Ion Pătraşcu

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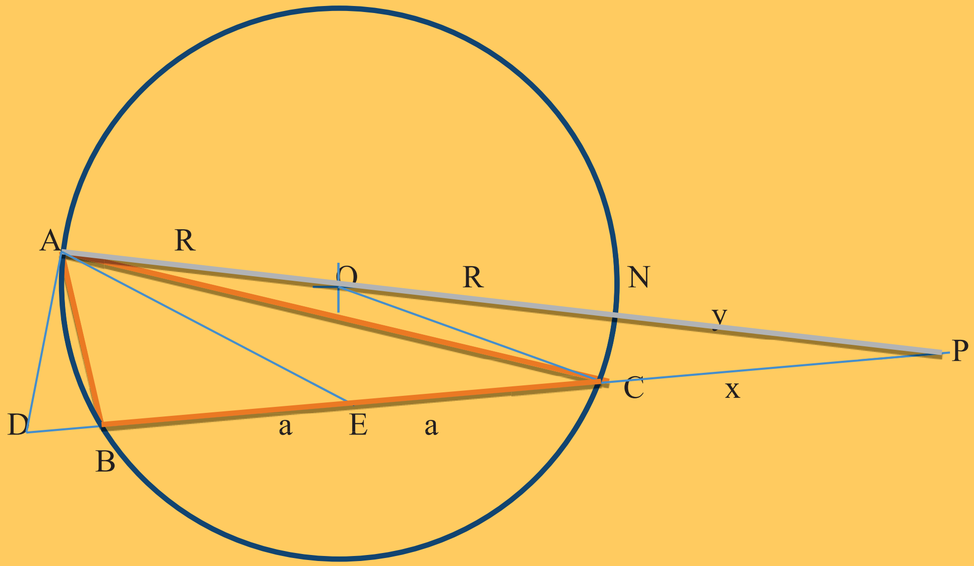

VARIANCE ON TOPICS OF PLANE GEOMETRY

Ion Pătraşcu Florentin Smarandache

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CONTENTS

VARIANCE ON TOPICS OF PLANE GEOMETRY

Preface

This book contains 21 papers of plane geometry.
It deals with various topics, such as: quasi-isogonal cevians, nedians, polar of a point with respect to a circle, anti-bisector, aalsonti-symmedian, anti-height and their isogonal.
A nedian is a line segment that has its origin in a triangle's vertex and divides the opposite side in n equal segments.

The papers also study distances between remarkable points in the 2D-geometry, the circumscribed octagon and the inscribable octagon, the circles adjointly ex-inscribed associated to a triangle, and several classical results such as: Carnot circles, Euler's line, Desargues theorem, Sondat's theorem, Dergiades theorem, Stevanovic's theorem, Pantazi's theorem, and Newton's theorem.

Special attention is given in this book to *orthological triangles, bi-orthological triangles, ortho-homological triangles,* and *tri-homological triangles.*

The notion of "ortho-homological triangles" was introduced by the Belgium mathematician Joseph Neuberg in 1922 in the journal Mathesis and it characterizes the triangles that are simultaneously orthogonal (i.e. the sides of one triangle are perpendicular to the sides of the other triangle) and homological. We call this "ortho-homological of first type" in order to distinguish it from our next notation.

In our articles, we gave the same denomination "ortho-homological triangles" to triangles that are simultaneously orthological and homological. We call it "ortho-homological of second type."

Each paper is independent of the others. Yet, papers on the same or similar topics are listed together one after the other.

This book is a continuation of the previous book *The Geometry of Homological Triangles*, by Florentin Smarandache and Ion Pătraşcu, Educ. Publ., Ohio, USA, 244 p., 2012.

The book is intended for College and University students and instructors that prepare for mathematical competitions such as National and International Mathematical Olympiads, or the AMATYC (American Mathematical Association for Two Year Colleges) student competition, or Putnam competition, Gheorghe Ţiteica Romanian student competition, and so on.

The book is also useful for geometrical researchers.

The authors

Quasi-Isogonal Cevians

Professor Ion Pătraşcu – National College Frații Buzeşti, Craiova, Romania
Professor Florentin Smarandache –University of New-Mexico, U.S.A.

In this article we will introduce the quasi-isogonal Cevians and we'll emphasize on triangles in which the height and the median are quasi-isogonal Cevians.

For beginning we'll recall:

Definition 1

In a triangle ABC the Cevians AD, AE are called isogonal if these are symmetric in rapport to the angle A bisector.

Observation

In figure 1, are represented the isogonal Cevians AD, AE

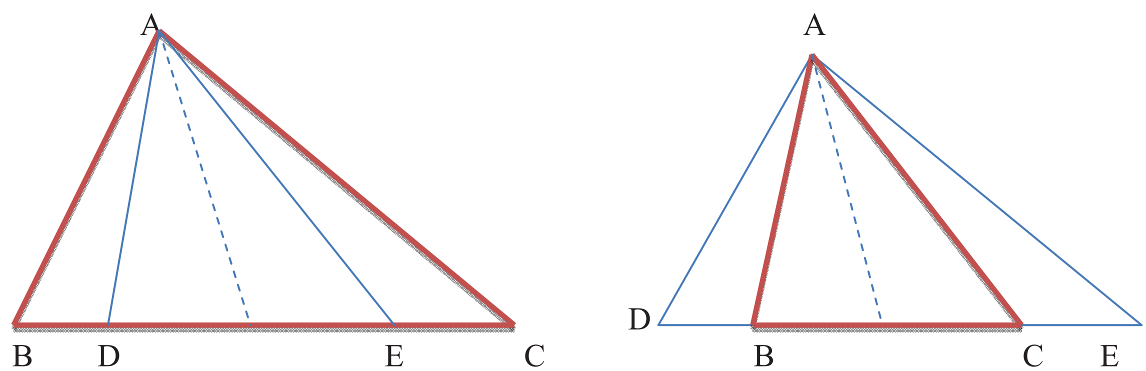

Fig. 1. Isogonal Cevians

Proposition 1.

In a triangle ABC, the height AD and the radius AO of the circumscribed circle are isogonal Cevians.

Definition 2.

We call the Cevians AD, AE in the triangle ABC quasi-isogonal if the point B is between the points D and E, the point E is between the points B and C, and $\sphericalangle DAB \equiv \sphericalangle EAC$.

Observation

In figure 2 we represented the quasi-isogonal Cevians AD, AE.

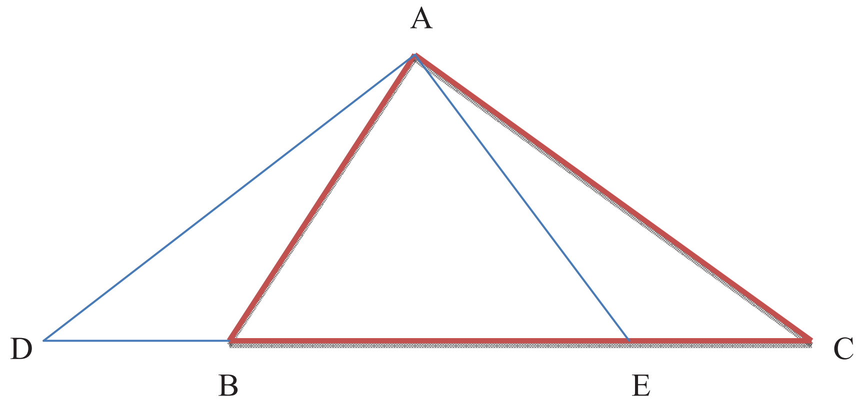

Fig. 2 quasi-isogonal Cevians

Proposition 2
There are triangles in which the height and the median are quasi-isogonal Cevians.

Proof
It is clear that if we look for triangles ABC for which the height and the median from the point A are quasi isogonal, then these must be obtuse-angled triangle. We'll consider such a case in which $m(\sphericalangle A) > 90°$ (see figure 3).

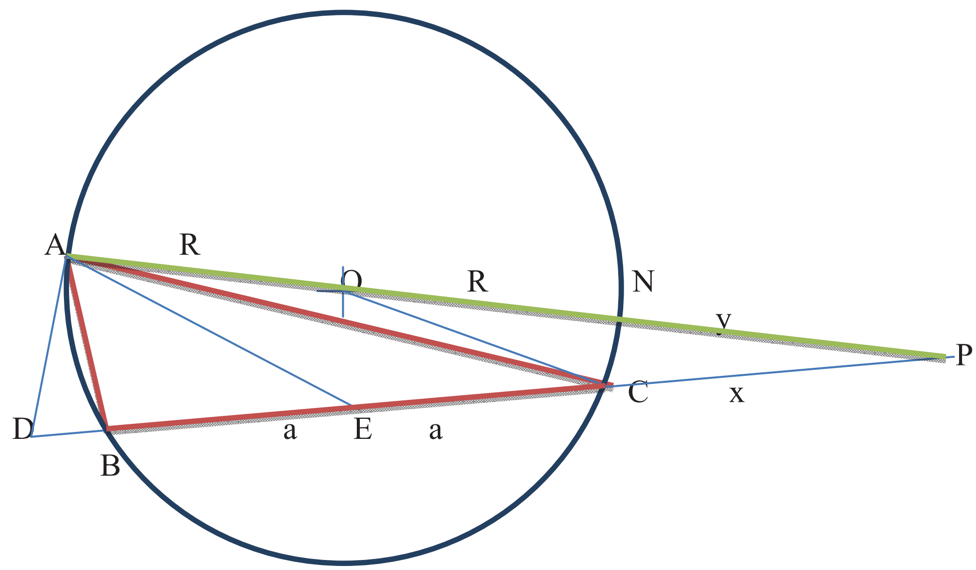

Fig. 3

Let O the center of the circumscribed triangle, we note with N the diametric point of A and with P the intersection of the line AO with BC.

We consider known the radius R of the circle and $BC = 2a$, $a < R$ and we try to construct the triangle ABC in which the height AD and the median AE are quasi isogonal Cevians; therefore $\sphericalangle DAB \equiv \sphericalangle EAC$. This triangle can be constructed if we find the lengths PC and PN in function of a and R. We note $PC = x,\ PN = y$.

We consider the power of the point P in function of the circle $\mathcal{C}(O,R)$. It results that

$$x\cdot(x+2a)=y\cdot(y+2R) \qquad (1)$$

From the Property 1 we have that $\sphericalangle DAB \equiv \sphericalangle OAC$. On the other side $\sphericalangle OAC \equiv \sphericalangle OCA$ and AD, AE are quasi isogonal, we obtain that $OC \parallel AE$.

The Thales' theorem implies that:

$$\frac{x}{a}=\frac{y+R}{R} \qquad (2)$$

Substituting x from (2) in (1) we obtain the equation:

$$\left(a^2-R^2\right)y^2-2R\left(R^2-2a^2\right)y+3a^2R^2=0 \qquad (3)$$

The discriminant of this equation is:

$$\Delta=4R^2\left(R^4-a^2R^2+a^4\right)$$

Evidently $\Delta>0$, therefore the equation has two real solutions.

Because the product of the solutions is $\dfrac{3a^2R^2}{a^2-R^2}$ and it is negative we obtain that one of solutions is strictly positive. For this positive value of y we find the value of x, consequently we can construct the point P, then the point N and at the intersection of the line PN we find A and therefore the triangle ABC is constructed.

For example, if we consider $R=\sqrt{2}$ and $a=1$, we obtain the triangle ABC in which $AB=\sqrt{2}$, $BC=2$ and $AC=1+\sqrt{3}$.

We leave to our readers to verify that the height and the median from the point A are quasi isogonal.

Nedians and Triangles with the Same Coefficient of Deformation

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Florentin Smarandache – University of New Mexico, Gallup, NM, USA

In [1] Dr. Florentin Smarandache generalized several properties of the nedians. Here, we will continue the series of these results and will establish certain connections with the triangles which have the same coefficient of deformation.

Definition 1

The line segments that have their origin in the triangle's vertex and divide the opposite side in n equal segments are called nedians.

We call the nedian AA_i being of order i ($i \in N^*$), in the triangle ABC, if A_i divides the side (BC) in the rapport $\frac{i}{n}$ ($\overrightarrow{BA_i} = \frac{i}{n} \cdot \overrightarrow{BC}$ or $\overrightarrow{CA_i} = \frac{i}{n} \cdot \overrightarrow{CB}$, $1 \le i \le n-1$)

Observation 1

The medians of a triangle are nedians of order 1, in the case when $n = 3$, these are called tertian.

We'll recall from [1] the following:

Proposition 1

Using the nedians of the same of a triangle, we can construct a triangle.

Proposition 2

The sum of the squares of the lengths of the nedians of order i of a triangle ABC is given by the following relation:

$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{i^2 - in + n^2}{n^2}\left(a^2 + b^2 + c^2\right) \tag{1}$$

We'll prove

Proposition3.

The sum of the squares of the lengths of the sides of the triangle $A_0B_0C_0$, determined by the intersection of the nedians of order i of the triangle ABC is given by the following relation:

$$A_0B_0^2 + B_0C_0^2 + C_0A_0^2 = \frac{\left(n-2i\right)^2}{i^2 - in + n^2}\left(a^2 + b^2 + c^2\right) \tag{2}$$

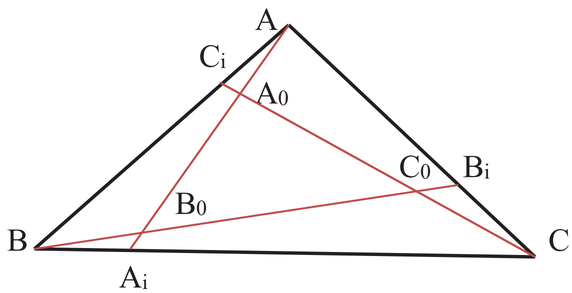

Fig. 1

We noted

$$\{A_0\} = CC_i \cap AA_i,\ \{B_0\} = AA_i \cap BB_i,\ \{C_0\} = BB_i \cap CC_i.$$

Proof

We'll apply the Menelaus 'theorem in the triangle AA_iC for the transversals $B - B_0 - B_i$, see Fig. 1.

$$\frac{BA_i}{BC} \cdot \frac{B_iC}{B_iA} \cdot \frac{B_0A}{B_0A_i} = 1 \tag{3}$$

Because $BA_i = \frac{ia}{n}$, $B_iC = \frac{ib}{n}$, $B_iA = \frac{(n-i)b}{n}$, from (3) it results that:

$$B_0A = \frac{n(n-i)}{i^2 - in + n^2} AA_i \tag{4}$$

The Menelaus 'theorem applied in the triangle AA_iB for the transversal $C - C_0 - C_i$ gives

$$\frac{CA_i}{CB} \cdot \frac{C_iB}{C_iA} \cdot \frac{A_0A}{A_0A_i} = 1 \tag{5}$$

But $CA_i = \frac{(n-i)a}{n}$, $C_iB = \frac{(n-i)c}{n}$, $C_iA = \frac{ic}{n}$, which substituted in (5), gives

$$AA_0 = \frac{in}{i^2 - in + n^2} AA_i \tag{6}$$

It is observed that $A_0B_0 = AB_0 - AA_0$ and using the relation (4) and (6) we find:

$$A_0B_0 = \frac{n(n-2i)}{i^2 - in + n^2} AA_i \tag{7}$$

Similarly, we obtain:

$$B_0C_0 = \frac{n(n-2i)}{i^2 - in + n^2} BB_i \tag{8}$$

$$C_0A_0 = \frac{n(n-2i)}{i^2 - in + n^2} CC_i \tag{9}$$

Using the relations (7), (8) and (9), after a couple of computations we obtain the relation (2).

Observation 2.

The triangle formed by the nedians of order i as sides is similar with the triangle formed by the intersections of the nedians of order i.

Indeed, the relations (7), (8) and (9) show that the sides A_0B_0, B_0C_0, C_0A_0 are proportional with AA_i, BB_i, CC_i

The Russian mathematician V. V. Lebedev introduces in [2] the notion of coefficient of deformation of a triangle. To define this notion we need a couple of definitions and observations.

Definition 2

If ABC is a triangle and in its exterior on its sides are constructed the equilateral triangles BCA_1, CAB_1, ABC_1, then the equilateral triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to the equilateral triangles, described above, is called the exterior triangle of Napoleon.

If the equilateral triangles BCA_1, CAB_1, ABC_1 intersect in the interior of the triangle ABC then the equilateral triangle $O_1'O_2'O_3'$ formed by the centers of the circumscribed circles to these triangles is called the interior triangle of Napoleon.

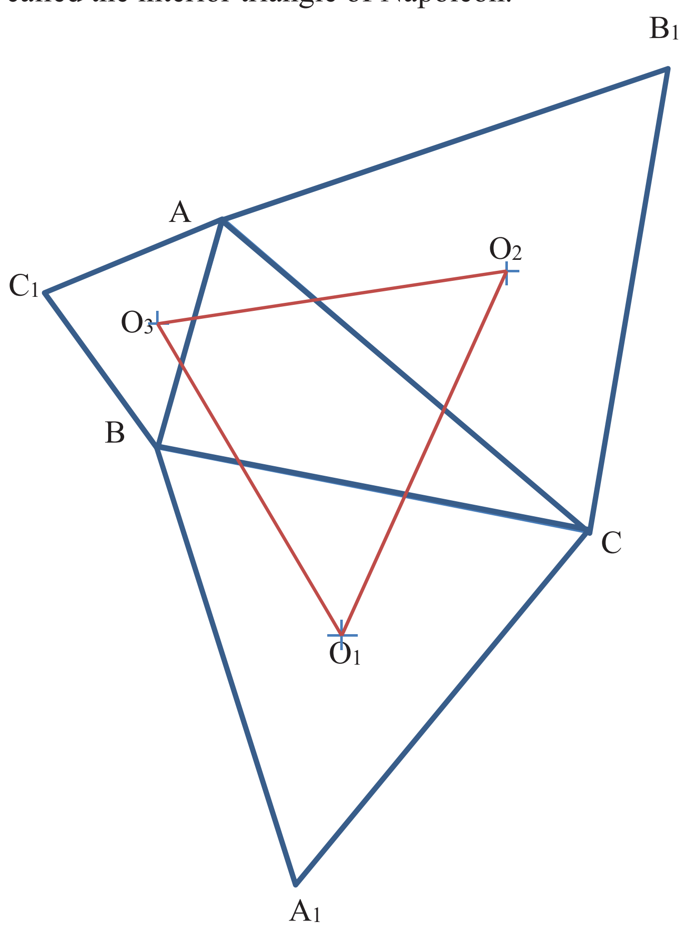

Fig. 2

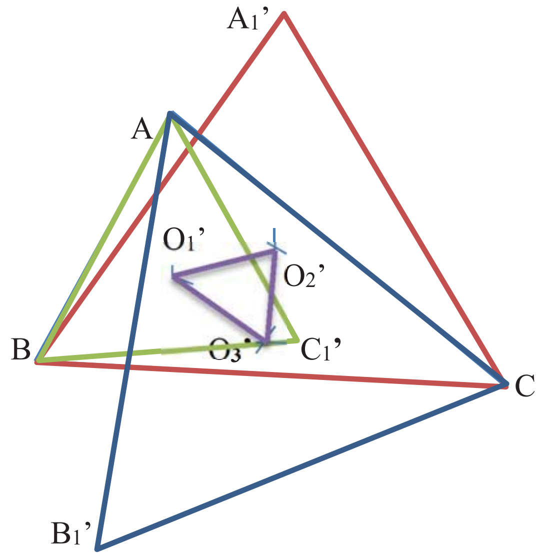

Fig. 3

Observation 3

In figure 2 is represented the external triangle of Napoleon and in figure 3 is represented the interior triangle of Napoleon.

Definition 3

A coefficient of deformation of a triangle is the rapport between the side of the interior triangle of Napoleon and the side of the exterior triangle of Napoleon corresponding to the same triangle.

Observation 4

The coefficient of deformation of the triangle ABC is

$$k = \frac{O_1'O_2'}{O_1O_2}$$

Proposition 4

The coefficient of deformation k of triangle ABC ha the following formula:

$$k = \left(\frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{a^2 + b^2 + c^2 + 4s\sqrt{3}} \right)^{\frac{1}{2}} \tag{10}$$

where s is the aria of the triangle ABC.

Proof

We'll apply the cosine theorem in the triangle $CO_1'O_2'$ (see Fig. 3), in which

$$CO_1' = \frac{a\sqrt{3}}{3},\ CO_2' = \frac{b\sqrt{3}}{3}\text{ , and } m(\sphericalangle O_1CO_2') = C - 60°.$$

We have

$$O_1'O_2'^2 = \frac{3a^2}{9} + \frac{3b^2}{9} - 2\frac{ab}{3}cos\left(C - 60°\right)$$

Because

$$cos\left(C - 60°\right) = cos\,C \cdot cos\,60° + sin\,60° \cdot sin\,C = \frac{1}{2}cos\,C + \frac{\sqrt{2}}{2}sin\,C \text{ and}$$

$$cosC = \frac{b^2 + a^2 - c^2}{2ab}, \text{ and}$$

$$ab\,sin\,C = 2s,$$

we obtain

$$O_1'O_2'^2 = \frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{6} \tag{11}$$

Similarly

$$O_1O_2^2 = \frac{a^2 + b^2 + c^2 + 4s\sqrt{3}}{6} \tag{12}$$

By dividing the relations (11) and (12) and resolving the square root we proved the proposition.

Observation 5

In an equilateral triangle the deformation coefficient is $k = 0$. In general, for a triangle ABC, $0 \le k < 1$.

Observation 6

From (11) it results that in a triangle is true the following inequality:

$$a^2 + b^2 + c^2 \ge 4s\sqrt{3} \tag{13}$$

which is the inequality Weitzeböck.

Observation 7

In a triangle there following inequality – stronger than (13) – takes also place:

$$a^2 + b^2 + c^2 \ge 4s\sqrt{3} + \left(a - b\right)^2 + \left(b - c\right)^2 + \left(c - a\right)^2 \tag{14}$$

which is the inequality of Finsher - Hadwiger.

Observation 8

It can be proved that in a triangle the coefficient of deformation can be defined by the

$$k = \frac{AA_1'}{AA_1} \tag{15}$$

Definition 4

We define the Brocard point in triangle ABC the point Ω from the triangle plane, with the property:

$$\sphericalangle \Omega AB \equiv \sphericalangle \Omega BC \equiv \sphericalangle \Omega CA \tag{16}$$

The common measure of the angles from relation (16) is called the Brocard angle and is noted

$$\sphericalangle \Omega AB = \omega$$

Observation 9

A triangle ABC has, in general, two points Brocard Ω and Ω' which are isogonal conjugated (see Fig. 4)

Proposition 5

In a triangle ABC takes place the following relation:

$$ctg\omega = \frac{a^2 + b^2 + c^2}{4s} \tag{17}$$

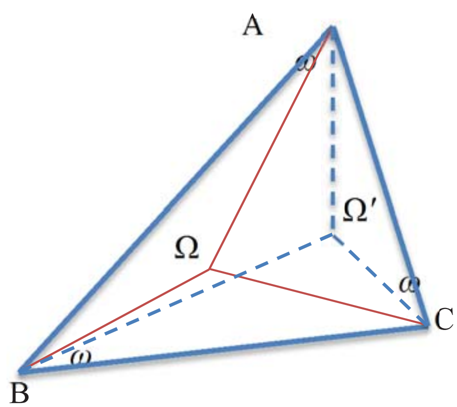

Fig. 4

Proof

We'll show, firstly, that in a non-rectangle triangle ABC is true the following relation:

$$ctg\omega = ctgA + ctgB + ctgC \tag{18}$$

Applying the sin theorem in triangle $A\Omega B$ and $A\Omega C$, we obtain

$$\frac{B\Omega}{sin\,\omega} = \frac{c}{sin\,B\Omega A} \text{ and } \frac{A\Omega}{sin\,\omega} = \frac{b}{sin\,A\Omega C}$$

Because $m(\sphericalangle B\Omega A) = 180° - m(\sphericalangle B)$ and $m(\sphericalangle A\Omega C) = 180° - m(\sphericalangle A)$ from the precedent relations we retain that

$$\frac{A\Omega}{B\Omega} = \frac{b}{c}\frac{sin\,B}{sin\,A} \tag{19}$$

On the other side also from the sin theorem in triangle $A\Omega B$, we obtain

$$\frac{A\Omega}{B\Omega} = \frac{sin(B - \omega)}{sin\,\omega} \tag{20}$$

Working out $sin(B - \omega)$, taking into account that $\frac{b}{c} = \frac{sin\,B}{sin\,C}$ and that $sin\,B = sin(A + C)$, we obtain (18).

In a triangle ABC is true the relation $ctgA = \frac{a^2 + b^2 + c^2}{4s}$ (19) and the analogues.

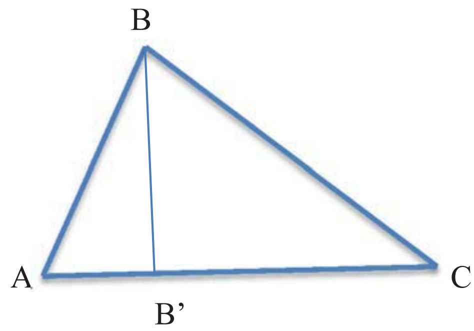

Fig. 5

Indeed, if $m(\sphericalangle A) < 90°$ and B' is the orthogonal projection of B on AC (see Fig. 5), then

$$ctgA = \frac{AB'}{BB'} = \frac{c \cdot cos\, A}{BB'}$$

Because $BB' = \frac{2s}{b}$ it results that $ctgA = \frac{2bc\, cos\, A}{4s}$

From the cosine theorem we get

$$2bc\, cos\, A = b^2 + c^2 - a^2$$

Replacing in (18) the $ctgA,\ ctgB,\ ctgC$, we obtain (17)

Observation 10

The coefficient of deformation k of triangle ABC is given by

$$k = \left(\frac{ctg\omega - \sqrt{3}}{ctg\omega + \sqrt{3}}\right)^{\frac{1}{2}} \tag{21}$$

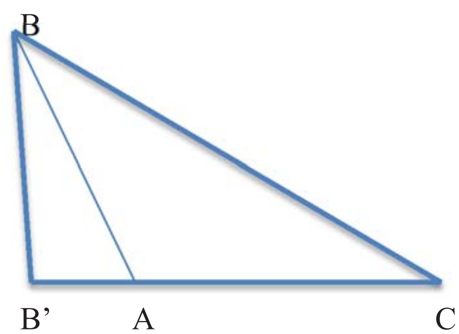

Fig. 6

Indeed, from (10) and (17), it results, without difficulties (21)

Proposition 6 (V.V. Lebedev)

The necessary and sufficient condition for two triangles to have the same coefficient of deformation is to have the same Brocard angle.

Proof

If the triangles ABC and $A_1B_1C_1$ have equal coefficients of deformation $k = k_1$ then from relation 21 it results

$$\frac{ctg\omega - \sqrt{3}}{ctg\omega + \sqrt{3}} = \frac{ctg\omega_1 - \sqrt{3}}{ctg\omega_1 + \sqrt{3}}$$

Which leads to $ctg\omega = ctg\omega_1$ with the consequence that $\omega = \omega_1$.

Reciprocal, if $\omega = \omega_1$, immediately results, using (21), that takes place $k = k_1$.

Proposition 7

Two triangles ABC and $A_1B_1C_1$ have the same coefficient of deformation if and only if

$$\frac{s_1}{s} = \frac{a_1^2 + b_1^2 + c_1^2}{a^2 + b^2 + c^2} \tag{22}$$

(s_1 being the aria of triangle $A_1B_1C_1$, with the sides a_1, b_1, c_1)

Proof

If ω, ω_1 are the Brocard angles of triangles ABC and $A_1B_1C_1$ then, taking into consideration (17) and Proposition 6, we'll obtain (22). Also from (22) taking into consideration of (17) and Proposition 6, we'll get $k = k_1$.

Proposition 8

Triangle $A_iB_iC_i$ formed by the legs of the nediands of order i of triangle ABC and triangle ABC have the same coefficient of deformation.

Proof

We'll use Proposition 7, applying the cosine theorem in triangle $A_iB_iC_i$, we'll obtain

$$B_iC_i^2 = AC_i^2 + AB_i^2 - 2AC_iAB_i \cos A$$

Because

$$AC_i = \frac{ic}{n},\ AB_i = \frac{(n-i)b}{n}$$

it results

$$B_iC_i^2 = \frac{i^2c^2}{n^2} + \frac{(n-i)^2 b^2}{n^2} - \frac{2i(n-i)bc\cos A}{n^2}$$

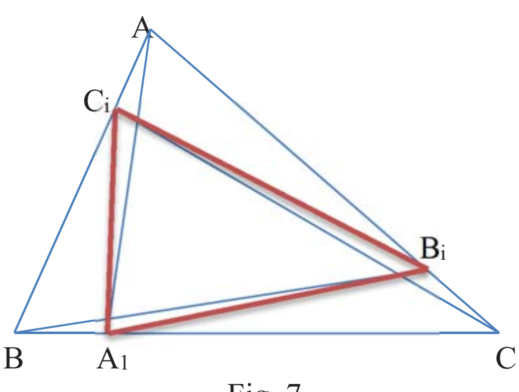

Fig. 7

The cosin theorem in the triangle ABC gives

$$2bc\cos A = b^2 + c^2 - a^2$$

which substituted above gives

$$B_iC_i^2 = \frac{i^2c^2 + (n-i)^2 b^2 + i(n-i)(a^2 - b^2 - c^2)}{n^2}$$

$$B_iC_i^2 = \frac{a^2(in - i^2) + b^2(n^2 - 3in + 2i^2) + c^2(2i^2 - in)}{n^2}$$

Similarly we'll compute $C_iA_i^2$ and $A_iB_i^2$

It results

$$\frac{A_iB_i^2 + B_iC_i^2 + C_iA_i^2}{a^2 + b^2 + c^2} = \frac{n^2 - 2in + 3i^2}{n^2} \tag{23}$$

If we note

$$s_i = Aria_{\vartriangle} A_iB_iC_i$$

We obtain

$$s_i = s - (Aria_{\vartriangle} AB_iC_i + Aria_{\vartriangle} BA_iC_i + Aria_{\vartriangle} CA_iB_i) \tag{24}$$

But

$$Aria_{\vartriangle} AB_iC_i = \frac{1}{2} AC_i \cdot AB_i \sin A$$

$$Aria_{\vartriangle} AB_iC_i = \frac{1}{2}\frac{i(n-i)b\cdot c}{n^2}\sin A = \frac{i(n-i)\cdot s}{n^2}$$

Similarly, we find that

$$Aria_{\vartriangle} BA_iC_i = Aria_{\vartriangle} CA_iB_i = \frac{i(n-i)\cdot s}{n^2}$$

Revisiting (23) we get that

$$s_i = \frac{sn^2 - 3in + 3i^2}{n^2}$$

therefore,

$$\frac{s_i}{s} = \frac{n^2 - 3in + 3i^2}{n^2} \tag{25}$$

The relations (23), (25) and Proposition 7 will imply the conclusion.

Proposition 9

The triangle formed by the medians of a given triangle, as sides, and the given triangle have the same coefficient of deformation.

Proof

The medians are nedians of order I. Using (1), it results

$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{3}{4}(a^2 + b^2 + c^2) \tag{26}$$

The proposition will be proved if we'll show that the rapport between the aria of the formed triangle with the medians of the given triangle and the aria of the given triangle is $\frac{3}{4}$.

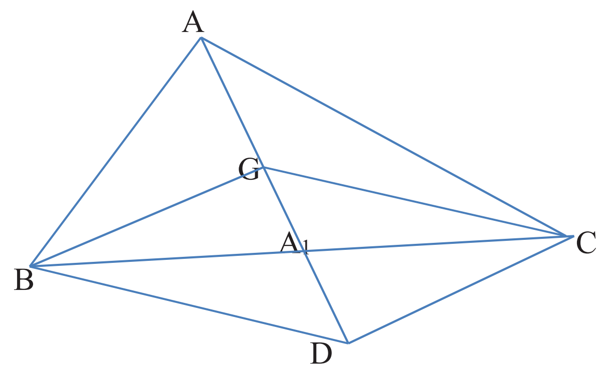

Fig. 9

If in triangle ABC we prolong the median AA_1 such that $A_1D = GA_1$ (G being the center of gravity of the triangle ABC), then the quadrilateral $BGCD$ is a parallelogram (see Fig. 9). Therefore $CD = BG$. It is known that $BG = \frac{2}{3}BB_1$, $CG = \frac{2}{3}CC_1$ and from construction we have that $GD = \frac{2}{3}AA_1$. Triangle GDC has the sides equal to $\frac{2}{3}$ from the length of the medians of the triangle ABC. Because the median of a triangle divides the triangle in two equivalent triangles and the gravity center of the triangle forms with the vertexes of the triangle three equivalent triangle, it results that $Aria_{\triangle}GDC = \frac{1}{3}s$. On the other side the rapport of the arias of two similar triangles is equal with the squared of their similarity rapport, therefore, if we note s_1 the aria of the triangle formed by the medians, we have $\frac{Aria_{\triangle}GDC}{s_1} = \left(\frac{2}{3}\right)^2$.

We find that $\frac{s_1}{s} = \frac{3}{4}$, which proves the proposition.

Proposition 10

The triangle formed by the intersections of the tertianes of a given triangle and the given triangle have the same coefficient of deformation.

Proof

If $A_0B_0C_0$ is the triangle formed by the intersections of the tertianes, from relation (2) we'll find

$$\frac{A_0B_0^2 + B_0C_0^2 + C_0A_0^2}{a^2 + b^2 + c^2} = \frac{1}{7}$$

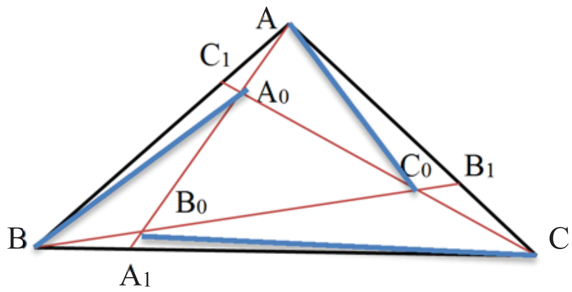

Fig 10

We note s_0 the aria of triangle $A_0B_0C_0$, we'll prove that $\frac{s_0}{s} = \frac{1}{7}$.

From the formulae (6) and (7), it is observed that $A_0 = A_0B_0$ and $CC_0 = C_0A_0$. Using the median's theorem in a triangle to determine that in that triangle two triangle are equivalent, we have that:

$$Aria_{\triangle}AA_0C_0 = Aria_{\triangle}AC_0C = Aria_{\triangle}A_0B_0C_0 =$$
$$= Aria_{\triangle}CB_0C_0 = Aria_{\triangle}CBB_0 = Aria_{\triangle}BB_0A_0 = Aria_{\triangle}ABA_0$$

Because the sum of the aria of these triangles is s, it results that $s_0 = \frac{1}{7}s$, which shows what we had to prove.

Proposition 11

We made the observation that the triangle $A_0B_0C_0$ and the triangle formed by the tertianes AA_1, BB_1, CC_1 as sides are similar. Two similar triangles have the same Brocard angle, therefore the same coefficient of deformation. Taking into account Proposition 10, we obtain the proof of the statement

Observation 11

From the precedent observations it results that being given a triangle, the triangles formed by the tertianes intersections with the triangle as sides, the intersections of the tertianes of the triangle have the same coefficient of deformation.

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From a problem of geometrical construction to the Carnot circles

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In this article we'll give solution to a problem of geometrical construction and we'll show the connection between this problem and the theorem relative to Carnot's circles.

Let ABC a given random triangle. Using only a compass and a measuring line, construct a point M in the interior of this triangle such that the circumscribed circles to the triangles MAB and MAC are congruent.

Construction

We'll start by assuming, as in many situations when we have geometrical constructions, that the construction problem is resolved.

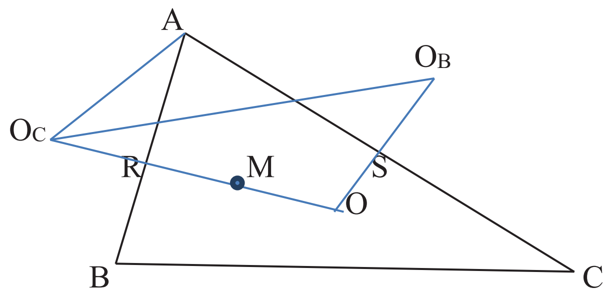

Fig. 1

Let M a point in the interior of the triangle ABC such that the circumscribed circles to the triangles MAB and MAC are congruent.

We'll note O_C and O_B the centers of these triangles, these are the intersections between the mediator of the segments $[AB]$ and $[AC]$. The quadrilateral AO_CMO_B is a rhomb (therefore M is the symmetrical of the point A in rapport to O_BO_C (see Fig. 1).

A. **Step by step construction**

We'll construct the mediators of the segments $[AB]$and$[AC]$, let R,S be their intersection points with $[AB]$respectively$[AC]$. (We suppose that $AB < AC$, therefore $AR < AS$.) With the compass in A and with the radius larger than AS we construct a circle which intersects OR in O_C and $O_{C'}$ respectively OS in O_B and $O_{B'}$ - O being the circumscribed circle to the triangle ABC.

Now we construct the symmetric of the point A in rapport to O_CO_B; this will be the point M, and if we construct the symmetric of the point A in rapport to $O_{C'}O_{B'}$ we obtain the point M'

Lazare Carnot (1753 – 1823), French mathematician, mechanical engineer and political personality (Paris).

B. **Proof of the construction**

Because $AO_C = AO_B$ and M is the symmetric of the point A in rapport of $O_C O_B$, it results that the quadrilateral $AO_C MO_B$ will be a rhombus, therefore $O_C A = O_C M$ and $O_B A = O_B M$. On the other hand, O_C and O_B being perpendicular points of AB respectively AC, we have $O_C A = O_C B$ and $O_B A = O_B C$, consequently

$$O_C A = O_C M = O_B A = O_B M = O_B C,$$

which shows that the circumscribed circles to the triangles MAB and MAC are congruent.

Similarly, it results that the circumscribed circles to the triangles ABM' and ACM' are congruent, more so, all the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ are congruent.

As it can be in the Fig. 2, the point M' is in the exterior of the triangle ABC.

Discussion

We can obtain, using the method of construction shown above, an infinity of pairs of points M and M', such that the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ will be congruent. It seems that the point M' is in the exterior of the triangle ABC

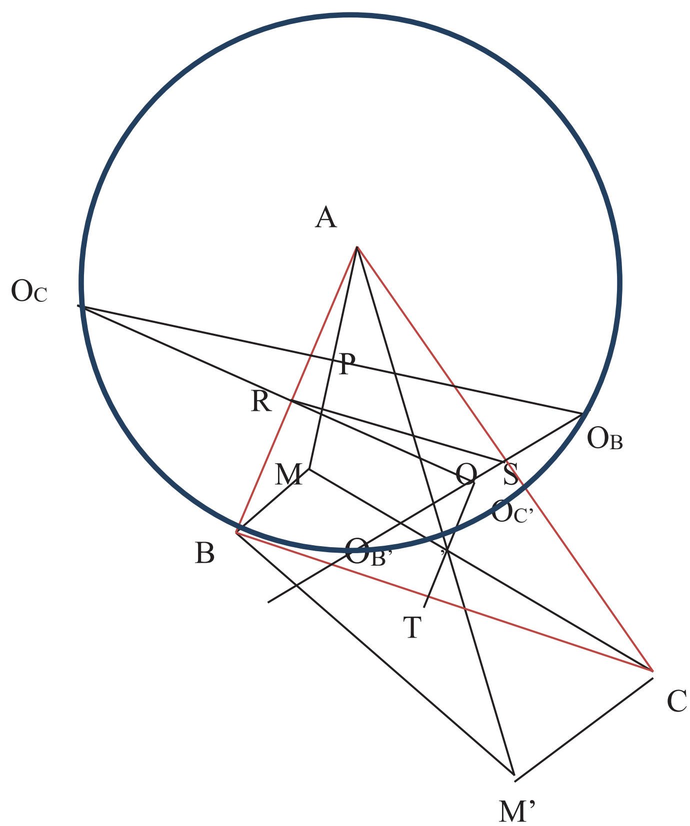

Fig. 2

Observation

The points M from the exterior of the triangle ABC with the property described in the hypothesis are those that belong to the arch $\overset{\frown}{BC}$, which does not contain the vertex A from the circumscribed circle of the triangle ABC.

Now, we'll try to answer to the following:

Questions

1. Can the circumscribed circles to the triangles MAB, MAC with M in the interior of the triangle ABC be congruent with the circumscribed circle of the triangle ABC
2. If yes, then, what can we say about the point M ?

Answers

1. The answer is positive. In this hypothesis we have $OA = AO_B = AO_C$ and it results also that O_C and O_B are the symmetrical of O in rapport to AB respectively AC The point M will be, as we showed, the symmetric of the point A in rapport to $O_C O_B$.

The point M will be also the orthocenter of the triangle ABC. Indeed, we prove that the symmetric of the point A in rapport to $O_C O_B$ is H which is the orthocenter of the triangle ABC Let RS the middle line of the triangle ABC. We observe that RS is also middle line in the triangle $OO_B O_C$, therefore $O_B O_C$ is parallel and congruent with BC, therefore it results that M belongs to the height constructed from A in the triangle ABC. We'll note T the middle of $[BC]$, and let R the radius of the circumscribed circle to the triangle ABC; we have

$$OT = \sqrt{R^2 - \frac{a^2}{4}}, \text{ where } a = BC.$$

If P is the middle of thesegment $[AM]$, we have

$$AP = \sqrt{R^2 - PO_B{}^2} = \sqrt{R^2 - \frac{a^2}{4}}.$$

From the relation $AM = 2 \cdot OT$ it results that M is the orthocenter of the triangle ABC, ($AH = 2OT$).

The answers to the questions 1 and 2 can be grouped in the following form:

Proposition

There is onlyone point in the interior of the triangle ABC such that the circumscribed circles to the triangles MAB, MAC and ABC are congruent. This point is the orthocenter of the triangle ABC.

Remark

From this proposition it practically results that the unique point M from the interior of the right triangle ABC with the property that the circumscribed circles to the triangles MAB, MAC, MBC are congruent with the circumscribed circle to the triangle is the point H, the triangle's orthocenter.

Definition

If in the triangle ABC, H is the orthocenter, then the circumscribed circles to the triangles HAB, HAC, HBC are called Carnot circles.

We can prove, without difficulty the following:

Theorem

The Carnot circles of a triangle are congruent with the circumscribed circle to the triangle.

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THE POLAR OF A POINT With Respect TO A CIRCLE

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In this article we establish a connection between the notion of the symmedian of a triangle and the notion of polar of a point in rapport to a circle

We'll prove for beginning two properties of the symmedians.

Lemma 1
If in triangle ABC inscribed in a circle, the tangents to this circle in the points B and C intersect in a point S, then AS is symmedian in the triangle ABC.

Proof
We'll note L the intersection point of the line AS with BC (see fig. 1).

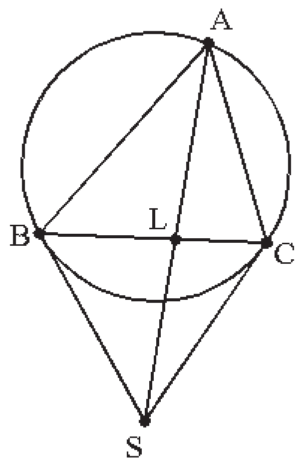

Fig. 1

We have

$$\frac{\text{Aria } \Delta ABL}{\text{Aria } \Delta ACL} = \frac{BL}{LC} = \frac{\text{Aria } \Delta BSL}{\text{Aria } \Delta CSL}$$

It result

$$\frac{\text{Aria } \Delta ABS}{\text{Aria } \Delta ACS} = \frac{BL}{LC} \tag{1}$$

We observe that

$$m(\angle ABS) = m\left(\hat{B}\right) + m\left(\hat{A}\right) \text{ and } m(\angle ACS) = m\left(\hat{C}\right) + m\left(\hat{A}\right)$$

We obtain that

$$\sin(\angle ABS) = \sin C \text{ and } \sin(\angle ACS) = \sin B$$

We have also

$$\frac{\text{Aria } \Delta ABS}{\text{Aria } \Delta ACS} = \frac{AB \cdot SB \cdot \sin C}{AC \cdot SC \cdot \sin B} = \frac{BL}{LC} \tag{2}$$

From the sinus' theorem it results

$$\frac{\sin C}{\sin B} = \frac{AB}{AC} \qquad (3)$$

The relations (2) and lead us to the relation

$$\frac{BL}{LC} = \left(\frac{AB}{AC}\right)^2,$$

which shows that AS is symmedian in the triangle ABC.

Observations

1. The proof is similar if the triangle ABC is obtuse.

2. If ABC is right triangle in A, the tangents in B and C are parallel, and the symmedian from A is the height from A, and, therefore, it is also parallel with the tangents constructed in B and C to the circumscribed circle.

Definition 1

The points A, B, C, D placed, in this order, on a line d form a harmonic division if and only if

$$\frac{AB}{AD} = \frac{CB}{CD}$$

Lemma 2

If in the triangle ABC, AL is the interior symmedian $L \in BC$, and AP is the external median $P \in BC$, then the points P, B, L, C form a harmonic division.

Proof

It is known that the external symmedian AP in the triangle ABC is tangent in A to the circumscribed circle (see fig. 2), also, it can be proved that:

$$\frac{PB}{PC} = \left(\frac{AB}{AC}\right)^2 \qquad (1)$$

but

$$\frac{LB}{LC} = \left(\frac{AB}{AC}\right)^2 \qquad (2)$$

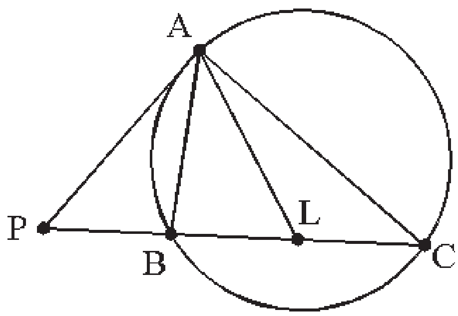

Fig. 2

From the relations (1) and (2) it results

$$\frac{PB}{PC} = \frac{LB}{LC},$$

Which shows that the points P, B, L, C form a harmonic division.

Definition 2

If P is a point exterior to circle $\mathcal{C}(0,r)$ and B, C are the intersection points of the circle with a secant constructed through the point P, we will say about the point Q∈(BC) with the property $\frac{PB}{PC}=\frac{QB}{QC}$ that it is the harmonic conjugate of the point P in rapport to the circle $\mathcal{C}(0,r)$.

Observation

In the same conjunction, the point P is also the conjugate of the point Q in rapport to the circle (see fig. 3).

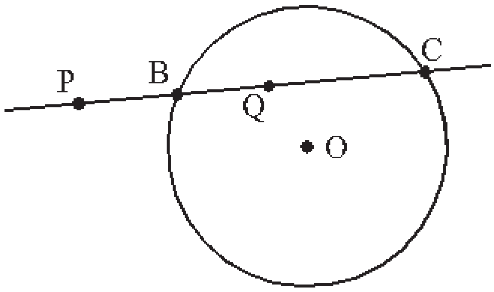

Fig. 3

Definition 3

The set of the harmonic conjugates of a point in rapport with a given circle is called the **polar** of that point in rapport to the circle.

Theorem

The polar of an exterior point to the circle is the circle's cord determined by the points of tangency with the circle of the tangents constructed from that point to the circle.

Proof

Let P an exterior point of the circle $\mathcal{C}(0,r)$ and M, N the intersections of the line PO with the circle (see fig. 4).

We note T and V the tangent points with the circle of the tangents constructed from the point P and let Q be the intersection between MN and TV.

Obviously, the triangle MTN is a right triangle in T, TQ is its height (therefore the interior symmedian, and TP is the exterior symmedian, and therefore the points P, M, Q, N form a harmonic division, (Lemma 2)). Consequently, Q is the harmonic conjugate of P in rapport to the circle and it belongs to the polar of P in rapport to the circle.

We'll prove that (TV) is the polar of P in rapport with the circle. Let M′N′ be the intersections of a random secant constructed through the point P with the circle, and X the intersection of the tangents constructed in M′ and N′ to the circle.

In conformity to Lemma 1, the line XT is for the triangle M′TN′ the interior symmedian, also TP is for the same triangle the exterior symmedian.

If we note Q′ the intersection point between XT and M′N′ it results that the point Q′ is the harmonic conjugate of the point P in rapport with the circle, and consequently, the point Q′ belongs to the polar P in rapport to the circle.

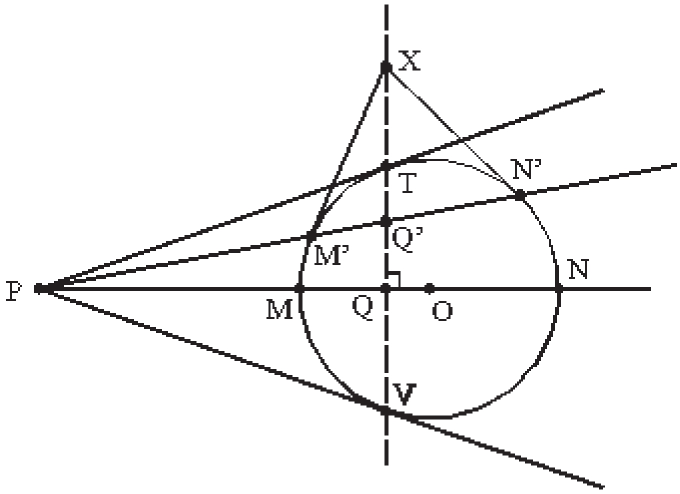

Fig. 4

For the triangle $VM'N'$, according to Lemma 1, the line VX is the interior symmedian and VP is for the same triangle the external symmedian. It will result, according to Lemma 2, that if $\{Q''\} = VX \cap M'N'$, the point Q'' is the harmonic conjugate of the point P in rapport to the circle. Because the harmonic conjugate of a point in rapport with a circle is a unique point, it results that $Q' = Q''$. Therefore the points V, T, X are collinear and the point Q' belongs to the segment (TV).

Reciprocal

If $Q_1 \in (TV)$ and PQ_1 intersect the circle in M_1 and N_1, we much prove that the point Q_1 is the harmonic conjugate of the point P in rapport to the circle.

Let X_1 the intersection point of the tangents constructed from M_1 and N_1 to the circle. In the triangle M_1TN_1 the line X_1T is interior symmedian, and the line TP is exterior symmedian. If $\{Q_1'\} = X_1T \cap M_1N_1$ then P, M_1, Q_1', N_1 form a harmonic division.

Similarly, in the triangle M_1VN_1 the line VX_1 is interior symmedian, and VP exterior symmedian. If we note $\{Q_1''\} = VX_1 \cap M_1N_1$, it results that the point Q_1'' is the harmonic conjugate of the point P in rapport to M_1 and N_1. Therefore, we obtain $Q_1' = Q_1''$. On the other side, X_1, T, Q_1' and V, X_1, Q_1'' are collinear, but $Q_1' = Q_1''$, it result that X_1, T, Q_1', V are collinear, and then $Q_1' = Q_1$, therefore Q_1 is the conjugate of P in rapport with the circle.

Several Metrical Relations Regarding the Anti-Bisector, the Anti-Symmedian, the Anti-Height and their Isogonal

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We suppose known the definitions of the isogonal cevian and isometric cevian; we remind that the anti-bisector, the anti-symmedian, and the anti-height are the isometrics of the bisector, of the symmedian and of the height in a triangle.

It is also known the following Steiner (1828) relation for the isogonal cevians AA_1 and AA_1':

$$\frac{BA_1}{CA_1}\cdot\frac{BA_1'}{CA_1'}=\left(\frac{AB}{AC}\right)^2$$

We'll prove now that there is a similar relation for the isometric cevians

Proposition

In the triangle ABC let consider AA_1 and AA_1' two isometric cevians, then there exists the following relation:

$$\frac{\sin\left(\widehat{BAA_1}\right)}{\sin\left(\widehat{CAA_1}\right)}\cdot\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)}=\left(\frac{\sin B}{\sin C}\right)^2 \tag{*}$$

Proof

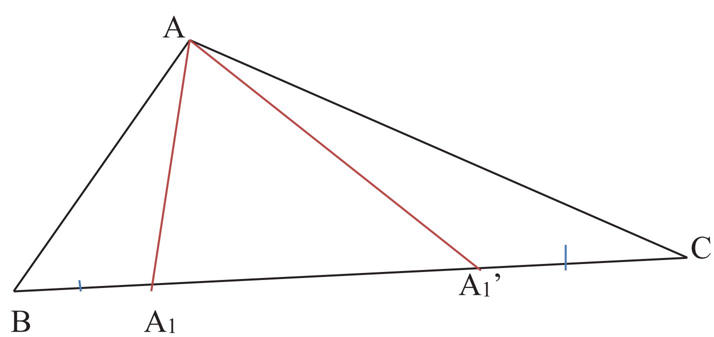

Fig. 1

The sinus theorem applied in the triangles ABA_1, ACA_1 implies (see above figure)

$$\frac{\sin\left(\widehat{BAA_1}\right)}{BA_1}=\frac{\sin B}{AA_1} \tag{1}$$

$$\frac{\sin\left(\widehat{CAA_1}\right)}{CA_1}=\frac{\sin C}{AA_1} \tag{2}$$

From the relations (1) and (2) we retain

$$\frac{sin\left(\widehat{BAA_1}\right)}{sin\left(\widehat{CAA_1}\right)}=\frac{sin\,B}{sin\,C}\cdot\frac{BA_1}{CA_1} \tag{3}$$

The sinus theorem applied in the triangles ACA_1', ABA_1' leads to

$$\frac{sin\left(\widehat{CAA_1'}\right)}{A_1'C}=\frac{sin\,C}{AA_1'} \tag{4}$$

$$\frac{sin\left(\widehat{BAA_1'}\right)}{BA_1'}=\frac{sin\,B}{AA_1'} \tag{5}$$

From the relations (4) and (5) we obtain:

$$\frac{sin\left(\widehat{BAA_1'}\right)}{sin\left(\widehat{CAA_1'}\right)}=\frac{sin\,B}{sin\,C}\cdot\frac{BA_1'}{CA_1'} \tag{6}$$

Because $BA_1=CA_1'$ and $A_1C=BA_1'$) the cevians being isometric), from the relations (3) and (6) we obtain relation (*) from the proposition's enouncement.

Applications

1. If AA_1 is the bisector in the triangle ABC and AA_1' is its isometric, that is an anti-bisector, then from (*) we obtain

$$\frac{sin\left(\widehat{BAA_1'}\right)}{sin\left(\widehat{CAA_1'}\right)}=\left(\frac{sin\,B}{sin\,C}\right)^2 \tag{7}$$

Taking into account of the sinus theorem in the triangle ABC we obtain

$$\frac{sin\left(\widehat{BAA_1'}\right)}{sin\left(\widehat{CAA_1'}\right)}=\left(\frac{AC}{AB}\right)^2 \tag{8}$$

2. If AA_1 is symmedian and AA_1' is an anti-symmedian, from (*) we obtain

$$\frac{sin\left(\widehat{BAA_1'}\right)}{sin\left(\widehat{CAA_1'}\right)}=\left(\frac{AC}{AB}\right)^3$$

Indeed, AA_1 being symmedian it is the isogonal of the median AM and

$$\frac{sin\left(\widehat{MAB}\right)}{sin\left(\widehat{MAC}\right)}=\frac{sin\,B}{sin\,C} \quad \text{and}$$

$$\frac{sin\left(\widehat{BAA_1'}\right)}{sin\left(\widehat{CAA_1'}\right)}=\frac{sin\left(\widehat{MAC}\right)}{sin\left(\widehat{MAB}\right)}=\frac{sin\,C}{sin\,B}=\frac{AB}{AC}$$

3. If AA_1 is a height in the triangle ABC, $A_1 \in (BC)$ and AA_1' is its isometric (anti-height), the relation (*) becomes.

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

Indeed

$$\sin\left(\widehat{BAA_1'}\right) = \frac{BA_1}{AB}; \ \sin\left(\widehat{CAA_1'}\right) = \frac{CA_1}{AC}$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{BA_1}{CA_1}$$

From (*) it results

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{CA_1}{BA_1}$$

or

$$CA_1 = AC \cdot \cos C \text{ and } BA_1 = AB \cdot \cos B$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

4. If AA_1'' is the isogonal of the anti-bisector AA_1' then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \quad \text{(Maurice D'Ocagne, 1883)}$$

Proof

The Steiner's relation for AA_1'' and AA_1' is

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{A_1'C} = \left(\frac{AB}{AC}\right)^2$$

But AA_1 is the bisector and according to the bisector theorem $\frac{BA_1}{CA_1} = \frac{AB}{AC}$ but $BA_1' = CA_1$ and $A_1'C = BA_1$ therefore

$$\frac{CA_1'}{BA_1'} = \frac{AB}{AC}$$

and we obtain the D'Ocagne relation

5. If in the triangle ABC the cevian AA_1'' is isogonal to the symmedian AA_1' then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

Proof

Because AA_1 is a symmedian, from the Steiner's relation we deduct that

$$\frac{BA_1}{CA_1} = \left(\frac{AB}{AC}\right)^2$$

The Steiner's relation for AA_1'', AA_1' gives us

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{CA_1'} = \left(\frac{AB}{AC}\right)^2$$

Taking into account the precedent relation, we obtain

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

6.

If AA_1'' is the isogonal of the anti-height AA_1' in the triangle ABC in which the height AA_1 has $A_1 \in (BC)$ then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \cdot \frac{\cos B}{\cos C}$$

Proof

If AA_1 is height in triangle ABC $A_1 \in (BC)$ then

$$\frac{BA_1}{A_1C} = \frac{AB}{AC} \cdot \frac{\cos B}{\cos C}$$

Because AA_1' is anti-median, we have $BA_1 = CA_1'$ and $A_1C = BA_1'$ then

$$\frac{BA_1''}{A_1''C} = \frac{AC}{AB} \cdot \frac{\cos C}{\cos B}$$

Observation

The precedent results can be generalized for the anti-cevians of rang k and for their isogonal.

An Important Application of the Computation of the Distances between Remarkable Points in the Triangle Geometry

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In this article we'll prove through computation the Feuerbach's theorem relative to the tangent to the nine points circle, the inscribed circle, and the ex-inscribed circles of a given triangle.

Let ABC a given random triangle in which we denote with O the center of the circumscribed circle, with I the center of the inscribed circle, with H the orthocenter, with I_a the center of the A ex-inscribed circle, with O_9 the center of the nine points circle, with $p = \frac{a+b+c}{2}$ the semi-perimeter, with R the radius of the circumscribed circle, with r the radius of the inscribed circle, and with r_a the radius of the A ex-inscribed circle.

Proposition

In a triangle ABC are true the following relations:

(i) $OI^2 = R^2 - 2Rr$ Euler's relation

(ii) $OI_a^2 = R^2 + 2Rr_a$ Feuerbach's relation

(iii) $OH^2 = 2r^2 - 2p^2 + 9R^2 + 8Rr$

(iv) $IH^2 = 3r^2 - p^2 + 4R^2 + 4Rr$

(v) $I_aH^2 = r^2 - p^2 + 2r_a^{\,2} + 4R^2 + 4Rr$

Proof

(i) The positional vector of the center I of the inscribed circle of the given triangle ABC is

$$\overrightarrow{PI} = \frac{1}{2p}\left(a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right)$$

For any point P in the plane of the triangle ABC.
We have

$$\overrightarrow{OI} = \frac{1}{2p}\left(a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}\right)$$

We compute $\overrightarrow{OI} \times \overrightarrow{OI}$, and we obtain:

$$OI^2 = \frac{1}{4p^2}\left(a^2OA^2 + b^2OB^2 + c^2OC^2 + 2ab\overrightarrow{OA}\times\overrightarrow{OB} + 2bc\overrightarrow{OB}\times\overrightarrow{OC} + 2ca\overrightarrow{OC}\times\overrightarrow{OA}\right)$$

From the cosin's theorem applied in the triangle OBC we get

$$\overrightarrow{OB} \times \overrightarrow{OC} = R^2 - \frac{a^2}{2}$$

and the similar relations, which substituted in the relation for OI^2 we find

$$OI^2 = \frac{1}{4p^2}\left(R^2 \cdot 4p^2 - abc \cdot 2p\right)$$

Because $abc = 4Rs$ and $s = pr$ it results (i)

(ii) The position vector of the center I_a of the A ex-inscribed circle is give by:

$$\overrightarrow{PI_a} = \frac{1}{2(p-a)}\left(-a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right)$$

We have:

$$\overrightarrow{OI_a} = \frac{1}{2(p-a)}\left(-a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}\right)$$

Computing $\overrightarrow{OI_a} \cdot \overrightarrow{OI_a}$ we obtain

$$\overrightarrow{OI_a}^2 = R^2 \cdot \frac{a^2+b^2+c^2}{2(p-a)^2} - \frac{ab}{2(p-a)^2}\overrightarrow{OA} \times \overrightarrow{OB} + \frac{bc}{2(p-a)^2}\overrightarrow{OB} \times \overrightarrow{OC} - \frac{ac}{2(p-a)^2}\overrightarrow{OA} \times \overrightarrow{OC}$$

Because $\overrightarrow{OB} \times \overrightarrow{OC} = R^2 - \frac{a^2}{2}$ and $s = r_a(p-a)$, executing a simple computation we obtain the Feuerbach's relation.

(iii) In a triangle it is true the following relation

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

This is the Sylvester's relation.

We evaluate $\overrightarrow{OH} \times \overrightarrow{OH}$ and we obtain:

$$OH^2 = 9R^2 - (a^2+b^2+c^2).$$

We'll prove that in a triangle we have:

$$ab + bc + ca = p^2 + r^2 + 4Rr$$

and

$$a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8Rr$$

We obtain

$$\frac{s^2}{p} = (p-a)(p-b)(p-c) = -p^3 + p(ab+bc+ca) - abc$$

Therefore

$$\frac{s^2}{p^2} = -p^2 + ab + bc + ca - \frac{4Rs}{p}$$

We find that

$$ab + bc + ca = p^2 + r^2 + 4Rr$$

Because

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$$

it results that

$$a^2+b^2+c^2=2p^2-2r^2-8Rr$$

which leads to (iii).

(iv) In the triangle ABC we have

$$\overrightarrow{IH}=\overrightarrow{OH}-\overrightarrow{OI}$$

We compute IH^2, and we obtain:

$$IH^2=OH^2+OI^2-2\overrightarrow{OH}\cdot\overrightarrow{OI}$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)\cdot\frac{1}{2p}\left(a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}\right)$$

$$\overrightarrow{OH}\times\overrightarrow{OI}=\frac{1}{2p}\left[R^2(a+b+c)+(a+b)\times\overrightarrow{OA}\times\overrightarrow{OB}+(b+c)\times\overrightarrow{OB}\times\overrightarrow{OC}+(c+a)\times\overrightarrow{OC}\times\overrightarrow{OA}\right]=$$

$$=3R^2-\frac{a^3+b^3+c^3}{2(a+b+c)}-\frac{a^2+b^2+c^2}{2}.$$

$$IH^2=4R^2-2Rr-\frac{a^3+b^3+c^3}{a+b+c}$$

To express $a^3+b^3+c^3$ in function of p,r,R we'll use the identity:

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right).$$

and we obtain

$$a^3+b^3+c^3=2p\left(p^2-3r^2-6Rr\right)$$

Substituting in the expression of IH^2, we'll obtain the relation (iv)

(v) We have

$$\overrightarrow{HI_a}=\frac{1}{2(p-a)}\left(-a\overrightarrow{HA}+b\overrightarrow{HB}+c\overrightarrow{HC}\right)$$

We'll compute $\vec{HI}_a\times\vec{HI}_a$

$$HI_a^{\ 2}=\frac{1}{4(p-a)^2}\left(a^2HA^2+b^2HB^2+c^2HC^2-2ab\overrightarrow{HA}\times\overrightarrow{HB}-2ac\overrightarrow{HA}\times\overrightarrow{HC}+2bc\overrightarrow{HB}\times\overrightarrow{HC}\right)$$

If A_1 is the middle point of (BC) it is known that $\overrightarrow{AH}=2\overrightarrow{OA_1}$, therefore

$$AH^2=4R^2-a^2$$

also,

$$\overrightarrow{HA}\times\overrightarrow{HB}=\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\left(\overrightarrow{OC}+\overrightarrow{OA}\right)$$

We obtain:

$$\overrightarrow{HA}\times\overrightarrow{HB}=4R^2-\frac{1}{2}\left(a^2+b^2+c^2\right)$$

Therefore

$$a^2+b^2+c^2=2\left(p^2-r^2-4Rr\right)$$

It results

$$\overrightarrow{HA}\times\overrightarrow{HB}=r^2-p^2+4R^2+4Rr$$

Similarly,

$$\overrightarrow{HB}\times\overrightarrow{HC}=\overrightarrow{HC}\times\overrightarrow{HA}=r^2-p^2+4R^2+4Rr$$

$$HI_a{}^2=\frac{1}{4(p-a)^2}\left[4R^2\left(a^2+b^2+c^2\right)-\left(a^4+b^4+c^4\right)+\left(r^2-p^2+4R^2+4Rr\right)(2bc-2ab-2ac)\right]$$

Because $b+c-a=2(p-a)$, it results

$$2bc-2ab-2ac=4(p-a)^2-\left(a^2+b^2+c^2\right)$$

$$HI_a{}^2=\frac{1}{4(p-a)^2}\left[\left(a^2+b^2+c^2\right)\left(p^2-r^2-4Rr\right)+4(p-a)^2\left(r^2-p^2+4R^2+4Rr\right)-\left(a^4+b^4+c^4\right)\right]$$

It is known that

$$16s^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$$

From which we find

$$a^2b^2+b^2c^2+c^2a^2=(ab+bc+ca)^2-2abc(a+b+c)=\left(r^2+p^2+4Rr\right)^2-4pabc$$

Substituting, and after several computations we obtain (v).

Theorem (K. Feuerbach)

In a given triangle the circle of the nine points is tangent to the inscribed circle and to the ex-inscribed circles of the triangle.

Proof

We apply the median's theorem in the triangle OIH and we obtain

$$4IO_9^2=2\left(OI^2+IH^2\right)-OH^2$$

We substitute OI^2, IH^2, OH^2 with the obtained formulae in function of r, R, p and after several simple computations we'll obtain

$$IO_9=\frac{R}{2}-r$$

This relation shows that the circle of the nine points (which has the radius $\frac{R}{2}$) is tangent to inscribed circle.

We apply the median's theorem for the triangle OI_aH, and we obtain

$$4I_aO_9^2=2\left(OI_a{}^2+I_aH^2\right)-OH^2$$

We substitute OI_a, I_aH, OH and we'll obtain

$$I_a O_9 = \frac{R}{2} + r_a$$

This relation shows that the circle of the nine points and the A- ex-inscribed circle are tangent in exterior.

Note

In an article published in the Gazeta Matematică, no. 4, from 1982, the late Romanian Professor Laurențiu Panaitopol asked for the finding of the strongest inequality of the type $kR^2 + hr^2 \geq a^2 + b^2 + c^2$ and proves that this inequality is

$$8R^2 + 4r^2 \geq a^2 + b^2 + c^2.$$

Taking into consideration that

$$IH^2 = 4R^2 + 2r^2 - \frac{a^2 + b^2 + c^2}{2}$$

and that $IH^2 \geq 0$ we re-find this inequality and its geometrical interpretation.

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The Duality and the Euler's Line

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In this article we'll discuss about a theorem which results from a duality transformation of a theorem and the configuration in relation to the Euler's line.

Theorem

Let ABC a given random triangle, I the center of its inscribed circle, and $A'B'C'$ its triangle of contact. The perpendiculars constructed in I on AI, BI, CI intersect BC, CA, AB respectively in the points A_1, B_1, C_1. The medians of the triangle of contact intersect the second time the inscribed circle in the points A_1', B_1', C_1', and the tangents in these points to the inscribed circle intersect the lines BC, CA, AB in the points A_2, B_2, C_2 respectively.

Then:

i) The points A_1, B_1, C_1 are collinear;

ii) The points A_2, B_2, C_2 are collinear;

iii) The lines A_1B_1, A_2B_2 are parallel.

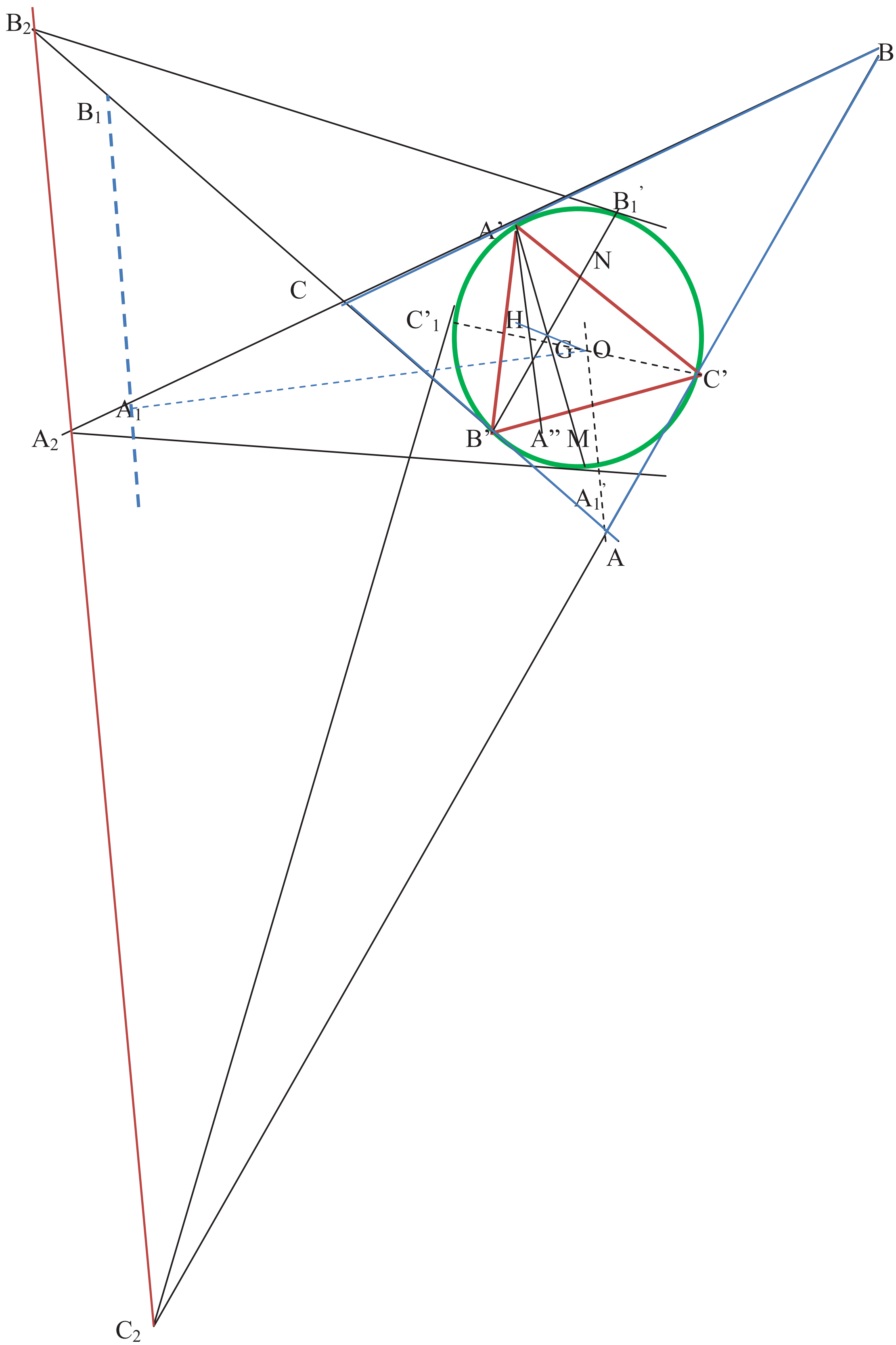
Proof

We'll consider a triangle $A'B'C'$ circumscribed to the circle of center O. Let $A'A'', B'B'', C'C''$ its heights concurrent in a point H and $A'M, B'N, C'P$ its medians concurrent in the weight center G. It is known that the points O, H, G are collinear; these are situated on the Euler's line of the triangle $A'B'C'$.

We'll transform this configuration (see the figure) through a duality in rapport to the circumscribed circle to the triangle $A'B'C'$.

To the points A', B', C' correspond the tangents in A', B', C' to the given circle, we'll note A, B, C the points of intersection of these tangents. For triangle ABC the circle $A'B'C'$ becomes inscribed circle, and $A'B'C'$ is the triangle of contact of ABC.

To the mediators $A'M, B'N, C'P$ will correspond through the considered duality, their pols, that is the points A_2, B_2, C_2 obtained as the intersections of the lines BC, CA, AB with the tangents in the points A_1', B_1', C_1' respectively to the circle $A'B'C'$ (A_1', B_1', C_1' are the intersection points with the circle $A'B'C'$ of the lines $(A'M, (B'N, (C'P$). To the height $A'M$ corresponds its pole noted A_1 situated on BC such that $m\left(\widehat{AOA_1}\right) = 90°$ (indeed the pole of $B'C'$ is the point A and because $A'M \perp B'C'$ we have $m\left(\widehat{AOA_1}\right) = 90°$), similarly to the height $B'N$ we'll correspond the point B_1 on AC such that $m\left(\widehat{BOB_1}\right) = 90°$, and to the height $C'N$ will correspond the point C_1 on AB such that $m\left(\widehat{COC_1}\right) = 90°$.



B_2
B
B_1
B_1'
A'
N
C
C'_1
H
G
O
C'
A_1
A_2
B'
A''
M
A_1'
A
C_2

Because the heights are concurrent in H it means that their poles, that is the points A_1, B_1, C_1 are collinear.

Because the medians are concurrent in the point G it means that their poles, that is the points A_2, B_2, C_2 are collinear.

The lines $A_1B_1C_1$ and $A_2B_2C_2$ are respectively the poles of the points H and G, because H, G are collinear with the point O; this means that these poles are perpendicular lines on OG respectively on OH; consequently these are parallel lines.

By re-denoting the point O with I we will be in the conditions of the propose theorem and therefore the proof is completed.

Note

This theorem can be proven also using an elementary method. We'll leave this task for the readers.

Two Applications of Desargues' Theorem

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In this article we will use the Desargues' theorem and its reciprocal to solve two problems.

For beginning we will enunciate and prove Desargues' theorem:

Theorem 1 (G.Desargues, 1636, the famous "perspective theorem": When two triangles are in perspective, the points where the corresponding sides meet are collinear.)

Let two triangle ABC and $A_1B_1C_1$ be in a plane such that $AA_1 \cap BB_1 \cap CC_1 = \{O\}$,

$$AB \cap A_1B_1 = \{N\}$$
$$BC \cap B_1C_1 = \{M\}$$
$$CA \cap C_1A_1 = \{P\}$$

then the points N, M, P are collinear.

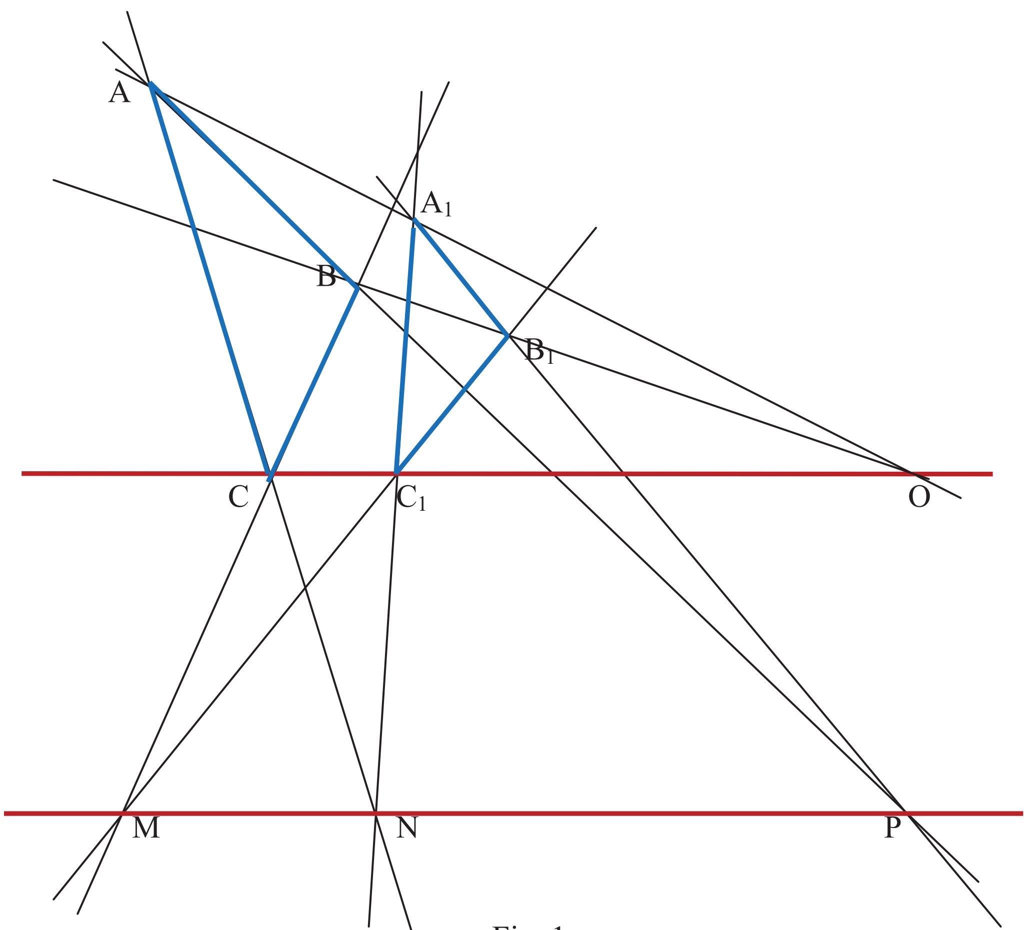

Fig. 1

Proof

Let $\{O\} = AA_1 \cap BB_1 \cap CC_1$, see Fig.1.. We'll apply the Menelaus' theorem in the triangles OAC; OBC; OAB for the transversals N, A_1, C_1; M, B_1, C_1; P, B_1, A_1, and we obtain

$$\frac{NA}{NC} \cdot \frac{C_1C}{C_1O} \cdot \frac{A_1O}{A_1A} = 1 \qquad (1)$$

$$\frac{MC}{MB} \cdot \frac{B_1B}{B_1O} \cdot \frac{C_1O}{C_1C} = 1 \qquad (2)$$

$$\frac{PB}{PA} \cdot \frac{B_1O}{B_1B} \cdot \frac{A_1A}{A_1O} = 1 \qquad (3)$$

By multiplying the relations (1), (2), and (3) side by side we obtain

$$\frac{NA}{NC} \cdot \frac{MC}{MB} \cdot \frac{PB}{PA} = 1.$$

This relation, shows that N, M, P are collinear (in accordance to the Menealaus' theorem in the triangle ABC).

Remark 1

The triangles ABC and $A_1B_1C_1$ with the property that AA_1, BB_1, CC_1 are concurrent are called homological triangles. The point of concurrency point is called the homological point of the triangles. The line constructed through the intersection points of the homological sides in the homological triangles is called the triangles' axes of homology.

Theorem 2 (The reciprocal of the Desargues' theorem)

` If two triangles ABC and $A_1B_1C_1$ are such that

$$AB \cap A_1B_1 = \{N\}$$

$$BC \cap B_1C_1 = \{M\}$$

$$CA \cap C_1A_1 = \{P\}$$

And the points N, M, P are collinear, then the triangles ABC and $A_1B_1C_1$ are homological.

Proof

We'll use the reduction ad absurdum method .

Let

$$AA_1 \cap BB_1 = \{O\}$$

$$AA_1 \cap CC_1 = \{O_1\}$$

$$BB_1 \cap CC_1 = \{O_2\}$$

We suppose that $O \neq O_1 \neq O_2 \neq O_3$.

The Menelaus' theorem applied in the triangles OAB, O_1AC, O_2BC for the transversals N, A_1, B_1; P, A_1, C_1; M, B_1, C_1, gives us the relations

$$\frac{NB}{NA} \cdot \frac{B_1O}{B_1B} \cdot \frac{AA_1}{A_1O} = 1 \qquad (4)$$

$$\frac{PA}{PC}\cdot\frac{A_1O_1}{A_1O}\cdot\frac{C_1C}{C_1O_1}=1 \qquad (5)$$

$$\frac{MC}{MB}\cdot\frac{B_1B}{B_1O}\cdot\frac{C_1O_2}{C_1C}=1 \qquad (6)$$

Multiplying the relations (4), (5), and (6) side by side, and taking into account that the points N, M, P are collinear, therefore

$$\frac{PA}{PC}\cdot\frac{MC}{MB}\cdot\frac{NB}{NA}=1 \qquad (7)$$

We obtain that

$$\frac{A_1O_1}{A_1O}\cdot\frac{B_1O}{B_1O_2}\cdot\frac{C_1O_2}{C_1O_2}=1 \qquad (8)$$

The relation (8) relative to the triangle $A_1B_1C_1$ shows, in conformity with Menelaus' theorem, that the points O,O_1,O_2 are collinear. On the other hand the points O,O_1 belong to the line AA_1, it results that O_2 belongs to the line AA_1. Because $BB_1\cap CC_1=\{O_2\}$, it results that $\{O_2\}=AA_1\cap BB_1\cap CC_1$, and therefore $O_2=O_1=O$, which contradicts the initial supposition.

Remark 2

The Desargues' theorem is also known as the theorem of the homological triangles.

Problem 1

If $ABCD$ is a parallelogram, $A_1\in(AB),B_1\in(BC),C_1\in(CD),D_1\in(DA)$ such that the lines A_1D_1,BD,B_1C_1 are concurrent, then:

a) The lines AC,A_1C_1 and B_1D_1 are concurrent

b) The lines A_1B_1,C_1D_1 and AC are concurrent.

Solution

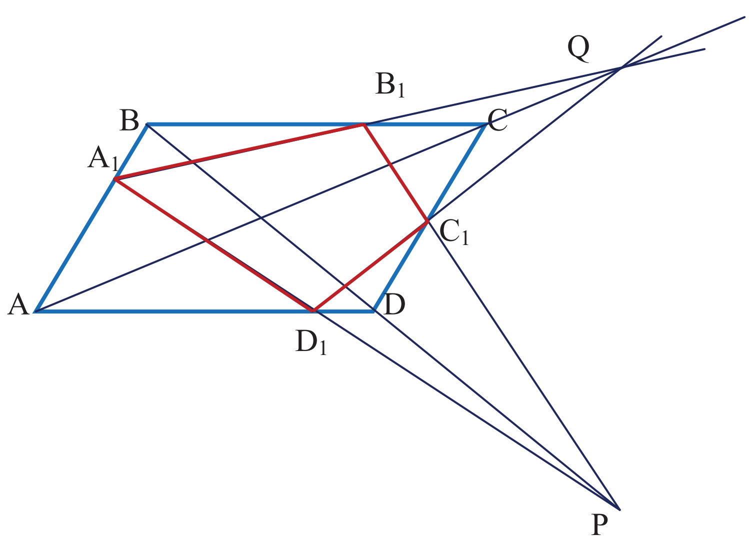

Fig. 2

Let $\{P\} = A_1D_1 \cap B_1C_1 \cap BD$ see Fig. 2. We observe that the sides A_1D_1 and B_1C_1; CC_1 and AD_1; A_1A and CB_1 of triangles AA_1D_1 and CB_1C_1 intersect in the collinear points P, B, D. Applying the reciprocal theorem of Desargues it results that these triangles are homological, that is, the lines: AC, A_1C_1 and B_1D_1 are collinear.

Because $\{P\} = A_1D_1 \cap B_1C_1 \cap BD$ it results that the triangles DC_1D_1 and BB_1A_1 are homological. From the theorem of the of homological triangles we obtain that the homological lines
DC_1 and BB_1**;** DD_1 and BA_1**;** D_1C_1 and A_1B_1 intersect in three collinear points, these are C, A, Q, where $\{Q\} = D_1C_1 \cap A_1B_1$. Because Q is situated on AC it results that A_1B_1, C_1D_1 and AC are collinear.

Problem 2

Let $ABCD$ a convex quadrilateral such that

$$AB \cap CB = \{E\}$$

$$BC \cap AD = \{F\}$$

$$BD \cap EF = \{P\}$$

$$AC \cap EF = \{R\}$$

$$AC \cap BD = \{O\}$$

We note with $G, H, I, J, K, L, M, N, Q, U, V, T$ respectively the middle points of the segments: $(AB),(BF),(AF),(AD),(AE),(DE),(CE),(BE),(BC),(CF),(DF),(DC)$. Prove that

i) The triangle POR is homological with each of the triangles: $GHI,\ JKL,\ MNQ,\ UVT$.
ii) The triangles GHI and JKL are homological.
iii) The triangles MNQ and UVT are homological.
iv) The homology centers of the triangles $GHI,\ JKL,\ POR$ are collinear.
v) The homology centers of the triangles $MNQ,\ UVT,\ POR$ are collinear.

Solution

i) when proving this problem we must observe that the $ABCDEF$ is a complete quadrilateral and if O_1, O_2, O_3 are the middle of the diagonals $(AC),(BD)$ respective EF, these point are collinear. The line on which the points O_1, O_2, O_3 are located is called the Newton-Gauss line [* for complete quadrilateral see [1]].

The considering the triangles POR and GHI we observe that $GI \cap OR = \{O_1\}$ because GI is the middle line in the triangle ABF and then it contains the also the middle of the segment (AC), which is O_1. Then $HI \cap PR = \{O_3\}$ because HI is middle line in the triangle AFB and O_3 is evidently on the line PR also. $GH \cap PO = \{O_2\}$ because GH is middle line in the triangle BAF and then it contains also O_2 the middle of the segment (BD).

The triangles GIH and ORP have as intersections of the homological lines the collinear points O_1, O_2, O_3, according to the reciprocal theorem of Desargues these are homological.

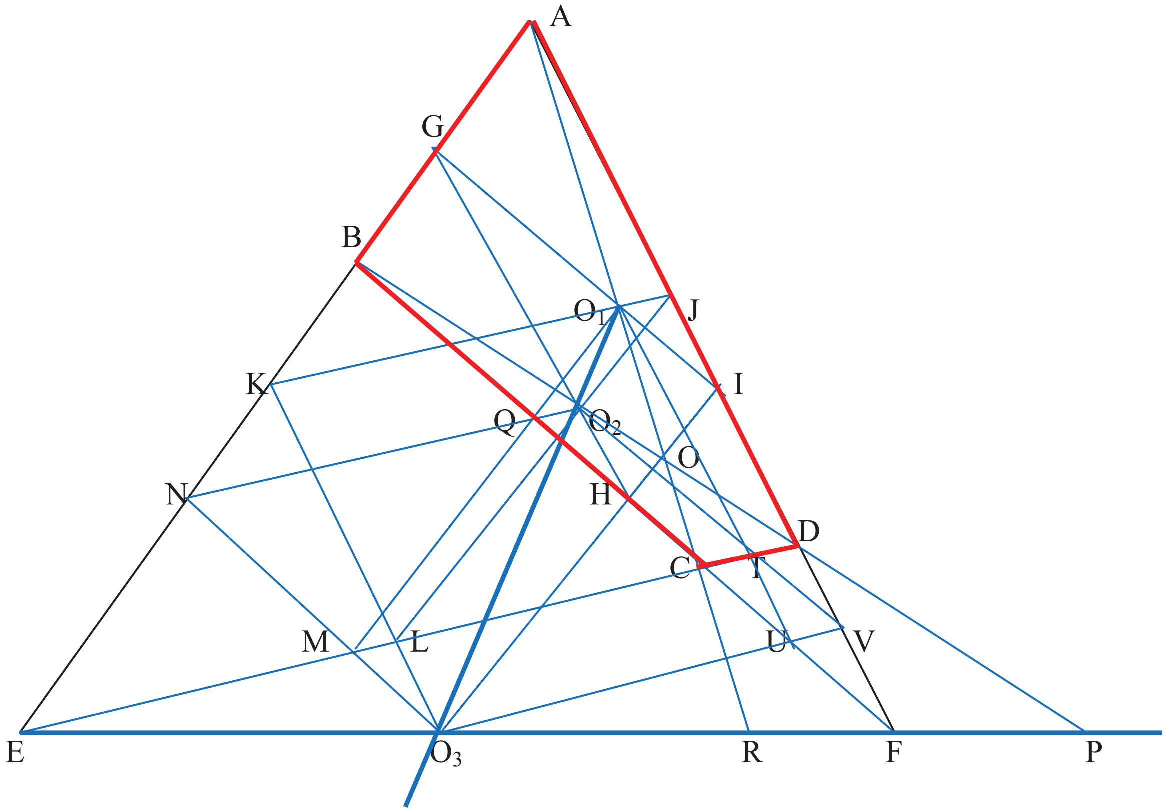

Fig. 3

Similarly, we can show that the triangle ORP is homological with the triangles JKL, MNQ, and UVT (the homology axes will be O_1, O_2, O_3).

ii) We observe that

$$GI \cap JK = \{O_1\}$$

$$GH \cap JL = \{O_2\}$$

$$HI \cap KL = \{O_3\}$$

then O_1, O_2, O_3 are collinear and we obtain that the triangles GIH and JKL are homological

iii) Analog with ii)

iv) Apply the Desargues' theorem. If three triangles are homological two by two, and have the same homological axes then their homological centers are collinear.

v) Similarly with iv).

Remark 3

The precedent problem could be formulates as follows:

The four medial triangles of the four triangles determined by the three sides of a given complete quadrilateral are, each of them, homological with the diagonal triangle of the complete quadrilateral and have as a common homological axes the Newton-Gauss line of the complete quadrilateral.

We mention that:

- The *medial triangle* of a given triangle is the triangle determined by the middle points of the sides of the given triangle (it is also known as the complementary triangle).
- The *diagonal triangle* of a complete quadrilateral is the triangle determined by the diagonals of the complete quadrilateral.

We could add the following comment:

Considering the four medial triangles of the four triangles determined by the three sides of a complete quadrilateral, and the diagonal triangle of the complete quadrilateral, we could select only two triplets of triangles homological two by two. Each triplet contains the diagonal triangle of the quadrilateral, and the triplets have the same homological axes, namely the Newton-Gauss line of the complete quadrilateral.

Open problems

1. What is the relation between the lines that contain the homology centers of the homological triangles' triplets defined above?
2. Desargues theorem was generalized in [2] in the following way: Let's consider the points $A_1,\ldots,A_n$ situated on the same plane, and $B_1,\ldots,B_n$ situated on another plane, such that the lines A_iB_i are concurrent. Then if the lines A_iA_j and B_iB_j are concurrent, then their intersecting points are collinear.
   Is it possible to generalize Desargues Theorem for two polygons both in the same plane?
3. What about Desargues Theorem for polyhedrons?

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An Application of Sondat's Theorem

Regarding the Ortho-homological Triangles

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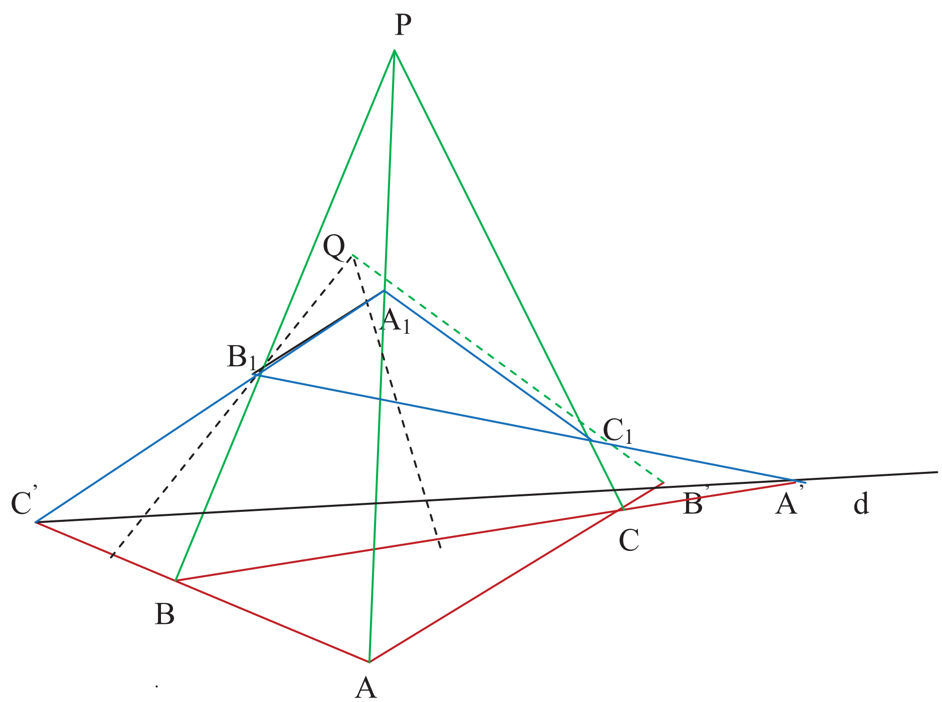
In this article we prove the Sodat's theorem regarding the ortho-homogolgical triangle and then we use this theorem along with Smarandache-Pătraşcu theorem to obtain another theorem regarding the ortho-homological triangles.

Theorem (P. Sondat)

Consider the ortho-homological triangles ABC, $A_1B_1C_1$. We note Q, Q_1 their orthological centers, P the homology center and d their homological axes. The points P, Q, Q_1 belong to a line that is perpendicular on d



Proof.

Let Q the orthologic center of the ABC the $A_1B_1C_1$ (the intersection of the perpendiculars constructed from $A_1,\ B_1,\ C_1$ respectively on $BC,\ CA,\ AB$), and Q_1 the other orthologic center of the given triangle.

We note $\{B'\} = CA \cap C_1A_1$, $\{A'\} = BC \cap B_1C_1$, $\{C'\} = AB \cap A_1B_1$.

We will prove that $PQ \perp d$ which is equivalent to

$$B'P^2 - B'Q^2 = C'P^2 - C'Q^2 \tag{1}$$

We have that

$$\overrightarrow{PA_1} = \alpha \overrightarrow{A_1A},\ \overrightarrow{PB_1} = \beta \overrightarrow{B_1B},\ \overrightarrow{PC_1} = \gamma \overrightarrow{C_1C}$$

From Menelaus' theorem applied in the triangle PAC relative to the transversals $B',\ C_1,\ A_1$ we obtain that

$$\frac{B'C}{B'A} = \frac{\alpha}{\gamma} \tag{2}$$

The Stewart's theorem applied in the triangle PAB' implies that

$$PA^2 \cdot CB' + PB'^2 \cdot AC - PC^2 \cdot AB' = AC \cdot CB' \cdot AB' \tag{3}$$

Taking into account (2), we obtain:

$$\gamma PC^2 - \alpha PA^2 = (\gamma - \alpha) PB'^2 - \alpha B'A^2 + \gamma B'C^2 \tag{4}$$

Similarly, we obtain:

$$\gamma QC^2 - \alpha QA^2 = (\gamma - \alpha) QB'^2 + \gamma B'C^2 - \alpha B'A^2 \tag{5}$$

Subtracting the relations (4) and (5) and using the notations:

$$PA^2 - QA^2 = u,\ PB^2 - QB^2 = v,\ PC^2 - QC^2 = t$$

we obtain:

$$PB'^2 - QB'^2 = \frac{\gamma t - \alpha u}{\gamma - \alpha} \tag{6}$$

The Menelaus' theorem applied in the triangle PAB for the transversal $C',\ B,\ A_1$ gives

$$\frac{C'B}{C'A} = \frac{\alpha}{\beta} \tag{7}$$

From the Stewart's theorem applied in the triangle $PC'A$ and the relation (7) we obtain:

$$\alpha PA^2 - \beta PB^2 = (\alpha - \beta) C'P^2 + \alpha C'A^2 - \beta C'B^2 \tag{8}$$

Similarly, we obtain:

$$\alpha QA^2 - \beta QB^2 = (\alpha - \beta) C'Q^2 + \alpha C'A^2 - \beta C'B^2 \tag{9}$$

From (8) and (9) it results

$$C'P^2 - C'Q^2 = \frac{\alpha u - \beta v}{\alpha - \beta} \tag{10}$$

The relation (1) is equivalent to:

$$\alpha\beta(u-v)+\beta\gamma(v-t)+\gamma\alpha(t-u)=0 \tag{11}$$

To prove relation (11) we will apply first the Stewart theorem in the triangle CAP, and we obtain:

$$CA^2 \cdot PA_1 + PC^2 \cdot A_1A - CA_1^2 \cdot PA = PA_1 \cdot A_1A \cdot PA \tag{12}$$

Taking into account the previous notations, we obtain:

$$\alpha CA^2 + PC^2 - CA_1^2(1+\alpha) = PA_1^2 + \alpha A_1A^2 \tag{13}$$

Similarly, we find:

$$\alpha BA^2 + PB^2 - BA_1^2(1+\alpha) = PA_1^2 + \alpha A_1A^2 \tag{14}$$

From the relations (13) and (14) we obtain:

$$\alpha BA^2 - \alpha CA^2 + PB^2 - PC^2 - (1+\alpha)\left(BA_1^2 - CA_1^2\right) = 0 \tag{15}$$

Because $A_1Q \perp BC$, we have that $BA_1^2 - CA_1^2 = QB^2 - QC^2$, which substituted in relation (15) gives:

$$BA^2 - CA^2 + QC^2 - QB^2 = \frac{t-v}{\alpha} \tag{16}$$

Similarly, we obtain the relations:

$$CB^2 - AB^2 + QA^2 - QC^2 = \frac{u-t}{\beta} \tag{17}$$

$$AC^2 - BC^2 + QB^2 - QA^2 = \frac{v-u}{\gamma} \tag{18}$$

By adding the relations (16), (17) and (18) side by side, we obtain

$$\frac{t-v}{\alpha} + \frac{u-t}{\beta} + \frac{v-u}{\gamma} = 0 \tag{19}$$

The relations (19) and (11) are equivalent, and therefore, $PQ \perp d$, which proves the Sondat's theorem.

Theorem (Smarandache – Pătraşcu)

Consider triangle ABC and the inscribed triangle $A_1B_1C_1$ ortho-homological, Q, Q_1 their centers of orthology, P the homology center and d their homology axes. If $A_2B_2C_2$ is the podar triangle of Q_1, P_1 is the homology center of triangles ABC and $A_2B_2C_2$, and d_1 their homology axes, then the points P, Q, Q_1, P_1 are collinear and the lines d and d_1 are parallel

Proof.

Applying the Sondat's theorem to the ortho-homological triangle ABC and $A_1B_1C_1$, it results that the points P, Q, Q_1 are collinear and their line is perpendicular on d. The same theorem applied to triangles ABC and $A_2B_2C_2$ shows the collinearity of the points P_1, Q, Q_1, and the conclusion that their line is perpendicular on d_1.

From these conclusions we obtain that the points P, Q, Q_1, P_1 are collinear and the parallelism of the lines d and d_1.

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Another proof of a theorem relative to the orthological triangles

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In [1] we proved, using barycentric coordinates, the following theorem:

Theorem: (generalization of the C. Coşniţă theorem)

If P is a point in the triangle's ABC plane, which is not on the circumscribed triangle, $A'B'C'$ is its pedal triangle and A_1, B_1, C_1 three points such that

$$\overrightarrow{PA'} \cdot \overrightarrow{PA_1} = \overrightarrow{PB'} \cdot \overrightarrow{PB_1} = \overrightarrow{PC'} \cdot \overrightarrow{PC_1} = k, \quad k \in R^*,$$

then the lines AA_1, BB_1, CC_1 are concurrent.

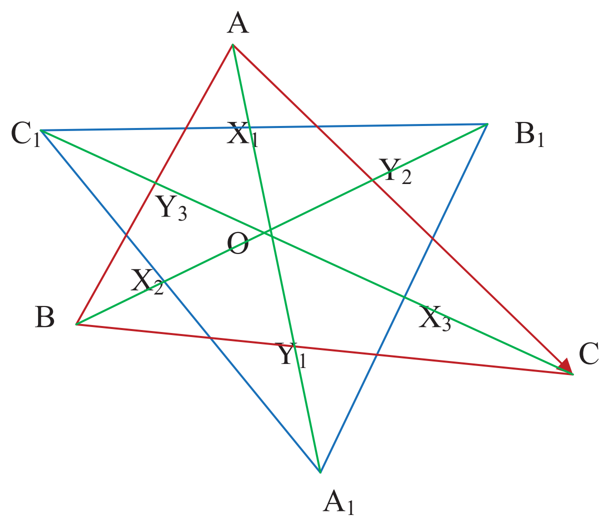
Bellow, will prove, using this theorem, the following:

Theorem

If the triangles ABC and $A_1B_1C_1$ are orthological and their orthological centers coincide, then the lines AA_1, BB_1, CC_1 are concurrent (the triangles ABC and $A_1B_1C_1$ are homological).

Proof:

Let O be the unique orthological center of the triangles ABC and $A_1B_1C_1$ and



$$\{X_1\} = AO \cap B_1C_1$$
$$\{X_2\} = BO \cap A_1C_1$$
$$\{X_3\} = CO \cap A_1B_1$$

We denote

$$\{Y_1\} = OA_1 \cap BC$$
$$\{Y_2\} = OB_1 \cap AC$$
$$\{Y_3\} = OC_1 \cap AB$$

We observe that $\sphericalangle OAY_3 = \sphericalangle OC_1X_1$ (angles with perpendicular sides).

Therefore:

$$\sin OAY_3 = \frac{OY_3}{OA}$$
$$\sin OC_1X_1 = \frac{OX_1}{OC_1},$$

then

$$OX_1 \cdot OA = OY_3 \cdot OC_1 \tag{1}$$

Also

$$\sphericalangle OC_1X_2 = \sphericalangle OBY_3$$

therefore

$$\sin OC_1X_2 = \frac{OX_2}{OC_1}$$
$$\sin OBY_3 = \frac{OY_3}{OB}$$

and consequently:

$$OX_2 \cdot OB = OY_3 \cdot OC_1 \tag{2}$$

Following the same path:

$$\sin OA_1X_2 = \frac{OX_2}{OC_1} = \sin OBY_1 = \frac{OY_1}{OB}$$

from which

$$OX_2 \cdot OB = OA_1 \cdot OY_1 \tag{3}$$

Finally

$$\sin OA_1X_3 = \frac{OX_3}{OA_1} = \sin OCY_1 = \frac{OY_1}{OC}$$

from which:

$$OX_3 \cdot OC = OA_1 \cdot OY_1 \tag{4}$$

The relations (1), (2), (3), (4) lead to

$$OX_1 \cdot OA = OX_2 \cdot OB = OX_3 \cdot OC \tag{5}$$

From (5) using the Coşniţă's generalized theorem, it results that $A_1A,\ B_1B,\ C_1C$ are concurrent.

Observation:

If we denote P the homology center of the triangles ABC and $A_1B_1C_1$ and d is the intersection of their homology axes, them in conformity with the Sondat's theorem, it results that $OP \perp d$.

References:

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Two Triangles with the Same Orthocenter and a Vectorial Proof of Stevanovic's Theorem

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Abstract. In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

Lemma 1

Let ABC an acute angle triangle, H its orthocenter, and A', B', C' the symmetrical points of H in rapport to the sides BC, CA, AB.

We denote by X, Y, Z the symmetrical points of A, B, C in rapport to $B'C', C'A', A'B'$ The orthocenter of the triangle XYZ is H.

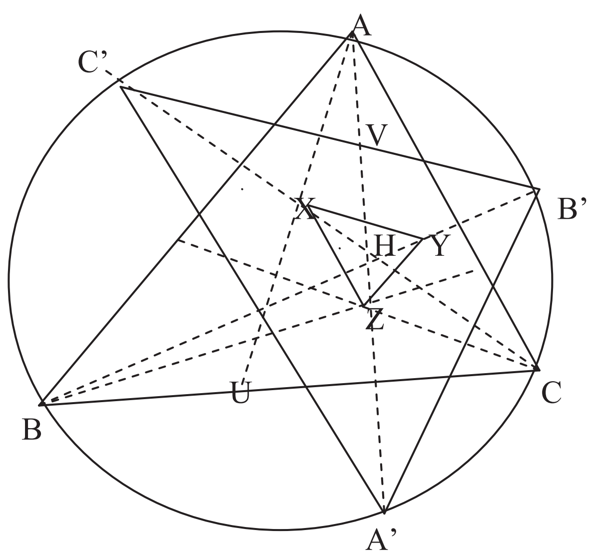

Fig. 1

Proof

We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.
We have (see Fig.1)

$$\overrightarrow{VH} = \overrightarrow{AH} - \overrightarrow{AX}$$
$$\overrightarrow{BC} = \overrightarrow{BY} + \overrightarrow{YZ} + \overrightarrow{ZC},$$

from here

$$\overrightarrow{YZ} = \overrightarrow{BC} - \overrightarrow{BY} - \overrightarrow{ZC}$$

Because Y is the symmetric of B in rapport to $A'C'$ and Z is the symmetric of C in rapport to $A'B'$, the parallelogram's rule gives us that:

$$\overrightarrow{BY} = \overrightarrow{BC'} + \overrightarrow{BA'}$$
$$\overrightarrow{CZ} = \overrightarrow{CB'} + \overrightarrow{CA'}.$$

Therefore

$$\overrightarrow{YZ} = \overrightarrow{BC} - \left(\overrightarrow{BC'} + \overrightarrow{BA'}\right) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But

$$\overrightarrow{BC'} = \overrightarrow{BH} + \overrightarrow{HC'}$$
$$\overrightarrow{BA'} = \overrightarrow{BH} + \overrightarrow{HA'}$$
$$\overrightarrow{CB'} = \overrightarrow{CH} + \overrightarrow{HB'}$$
$$\overrightarrow{CA'} = \overrightarrow{CH} + \overrightarrow{HA'}$$

By substituting these relations in the $\overrightarrow{YZ}$, we find:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \left(\overrightarrow{AH} - \overrightarrow{AX}\right) \cdot \left(\overrightarrow{BC} + \overrightarrow{C'B'}\right) = \overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$$

Because

$$AH \perp BC$$

we have

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0,$$

also

$$AX \perp B'C'$$

and therefore

$$\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0.$$

We need to prove also that

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$$

We note:

$$\{U\} = AX \cap BC \text{ and } \{V\} = AH \cap B'C'$$
$$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot cox\sphericalangle\left(AX, BC\right) = AX \cdot BC \cdot cox\left(\sphericalangle AUC\right)$$
$$\overrightarrow{AH} \cdot \overrightarrow{C'B'} = AH \cdot C'B' \cdot cox\sphericalangle\left(AH, C'B'\right) = AH \cdot C'A' \cdot cox\left(\sphericalangle AVC'\right)$$

We observe that

$\sphericalangle AUC \equiv \sphericalangle AVC'$ (angles with the sides respectively perpendicular).

The point B' is the symmetric of H in rapport to AC, consequently

$$\sphericalangle HAC \equiv \sphericalangle CAB',$$

also the point C' is the symmetric of the point H in rapport to AB, and therefore

$$\sphericalangle HAB \equiv \sphericalangle BAC'.$$

From these last two relations we find that

$$\sphericalangle B'AC' = 2\sphericalangle A.$$

The sinus theorem applied in the triangles $AB'C'$ and ABC gives:

$$B'C' = 2R \cdot \sin 2A$$
$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R \sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH \cos A$$

We noticed that

$$\sphericalangle B'AC' = 2A,$$

Because

$$AX \perp B'C',$$

it results that

$$\sphericalangle TAB \equiv \sphericalangle A,$$

we noted $\{T\} = AX \cap B'C'$.

On the other side

$$AC' = AH,\ \ AT = \frac{1}{2}AY,$$

and

$$AT = AC' \cos A = AH \cos A,$$

therefore

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0.$$

Similarly, we prove that

$$YH \perp XZ,$$

and therefore H is the orthocenter of triangle XYZ.

Lemma 2

Let ABC a triangle inscribed in a circle, I the intersection of its bisector lines, and A', B', C' the intersections of the circumscribed circle with the bisectors AI, BI, CI respectively.

The orthocenter of the triangle $A'B'C'$ is I.

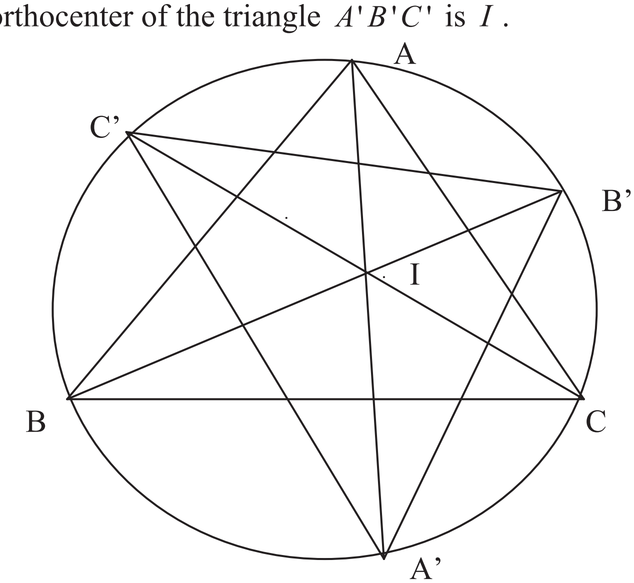

Fig. 2

Proof

We'll prove that $A'I \perp B'C'$.

Let

$$\alpha = m\left(\overset{\frown}{A'C}\right) = m\left(\overset{\frown}{A'B}\right),$$

$$\beta = m\left(\overset{\frown}{B'C}\right) = m\left(\overset{\frown}{B'A}\right)$$

$$\gamma = m\left(\overset{\frown}{C'A}\right) = m\left(\overset{\frown}{C'B}\right)$$

Then

$$m\sphericalangle\left(A'IC'\right) = \frac{1}{2}(\alpha+\beta+\gamma)$$

Because

$$2(\alpha+\beta+\gamma) = 360°$$

it results

$$m\sphericalangle\left(A'IC'\right) = 90°,$$

therefore

$$A'I \perp B'C'.$$

Similarly, we prove that

$$B'I \perp A'C',$$

and consequently the orthocenter of the triangle $A'B'C'$ is I, the center of the circumscribed circle of the triangle ABC.

Definition

Let ABC a triangle inscribed in a circle with the center in O and A', B', C' the middle of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$ respectively. The triangle XYZ formed by the symmetric of the points A', B', C' respectively in rapport to BC, CA, AB is called the Fuhrmann triangle of the triangle ABC.

Note

In 2002 the mathematician Milorad Stevanovic proved the following theorem:

Theorem (M. Stevanovic)

In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.

Proof

We note $A'B'C'$ the given triangle and let A, B, C respectively the middle of the arcs $\overset{\frown}{B'C'}$, $\overset{\frown}{C'A'}$, $\overset{\frown}{A'B'}$ (see Fig. 1). The lines AA', BB', CC' being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note H, and which, in conformity with Lemma 2 is the orthocenter of the triangle ABC. Let XYZ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of XYZ coincides with H the orthocenter of ABC, therefore with the center of the inscribed circle in the given triangle $A'B'C'$.

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2. Stevanovic M., *Orthocentrer of Fuhrmann triangle*, Message Hyacinthos # du 20/09/2002, http://group.yahoo.com/group/Hyacinthos
3. Barbu, Cătălin, *Teoreme fundamentale din geometria triunghiului*, Editura Unique, Bacău, Romania, 2008.

Two Remarkable Ortho-Homological Triangles

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In a previous paper [5] we have introduced the ortho-homological triangles, which are triangles that are orthological and homological simultaneously.

In this article we call attention to two remarkable ortho-homological triangles (the given triangle ABC and its first Brocard's triangle), and using the Sondat's theorem relative to orthological triangles, we emphasize on four important collinear points in the geometry of the triangle. Orthological / homological / orthohomological triangles in the 2D-space are generalized to orthological / homological / orthohomological polygons in 2D-space, and even more to orthological / homological / orthohomological triangles, polygons, and polyhedrons in 3D-space.

Definition 1

The first Brocard triangle of a given triangle ABC is the triangle formed by the projections of the symmedian center of the triangle ABC on its perpendicular bisectors.

Observation

In figure 1 we note with K the symmedian center, OA', OB', OC' the perpendicular bisectors of the triangle ABC and $A_1B_1C_1$ the first Brocard's triangle.

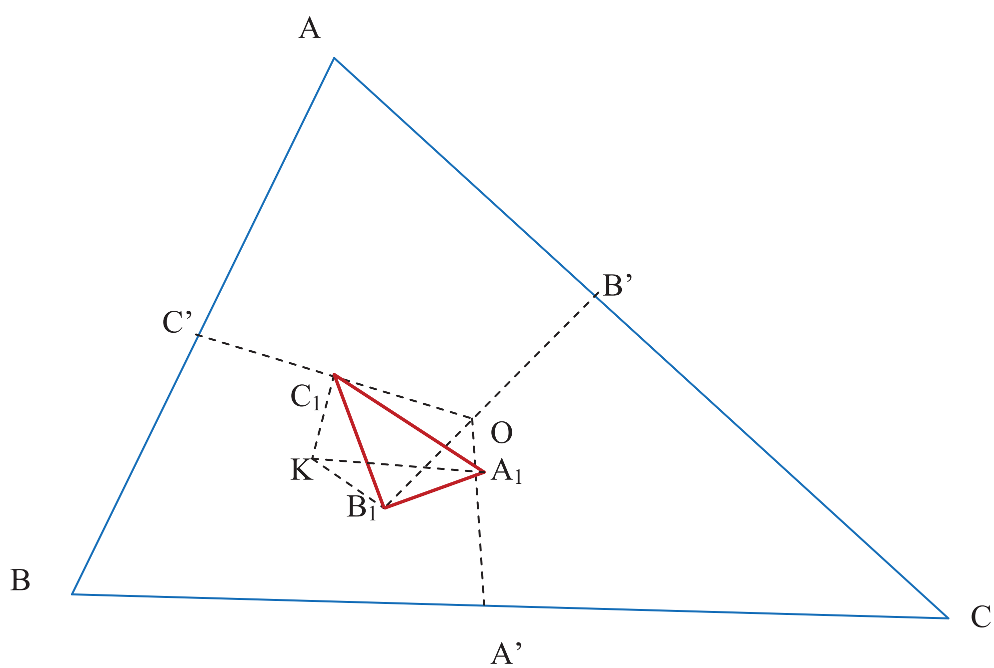

Fig. 1

Theorem 1

If ABC is a given triangle and $A_1B_1C_1$ is its first triangle Brocard, then the triangles ABC and $A_1B_1C_1$ are ortho-homological.

We'll perform the proof of this theorem in two stages.

I. We prove that the triangles $A_1B_1C_1$ and ABC are orthological.
The perpendiculars from A_1, B_1, C_1 on BC, CA respective AB are perpendicular bisectors in the triangle ABC, therefore are concurrent in O, the center of the circumscribed circle of triangle ABC which is the orthological center for triangles $A_1B_1C_1$ and ABC.

II. We prove that the triangles $A_1B_1C_1$ and ABC are homological, that is the lines AA_1, BB_1, CC_1 are concurrent.

To continue with these proves we need to refresh some knowledge and some helpful results.

Definition 2

In any triangle ABC there exist the points Ω and Ω' and the angle ω such that:

$$m(\sphericalangle \Omega AB) = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$$

$$m(\sphericalangle \Omega' BA) = \sphericalangle \Omega' CA = \sphericalangle \Omega' AB = \omega$$

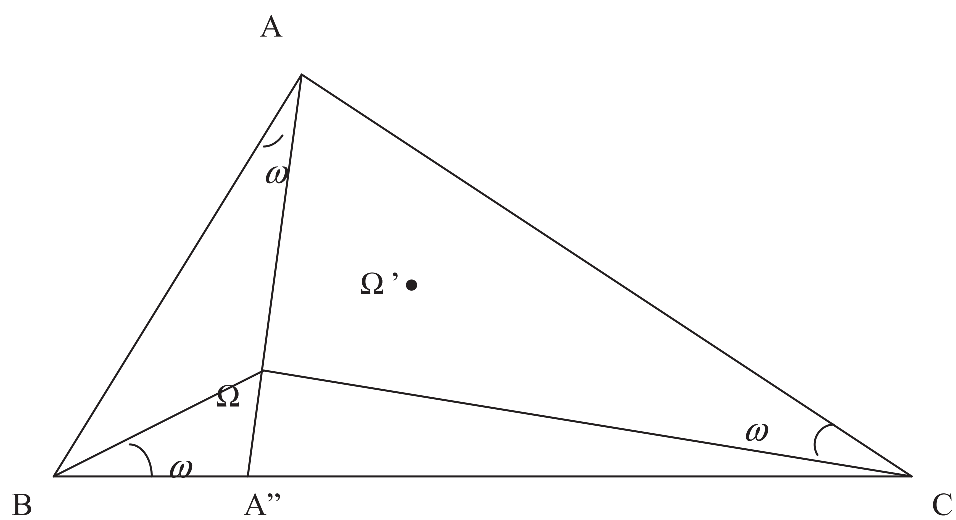

Fig. 2

The points Ω and Ω' are called the first, respectively the second point of Brocard and ω is called the Brocard's angle.

Lemma 1

In the triangle ABC let Ω the first point of Brocard and $\{A''\}$ $\{A''\} = A\Omega \cap BC$, then:

$$\frac{BA''}{CA''}=\frac{c^2}{a^2}$$

Proof

$$Aria_{\triangle}ABA''=\frac{1}{2}AB\cdot AA''\sin\omega \quad (1)$$

$$Aria_{\triangle}ACA''=\frac{1}{2}AC\cdot AA''\sin(A-\omega) \quad (2)$$

From (1) and (2) we find:

$$\frac{Aria_{\triangle}ABA''}{Aria_{\triangle}ACA''}=\frac{AB\cdot\sin\omega}{AC\cdot\sin(A-\omega)} \quad (3)$$

On the other side, the mentioned triangles have the same height built from A, therefore:

$$\frac{Aria_{\triangle}ABA''}{Aria_{\triangle}ACA''}=\frac{BA''}{CA''} \quad (4)$$

From (3) and (4) we have:

$$\frac{BA''}{CA''}=\frac{AB\cdot\sin\omega}{AC\cdot\sin(A-\omega)} \quad (5)$$

Applying the sinus theorem in the triangle $A\Omega C$ and in the triangle $B\Omega C$, it results:

$$\frac{C\Omega}{\sin(A-\omega)}=\frac{AC}{\sin A\Omega C} \quad (6)$$

$$\frac{C\Omega}{\sin\omega}=\frac{BC}{\sin B\Omega C} \quad (7)$$

Because

$$m(\measuredangle A\Omega C)=180^o-A$$

$$m(\measuredangle B\Omega C)=180^o-C$$

From the relations (6) and (7) we find:

$$\frac{\sin\omega}{\sin(A-\omega)}=\frac{AC}{BC}\cdot\frac{\sin C}{\sin A} \quad (8)$$

Applying the sinus theorem in the triangle ABC leads to:

$$\frac{\sin C}{\sin A}=\frac{AB}{BC} \quad (9)$$

The relations (5), (8), (9) provide us the relation:

$$\frac{BA''}{CA''}=\frac{c^2}{a^2}$$

Remark 1

By making the notations: $\{B''\}=B\Omega C\cap AC$ and $\{C''\}=C\Omega A\cap AB$ we obtain also the relations:

$$\frac{CB''}{AB''}=\frac{a^2}{b^2} \text{ and } \frac{AC''}{BC''}=\frac{b^2}{c^2}$$

Lemma 2

In a triangle ABC, the Brocard's Cevian $B\Omega$, symmedian from C and the median from A are concurrent.

Proof

It is known that the symmedian CK of triangle ABC intersects AB in the point C_2 such that $\frac{AC_2}{BC_2}=\frac{b^2}{c^2}$. We had that the Cevian $B\Omega$ intersects AC in $B"$ such that $\frac{BC"}{B"A}=\frac{a^2}{b^2}$.
The median from A intersects BC in A' and $BA'=CA'$.

Because $\frac{A'B}{A'C}\cdot\frac{B"C}{B"A}\cdot\frac{C_2A}{C_2B}=1$, the reciprocal of Ceva's theorem ensures the concurrency of the lines $B\Omega,\ CK$ and AA'.

Lemma 3

Give a triangle ABC and ω the Brocard's angle, then

$$ctg\omega = ctgA + ctgB + ctgC \tag{9}$$

Proof

From the relation (8) we find:

$$\sin(A-\omega)=\frac{a}{b}\cdot\frac{\sin A}{\sin C}\cdot\sin\omega \tag{10}$$

From the sinus' theorem in the triangle ABC we have that

$$\frac{a}{b}=\frac{\sin A}{\sin B}$$

Substituting it in (10) it results: $\sin(A-\omega)=\frac{\sin^2 A\cdot\sin\omega}{\sin B\cdot\sin C}$

Furthermore we have:

$$\sin(A-\omega)=\sin A\cdot\cos\omega-\sin\omega\cdot\cos A$$

$$\sin A\cdot\cos\omega-\sin\omega\cdot\cos A=\frac{\sin^2 A\cdot\sin\omega}{\sin B\cdot\sin C} \tag{11}$$

Dividing relation (11) by $\sin A\cdot\sin\omega$ and taking into account that $\sin A=\sin(B+C)$, and $\sin(B+C)=\sin B\cdot\cos C+\sin C\cdot\cos B$ we obtain relation (5)

Lemma 4

If in the triangle ABC, K is the symmedian center and K_1, K_2, K_3 are its projections on the sides $BC,\ CA,\ AB$, then:

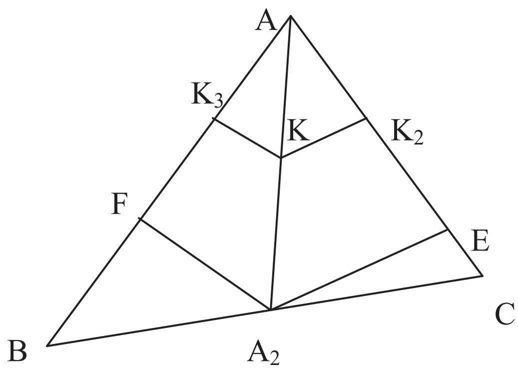

Fig. 3

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} = \frac{1}{2}tg\omega$$

Proof:

Let AA_2 the symmedian in the triangle ABC, we have:

$$\frac{BA_2}{CA_2} = \frac{Aria\vartriangle BAA_2}{Aria\vartriangle CAA_2},$$

where E and F are the projection of A_2 on AC respectively AB.

It results that $\frac{A_2F}{A_2E} = \frac{c}{b}$

From the fact that $\vartriangle AKK_3 \sim \vartriangle AA_2F$ and $\vartriangle AKK_2 \sim \vartriangle AA_2E$ we find that $\frac{KK_3}{KK_2} = \frac{A_2F}{A_2E}$

Also: $\frac{KK_2}{b} = \frac{KK_3}{c}$, and similarly: $\frac{KK_1}{a} = \frac{KK_2}{b}$, consequently:

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} \qquad (12)$$

The relation (12) is equivalent to:

$$\frac{aKK_1}{a^2} = \frac{bKK_2}{b^2} = \frac{cKK_3}{c^2} = \frac{aKK_1 + bKK_2 + cKK_3}{a^2 + b^2 + c^2}$$

Because

$$aKK_1 + bKK_2 + cKK_3 = 2Aria\vartriangle ABC = 2S,$$

we have:

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} = \frac{2S}{a^2 + b^2 + c^2}$$

If we note H_1, H_2, H_3 the projections of A, B, C on BC, CA, AB, we have

$$ctgA = \frac{H_2A}{BH_2} = \frac{bc\cos A}{2S}$$

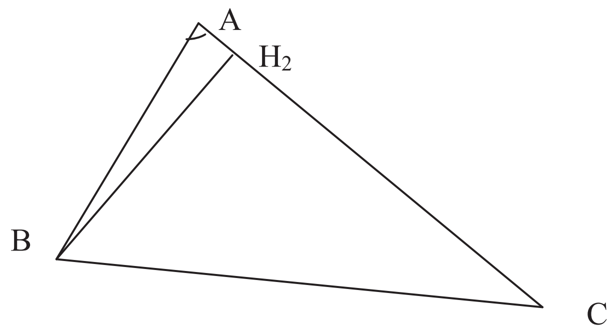

Fig. 4

From the cosine's theorem it results that : $b \cdot c \cdot \cos A = \frac{b^2 + c^2 - a^2}{2}$, and therefore

$$ctgA = \frac{b^2 + c^2 - a^2}{4S}$$

Taking into account the relation (9), we find:

$$ctg\omega = \frac{a^2 + b^2 + c^2}{4S},$$

then

$$tg\omega = \frac{4S}{a^2 + b^2 + c^2}$$

and then

$$\frac{KK_1}{a} = \frac{2S}{a^2 + b^2 + c^2} = \frac{1}{2} tg\omega .$$

Lemma 5

The Cevians AA_1, BB_1, CC_1 are the isotomics of the symmedians AA_2, BB_2, CC_2 in the triangle ABC.

Proof:

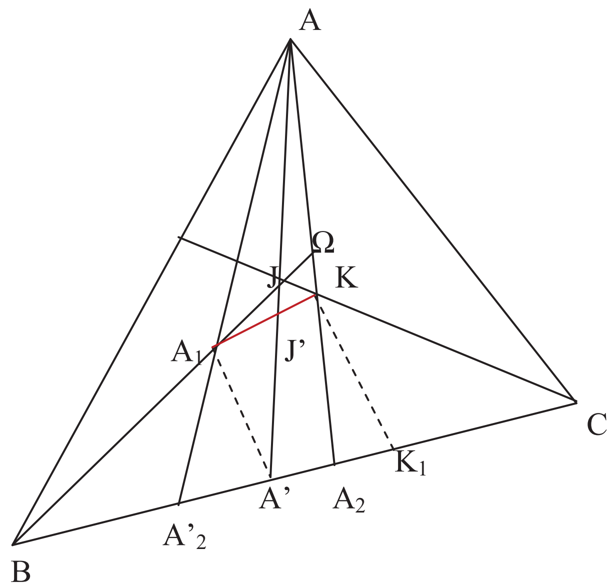

Fig. 5

In figure 5 we note J the intersection point of the Cevians from the Lemma 2.

Because $KA_1 \parallel BC$, we have that $A_1A' = KK_1 = \frac{1}{2} a tg\omega$. On the other side from the right triangle $A'A_1B$ we have: $tg\sphericalangle A_1BA' = \frac{A_1A'}{BA'} = tg\omega$, consequently the point A_1, the vertex of the first triangle of Brocard belongs to the Cevians $B\Omega$.

We note $\{J'\} = A_1K \cap AA'$, and evidently from $A_1K \parallel BC$ it results that JJ' is the median in the triangle JA_1K, therefore $A_1J' = J'K$.

We note with A'_2 the intersection of the Cevians AA_1 with BC, because $A_1K \parallel A'_2A_2$ and AJ' is a median in the triangle AA_1K it results that AA' is a median in triangle AA'_2A_2 therefore the points A'_2 and A_2 are isometric.

Similarly it can be shown that BB_2' and CC_2' are the isometrics of the symmedians BB_2 and CC_2.

The second part of this proof: Indeed it is known that the isometric Cevians of certain concurrent Cevians are concurrent and from Lemma 5 along with the fact that the symmedians of a triangle are concurrent, it results the concurrency of the Cevians AA_1, BB_1, CC_1 and therefore the triangle ABC and the first triangle of Brocard are homological. The homology's center (the concurrency point) of these Cevians is marked in some works with Ω'' with and it is called the third point of Brocard.

From the previous proof, it results that Ω'' is the isotomic conjugate of the symmedian center K.

Remark 2

The triangles ABC and $A_1B_1C_1$ (first Brocard triangle) are *triple-homological*, since first time the Cevians AB_1, BC_1, CA_1 are concurrent (in a Brocard point), second time the Cevians

AC_1, BA_1, CB_1 are also concurrent (in the second Brocard point), and third time the Cevians AA_1, BB_1, CC_1 are concurrent as well (in the third point of Brocard).

Definition 3

It is called the Tarry point of a triangle ABC, the concurrency point of the perpendiculars from A, B, C on the sides B_1C_1, C_1A_1, A_1B_1 of the Brocard's first triangle.

Remark 3

The fact that the perpendiculars from the above definition are concurrent results from the theorem 1 and from the theorem that states that the relation of triangles' orthology is symmetric.

We continue to prove the concurrency using another approach that will introduce supplementary information about the Tarry's point.

We'll use the following:

Lemma 6:

The first triangle Brocard of a triangle and the triangle itself are similar.

Proof

From $KA_1 \parallel BC$ and $OA' \perp BC$ it results that

$$m\left(\sphericalangle KA_1O\right) = 90^\circ$$

(see Fig. 1), similarly

$$m\left(\sphericalangle KB_1O\right) = m\left(\sphericalangle KC_1O\right) = 90^\circ$$

and therefore the first triangle of Brocard is inscribed in the circle with OK as diameter (this circle is called the Brocard circle).

Because

$$m\left(\sphericalangle A_1OC_1\right) = 180^\circ - B$$

and A_1, B_1, C_1, O are concyclic, it results that $\sphericalangle A_1B_1C_1 = \sphericalangle B$, similarly

$$m\left(\sphericalangle B'OC'\right) = 180^\circ - A,$$

it results that

$$m\left(\sphericalangle B_1OC_1\right) = m(A)$$

but

$$\sphericalangle B_1OC_1 \equiv \sphericalangle B_1A_1C_1,$$

therefore

$$\sphericalangle B_1A_1C_1 = \sphericalangle A$$

and the triangle $A_1B_1C_1$ is similar wit the triangle ABC.

Theorem 2

The orthology center of the triangle ABC and of the first triangle of Brocard is the Tarry's point T of the triangle ABC, and T belongs to the circumscribed circle of the triangle ABC.

Proof

We mark with T the intersection of the perpendicular raised from B on A_1C_1 with the perpendicular raised from C on A_1B_1 and let

$$\{B_1'\} = BT \cap A_1C_1,\ A_1\{C_1'\} = A_1B_1 \cap CT.$$

We have

$$m(\triangle B_1'TC_1') = 180° - m(\triangle C_1A_1B_1)$$

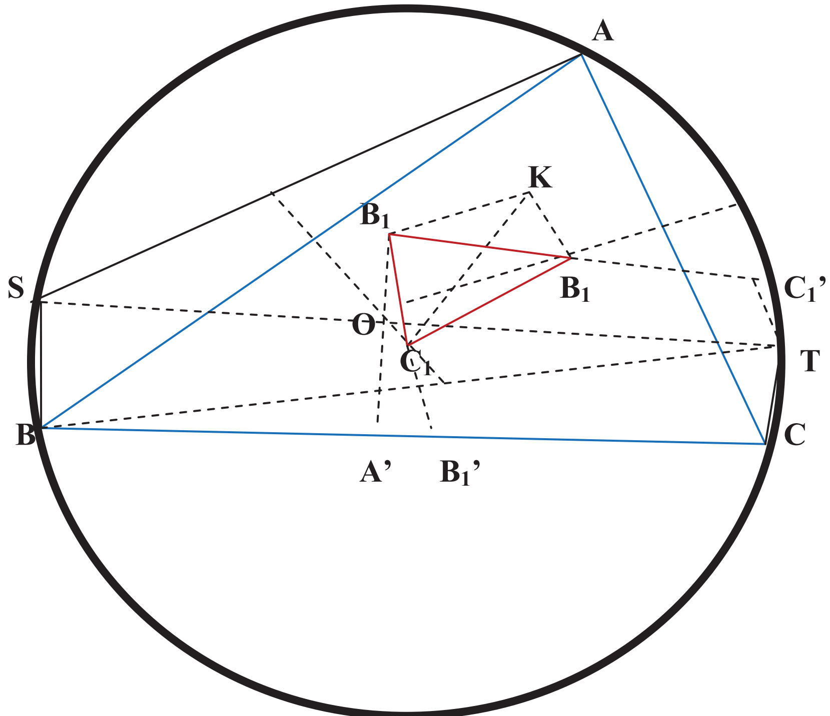

Fig 6

But because of Lemma 6 $\sphericalangle C_1A_1B_1 = \sphericalangle A$.

It results that $m(\sphericalangle B_1'TC_1') = 180° - A$, therefore

$$m(\sphericalangle BTC') + m(\sphericalangle BAC) = 180°$$

Therefore T belongs to the circumscribed circle of triangle ABC

If $\{A_1'\} = B_1C_1 \cap AT$ and if we note with T' the intersection of the perpendicular raised from A on B_1C_1 with the perpendicular raised from B on A_1C_1, we observe that

$$m(\sphericalangle B_1'T'A_1') = m(\sphericalangle A_1C_1B_1)$$

therefore

$$m(\sphericalangle BT'A) + m(\sphericalangle BCA)$$

and it results that T' belongs to the circumscribed triangle ABC.

Therefore $T = T'$ and the theorem is proved.

Theorem 3

If through the vertexes A, B, C of a triangle are constructed the parallels to the sides B_1C_1, C_1A_1 respectively A_1B_1 of the first triangle of Brocard of this triangle, then these lines are concurrent in a point S (the Steiner point of the triangle)

Proof

We note with S the polar intersection constructed through A to B_1C_1 with the polar constructed through B to A_1C_1 (see Fig. 6).

We have

$$m(\sphericalangle ASB) = 180° - m(\sphericalangle A_1C_1B_1) \text{ (angles with parallel sides)}$$

because

$$m(\sphericalangle A_1C_1B_1) = m\sphericalangle C,$$

we have

$$m(\sphericalangle ASB) = 180° - m\sphericalangle C,$$

therefore A_1SB_1C are concyclic.

Similarly, if we note with S' the intersection of the polar constructed through A to B_1C_1 with the parallel constructed through C to A_1B_1 we find that the points $A_1S_1'B_1C$ are concyclic.

Because the parallels from A to B_1C_1 contain the points A, S, S' and the points S, S', A are on the circumscribed circle of the triangle, it results that $S = S'$ and the theorem is proved.

Remark 4

Because $SA \parallel B_1C_1$ and $B_1C_1 \perp AT$, it results that

$$m(\sphericalangle SAT) = 90°,$$

but S and T belong to the circumscribed circle to the triangle ABC, consequently the Steiner's point and the Tarry point are diametric opposed.

Theorem 4

In a triangle ABC the Tarry point T, the center of the circumscribed circle O, the third point of Brocard Ω'' and Steiner's point S are collinear points

Proof

The P. Sondat's theorem relative to the orthological triangles (see [4]) says that the points T, O, Ω'' are collinear, therefore the points: T, O, Ω'', S are collinear.

Open Questions

1) Is it possible to have two triangles which are four times, five times, or even six times orthological? But triangles which are four times, five times, or even six times homological? What about orthohomological? What is the largest such rank?
   For two triangles $A_1B_1C_1$ and $A_2B_2C_2$, we can have (in the case of orthology, and similarity in the cases of homology and orthohomology) the following 6 possibilities:
   1) the perpendicular from A_1 onto B_2C_2, the perpendicular from B_1 onto C_2A_2, and the perpendicular from C_1 onto A_2B_2 concurrent;

2) the perpendicular from A_1 onto B_2C_2, the perpendicular from B_1 onto A_2B_2, and the perpendicular from C_1 onto C_2A_2 concurrent;
3) the perpendicular from B_1 onto B_2C_2, the perpendicular from A_1 onto C_2A_2, and the perpendicular from C_1 onto A_2B_2 concurrent;
4) the perpendicular from B_1 onto B_2C_2, the perpendicular from A_1 onto A_2B_2, and the perpendicular from C_1 onto C_2A_2 concurrent;
5) the perpendicular from C_1 onto B_2C_2, the perpendicular from B_1 onto C_2A_2, and the perpendicular from A_1 onto A_2B_2 concurrent;
6) the perpendicular from C_1 onto B_2C_2, the perpendicular from B_1 onto A_2B_2, and the perpendicular from A_1 onto C_2A_2 concurrent.

2) We generalize the orthological, homological, and orthohomological triangles to respectively orthological, homological, and orthohomological polygons and polyhedrons. Can we have double, triple, etc. orthological, homological, or orthohomological polygons and polyhedrons? What would be the largest rank for each case?

3) Let's have two triangles in a plane. Is it possible by changing their positions in the plane and to have these triangles be orthological, homological, orthohomological? What is the largest rank they may have in each case?

4) Study the orthology, homology, orthohomology of triangles and poligons in a 3D space.

5) Let's have two triangles, respectively two polygons, in a 3D space. Is it possible by changing their positions in the 3D space to have these triangles, respectively polygons, be orthological, homological, or orthohological?
Similar question for two polyhedrons?

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A Generalization of Certain Remarkable Points of the Triangle Geometry

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In this article we prove a theorem that will generalize the concurrence theorems that are leading to the Franke's point, Kariya's point, and to other remarkable points from the triangle geometry.

Theorem 1:

Let $P(\alpha,\beta,\gamma)$ and A', B', C' its projections on the sides BC, CA respectively AB of the triangle ABC.

We consider the points A'', B'', C'' such that $\overrightarrow{PA''}=k\overrightarrow{PA'}$, $\overrightarrow{PB''}=k\overrightarrow{PB'}$, $\overrightarrow{PC''}=k\overrightarrow{PC'}$, where $k\in R^*$. Also we suppose that AA', BB', CC' are concurrent. Then the lines AA'', BB'', CC'' are concurrent if and only if are satisfied simultaneously the following conditions:

$$\alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0$$

$$\frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0$$

Proof:

We find that

$$A'\left(0,\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

$$\overrightarrow{PA''}=k\overrightarrow{PA'}=k\left[-\alpha\overrightarrow{r_A}+\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)\overrightarrow{r_B}+\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)\overrightarrow{r_C}\right]$$

$$\overrightarrow{PA''}=\left(\alpha''-\alpha\right)\overrightarrow{r_A}+\left(\beta''-\beta\right)\overrightarrow{r_B}+\left(\gamma''-\gamma\right)\overrightarrow{r_C}$$

We have:

$$\begin{cases}\alpha''-\alpha=-k\alpha\\ \beta''-\beta=\dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right),\\ \gamma''-\gamma=\dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)\end{cases}$$

Therefore:

$$\begin{cases} \alpha'' = (1-k)\alpha \\ \beta'' = \dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta \\ \gamma'' = \dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma \end{cases}$$

Hence:

$$A''\left((1-k)\alpha,\ \frac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

Similarly:

$$B'\left(-\frac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ 0,\ -\frac{\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$B''\left(-\frac{k\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ (1\text{-}k)\beta,\ -\frac{k\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$C'\left(-\frac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ 0\right)$$

$$C''\left(-\frac{k\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{k\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ (1\text{-}k)\gamma\right)$$

Because $AA',\ BB',\ CC'$ are concurrent, we have:

$$\frac{-\dfrac{\alpha}{2a^2}\left(-a^2-b^2+c^2\right)+\beta}{-\dfrac{\alpha}{2a^2}\left(-a^2+b^2-c^2\right)+\gamma}\cdot\frac{-\dfrac{\beta}{2b^2}\left(-a^2-b^2-c^2\right)+\gamma}{-\dfrac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha}\cdot\frac{-\dfrac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha}{-\dfrac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta}=1$$

We note

$$M=\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)=\frac{\alpha}{a}\cdot b\cos C$$

$$N=\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)=\frac{\alpha}{a}\cdot c\cos B$$

$$P=\frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)=\frac{\beta}{b}\cdot c\cos A$$

$$Q=\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)=\frac{\beta}{b}\cdot a\cos C$$

$$R=\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos B$$

$$S=\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos A$$

The precedent relation becomes

$$\frac{M+\beta}{N+\gamma}\cdot\frac{P+\gamma}{Q+\alpha}\cdot\frac{R+\alpha}{S+\beta}=1$$

The coefficients M,N,P,Q,R,S verify the following relations:

$$M+N=\alpha$$
$$P+Q=\beta$$
$$R+S=\gamma$$
$$\frac{M}{Q}=\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}=\frac{\frac{\alpha}{a^2}}{\frac{\beta}{b^2}}$$
$$\frac{P}{S}=\frac{\beta}{\gamma}\cdot\frac{c^2}{b^2}=\frac{\frac{\beta}{b^2}}{\frac{\gamma}{c^2}}$$
$$\frac{R}{N}=\frac{\gamma}{\alpha}\cdot\frac{a^2}{c^2}=\frac{\frac{\gamma}{c^2}}{\frac{\alpha}{a^2}}$$

Therefore $\frac{M}{Q}\cdot\frac{P}{S}\cdot\frac{R}{N}=1$

$$(M+\beta)(P+\gamma)(R+\alpha)=\alpha\beta\gamma+\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM+MPR$$
$$(N+\gamma)(Q+\alpha)(S+\beta)=\alpha\beta\gamma+\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS\,.$$

We deduct that:

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM=\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS \quad (1)$$

We apply the theorem:

Given the points $Q_i(a_i,b_i,c_i),\ i=\overline{1,3}$ in the plane of the triangle ABC, the lines AQ_1, BQ_2, CQ_3 are concurrent if and only if $\frac{b_1}{c_1}\cdot\frac{c_2}{a_2}\cdot\frac{a_3}{b_3}=1$.

For the lines AA'', BB'', CC'' we obtain

$$\frac{kM+\beta}{kN+\gamma}\cdot\frac{kP+\alpha}{kS+\beta}\cdot\frac{kR+\alpha}{kS+\beta}=1\,.$$

It result that

$$k^2(\alpha\beta P+\beta\gamma R+\gamma\alpha M)+k(\alpha MP+\beta PR+\gamma RM)=$$
$$=k^2(\alpha\beta N+\beta\gamma Q+\gamma\alpha S)+k(\alpha NS+\beta NQ+\gamma QS) \quad (2)$$

For relation (1) to imply relation (2) it is necessary that

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M=\alpha\beta N+\beta\gamma Q+\gamma\alpha S$$

and

$$\alpha NS+\beta NQ+\gamma QS=\alpha MP+\beta PR+\gamma RM$$

or

$$\begin{cases} \alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0 \\ \frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0 \end{cases}$$

As an open problem, we need to determine the set of the points from the plane of the triangle ABC that verify the precedent relations.

We will show that the points I and O verify these relations, proving two theorems that lead to Kariya's point and Franke's point.

Theorem 2 (Kariya -1904)

Let I be the center of the circumscribe circle to triangle ABC and A', B', C' its projections on the sides BC, CA, AB. We consider the points A'', B'', C'' such that:

$$\overrightarrow{IA''}=k\overrightarrow{IA'},\ \overrightarrow{IB''}=k\overrightarrow{IB'},\ \overrightarrow{IC''}=k\overrightarrow{IC'},\ k\in R^*.$$

Then AA'', BB'', CC'' are concurrent (the Kariya's point)

Proof:

The barycentric coordinates of the point I are $I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)$.

Evidently:

$$abc(\cos A-\cos B)+abc(\cos B-\cos C)+abc(\cos C-\cos A)=0$$

and

$$\cos A(\cos B-\cos C)+\cos B(\cos C-\cos A)+\cos C(\cos A-\cos B)=0.$$

In conclusion AA'', BB'', CC'' are concurrent.

Theorem 3 (de Boutin - 1890)

Let O be the center of the circumscribed circle to the triangle ABC and A', B', C' its projections on the sides BC, CA, AB. Consider the points A'', B'', C'' such that $\frac{OA'}{OA''}=\frac{OB'}{OB''}=\frac{OC'}{OC''}=k,\ k\in R^*$. Then the lines AA'', BB'', CC'' are concurrent (The point of Franke – 1904).

Proof:

$$O\left(\frac{R^2}{2S}\sin 2A,\frac{R^2}{2S}\sin 2B,\frac{R^2}{2S}\sin 2C\right),\ P=N \text{, because } \frac{\sin 2B\cos A}{\sin B}-\frac{\sin 2A\cos B}{\sin A}=0.$$

Similarly we find that $R=Q$ and $M=S$.

Also $\alpha MP=\alpha NS$, $\beta PR=\beta NQ$, $\gamma RM=\gamma QS$. It is also verified the second relation from the theorem hypothesis. Therefore the lines AA'', BB'', CC'' are concurrent in a point called the Franke's point.

Remark 1:

It is possible to prove that the Franke's points belong to Euler's line of the triangle ABC.

Theorem 4:

Let I_a be the center of the circumscribed circle to the triangle ABC (tangent to the side BC) and A', B', C' its projections on the sites BC, CA, AB. We consider the points A'', B'', C'' such that $\overrightarrow{IA''} = k\overrightarrow{IA'},\ \overrightarrow{IB''} = k\overrightarrow{IB'},\ \overrightarrow{IC''} = k\overrightarrow{IC'},\ k \in R^*$. Then the lines AA'', BB'', CC'' are concurrent.

Proof

$$I_a\left(\frac{-a}{2(p-a)}, \frac{b}{2(p-a)}, \frac{c}{2(p-a)}\right);$$

The first condition becomes:

$-abc(\cos A + \cos B) + abc(\cos B - \cos C) - abc(-\cos C - \cos A) = 0$, and the second condition:

$$\cos A(\cos B - \cos C) + \cos B(-\cos C - \cos A) + \cos C(\cos A + \cos B) = 0$$

Is also verified.

From this theorem it results that the lines AA'', BB'', CC'' are concurrent.

Observation 1:

Similarly, this theorem is proven for the case of I_b and I_c as centers of the ex-inscribed circles.

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Generalization of a Remarkable Theorem

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In [1] Professor Claudiu Coandă proved, using the barycentric coordinates, the following remarkable theorem:

Theorem (C. Coandă)

Let ABC be a triangle, where $m(\sphericalangle A) \neq 90°$ and Q_1, Q_2, Q_3 are three points on the circumscribed circle of the triangle ABC. We'll note $BQ_i \cap AC = \{B_i\}$, $i = \overline{1,3}$. Then the lines B_1C_1, B_2C_2, B_3C_3 are concurrent.

We will generalize this theorem using some results from projective geometry relative to the pole and polar notions.

Theorem (Generalization of C. Coandă theorem)

Let ABC be a triangle where $m(\sphericalangle A) \neq 90°$ and $Q_1, Q_2, \dots, Q_n$ points on its circumscribed circle $(n \in N,\ n \geq 3)$, $i = \overline{1,n}$. Then the lines B_1C_1, $B_2C_2, \dots, B_nC_n$ are concurrent in fixed point.

To prove this theorem we'll utilize the following lemmas:

Lemma 1

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, then the polar of the point P in rapport with the circle is the line EF, where $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$

Lemma 2

The pole of a line is the intersection of the corresponding polar to any two points of the line.

The pols of concurrent lines in rapport to a given circle are collinear points and the reciprocal is also true: the polar of collinear points, in rappoer with a given circle, are concurrent lines.

Lemma 3

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$, then the polar of point E in rapport to the circle is the line PF.

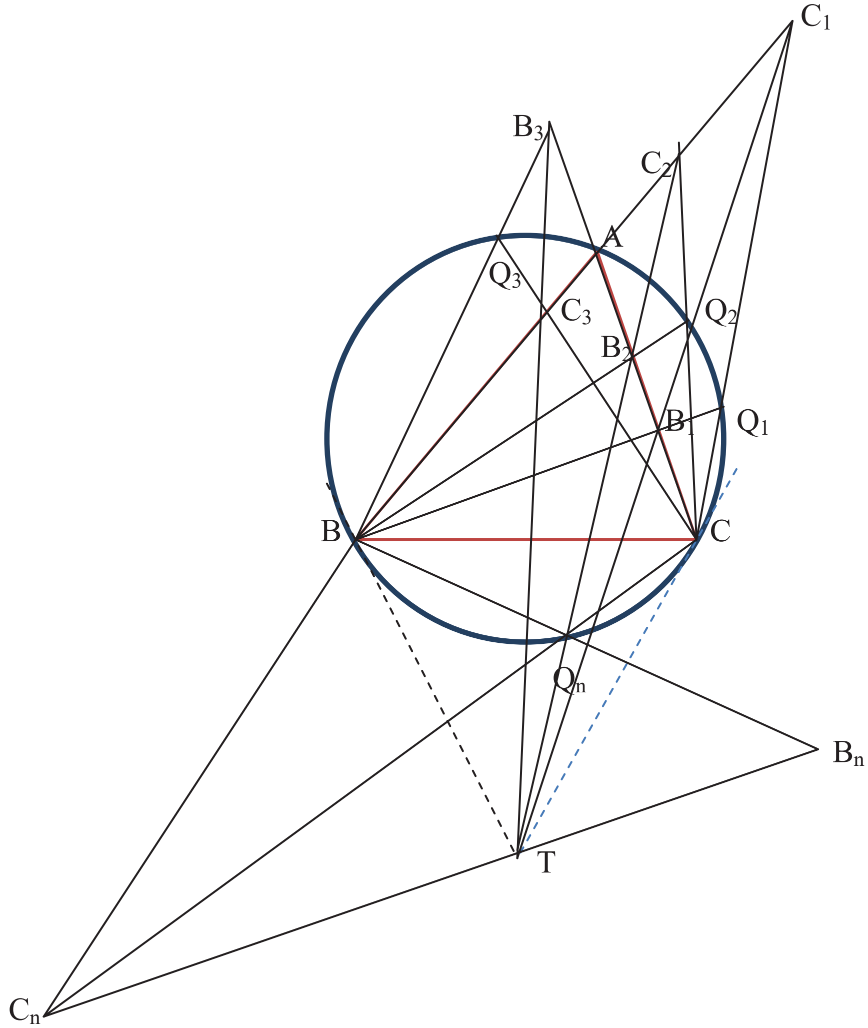
The proof for the Lemmas 1 - 3 and other information regarding the notions of pole and polar in rapport to a circle can be found in [2] or [3].

Proof of the generalized theorem of C. Coandă

Let $Q_1, Q_2, ..., Q_n$ points on the circumscribed circle to the triangle ABC (see the figure)

We'll consider the inscribed quadrilaterals $ABCQ_n$, $i = \overline{1,n}$ and we'll note $\{T_i\} = AQ_i \cap BC$.

In accordance to Lemma 1 and Lemma 3, the lines B_iC_i are the respectively polar



(in rapport with the circumscribed circle to the triangle ABC) to the points T_i.

Because the points T_i are collinear (belonging to the line BC), from Lemma 2 we'll obtain that their polar, that is the lines B_iC_i, are concurrent in a point T.

Remark

The concurrency point T is the harmonic conjugate in rapport with the circle of the symmedian center K of the given triangle.

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Pantazi's Theorem Regarding the Bi-orthological Triangles

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In this article we'll present an elementary proof of a theorem of Alexandru Pantazi (1896-1948), Romanian mathematician, regarding the bi-orthological triangles.

1. Orthological triangles

Definition

The triangle ABC is orthologic in rapport to the triangle $A_1B_1C_1$ if the perpendiculars constructed from A, B, C respectively on B_1C_1, C_1A_1 and A_1B_1 are concurrent. The concurrency point is called the orthology center of the triangle ABC in rapport to triangle $A_1B_1C_1$.

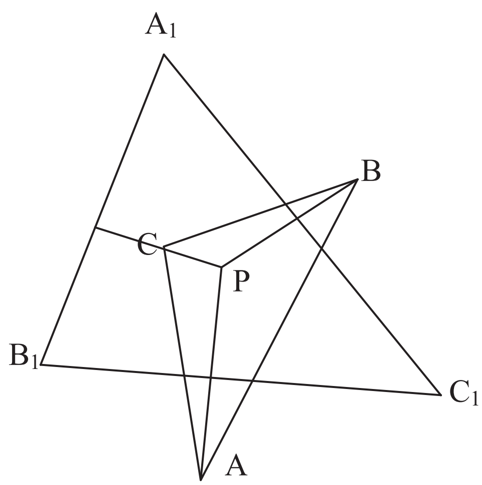

Fig. 1

In figure 1 the triangle ABC is orthologic in rapport with $A_1B_1C_1$, and the orthology center is P.

2. Examples

a) The triangle ABC and its complementary triangle $A_1B_1C_1$ (formed by the sides' middle) are orthological, the orthology center being the orthocenter H of the triangle ABC.
Indeed, because B_1C_1 is a middle line in the triangle ABC, the perpendicular from A on B_1C_1 will be the height from A. Similarly the perpendicular from B on C_1A_1 and the perpendicular from C on A_1B_1 are heights in ABC, therefore concurrent in H (see Fig. 2)

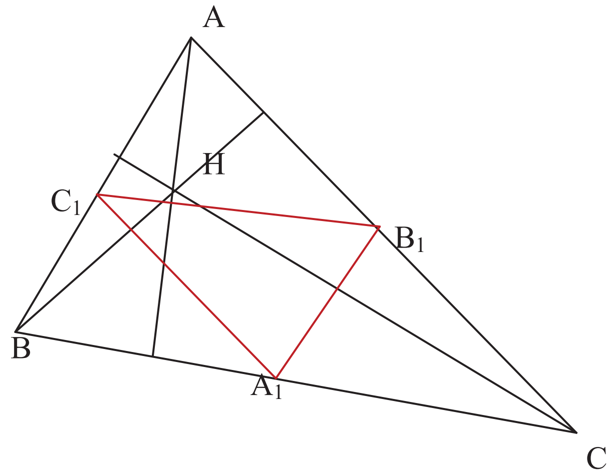

Fig. 2

b) **Definition**

Let D a point in the plane of triangle ABC. We call the circum-pedal triangle (or meta-harmonic) of the point D in rapport to the triangle ABC, the triangle $A_1B_1C_1$ of whose vertexes are intersection points of the Cevianes AD, BD, CD with the circumscribed circle of the triangle ABC.

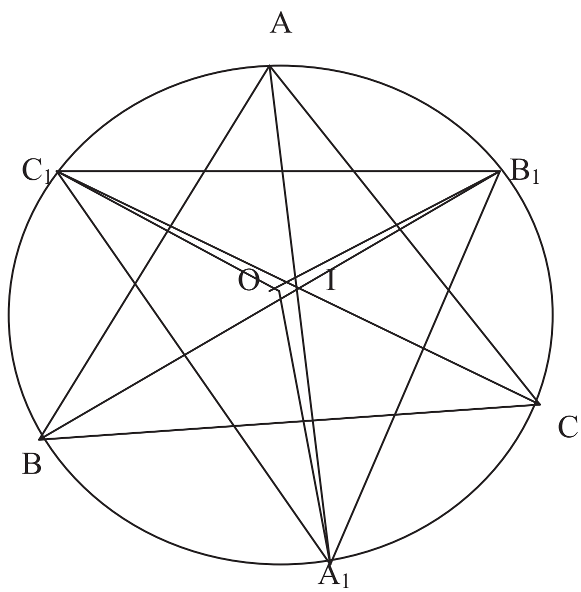

Fig. 3

The triangle circum-pedal $A_1B_1C_1$ of the center of the inscribed circle in the triangle ABC and the triangle ABC are orthological (Fig. 3).

The points A_1, B_1, C_1 are the midpoints of the arcs $\overset{\frown}{BC}, \overset{\frown}{CA}$ respectively $\overset{\frown}{AB}$. We have $\overset{\frown}{A_1B} \equiv \overset{\frown}{A_1C}$, it results that $A_1B = A_1C$, therefore A_1 is on the perpendicular bisector of BC, and therefore the perpendicular raised from A_1 on BC passes through O,the center of the circumscribed circle to triangle ABC. Similarly the perpendiculars raised from B_1, C_1 on AC respectively AB pass through O. The orthology center of triangle $A_1B_1C_1$ in rapport to ABC is O

3. The characteristics of the orthology property

The following Lemma gives us a necessary and sufficient condition for the triangle ABC to be orthologic in rapport to the triangle $A_1B_1C_1$.

Lemma

The triangle ABC is orthologic in rapport with the triangle $A_1B_1C_1$ if and only if:

$$\overrightarrow{MA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0 \qquad (1)$$

for any point M from plane.

Proof

In a first stage we prove that the relation from the left side, which we'll note $E(M)$ is independent of the point M.

Let $N \neq M$ and $E(N)=\overrightarrow{NA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{NB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{NC}\cdot\overrightarrow{A_1B_1}$

Compute $E(M)-E(N)=\overrightarrow{MN}\cdot\left(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}\right)$.

Because $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=0$ we have that $E(M)-E(N)=\overrightarrow{MN}\cdot\vec{0}=0$.

If the triangle ABC is orthologic in rapport to $A_1B_1C_1$, we consider M their orthologic center, it is obvious that (1) is verified. If (1) is verified for a one point, we proved that it is verified for any other point from plane.

Reciprocally, if (1) is verified for any point M, we consider the point M as being the intersection of the perpendicular constructed from A on B_1C_1 with the perpendicular constructed from B on C_1A_1. Then (1) is reduced to $\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0$, which shows that the perpendicular constructed from C on $\overrightarrow{A_1B_1}$ passes through M. Consequently, the triangle ABC is orthologic in rapport to the triangle $A_1B_1C_1$.

4. The symmetry of the orthology relation of triangles

It is natural to question ourselves that given the triangles ABC and $A_1B_1C_1$ such that ABC is orthologic in rapport to $A_1B_1C_1$, what are the conditions in which the triangle $A_1B_1C_1$ is orthologic in rapport to the triangle ABC.

The answer is given by the following

Theorem (The relation of orthology of triangles is symmetric)

If the triangle ABC is othologic in rapport with the triangle $A_1B_1C_1$ then the triangle $A_1B_1C_1$ is also orthologic in rapport with the triangle ABC.

Proof

We'll use the lemma. If the triangle ABC is orthologic in rapport with $A_1B_1C_1$ then

$$\overrightarrow{MA} \cdot \overrightarrow{B_1C_1} + \overrightarrow{MB} \cdot \overrightarrow{C_1A_1} + \overrightarrow{MC} \cdot \overrightarrow{A_1B_1} = 0$$

for any point M . We consider $M = A$, then we have

$$\overrightarrow{AA} \cdot \overrightarrow{B_1C_1} + \overrightarrow{AB} \cdot \overrightarrow{C_1A_1} + \overrightarrow{AC} \cdot \overrightarrow{A_1B_1} = 0 .$$

This expression is equivalent with

$$\overrightarrow{A_1A_1} \cdot \overrightarrow{BC} + \overrightarrow{A_1B_1} \cdot \overrightarrow{CA} + \overrightarrow{A_1C_1} \cdot \overrightarrow{AB} = 0$$

That is with (1) in which $M = A_1$, which shows that the triangle $A_1B_1C_1$ is orthologic in rapport to triangle ABC .

Remarks

1. We say that the triangles ABC and $A_1B_1C_1$ are orthological if one of the triangle is orthologic in rapport to the other.
2. The orthology centers of two triangles are, in general, distinct points.
3. The second orthology center of the triangles from a) is the center of the circumscribed circle of triangle ABC .
4. The orthology relation of triangles is reflexive. Indeed, if we consider a triangle, we can say that it is orthologic in rapport with itself because the perpendiculars constructed from A, B, C respectively on BC, CA, AB are its heights and these are concurrent in the orthocenter H .

5. Bi-orthologic triangles

Definition

If the triangle ABC is simultaneously orthologic to triangle $A_1B_1C_1$ and to triangle $B_1C_1A_1$, we say that the triangles ABC and $A_1B_1C_1$ are bi-orthologic.

Pantazi's Theorem

If a triangle ABC is simultaneously orthologic to triangle $A_1B_1C_1$ and $B_1C_1A_1$, then the triangle ABC is orthologic also with the triangle $C_1A_1B_1$.

Proof

Let triangle ABC simultaneously orthologic to $A_1B_1C_1$ and to $B_1C_1A_1$, using lemma, it results that

$$\overrightarrow{MA} \cdot \overrightarrow{B_1C_1} + \overrightarrow{MB} \cdot \overrightarrow{C_1A_1} + \overrightarrow{MC} \cdot \overrightarrow{A_1B_1} = 0 \qquad (2)$$

$$\overrightarrow{MA} \cdot \overrightarrow{C_1A_1} + \overrightarrow{MB} \cdot \overrightarrow{A_1B_1} + \overrightarrow{MC} \cdot \overrightarrow{B_1C_1} = 0 \qquad (3)$$

For any M from plane.

Adding the relations (2) and (3) side by side, we have:

$$\overrightarrow{MA} \cdot \left(\overrightarrow{B_1C_1} + \overrightarrow{C_1A_1}\right) + \overrightarrow{MB} \cdot \left(\overrightarrow{C_1A_1} + \overrightarrow{A_1B_1}\right) + \overrightarrow{MC} \cdot \left(\overrightarrow{A_1B_1} + \overrightarrow{B_1C_1}\right) = 0$$

Because

$$\overrightarrow{B_1C_1} + \overrightarrow{C_1A_1} = \overrightarrow{B_1A_1},\ \overrightarrow{C_1A_1} + \overrightarrow{A_1B_1} = \overrightarrow{C_1B_1},\ \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} = \overrightarrow{A_1C_1}$$

(Chasles relation), we have:

$$\overrightarrow{MA} \cdot \overrightarrow{B_1A_1} + \overrightarrow{MB} \cdot \overrightarrow{C_1B_1} + \overrightarrow{MC} \cdot \overrightarrow{A_1C_1} = 0$$

for any M from plane, which shows that the triangle ABC is orthologic with the triangle $C_1A_1B_1$ and the Pantazi's theorem is proved.

Remark

The Pantazi's theorem can be formulated also as follows: If two triangles are bi-orthologic then these are tri-orthologic.

Open Questions

1) Is it possible to extend Pantazi's Theorem (in 2D-space) in the sense that if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are bi-orthological, then they are also k-orthological, where $k = 4, 5,$ or 6?

2) Is it true a similar theorem as Pantazi's for two bi-homological triangles and bi-orthohomological triangles (in 2D-space)? We mean, if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are bi-homological (respectively bi-orthohomological), then they are also k-homological (respectively k-orthohomological), where $k = 4, 5,$ or 6?

3) How the Pantazi Theorem behaves if the two bi-orthological non-coplanar triangles $A_1B_1C_1$ and $A_2B_2C_2$ (if any) are in the 3D-space?

4) Is it true a similar theorem as Pantazi's for two bi-homological (respectively bi-orthohomological) non-coplanar triangles $A_1B_1C_1$ and $A_2B_2C_2$ (if any) in the 3D-space?

5) Similar questions as above for bi-orthological / bi-homological / bi-orthohomological polygons (if any) in 2D-space, and respectively in 3D-space.

6) Similar questions for bi-orthological / bi-homological / bi-orthohomological polyhedrons (if any) in 3D-space.

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A New Proof and an Application of Dergiades' Theorem

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In this article we'll present a new proof of Dergiades' Theorem, and we'll use this theorem to prove that the orthological triangles with the same orthological center are homological triangles.

Theorem 1 (Dergiades)

Let $C_1(O_1,R_1)$, $C_2(O_2,R_2)$, $C_3(O_3,R_3)$ three circles which pass through the vertexes B and C, C and A, A and B respectively of a given triangle ABC. We'll note D,E,F respectively the second point of intersection between the circles (C_1) and (C_3), (C_3) and (C_2), (C_1) and (C_2). The perpendiculars constructed in the points D,E,F on AD, BE respectively CF intersect the sides BC, CA, AB in the points X, Y, Z. Then the points X, Y, Z are collinear

Proof

To prove the collinearity of the points X, Y, Z, we will use the reciprocal of the Menelaus Theorem (see *Fig. 1*).

We have

$$\frac{XB}{XC}=\frac{Aria\Delta XDB}{Aria\Delta XDC}=\frac{DB\cdot sin\,\widehat{XDB}}{DC\cdot sin\,\widehat{XDC}}=\frac{DB\cdot cos\,\widehat{ADB}}{DC\cdot cos\,\widehat{ADC}}$$

Similarly we find

$$\frac{YC}{YA}=\frac{EC\cdot cos\,\widehat{BEC}}{EA\cdot cos\,\widehat{BEA}}$$

$$\frac{ZA}{ZB}=\frac{FA\cdot cos\,\widehat{CFA}}{FB\cdot cos\,\widehat{CFB}}$$

From the inscribed quadrilaterals $ADEB; BEFC; ADFC$, we can observe that

$$\sphericalangle ADB\equiv\sphericalangle BEA;\ \sphericalangle BEC\equiv\sphericalangle CFB;\ \sphericalangle CFA\equiv\sphericalangle ADC$$

Consequently,

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB}\qquad(1)$$

On the other side $DB=2R_3\,sin\,\widehat{BAD}$; $EA=2R_3\,sin\,\widehat{ABE}$; $DC=2R_2\,sin\,\widehat{CAD}$; $FA=2R_2\,sin\,\widehat{ACF}$; $FB=2R_1\,sin\,\widehat{BCF}$; $EC=2R_1\,sin\,\widehat{CBE}$.

Using these relations in (1), we obtain

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{sin\,\widehat{BAD}}{sin\,\widehat{CAD}}\cdot\frac{sin\,\widehat{CBE}}{sin\,\widehat{ABE}}\cdot\frac{sin\,\widehat{ACF}}{sin\,\widehat{BCF}}\qquad(2)$$

According to one of Carnot's theorem, the common strings of the circles (C_1), (C_2), (C_3) are concurrent, that is $AD\cap BE\cap CF=\{P\}$ (the point P is the radical center of the circles (C_1), (C_2), (C_3)).

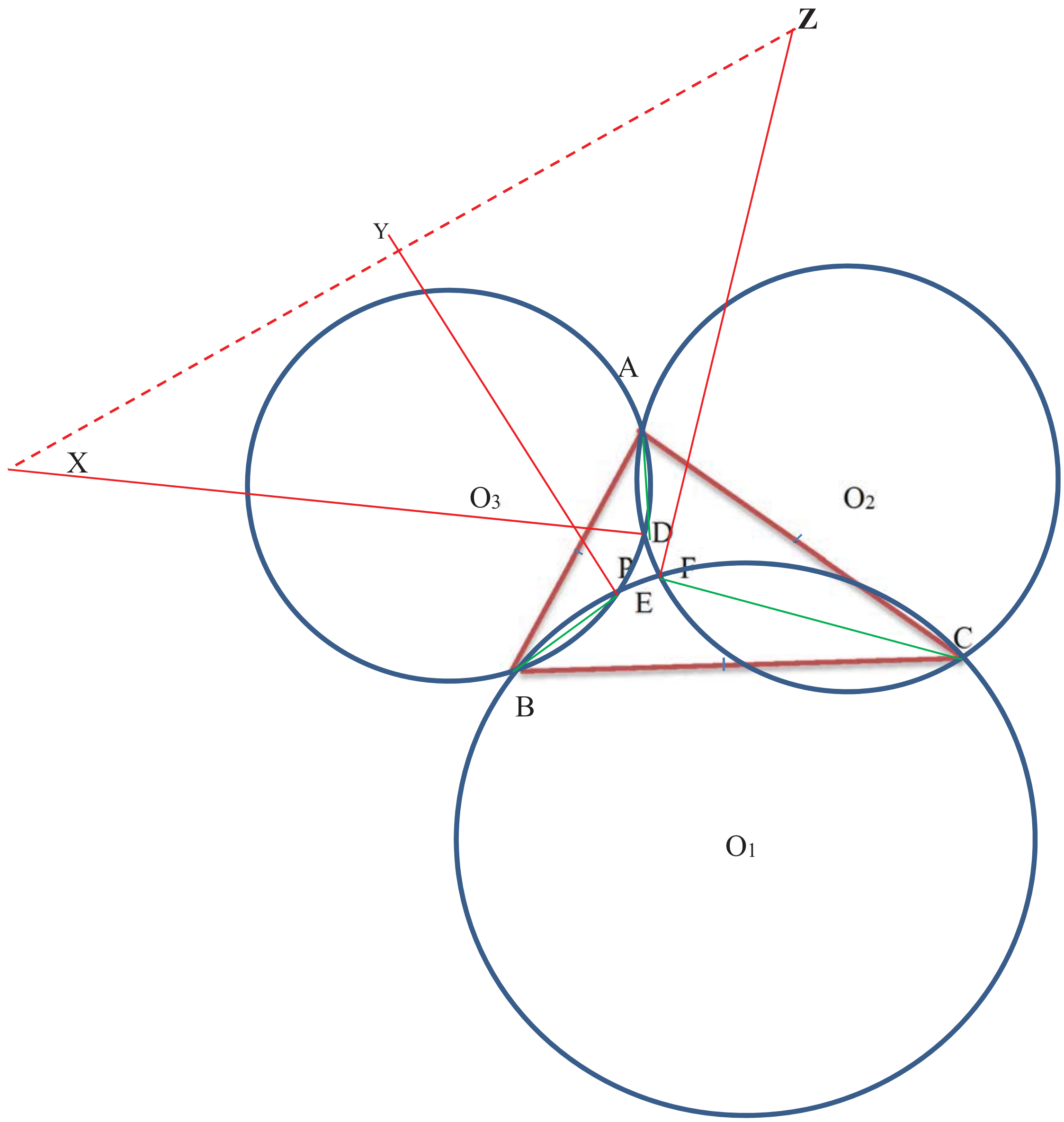

Fig 1.

In triangle ABC, the cevians AD, BE, CF being concurrent, we can use for them the trigonometrically form of the Ceva's theorem as follows

$$\frac{\sin \widehat{BAD}}{\sin \widehat{CAD}} \cdot \frac{\sin \widehat{CBE}}{\sin \widehat{ABE}} \cdot \frac{\sin \widehat{ACF}}{\sin \widehat{BCF}} = 1 \tag{3}$$

The relations (2) and (3) lead to

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

Relation, which in conformity with Menelaus theorem proves the collinearity of the points X, Y, Z.

Definition 1

Two triangles ABC and $A'B'C'$ are called orthological if the perpendiculars constructed from A on $B'C'$, from B on $C'A'$ and from C on $A'B'$ are concurrent. The concurrency point of these perpendiculars is called the orthological center of the triangle ABC in rapport to triangle $A'B'C'$.

Theorem 2 (The theorem of orthological triangle of J. Steiner)

If the triangle ABC is orthological with the triangle $A'B'C'$, then the triangle $A'B'C'$ is also orthological in rapport to triangle ABC.

For the proof of this theorem we recommend [1].

Observation

A given triangle and its contact triangle are orthological triangles with the same orthological center. Their common orthological center is the center of the inscribed circle of the given triangle.

Definition 3

Two triangles ABC and $A'B'C'$ are called homological if and only if the lines AA', BB', CC' are concurrent. The congruency point is called the homological center of the given triangles.

Theorem 3 (Desargues – 1636)

If ABC and $A'B'C'$ are two homological triangles, then the lines $(BC, B'C');\ (CA, C'A');\ (AB, A'B')$ are concurrent respectively in the points X, Y, Z, and these points are collinear. The line that contains the points X, Y, Z is called the homological axis of the triangles ABC and $A'B'C'$.

For the proof of Desargues theorem see [3].

Theorem 4

Two orthological triangles that have a common orthological center are homological triangles.

Lemma 1

Let ABC and $A'B'C'$ two orthological triangles. The orthogonal projections of the vertexes B and C on the sides $A'C'$ respectively $A'B'$ are concyclic.

Proof

We note with E, F the orthogonal projections f the vertexes B and C on $A'C'$ respectively $A'B'$ (see Fig. 2). Also, we'll note O the common orthological center of the orthological triangles ABC and $A'B'C'$ and $\{B''\} = EO \cap AC$, $\{C''\} = FO \cap AB$. In the triangle $A'B''C''$, O being the intersection of the heights constructed from B'', C'', is the orthocenter of this triangle, consequently, it results that $A'O \perp B''C''$. On the other side $A'O \perp BC$; we obtain, therefore that $B''C'' \parallel BC$. Taking into consideration that EF and $B''C''$ are antiparallel in rapport to $A'B'$ and $A'C'$, we obtain that EF is antiparallel with BC, fact that shows that the quadrilateral $BCFE$ is inscribable.

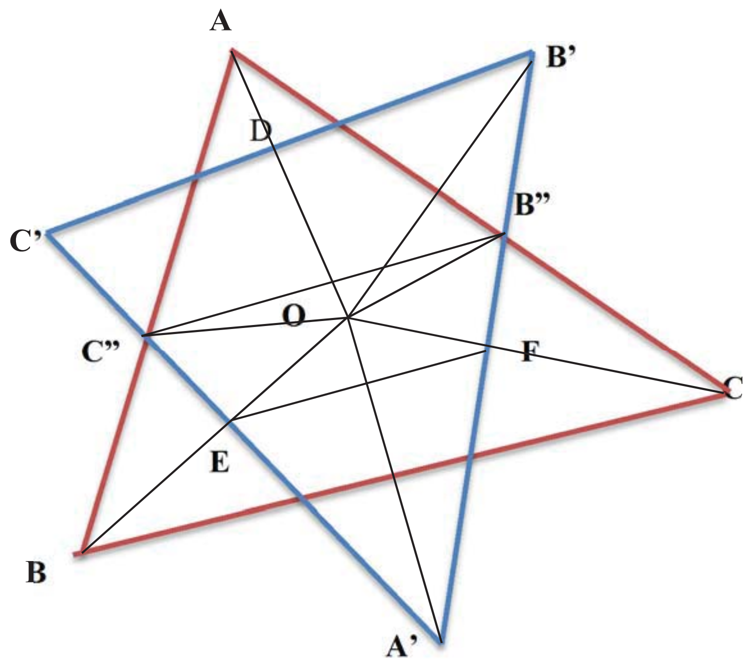

Fig. 2

Observation

If we denote with D the projection of A on $B'C'$, similarly, it will result that the points A, D, F, C respectively A, D, E, B are concyclic.

Proof of Theorem 4

The quadrilaterals $BCFE$, $CFDA$, $ADEB$ being inscribable, it result that their circumscribed circles satisfy the Dergiades theorem (Fig. 2). Applying this theorem it results that the pairs of lines $(BC, B'C')$; $(CA, C'A')$; $(AB, A'B')$ intersect in the collinear points X, Y, Z, respectively. Using the reciprocal theorem of Desargues, it result that the lines AA', BB', CC' are concurrent and consequently the triangles ABC and $A'B'C'$ are homological.

Observations

1 Triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to quadrilaterals $BCFE$, $CFDA$, $ADEB$ and the triangle ABC are orthological triangles.
The orthological centers are the points P - the radical center of the circles $(O_1),(O_2),(O_3)$ and O - the center of the circumscribed circle of the triangle ABC.

2 The triangles $O_1O_2O_3$ and DEF (formed by the projections of the vertexes A, B, C on the sides of the triangle $A'B'C'$) are orthological. The orthological centers are the center of the circumscribed circle to triangle DEF and P the radical center of the circles $(O_1),(O_2),(O_3)$.

Indeed, the perpendiculars constructed from O_1, O_2, O_3 on EF, FD, DA respectively are the mediators of these segments and, therefore, are concurrent in the center of the circumscribed circle to triangle DEF, and the perpendiculars constructed from D, E, F on the sides of the triangle $O_1O_2O_3$ are the common strings AD, BE, CF, which, we observed above, are concurrent in the radical center P of the circles with the centers in O_1, O_2, O_3.

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Mixt-Linear Circles Adjointly Ex-Inscribed Associated to a Triangle

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Abstract

In [1] we introduced the mixt-linear circles adjointly inscribed associated to a triangle, with emphasizes on some of their properties. Also, we've mentioned about mixt-linear circles adjointly ex-inscribed associated to a triangle.

In this article we'll show several basic properties of the mixt-linear circles adjointly ex-inscribed associate to a triangle.

Definition 1

We define a mixt-linear circle adjointly ex-inscribed associated to a triangle, the circle tangent exterior to the circle circumscribed to a triangle in one of the vertexes of the triangle, and tangent to the opposite side of the vertex of that triangle.

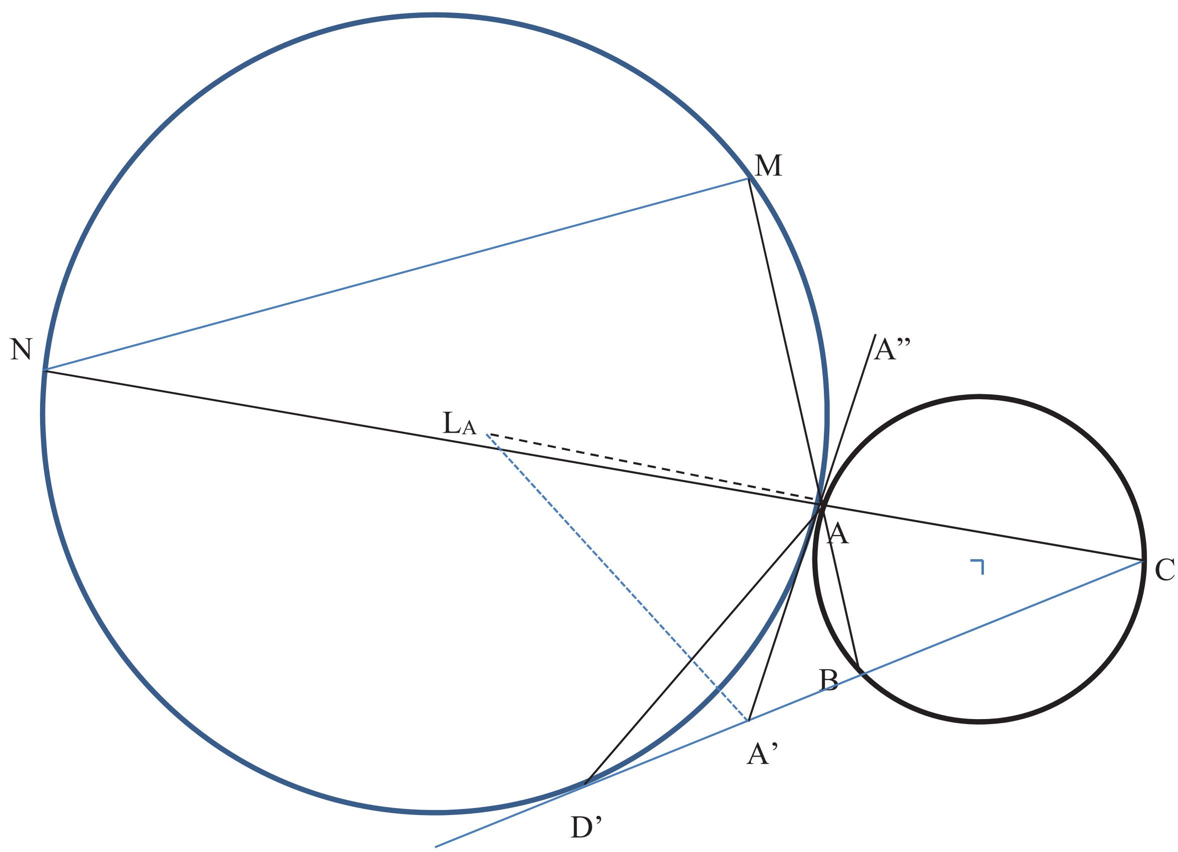

Fig. 1

Observation

In Fig.1 we constructed the mixt-linear circle adjointly ex-inscribed to triangle ABC, which is tangent in A to the circumscribed circle of triangle ABC, and tangent to the side BC. Will call this the A-mixt-linear circle adjointly ex-inscribed to triangle ABC. We note L_A the center of this circle.

Remark

In general, for a triangle exists three mixt-linear circles adjointly ex-inscribed. If the triangle ABC is isosceles with the base BC, then we cannot talk about mixt-linear circles adjointly ex-inscribed associated to the isosceles triangle.

Proposition 1

The tangency point with the side BC of the A-mixt-linear circle adjointly ex-inscribed associated to the triangle is the leg of the of the external bisectrix of the angle BAC

Proof

Let D' the contact point with the side BC of the A-mixt-linear circle adjointly ex-inscribed and let A' the intersection of the tangent in the point A to the circumscribed circle to the triangle ABC with BC (see Fig. 1)

We have

$$m\left(\sphericalangle AA'B\right)=\frac{1}{2}\left[m\left(\hat{B}\right)-m\left(\hat{C}\right)\right],$$

(we supposed that $m\left(\hat{B}\right)>m\left(\hat{C}\right)$). The tangents $AA', A'D'$ to the A-mixt-linear circle adjointly ex-inscribed are equal, therefore

$$m\left(\sphericalangle D'AA'\right)=\frac{1}{4}m\left(\hat{B}-\hat{C}\right).$$

Because

$$m\left(\sphericalangle A'AB\right)=\frac{1}{2}m\left(\hat{C}\right)$$

we obtain that

$$m\left(\sphericalangle D'AB\right)=\frac{1}{2}\left[m\left(\hat{B}\right)+m\left(\hat{C}\right)\right]$$

This relation shows that D' is the leg of the external bisectrix of the angle BAC.

Proposition 2

The A-mixt-linear circle adjointly ex-inscribed to triangle ABC intersects the sides AB, AC, respectively, in two points of a cord which is parallel to BC.

Proof

We'll note with M, N the intersection points with AB respectively AC of the A-mixt-linear circle adjointly ex-inscribed. We have $\sphericalangle BCA\equiv\sphericalangle BAA'$ and $\sphericalangle A'AB\equiv\sphericalangle A''AM$ (see Fig.1).

Because $\sphericalangle A''AM=\sphericalangle ANM$, we obtain $\sphericalangle ANM\equiv\sphericalangle ACB$ which implies that MN is parallel to BC.

Proposition 3

The radius R_A of the A-mixt-linear circle adjointly ex-inscribed to triangle ABC is given by the following formula

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}$$

Proof

The sinus theorem in the triangle AMN implies

$$R_A = \frac{MN}{2\sin A}$$

We observe that the triangles AMN and ABC are similar; it results that

$$\frac{MN}{a} = \frac{AM}{c}.$$

Considering the power of the point B in rapport to the A-mixt-linear circle adjointly ex-inscribed of triangle ABC, we obtain

$$BA \cdot BM = BD'^2.$$

From the theorem of the external bisectrix we have $\frac{D'B}{D'C} = \frac{c}{b}$ from which we retain $D'B = \frac{ac}{b-c}$. We obtain then $BM = \frac{a^2 c}{(b-c)^2}$, therefore

$$AM = \frac{c(a-b+c)(a+b-c)}{(b-c)^2} = \frac{4c(p-b)(p-c)}{(b-c)^2}$$

and

$$MN = \frac{4a(p-b)(p-c)}{(b-c)^2}$$

From the sinus theorem applied in the triangle ABC results that $\frac{a}{2\sin A} = R$ and we obtain that

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}.$$

Remark

If we note $P \in L_A A' \cap AD'$ and $AD' = l_a'$ (the length of the exterior bisectrix constructed from A) in triangle $L_A P A'$, we find

$$R_A = \frac{l_a'}{2\sin\frac{B-C}{2}}.$$

We'll remind here several results needed for the remaining of this presentation.

Definition 2

We define an adjointly circle of triangle ABC a circle which contains two vertexes of the triangle and in one of these vertexes is tangent to the respective side.

Theorem 1

The adjointly circles $A\overline{B}, B\overline{C}, C\overline{A}$ have a common point Ω; similarly, the circles $B\overline{A}, C\overline{B}, A\overline{C}$ have a common point Ω'.

The points Ω and Ω' are called the points of Brocard: Ω is the direct point of Brocard and Ω' is called the retrograde point.

The points Ω and Ω' are conjugate isogonal

$$\measuredangle \Omega AB = \measuredangle \Omega BC = \measuredangle \Omega CA = \omega$$

$$\measuredangle \Omega' AC = \measuredangle \Omega' CB = \measuredangle \Omega' BA = \omega$$

(see Fig. 2).

The angle ω is called the Brocard angle. More information can be found in [3].

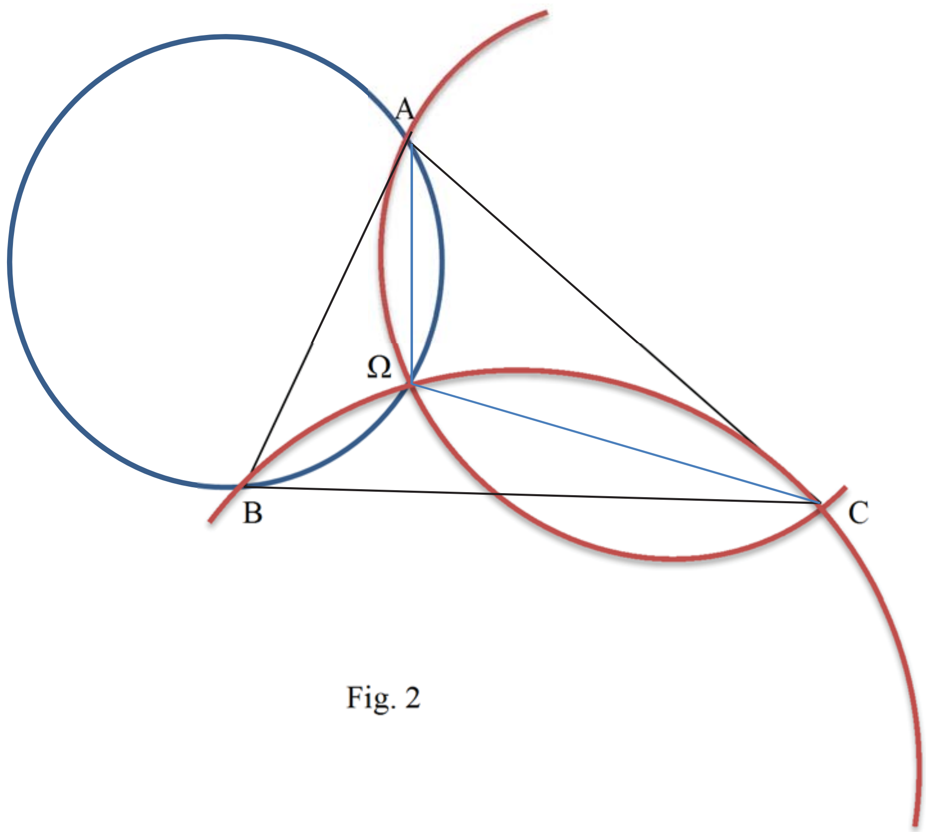

Fig. 2

Proposition 4

In triangle ABC in which D' is the leg of the external bisectrix of the angle BAC, the A-mixt-linear circle adjointly ex-inscribed to triangle ABC is an adjointly circle of triangles $AD'B, AD'C$.

Proposition 5

In a triangle ABC in which D' is the leg of the external bisectrix of the angle BAC, the direct points of Brocard corresponding to triangles $AD'B$, $AD'C$, A, D' are concyclic.

The following theorems show remarkable properties of the mixt-linear circles adjointly ex-inscribed associated to a triangle ABC.

Theorem 2

The triangle $L_A L_B L_C$ determined by the centers of the mixt-linear circles adjointly ex-inscribed to triangle ABC and the tangential triangle $T_a T_b T_c$ corresponding to ABC are orthological. Their orthological centers are O the center of the circumscribed circle to triangle ABC and the radical center of the mixt-linear circles adjointly ex-inscribed associated to triangle ABC.

Proof

The perpendiculars constructed from L_A, L_B, L_C on the corresponding sides of the tangential triangle contain the radiuses OA, OB, OC respectively of the circumscribed circle.

Consequently, O is the orthological center of triangles $L_A L_B L_C$ and $T_a T_b T_c$.

In accordance to the theorem of orthological triangles and the perpendiculars constructed from T_a, T_b, T_c respectively on the sides of the triangle $L_A L_B L_C$ are concurrent.

The point T_a belongs to the radical axis of the circumscribed circles to triangle ABC and the C-mixt-linear circle adjointly ex-inscribed to triangle ABC (belongs to the common tangent constructed in C to these circles).

On the other side T_a belongs to the radical axis of the B and C-mixt-linear circle adjointly ex-inscribed, which means that the perpendicular constructed from T_a on the $L_B L_C$ centers line passes through the radical center of the mixt-linear circle adjointly ex-inscribed associated to the triangle; which is the second orthological center of the considered triangles.

Proposition 6

The triangle $L_a L_b L_c$ (determined by the centers of the mixt-linear circles adjointly inscribed associated to the triangle ABC) and the triangle $L_A L_B L_C$ (determined by the centers of the mixt-linear circles adjointly ex-inscribed associated to the triangle ABC) are homological. The homological center is the point O, which is the center of the circumscribed circle of triangle ABC.

The proof results from the fact that the points L_A, A, L_a, O are collinear. Also, L_B, B, L_b, O and L_C, C, L_c, O are collinear.

Definition 3

Given three circles of different centers, we define their Apollonius circle as each of the circles simultaneous tangent to three given circles.

Observation

The circumscribed circle to the triangle ABC is the Apollonius circle for the mixt-linear circles adjointly ex-inscribed associated to ABC.

Theorem 3

The Apollonius circle which has in its interior the mixt-linear circles adjointly ex-inscribed to triangle ABC is tangent with them in the points T_1, T_2, T_3 respectively. The lines AT_1, BT_2, CT_3 are concurrent.

Proof

We'll use the D'Alembert theorem: Three circles non-congruent whose centers are not collinear have their six homothetic centers placed on four lines, three on each line.

The vertex A is the homothety inverse center of the circumscribed circle (O) and of the A-mixt-linear circle adjointly ex-inscribed (L_A); T_1 is the direct homothety center of the Apollonius circle which is tangent to the mixt-linear circles adjointly ex-inscribed and of circle (L_A), and J is the center of the direct homothety of the Apollonius circle and of the circumscribed circle (O).

According to D'Alembert theorem, it results that the points A, J, T_1 are collinear. Similarly is shown that the points B, J, T_2 and C, J, T_3 are collinear.

Consequently, J is the concurrency point of the lines AT_1, BT_2, CT_3.

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A PROPERTY OF THE CIRCUMSCRIBED OCTAGON

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Abstract

In this article we’ll obtain through the duality method a property in relation to the contact cords of the opposite sides of a circumscribable octagon.

In an inscribed hexagon the following theorem proved by Blaise Pascal in 1640 is true.

Theorem 1 (Blaise Pascal)

The opposite sides of a hexagon inscribed in a circle intersect in collinear points.

To prove the Pascal theorem one may use [1].

In [2] there is a discussion that the Pascal’s theorem will be also true if two or more pairs of vertexes of the hexagon coincide. In this case, for example the side AB for $B \to A$ must be substituted with the tangent in A. For example we suppose that two pairs of vertexes coincide. The hexagon $AA'BCC'D$ for A’→ A, C’→ C becomes the inscribed quadrilateral $ABCD$. This quadrilateral viewed as a degenerated hexagon of sides AB, BC, CC' → the tangent in $C, C'D \to CD, D'A \to DA, AA'$ → the tangent in A and the Pascal theorem leads to:

Theorem 2

In an inscribed quadrilateral the opposite sides and the tangents in the opposite vertexes intersect in four collinear points.

Remark 1

In figure 1 is presented the corresponding configuration of theorem 2.

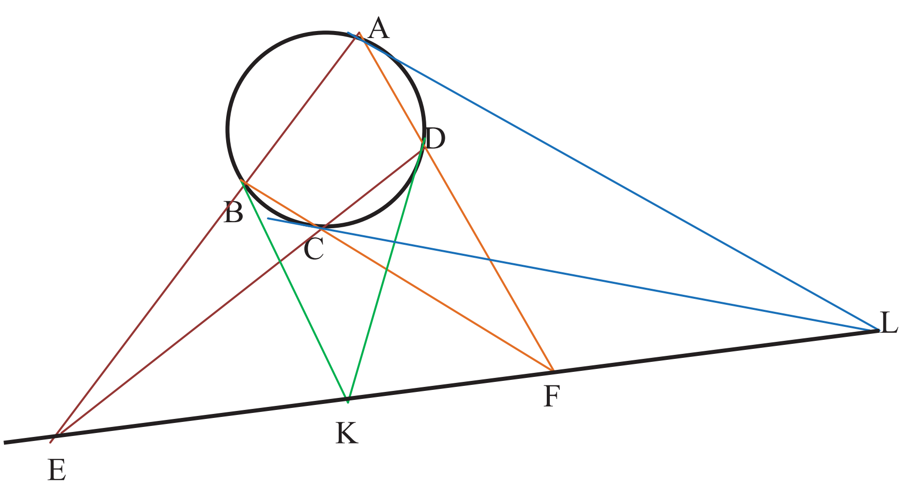

Fig. 1

For the tangents constructed in B and D the property is also true if we consider the $ABCD$ as a degenerated hexagon $ABB'CDD'A$.

Theorem 3

In an inscribed octagon the four cords determined by the contact points with the circle of the opposite sides are concurrent.

Proof

We'll transform through reciprocal polar the configuration from figure 1. To point E will correspond, through this transformation the line determined by the tangent points with the circle of the tangents constructed from E (its polar). To point K corresponds the side BD.

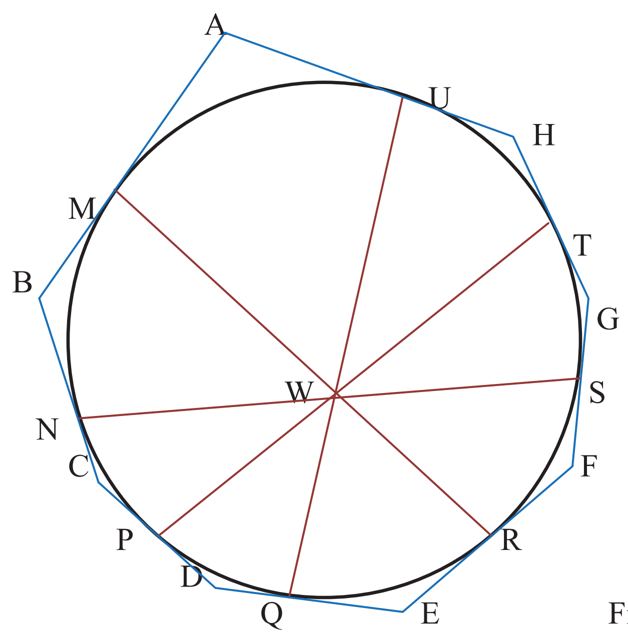

Fig.2

To point F corresponds the line determined by the contact points of the tangents constructed from F to the circle. To point L corresponds its polar AC. To point A corresponds, by duality, the tangent AL, also to points B, C, D correspond the tangents BK, CL, DK. These four tangents together with the tangents constructed from E and F (also four) will contain the sides of an octagon circumscribed to the given circle.

In this octagon (AC) and (BD) will connect the contact points of two pairs of opposite sides with the circle; the other two lines determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points E and F. Because the polar transformation through reciprocal polar leads to the fact that to collinear points correspond concurrent lines; the points' polar E, K, F, L are concurrent; these lines are the cords to which the theorem refers to.

Remark 2

In figure 2 we represented an octagon circumscribed $ABCDEFGH$. As it can be seen the cords MR, NS, PT, QU are concurrent in the point W.

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From Newton's Theorem to a Theorem of the Inscribable Octagon

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In this article we'll prove the Newton's theorem relative to the circumscribed quadrilateral, we'll transform it through duality, and we obtain another theorem which is true for an inscribable quadrilateral, which transformed through duality, we'll obtain a theorem which is true for a circumscribable octagon.

Theorem 1 (I. Newton)
In a circumscribable quadrilateral its diagonals and the cords determined by the contact points of the opposite sides of the quadrilateral with the circumscribed circle are four concurrent lines.

Proof

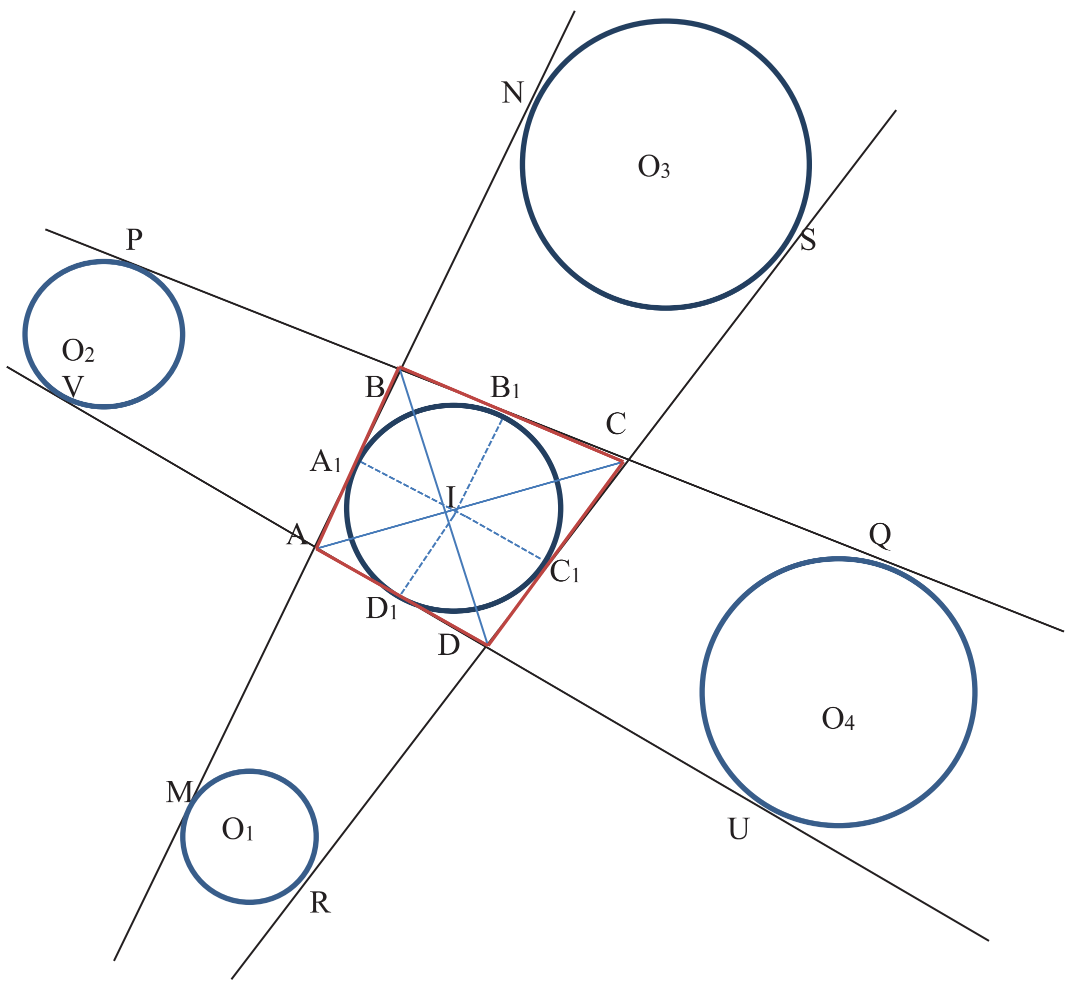

Fig. 1

We constructed the circles O_1, O_2, O_3, O_4 tangent to the extensions of the quadrilateral $ABCD$ such that

$$A_1M = A_1N = B_1P = B_1Q = C_1R = C_1S = D_1U = D_1V$$

See Fig. 1.

From $A_1M = A_1N = C_1R = C_1S$ it results that the points A_1 and C_1 have equal powers in relation to the circles O_1 and O_3, therefore A_1C_1 is the radical axis of these circles. Similarly B_1D_1 is the radical axis of the circles O_2 and O_4.

Let $I \in A_1C_1 \bigcap B_1D_1$. The point I has equal powers in rapport to circles O_1, O_2, O_3, O_4. Because $BA_1 = BB_1$ from $B_1P = A_1N$ it results that $BP = BN$, similarly, from $DD_1 = DC_1$ and $D_1V = C_1S$ it results that $DV = DS$, therefore B and D have equal powers in rapport with the circles O_3 and O_4, which shows that BD is the radical axis of these circles. Consequently, $I \in BD$, similarly it results that $I \in AC$, and the proof is complete.

Theorem 2.

In an inscribed quadrilateral in which the opposite sides intersect, the intersection points of the tangents constructed to the circumscribed circle with the opposite vertexes and the points of intersection of the opposite sides are collinear.

Proof

We'll prove this theorem applying the configuration from the Newton theorem, o transformation through duality in rapport with the circle inscribed in the quadrilateral. Through this transformation to the lines AB, BC, CD, DA will correspond, respectively, the points A_1, B_1, C_1, D_1 their pols. Also to the lines $A_1B_1, B_1C_1, C_1D_1, D_1A_1$ correspond, respectively, the points B, C, D, A. We note $X \in AB \bigcap CD$ and $Y \in AD \bigcap BC$, these points correspond, through the considered duality, to the lines A_1C_1 respectively B_1D_1. If $I \in A_1C_1 \bigcap B_1D_1$ then to the point I corresponds line XY, its polar.

To line BD corresponds the point $Z \in A_1D_1 \bigcap C_1B_1$.

To line AC corresponds the point $T \in A_1D_1 \bigcap C_1B_1$.

To point $\{I\} = BD \bigcap AC$ corresponds its polar ZT.

We noticed that to the point I corresponds the line XY, consequently the points X, Y, Z, T are collinear.

We obtained that the quadrilateral $A_1B_1C_1D_1$ inscribed in a circle has the property that if $A_1D_1 \bigcap C_1B_1 = \{Z\}$, $A_1D_1 \bigcap C_1B_1 = \{T\}$, the tangent in A_1 and the tangent in C_1 intersect in the point X; the tangent in B_1 and the tangent in D_1 intersect in Y, then X, Y, Z, T are collinear (see Fig. 2).

Theorem 3.

In a circumscribed octagon, the four cords, determined by the octagon's contact points with the circle of the octagon opposite sides, are concurrent.

Proof

We'll transform through reciprocal polar the configuration in figure 3.

To point Z corresponds through this transformation the line determined by the tangency points with the circle of the tangents constructed from Z - its polar; to the point Y it corresponds the line determined by the contact points of the tangents constructed from T at the circle; to the point X corresponds its polar A_1C_1.

To point A_1 corresponds through duality the tangent A_1X, also to the points B_1, C_1, D_1 correspond the tangents B_1Y, C_1T, D_1Z.

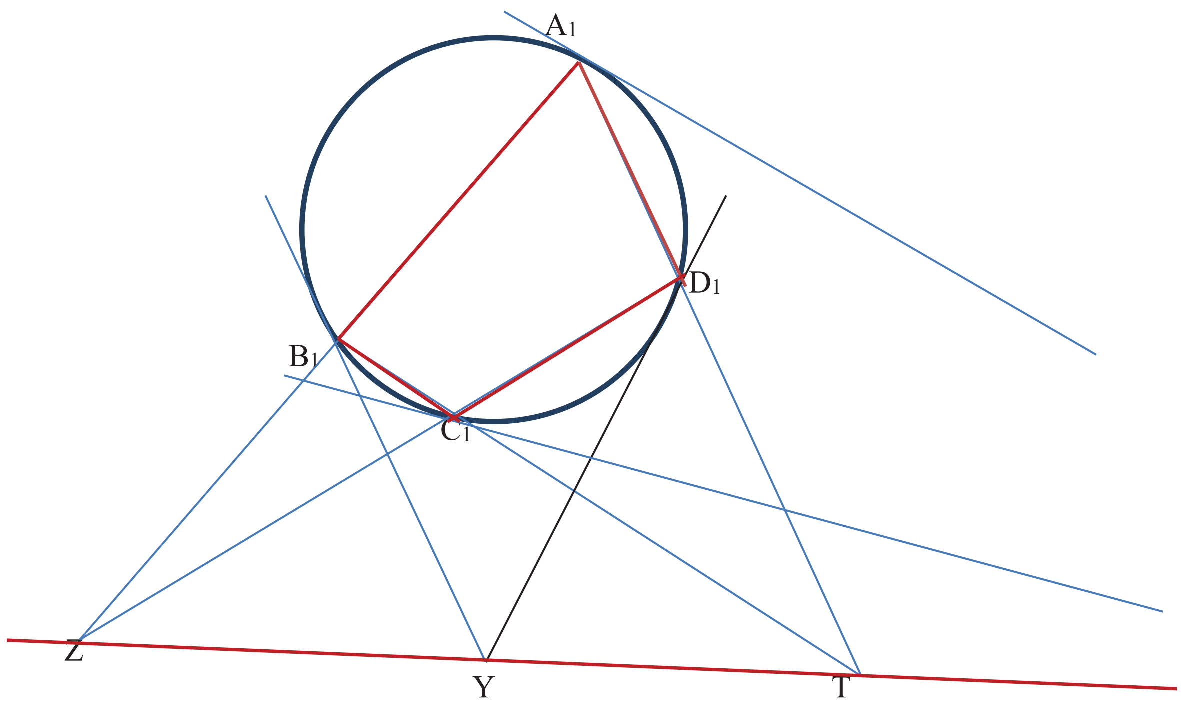

Fig. 2

These four tangents together with the tangents constructed from X and Y (also four) will contain the sides of an octagon circumscribed to the given circle.

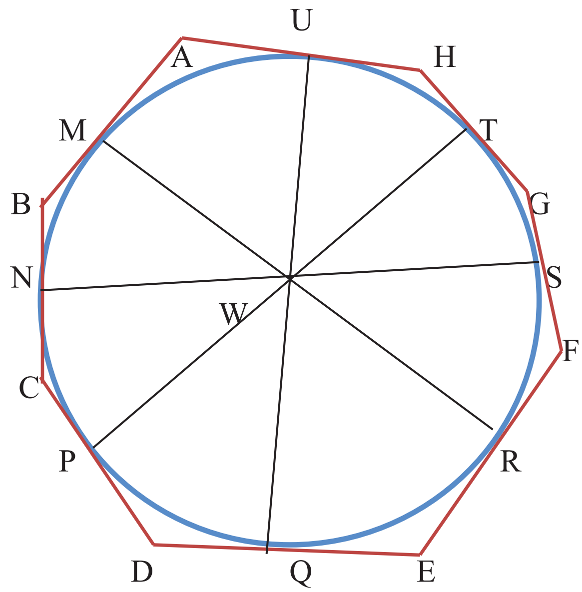

Fig .3

In this octagon A_1C_1 and B_1D_1 will connect the contact points of two pairs of sides opposed to the circle, the other two cords determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points Z and T.

Because the transformation through reciprocal polar will make that to collinear points will correspond concurrent lines, these lines are the cords from our initial statement.

Observation

In figure 3 we represented an octagon $ABCDEFGH$ circumscribed to a circle.

As it can be observed the cords MR, NS, PT, QU are concurrent in a point notated W

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TRIPLETS OF TRI-HOMOLOGICAL TRIANGLES

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In this article will prove some theorems in relation to the triplets of homological triangles two by two. These theorems will be used later to build triplets of triangles two by two tri-homological.

I Theorems on the triplets of homological triangles

Theorem 1

Two triangles are homological two by two and have a common homological center (their homological centers coincide) then their homological axes are concurrent.

Proof

Let's consider the homological triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ whose common homological center is O (see figure 1.)

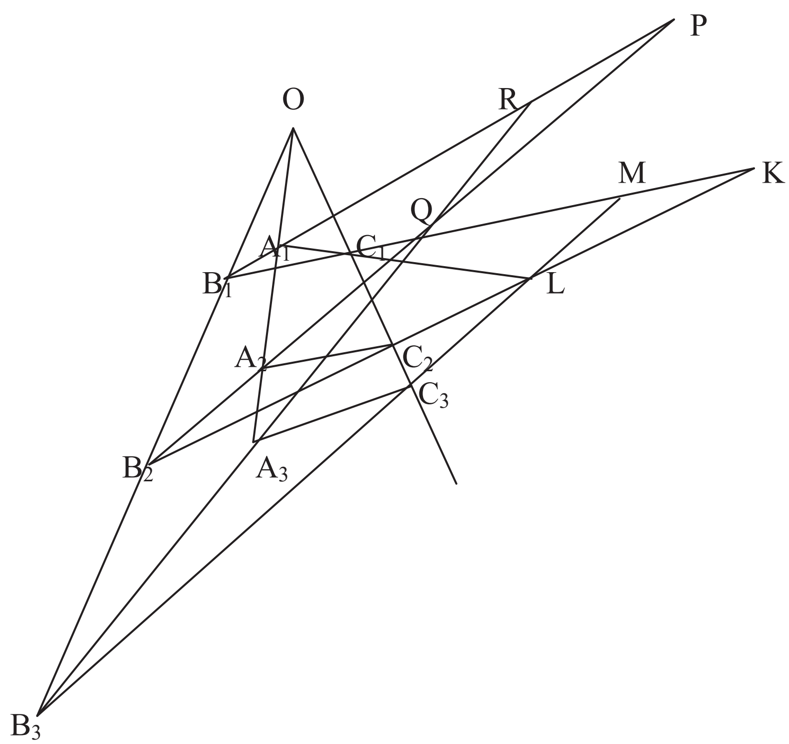

Fig. 1

We consider the triangle formed by the intersections of the lines: A_1B_1, A_2B_2, A_3B_3 and we note it PQR and the triangle formed by the intersection of the lines B_1C_1, B_2C_2, B_3C_3 and we'll note it KLM .

We observe that $PR \cap KM = \{B_1\}, RQ \cap ML = \{B_2\}$, PQ$\cap$ KL=$\{B_3\}$ and because B_1, B_2, B_3 are collinear it results, according to the Desargues reciprocal theorem that the triangles PQR and KLM are homological, therefore PK, RM, QL are concurrent lines.

The line PK is the homological axes of triangles $A_1B_1C_1$ and $A_2B_2C_2$, the line RM is the homological axis for triangles $A_1B_1C_1$ and $A_3B_3C_3$, and the line QL is the homological axis for triangles $A_2B_2C_2$ and $A_3B_3C_3$, which proves the theorem.

Remark 1

Another proof of this theorem can be done using the spatial vision; if we imagine figure 1 as being the correspondent of a spatial figure, we notice that the planes $(A_1B_1C_1)$ and $(A_2B_2C_2)$ have in common the linePK, similarly the planes $(A_1B_1C_1)$ and $(A_3B_3C_3)$ have in common the line QL. If $\{O'\} = PK \cap$LQ then O' will be in the plane $(A_2B_2C_2)$ and in the plane $(A_3B_3C_3)$, but these planes intersect by the line RM, therefore O' belongs to this line as well. The lines PK, RM, QL are the homological axes of the given triangles and therefore these are concurrent in O'.

Theorem 2

If three triangles are homological two by two and have the same homological axis (their homological axes coincide) then their homological axes are collinear.

Proof

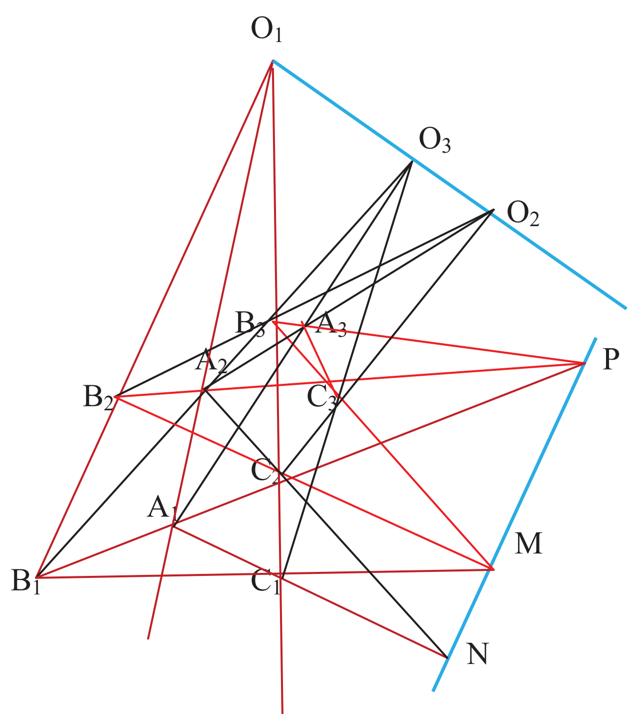

Fig. 2

Let's consider the homological triangles two by two $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$. We note M, N, P their common homological axis (see figure 2). We note O_1 the homological center of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, with O_2 the homological center of the triangles $A_2B_2C_2$ and $A_3B_3C_3$ and with O_3 the homological center of the triangles $A_3B_3C_3$ and $A_1B_1C_1$.

We consider the triangles $A_1A_2A_3$ and $B_1B_2B_3$, and we observe that these are homological because A_1B_1, A_2B_2, A_3B_3 intersect in the point P which is their homological center. The homological axis of these triangles is determined by the points

, $\{O_1\}=A_1A_2 \cap B_1B_2$, $\{O_2\} = A_2A_3 \cap B_2B_3$, $\{O_3\} = A_1A_3 \cap B_1B_3$

therefore the points O_1, O_2, O_3 are collinear and this concludes the proof of this theorem.

Theorem 3 (The reciprocal of theorem 2)

If three triangles are homological two by two and have their homological centers collinear, then these have the same homological axis.

Proof

We will use the triangles from figure 2. Let therefore O_1, O_2, O_3 the three homological collinear points. We consider the triangles $B_1B_2B_3$ and $C_1C_2C_3$, we observe that these admit as homological axis the line $O_1O_2O_3$.

Because

$$\{O_1\} = B_1B_2 \cap C_1C_2, \{O_2\} = B_2B_3 \cap C_2C_3, \{O_3\} = B_1B_3 \cap C_1C_3,$$

It results that these have as homological center the point $\{M\} = B_1C_1 \cap B_2C_2 \cap B_3C_3$.

Similarly for the triangles $A_1A_2A_3$ and $C_1C_2C_3$ have as homological axis $O_1O_2O_3$ and the homological center M. We also observe that the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are homological and $O_1O_2O_3$ is their homological axis, and their homological center is the point P. Applying the theorem 2, it results that the points M, N, P are collinear, and the reciprocal theorem is then proved.

Theorem 4 (The Veronese theorem)

If the triangles $A_1B_1C_1$, $A_2B_2C_2$ are homological and

$\{A_3\} = B_1C_2 \cap B_2C_1$, $\{B_3\} = A_1C_2 \cap A_2C_1$, $\{C_3\} = A_1B_2 \cap A_2B_1$

then the triangle $A_3B_3C_3$ is homological with each of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, and their homological centers are collinear.

Proof

Let O_1 be the homological center of triangles $A_1B_1C_1$ and $A_2B_2C_2$ (see figure 3) and A', B', C' their homological axis.

We observe that O_1 is a homological center also for the triangles $A_1B_1C_2$ and $A_2B_2C_1$. The homological axis of these triangles is C', A_3, B_3. Also O_1 is the homological center for the triangles

$B_1C_1A_2$ and $B_2C_2A_1$, it results that their homological axis is A', B_3, C_3

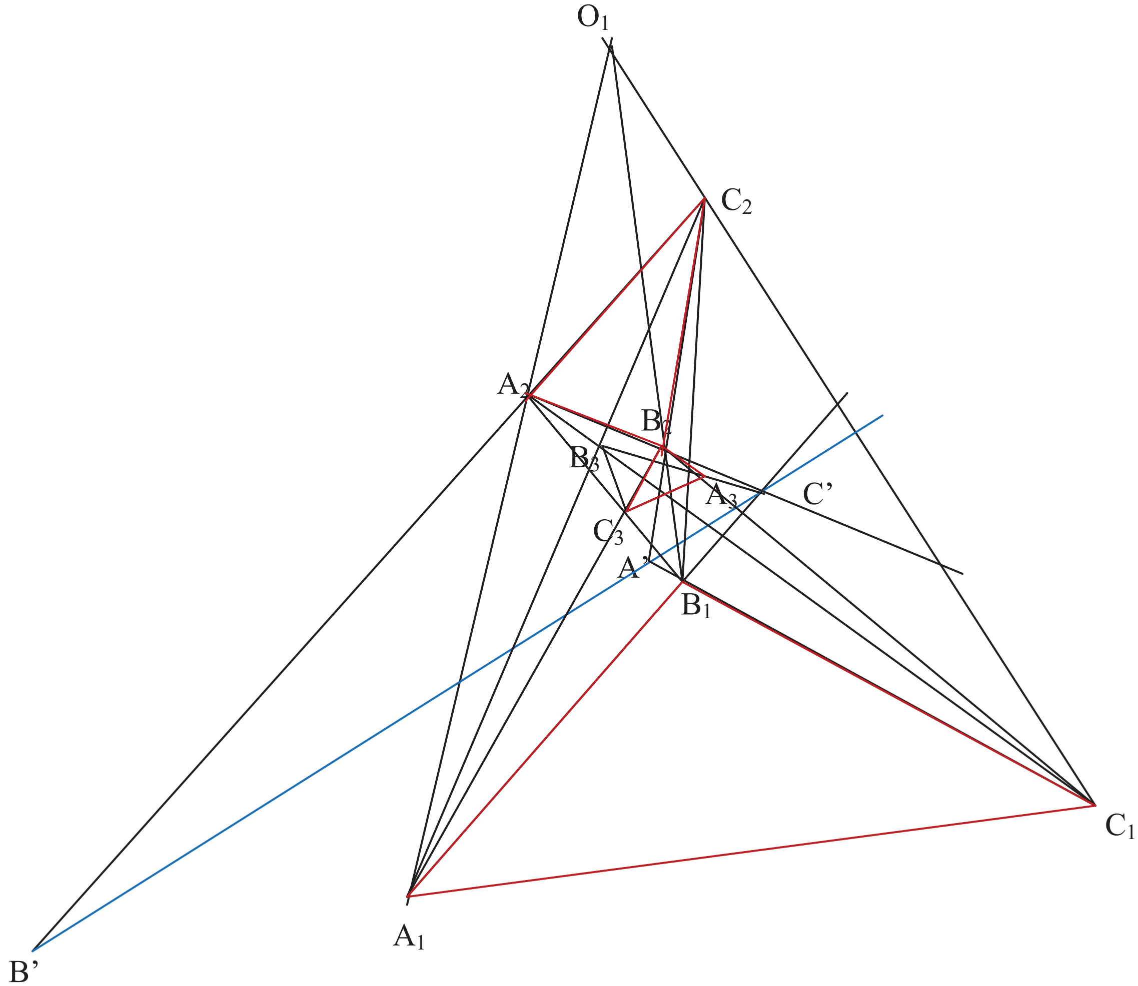

Fig. 3

Similarly, we obtain that the points B', A_3, C_3 are collinear, these being on a homological axis of triangle $C_1A_1B_2$ and $C_2A_2B_1$. The triplets of the collinear points (C', A_3, B_3), (B', A_3, C_3) and (A', B_3, C_3) show that the triangle $A_3B_3C_3$ is homological with triangle $A_1B_1C_1$ and with the triangle $A_2B_2C_2$.

The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are homological two by two and have the same homological axis A', B', C'. Using theorem 3, it results that their homological centers are collinear points.

II. Double-homological triangles

Definition 1

We say that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are double-homological or bi- homological if these are homological in two modes.

Theorem 5

Let's consider the triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that

$$B_1C_1 \cap B_2C_2 = \{P_1\}, B_1C_1 \cap A_2C_2 = \{Q_1\}, B_1C_1 \cap A_2B_2 = \{R_1\}$$

$$A_1C_1 \cap A_2C_2 = \{P_2\}, A_1C_1 \cap A_2B_2 = \{Q_2\}, A_1C_1 \cap B_2C_2 = \{R_2\}$$

$$A_1B_1 \cap A_2B_2 = \{P_3\}, A_1B_1 \cap B_2C_2 = \{Q_3\}, A_1B_1 \cap C_2A_2 = \{R_3\}$$

Then:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} \cdot \frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} \cdot \frac{R_1B_1 \cdot R_2C_1 \cdot R_3A_1}{R_1C_1 \cdot R_2A_1 \cdot R_3B_1} = 1 \qquad (1)$$

Proof

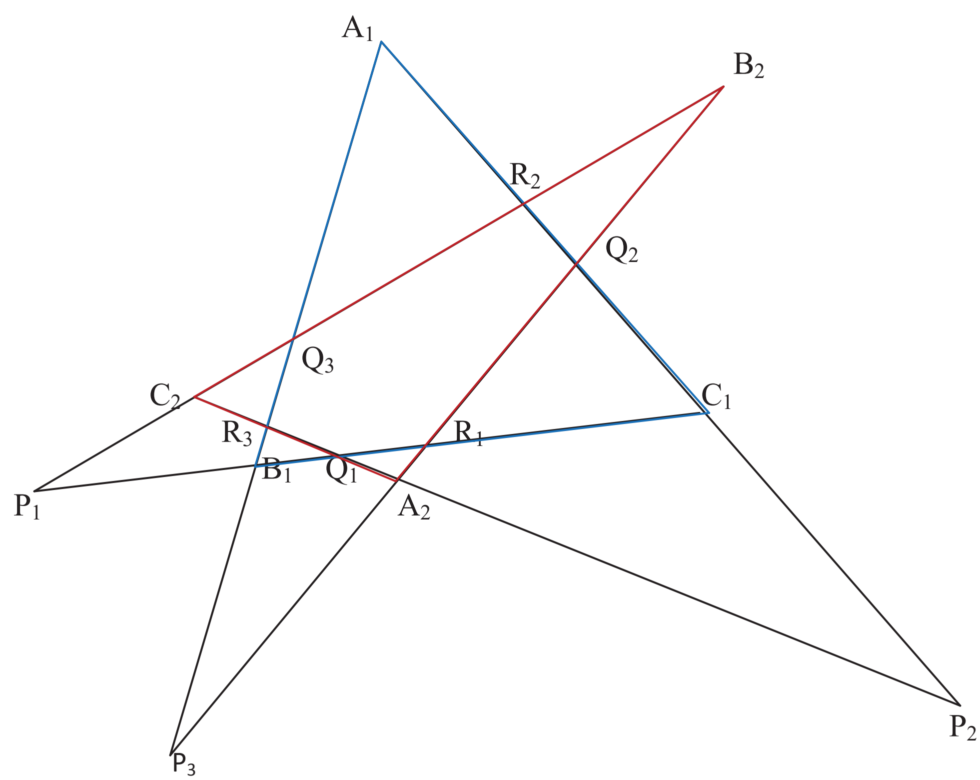

Fig. 4

We'll apply the Menelaus' theorem in the triangle $A_1B_1C_1$ for the transversals $P_1Q_3R_2$, $P_2Q_1R_3$, $P_3Q_2R_1$, (see figure 4).

We obtain

$$\frac{P_1B_1 \cdot R_2C_1 \cdot Q_3A_1}{P_1C_1 \cdot R_2A_1 \cdot Q_3B_1} = 1$$

$$\frac{P_2C_1 \cdot Q_1B_1 \cdot R_3A_1}{P_2A_1 \cdot Q_1C_1 \cdot R_3B_1} = 1$$

$$\frac{P_3A_1 \cdot R_1B_1 \cdot Q_2C_1}{P_3B_1 \cdot R_1C_1 \cdot Q_2A_1} = 1$$

Multiplying these relations side by side and re-arranging the factors, we obtain relation (1).

Theorem 6

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological (the lines A_1A_2, B_1B_2, C_1C_2 are concurrent) if and only if:

$$\frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} = \frac{R_1C_1 \cdot R_2A_1 \cdot R_3B_1}{R_1B_1 \cdot R_2C_1 \cdot R_3A_1} \tag{2}$$

Proof

Indeed, if A_1A_2, B_1B_2, C_1C_2 are concurrent then the points P_1, P_2, P_3 are collinear and the Menelaus' theorem for the transversal $P_1P_2P_3$ in the triangle $A_1B_1C_1$ gives:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} = 1 \tag{3}$$

This relation substituted in (1) leads to (2)

Reciprocal

If the relation (2) takes place then substituting it in the relation (1) we obtain (3) which shows that P_1, P_2, P_3 is the homology axis of the triangles $A_1B_1C_1$ and $A_2B_2C_2$.

Remark 2

If in relation (1) two fractions are equal to 1, then the third fraction will be equal to 1, and this leads to the following:

Theorem 7

If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological in two modes (are double-homological) then these are homological in three modes (are tri-homological).

Remark 3

The precedent theorem can be formulated in a different mod that will allow us to construct tri-homological triangles with a given triangle and of some tri-homological triangles.

Here is the theorem that will do this:

Theorem 8

(i) Let ABC a given triangle and P, Q two points in its plane such that BP intersects CQ in A_1, CP intersects AQ in B_1 and AP intersects BQ in C_1.

Then AA_1, BB_1, CC_1 intersect in a point R.

(ii) If $\cap CP = \{A_2\}, CQ \cap AP = \{B_2\}$, BP$\cap AQ = \{C_2\}$then the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ are two by two homological and their homological centers are collinear.

Proof

(i). From the way how we constructed the triangle $A_1B_1C_1$, we observe that ABC and $A_1B_1C_1$ are double homological, their homology centers being two given points P,Q (see figure 5). Using theorem 7 it results that the triangles ABC, $A_1B_1C_1$ are tri-homological, therefore AA_1, BB_1, CC_1 are concurrent in point noted R.

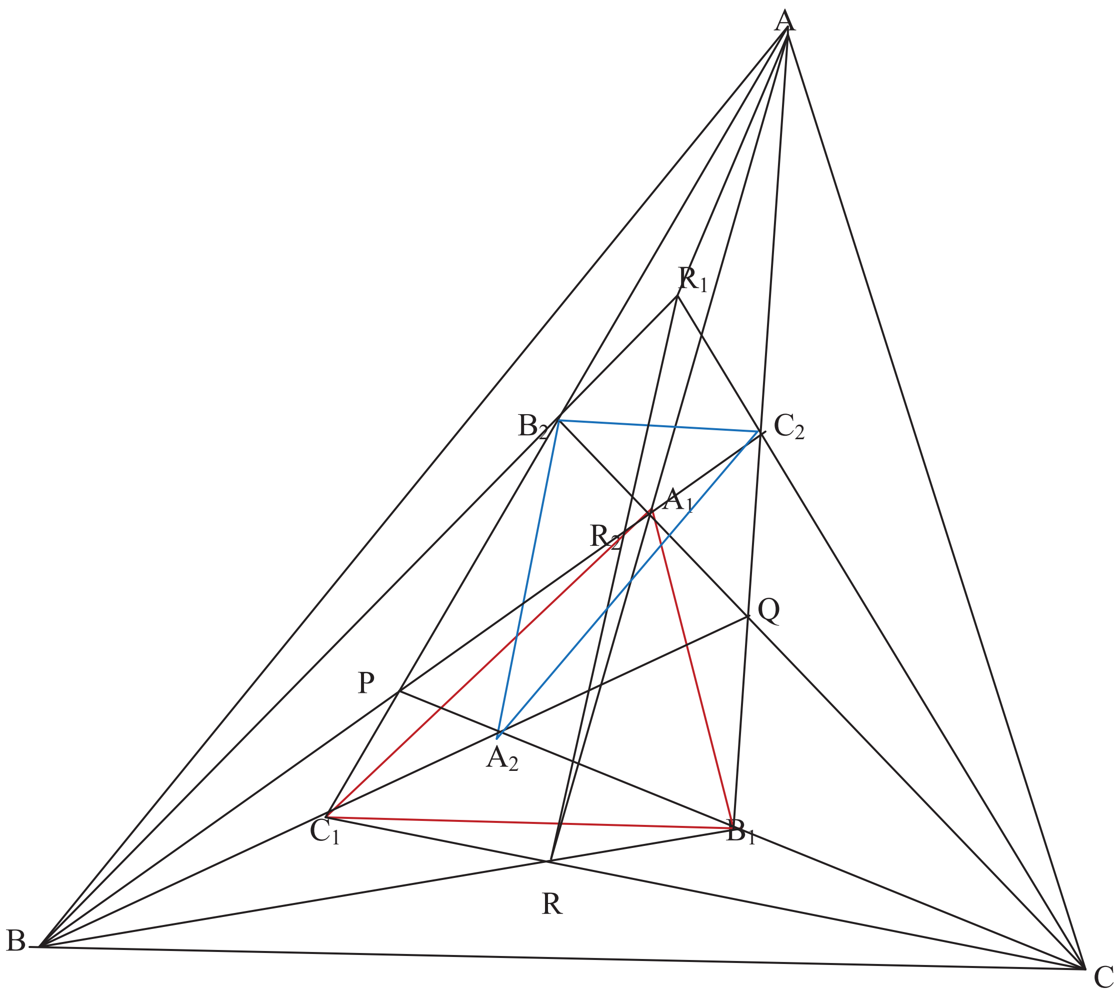

Fig. 5

(ii) The conclusion results by applying the Veronese theorem for the homological triangles ABC, $A_1B_1C_1$ that have as homological center the point R.

Remark 4

We observe that the triangles ABC and $A_2B_2C_2$ are bi-homological, their homological centers being the given points P,Q. It results that these are tri-homological and therefore AA_2, BB_2, CC_2 are concurrent in the third homological center of these triangle, which we'll note R_1.

Similarly we observe that the triangles $A_1B_1C_1,\ A_2B_2C_2$ are double homological with the homological centers P,Q; it results that these are tri-homological, therefore A_1A_2, B_2B_2, C_2C_2 are concurrent, their concurrence point being notated with R_2. In accordance to the Veronese's theorem, applied to any pair of triangles from the triplet $\left(ABC,\ A_1B_1C_1,\ A_2B_2C_2\right)$ we find that the points $R,\ R_1,\ R_2$ are collinear.

Remark 5

Considering the points P,R and making the same constructions as in theorem 8 we obtain the triangle $A_3B_3C_3$ which along with the triangles $ABC,\ A_1B_1C_1$ will form another triplet of triangles tri-homological two by two.

Remark 6

The theorem 8 provides us a process of getting a triplet of tri-homological triangles two by two beginning with a given triangle and from two given points in its plane. Therefore if we consider the triangle ABC and as given points the two points of Brocard $\Omega\Omega$ and Ω', the triangle $A_1B_1C_1$ constructed as in theorem 8 will be the first Brocard's triangle and we'll find that this is a theorem of J. Neuberg: the triangle ABC and the first Brocard triangle are tri-homological. The third homological center of these triangles is noted $\Omega''$and it is called the Borcard's third point and Ω'' is the isometric conjugate of the simedian center of the triangle ABC

Open problems

1) If T_1,T_2,T_3 are triangles in a plane, such that $\left(T_1,T_2\right)$ are tri-homological, $\left(T_2,T_3\right)$ are tri-homological, then are the $\left(T_1,T_3\right)$ tri-homological?

2) If T_1,T_2,T_3 are triangles in a plane such that $\left(T_1,T_2\right)$ are tri-homological, $\left(T_2,T_3\right)$ are tri-homological, $\left(T_1,T_3\right)$ are tri-homological and these pairs of triangles have in common two homological centers, then are the three remaining non-common homological centers collinear?

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A Class of Orthohomological Triangles

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Abstract.

In this article we propose to determine the triangles' class $A_iB_iC_i$ orthohomological with a given triangle ABC, inscribed în the triangle ABC ($A_i \in BC,\ B_i \in AC,\ C_i \in AB$).

We'll remind, here, the fact that if the triangle $A_iB_iC_i$ inscribed in ABC is orthohomologic with it, then the perpendiculars in A_i, B_i, respectively in C_i on BC, CA, respectively AB are concurrent in a point P_i (the orthological center of the given triangles), and the lines AA_i, BB_i, CC_i are concurrent in point (the homological center of the given triangles).

To find the triangles $A_iB_iC_i$, it will be sufficient to solve the following problem.

Problem.

Let's consider a point P_i in the plane of the triangle ABC and $A_iB_iC_i$ its pedal triangle. Determine the locus of point P_i such that the triangles ABC and $A_iB_iC_i$ to be homological.

Solution.

Let's consider the triangle ABC, $A(1,0,0),\ B(0,1,0),\ C(0,0,1)$, and the point $P_i(\alpha,\beta,\gamma),\ \alpha+\beta+\gamma=0$.

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp}\left(2a^2,\ -a^2-b^2+c^2,\ -a^2+b^2-c^2\right)$$

$$U_{CA}^{\perp}\left(-a^2-b^2+c^2,\ 2b^2,\ a^2-b^2-c^2\right)$$

$$U_{AB}^{\perp}\left(-a^2+b^2-c^2,\ a^2-b^2-c^2,\ 2c^2\right)$$

The coordinates of the vector $\overrightarrow{BC}$ are $(0,-1,1)$, and the line BC has the equation $x=0$.

The equation of the perpendicular raised from point P_i on BC is:

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0$$

We note $A_i(x,y,z)$, because $A_i \in BC$ we have:

$x=0$ and $y+z=1$.

The coordinates y and z of A_i can be found by solving the system of equations

$$\begin{cases} \begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0 \\ y+z=0 \end{cases}$$

We have:

$$y \times \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2-c^2 \end{vmatrix} = z \times \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix},$$

$$y\left[\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2\right]=z\left[\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2\right],$$

$$y+y\times\frac{\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2}=1,$$

$$y\times\frac{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2+\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2}=1,$$

$$y\times\frac{-2a^2\left(\alpha+\beta+\gamma\right)}{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2}=1,$$

it results

$$y=\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta$$

$$z=1-y=1-\beta-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)=\alpha+\gamma-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right).$$

Therefore,

$$A_i\left(0,\ \ \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right).$$

Similarly we find:

$$B_i\left(\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha,\ 0,\ \frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)+\gamma\right),$$

$$C_i\left(\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha,\ \frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta,\ \ 0\right).$$

We have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}}=-\frac{\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma}{\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta}=-\frac{\alpha c\cos B+\gamma a}{\alpha b\cos C+\beta a}$$

$$\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}}=-\frac{\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha}{\frac{\alpha}{2a^2}\left(-a^2+b^2+c^2\right)+\gamma}=-\frac{\beta a\cos C+\alpha b}{\beta c\cos A+\gamma b}.$$

$$\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}}=-\frac{\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta}{\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha}=-\frac{\gamma b\cos A+\beta c}{\gamma a\cos B+\alpha c}$$

(We took into consideration the cosine's theorem: $a^2=b^2+c^2-2bc\cos A$).
In conformity with Ceva's theorem, we have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}}\times\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}}\times\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}}=-1.$$

$$\left(a\gamma+\alpha c\cos B\right)\left(b\alpha+\beta a\cos C\right)\left(c\beta+\gamma b\cos A\right)=$$
$$=\left(a\beta+\alpha b\cos C\right)\left(b\gamma+\beta c\cos A\right)\left(c\alpha+\gamma a\cos B\right)$$
$$a\alpha\left(b^2\gamma^2-c^2\beta^2\right)\left(\cos A-\cos B\cos C\right)+b\beta\left(c^2\alpha^2-a^2\gamma^2\right)\left(\cos B-\cos A\cos C\right)+$$
$$+c\gamma\left(a^2\beta^2-b^2\alpha^2\right)\left(\cos C-\cos A\cos B\right)=0.$$

Dividing it by $a^2b^2c^2$, we obtain that the equation in barycentric coordinates of the locus $\mathfrak{L}$ of the point P_i is:

$$\frac{\alpha}{a}\left(\frac{\gamma^2}{c^2}-\frac{\beta^2}{b^2}\right)\left(\cos A-\cos B\cos C\right)+\frac{\beta}{b}\left(\frac{\alpha^2}{a^2}-\frac{\gamma^2}{c^2}\right)\left(\cos B-\cos A\cos C\right)+$$
$$+\frac{\gamma}{c}\left(\frac{\beta^2}{b^2}-\frac{\alpha^2}{a^2}\right)\left(\cos C-\cos A\cos B\right)=0.$$

We note $\overline{d}_A$, $\overline{d}_B$, $\overline{d}_C$ the distances oriented from the point P_i to the sides BC, CA respectively AB, and we have:

$$\frac{\alpha}{a}=\frac{\overline{d}_A}{2s},\ \frac{\beta}{b}=\frac{\overline{d}_B}{2s},\ \frac{\gamma}{c}=\frac{\overline{d}_C}{2s}.$$

The locus' $\mathfrak{L}$ equation can be written as follows:

$$\overline{d}_A\left(\overline{d}_C^2-\overline{d}_B^2\right)\left(\cos A-\cos B\cos C\right)+\overline{d}_B\left(\overline{d}_A^2-\overline{d}_C^2\right)\left(\cos B-\cos A\cos C\right)+$$
$$+\overline{d}_C\left(\overline{d}_B^2-\overline{d}_A^2\right)\left(\cos C-\cos A\cos B\right)=0$$

Remarks.

1. It is obvious that the triangle's ABC orthocenter belongs to locus $\mathcal{L}$. The orthic triangle and the triangle ABC are orthohomologic; a orthological center is the orthocenter H, which is the center of homology.
2. The center of the inscribed circle in the triangle ABC belongs to the locus $\mathcal{L}$, because $\overline{d}_A = \overline{d}_B = \overline{d}_C = r$ and thus locus' equation is quickly verified.

Theorem (Smarandache-Pătraşcu).

If a point P belongs to locus $\mathcal{L}$, then also its isogonal P' belongs to locus $\mathcal{L}$.

Proof.

Let $P(\alpha,\beta,\gamma)$ a point that verifies the locus' $\mathcal{L}$ equation, and $P'(\alpha',\beta',\gamma')$ its isogonal in the triangle ABC. It is known that $\frac{\alpha\alpha'}{a^2} = \frac{\beta\beta'}{b^2} = \frac{\gamma\gamma'}{c^2}$. We'll prove that $P' \in \mathcal{L}$, i.e.

$$\sum \frac{\alpha'}{a}\left(\frac{\gamma'^2}{c^2} - \frac{\beta'^2}{b^2}\right)(\cos A - \cos B\cos C) = 0$$

$$\sum \frac{\alpha'}{a}\left(\frac{\gamma'^2 b^2 - \beta'^2 c^2}{b^2c^2}\right)(\cos A - \cos B\cos C) = 0$$

$$\sum \frac{\alpha'}{ab^2c^2}\left(\gamma'^2 b^2 - \beta'^2 c^2\right)(\cos A - \cos B\cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'}{ab^2c^2}\left(\frac{\gamma'\beta\beta' c^2}{\gamma} - \frac{c^2\gamma\gamma'\beta'}{\beta}\right)(\cos A - \cos B\cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta c^2}{\gamma} - \frac{\gamma b^2}{\beta}\right)(\cos A - \cos B\cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta^2c^2 - \gamma^2b^2}{\beta\gamma}\right)(\cos A - \cos B\cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha}{a}\left(\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\right)\frac{1}{b^2c^2} \times b^2c^2\left(\frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}\right)(\cos A - \cos B\cos C) = 0.$$

We obtain that:

$$\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\sum \frac{\alpha}{a}\left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{b^2}\right)(\cos A - \cos B\cos C) = 0,$$

this is true because $P \in \mathcal{L}$.

Remark.

We saw that the triangle 's ABC orthocenter H belongs to the locus, from the precedent theorem it results that also O, the center of the circumscribed circle to the triangle ABC (isogonable to H), belongs to the locus.

Open problem:

What does it represent from the geometry's point of view the equation of locus $\mathcal{L}$?

In the particular case of an equilateral triangle we can formulate the following:

Proposition:

The locus of the point P from the plane of the equilateral triangle ABC with the property that the pedal triangle of P and the triangle ABC are homological, is the union of the triangle's heights.

Proof:

Let $P(\alpha,\beta,\gamma)$ a point that belongs to locus $\mathcal{L}$. The equation of the locus becomes:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=0$$

Because:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2=$$
$$=\alpha\beta\gamma+\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2-\alpha\beta\gamma=$$
$$=\alpha\beta(\gamma-\beta)+\alpha\gamma(\gamma-\beta)-\alpha^2(\gamma-\beta)-\beta\gamma(\gamma-\beta)=$$
$$=(\gamma-\beta)\left[\alpha(\beta-\alpha)-\gamma(\beta-\alpha)\right]=(\beta-\alpha)(\alpha-\gamma)(\gamma-\beta).$$

We obtain that $\alpha=\beta$ or $\beta=\gamma$ or $\gamma=\alpha$, that shows that P belongs to the medians (heights) of the triangle ABC.

References:

[1] C. Coandă , Geometrie analitică în coordanate baricentrice, Editura Reprograph, Craiova, 2005.

[2] Multispace & Multistructure. Neutrosophic Trandisciplinarity (100 Collected Papers of Sciences), vol. IV, North European Scientific Publishers, Hanko, Finland, 2010.

This book contains 21 papers of plane geometry.

It deals with various topics, such as: quasi-isogonal cevians, nedians, polar of a point with respect to a circle, anti-bisector, aalsonti-symmedian, anti-height and their isogonal.

A *nedian* is a line segment that has its origin in a triangle's vertex and divides the opposite side in n equal segments.

The papers also study distances between remarkable points in the 2D-geometry, the circumscribed octagon and the inscribable octagon, the circles adjointly ex-inscribed associated to a triangle, and several classical results such as: Carnot circles, Euler's line, Desargues theorem, Sondat's theorem, Dergiades theorem, Stevanovic's theorem, Pantazi's theorem, and Newton's theorem.

Special attention is given in this book to *orthological triangles, bi-orthological triangles, ortho-homological triangles,* and *tri-homological triangles.*

Each paper is independent of the others. Yet, papers on the same or similar topics are listed together one after the other.

The book is intended for College and University students and instructors that prepare for mathematical competitions such as National and International Mathematical Olympiads, or for the AMATYC (American Mathematical Association for Two Year Colleges) student competition, Putnam competition, Gheorghe Ţiţeica Romanian competition, and so on.

The book is also useful for geometrical researchers.

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