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A Fixed Point Approach on Bipolar Neutrosophic Soft Metric Space

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Abstract. The main aim of this paper is to premise bipolar neutrosophic soft metric space (BNSMS) in terms of bipolar neutrosophic soft points. In addition, we define convergence of sequence, Cauchy sequence and completeness in BNSMS with appropriate examples. Further, we represent bipolar neutrosophic soft mappings using a Cartesian product with relations on bipolar neutrosophic soft sets and developed a fixed point theorem for self maps with contractive conditions using BNS mappings.

INTRODUCTION

In 1965, Zadeh [19] originated fuzzy set theory which deals with uncertainities and it is represented by a membership degree with range in the unit interval [0, 1]. In 1999, soft set theory was initiated by Molodtsov [11] for modeling vagueness. Moreover, Beaula et al. [2, 3] and Yazar et al. [18, 7, 16, 17] developed the the concept of soft normed spaces and soft metric spaces using the idea of soft sets. In order to generalize intuitionistic fuzzy set, Smarandache [13, 14] introduced Neutrosophic set. Later, Maji [10] established 'Neutrosophic soft set' by combining neutrosophic set with soft set theory. For further details on single valued neutrosophic sets, the reader may refer to [12, 15].

In recent years, Bosc and Pivert [5] conveyed bipolarity which furnishes the human mind to recognize decisions based on positive and negative effects. To employ this idea, Lee [9, 8] extrapolate bipolar fuzzy sets which generalizes fuzzy sets. After Deli et al. [6] proposed bipolar neutrosophic sets and studied applications in decicion making. Furthermore, Mumtaz Ali et al. [1] propound the concept of bipolar neutrosophic soft set which is a hybrid structure of bipolar neutrosophic set and soft set.

In this article, we construct the concepts of convergence, Cauchy sequence and completeness in bipolar neutrosophic soft metric space and prove fixed point theorem for self maps with certain contractive conditions. An example is given to frame up the developed approach.

Preliminaries

In this section, we give some basic definitions of neutrosophic soft set and bipolar neutrosophic soft set that are useful for subsequent discussions.

Definition 1. *[4] Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,*

$$N = \{(e, \{< x,\ T_{f_N(e)}(x),\ I_{f_N(e)}(x),\ F_{f_N(e)}(x) >: x \in U\}) :\ e \in E\}$$

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where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0,1]$, *respectively called the truth-membership, indeterminacy-membership, falsity-membership function of* $f_N(e)$. *Since supremum of each* T, I, F *is 1 so the inequality* $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ *is obvious.*

Definition 2. *[1] Let* U *be a universe. A bipolar neutrosophic soft set* B *in* U *is defined as*

$$B = \{(e, \{< u,\ T^+(u),\ I^+(u),\ F^+(u),\ T^-(u),\ I^-(u),\ F^-(u) >: u \in U\}) : \ e \in E\}$$

where $T^+, I^+, F^+ \to [0,1]$ *and* $T^-, I^-, F^- \to [-1,0]$. *The positive membership degree* $T^+(u)$, $I^+(u)$, $F^+(u)$ *denotes the truth membership, indeterminate membership and false membership of an element corresponding to a bipolar neutrosophic soft set* B *and the negative membership degree* $T^-(u)$, $I^-(u)$, $F^-(u)$ *denotes the truth membership, indeterminate membership and false membership of an element* $u \in U$ *to some implicit counter-property corresponding to a bipolar neutrosophic soft set* B.

Definition 3. *[1] Let* $B_i = \{(e, \{(u, T_i^+(u), I_i^+(u), F_i^+(u), T_i^-(u), I_i^-(u), F_i^-(u)) : u \in U\}) : e \in E\}$ *for* $i = 1,2$ *be two bipolar neutrosophic soft sets over* U. *Then* B_1 *is bipolar neutrosophic soft subset of* B_2, *is denoted by* $B_1 \subseteq B_2$, *if* $T_1^+(u) \leq T_2^+(u)$, $I_1^+(u) \geq I_2^+(u)$, $F_1^+(u) \geq F_2^+(u)$, $T_1^-(u) \geq T_2^-(u)$, $I_1^-(u) \leq I_2^-(u)$ *and* $F_1^-(u) \leq F_2^-(u)$ *for all* $(e,u) \in E \times U$.

Definition 4. *[1] Let* B *be a bipolar neutrosophic soft set over* U. *Then, the complement of a bipolar neutrosophic soft set* B, *is denoted by* B^c, *is defined as:*

$$B^c = \{(e, \{(u, F^+(u), 1 - I^+(u),\ T^+(u), F^-(u), -1 - I^-(u), T^-(u)) : u \in U\}) : e \in E\}.$$

Definition 5. *[1] Let* $B_i = \{(e, \{(u, T_i^+(u), I_i^+(u), F_i^+(u), T_i^-(u), I_i^-(u), F_i^-(u)) : u \in U\}) : e \in E\}$ *for* $i = 1,2$ *be two bipolar neutrosophic soft sets over* U. *Then the union of* B_1 *and* B_2 *is denoted by* $B_1 \cup B_2$, *is defined as:*

$$B_1 \cup B_2 = \{(e, \{(u, max_i\{T_i^+(u)\},\ min_i\{I_i^+(u)\},\ min_i\{F_i^+(u)\},\ min_i\{T_i^-(u)\},$$
$$max_i\{I_i^-(u)\},\ max_i\{F_i^-(u)\}) : \ u \in U\}) : e \in E, and\ i = 1,2\}.$$

Definition 6. *[1] Let* $B_i = \{(e, \{(u, T_i^+(u), I_i^+(u), F_i^+(u), T_i^-(u), I_i^-(u), F_i^-(u)) : u \in U\}) : e \in E\}$ *for* $i = 1,2$ *be two bipolar neutrosophic soft sets over* U. *Then the intersection of* B_1 *and* B_2 *is denoted by* $B_1 \cap B_2$, *is defined as:*

$$B_1 \cap B_2 = \{(e, \{(u, min_i\{T_i^+(u)\},\ max_i\{I_i^+(u)\},\ max_i\{F_i^+(u)\},\ max_i\{T_i^-(u)\},$$
$$min_i\{I_i^-(u)\},\ min_i\{F_i^-(u)\}) : \ u \in U\}) : e \in E, and\ i = 1,2\}.$$

Definition 7. *[1] An empty bipolar neutrosopohic soft set* $B^\emptyset$ *in* U *is defined as:*

$$B^\emptyset = \{(e, \{(u, 0, 0, 1, -1, 0, 0)) : u \in U\}) : e \in E\}.$$

Definition 8. *[1] An absolute bipolar neutrosopohic soft set* B^U *in* U *is defined as:*

$$B^U = \{(e, \{(u, 1, 1, 0, 0, -1, -1)) : u \in U\}) : e \in E\}.$$

Bipolar neutrosophic soft points

In this main section, we discuss the notion of bipolar neutrosophic soft metric spaces using bipolar neutrosophic soft points.

Definition 9. *A bipolar neutrosophic soft point in an bipolar neutrosophic soft set* B *is defined as an element* $(e, f_B(e))$ *of* B, *for* $e \in E$ *and is denoted by* e_B, *if* $f_B(e) \notin B^\emptyset$ *and* $f_B(e') \in B^\emptyset,\ \forall\ e' \in E - \{e\}$.

Definition 10. *The complement of a bipolar neutrosophic soft point* e_B *is a bipolar neutrosophic soft point* e_B^c *such that* $f_B^c(e) = (f_B(e))^c$.

Definition 11. *A bipolar neutrosophic soft point* $e_B \in M$, M *being an bipolar neutrosophic soft set if for* $e \in E$, $f_B(e) \leq f_M(e)$ *i.e.,*

$$T^+_{f_B(e)}(x) \leq T^+_{f_M(e)}(x),\ I^+_{f_B(e)}(x) \geq I^+_{f_M(e)}(x),\ F^+_{f_B(e)}(x) \geq F^+_{f_M(e)}(x),$$
$$T^-_{f_B(e)}(x) \geq T^-_{f_M(e)}(x),\ I^-_{f_B(e)}(x) \leq I^-_{f_M(e)}(x),\ F^-_{f_B(e)}(x) \leq F^-_{f_M(e)}(x).$$

Example 1. *Let $U = \{x, y, z\}$ and $E = \{e_1, e_2\}$. Then,*

$$e_{1B} = \{< x, 0.8, 0.7, 0.7, -0.3, -0.5, -0.3) >, < y, 0.6, 0.8, 0.5, -0.5, -0.3, -0.3) >, \\ < z, 0.7, 0.6, 0.6, -0.3, -0.4, -0.7) >\}$$

is a bipolar neutrosophic soft point. Its complement is given by:

$$e^c_{1B} = \{< x, 0.7, 0.3, 0.8, -0.3, -0.5, -0.3) >, < y, 0.5, 0.2, 0.6, -0.3, -0.7, -0.5) >, \\ < z, 0.6, 0.4, 0.7, -0.7, -0.6, -0.3) >\}$$

For another bipolar neutrosophic soft set M defined on same (U, E), let

$$f_M(e_1) = \{< x, 0.8, 0.6, 0.6, -0.4, -0.2, -0.1) >, < y, 0.7, 0.6, 0.4, -0.6, -0.2, -0.1) >, \\ < z, 0.9, 0.5, 0.3, -0.4, -0.2, -0.5) >\}$$

Then $f_B(e_1) \leq f_M(e_1)$ i.e., $e_{1B} \in M$.

Definition 12. *Let $BNS(U_E)$ be the collection of all bipolar neutrosophic soft points over (U, E). Then the bipolar neutrosophic soft metric interms of bipolar neutrosophic soft points is defined by a mapping $d : BNS(U_E) \times BNS(U_E) \to [0, 6]$ satisfying the following conditions:*

$$BNSM1 : d(e_B, e_M) \geq 0, \forall\, e_B, e_M \in BNS(U_E).$$
$$BNSM2 : d(e_B, e_M) = 0 \Leftrightarrow e_B = e_M.$$
$$BNSM3 : d(e_B, e_M) = d(e_M, e_B).$$
$$BNSM4 : d(e_B, e_M) \leq d(e_B, e_p) + d(e_p, e_M),\ \forall\, e_M, e_p, e_B \in BNS(U_E).$$

Then BNS(U_E) is said to form an BNSMS with respect to the BNSMS 'd' over (U, E) and is denoted by $(BNS(U_E), d)$. Here $e_M = e_B$ means that,

$$T^+_{e_M}(x_i) = T^+_{e_B}(x_i),\ I^+_{e_M}(x_i) = I^+_{e_B}(x_i),\ F^+_{e_M}(x_i) = F^+_{e_B}(x_i),$$
$$T^-_{e_M}(x_i) = T^-_{e_B}(x_i),\ I^-_{e_M}(x_i) = I^-_{e_B}(x_i),\ F^-_{e_M}(x_i) = F^-_{e_B}(x_i),\quad \forall x_i \in U.$$

Example 2. *Define*

$$d(e_B, e_M) = min_{x_i}\{|T^+_{e_B}(x_i) - T^+_{e_M}(x_i)| + |I^+_{e_B}(x_i) - I^+_{e_M}(x_i)| + |F^+_{e_B}(x_i) - F^+_{e_M}(x_i)| + \\ |T^-_{e_B}(x_i) - T^-_{e_M}(x_i)| + |I^-_{e_B}(x_i) - I^-_{e_M}(x_i)| + |F^-_{e_B}(x_i) - F^-_{e_M}(x_i)|\} \tag{1}$$

Clearly, $d(e_B, e_M) \geq 0$ and $d(e_B, e_M) = 0 \Leftrightarrow e_B = e_M$. Also $d(e_B, e_M) = d(e_M, e_B)$. To verify the final condition, we use the triangle inequality

$$\begin{aligned} d(e_B, e_M) = {} & min_{x_i}\{|T^+_{e_B}(x_i) - T^+_{e_M}(x_i)| + |I^+_{e_B}(x_i) - I^+_{e_M}(x_i)| + |F^+_{e_B}(x_i) - F^+_{e_M}(x_i)| + \\ & |T^-_{e_B}(x_i) - T^-_{e_M}(x_i)| + |I^-_{e_B}(x_i) - I^-_{e_M}(x_i)| + |F^-_{e_B}(x_i) - F^-_{e_M}(x_i)|\} \\ = {} & min_{x_i}\{|T^+_{e_B}(x_i) - T^+_{e_p}(x_i) + T^+_{e_p}(x_i) - T^+_{e_M}(x_i)| + |I^+_{e_B}(x_i) - I^+_{e_p}(x_i) + \\ & I^+_{e_p}(x_i) - I^+_{e_M}(x_i)| + |F^+_{e_B}(x_i) - F^+_{e_p}(x_i) + F^+_{e_p}(x_i) - F^+_{e_M}(x_i)| + \\ & |T^-_{e_B}(x_i) - T^-_{e_p}(x_i) + T^-_{e_p}(x_i) - T^-_{e_M}(x_i)| + |I^-_{e_B}(x_i) - I^-_{e_p}(x_i) + \\ & I^-_{e_p}(x_i) - I^-_{e_M}(x_i)| + |F^-_{e_B}(x_i) - F^-_{e_p}(x_i) + F^-_{e_p}(x_i) - F^-_{e_M}(x_i)|\} \\ \leq {} & min_{x_i}\{|T^+_{e_B}(x_i) - T^+_{e_p}(x_i)| + |I^+_{e_B}(x_i) - I^+_{e_p}(x_i)| + |F^+_{e_B}(x_i) - F^+_{e_p}(x_i)| + \\ & |T^-_{e_B}(x_i) - T^-_{e_p}(x_i)| + |I^-_{e_B}(x_i) - I^-_{e_p}(x_i)| + |F^-_{e_B}(x_i) - F^-_{e_p}(x_i)|\} + \\ & min_{x_i}\{|T^+_{e_p}(x_i) - T^+_{e_M}(x_i)| + |I^+_{e_p}(x_i) - I^+_{e_M}(x_i)| + |F^+_{e_p}(x_i) - F^+_{e_M}(x_i)| + \\ & |T^-_{e_p}(x_i) - T^-_{e_M}(x_i)| + |I^-_{e_p}(x_i) - I^-_{e_M}(x_i)| + |F^-_{e_p}(x_i) - F^-_{e_M}(x_i)|\} \\ = {} & d(e_B, e_p) + d(e_p, e_M). \end{aligned}$$

Thus d defined above is called a bipolar neutrosophic soft metric over (U, E).

Definition 13. *A sequence of bipolar neutrosophic soft points $\{e_{nB}\}$ in an BNSMS $(BNS(U_E), d)$ is said to converge in $(BNS(U_E), d)$ if there exists a bipolar neutrosophic soft point $e_B \in BNS(U_E)$ such that $d(e_{nB}, e_B) \to 0$ as $n \to \infty$ or $e_{nB} \to e_B$ as $n \to \infty$. Analytically for every $\epsilon > 0$ there exists a natural number n_0 such that $d(e_{nB}, e_B) < \epsilon$, $\forall\ n \geq n_0$.*

Example 3. *Let $E = \mathbb{N}$ be the parametric set and $U = \mathbb{Z}$ be the universal set. Define a mapping $f_B : \mathbb{N} \to NS(\mathbb{Z})$ where, for any $n \in \mathbb{N}$ and $x \in \mathbb{Z}$,*

$$T^+_{f_B(n)}(x) = \begin{cases} 0, & \text{if } x \text{ is odd} \\ \frac{1}{n+1}, & \text{if } x \text{ is even} \end{cases}$$

$$I^+_{f_B(n)}(x) = \begin{cases} \frac{1}{2n}, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases}$$

$$F^+_{f_B(n)}(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases}$$

$$T^-_{f_B(n)}(x) = \begin{cases} 0, & \text{if } x \text{ is odd} \\ \frac{-1}{n+2}, & \text{if } x \text{ is even} \end{cases}$$

$$I^-_{f_B(n)}(x) = \begin{cases} \frac{-1}{3n}, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases}$$

$$F^-_{f_B(n)}(x) = \begin{cases} -1 + \frac{1}{n}, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases}$$

Notice that, $\{e_{nB}\} \to (0, 0, 1, 0, 0, -1)$ for odd integers and $\{e_{nB}\} \to (0, 0, 0, 0, 0, 0)$ for even integers. Hence, $\{e_{nB}\}$ is divergent bipolar neutrosophic soft sequence over $(\mathbb{Z}, \mathbb{N})$.

Definition 14. *A sequence $\{e_{nB}\}$ of bipolar neutrosophic soft point in an BNSMS $(BNS(U_E), d)$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(e_{mB}, e_{nB}) < \epsilon, \forall\, m, n \geq n_0$ i.e., $d(e_{mB}, e_{nB}) \to 0$ as $m, n \to \infty$.*

Example 4. *Let $E = \mathbb{N}$ be the parametric set and $U = \mathbb{Z}$ be the universal set. Define a mapping $f_B : \mathbb{N} \to NS(\mathbb{Z})$ where, for any $n \in \mathbb{N}$ and $x \in \mathbb{Z}$,*

$$T^+_{f_B(n)}(x) = \frac{1}{n+1},\ I^+_{f_B(n)}(x) = \frac{1}{2n},\ F^+_{f_B(n)}(x) = \frac{1}{3n},$$
$$T^-_{f_B(n)}(x) = \frac{-1}{n+2},\ I^-_{f_B(n)}(x) = \frac{-1}{4n},\ F^-_{f_B(n)}(x) = \frac{-1}{6n}.$$

Define the distance function as

$$\begin{aligned} d(e_{mB}, e_{nB}) = min_{x_i}\{&|T^+_{e_{mB}}(x_i) - T^+_{e_{nB}}(x_i)| + |I^+_{e_{mB}}(x_i) - I^+_{e_{nB}}(x_i)| + |F^+_{e_{mB}}(x_i) - F^+_{e_{nB}}(x_i)| \\ &+ |T^-_{e_{mB}}(x_i) - T^-_{e_{nB}}(x_i)| + |I^-_{e_{mB}}(x_i) - I^-_{e_{nB}}(x_i)| + |F^-_{e_{mB}}(x_i) - F^-_{e_{nB}}(x_i)|\} \\ = &|T^+_{e_{mB}}(x) - T^+_{e_{nB}}(x)| + |I^+_{e_{mB}}(x) - I^+_{e_{nB}}(x)| + |F^+_{e_{mB}}(x) - F^+_{e_{nB}}(x)| + \\ &|T^-_{e_{mB}}(x) - T^-_{e_{nB}}(x)| + |I^-_{e_{mB}}(x) - I^-_{e_{nB}}(x)| + |F^-_{e_{mB}}(x) - F^-_{e_{nB}}(x)| \\ = &|\frac{1}{m+1} - \frac{1}{n+1}| + |\frac{1}{2m} - \frac{1}{2n}| + |\frac{1}{3m} - \frac{1}{3n}| + \\ &|\frac{-1}{m+2} + \frac{1}{n+2}| + |\frac{-1}{4m} + \frac{1}{4n}| + |\frac{-1}{6m} + \frac{1}{6n}| \\ \to &\ 0 \ \text{ as } \ m, n \to \infty \end{aligned}$$

Hence $\{e_{nB}\}$ is a Cauchy sequence.

Definition 15. *An BNSMS $(BNS(U_E), d)$ is said to be complete if every Cauchy sequence converges to a bipolar neutrosophic soft point of $BNS(U_E)$.*

Definition 16. *Let B_1 and B_2 be two bipolar neutrosophic soft sets over a common universe (U, E). Then their Cartesian product is $B_1 \times B_2 = B_3$ where $f_{B_3}(a,b) = f_{B_1}(a) \times f_{B_2}(b)$ for all $(a,b) \in E \times E$. Analytically,*

$$f_{B_3}(a,b) = \{ < (x,y),\ T^+_{f_{B_3}(a,b)}(x,y),\ I^+_{f_{B_3}(a,b)}(x,y),\ F^+_{f_{B_3}(a,b)}(x,y),\ T^-_{f_{B_3}(a,b)}(x,y), \\ I^-_{f_{B_3}(a,b)}(x,y),\ F^-_{f_{B_3}(a,b)}(x,y) >:\ (x,y) \in U \times U\} \text{ with}$$

$$T^+_{f_{B_3}(a,b)}(x,y) = min\{T^+_{f_{B_1}(a)}(x),\ T^+_{f_{B_2}(b)}(y)\}$$

$$I^+_{f_{B_3}(a,b)}(x,y) = max\{I^+_{f_{B_1}(a)}(x),\ I^+_{f_{B_2}(b)}(y)\}$$

$$F^+_{f_{B_3}(a,b)}(x,y) = max\{F^+_{f_{B_1}(a)}(x),\ F^+_{f_{B_2}(b)}(y)\}$$

$$T^-_{f_{B_3}(a,b)}(x,y) = max\{T^-_{f_{B_1}(a)}(x),\ T^-_{f_{B_2}(b)}(y)\}$$

$$I^-_{f_{B_3}(a,b)}(x,y) = min\{I^-_{f_{B_1}(a)}(x),\ I^-_{f_{B_2}(b)}(y)\}$$

$$F^-_{f_{B_3}(a,b)}(x,y) = min\{F^-_{f_{B_1}(a)}(x),\ F^-_{f_{B_2}(b)}(y)\}$$

Example 5. *Let $U = \{x, y\}$ and $E = \{e_1, e_2\}$. Define bipolar neutrosophic soft sets B_1 and B_2 as follows:*

$$\begin{aligned} B_1 &= \{e_{1B_1}, e_{2B_1}\} \\ &= \{\{< x, (0.7, 0.8, 0.2, -0.5, -0.9, -0.3) >, < y, (0.8, 0.8, 0.7, -0.5, -0.7, -0.2) >\}, \\ &\quad \{< x, (0.1, 0.2, 0.7, -0.4, -0.3, -0.4) >, < y, (0.4, 0.8, 0.7, -0.4, -0.6, -0.2 >\}\}. \\ B_2 &= \{e_{1B_2}, e_{2B_2}\} \\ &\quad \{\{< x, (0.7, 0.2, 0.6, -0.2, -0.3, -0.5) >, < y, (0.1, 0.2, 0.6, -0.8, -0.2, -0.5) >\}, \\ &\quad \{< x, (0.7, 0.6, 0.1, -0.4, -0.7, -0.3) >, < y, (0.4, 0.9, 0.7, -0.5, -0.7, -0.2) >\}\}. \end{aligned}$$

Then $f_{B_3}(a,b) = f_{B_1}(a) \times f_{B_2}(b)$ where $(a,b) \in E \times E$ is calculated as

$f_{B_3}(a,b)$	(e_1, e_1)	(e_1, e_2)
(x,x)	$(0.7, 0.8, 0.6, -0.2, -0.9, -0.5)$	$(0.7, 0.8, 0.2, -0.4, -0.9, -0.3)$
(x,y)	$(0.1, 0.8, 0.6, -0.5, -0.9, -0.5)$	$(0.4, 0.9, 0.7, -0.5, -0.9, -0.3)$
(y,x)	$(0.7, 0.8, 0.7, -0.2, -0.7, -0.5)$	$(0.7, 0.8, 0.7, -0.4, -0.7, -0.3)$
(y,y)	$(0.1, 0.8, 0.7, -0.5, -0.7, -0.5)$	$(0.4, 0.9, 0.7, -0.1, -0.9, -0.4)$

$f_{B_3}(a,b)$	(e_2, e_1)	(e_2, e_2)
(x,x)	$(0.1, 0.2, 0.7, -0.2, -0.3, -0.5)$	$(0.1, 0.6, 0.7, -0.4, -0.7, -0.4)$
(x,y)	$(0.1, 0.2, 0.7, -0.4, -0.3, -0.5)$	$(0.1, 0.9, 0.7, -0.4, -0.7, -0.4)$
(y,x)	$(0.4, 0.8, 0.7, -0.2, -0.6, -0.5)$	$(0.4, 0.8, 0.7, -0.4, -0.7, -0.3)$
(y,y)	$(0.1, 0.8, 0.7, -0.4, -0.6, -0.5)$	$(0.4, 0.9, 0.7, -0.4, -0.7, -0.2)$

Definition 17. *Let B_1 and B_2 be bipolar neutrosophic soft sets over a common universe (U, E). A bipolar neutrosophic soft set R is said to be an bipolar neutrosophic soft relation(BNS) from B_1 to B_2 if $f_R : \mathbb{E} \rightarrow BNS(U_E)$ where $\mathbb{E} \subseteq E \times E$.*

Example 6. *Let B_1 and B_2 be bipolar neutrosophic soft sets in Example 5. Then $R = \{f_{B_1}(e_1) \times f_{B_2}(e_2),\ f_{B_1}(e_2) \times f_{B_2}(e_1),\ f_{B_1}(e_2) \times f_{B_1}(e_2)\}$ is a bipolar neutrosophic soft relation from B_1 to B_2 which itself is an BNS set with $\{(e_1, e_2),\ (e_2, e_1),\ (e_2, e_2)\}$ as a set of parameters.*

Definition 18. *Let B_1 and B_2 be bipolar neutrosophic soft sets in $BNS(U_E)$. A BNS-relation from B_1 to B_2 is said to be an BNS-mapping from B_1 to B_2 symbolized by $f : B_1 \rightarrow B_2$ if these properties are satisfied:*

- *For every element $e_{B_1} \in B_1$, there exists only one element $e_{B_2} \in B_2$ such that $f(e_{B_1}) = e_{B_2}$.*
- *For each empty BNS element $\emptyset_u \in B_1$, $f(\emptyset_u)$ is an empty BNS element for B_2.*

Definition 19. *Let B_1 and B_2 be bipolar neutrosophic soft sets in BNS(U_E) and $f : B_1 \rightarrow B_2$ be an BNS-mapping. The image of $C \subseteq B_1$ under BNS-mapping $'f'$ is the bipolar neutrosophic soft set $f(C)$ defined by $f(C) = \{\cup_{e_{B_1} \in C} f(e_{B_1}) : e \in E\}$. It is obvious that $f(\emptyset_u) = \emptyset_u$ for every BNS mapping $'f'$.*

Definition 20. *Let B_1 and B_2 be bipolar neutrosophic soft sets in BNS(U_E) and $f : B_1 \rightarrow B_2$ be an BNS-mapping. The inverse image of $D \subseteq B_2$ under BNS-mapping $'f'$ is the bipolar neutrosophic soft set $f^{-1}(D)$ defined by $f^{-1}(D) = \{\{\cup_{e_{B_1} \in B_1} e_{B_1} : e \in E\} : f(e_{B_1}) \in D$ for each $e \in E\}$.*

Example 7. *Let B_1 and B_2 be given in Example 5. Define $'f'$ as $f(e_{B_1}) = \tilde{e}_{B_2}$ for each $e \in E$, where $\tilde{e}_{B_2}$ is the greatest bipolar neutrosophic soft element for every parameter $e \in E$ i.e., if e_{B_2} is an arbitrary bipolar neutrosophic soft element in B_2 then $e_{B_2} \subseteq \tilde{e}_{B_2}$. Then*

$$\begin{aligned} f(e_{1B_1}) &= \tilde{e_1}_{B_2} \\ &= \{< x, (0.7, 0.2, 0.6, -0.2, -0.3, -0.5) >, < y, (0.1, 0.2, 0.6, -0.8, -0.2, -0.5) >\} \end{aligned}$$

for all $e_{1B_1} \in B_1$. Similarly

$$\begin{aligned} f(e_{2B_1}) &= \tilde{e_2}_{B_2} \\ &= \{< x, (0.7, 0.6, 0.1, -0.4, -0.7, -0.3) >, < y, (0.4, 0.9, 0.7, -0.5, -0.7, -0.2) >\} \end{aligned}$$

for all $e_{2B_1} \in B_1$. The image of $'f'$ can be obtained as follows:

$$\begin{aligned} f(C) &= \{\cup_{e_{B_1} \in C} f(e_{B_1}) : e \in E\} \\ &= \{\{\cup_{e_{1B_1} \in C} f(e_{1B_1})\}, \{\cup_{e_{2B_1} \in C} f(e_{2B_1})\}\} \\ &= \{\tilde{e_1}_{B_2},\ \tilde{e_2}_{B_2}\} = B_2. \end{aligned}$$

For $D = B_2$, the inverse image of $'f'$ can be obtained as follows:

$$\begin{aligned} f^{-1}(D) &= \{\{\cup_{e_{B_1} \in B_1} e_{B_1} : e \in E\} : f(e_{B_1}) \in D \text{ for each } e \in E\} \\ &= \{\{\cup_{e_{1B_1} \in B_1} e_{1B_1})\}, \{\cup_{e_{2B_1} \in B_1} e_{2B_1}\}\} \\ &= \{e_{1B_1},\ e_{2B_1}\} = B_1. \end{aligned}$$

Definition 21. *Let $(BNS(U_E), d)$ be an BNSMS. Let $f : BNS(U_E) \rightarrow BNS(U_E)$ be a function from a metric space $(BNS(U_E), d)$ into itself. A bipolar neutrosophic soft point $e_B \in BNS(U_E)$ is called a fixed point of $'f'$ if $f(e_B) = e_B$ i.e.,*

$$T^+_{f(e_B)}(x) = T^+_{e_B}(x),\ I^+_{f(e_B)}(x) = I^+_{e_B}(x),\ F^+_{f(e_B)}(x) = F^+_{e_B}(x),$$

$$T^-_{f(e_B)}(x) = T^-_{e_B}(x),\ I^-_{f(e_B)}(x) = I^-_{e_B}(x),\ F^-_{f(e_B)}(x) = F^-_{e_B}(x).$$

Example 8. *Let $BNS(U_E) = \{e_{1B},\ e_{2B},\ e_{3B}\}$ where*

$$\begin{aligned} e_{1B} =& \{< x, (0.5, 0.6, 0.3, -0.2, -0.4, -0.8) >,\ < y, (0.8, 0.3, 0.5, -0.3, -0.7, -0.6) >, \\ & < z, (0.6, 0.4, 0.8, -0.5, -0.6, -0.5) >\} \\ e_{2B} =& \{< x, (0.4, 0.6, 0.3, -0.3, -0.5, -0.1) >,\ < y, (0.5, 0.1, 0.4, -0.2, -0.3, -0.3) >, \\ & < z, (, 0.5, 0.3, 0.1, -0.6, -0.4, -0.1) >\} \\ e_{3B} =& \{< x, (0.5, 0.4, 0.1, -0.6, -0.5, -0.1) >,\ < y, (0.8, 0.4, 0.6, -0.4, -0.5, -0.7) >, \\ & < z, (0.5, 0.1, 0.7, -0.2, -0.1, -0.5) >\}. \end{aligned}$$

Let $f : BNS(U_E) \rightarrow BNS(U_E)$ be defined as follows:

$$f(e_{iB}) = \begin{cases} e_{iB}, & \text{for } i = 1 \\ e_{(i-1)B}, & \text{Otherwise} \end{cases} \tag{2}$$

Then we obtain,

$$\begin{aligned} f(e_{1B}) = e_{1B} =&\{< x,(0.5,0.6,0.3,-0.2,-0.4,-0.8)>,< y,(0.8,0.3,0.5,-0.3,-0.7,-0.6)>,\\ &< z,(0.6,0.4,0.8,-0.5,-0.6,-0.5)>\}\\ f(e_{2B}) = e_{1B} =&\{< x,(0.5,0.6,0.3,-0.2,-0.4,-0.8)>,< y,(0.8,0.3,0.5,-0.3,-0.7,-0.6)>,\\ &< z,(0.6,0.4,0.8,-0.5,-0.6,-0.5)>\}\\ f(e_{3B}) = e_{2B} =&\{< x,(0.4,0.6,0.3,-0.3,-0.5,-0.1)>,< y,(0.5,0.1,0.4,-0.2,-0.3,-0.3)>,\\ &< z,(,0.5,0.3,0.1,-0.6,-0.4,-0.2)>\}. \end{aligned}$$

Hence e_{1B} is the fixed point of $'f'$.

Definition 22. *Let $(BNS(U_E), d)$ be an BNSMS. A contraction mapping on a metric space $(BNS(U_E), d)$ is a function from $BNS(U_E)$ to itself, with the property that there is some non-negative real number $0 < \alpha < 1$ such that for all $e_B, e_p \in BNS(U_E)$,*

$$d(f(e_B), f(e_p)) \leq \alpha\, d(e_B, e_p).$$

Theorem 1. *If $(BNS(U_E), d)$ is a complete BNSMS and $T : BNS(U_E) \to BNS(U_E)$ is a contractive mapping, then there exists a unique fixed bipolar neutrosophic soft point in $BNS(U_E)$.*

Proof. Choose an element $e_{0B} \in \text{BNS}(U_E)$ and set $e_{(n+1)B} = Te_{nB} = \ldots = T^{n+1}e_{0B}, n = 1, 2, \ldots$. For convenience, by k we denote the element $d(e_{1B}, e_{0B})$ in $[0, 6]$.

$$\begin{aligned} d(e_{(n+1)B}, e_{nB}) &= d(Te_{nB,}Te_{(n-1)B}) \leq \alpha d(e_{nB}, e_{(n-1)B})\\ &\leq \alpha^2 d(e_{(n-1)B}, e_{(n-2)B})\\ &\ \ \vdots\\ &\leq \alpha^n d(e_{1B}, e_{0B})\\ &= \alpha^n k. \end{aligned} \tag{3}$$

For $n + 1 > m$, it follows that

$$\begin{aligned} d(e_{(n+1)B}, e_{mB}) &\leq d(e_{(n+1)B}, e_{nB}) + d(e_{nB}, e_{mB})\\ &\leq d(e_{(n+1)B}, e_{nB}) + d(e_{nB}, e_{(n-1)B}) + d(e_{(n-1)B}, e_{mB})\\ &\ \ \vdots\\ &\leq d(e_{(n+1)B}, e_{nB}) + d(e_{nB}, e_{(n-1)B}) + \ldots + d(e_{(m+1)B}, e_{mB})\\ &\leq \alpha^n k + \alpha^{n-1}k + \ldots \alpha^m k\\ &= \alpha^m k\Big[\frac{1-\alpha^{n-m}}{1-\alpha}\Big]\\ &\leq \alpha^m k\Big[\frac{1}{1-\alpha}\Big]\\ &\to 0 \text{ (as } m \to \infty). \end{aligned}$$

Therefore $\{e_{nB}\}$ is a Cauchy sequence. By the completeness of $(BNS(U_E), d)$, for every $\epsilon > 0$ there exists a natural number n_0 and an element $e_B \in BNS(U_E)$ such that $d(e_{nB}, e_B) < \epsilon, \forall n \geq n_0$.

$$\begin{aligned} 0 \leq d(Te_B, e_B) &\leq d(Te_B, Te_{nB}) + d(Te_{nB}, e_B)\\ &\leq \alpha\, d(e_B, e_{nB}) + d(e_{(n+1)B}, e_B)\\ &\to 0 \text{ as n} \to \infty. \end{aligned}$$

Therefore $d(Te_B, e_B) = 0$ which implies $Te_B = e_B$ i.e., e_B is a fixed point of T. Now suppose that $e_M(\neq e_B)$ is another fixed point of T. Since,

$$0 \leq d(e_B, e_M) = d(Te_B, Te_M)$$
$$\leq \alpha\, d(e_B, e_M) < d(e_B, e_M), \text{ which is impossible.}$$

Hence $d(e_B, e_M) = 0$ and $e_B = e_M$, which implies that the fixed point is unique. □

Example 9. *Consider an BNSMS $(BNS(U_E), d)$ with respect to the distance function $'d'$ defined in (1) and the function $'f'$ defined in (2) where $BNS(U_E) = \{e_{1B}, e_{2B}, e_{3B}\}$ are given as follows:*

$$e_{1B} = \{< x, (0.4, 0.6, 0.7, -0.2, -0.9, -0.3) >, < y, (0.4, 0.8, 0.7, -0.6, -0.7, -0.1) >\}$$

$$e_{2B} = \{< x, (0.3, 0.6, 0.8, -0.4, -0.8, -0.5) >, < y, (0.7, 0.6, 0.1, -0.3, -0.9, -0.4) >\}$$

$$e_{3B} = \{< x, (0.1, 0.8, 0.7, -0.9, -0.2, -0.1) >, < y, (0.6, 0.2, 0.6, -0.9, -0.1, -0.8) >\}$$

Here we see that $d(f(e_{iB}), f(e_{jB})) \leq \alpha\, d(e_{iB}, e_{jB})$ for all $e_{iB}, e_{jB} \in BNS(U_E)$ and $i, j = 1, 2, 3$ with $\alpha = \frac{1}{2}$. Hence by the above theorem $'f'$ has a unique fixed bipolar neutrosophic soft point in BNS(U_E) which is e_{1B}.

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