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A New Hesitant Fuzzy Linguistic TOPSIS Method for Group Multi-Criteria Linguistic Decision Making

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Abstract: Hesitant fuzzy linguistic decision making is a focus point in linguistic decision making, in which the main method is based on preference ordering. This paper develops a new hesitant fuzzy linguistic TOPSIS method for group multi-criteria linguistic decision making; the method is inspired by the TOPSIS method and the preference degree between two hesitant fuzzy linguistic term sets (HFLTSs). To this end, we first use the preference degree to define a pseudo-distance between two HFLTSs and analyze its properties. Then we present the positive (optimistic) and negative (pessimistic) information of each criterion provided by each decision maker and aggregate these by using weights of decision makers to obtain the hesitant fuzzy linguistic positive and negative ideal solutions. On the basis of the proposed pseudo-distance, we finally obtain the positive (negative) ideal separation matrix and a new relative closeness degree to rank alternatives. We also design an algorithm based on the provided method to carry out hesitant fuzzy linguistic decision making. An illustrative example shows the elaboration of the proposed method and comparison with the symbolic aggregation-based method, the hesitant fuzzy linguistic TOPSIS method and the hesitant fuzzy linguistic VIKOR method; it seems that the proposed method is a useful and alternative decision-making method.

Keywords: group multi-criteria decision making; the TOPSIS method; hesitant fuzzy linguistic term set; pseudo-distance; aggregation operator

1. Introduction

In real-world practices, we always face tasks and activities in which it is necessary to use decision-making processes. Generally, decision making is a cognitive process based on different mental and reasoning processes that lead to the choice of a suitable alternative from a set of possible alternatives in a decision situation [1–5]. Because of the inherent complexity and uncertainty of the decision situation or the existence of multiple and conflicting objectives, decision-making problems are complex and difficult; particularly in the era of big data, decision making becomes more complicated because the huge amounts of decision information and alternatives are continuously growing. Many new decision-making methods, such as granular computing techniques [1,6–10], have been proposed for expressing complex or uncertain information in decision-making processes and solving decision-making problems [11–23].

Nowadays, decision-making methods with hesitant fuzzy linguistic term sets (HFLTSs) are a focus point in linguistic decision making (LDM). In many qualitative decision environments, experts think of several possible linguistic values or richer expressions than a single term for an indicator, alternative, variable, and so forth. Accordingly, Rodríguez et al. [24] proposed the concept of HFLTSs to overcome the drawback of existing fuzzy linguistic approaches: the elicitation of single and very simple terms to encompass and express the qualitative information. Formally, by taking into account the idea of hesitant fuzzy sets [25] and using extended context-free grammars of a linguistic variable [26], HFLTSs allow us to use different and great flexible forms to represent decision makers' knowledge and preferences in LDM. To make a multi-criteria linguistic decision with HFLTSs, Rodríguez et al. developed the "min_upper" and "max_lower" operators to combine HFLTSs and obtain a linguistic interval for each alternative; then the linguistic intervals are used to build a preference relation between alternatives, and a nondominance choice degree is applied to obtain a solution set of alternatives for the decision problem. The use of the min_upper and max_lower operators produced the first method to deal with hesitant fuzzy linguistic information; since then, many researchers have paid attention to linguistic decision making with HFLTSs, such as in [27], where Lee and Chen proposed likelihood-based comparison relations of HFLTSs and several hesitant fuzzy linguistic aggregation operators to overcome the drawbacks of the methods in [24,28]. In [29], Liu and Rodríguez proposed a fuzzy envelope of HFLTSs for linguistic decision making with HFLTSs. In [30], Montserrat-Adell et al. provided a lattice structure of the set of HFLTSs by means of the operations intersection and connected union, and presented two distances between hesitant fuzzy linguistic sets in the lattice structure, which can be used in linguistic decision making with HFLTSs. In [31], Rodríguez et al. presented a group decision-making model based on HFLTSs. In [28], Wei et al. defined new negation, max-union and min-intersection closed operations for HFLTSs; then they proposed a hesitant fuzzy linguistic weighting averaging operator and a hesitant fuzzy linguistic ordered weighting averaging operator to deal with multi-criteria decision-making problems with HFLTSs. Up to now, operations and extensions of HFLTSs [32–38], hesitant fuzzy linguistic measures and aggregation operators [39–43], and HFLTSs in decision making [44–47] have been widely studied.

In the existing decision-making methods, despite the existence of different decision-making processes in the literature that are composed of different phases, the TOPSIS method proposed in [5] is a useful, important and widely studied multiple-attribute group decision-making method; formally, the TOPSIS method originates from the concept that the selected alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. Its decision-making process can be expressed in the following five steps [48]: (1) The normalization of the decision matrix; (2) the construction of the weighted normalized decision matrix; (3) the determination of positive and negative ideal solutions; (4) the calculation of separation measures and relative closeness; (5) the ranking of alternatives. Since then, many extended TOPSIS methods have been applied to different multiple-attribute decision making scenarios [49–55]; for example, Chen [56] proposed an extended TOPSIS method for multiple-attribute decision making by considering triangular fuzzy numbers and defining the crisp Euclidean distance between two fuzzy numbers. Similarly, Ashtiani et al. [57] extended the TOPSIS method to solve a multiple-attribute decision-making problem with interval-valued fuzzy sets. He and Gong [58] provided a natural generalization of the TOPSIS method to solve a multiple-attribute decision-making problem with intuitionistic fuzzy sets. Liu et al. [59] developed a new TOPSIS method for decision-making problems with interval-valued intuitionistic fuzzy data. Yue [60] presented a method for solving decision-making problems with an interval number and extended his method to intuitionistic fuzzy sets. In [61], Liang et al. proposed an extended TOPSIS method with linguistic neutrosophic numbers to evaluate investment risks of metallic mines. In [62], Sałabun proposed a new method to estimate the mean error of TOPSIS with the use of a fuzzy reference model.

In [63], Beg and Rashid firstly proposed the TOPSIS method for HFLTSs, in which, the one decision matrix X is calculated by aggregating the opinions of decision makers; the HFLTS positive- and negative-ideal solutions are obtained by the minimization of the minimal and maximal assessments of cost criteria and the maximization of the minimal and maximal assessments of benefit criteria; then the positive-ideal separation matrix (negative-ideal separation matrix) is constructed by distances between X and the positive-ideal (negative-ideal) solution, which can be used to obtain the relative closeness of each alternative and rank all the alternatives. In this paper, we develop a new hesitant fuzzy linguistic TOPSIS method for group multi-criteria linguistic decision making, in which, we use the preference degree to define a pseudo-distance between two HFLTSs, and we present the positive and negative information of each criterion provided by each decision maker. Making use of the weighted 2-tuple linguistic aggregation operator, we aggregate the positive and negative information provided by m decision makers to obtain the HFLTS positive- and negative-ideal solutions. Finally, we utilize the pseudo-distance to calculate distances between the assessments of the decision maker and the HFLTS positive- and negative-ideal solutions, and provide a new relative closeness degree of each alternative to rank all the alternatives. The rest of this paper is structured as follows: In Section 2, we briefly review basic concepts and operations of HFLTSs and Beg and Rashid's TOPSIS method. In Section 3, we define the pseudo-distance between two HFLTSs and analyze its properties. We provide the positive and negative information of each criterion and aggregate these to obtain the HFLTS positive- and negative-ideal solutions. Accordingly, we propose the new hesitant fuzzy linguistic TOPSIS method for group multi-criteria linguistic decision making and design an algorithm to carry out hesitant fuzzy linguistic decision making. In Section 4, we utilize an example to illustrate the practicality of the new hesitant fuzzy linguistic TOPSIS method and compare the method with Rodriguez's method [24], Beg and Rashid's method [63] and Liao's method [45]. We conclude the paper in Section 5.

2. Preliminaries

In this section, we briefly review concepts and operators of HFLTSs and the TOPSIS method for HFLTSs, and we present the two important hesitant fuzzy linguistic decision-making methods, that is, Rodriguez's method and Beg and Rashid's method.

Definition 1. *[24] Let S be a linguistic term set, $S = \{s_0; \cdots, s_g\}$, a HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.*

The basic operations on HFLTSs are as follows [24]:

1. Lower bound: $H_{S^-} = min(s_i) = s_j$, $s_i \in H_S$ and $s_i \geq s_j \forall i$;
2. Upper bound: $H_{S^+} = max(s_i) = s_j$, $s_i \in H_S$ and $s_i \leq s_j \forall i$;
3. Complement: $H_S^c = S - H_S = \{s_i | s_i \in S$ and $s_i \notin H_S\}$;
4. Union: $H_S^1 \cup H_S^2 = \{s_i | s_i \in H_S^1$ or $s_i \in H_S^2\}$;
5. Intersection: $H_S^1 \cap H_S^2 = \{s_i | s_i \in H_S^1$ and $s_i \in H_S^2\}$;
6. Envelope: $env(H_S) = [H_{S^-}, H_{S^+}]$.

Rodríguez et al. [24] proposed the min_upper and max_lower operators to obtain the core information of hesitant fuzzy linguistic assessments of each alternative; then preference degrees [64] are used to deal with multi-criteria linguistic decision making with HFLTSs. Formally, the min_upper and max_lower operators are as follows: Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives, $C = \{c_1, \ldots, c_m\}$ be a set of criteria, $S = \{s_0, \cdots, s_g\}$ be a linguistic term set, and $\{H_S^j(x_i) | i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ be a set of HFLTSs. The min_upper operator consists of the following two steps:

1. Apply the upper bound H_{S^+} for each HFLTS that is associated with each alternative:

$$H_{S^+}(x_i) = \{H_{S^+}^1(x_i), \ldots, H_{S^+}^m(x_i)\}, i \in \{1, \ldots, n\}$$

2. Obtain the minimum linguistic term for each alternative:

$$H_{S^+_{min}}(x_i) = min\{H_{S^+}^j(x_i) | j \in \{1, \ldots, m\}\}, i \in \{1, \ldots, n\}$$

The max_lower operator consists of the following two steps:

1. Apply the lower bound H_{S^-} for each HFLTS that is associated with each alternative:

$$H_{S^-}(x_i) = \{H^1_{S^-}(x_i), \ldots, H^m_{S^-}(x_i)\}, i \in \{1, \ldots, n\}$$

2. Obtain the maximum linguistic term for each alternative:

$$H_{S^-_{max}}(x_i) = max\{H^j_{S^-}(x_i)|j \in \{1, \ldots, m\}\}, i \in \{1, \ldots, n\}$$

Let $H'_{max}(x_i) = max\{H_{S^+_{min}}(x_i), H_{S^-_{max}}(x_i)\}$ and $H'_{min}(x_i) = min\{H_{S^+_{min}}(x_i), H_{S^-_{max}}(x_i)\}$; then the core information of hesitant fuzzy linguistic assessments of alternative x_i is

$$H'(x_i) = [H'_{min}(x_i), H'_{max}(x_i)]$$

On the basis of the core information of each alternative and the preference degrees [64] between two sets of core information, the nondominance degree NDD_i of each alternative can be calculated, and the best alternatives are the set of nondominated alternatives $X^{ND} = \{x_i|x_i \in X, \text{NDD}_i = max_{x_j \in X}\{\text{NDD}_j\}\}$.

Example 1. *[24] Let $X = \{x_1, x_2, x_3\}$, $C = \{c_1, c_2, c_3\}$ and $S = \{$nothing (s_0), very low (s_1), low (s_2), medium (s_3), high (s_4), very high (s_5), perfect (s_6)$\}$. Assessments provided by the decision maker are shown in Table 1.*

Table 1. Assessments of X with respect to criteria C.

	c_1	c_2	c_3
x_1	$\{s_1, s_2, s_3\}$	$\{s_4, s_5\}$	$\{s_4\}$
x_2	$\{s_2, s_3\}$	$\{s_3\}$	$\{s_0, s_1, s_2\}$
x_3	$\{s_4, s_5, s_6\}$	$\{s_1, s_2\}$	$\{s_4, s_5, s_6\}$

The min_upper operator and the max_lower operator are used to obtain the core information of each alternative, such as for alternative x_1, $H_{S^+}(x_1) = \{H^1_{S^+}(x_1), H^2_{S^+}(x_1), H^3_{S^+}(x_1)\} = \{s_3, s_5, s_4\}$, $H_{S^-}(x_1) = \{H^1_{S^-}(x_1), H^2_{S^-}(x_1), H^3_{S^-}(x_1)\} = \{s_1, s_4\}$, $H_{S^+_{min}}(x_1) = min\ \{s_3, s_5, s_4\} = s_3$, $H_{S^-_{max}}(x_1) = max\{s_1, s_4\} = s_4$, $H'_{max}(x_1) = max\ \{H_{S^+_{min}}(x_1), H_{S^-_{max}}(x_1)\} = s_4$, $H'_{min}(x_i) = min\{H_{S^+_{min}}(x_1), H_{S^-_{max}}(x_1)\} = s_3$ and $H'(x_1) = [H'_{min}(x_1), H'_{max}(x_1)] = [s_3, s_4]$. Similarly, $H'(x_2) = [s_2, s_3]$ and $H'(x_3) = [s_2, s_4]$.

On the basis of $H'(x_1)$, $H'(x_2)$ and $H'(x_3)$, preference degrees [64] between them can be calculated; for example, for $H'(x_1)$ and $H'(x_2)$, their preference degree is $p_{12} = P(a_1 > a_2) = \frac{max\{4-2,0\}-max\{3-3,0\}}{(4-3)+(3-2)} = 1$ and the binary preference relation between the three alternatives is

$$P = [p_{jj'}]_{3\times3} = \begin{pmatrix} - & 1 & 0.667 \\ 0 & - & 0.333 \\ 0.333 & 0.667 & - \end{pmatrix}$$

Then the nondominance degree NDD_i of x_i can be calculated, that is, $NDD_i = min\{1 - p^S_{ji}|j \neq i\}$ and $p^S_{ji} = max\{p_{ji} - p_{ij}, 0\}$. Such as $NDD_1 = min\{1 - p^S_{21}, 1 - p^S_{31}\} = min\{1 - max\{0 - 1, 0\}, 1 - max\{0.333 - 0.667, 0\}\} = 1$, similarly, $NDD_2 = 0$ and $NDD_3 = 0.666$; x_1 is selected.

Beg and Rashid proposed an alternative hesitant fuzzy linguistic group decision method, that is, the TOPSIS method for HFLTSs [63]. In the TOPSIS method, the main concepts are a distance between two HFLTSs and the HFLTS positive- and negative-ideal solutions, which can be formalized as follows: Let H^1_S and H^2_S be the two HFLTSs on $S = \{s_0, \cdots, s_g\}$, $env(H^1_S) = [s_p, s_q]$ and $env(H^2_S) = [s_{p'}, s_{q'}]$; then the distance between H^1_S and H^2_S is

$$d(H_S^1, H_S^2) = |q' - q| + |p' - p|$$

Let $\{X^1, \cdots, X^m\}$ be m HFLTS decision matrices provided by m decision makers; then the one decision matrix formed by aggregating the opinions of m decision makers is $X = [x_{ij}]$, where $x_{ij} = [s_{p_{ij}}, s_{q_{ij}}]$ and

$$s_{p_{ij}} = min\{min_{l=1}^m(maxH_{S_{ij}}^l), max_{l=1}^m(minH_{S_{ij}}^l)\} \tag{1}$$

$$s_{q_{ij}} = max\{min_{l=1}^m(maxH_{S_{ij}}^l), max_{l=1}^m(minH_{S_{ij}}^l)\} \tag{2}$$

Let Ω_b and Ω_c be collections of benefit and cost criteria, respectively. The HFLTS positive-ideal (negative-ideal) solution $A^+(A^-)$ is defined as follows:

$$\begin{aligned} A^+ = {} & [((max_{l=1}^m(max_i H_{S_{ij}}^l))|j \in \Omega_b, (min_{l=1}^m(min_i H_{S_{ij}}^l))|j \in \Omega_c) \\ & ((max_{l=1}^m(max_i H_{S_{ij}}^l))|j \in \Omega_b, (min_{l=1}^m(min_i H_{S_{ij}}^l))|j \in \Omega_c)] \end{aligned} \tag{3}$$

$$\begin{aligned} A^- = {} & [((min_{l=1}^m(min_i H_{S_{ij}}^l))|j \in \Omega_b, (max_{l=1}^m(max_i H_{S_{ij}}^l))|j \in \Omega_c) \\ & ((min_{l=1}^m(min_i H_{S_{ij}}^l))|j \in \Omega_b, (max_{l=1}^m(max_i H_{S_{ij}}^l))|j \in \Omega_c)] \end{aligned} \tag{4}$$

where $i = 1, \cdots, n$ is the ith considered alternative; $j = 1, \cdots, r$ is the jth criterion used for evaluating the alternatives; $A^+ = [V_1^+, \cdots, V_r^+]$; $A^- = [V_1^-, \cdots, V_r^-]$; and V_j^+ or V_j^- have the form $[v_{p_j}, v_{q_j}]$. On the basis of the distance between two HFLTSs and the HFLTS positive- and negative-ideal solutions, the positive-ideal separation matrix D^+ and negative-ideal separation matrix D^- between X and the positive- and negative-ideal solutions can be calculated as follows:

$$D^+ = \begin{pmatrix} d(x_{11}, V_1^+) + \cdots + d(x_{1r}, V_r^+) \\ \vdots \\ d(x_{n1}, V_1^+) + \cdots + d(x_{nr}, V_r^+) \end{pmatrix}, \quad D^- = \begin{pmatrix} d(x_{11}, V_1^-) + \cdots + d(x_{1r}, V_r^-) \\ \vdots \\ d(x_{n1}, V_1^-) + \cdots + d(x_{nr}, V_r^-) \end{pmatrix} \tag{5}$$

Accordingly, the relative closeness (RC) of each alternative to the ideal solution is as follows:

$$RC(A_i) = \frac{D_i^-}{D_i^+ + D_i^-} \tag{6}$$

where $D_i^+ = \sum_{j=1}^r d(x_{ij}, V_j^+)$ and $D_i^- = \sum_{j=1}^r d(x_{ij}, V_j^-)$. Ranking alternatives are carried out by using the following rule: the greater the value of $RC(A_i)$, the better the alternative A_i.

3. The Proposed TOPSIS for HFLTSs

In this section, we develop a new hesitant fuzzy linguistic TOPSIS method for linguistic decision-making problems. Compared with Beg and Rashid's TOPSIS method, there are three different aspects: (1) We use the preference degree to define a pseudo-distance between two HFLTSs; (2) We present the positive and negative information of each criterion provided by each decision maker; considering weights of decision makers, we aggregate the positive and negative information provided by all decision makers to obtain the HFLTS positive- and negative-ideal solutions, respectively; (3) We propose a new relative closeness degree to rank alternatives. All of these are elaborated on in the following subsections.

3.1. A Pseudo-Distance between Two HFLTSs

The preference degree between two HFLTSs has been studied by many researchers [24,27,28,65]; generally, we let H_S^1 and H_S^2 be the two HFLTSs on $S = \{s_0, \cdots, s_g\}$, $env(H_S^1) = [s_p, s_q]$ and $env(H_S^2) = [s_{p'}, s_{q'}]$. Then the preference degree $p(H_S^1 \geq H_S^2)$ between H_S^1 and H_S^2 is as follows:

$$p(H_S^1 \geq H_S^2) = max\{1 - max\{\frac{q' - p}{(q - p) + (q' - p')}, 0\}, 0\} \tag{7}$$

For example, let $S = \{$nothing (s_0), very low (s_1), low (s_2), medium (s_3), high (s_4), very high (s_5), perfect $(s_6)\}$. For HFLTSs $H_S^1 = \{s_3, s_4, s_5\}$ and $H_S^2 = \{s_2, s_3, s_4\}$, $env(H_S^1) = [s_3, s_5]$ and $env(H_S^2) = [s_2, s_4]$; then

$$p(H_S^1 \geq H_S^2) = max\{1 - max\{\frac{4 - 3}{2 + 2}, 0\}, 0\} = \frac{3}{4}$$

Formally, the preference degree between two HFLTSs has the following properties: (1) $0 \leq p(H_S^1 \geq H_S^2) \leq 1$; (2) $p(H_S^1 \geq H_S^2) + p(H_S^2 \geq H_S^1) = 1$; (3) if $q' \leq p$, then $p(H_S^1 \geq H_S^2) = 1$; (4) if $q \leq p'$, then $p(H_S^1 \geq H_S^2) = 0$.

Definition 2. *Let H_S^1 and H_S^2 be any two HFLTSs on $S = \{s_0, \cdots, s_g\}$ and the HFLTS H_S on S be the reference set; then we define the following:*

$$d(H_S^1, H_S^2) = |p(H_S^1 \geq H_S) - p(H_S^2 \geq H_S)| \tag{8}$$

Intuitively, $d(H_S^1, H_S^2)$ is the difference of preference degrees between two HFLTSs (H_S^1 and H_S^2) and the the reference set H_S. According to Equations (7) and (8), we have the following property:

Proposition 1. *Let H_S^1, H_S^2 and H_S^3 be HFLTSs on $S = \{s_0, \cdots, s_g\}$ and the HFLTS H_S on S be the reference set; then*

1. $d(H_S^1, H_S^2) \geq 0$;
2. $d(H_S^1, H_S^2) = d(H_S^2, H_S^1)$;
3. $d(H_S^1, H_S^2) \leq d(H_S^1, H_S^3) + d(H_S^3, H_S^2)$.

Proof. According to Equation (8), $d(H_S^1, H_S^2) \geq 0$ and $d(H_S^1, H_S^2) = d(H_S^2, H_S^1)$ is apparent. Here, we prove Proposition 1. as follows:

$$\begin{aligned} d(H_S^1, H_S^2) &= |p(H_S^1 \geq H_S) - p(H_S^2 \geq H_S)| \\ &= |p(H_S^1 \geq H_S) - p(H_S^3 \geq H_S) + p(H_S^3 \geq H_S) - p(H_S^2 \geq H_S)| \\ &\leq |p(H_S^1 \geq H_S) - p(H_S^3 \geq H_S)| + |p(H_S^3 \geq H_S) - p(H_S^2 \geq H_S)| \\ &= d(H_S^1, H_S^3) + d(H_S^3, H_S^2) \end{aligned}$$

That is, $d(H_S^1, H_S^2) \leq d(H_S^1, H_S^3) + d(H_S^3, H_S^2)$ holds. □

Proposition 1 means that $d(H_S^1, H_S^2)$ is the pseudo-distance between HFLTSs H_S^1 and H_S^2 on $S = \{s_0, \cdots, s_g\}$. In fact, we let $env(H_S^1) = [s_p, s_q]$, $env(H_S^2) = [s_{p'}, s_{q'}]$ and $env(H_S) = [s_{p_0}, s_{q_0}]$. According to Equation (8), if $H_S^1 = H_S^2$, then $d(H_S^1, H_S^2) = 0$ is apparent. If $d(H_S^1, H_S^2) = 0$, we have $p(H_S^1 \geq H_S) = p(H_S^2 \geq H_S)$, and according to Equation (7), we have

$$max\{1 - max\{\frac{q_0 - p}{(q - p) + (q_0 - p_0)}, 0\}, 0\} = max\{1 - max\{\frac{q_0 - p'}{(q' - p') + (q_0 - p_0)}, 0\}, 0\}$$

If $p \neq p'$, $q_0 \leq p$ and $q_0 \leq p'$, then $p(H_S^1 \geq H_S) = p(H_S^2 \geq H_S) = 1$; this means that $d(H_S^1, H_S^2) = 0$ if and only if $H_S^1 = H_S^2$ does not always hold. As a special case, in Equation (8), we consider the condition $H_S^2 = H_S$, that is, the pseudo-distance between HFLTS H_S^1 and the reference set H_S. According to property 2 of the preference degree, $p(H_S^2 \geq H_S) + p(H_S \geq H_S^2) = 1$; thus $2p(H_S \geq H_S) = 1$, that is,

$p(H_S \geq H_S) = 0.5$, and hence the pseudo-distance between H_S^1 and the reference set H_S is reduced as follows:

$$d(H_S^1, H_S) = |p(H_S^1 \geq H_S) - p(H_S \geq H_S)| = |p(H_S^1 \geq H_S) - 0.5| = \begin{cases} 0.5, & \text{if } q_0 \leq p \text{ or } q \leq p_0 \\ |\frac{q-p_0}{(q-p)+(q_0-p_0)} - 0.5|, & \text{otherwise} \end{cases} \tag{9}$$

Accordingly, we can develop an ordering of HFLTSs on $S = \{s_0, \cdots, s_g\}$ on the basis of the reference set; that is, for any two HFLTSs H_S^1 and H_S^2 on $S = \{s_0, \cdots, s_g\}$ and the reference set H_S, $H_S^1 \succeq_{H_S} H_S^2$ if and only if $d(H_S^1, H_S) \leq d(H_S^2, H_S)$. Intuitively, the order $\succeq_{H_S}$ on HFLTSs means that the closer the HFLTS H_S^1 is to the reference set H_S, the bigger H_S^1 is. According to Equation (9), it can be easily proved that the order $\succeq_{H_S}$ on HFLTSs is a pre-order, that is, $\succeq_{H_S}$ satisfies the following:

1. The reflexive property: $H_S^1 \succeq_{H_S} H_S^1$.
2. Transitivity: if $H_S^1 \succeq_{H_S} H_S^2$ and $H_S^2 \succeq_{H_S} H_S^3$, then $H_S^1 \succeq_{H_S} H_S^3$.

We note that if HF_S is the set of all HFLTSs on $S = \{s_0, \cdots, s_g\}$, then $(HF_S, \succeq_{H_S})$ is a pre-order set.

Example 2. *Let $S = \{nothing\ (s_0), very\ low\ (s_1), low\ (s_2), medium\ (s_3), high\ (s_4), very\ high\ (s_5), perfect\ (s_6)\}$. For the HFLTSs $H_S^1 = \{s_3, s_4, s_5\}$ and $H_S^2 = \{s_2, s_3, s_4\}$, and $env(H_S^1) = [s_3, s_5]$ and $env(H_S^2) = [s_2, s_4]$, suppose the reference set $H_S = \{s_4, s_5\}$ and $env(H_S) = [s_4, s_5]$. According to Equation (9), we have $d(H_S^1, H_S) = |\frac{5-4}{(5-3)+(5-4)} - 0.5| = \frac{1}{6}$ and $d(H_S^2, H_S) = 0.5$, that is, $H_S^1 \succeq_{H_S} H_S^2$ as a result of $d(H_S^1, H_S) = \frac{1}{6} < d(H_S^2, H_S) = 0.5$.*

3.2. The HFLTS Positive- and Negative-Ideal Solutions

A group multi-criteria hesitant fuzzy linguistic decision-making problem is described as follows: m decision makers $M = \{d_1, \cdots, d_m\}$ are asked to assess n alternatives $A = \{a_1, \cdots, a_n\}$ with respect to r criteria $C = \{c_1, \cdots, c_r\}$ by using HFLTSs on $S = \{s_0, \cdots, s_g\}$; formally, decision maker $d_i (i = 1, \cdots, m)$ provides the decision matrix to express his or her assessments, that is,

$$D_i = (e_{jk}^i)_{n \times r} = \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & \cdots & c_r \end{array} \\ \begin{pmatrix} e_{11}^i & \cdots & e_{1r}^i \\ \vdots & \vdots & \vdots \\ e_{n1}^i & \cdots & e_{nr}^i \end{pmatrix} \end{array} \tag{10}$$

where $e_{jk}^i \in HF_S (j \in \{1, \cdots, n\}, k \in \{1, \cdots, r\})$ means that decision maker $d_i (i = 1, \cdots, m)$ assesses alternative a_j with respect to criterion c_k by using the HFLTS e_{jk}^i on $S = \{s_0, \cdots, s_g\}$. On the basis of the decision matrix D_i, we provide the following definitions.

Definition 3. *In the decision matrix D_i, the positive information of each c_k provided by decision maker d_i is*

$$C_i(c_k) = [max\{min(e_{jk}^i) | j = 1, \cdots, n\}, max\{max(e_{jk}^i) | j = 1, \cdots, n\}] \tag{11}$$

The negative information of c_k provided by decision maker d_i is

$$H_i(c_k) = [min\{min(e_{jk}^i) | j = 1, \cdots, n\}, min\{max(e_{jk}^i) | j = 1, \cdots, n\}] \tag{12}$$

Example 3. *Let $A = \{a_1, a_2, a_3\}$ be a set of three alternatives, $C = \{c_1, c_2, c_3\}$ be a set of criteria defined for each alternative and $S = \{nothing\ (s_0), very\ low\ (s_1), low\ (s_2), medium\ (s_3), high\ (s_4), very\ high\ (s_5), perfect\ (s_6)\}$ be the linguistic term set. The assessments provided by decision maker d_i are shown in Table 2.*

Table 2. Assessments of A with respect to criteria C provided by decision maker d_i.

		c_1	c_2	c_3
	a_1	$\{s_1, s_2, s_3\}$	$\{s_4, s_5\}$	$\{s_3, s_4\}$
d_i	a_2	$\{s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_0, s_1, s_2\}$
	a_3	$\{s_4, s_5, s_6\}$	$\{s_1, s_2\}$	$\{s_4, s_5, s_6\}$

For the criterion c_1, we have $e^i_{11} = \{s_1, s_2, s_3\}$, $e^i_{21} = \{s_2, s_3\}$ and $e^i_{31} = \{s_4, s_5, s_6\}$; hence $max\{min(e^i_{j1})|j = 1,2,3\} = max\{min\{s_1, s_2, s_3\}, min\{s_2, s_3\}, min\{s_4, s_5, s_6\}\} = max\{s_1, s_2, s_4\} = s_4$, $max\{max(e^i_{j1})|j = 1,2,3\} = max\{max\{s_1, s_2, s_3\}, max\{s_2, s_3\}, max\{s_4, s_5, s_6\}\} = max\{s_3, s_6\} = s_6$, $min\{min(e^i_{j1})|j = 1,2,3\} = min\{min\{s_1, s_2, s_3\}, min\{s_2, s_3\}, min\{s_4, s_5, s_6\}\} = min\{s_1, s_2, s_4\} = s_1$ and $min\{max(e^i_{j1})|j = 1,2,3\} = min\{max\{s_1, s_2, s_3\}, max\{s_2, s_3\}, max\{s_4, s_5, s_6\}\} = min\{s_3, s_6\} = s_3$; that is, the positive information and negative information of c_1 provided by decision maker d_i are $C_i(c_1) = [s_4, s_6]$ and $H_i(c_1) = [s_1, s_3]$, respectively.

We can notice from Example 3 that the positive information of c_k is the optimistic information according to assessments of all alternatives provided by decision maker d_i; the negative information of c_k is the pessimistic information according to assessments of all alternatives provided by decision maker d_i. Compared with Beg and Rashid's method [63], Equations (1) and (2) are aimed at aggregating the opinions of m decision makers; the result is the one decision matrix $X_{n\times r}$. However, Equations (11) and (12) are used to aggregate the opinions of n alternatives provided by decision maker d_i with respect to the criterion c_k; the results are the optimistic information vector $(C_i(c_1), \cdots, C_i(c_r))$ and the pessimistic information vector $(H_i(c_1), \cdots, H_i(c_r))$ provided by decision maker d_i with respect to the criteria.

In a group multi-criteria hesitant fuzzy linguistic decision-making problem, for each criterion c_k, we denote the positive and negative information of c_k as $C_i(c_k) = [s_{p_{ki}}, s_{q_{ki}}]$ and $H_i(c_k) = [s_{p'_{ki}}, s_{q'_{ki}}]$ provided by decision maker $d_i (i = 1, \cdots, m)$. We suppose weights $W = \{w_1, \cdots, w_m\}$ of m decision makers; then we can use the weighted 2-tuple linguistic aggregation operator [66,67] to obtain the positive and negative information of c_k provided by m decision makers, that is,

$$C(c_k) = \sum_{i=1}^{m} w_i C_i(c_k) = \sum_{i=1}^{m} w_i [s_{p_{ki}}, s_{q_{ki}}] = [s_{p_k}, s_{q_k}] \tag{13}$$

$$H(c_k) = \sum_{i=1}^{m} w_i H_i(c_k) = \sum_{i=1}^{m} w_i [s_{p'_{ki}}, s_{q'_{ki}}] = [s_{p'_k}, s_{q'_k}] \tag{14}$$

where $p_k = round(\sum_{i=1}^m w_i p_{ki})$, $q_k = round(\sum_{i=1}^m w_i q_{ki})$, $p'_k = round(\sum_{i=1}^m w_i p'_{ki})$ and $q'_k = round(\sum_{i=1}^m w_i q'_{ki})$, and $round(\cdot)$ is the usual round operation. For example, let the positive information of c_1 be $C_1(c_1) = [s_3, s_4]$, $C_2(c_1) = [s_4, s_6]$ and $C_3(c_1) = [s_2, s_5]$, which are provided by three decision makers with weights $W = \{0.3, 0.5, 0.2\}$ in a group multi-criteria hesitant fuzzy linguistic decision-making problem; then the positive information of c_1 provided by the three decision makers is

$$\begin{aligned} C(c_1) &= \sum_{i=1}^{3} w_i C_i(c_k) = \sum_{i=1}^{m} w_i [s_{p_{ki}}, s_{q_{ki}}] = [s_{round(\sum_{i=1}^3 w_i p_{ki})}, s_{round(\sum_{i=1}^3 w_i q_{ki})}] \\ &= [s_{round(0.3\times3+0.5\times4+0.2\times2)}, s_{round(0.3\times4+0.5\times6+0.2\times5)}] = [s_{round(3.3)}, s_{round(5.2)}] = [s_3, s_5] \end{aligned}$$

Definition 4. *In a group multi-criteria hesitant fuzzy linguistic decision-making problem, $(C(c_1), \cdots, C(c_r))$ and $(H(c_1), \cdots, H(c_r))$ are called the HFLTS positive-ideal solution (HPIS) and the HFLTS negative-ideal solution (HNIS), where $C(c_k)$ and $H(c_k)$ $(k = 1, \cdots, r)$ are decided by Equations (13) and (14), respectively.*

3.3. The New Hesitant Fuzzy Linguistic TOPSIS Method

On the basis of Sections 3.1 and 3.2, we propose a hesitant fuzzy linguistic TOPSIS method that involves the following steps:

Step 1: Let m decision makers $M = \{d_1, \cdots, d_m\}$ be asked to assess n alternatives $A = \{a_1, \cdots, a_n\}$ with respect to r criteria $C = \{c_1, \cdots, c_r\}$ by using HFLTSs on $S = \{s_0, \cdots, s_g\}$; decision maker $d_i (i = 1, \cdots, m)$ with weight w_i provides the decision matrix $D_i = (e^i_{jk})_{n\times r}$ to express his or her assessments, where $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$.

Step 2: For each decision matrix $D_i = (e^i_{jk})_{n\times r}$, making use of Equations (11) and (12), we obtain the positive information $C_i(c_k) = [s_{p_{ki}}, s_{q_{ki}}]$ and the negative information $H_i(c_k) = [s_{p'_{ki}}, s_{q'_{ki}}]$ of $c_k (k = 1, \cdots, r)$. Then we utilize weight $w_i (i = 1, \cdots, m)$ and Equations (13) and (14) to calculate the positive and negative information $C(c_k) = [s_{p_k}, s_{q_k}]$ and $H(c_k) = [s_{p'_k}, s_{q'_k}]$ of each $c_k (k = 1, \cdots, r)$ provided by m decision makers; we can obtain the HFLTS positive- and negative-ideal solutions as follows:

$$HPIS = (C(c_1), \cdots, C(c_r)),\ HNIS = (H(c_1), \cdots, H(c_r)) \tag{15}$$

Step 3: We calculate the one decision matrix D by aggregating assessments of decision makers; that is, we use weights $(w_1, \cdots, w_m)$ and the weighted 2-tuple linguistic aggregation operator to aggregate m decision matrices $(D_1, \cdots, D_m)$:

$$\begin{aligned} D &= (e_{jk})_{n\times r} = \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & \cdots & c_r \end{array} \\ \begin{pmatrix} [s_{p_{11}}, s_{q_{11}}] & \cdots & [s_{p_{1r}}, s_{q_{1r}}] \\ \vdots & \vdots & \vdots \\ [s_{p_{n1}}, s_{q_{n1}}] & \cdots & [s_{p_{nr}}, s_{q_{nr}}] \end{pmatrix} \end{array} = \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & \cdots & c_r \end{array} \\ \begin{pmatrix} \sum_{i=1}^m w_i e^i_{11} & \cdots & \sum_{i=1}^m w_i e^i_{1r} \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^m w_i e^i_{n1} & \cdots & \sum_{i=1}^m w_i e^i_{nr} \end{pmatrix} \end{array} \\ &= \begin{array}{c} \\ a_1 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & \cdots & c_r \end{array} \\ \begin{pmatrix} [\sum_{i=1}^m w_i s_{p^i_{11}}, \sum_{i=1}^m w_i s_{q^i_{11}}] & \cdots & [\sum_{i=1}^m w_i s_{p^i_{1r}}, \sum_{i=1}^m w_i s_{q^i_{1r}}] \\ \vdots & \vdots & \vdots \\ [\sum_{i=1}^m w_i s_{p^i_{n1}}, \sum_{i=1}^m w_i s_{q^i_{n1}}] & \cdots & [\sum_{i=1}^m w_i s_{p^i_{nr}}, \sum_{i=1}^m w_i s_{q^i_{nr}}] \end{pmatrix} \end{array} \end{aligned} \tag{16}$$

where $p_{jk} = round(\sum_{i=1}^m w_i p^i_{jk})$ and $q_{jk} = round(\sum_{i=1}^m w_i q^i_{jk})$ for every $j = 1, \cdots, n$ and $k = 1, \cdots, r$, respectively, and $round(\cdot)$ is the usual round operation.

Step 4: On the basis of Equation (9) and the HFLTS positive- and negative-ideal solutions of Equation (15), we calculate the positive-ideal separation matrix D^+ and the negative-ideal separation matrix D^- between assessments of decision makers and the HFLTS positive- and negative-ideal solutions, that is,

$$D^+ = \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \begin{pmatrix} d(e_{11}, C(c_1)) + \cdots + d(e_{1r}, C(c_r)) \\ \vdots \\ d(e_{n1}, C(c_1)) + \cdots + d(e_{nr}, C(c_r)) \end{pmatrix} \tag{17}$$

$$D^- = \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \begin{pmatrix} d(e_{11}, H(c_1)) + \cdots + d(e_{1r}, H(c_r)) \\ \vdots \\ d(e_{n1}, H(c_1)) + \cdots + d(e_{nr}, H(c_r)) \end{pmatrix} \tag{18}$$

where $d(e_{jk}, C(c_k))$ is the pseudo-distance between $e_{jk} = [s_{p_{jk}}, s_{q_{jk}}]$ and the reference set $C(c_k)$, and $d(e_{jk}, H(c_k))$ is the pseudo-distance between e_{jk} and the reference set $H(c_k)$.

If we consider weights $V = \{v_1, \cdots, v_r\}$ of r criteria such that $v_k \geq 0$ and $\sum_{k=1}^r v_k = 1$, then the positive-ideal separation matrix D^+ and the negative-ideal separation matrix D^- have the following forms:

$$D^{+} = \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \begin{pmatrix} v_1 d(e_{11}, C(c_1)) + \cdots + v_r d(e_{1r}, C(c_r)) \\ \vdots \\ v_1 d(e_{n1}, C(c_1)) + \cdots + v_r d(e_{nr}, C(c_r)) \end{pmatrix} \tag{19}$$

$$D^{-} = \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \begin{pmatrix} v_1 d(e_{11}, H(c_1)) + \cdots + v_r d(e_{1r}, H(c_r)) \\ \vdots \\ v_1 d(e_{n1}, H(c_1)) + \cdots + v_r d(e_{nr}, H(c_r)) \end{pmatrix} \tag{20}$$

Step 5: The ranking of alternatives in the original TOPSIS method is based on "the shortest distance from the positive-ideal solution and the farthest from the negative-ideal solution"; formally, this is also fulfilled by the relative closeness degree of each alternative in the existing TOPSIS methods. In the paper, on the basis of D^+ and D^-, we provide the following relative closeness degree $RC(a_j)$ of each alternative:

$$D_j^{+} = d(e_{j1}, C(c_1)) + \cdots + d(e_{jr}, C(c_r)) = \sum_{k=1}^{r} d(e_{jk}, C(c_k)) \tag{21}$$

$$D_j^{-} = d(e_{j1}, H(c_1)) + \cdots + d(e_{jr}, H(c_r)) = \sum_{k=1}^{r} d(e_{jk}, H(c_k)) \tag{22}$$

$$D_{min}^{+} = min\{D_1^{+}, \cdots, D_n^{+}\}, D_{max}^{-} = max\{D_1^{-}, \cdots, D_n^{-}\} \tag{23}$$

$$RC(a_j) = \frac{D_{min}^{+}}{D_j^{+}} + \frac{D_j^{-}}{D_{max}^{-}} \tag{24}$$

Formally, the relative closeness degree $RC(a_j)$ of each alternative a_j is in $[0, 2]$. More importantly, $RC(a_j)$ is a monotone function in its components; that is, $RC(a_j)$ is increasing for D_j^- and decreasing for D_j^+. This is coincidental with "the shortest distance from the positive-ideal solution and the farthest from the negative-ideal solution".

Step 6: Rank all the alternatives $a_j (j = 1, \cdots, n)$ according to the relative closeness degree $RC(a_j)$. The greater the value $RC(a_j)$, the better the alternative a_j; that is, for any $j, j' \in \{1, \cdots, n\}$, $a_j \succeq a_{j'}$ if and only if $RC(a_j) \geq RC(a_{j'})$.

On the basis of the above-mentioned six steps, we provide the following algorithm to implement the new hesitant fuzzy linguistic TOPSIS method to solve hesitant fuzzy linguistic group multi-criteria decision-making problems.

4. Numerical Example

In this section, we utilize an example to illustrate the practicality of the new hesitant fuzzy linguistic TOPSIS method, and compare it with Rodriguez's method [24], Beg and Rashid's method [63] and Liao's method [45].

Example 4. *Let $A = \{a_1, a_2, a_3\}$ be a set of three alternatives and $C = \{c_1, c_2, c_3\}$ be a set of criteria defined for each alternative. Three decision makers $M = \{d_1, d_2, d_3\}$ with weights $W = \{0.3, 0.5, 0.2\}$ use linguistic terms $S = \{$nothing (s_0), very low (s_1), low (s_2), medium (s_3), high (s_4), very high (s_5), perfect (s_6)$\}$ to assess three alternatives with respect to three criteria (shown in Table 3).*

Table 3. The hesitant fuzzy linguistic assessments of alternatives provided by decision makers.

		c_1	c_2	c_3
$d_1(0.3)$	a_1	$\{s_4, s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_4, s_5, s_6\}$
	a_2	$\{s_4, s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_1, s_2, s_3\}$
	a_3	$\{s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_6\}$
$d_2(0.5)$	a_1	$\{s_4, s_5\}$	$\{s_5, s_6\}$	$\{s_3, s_4, s_5\}$
	a_2	$\{s_3, s_4, s_5\}$	$\{s_4, s_5\}$	$\{s_2, s_3\}$
	a_3	$\{s_2, s_3, s_4\}$	$\{s_3, s_4\}$	$\{s_4, s_5\}$
$d_3(0.2)$	a_1	$\{s_3, s_4\}$	$\{s_4, s_5\}$	$\{s_5, s_6\}$
	a_2	$\{s_5, s_6\}$	$\{s_3, s_4\}$	$\{s_3, s_4\}$
	a_3	$\{s_4, s_5\}$	$\{s_3, s_4, s_5\}$	$\{s_5, s_6\}$

By using the new hesitant fuzzy linguistic TOPSIS method (Algorithm 1), the hesitant fuzzy linguistic group multi-criteria decision making can be carried out as follows:

(1) *On the basis of Table 3, we can obtain three decision matrices provided by the three decision makers, as follows:*

$$D_1 = \begin{pmatrix} [s_4, s_6] & [s_5, s_6] & [s_4, s_6] \\ [s_4, s_6] & [s_4, s_6] & [s_1, s_3] \\ [s_5, s_6] & [s_4, s_6] & [s_6, s_6] \end{pmatrix}, \; D_2 = \begin{pmatrix} [s_4, s_5] & [s_5, s_6] & [s_3, s_5] \\ [s_3, s_5] & [s_4, s_5] & [s_2, s_3] \\ [s_2, s_4] & [s_3, s_4] & [s_4, s_5] \end{pmatrix}$$

$$D_3 = \begin{pmatrix} [s_3, s_4] & [s_4, s_5] & [s_5, s_6] \\ [s_5, s_6] & [s_3, s_4] & [s_3, s_4] \\ [s_4, s_5] & [s_3, s_5] & [s_5, s_6] \end{pmatrix}$$

(2) *On the basis of Equations (11)–(14), we can calculate the positive and negative information of each criterion provided by the three decision makers. For example, for criterion c_1, the positive and negative information provided by decision maker d_1 are $C_1(c_1) = [max\{s_4, s_5\}, max\{s_6\}] = [s_5, s_6]$ and $H_1(c_1) = [min\{s_4, s_5\}, min\{s_6\}] = [s_4, s_6]$; similarly, $C_2(c_1) = [s_4, s_5]$, $H_2(c_1) = [s_2, s_4]$, $C_3(c_1) = [s_5, s_6]$ and $H_3(c_1) = [s_3, s_4]$. Making use of the weights $(0.3, 0.5, 0.2)$, we obtain $C(c_1) = [s_{round(0.3\times5+0.5\times4+0.2\times5)}, s_{round(0.3\times6+0.5\times5+0.2\times6)}] = [s_5, s_6]$ and $H(c_1) = [s_{round(0.3\times4+0.5\times2+0.2\times3)}, s_{round(0.3\times6+0.5\times4+0.2\times4)}] = [s_3, s_5]$; the others are shown in Table 4.*

Table 4. The positive and negative information of each criterion provided by decision makers.

		c_1	c_2	c_3
$d_1(0.3)$	C_1	$[s_5, s_6]$	$[s_5, s_6]$	$[s_6, s_6]$
	H_1	$[s_4, s_6]$	$[s_4, s_6]$	$[s_1, s_3]$
$d_2(0.5)$	C_2	$[s_4, s_5]$	$[s_5, s_6]$	$[s_4, s_5]$
	H_2	$[s_2, s_4]$	$[s_3, s_4]$	$[s_2, s_3]$
$d_3(0.2)$	C_3	$[s_5, s_6]$	$[s_4, s_5]$	$[s_5, s_6]$
	H_3	$[s_3, s_4]$	$[s_3, s_4]$	$[s_3, s_4]$
	C	$[s_5, s_6]$	$[s_5, s_6]$	$[s_5, s_6]$
	H	$[s_3, s_5]$	$[s_3, s_5]$	$[s_2, s_3]$

(3) *On the basis of the weights $(0.3, 0.5, 0.2)$, we aggregate D_1, D_2 and D_3 to obtain the one decision matrix D, that is,*

$$D = 0.3D_1 + 0.5D_2 + 0.2D_3 = \begin{pmatrix} [s_4, s_5] & [s_5, s_6] & [s_4, s_6] \\ [s_4, s_6] & [s_4, s_5] & [s_2, s_3] \\ [s_3, s_5] & [s_3, s_5] & [s_5, s_6] \end{pmatrix}$$

(4) *On the basis of the one decision matrix D and the HFLTS positive- and negative-ideal solutions* $HPIS = ([s_5, s_6], [s_5, s_6], [s_5, s_6])$ *and* $HNIS = ([s_3, s_5], [s_3, s_5], [s_2, s_3])$, *we use Equations (9), (17) and (18) to calculate the positive- and negative-ideal separation matrices* D^+ *and* D^-, *that is,*

$$\begin{aligned} D^+ &= \begin{pmatrix} d([s_4, s_5], [s_5, s_6]) + d([s_5, s_6], [s_5, s_6]) + d([s_4, s_6], [s_5, s_6]) \\ d([s_4, s_6], [s_5, s_6]) + d([s_4, s_5], [s_5, s_6]) + d([s_2, s_3], [s_5, s_6]) \\ d([s_3, s_5], [s_5, s_6]) + d([s_3, s_5], [s_5, s_6]) + d([s_5, s_6], [s_5, s_6]) \end{pmatrix} \\ &\doteq \begin{pmatrix} 0.5 + 0 + 0.17 \\ 0.17 + 0.5 + 0.5 \\ 0.5 + 0.5 + 0 \end{pmatrix} = \begin{pmatrix} 0.67 \\ 1.17 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} D^- &= \begin{pmatrix} d([s_4, s_5], [s_3, s_5]) + d([s_5, s_6], [s_3, s_5]) + d([s_4, s_6], [s_2, s_3]) \\ d([s_4, s_6], [s_3, s_5]) + d([s_4, s_5], [s_3, s_5]) + d([s_2, s_3], [s_2, s_3]) \\ d([s_3, s_5], [s_3, s_5]) + d([s_3, s_5], [s_3, s_5]) + d([s_5, s_6], [s_2, s_3]) \end{pmatrix} \\ &= \begin{pmatrix} 0.17 + 0.5 + 0.5 \\ 0.25 + 0.17 + 0 \\ 0 + 0 + 0.5 \end{pmatrix} = \begin{pmatrix} 1.17 \\ 0.42 \\ 0.5 \end{pmatrix} \end{aligned}$$

(5) *On the basis of Equations (21)–(24), we obtain the relative closeness degrees* $RC(a_j)$ *of each alternative, which are shown in Table 5.*

Table 5. The relative closeness degrees of alternatives.

	D_j^+	D_j^-	$RC(a_j)$
a_1	0.67	1.17	$\frac{0.67}{0.67} + \frac{1.17}{1.17} = 2$
a_2	1.17	0.42	$\frac{0.67}{1.17} + \frac{0.42}{1.17} \doteq 0.92$
a_3	1	0.5	$\frac{0.67}{1} + \frac{0.5}{1.17} \doteq 1.1$
	$D_{min}^+ = 0.67$	$D_{max}^- = 1.17$	

(6) *According to* $RC(a_j)$ *of each alternative in Table 5, we obtain that the ranking of alternatives is* $a_1 \succeq a_3 \succeq a_2$, *given that* $RC(a_1) > RC(a_3) > RC(a_2)$, *and that* a_1 *is the the most satisfying alternative.*

In the following, we compare the new hesitant fuzzy linguistic TOPSIS method with Rodriguez's method [24], Beg and Rashid's method [63] and Liao's method [45] by using Example 4, in which, because Rodriguez's method and Liao's method are used to carry out the hesitant fuzzy linguistic multi-criteria decision making, we use the decision matrix D_1 provided by decision maker d_1 as the hesitant fuzzy linguistic multi-criteria of decision making for comparison with the three methods. Then we use Beg and Rashid's method in Example 4 and compare this with the proposed method.

Algorithm 1: The new hesitant fuzzy linguistic TOPSIS method

Input The decision matrix $D_i = (e^i_{jk})_{n\times r}(i = 1, \cdots, m)$ and weights $(w_1, \cdots, w_m)$ of m decision makers.
Output The ranking of n alternatives $A = \{a_1, \cdots, a_n\}$ and the most satisfying alternative A.
Begin
for each $i = 1, \cdots, m$ and $k = 1, \cdots, r$ **do**
$C_i(c_k) = [s_{p_{ki}}, s_{q_{ki}}]$ and $H_i(c_k) = [s_{p'_{ki}}, s_{q'_{ki}}]$ (in each D_i by using Equations (11) and (12) to obtain the positive and negative information)
end
for $i = 1 : m$ and each $k = 1, \cdots, r$ **do**
$C(c_k) = [s_{p_k}, s_{q_k}]$ and $H(c_k) = [s_{p'_k}, s_{q'_k}]$ (using weight $w_i(i = 1, \cdots, m)$, and Equations (13) and (14) to obtain the positive and negative information of each c_k)
$HPIS = (C(c_1), \cdots, C(c_r))$ and $HNIS = (H(c_1), \cdots, H(c_r))$ (the HFLTS positive- and negative-ideal solutions)
end
for $i = 1 : m$ **do**
$D = \sum_{i=1}^{m} w_i D_i$ (using weight w_i and Equation (16) to obtain the one decision matrix)
end
for $k = 1 : r$ **do**
$D^+ = d(D, HPIS)$ and $D^- = d(D, HNIS)$ (using Equations (9), (17) and (18) or (19) and (20) to obtain positive- and negative-ideal separation matrices)
end
for $j = 1 : n$ **do**
D^+_{min} and D^-_{max} (in D^+ and D^- using Equation (23))
$RC(a_j) = \frac{D^+_{min}}{D^+_j} + \frac{D^-_j}{D^-_{max}}$ (using Equation (24) to obtain the relative closeness degree of each alternative)
end
Output $A = \{a_j | \forall j' \in \{1, \cdots, n\}, RC(a_j) \geq RC(a_{j'})\}$
end

4.1. Comparison with Rodriguez's and Liao's Methods

Rodriguez's method is also called the symbolic aggregation-based method [24] and Liao's method is called the hesitant fuzzy linguistic VIKOR (HFL-VIKOR) method [45]; here, we use the decision matrix D_1 provided by decision maker d_1 in Example 4 to show the symbolic aggregation-based method, the HFL-VIKOR method and the proposed method, and carry out a comparison of the three methods, as follows:

(1) The positive- and negative-ideal solutions: The symbolic aggregation-based method utilizes min_upper and max_lower operators to construct the core information of each alternative. For example, for a_1 of the decision matrix D_1, the min bounds of c_1, c_2 and c_3 are s_4, s_5 and s_4; thus the min_upper of a_1 is s_5. The max bounds of c_1, c_2 and c_3 are s_6, s_6 and s_6; thus the max_lower of a_1 is s_6, and hence the core information of a_1 is $[s_5, s_6]$. Intuitively, the core information reduces HFLTSs of each alternative with respect to the criteria into a linguistic interval.
The HFL-VIKOR method utilizes the score function and the variance function of HFLTSs [45] to rank HFLTSs of all alternatives with respect to each criterion; for example, for c_3 of the decision matrix D_1, according to the score functions and the variance functions of $[s_4, s_6]$, $[s_1, s_3]$ and $[s_6, s_6]$, we obtain $[s_6, s_6] > [s_4, s_6] > [s_1, s_3]$; hence the positive- and negative-ideal solutions of c_3 in the decision matrix D_1 are $[s_6, s_6]$ and $[s_1, s_3]$, respectively.
The proposed method uses Equations (11) and (12) to obtain the positive and negative information of each criterion. Intuitively, the positive information of each criterion in the decision matrix D_1 is also the optimistic information of all the alternatives provided by decision maker d_1, and the negative information of each criterion is the pessimistic information of all the alternatives, which can be understood as the positive- and negative-ideal solutions provided by decision maker d_1. Table 6 shows the comparison of the three methods.

Table 6. The positive- and negative-ideal solutions of the three methods.

	The Positive-Ideal Solution	The Negative-Ideal Solution	The Core Information
The method [24]	–	–	$([s_5,s_6],[s_3,s_4],[s_6,s_6])$
The method [45]	$([s_5,s_6],[s_5,s_6],[s_6,s_6])$	$([s_4,s_6],[s_6,s_6],[s_1,s_3])$	–
The proposed method	$([s_5,s_6],[s_5,s_6],[s_6,s_6])$	$([s_4,s_6],[s_4,s_6],[s_1,s_3])$	–

(2) The ranking of alternatives: In the symbolic aggregation-based method, on the basis of the core information of each alternative, a binary preference relation $p(a_j > a_{j'})$ between two alternatives is calculated on the basis of Equation (7); then the nondominance degree (NDD_j) of each alternative is used to obtain the set of nondominated alternatives, which indicates the degree to which alternative a_j is not dominated by the remaining alternatives.
In the HFL-VIKOR method, the hesitant fuzzy linguistic group utility measure HFLGU_j and the hesitant fuzzy individual regret measure HFLIR_j for the alternative a_j are defined by the hesitant fuzzy linguistic Euclidean L_p-metric; then the hesitant fuzzy linguistic compromise measure HFLC_j is established, that is,

$$\mathrm{HFLC}_j = \theta\frac{\mathrm{HFLGU}_j-\mathrm{HFLGU}^+}{\mathrm{HFLGU}^- - \mathrm{HFLGU}^+}+(1-\theta)\frac{\mathrm{HFLIR}_j-\mathrm{HFLIR}^+}{\mathrm{HFLIR}^- - \mathrm{HFLIR}^+}$$

in which, $\mathrm{HFLGU}^+ = min\{\mathrm{HFLGU}_1, \mathrm{HFLGU}_2, \mathrm{HFLGU}_3\}$, $\mathrm{HFLGU}^- = max\{\mathrm{HFLGU}_1, \mathrm{HFLGU}_2, \mathrm{HFLGU}_3\}$, $\mathrm{HFLIR}^+ = min\{\mathrm{HFLIR}_1, \mathrm{HFLIR}_2, \mathrm{HFLIR}_3\}$ and $\mathrm{HFLIR}^- = max\{\mathrm{HFLIR}_1, \mathrm{HFLIR}_2, \mathrm{HFLIR}_3\}$; $\theta \in [0,1]$ is the weight of the strategy of the majority of the criteria or the maximum overall utility. By ranking HFLGU_j, HFLIR_j and HFLC_j in descending order, the final optimal solution should be that which makes these measures attain the minimum values.

In the proposed method, positive- and negative-ideal separation matrices D^+ and D^- are used to obtain the relative closeness degree $RC(a_j)$ of each alternative, that is,

$$D^+ = \begin{pmatrix} d([s_4,s_6],[s_5,s_6])+d([s_5,s_6],[s_5,s_6])+d([s_4,s_6],[s_6,s_6]) \\ d([s_4,s_6],[s_5,s_6])+d([s_4,s_6],[s_5,s_6])+d([s_1,s_3],[s_6,s_6]) \\ d([s_5,s_6],[s_5,s_6])+d([s_4,s_6],[s_5,s_6])+d([s_6,s_6],[s_6,s_6]) \end{pmatrix} \doteq \begin{pmatrix} 0.67 \\ 0.84 \\ 0.17 \end{pmatrix}$$

$$D^- = \begin{pmatrix} d([s_4,s_6],[s_4,s_6])+d([s_5,s_6],[s_4,s_6])+d([s_4,s_6],[s_1,s_3]) \\ d([s_4,s_6],[s_4,s_6])+d([s_4,s_6],[s_4,s_6])+d([s_1,s_3],[s_1,s_3]) \\ d([s_5,s_6],[s_4,s_6])+d([s_4,s_6],[s_4,s_6])+d([s_6,s_6],[s_1,s_3]) \end{pmatrix} \doteq \begin{pmatrix} 0.67 \\ 0 \\ 0.67 \end{pmatrix}$$

$$RC(a_1) = \frac{0.17}{0.67}+\frac{0.67}{0.67} \doteq 1.25, RC(a_2) = \frac{0.17}{0.84}+0 \doteq 0.20, RC(a_3) = 2$$

Table 7 shows the ranking comparison of the three methods.

Table 7. The ranking comparison of the three methods.

	NDD_j, HFLC_j or $RC(a_j)$	The Ranking	The Best
The method [24]	$(0, 0.5, 1)$	$a_1 \prec a_2 \prec a_3$	a_3
The method [45]	$(0^*, 1^-, 0.6074)$	$a_2 \prec a_3 \prec a_1$	a_1
The proposed method	$(1.25, 0.20, 2)$	$a_2 \prec a_1 \prec a_3$	a_3

4.2. Comparison with Beg and Rashid's Method

Beg and Rashid's method is also called the hesitant fuzzy linguistic TOPSIS (HFL-TOPSIS) method [63] and was used to carry out Example 4, as follows:

(1) On the basis of Equations (1) and (2), we aggregate D_1, D_2 and D_3 to obtain the one decision matrix D:

$$\begin{aligned} D &= ([min\{max_{i=1}^3(mine_{jk}^i), min_{i=1}^3(maxe_{jk}^i)\}, max\{max_{i=1}^3(mine_{jk}^i), min_{i=1}^3(maxe_{jk}^i)\}])_{3\times 3} \\ &= \begin{pmatrix} [s_4, s_4] & [s_5, s_5] & [s_5, s_5] \\ [s_5, s_5] & [s_4, s_4] & [s_3, s_3] \\ [s_4, s_5] & [s_4, s_4] & [s_5, s_6] \end{pmatrix} \end{aligned}$$

(2) On the basis of Equations (3) and (4), we calculate the HFLTS positive- and negative-ideal solutions A^+ and A^-; here, we suppose that the criteria are beneficial, that is,

$$A^+ = ([s_5, s_6], [s_5, s_6], [s_6, s_6]), A^- = ([s_2, s_4], [s_3, s_4], [s_1, s_3])$$

(3) On the basis of the distance $d(H_S^1, H_S^2) = |q' - q| + |p' - p|$ between H_S^1 and H_S^2, we obtain the positive (negative)-ideal matrices D^+ (D^-) between D and A^+ (A^-), that is,

$$\begin{aligned} D^+ &= \begin{pmatrix} d([s_4, s_4], [s_5, s_6]) + d([s_5, s_5], [s_5, s_6]) + d([s_5, s_5], [s_6, s_6]) \\ d([s_5, s_5], [s_5, s_6]) + d([s_4, s_4], [s_5, s_6]) + d([s_3, s_3], [s_6, s_6]) \\ d([s_4, s_5], [s_5, s_6]) + d([s_4, s_4], [s_5, s_6]) + d([s_5, s_6], [s_6, s_6]) \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 6 \end{pmatrix} \\ D^- &= \begin{pmatrix} d([s_4, s_4], [s_2, s_4]) + d([s_5, s_5], [s_3, s_4]) + d([s_5, s_5], [s_1, s_3]) \\ d([s_5, s_5], [s_2, s_4]) + d([s_4, s_4], [s_3, s_4]) + d([s_3, s_3], [s_1, s_3]) \\ d([s_4, s_5], [s_2, s_4]) + d([s_4, s_4], [s_3, s_4]) + d([s_5, s_6], [s_1, s_3]) \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 9 \end{pmatrix} \end{aligned}$$

According to D^+ and D^-, we can obtain the relative closeness (Equation (6)) of each alternative to the ideal solution; Table 8 shows the comparison of the HFL-TOPSIS method and the proposed method.

Table 8. The comparison of the HFL-TOPSIS method and the proposed method.

	Weights	HPIS and HNIS	$RC(a_j)$	The Ranking	The Best
The method [63]	−	$([s_5, s_6], [s_5, s_6], [s_6, s_6])$ $([s_2, s_4], [s_3, s_4], [s_1, s_3])$	$(0.65, 0.41, 0.6)$	$a_2 \prec a_3 \prec a_1$	a_1
The proposed method	$\surd$	$([s_5, s_6], [s_5, s_6], [s_5, s_6])$ $([s_3, s_5], [s_3, s_5], [s_2, s_3])$	$(2, 0.92, 1.1)$	$a_2 \prec a_3 \prec a_1$	a_1

Comparing the above-mentioned steps (1)–(3) with steps (1)–(6) of Example 4, we can notice the following differences between the HFL-TOPSIS method and the proposed method: (a) The HFLTS positive- and negative-ideal solutions are different; in the HFL-TOPSIS method, the decision matrices of decision making are first aggregated by using operations *min* and *max* (Equations (1) and (2)); then the HFLTS positive- and negative-ideal solutions A^+ and A^- are obtained by using operations *min* and *max* (Equations (3) and (4)), for which, the weights of decision makers or criteria are not considered. In the proposed method, the weights of decision makers are used to aggregate the positive and negative information of decision makers and obtain the HFLTS positive- and negative-ideal solutions (Equations (11)–(14)). (b) The positive- and negative-ideal separation matrices are different; in the HFL-TOPSIS method, $d(H_S^1, H_S^2) = |q' - q| + |p' - p|$ between H_S^1 and H_S^2 is used to obtain the positive- and negative-ideal separation matrices, which satisfy three properties of the distance. In the proposed method, the pseudo-distance between HFLTSs (Equation (8)) is used to obtain the positive- and negative-ideal separation matrices, which are based on preference degrees between two HFLTSs. (c) The relative closeness degrees of the HFL-TOPSIS method and the proposed method are different.

5. Conclusions

Motivated by the TOPSIS method in decision making, in this paper, we have developed a new hesitant fuzzy linguistic TOPSIS method for group multi-criteria hesitant fuzzy linguistic decision making. In the proposed method, we presented the positive and negative information of c_k provided by decision maker d_i to express the optimistic and pessimistic information of all alternatives provided by each decision maker. Making use of the weighted 2-tuple linguistic aggregation operator, we aggregated the positive and negative information of c_k provided by m decision makers to obtain the HFLTS positive- and negative-ideal solutions. We defined the pseudo-distance between two HFLTSs and used this to measure the distance between assessments of the decision maker and the HFLTS positive- and negative-ideal solutions. On the basis of the obtained positive- and negative-ideal separation matrices, we proposed a new relative closeness degree of each alternative, which could be used to rank all the alternatives; intuitively, the greater the value of the relative closeness degree, the better the alternative. We utilized an example to illustrate the performance, usefulness and effectiveness of the new hesitant fuzzy linguistic TOPSIS method, and compared it with the symbolic aggregation-based method, the HFL-TOPSIS method and the HFL-VIKOR method.

It seems that the pseudo-distance between two HFLTSs and the relative closeness degree of the alternative are useful and alternative tools in hesitant fuzzy linguistic decision making. We will use the two concepts in the other decision making method and consider the proposed hesitant fuzzy linguistic TOPSIS to carry out hesitant fuzzy linguistic decision making with huge amounts of decision information and alternatives in the future works.

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