membership and falsity-membership functions in the interval numbers (INs) form, are feasible in modeling more uncertainty of reality. In this paper, a newly extended weighted aggregated sum product assessment (WASPAS) technique, which involves novel three procedures, is utilized to handle MCDM issues under INSs environment. Some modifications are conducted in the extended method comparing with the classical WASPAS method. The most obvious improvement of the extended method relies on that it can generate more realistic criteria weight information by an objective and subjective integrated criteria weight determination method. A case study concerning solar-wind power station location selection is implemented to demonstrate the applicability and rationality of the proposed method in practice. Its validity and feasibility are further verified by a sensitivity analysis and a comparative analysis. These analyses effectively reveal that the extended WASPAS technique can well match the reality and appropriately handle the solar-wind power station location selection problem.

Keywords: multiple criteria decision-making (MCDM); interval neutrosophic sets (INSs); weighted aggregated sum product assessment (WASPAS); integrated criteria weight information; solar-wind power station location selection

1. Introduction

Remarkable growth of urbanization and industrialization make it imperative to increase widespread useable electricity for facilitating regional economy development [1]. Due to the increasing awareness of the global climate degradation, conventional energy resources cannot simultaneously meet the environmental challenge and energy demand [2]. The solar-wind energy system has gradually substituted the status of traditional energy for the friendly environment concern [3]. Solar-wind power station devotes to convert renewable resources such as solar and wind energy into power for supporting socio-economic requirement [4]. For better contributing to regional competition and determining future energy generation, it is significant to seek a good solar-wind power station location [5]. Considerable researches have increasingly emerged concerning selecting the renewable energy location [6]. It is well known that the usage starting point of multiple criteria decision analysis (MCDA) is a definition of the set of decision variants [7]. The aim of MCDA is to select a good solution according to decision makers' (DMs') preferences [8,9]. As numerous influential factors need to be considered in the selecting procedure, the location selection problem can be treated as a complex multiple criteria decision-making (MCDM) problem [6]. In addition, MCDM methods have been adopted as effective instruments in this field. Incorporating analytic hierarchy process (AHP) to GIS environment, the most desirable nuclear power plant location was determined by MCDM analysis technique [10]. The framework of MCDM analysis based on GIS was also identified as an effective decision tool for wind-farm location planning area [11]. Furthermore, grey cumulative prospect theory and cloud decision framework were gradually employed into power location selection [12,13]. To manage inevitable fuzziness and uncertainty in realistic application, ELECTRE-III as one of the most commonly used MCDM technique was extended into intuitionistic fuzzy circumstances for determining a good offshore wind power station location [14]. The selection of the solar-wind power station location can be regarded as another complex MCDM problem involving many influential factors such as economic factors, traffic factors, natural resources, environmental factors and social factors [15]. Though MCDM methods such as ELECTRE-II has been widely employed into this field [15], existing researches concerning solar-wind power station location selection neglect the impact of uncertainty and only evaluate the alternatives in the real number context [16]. Effective information expression form should be introduced into the selection problem for representing more abundant and realistic information.

Along with the development of fuzzy set theory, fuzzy set (FSs) [17] as well as interval valued fuzzy sets (IVFSs) [18], intuitionistic fuzzy sets [19] and interval-valued intuitionistic fuzzy sets (IVIFSs) [20] were extensively applied into dealing with uncertain situation. Owing to multiple uncertainty and information existing in practice, above traditional types of FSs are unable to adequately express indeterminate information and inconsistent information [21]. Neutrosophic sets (NSs) [22] were initially presented by Smarandache to feasibly manage these uncertain information via considering the truth membership, indeterminacy membership and falsity membership functions simultaneously [23,24]. However, NS was lacking of specific description to adequately express actual issues. As a particular case of NSs, single-valued neutrosophic sets (SVNSs) were introduced for the first time by Smarandache in 1998 in his book [22], reviewed in [25], which is also mentioned by Denis Howe, from England, in The Free Online Dictionary of Computing, 1999. Since more information can be described by interval numbers (INs), interval neutrosophic set (INS) [26] as a further extension of NS was utilized in various fields such as trustworthy cloud service selection [27], lean supplier selection [28] and medical diagnosis [29]. Relative basic theories of INS which briefly focus on its aggregation operators and MCDM methods have already developed maturity [30]. Particularly, MCDM methods including outranking approach [31], cross-entropy measure [32], and correlation coefficient measure [33] can effectively deal with problems under INS environment. This paper aims to develop a novel MCDM technique with flexibility and applicability under INSs circumstances for matching the solar-wind power station location selection case.

The weighted aggregated sum product assessment (WASPAS) method was originally presented in 2012 regarded as an effective extension of Weighted Product Model (WPM) and Weighted Sum Model (WSM) [34]. Its accuracy in dealing with MCDM problems has been proven by comparing with the simple utilization one of WPM or WSM [34]. This method has been extensively employed into MCDM situation and various applications [35], especially in the site selection field. The complex circuit design of lead-zinc froth flotation selection regarded as a MCDM procedure was adequately settled by WASPAS method for fully processing costs and reinforcing the utilization [36]. Selection of the best wind farm location was feasibly handled and assessed by utilizing the WASPAS method [37]. The selection of the construction site for a waste incineration plant plays a critical role in public health and city development. It can be effectively solved by means of the WASPAS method [38]. Within the MCDM framework, relevant mathematical model and AHP approach were incorporated into the WASPAS method, by which the location selection problem of the garage was explicitly formulated [39]. Its robustness was verified by comparing with other MCDM methods when finding a good solution [40]. Referring to specific application setting, WASPAS method has been generalized into a variety of assessment fields such as energy supply system [41], solar projects [42], third-party logistics providers [43] and indoor environment [35]. The combination of the WASPAS method with fuzzy set theory has been introduced into dealing with uncertainty under grey values [44], interval type-2 fuzzy sets [43] or SVNSs environment [45]. Incorporating IVIFSs into WASPAS method, it can effectively deal with the MCDM problems [46]. Relative research concerning the WASPAS technique with INSs needs to be further investigated to enrich theory basics and represent more uncertain information. Consequently, this paper generalizes the WASPAS technique into INSs circumstances for matching the solar-wind power station location selection problem.

Previous researches have revealed that incomplete or unknown weight information commonly exists when applying the MCDM methods into assessment [47]. From the perspective of the objective weight determination methods, maximizing derivation method [48], the most widely utilized method, can generate the criteria weights under weights partly known or completely unknown circumstances. Corresponding mathematical programming models are constructed according to different circumstances such as hesitant fuzzy sets [49,50], IFSs [51] and multiple types of linguistic circumstances [52,53]. As one of the representative subjective weight determination methods, order relation analysis method (G1) determines the weight information by virtue of DMs' experience judgement [54]. G1 not only reflects the subjective judgment of DMs, but also possesses convenience and feasibility comparing to AHP method. These advantages are due to its simple acquisition process, and the fact that there is no need to construct judgement matrix [54]. To adequately reflect more realistic information from both the subjective and objective aspects, this paper investigates an integrated criteria weight utilizing the combination strengths of above determination methods, and further employs it to the WASPAS technique under INSs environment.

The reminder of the paper is structured as follows. In Section 2, some basic concepts concerning INs and INSs are roughly reviewed. In Section 3, the framework of the extended WASPAS technique is constructed based on the objective weight and subjective weight integrated criteria weight information. Subsequently, to verify its applicability within INSs environment, the extended WASPAS technique is employed into practical solar-wind power station location case in Section 4. In Section 5, a sensitivity analysis and a comparative analysis are conducted to further demonstrate the rationality of the extended WASPAS technique. Finally, conclusions are summarized in Section 6.

2. Background

This section briefly reviews some basics concerning INSs for the latter discussion.

2.1. INs

Some definitions and concepts of INs are recalled.

Definition 1. *[55] Let* $a = [a^L, a^U] = \{x | a^L \le x \le a^U\}$*, then a is said to be an IN. Particularly,* $a = [a^L, a^U]$ *will be deduced to a real number if* $a^L = a^U$*.*

Assume that there are two nonnegative INs $a_1 = [a_1^L, a_1^U]$ *and* $a_2 = [a_2^L, a_2^U]$*. Then, their operations are defined as follows [56]:*

1. $a_1 + a_2 = [a_1^L + a_2^L, a_1^U + a_2^U]$,
2. $\lambda a_1 = [\lambda a_1^L, \lambda a_1^U], \lambda > 0$.

Definition 2. *[30] Let $a_1 = [a_1^L, a_1^U]$ and $a_2 = [a_2^L, a_2^U]$ be two INs, $L(a_1) = a_1^u - a_1^L$ and $L(a_2) = a_2^u - a_2^L$, then the possibility degree of $a_1 \geq a_2$ is denoted as:*

$$p(a_1 \geq a_2) = \max\left\{1 - \max\left\{\frac{a_2^U - a_1^L}{L(a_1) + L(a_2)}, 0\right\}, 0\right\}, \tag{1}$$

Consider there exist m INs $a_i = [a_i^L, a_i^U]$ $(i = 1, 2, \ldots, m)$. The possibility degree of Equation (1) can be denoted as

$$p(a_i \geq a_j) = \max\left\{1 - \max\left\{\frac{a_j^U - a_i^L}{L(a_i) + L(a_j)}, 0\right\}, 0\right\}, \tag{2}$$

when comparing each IN a_i to all INs $a_j = \left[a_j^L, a_j^U\right]$ $(j = 1, 2, \ldots, n)$. Then, a complementary possibility degree matrix can be established as

$$P(a_i \geq a_j) = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}_{n \times n}, \tag{3}$$

whose elements satisfy the conditions $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$ and $p_{ii} = 0.5$. Its ranking vector can be derived from the equation as follows:

$$p_i = \frac{\sum_{j=1}^{n} p_{ij} + \frac{m}{2} - 1}{m(m-1)}, (i = 1, 2, \ldots, m), \tag{4}$$

2.2. INSs

Due to the fact that indeterminacy and inconsistency information commonly exist in the daily life, numerous researches tackled the NSs as the instrument to manage that issue [57,58]. However, to fully and adequately indicate uncertainty and fuzziness in the reality, IN is utilized as the form to depict the truth-membership, indeterminacy-membership and falsity-membership information of NSs rather than crisp values [59]. In this section, we briefly review some basics of INSs, which involves operational laws, aggregation operators and score functions. For the convenience of expressing the reality, INS is defined motivated by the definition of SVNS as follows:

Definition 3. *[60] Let X be an arbitrary universe of discourse whose generic element can be denoted by x. Then, an INS A in X is*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}, \tag{5}$$

which is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. For each point of x in X, there exists the conditions that $T_A(x) = [T_A^L(x), T_A^U(x)]$, $I_A(x) = [I_A^L(x), I_A^U(x)]$, $F_A(x) = [F_A^L(x), F_A^U(x)]$ and $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$ and $0 \leq T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3$.

For notation simplification, we adopt $a = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U] \rangle$ as the representation of an INS in this paper.

Definition 4. *[61] Let $a_1 = \langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle$ and $a_2 = \langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle$ be two arbitrary INSs, then, its operational laws can be defined as*

(1) *The complement of a_1 is $\bar{a}_1 = \langle [F_1^L, F_1^U], [1 - I_1^U, 1 - I_1^L], [T_1^L, T_1^U] \rangle$,*
(2) $a_1 + a_2 = \langle [T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U] \rangle$,

(3) $a_1 \times a_2 = \langle [T_1^L T_2^L, T_1^U T_2^U], [I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U] \rangle$,

(4) $\eta a_1 = \left\langle \left[1-(1-T_1^L)^\eta, 1-(1-T_1^U)^\eta\right], \left[(I_1^L)^\eta, (I_1^U)^\eta\right], \left[(F_1^L)^\eta, (F_1^U)^\eta\right] \right\rangle$, $\eta > 0$,

(5) $a_1{}^\eta = \left\langle \left[(T_1^L)^\eta, (T_1^U)^\eta\right], \left[1-(1-I_1^L)^\eta, 1-(1-I_1^U)^\eta\right], \left[1-(1-F_1^L)^\eta, 1-(1-F_1^U)^\eta\right] \right\rangle$, $\eta > 0$.

Definition 5. [61] *Let* $a_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$ $(j = 1, 2, \ldots, n)$ *be a permutation of the INSs. Then, the interval neutrosophic power generalized weighted aggregation operator is defined as*

$$INPGWA(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \frac{\omega_j (1 + T(a_j)) a_j{}^\eta}{\sum_{j=1}^{n} \omega_j (1 + T(a_j))} \right)^{1/\eta}, \tag{6}$$

in which $\omega_j = (\omega_1, \omega_2, \cdots, \omega_n)$ *is the associated weight vector of* $a_j (j = 1, 2, \ldots, n)$, $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$; $T(a_j) = \sum_{i=1, i \neq j}^{n} Sup(a_j, a_i)$, $Sup(a_j, a_i)$ *is the support degree for* a_j *from* a_i; η *is a parameter belonging to* $(0, +\infty)$.

Theorem 1. [61] *Let* $a_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$ $(j = 1, 2, \ldots, n)$ *be a permutation of the INSs, the aggregated result utilizing the interval neutrosophic power generalized weighted aggregation operator derived from Definition 4 is shown as*

$$\begin{aligned} INPGWA(a_1, a_2, \cdots, a_n) = & \left\langle \left[\left(1 - \prod_{j=1}^{n} \left(1 - \left(T_j^L\right)^\eta\right)^{w_j}\right)^{1/\eta}, \left(1 - \prod_{j=1}^{n} \left(1 - \left(T_j^U\right)^\eta\right)^{w_j}\right)^{1/\eta} \right], \right. \\ & \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(I_j^L\right)^\eta\right)^{w_j}\right)^{1/\eta}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(I_j^U\right)^\eta\right)^{w_j}\right)^{1/\eta} \right], \\ & \left. \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(F_j^L\right)^\eta\right)^{w_j}\right)^{1/\eta}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(F_j^U\right)^\eta\right)^{w_j}\right)^{1/\eta} \right] \right\rangle, \end{aligned} \tag{7}$$

in which $w_j = \frac{\omega_j (1 + T(a_j))}{\sum_{j=1}^{n} \omega_j (1 + T(a_j))}$. *Particularly, when* $\eta \to 0$, *the interval neutrosophic power generalized weighted aggregation operator reduces to an interval neutrosophic power geometric weighted aggregation (INPGWA) operator, which is shown as*

$$\begin{aligned} INPGWA(a_1, a_2, \cdots, a_n) = & \left\langle \left[\prod_{j=1}^{n} \left(T_j^L\right)^{w_j}, \prod_{j=1}^{n} \left(T_j^U\right)^{w_j} \right], \left[1 - \prod_{j=1}^{n} \left(1 - I_j^L\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_j^U\right)^{w_j} \right], \right. \\ & \left. \left[1 - \prod_{j=1}^{n} \left(1 - F_j^L\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - F_j^U\right)^{w_j} \right] \right\rangle; \end{aligned} \tag{8}$$

When $\eta \to 1$, the interval neutrosophic power generalized weighted aggregation operator reduces to an interval neutrosophic power weighted aggregation (INPWA) operator, which is shown as

$$INPWA(a_1, a_2, \cdots, a_n) = \left\langle \left[1 - \prod_{j=1}^{n}\left(1 - T_j^L\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - T_j^U\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(I_j^L\right)^{w_j}, \prod_{j=1}^{n}\left(I_j^U\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(F_j^L\right)^{w_j}, \prod_{j=1}^{n}\left(F_j^U\right)^{w_j}\right]\right\rangle. \tag{9}$$

Definition 6. *[62] Let $a_1 = \langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle$ and $a_2 = \langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle$ be two arbitrary INSs. Then, the normalized Hamming distance between a_1 and a_2 can be defined as*

$$d(a_1, a_2) = \frac{1}{6}\left(\left|T_1^L - T_2^L\right| + \left|T_1^U - T_2^U\right| + \left|I_1^L - I_2^L\right| + \left|I_1^U - I_2^U\right| + \left|F_1^L - F_2^L\right| + \left|F_1^U - F_2^U\right|\right). \tag{10}$$

Definition 7. *[63] Let $a = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U] \rangle$ be an INS, then, its score function as well as accuracy function and certainty function are defined as*

(1) $S(a) = [T^L + 1 - I^U + 1 - F^U, T^U + 1 - I^L + 1 - F^L]$,
(2) $H(a) = [\min\{T^L - F^L, T^U - F^U\}, \max\{T^L - F^L, T^U - F^U\}]$,
(3) $B(a) = [T^L, T^U]$.

Its comparison rules are specifically introduced in [63].

3. The Framework of an Extended WASPAS Technique

In this section, an extended WASPAS technique is newly investigated to match solar-wind power station location selection issue with completely unknown criteria weight information.

3.1. Maximizing Deviation Method for Objective Weight Estimating

The maximizing deviation method was initially presented by Wang [64] for managing MCDM problems in numerical context. Its main ideal relies on the performance value of each alternative differs under certain criteria. Thus, it can be inferred that if certain criteria makes the performance values concerning all the alternatives apparently different, the criteria plays a critical role in seeking a good alternative under the MCDM context. Therefore, by virtue of this ideal, criteria with similar performance value with respect to all the alternatives should be allocated small weight; otherwise, the criteria makes huge differences over alternatives should be allocated bigger weight. By above analysis, the maximizing deviation method can be applied into specific MCDM application as an effective tool in deriving completely unknown criteria weight information. Specially, the model, which reveals the differences of the performance value for each alternative, can be established in the following within the INS context.

With respect to certain criteria $C_j \in C$, the performance derivation values of alternative x_i to all the other alternatives can be established as follows:

$$D_{ij}(\omega') = \sum_{k=1}^{m} d\left(a_{ij}, a_{kj}\right)\omega_j', \ i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, n \tag{11}$$

in which a_{ij} and a_{kj} represent the performance value of alternative i and alternative k under criteria j, and characterized by $a_{ij} = \left\langle \left[T_{ij}^L, T_{ij}^U\right], \left[I_{ij}^L, I_{ij}^U\right], \left[F_{ij}^L, F_{ij}^U\right]\right\rangle$ and $a_{kj} = \left\langle \left[T_{kj}^L, T_{kj}^U\right], \left[I_{kj}^L, I_{kj}^U\right], \left[F_{kj}^L, F_{kj}^U\right]\right\rangle$, respectively.

Furthermore, let

$$D_j(\omega') = \sum_{i=1}^{m} D_{ij}(\omega') = \sum_{i=1}^{m}\sum_{k=1}^{m} d\left(a_{ij}, a_{kj}\right)\omega'_j \tag{12}$$

$D_j(\omega)$ represents the derivation performance value of all the alternatives to the others under criteria $C_j \in C$.

From above analysis, the determination of weight vector ω' can maximize the collective derivation performance value for all the criteria. A linear programming model can be established to derive the optimal weight vector solution $\omega' = \left(\omega'_1, \omega'_2, \cdots, \omega'_n\right)$ which is utilized as the criteria weight vector, shown as follows:

$$\begin{cases} \max D(\omega') = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} D_{ij}(\omega') = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m} d\left(a_{ij}, a_{kj}\right)\omega'_j \\ s.t.\, \omega' \in \Omega, \quad \sum\limits_{j=1}^{n} \omega'_j = 1, \quad \omega'_j \geq 0, \quad j = 1, 2, \cdots, n, \end{cases} \tag{13}$$

in which specific distance equation $d\left(a_{ij}, a_{kj}\right)$ refers to Definition 6. Above model is designed to solve the decision-making problem with partial known criteria weight information in Ω. However, due to the complexity in practice, criteria weight information tends to be completely unknown, and cannot be predefined in MCDM problems in most cases. Another programming model is established in the following to derive the criteria weight vector within the completely unknown weight information.

$$\begin{cases} \max D(\omega') \\ = \sum\limits_{i=1}^{m} D_i(\omega') \\ = \frac{1}{6}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m} \left(\left|T_{ij}^{L} - T_{kj}^{L}\right| + \left|T_{ij}^{U} - T_{kj}^{U}\right| + \left|I_{ij}^{L} - I_{kj}^{L}\right| + \left|I_{ij}^{U} - I_{kj}^{U}\right| + \left|F_{ij}^{L} - F_{kj}^{L}\right| + \left|F_{ij}^{U} - F_{kj}^{U}\right|\right)\omega'_j \\ s.t. \sum\limits_{j=1}^{n} \left(\omega'_j\right)^2 = 1, \quad \omega'_j \geq 0, \quad j = 1, 2, \cdots, n. \end{cases} \tag{14}$$

A Lagrange function can be established to solve this model, and shown as

$$\begin{aligned} L(\omega', \gamma) = \frac{1}{6}\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} & \left(\left|T_{ij}^{L} - T_{kj}^{L}\right| + \left|T_{ij}^{U} - T_{kj}^{U}\right| + \left|I_{ij}^{L} - I_{kj}^{L}\right| + \left|I_{ij}^{U} - I_{kj}^{U}\right|\right. \\ & \left. + \left|F_{ij}^{L} - F_{kj}^{L}\right| + \left|F_{ij}^{U} - F_{kj}^{U}\right|\right)\omega'_j + \frac{\gamma}{12}\left(\sum_{j=1}^{n}\left(\omega'_j\right)^2 - 1\right), \end{aligned} \tag{15}$$

in which γ is the Lagrange multiplier. Differentiating Equation (15) concerning ω'_j $(j = 1, 2, \ldots, n)$ and γ, respectively. Let these partial derivations equal to 0 value, these equations can be derived as follows:

$$\begin{cases} \frac{\partial L}{\partial \omega'_j} = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m} \left(\left|T_{ij}^{L} - T_{kj}^{L}\right| + \left|T_{ij}^{U} - T_{kj}^{U}\right| + \left|I_{ij}^{L} - I_{kj}^{L}\right| + \left|I_{ij}^{U} - I_{kj}^{U}\right| + \left|F_{ij}^{L} - F_{kj}^{L}\right| + \left|F_{ij}^{U} - F_{kj}^{U}\right|\right) + \gamma\omega'_j, \\ \frac{\partial L}{\partial \gamma} = \sum\limits_{j=1}^{n} \left(\omega'_j\right)^2 - 1 = 0. \end{cases}$$

Then, to determine the criteria weight vector, a simple equation is generated as follows:

$$\omega'_j = \frac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\left(\left|T_{ij}^{L}-T_{kj}^{L}\right| + \left|T_{ij}^{U}-T_{kj}^{U}\right| + \left|I_{ij}^{L}-I_{kj}^{L}\right| + \left|I_{ij}^{U}-I_{kj}^{U}\right| + \left|F_{ij}^{L}-F_{kj}^{L}\right| + \left|F_{ij}^{U}-F_{kj}^{U}\right|\right)}{\sqrt{\sum\limits_{j=1}^{n}\left(\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\left(\left|T_{ij}^{L}-T_{kj}^{L}\right| + \left|T_{ij}^{U}-T_{kj}^{U}\right| + \left|I_{ij}^{L}-I_{kj}^{L}\right| + \left|I_{ij}^{U}-I_{kj}^{U}\right| + \left|F_{ij}^{L}-F_{kj}^{L}\right| + \left|F_{ij}^{U}-F_{kj}^{U}\right|\right)\right)^2}} \tag{16}$$

Consequently, the normalized weight vector of criteria can be derived as

$$\omega'^{*}_{j} = \frac{\omega'_j}{\sum\limits_{j=1}^{n} \omega'_j}, (j = 1, 2, \ldots, n). \tag{17}$$

Based on aforementioned discussion, we can obtain criteria weight vector via these models and equations with incompletely or completely unknown weight information under INS context.

3.2. G1 for Subjective Weight Estimating

G1 [54] is one of the subjective weight estimation methods, in which all the weight information index derived from subjective evaluation of DM according to their experience. Owing to its practical applicability, G1 was adopted to dispose the weight information in the assessment of electric vehicle sharing programs [65]. Specific acquisition process within the location selection case is outlined as follows:

Step 1 Determine the criteria ranking order relation.

Let DMs provide the order relation of the set $C = \{C_1, \cdots, C_j, \cdots, C_n\}$ according to the importance of the criteria judging from their experience.

Step 2 Assign the relative importance degree index of adjacent criteria.

Determine the relative importance degree index $r_j = \omega''_{j-1}/\omega''_j$ of the adjacent criteria C_{j-1} and C_j according to Table 1.

Step 3 Calculate the subjective weights of criteria by Equations (18) and (19).

$$\omega''_n = \left(1 + \sum_{i=2}^{n} \prod_{j=i}^{n} r_j\right)^{-1}, \tag{18}$$

$$\omega''_j = \prod_{k=j+1}^{n} r_k \omega''_n. \tag{19}$$

Table 1. The relative importance degree index among adjacent criteria.

r_j	Description
1.0	C_{j-1} is equally important as C_j
1.2	C_{j-1} is slightly more important than C_j
1.4	C_{j-1} is obviously more important than C_j
1.6	C_{j-1} is strongly more important than C_j
1.8	C_{j-1} is extremely more important than C_j

3.3. An Extended WASPAS Technique with Integrated Criteria Weight Information

The WASPAS method [34] is a well-known decision-making technique which can effectively increase the ranking accuracy by integrating WSM and WPM. It has better accuracy than only using one of WSM or WPM, which has been proved in [34]. Total importance of an alternative is determined by the aggregated WASPAS measure, which is in essence a joint criterion derived from the use of weighted arithmetic and geometric averaging operators simultaneously [66]. The feasible ranking order can be ensured by altering the parameter between the sum total relative importance and the product total relative importance of alternative computed by theses operators within the completely unknown criteria information.

In most practical cases, the criteria weight information tends to be completely unknown, and there commonly exists relationship among alternatives. To this end, this paper combines the objective weight and subjective weight acquisition methods defined above to reflect more realistic weight information in the integrated weight information estimation procedure of the newly extended WASPAS technique. It is well known that a PA operator can effectively reveal the relationship among alternatives by calculating support degrees from others [67]. The PA operator is generalized into the WASPAS method to optimize the aggregated WASPAS measure determination procedure in this paper. Consider the human beings' expression preference and the uncertainty they faced with, this paper proposes an extended WASPAS technique in which three procedures are feasibly implemented under INSs context. The framework of the extended technique is displayed in Figure 1.

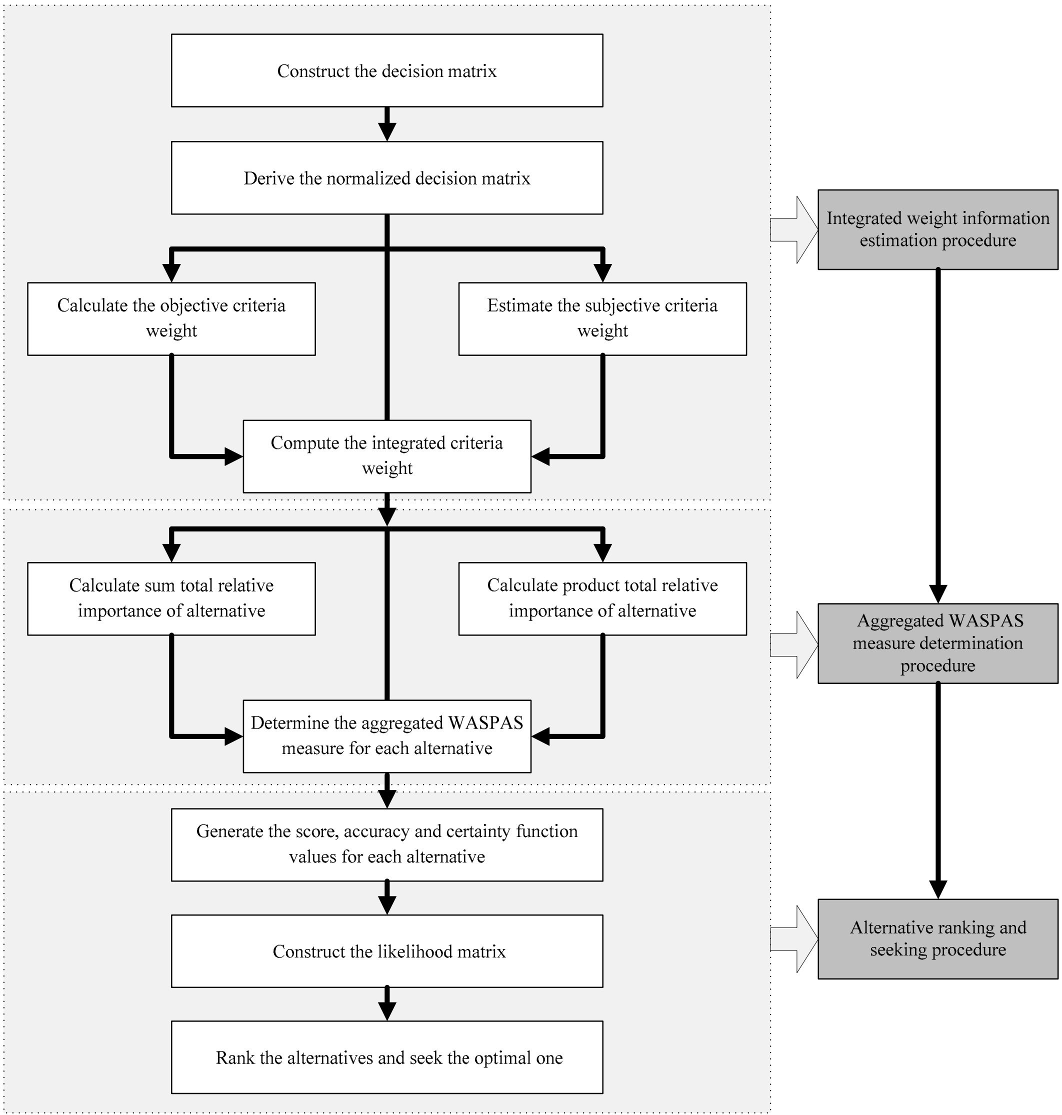

Figure 1. The framework of the extended WASPAS technique.

Assume that a MCDM problem in which a permutation of m alternatives $\{A_1, \cdots, A_i, \cdots, A_m\}$ are evaluated under a permutation of n criteria $\{C_1, \cdots, C_j, \cdots, C_n\}$. The performance evaluation of the i th alternative on the j th criteria is assessed by the INSs denoted by $a_{ij} = \left\langle \left[T_{ij}^L, T_{ij}^U\right], \left[I_{ij}^L, I_{ij}^U\right], \left[F_{ij}^L, F_{ij}^U\right] \right\rangle$. The main procedures are outlined as follows:

Step 1. Construct the decision matrix.

Let a DM provide performance estimation of every alternative with respect to all the criteria, which is shown as

$$A = (A_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{m\times n}$$

Step 2. Derive the normalized decision matrix.

Utilize Equation (1) in Definition 4 to convert the evaluation under cost criteria to benefit criteria. For convenience, the normalized evaluations for the ith alternative with respect to the jth cost criteria are also denoted by $a_{ij} = \left\langle \left[T_{ij}^L, T_{ij}^U\right], \left[I_{ij}^L, I_{ij}^U\right], \left[F_{ij}^L, F_{ij}^U\right] \right\rangle$.

Step 3. Calculate the objective criteria weight.

Use Equations (16) and (17) to calculate the objective weight $\omega_j'^*$ for each criteria by the maximizing deviation method.

Step 4. Estimate the subjective criteria weight.

Conduct the procedures proposed in Section 3.2, and estimate the subjective criteria weight ω_j'' for each criteria.

Step 5. Compute the integrated criteria weight.

Combine the objective and subjective weights generated from Step 3 and Step 4, the integrated criteria weight ω_j is shown as

$$\omega_j = \lambda\omega_j'^* + (1-\lambda)\omega_j'', \tag{20}$$

in which λ is the aggregation parameter altering in $[0,1]$.

Step 6. Calculate sum total relative importance of alternative.

Incorporate the INPWA operator defined in Equation (9), the sum total relative importance of alternative i is calculated by Equation (21).

$$Q_i' = INPWA(a_1, a_2, \cdots, a_n). \tag{21}$$

Step 7. Calculate product total relative importance of alternative.

Refer to the INPGWA operator in Equation (8), the product total relative importance for alternative i is defined as

$$Q_i'' = INPGWA(a_1, a_2, \cdots, a_n). \tag{22}$$

Step 8. Determine the aggregated WASPAS measure for each alternative.

Aggregate Q_i' and Q'', the final WASPAS measure can be determined by the equation as follows:

$$Q_j = \theta Q_i' + (1-\theta)Q_i'', \tag{23}$$

in which θ is the parameter to adjust the proportion of WSM and WPM in the WASPAS technique altering in $[0,1]$. When $\theta = 1$, the WASPAS technique is degenerated to WSM. When $\theta = 0$, the WASPAS technique is degenerated to WPM.

Step 9. Generate the score, accuracy and certainty function values for each alternative.

Obtain the score, accuracy and certainty function values $S(a_i)$, $H(a_i)$ and $B(a_i)$ for each alternative utilizing Equation (1) in Definition 7.

Step 10. Construct the likelihood matrix.

Construct the possibility matrix of the score function value $S(a_i)$ according to Equation (2), which is shown as follows:

$$P^S\left(S(a_i) \geq S\left(a_j\right)\right) = \begin{bmatrix} p_{11}^S & p_{12}^S & \cdots & p_{1n}^S \\ p_{21}^S & p_{22}^S & \cdots & p_{2n}^S \\ & & \vdots & \\ p_{n1}^S & p_{n2}^S & \cdots & p_{nn}^S \end{bmatrix}_{n\times n},$$

whose elements p_{ij}^S represents the possibility degree of $S(a_i) \geq S\left(a_j\right)$. If $p_{ij}^S = 0.5$, then calculate the possibility degree of $H(a_i) \geq H\left(a_j\right)$ characterized by p_{ij}^H. If $p_{ij}^H = 0.5$, then calculate the possibility degree of $B(a_i) \geq B\left(a_j\right)$ characterized by p_{ij}^B. Obtain the ranking vector p_i according to Equation (4).

Step 11. Rank the alternatives and select the good location.

Rank all the alternatives and select the good location according to the descending order of $p_i\,(i = 1, 2, \cdots, m)$.

4. Case Study

4.1. Problem Description

With the increasing concern of the regional competition and circumstance protection, solar-wind power station contributes to converting solar or wind resources into power, and further generating electric energy. When assuming that government planners want to build a solar-wind power station for better serving regional socio-economy, a good station construction location should be selected. A group of experts are invited to consist a working team for evaluating four locations $A_i(i = 1, 2, \cdots, 4)$ with respect to various influential factors utilizing numerical rating in the range from 0 to 1. According to relative literature research, this paper concludes its influential factors into five criteria in Table 2. In practical assessment procedure, the evaluation rating not only contains the numerical rating but also the emotional tendency of DMs. For a candidate location i under certain criteria j, the positive maximum evaluation among all the DMs is treated as the T_{ij}^U in the INS $a_{ij} = \left\langle \left[T_{ij}^L, T_{ij}^U\right], \left[I_{ij}^L, I_{ij}^U\right], \left[F_{ij}^L, F_{ij}^U\right] \right\rangle$, and the negative minimum evaluation of the entire working team is treated as F_{ij}^L in the INS. For instance, when rating for location A_1 with respect to economic factors C_2, the maximum and minimum evaluation from all the experts, who deem it is appropriate to construct the station here, are 0.5, 0.3, respectively; the maximum and minimum of inappropriate evaluation are 0.8, 0.2; the maximum and minimum of the other experts are 0.3, 0.2; then, the INS can be expressed as $a_{12} = \langle[0.3, 0.5], [0.2, 0.8], [0.2, 0.3]\rangle$. Based on this principle, final decision matrix $A = \left(A_{ij}\right)_{m\times n}$ involving the synthetic evaluation information from all the DMs in the working group is derived as follows:

$$A = \left(A_{ij}\right)_{4\times 5} = \begin{bmatrix} \langle[0.7,0.8],[0.5,0.7],[0.1,0.2]\rangle & \langle[0.3,0.5],[0.2,0.8],[0.2,0.3]\rangle & \langle[0.4,0.6],[0.2,0.2],[0.2,0.4]\rangle & \langle[0.4,0.5],[0.5,0.6],[0.4,0.4]\rangle & \langle[0.6,0.7],[0.4,0.5],[0.4,0.5]\rangle \\ \langle[0.6,0.8],[0.4,0.5],[0.3,0.3]\rangle & \langle[0.5,0.7],[0.3,0.5],[0.1,0.3]\rangle & \langle[0.6,0.7],[0.4,0.6],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.3,0.4],[0.4,0.5]\rangle & \langle[0.8,0.9],[0.3,0.4],[0.1,0.2]\rangle \\ \langle[0.8,0.8],[0.4,0.6],[0.1,0.2]\rangle & \langle[0.6,0.6],[0.2,0.3],[0.4,0.5]\rangle & \langle[0.7,0.8],[0.6,0.7],[0.1,0.2]\rangle & \langle[0.6,0.7],[0.7,0.8],[0.2,0.3]\rangle & \langle[0.7,0.8],[0.5,0.6],[0.1,0.2]\rangle \\ \langle[0.7,0.9],[0.3,0.4],[0.2,0.2]\rangle & \langle[0.6,0.8],[0.4,0.4],[0.2,0.4]\rangle & \langle[0.5,0.6],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.8,0.9],[0.3,0.4],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle \end{bmatrix}_{4\times 5}$$

Table 2. Criteria information and relative description.

Criteria C_j	Description
Natural resources C_1	Natural resources include various indicators related to wind and solar resources in the location.
Economic factors C_2	Economic factors briefly measure the cost during the engineering construction, operation and maintenance procedures.
Traffic conditions C_3	Traffic conditions reflect the traffic convenience to the location during the engineering construction, operation and maintenance procedures.
Environmental factors C_4	Environmental factors reflect the environment destruction during the engineering construction and operation procedures.
Social factors C_5	Social factors reflect the attitude of the local residents to the engineering.

4.2. The Selection of Solar-Wind Power Station Location

The specific procedures of seeking a good solar-wind power station location with the extended WASPAS technique are implemented as follows:

Step 1. Construct the decision matrix.

The decision matrix is constructed by the illustration in Section 4.1, which is shown as $A = (A_{ij})_{4\times 5}$ above.

Step 2. Derive the normalized decision matrix.

Referring to the criteria description in Table 2, C_2 and C_4 are cost criteria. Utilize Equation (1) in Definition 4, the normalized decision matrix can be obtained as

$$A = (A_{ij})_{4\times 5} = \begin{bmatrix} \langle[0.7,0.8],[0.5,0.7],[0.1,0.2]\rangle & \langle[0.2,0.3],[0.2,0.8],[0.3,0.5]\rangle & \langle[0.4,0.6],[0.2,0.2],[0.2,0.4]\rangle & \langle[0.4,0.4],[0.4,0.5],[0.4,0.5]\rangle & \langle[0.6,0.7],[0.4,0.5],[0.4,0.5]\rangle \\ \langle[0.6,0.8],[0.4,0.5],[0.3,0.3]\rangle & \langle[0.1,0.3],[0.5,0.7],[0.5,0.7]\rangle & \langle[0.6,0.7],[0.4,0.6],[0.3,0.4]\rangle & \langle[0.4,0.5],[0.6,0.7],[0.5,0.6]\rangle & \langle[0.8,0.9],[0.3,0.4],[0.1,0.2]\rangle \\ \langle[0.8,0.8],[0.4,0.6],[0.1,0.2]\rangle & \langle[0.4,0.5],[0.7,0.8],[0.6,0.6]\rangle & \langle[0.7,0.8],[0.6,0.7],[0.1,0.2]\rangle & \langle[0.2,0.3],[0.2,0.3],[0.6,0.7]\rangle & \langle[0.7,0.8],[0.5,0.6],[0.1,0.2]\rangle \\ \langle[0.7,0.9],[0.3,0.4],[0.2,0.2]\rangle & \langle[0.2,0.4],[0.6,0.6],[0.6,0.8]\rangle & \langle[0.5,0.6],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.1,0.2],[0.6,0.7],[0.8,0.9]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle \end{bmatrix}_{4\times 5}$$

Step 3. Calculate the objective criteria weight.

Use Equations (16) and (17), then the objective weight $\omega_j'^*$ for each criteria can be calculated as

$$\omega_1'^* = 0.1259, \omega_2'^* = 0.2122, \ \omega_3'^* = 0.2086, \ \omega_4'^* = 0.2698, \ \omega_5'^* = 0.1835.$$

Step 4. Estimate the subjective criteria weight.

Assume that the aggregation parameter $\lambda = 0.5$, and the order relation of all the criteria is $C_4 \succ C_2 \succ C_1 \succ C_5 \succ C_3$ judging from DMs' subjective experience. Referring to the relative importance degree index among adjacent criteria in Table 1, the subjective criteria weight for each criteria is estimated as $\omega_1'' = 0.3533$, $\omega_2'' = 0.2945$, $\omega_3'' = 0.1840$, $\omega_4'' = 0.1022$ and $\omega_5'' = 0.0639$.

Step 5. Compute the integrated criteria weight.

Combine the objective and subjective weights generated from Step 3 and Step 4, the integrated weight of criteria is shown as $\omega_1 = 0.2396$, $\omega_2 = 0.2534$, $\omega_3 = 0.1963$, $\omega_4 = 0.1860$ and $\omega_5 = 0.1237$.

Step 6. Calculate sum total relative importance of alternative.

Incorporate the INPWA operator defined in Equation (9), the sum total relative importance of all alternatives are

$$Q_1' = \langle [0.4986, 0.6171], [0.3084, 0.6468], [0.1866, 0.3342] \rangle,$$

$$Q_2' = \langle [0.4236, 0.6255], [0.4479, 0.5977], [0.3746, 0.4618] \rangle,$$

$$Q_3' = \langle [0.6136, 0.6642], [0.5212, 0.6680], [0.2643, 0.3665] \rangle$$

$$Q_4' = \langle [0.4867, 0.7214], [0.4433, 0.5128], [0.3468, 0.4122] \rangle$$

Step 7. Calculate product total relative importance of alternative.

Refer to the INPGWA operator in Equation (8), the product total relative importance for alternatives are

$$Q_1'' = \langle [0.3816, 0.4983], [0.3562, 0.7138], [0.2219, 0.3815] \rangle,$$

$$Q_2'' = \langle [0.2638, 0.4970], [0.4566, 0.6189], [0.4046, 0.5409] \rangle,$$

$$Q_3'' = \langle [0.5184, 0.6005], [0.5784, 0.7099], [0.4203, 0.4605] \rangle$$

$$Q_4'' = \langle [0.3533, 0.5616], [0.4852, 0.5355], [0.4509, 0.6134] \rangle$$

Step 8. Determine the aggregated WASPAS measure for each alternative.

Let $\theta = 0.5$, then the final WASPAS measure can be derived as

$$Q_1 = \langle [0.4401, 0.5577], [0.3323, 0.6803], [0.2043, 0.3579] \rangle,$$

$$Q_2 = \langle [0.3437, 0.5613], [0.4523, 0.6083], [0.3896, 0.5014] \rangle,$$

$$Q_3 = \langle [0.5660, 0.6324], [0.5498, 0.6889], [0.3423, 0.4135] \rangle$$

$$Q_4 = \langle [0.4200, 0.6415], [0.4643, 0.5241], [0.3989, 0.5129] \rangle$$

Step 9. Generate the score, accuracy and certainty function values for each alternative.

For each alternative, the score, accuracy and certainty function values $S(a_i)$, $H(a_i)$ and $B(a_i)$ are shown in Table 3.

Table 3. Relative function values of alternatives.

A_i	$S(a_i)$	$H(a_i)$	$B(a_i)$
A_1	$[1.4020, 2.0212]$	$[0.1998, 0.2359]$	$[0.4401, 0.5577]$
A_2	$[1.2341, 1.7194]$	$[-0.0459, 0.0599]$	$[0.3437, 0.5613]$
A_3	$[1.4636, 1.7402]$	$[0.2189, 0.2237]$	$[0.5660, 0.6324]$
A_4	$[1.3831, 1.7783]$	$[0.0212, 0.1286]$	$[0.4200, 0.6415]$

Step 10. Construct the likelihood matrix.

Construct the possibility matrix of the score function value $S(a_i)$ according to Equation (2), which is shown as follows:

$$P^S\left(S(a_i) \geq S(a_j)\right) = \begin{bmatrix} 0.5 & 0.7126 & 0.6224 & 0.6290 \\ 0.7126 & 0.5 & 0.3358 & 0.3820 \\ 0.6224 & 0.3358 & 0.5 & 0.5315 \\ 0.6290 & 0.3820 & 0.5315 & 0.5 \end{bmatrix}_{4\times 4}.$$

And the ranking vector can be obtained as $p_1 = 2.5981$, $p_2 = 2.1977$, $p_3 = 2.2423$ and $p_4 = 2.2819$.

Step 11. Rank the alternatives and select the good location.

The ranking order of all the alternatives is $A_1 \succ A_4 \succ A_3 \succ A_2$ and the good location is A_1.

5. Sensitivity Analysis and Comparison Analysis

In this section, comparison analysis and sensitivity analysis are conducted to further testify the effectiveness and reliability of the extended WASPAS technique with existing methods on the same example.

5.1. Sensitivity Analysis and Discussion

To demonstrate the impact of different aggregation parameter λ and the proportion adjustment parameter θ on the final location selection results, we conduct the sensitivity analysis on the same example with five $\lambda(0, 0.2, 0.5, 0.8, 1)$ values and five $\theta(0, 0.2, 0.5, 0.8, 1)$ values simultaneously. The ranking results for different λ and θ are displayed in Table 4.

Table 4. Ranking results for different λ and θ.

λ	$\theta = 0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.8$	$\theta = 1$
0	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$
0.2	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$
0.5	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$
0.8	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$	$A_1 \succ A_4 \succ A_3 \succ A_2$
1	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_4 \succ A_1 \succ A_3 \succ A_2$

The variation of λ represents the influence of criteria weight integrated by objective and subjective weights to final selection results. In addition, altering the θ values means that the change from sum total relative importance to product total relative importance. Synthetizing what is shown in Table 4, it can be inferred that different λ and θ values indeed affect final selection results. When $\lambda = 0$ and $\lambda = 1$, the integrated criteria weight is complete subjective or objective weight, and the ranking order has different selection results A_1 and A_4 under these opposite context. Furthermore, no matter how the variation of θ values, the ranking order remains all the same in these circumstances. When $\lambda = 0.2$ and $\lambda = 0.8$, the proportion adjustment parameter θ values make huge differences to the ranking order. The ranking result yields $A_4 \succ A_1 \succ A_3 \succ A_2$ when $0 \leq \theta < 0.5$, while it yields $A_1 \succ A_4 \succ A_3 \succ A_2$ when $0.5 \leq \theta \leq 1$. In view of the identical changeable tendency of ranking order, it is apparent that the gradual increase of the adjustment proportion θ clearly affects the order when different importance of the objective and subjective weights in the integrated weight. However, when the importance of the objective and subjective weights in the integrated weight are totally equivalent, the ranking order keeps $A_4 \succ A_1 \succ A_3 \succ A_2$ all the time.

Obviously, these ranking order and selection results reveal that different λ and θ values have an effect on the decision-making procedure. It further indicates that the combination of objective and subjective weights, and the integration of sum total relative importance and product total relative importance emphasize the influence to the selection procedure. Integrated the specific meaning of λ and θ with their role playing in the decision-making procedures, the location selection becomes a dynamic procedure by setting different parameter values derived from practical context.

5.2. Comparison Analysis and Discussion

This subsection further validates the effectiveness and reliability of the newly proposed method by comparing it with different existing methods on the identical illustrative example. The ranking results are shown in Table 5. As there exists no relative criteria weight obtainment method in [34,62,68] chosen for comparison, we use the integrated criteria weight obtained in this paper as the criteria weight information for the comparison convenience.

Table 5. Ranking results with different existing methods.

Methods	Ranking Results
Similarity measure in [62]	$A_4 \succ A_2 \succ A_3 \succ A_1$
An extended TOPSIS method in [69]	$A_3 \succ A_4 \succ A_2 \succ A_1$
Method used weighted arithmetic aggregation operator in [68]	$A_4 \succ A_1 \succ A_3 \succ A_2$
Method used weighted geometric aggregation operator in [68]	$A_4 \succ A_1 \succ A_3 \succ A_2$
Classical WASPAS method in [34]	$A_4 \succ A_1 \succ A_3 \succ A_2$
The proposed method	$A_1 \succ A_4 \succ A_3 \succ A_2$

(1) The method in [62] generates the selection result by implementing two steps. Firstly, confirm the ideal alternatives for different type of criteria. Subsequently, derive the result by similarity measures. According to the first similarity measure in that literature, the similarity measures are obtained as $S_1^*(A^*, A_1) = 0.8178$, $S_1^*(A^*, A_2) = 0.8705$, $S_1^*(A^*, A_3) = 0.8692$ and $S_1^*(A^*, A_4) = 0.9057$.

(2) In the method of [69], maximizing deviation method is utilized to derive objective weights. Then, based on the ideal of TOPSIS method, alternatives are ranked by the relative closeness coefficient. Conduct these procedures, relative closeness coefficient can be calculated as $RCC_1 = 0.5226$, $RCC_2 = 0.5186$, $RCC_3 = 0.5190$ and $RCC_4 = 0.5189$.

(3) The procedure of the method in [68] can be briefly classified into aggregation process and ranking process. The ranking procedure in [68] is identical with our proposed method. Based on weighted arithmetic aggregation operator or weighted geometric aggregation operator, the total score are obtained as $p_s = [2.2997, 2.0098, 2.0589, 2.5926]$ or $p_s = [2.5804, 2.2751, 2.3728, 2.9288]$ with the same ranking order $A_4 \succ A_1 \succ A_3 \succ A_2$.

(4) Classical WASPAS method in [34] generates the final WASPAS measure by aggregating weighted arithmetic aggregation operator and weighted geometric aggregation operator. To better compare the classical WASPAS method with our method, the alternatives are ranked by the ranking procedure in our paper. Then, ranking vector can be obtained as $p_1 = 2.4449$, $p_2 = 2.1483$, $p_3 = 2.2189$ and $p_4 = 2.7681$.

It can be easily inferred from Table 5 that different methods generate totally distinct selection results. Especially, the order sequence derived from the methods in [62,69] is entirely opposite to ours. The good location selection for [62,69] and our paper are A_4, A_3 and A_1, respectively. The primary reason for that phenomenon relies on that these methods were proposed based on completely different ideal. It results in the inconsistent sequence between each pair of alternatives in these methods. Both the similarity measures and TOPSIS method rank the alternatives on the ideal of distance. For a better comparison analysis, we adopt Hamming distance during the calculation procedure. Moreover, the method in [62] neglects the relation of criteria to each other, while the method in [69] only considers the objective criteria weight without focusing on the subjective preference of DMs. The proposed method emphasizes the support relation from other criteria in the course of WASPAS measure determination. It adequately considers the objective criteria weights and subjective experience judgement from DMs by integrated criteria weight. The selection A_4 and A_3 cannot reflect realistic situation without considering the subjective criteria during the MCDM procedure. In addition, these characteristics ensure the effectiveness and reliability of the location selection A_1 derived from our method. Although both the method in [68] and our method utilize an identical ranking method, the aggregation operators cause huge differences to the final location selection. When using one of the weighted arithmetic aggregation operator or weighted geometric aggregation operator in [68], it yields identical selection result A_4. However, the proposed method not only contains the advantage of PA operator, but also effectively utilizes the combination of INPGWA and INPWA rather than applying them respectively.

Besides, the extended WASPAS technique reveals its superiority by implementing three procedures to assure the selection result A_1 rational and convincing.

Utilizing identical aggregation operator, methods in [34,68] provide the totally same ranking order $A_4 \succ A_1 \succ A_3 \succ A_2$ with identical location selection A_4. It indicates that the superiority of the classical WASPAS method cannot be inferred because there exists no ranking or selection discrepancy when using completely different methods. It is obvious that the ranking and selection results of the classical WASPAS method and the proposed method exists discrepancy. The order of A_1 and A_4 is opposite in those methods. That is, though both the classical and extended WASPAS methods have the same procedure in which the WASPAS measure are aggregated by two kinds of operators, the extended WASPAS method reflects a more rational selection result by newly incorporating the advantage of PA operators in that. Furthermore, the integrated weight information estimation procedure further ensures the selection of the extended WASPAS method scientific. The identical order between alternatives A_3 and A_2 in these methods can verify the effectiveness of the extended one.

From above discussion, primary highlights of the newly extended WASPAS technique can be simply summarized into the following points.

(1) It can effectively manage the solar-wind power station location problem via embedding three procedures into the newly extended WASPAS technique. During the WASPAS technique implement process, a rational location selection result will be generated by incorporating the advantages of relevant methods in these procedures.

(2) With the maximizing derivation method, objective criteria weights can be simply determined no matter under the criteria weights completely unknown or incomplete circumstances. Apart from the objective criteria weights, subjective weights, which fully reflect the subjective preference under practice, can be obtained with G1. The integrated criteria weight is the combination of the objective and subjective weights, and can adequately represent more realistic situation.

(3) Different aggregation parameter λ and the proportion adjustment parameter θ facilitate the whole procedures a dynamic selection. The parameter setting is based on the requirement of real application and subjective preference of DMs, which makes the extended WASPAS technique feasible in dealing with the reality.

6. Conclusions

A good location of the solar-wind power station can affect regional competitiveness and direct future development to a great extent. Faced with multiple uncertainties in reality, the location selection case is considered within the INSs circumstances for tackling such challenges. Recognized as a complex MCDM procedure, the selection in this paper is settled by an extended WASPAS technique containing three procedures to reinforce its applicability to real situation. For modelling more realistic information, some modifications are made in the classical WASPAS method especially utilizing the objective and subjective criteria weight integrated weight information. Its strengths have been adequately discussed via comparison analysis and sensitivity analysis.

Highlights of the extended technique can be briefly summarized in three aspects. Firstly, it ensures a relatively rational and scientific result by incorporating three procedures into the framework of the technique. Secondly, the integrated weight information reflects more realistic weight information with the combination of the objective and subjective criteria weight information. Thirdly, more practical contexts can be reflected by altering the aggregation parameter and the proportion adjustment parameter during performing relevant procedures. Although the location selection case can be well measured by the criteria dimensions in this paper, potential work should also be focused on specific sub-criteria information for reaching a more promising solution.

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