Smarandache Idempotents in Loop Rings $Z_tL_n(m)$ of the Loops $L_n(m)$:

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Abstract

In this paper we establish the existance of S-idempotents in case of loop rings $Z_tL_n(m)$ for a special class of loops $L_n(m)$; over the ring of modulo integers Z_t for a specific value of t.These loops satisfy the conditions $g_i^2 = 1$ for every $g_i \in L_n(m)$. We prove $Z_tL_n(m)$ has an S-idempotent when t is a perfect number or when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$ (p_1 and p_2 are distinct odd primes). It is important to note that we are able to prove only the existance of a single S-idempotent ; however we leave it as an open problem wheather such loop rings have more than one S-idempotent.

This paper has three sections. In section one, we give the basic notions about the loops $L_n(m)$ and recall the definition of S-idempotents in rings. In section two, we establish the existance of S-idempotents in the loop ring $Z_tL_n(m)$. In the final section, we suggest some interesting problems based on our study.

§ 1 : Basic Results

Here we just give the basic notions about the loops $L_n(m)$ and the definition of S-idempotents in rings.

Definition 1.1 [**4**]: Let R be a ring. An element $x \in R \setminus \{0\}$ is said to be a Smarandache idempotent (S-idempotent) of R if $x^2 = x$ and there exist $a \in R \setminus \{x, 0\}$ such that

$$i. \quad a^2 = x$$

$$ii. \quad xa = x \quad or \quad ax = a.$$

For more about S-idempotent please refer [4].

Definition 1.2 [**2**]: A positive integer n is said to be a perfect number if n is equal to the sum of all its positive divisors, excluding n itself. e.g. 6 is a perfect number. As

$6 = 1 + 2 + 3$.

Definition 1.3 [1]: A non-empty set L is said to form a loop, if in L is defined a binary operation, called product and denoted by $'.'$ such that

1. For $a, b \in L$ we have $a.b \in L$. (closure property.)

2. There exists an element $e \in L$ such that $a.e = e.a = a$ for all $a \in L$. (e is called the identity element of L.)

3. For every ordered pair $(a, b) \in L \times L$ there exists a unique pair $(x, y) \in L \times L$ such that $ax = b$ and $ya = b$.

Definition 1.4 [3]: Let $L_n(m) = \{e, 1, 2, 3..., n\}$ be a set where $n > 3, n$ is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as follows:

$$i.\ \ e.i = i.e = i \quad for\ all\ \ i \in L_n(m)\backslash\{e\}$$
$$ii.\ \ i^2. = e \quad for\ all\ \ i \in L_n(m)$$
$$iii.\ i.j = t, \quad where\ \ t \equiv (mj - (m-1)i)(mod\, n)\ \ \ for\ all\ i, j \in L_n(m),$$
$$i \neq e\ and\ j \neq e.$$

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by

$$L_n = \{L_n(m)|\ \ n > 3, n \text{ is odd and } (m, n) = 1, (m - 1, n) = 1 \text{ with } m < n\}.$$

Example 1.1 [3]: Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

.	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

§ 2 : Existance of S-idempotents in the Loop Rings $Z_tL_n(m)$:

In this section we will prove the existence of an S-idempotent for the loop ring $Z_tL_n(m)$ when t is an even perfect number. Also we will prove that the loop ring $Z_tL_n(m)$ has an S-idempotent when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$ (p_1 and p_2 are distinct odd primes).

Theorem 2.1: Let $Z_tL_n(m)$ be a loop ring, where t is an even perfect number of the form $t = 2^s(2^{s+1}-1)$ for some $s > 1$, then $\alpha = 2^s + 2^sg_i \in Z_tL_n(m)$ is an S-idempotent.

Proof: As t is an even perfect number, t must be of the form

$$t = 2^s(2^{s+1} - 1), \quad \text{for some } s > 1$$

where $2^{s+1} - 1$ is a prime.
Consider

$$\alpha = 2^s + 2^sg_i \in Z_tL_n(m).$$

Choose

$$\beta = (t - 2^s) + (t - 2^s)g_i \in Z_tL_n(m).$$

Clearly

$$\begin{aligned} \alpha^2 &= (2^s + 2^sg_i)^2 \\ &= 2.2^{2s}(1 + g_i) \\ &\equiv 2^s(1 + g_i) \qquad [\because 2^s.2^{s+1} \equiv 2^s(\text{mod} \quad \text{t})] \\ &= \alpha. \end{aligned}$$

Now

$$\begin{aligned} \beta^2 &= [(t - 2^s) + (t - 2^s)g_i]^2 \\ &= 2.(t - 2^s)^2(1 + g_i) \\ &\equiv 2^s(1 + g_i) \\ &= \alpha. \end{aligned}$$

Also

$$\begin{aligned} \alpha\beta &= [2^s + 2^sg_i][(t - 2^s) + (t - 2^s)g_i] \\ &= 2^s(1 + g_i)(t - 2^s)(1 + g_i) \end{aligned}$$

$$
\begin{aligned}
&\equiv -2.2^s.2^s(1+g_i)\\
&\equiv (t-2^s)(1+g_i)\\
&= \beta.
\end{aligned}
$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{and} \ \ \alpha\beta = \beta.$$

Therefore $\alpha = 2^s + 2^s g_i$ is an S-idempotent.

Example:2.1 Take the loop ring $Z_6 L_n(m)$. Here 6 is an even perfect number. As $6 = 2.(2^2-1)$, so $\alpha = 2 + 2g_i$ is an S-idempotent.
For

$$
\begin{aligned}
\alpha^2 &= (2+2g_i)^2\\
&\equiv 2+2g_i\\
&= \alpha.
\end{aligned}
$$

Choose now

$$\beta = (6-2) + (6-2)g_i$$

then

$$
\begin{aligned}
\beta^2 &= (4+4g_i)^2\\
&\equiv 2+2g_i\\
&= \alpha.
\end{aligned}
$$

And

$$
\begin{aligned}
\alpha\beta &= (2+2g_i)(4+4g_i)\\
&= 8+8g_i+8g_i+8\\
&\equiv 4+4g_i\\
&= \beta.
\end{aligned}
$$

So $\alpha = 2 + 2g_i$ is an S-idempotent.

Theorem 2.2: Let $Z_{2p}L_n(m)$ be a loop ring where p is an odd prime such that $p|2^{t_0+1}-1$ for some $t_0 \geq 1$, then $\alpha = 2^{t_0} + 2^{t_o} g_i \in Z_{2p}L_n(m)$ is an S-idempotent.

Proof: Suppose $p|2^{t_0+1}-1$ for some $t_0 \geq 1$. Take $\alpha = 2^{t_0} + 2^{t_o} g_i \in Z_{2p}L_n(m)$ and $\beta = (2p - 2^{t_0}) + (2p - 2^{t_0})g_i \in Z_{2p}L_n(m)$.
Clearly

$$\alpha^2 = (2^{t_0} + 2^{t_o} g_i)^2$$

$$
\begin{aligned}
&= 2.2^{2t_o}(1+g_i)\\
&= 2^{t_o+1}.2^{t_0}(1+g_i)\\
&\equiv 2^{t_0}(1+g_i)\\
&= \alpha.
\end{aligned}
$$

As

$$2^{t_0}.2^{t_{n_0+1}} \equiv 2^{t_0}(\text{mod } \ 2\text{p}).$$

Since

$$
\begin{aligned}
2^{t_0+1} &\equiv \ 1(\text{mod } \ \text{p})\\
\Leftrightarrow 2^{t_0}.2^{t_0+1} &\equiv \ 2^{t_0}(\text{mod } \ 2\text{p}) \quad \text{for gcd}\,(2^{t_0}, 2p) = 2, \ \ t_0 \geq 1.
\end{aligned}
$$

Also

$$
\begin{aligned}
\beta^2 &= [(2p-2^{t_0}) + (2p-2^{t_0})g_i]^2\\
&= 2(2p-2^{t_0})^2(1+g_i)\\
&\equiv 2.2^{2t_0}(1+g_i)\\
&= 2^{t_0+1}2^{t_0}(1+g_i)\\
&\equiv 2^{t_0}(1+g_i)\\
&= \alpha.
\end{aligned}
$$

And

$$
\begin{aligned}
\alpha\beta &= [2^{t_0} + 2^{t_0}g_i].[(2p-2^{t_0}) + (2p-2^{t_0})g_i]^2\\
&\equiv -2^{t_0}(1+g_i)2^{t_0}(1+g_i)\\
&= -2.2^{2t_0}(1+g_i)\\
&\equiv (2p-2^{t_0})(1+g_i)\\
&= \beta.
\end{aligned}
$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{ and } \ \alpha\beta = \beta.$$

Hence $\alpha = 2^{t_0} + 2^{t_0}g_i$ is an S-idempotent.

Example:2.2 Consider the loop ring $Z_{10}L_n(m)$. Here $5|2^{3+1}-1, \quad$ so $t_0 = 3$. Take

$$\alpha = 2^3 + 2^3 g_i \quad \text{ and } \quad \beta = 2 + 2g_i.$$

Now

$$\alpha^2 = (8+8g_i)^2$$

$$
\begin{aligned}
&= 64 + 128g_i + 64 \\
&\equiv 8 + 8g_i \\
&= \alpha.
\end{aligned}
$$

And

$$
\begin{aligned}
\beta^2 &= (2 + 2g_i)^2 \\
&= 4 + 8g_i + 4 \\
&\equiv 8 + 8g_i \\
&= \alpha.
\end{aligned}
$$

Also

$$
\begin{aligned}
\alpha\beta &= (8 + 8g_i)(2 + 2g_i) \\
&= 16 + 16g_i + 16g_i + 16 \\
&\equiv 2 + 2g_i \\
&= \beta.
\end{aligned}
$$

So $\alpha = 8 + 8g_i$ is an S-idempotent.

Theorem 2.3: Let $Z_{2^i p}L_n(m)$ be the loop ring where p is an odd prime such that $p|2^{t_0+1} - 1$ for some $t_0 \geq i$, then $\alpha = 2^{t_0} + 2^{t_o}g_i \in Z_{2^i p}L_n(m)$ is an S-idempotent.

Proof: Note that $p|2^{t_0+1} - 1$ for some $t_0 \geq i$.
Therefore

$$
\begin{aligned}
2^{t_0+1} &\equiv 1 (\text{mod } \ \text{p}) \quad \text{for some} \ \ t_0 \geq i \\
\Leftrightarrow 2^{t_0}.2^{t_0+1} &\equiv 2^{t_0} (\text{mod } \ 2^i p) \quad \text{as gcd } (2^{t_0}, 2^i p) = 2^i, \ \ t_0 \geq i.
\end{aligned}
$$

Now take $\alpha = 2^{t_0} + 2^{t_o}g_i \in Z_{2^i p}L_n(m)$ and $\beta = (2^i p - 2^{t_0}) + (2^i p - 2^{t_0})g_i \in Z_{2^i p}L_n(m)$.

Then it is easy to see that

$$
\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{and} \ \ \alpha\beta = \beta.
$$

Hence $\alpha = 2^{t_0} + 2^{t_0}g_i$ is an S-idempotent.

Example:2.3 Take the loop ring $Z_{2^3.7}L_n(m)$. Here $7|2^{5+1} - 1$, so $t_0 = 5$.
Take

$$
\alpha = 2^5 + 2^5 g_i \quad \text{and} \quad \beta = (2^3.7 - 2^5) + (2^3.7 - 2^5)g_i.
$$

Now

$$
\alpha^2 = (32 + 32g_i)^2
$$

$$= 1024 + 2048g_i + 1024$$
$$\equiv 32 + 32g_i$$
$$= \alpha.$$

And

$$\beta^2 = (24 + 24g_i)^2$$
$$= 576 + 1152g_i + 576$$
$$\equiv 24 + 24g_i$$
$$= \alpha.$$

Also

$$\alpha\beta = (32 + 32g_i)(24 + 24g_i)$$
$$\equiv 24 + 24g_i$$
$$= \beta.$$

So $\alpha = 32 + 32g_i$ is an S-idempotent.

Theorem 2.4: Let $Z_{3^ip}L_n(m)$ be the loop ring where p is an odd prime such that $p|2.3^{t_0} - 1$ for some $t_0 \geq i$, then $\alpha = 3^{t_0} + 3^{t_o}g_i \in Z_{3^ip}L_n(m)$ is an S-idempotent.

Proof: Suppose $p|2.3^{t_0} - 1$ for some $t_0 \geq i$.

Take $\alpha = 3^{t_0} + 3^{t_o}g_i \in Z_{3^ip}L_n(m)$ and $\beta = (3^ip - 3^{t_0}) + (3^ip - 3^{t_0})g_i \in Z_{3^ip}L_n(m)$. Then

$$\alpha^2 = (3^{t_0} + 3^{t_o}g_i)^2$$
$$= 2.3^{2t_o}(1 + g_i)$$
$$= 2.3^{t_0}3^{t_0}(1 + g_i)$$
$$\equiv 3^{t_0}(1 + g_i)$$
$$= \alpha.$$

As

$$2.3^{t_0} \equiv 1(\text{mod p}) \quad \text{for some } t_0 \geq i.$$
$$\Leftrightarrow 2.3^{t_0}.3^{t_0} \equiv 3^{t_0}(\text{mod } 3^ip) \quad \text{as gcd } (3^{t_0}, 3^ip) = 3^i, \quad t_0 \geq i.$$

Similarly

$$\beta^2 = \alpha \quad \text{and } \alpha\beta = \beta.$$

So $\alpha = 3^{t_0} + 3^{t_0}g_i$ is an S-idempotent.

Example:2.4 Take the loop ring $Z_{3^2.5}L_n(m)$. Here $5|2.3^5-1, \quad$ so $t_0=5$. Take

$$\alpha = 3^5+3^5g_i \quad \text{and} \quad \beta = (3^2.5-3^5)+(3^2.5-3^5)g_i$$

Now

$$\begin{aligned}\alpha^2 &= (18+18g_i)^2\\ &\equiv 18+18g_i\\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\beta^2 &= (27+27g_i)^2\\ &\equiv 18+18g_i\\ &= \alpha.\end{aligned}$$

Also

$$\alpha\beta = \beta$$

So $\alpha = 3^5+3^5g_i$ is an S-idempotent.

We can generalize Theorem 2.3 and Theorem 2.4 as follows:

Theorem 2.5: Let $Z_{p_1^ip_2}L_n(m)$ be a loop ring where p_1 and p_2 are distinct odd primes and $p_2|2.p_1^{t_0}-1$ for some $t_0 \geq i$, then $\alpha = p_1^{t_0}+p_1^{t_o}g_i \in Z_{p_1^ip_2}L_n(m)$ is an S-idempotent.

Proof: Suppose $p_2|2.p_1^{t_0}-1$ for some $t_0 \geq i$.

Take $\alpha = p_1^{t_0}+p_1^{t_o}g_i \in Z_{p_1^ip_2}L_n(m)$ and $\beta = (p_1^ip_2-p_1^{t_0})+(p_1^ip_2-p_1^{t_0}) \in Z_{p_1^ip_2}L_n(m)$. Then

$$\begin{aligned}\alpha^2 &= (p_1^{t_0}+p_1^{t_o}g_i)^2\\ &= 2.p_1^{2t_o}(1+g_i)\\ &= 2.p_1^{t_0}p_1^{t_0}(1+g_i)\\ &\equiv p_1^{t_0}(1+g_i)\\ &= \alpha.\end{aligned}$$

As

$$2.p_1^{t_0} \equiv \ 1(\text{mod} \ \ p_2) \quad \text{for some} \ \ t_0 \geq i.$$

$$\Leftrightarrow 2.p_1^{t_0}.p_1^{t_0} \equiv \ p_1^{t_0}(\text{mod} \ \ p_1^ip_2) \quad \text{as gcd } (p_1^{t_0}, p_1^ip_2) = p_1^i, \ \ t_0 \geq i.$$

Similarly

$$\beta^2 = \alpha \quad \text{and } \alpha\beta = \beta.$$

So $\alpha = p_1^{t_0} + p_1^{t_0} g_2$ is an S-idempotent.

§ 3 : **Conclusions:**

We see in all the 5 cases described in the Theorem 2.1 to 2.5 we are able to establish the existence of one non-trivial S-idempotent. However we are not able to prove the uniqueness of this S-idempotent. Hence we suggest the following problems:

- Does the loop rings described in the Theorems 2.1 to 2.5 can have more than one S-idempotent ?

- Does the loop rings $Z_t L_n(m)$ have S-idempotents when t is of the form $t = p_1 p_2 ... p_s$ where $p_1, p_2, ... p_s$ are distinct odd primes ?.

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