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A new perspective on traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets

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ABSTRACT

Controlling traffic flow on roads is an important traffic management task necessary to ensure a peaceful and safe environment for people. The number of cars on roads at any given time is always unknown. Type-2 fuzzy sets and neutrosophic sets play a vital role in dealing efficiently with such uncertainty. In this paper, a triangular interval type-2 Schweizer and Sklar weighted arithmetic (TIT2SSWA) operator and a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on Schweizer and Sklar triangular norms have been studied, and the validity of these operators has been checked using a numerical example and extended to an interval neutrosophic environment by proposing interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) and interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operators. Furthermore, their properties have been examined; some of the more important properties are examined in detail. Moreover, we proposed an improved score function for interval neutrosophic numbers (INNs) to control traffic flow that has been analyzed by identifying the junction that has more vehicles. This improved score function uses score values of triangular interval type-2 fuzzy numbers (TIT2FNs) and interval neutrosophic numbers.

1. Introduction

Issues related to traffic congestion is regularly experienced in daily life. Controlling traffic signals is one of the areas in which fuzzy logic is most popularly employed in transportation engineering. Traffic congestion affects the safety of the people, disrupts routine (daily/everyday) activities and the quality of lifestyle and leads to a commercial, natural and health burden for the government and related organizations. Traffic control aims to reduce the negative effects of traffic by establishing intelligent models to correct state calculation, control and forecasting. The theory of triangular norms provides the mathematical properties, and these properties represent the crucial qualities of the control system, such as stability. Control problems have attracted considerable attention in the control community [1,4,11].

As real-world problems in nature often involve uncertainty, fuzzy logic has been applied successfully to deal with impreciseness. This theory is based on the concepts of degree to deal with uncertainties in a field of knowledge. This logic agrees with linguistic and imprecise traffic data as well as in modeling signal timings. Modeling the control is the basic principle of fuzzy signal control with respect to a human expert's knowledge. The model of the fuzzy controller needs an expert's knowledge and experience in the traffic control field in developing the linguistic protocol that produces the input of the control in the system. As fuzzy logic exploits linguistic information, reproduces human thinking and captures the uncertainty of the real-world problems, it is successful in producing good performance for various practical problems [2].

A fuzzy logic system works with the use of IF-THEN rules, where the knowledge will be often uncertain. It is very useful for decoy approximation. If the antecedent and consequent parts are type-1 fuzzy sets (T1FSs), then the system is called type-1 fuzzy logic system (T1FLS), whereas in type-2 fuzzy logic system (T2FLS), the antecedent or consequent set will be of type-2 fuzzy sets (T2FSs). The membership function of T2FS is a three-dimensional one, which includes upper and lower membership functions, and the area between them is called the footprint of uncertainty (FOU) [3,6,13,18]. For a set of regional linear models, the Takagi Sugeno model will be used for an optimized output [14,30–31].

As noise is nonlinear, systems that use traditional logic and electronics fail to deal with the complex nature of the signal in terms of

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algorithms and circuits. Fuzzy logic is the most appropriate method to describe imprecise characteristics accurately [12]. Traffic congestion problems may arise owing to different conditions such as insufficient number of lanes, broken road surface, high volume of vehicles, irrational allocation of signaling system and poor visibility of the road. Furthermore, traffic congestion increases the level of pollution, as in most of the cases, the engines of the vehicles are left running.

To mitigate these problems, a new methodology has been implemented by accompanying the automated sensor approach in the system of traffic signaling. In early years, and at present, in some places, traffic is controlled by the usage of hand signs by the traffic police, signals and markings. This impreciseness cannot be dealt with by type-1 fuzzy as it is precise in nature, whereas type-2 fuzzy, an extension of type-1 fuzzy, can handle a high level of uncertainty [17]. The approach of type-2 fuzzy will overcome the consequence of time delay in control systems [19,20]. Hence, fuzzy logic controllers have been applied for controlling several physical processes successfully [15,22].

Type-1 fuzzy set induces a unique membership value to every set element between 0 and 1 and is useful to model knowledge, but it fails to deal with special uncertainties such as different opinions of experts on the same concept. It also cannot deal with only degree of truth and is not able to minimize the noise. In spite of these shortcomings, different opinions may produce different membership functions, therefore a model can be designed using T2FS [23].

T-norms and *t*-conorms are the triangular norms and preferable operators for controlling the system as it satisfies Commutativity, Idempotency, monotonicity and boundary conditions, which represent the qualities of a good system and generalize conjunction and disjunction respectively. Fuzzy conjunction is used for the system to decide the particular decision in a given period of time. Additionally, it is an operation between two membership degrees, which describes two fuzzy sets treated as a premise in an inference system. *T*-norms use some parameters of the system to control the inference, and may have different behavior based on the parameters [24].

Interval-valued neutrosophic sets (IVNSs) are determined by an interval membership grade, interval indeterminacy grade and interval non-membership grade [32,33]. The generalization of intuitionistic fuzzy is the neutrosophic set with the indeterminate reasoning, and interval-valued neutrosophic set is the general case of single-valued neutrosophic environment. Using these concepts, the uncertainty of the problem can be dealt with effectively, as the neutrosophic concept deals with indeterminacy also. Interval-valued sets, especially neutrosophic sets, handle indeterminacy with the lower and upper membership functions, and hence uncertainty in a real- world problem can be solved in an optimized way [26,35–37]. Many aggregation operators have been proposed under single-valued and interval-valued neutrosophic environments and applied in decision-making problems to choose the best option [25,27,34,38,39].

By considering the determinate part and indeterminacy, a neutrosophic number can be formed and interchanged in the form of an interval number. Using this concept and the operational laws of neutrosophic matrices, traffic flow can be identified in each intersection by considering neutrosophic linear equations, and is an effective way of finding traffic flow [43]. In addition, some models have been designed to avoid accidents, and unwanted situations while inspecting and collecting information about individuals [49–51].

The rest of the paper is arranged as follows. In, Section 2, a literature review is provided for the proposed concept, and this will show the novelty of the methods proposed in this paper. In Section 3 and Section 5.1, basic concepts have been given. In Section 4, operational laws, aggregation operators and their properties are examined with a numerical example and applied in traffic flow control under triangular interval type-2 environment. In Section 5, the proposed concepts in Section 4 have been extended to an interval neutrosophic environment. In Section 6, traffic flow analysis using the proposed operators are listed. In Section 7, qualitative analysis is provided for different fuzzy environments and crisp set as well. In Section 8, conclusion of the present work is given.

2. Review of literature

The authors of Gupta and Qi [1] proposed the theoretical concepts of *t*-norms and methods of fuzzy reasoning. Castro [2] proved that fuzzy logic controllers (FLCs) are global approximations. Karnik et al. [3] presented type-2 fuzzy logic systems, which can deal with more uncertainties. Niittymaki and Pursula [4] proved that signal control using fuzzy logic can be an efficient controlling method for signalized intersections. Wei et al. [5] presented traffic signal control management using fuzzy logic and MOGA. Wu and Mendel [6] applied imprecise bounds in the model of interval type-2 fuzzy logic systems (IT2FLSs).

Wang et al. [7] introduced interval neutrosophic sets based on truth value. Aguero and Vargas [8] concluded the dynamic structure of distribution networks using T2FLSs. Wang et al. [9] presented, in detail, the theoretical concepts about interval based neutrosophic set and its application in computing. Smarandache [10] proved that a neutrosophic set is the logical reasoning and generalization of intuitionistic fuzzy set. Li et al. [11] proposed a different method for predicting traffic using type-2 fuzzy logic. Jarrah and Shaout [12] proposed volume control of motor vehicles using fuzzy logic. Ozek and Akpolat [13] proposed an operating system for type-2 fuzzy logic tool box.

Petrescu et al. [14] proposed a fuzzy control design for an independent vehicle governing system where the design is replaced by some local linear systems, which are defined over the given points and the union of these systems inclined a Takagi Sugeno model. Algreer and Ali [15] accomplished position control using fuzzy logic. Wang et al. [16] introduced single-valued neutrosophic sets (SVNSs). Almaraaashi et al. [17] constructed generalized T2FLSs using interval type-2 setting and artificial strengthening. Tellez et al. [18] proposed T2FLSs using parametric representation. Li et al. [19] described mathematical properties such as monotonicity of IT2FLSs.

Blaho et al. [20] used type-2 fuzzy logic in diminishing the collision of impreciseness in a chain of control systems. Singhala et al. [21] developed a temperature control system using fuzzy logic. Patel [22] explained the situations and methods in which fuzzy logic can be applied. Comas et al. [23] defined measures to determine the degree of truth and the theoretical background of the decision support system. Qin and Liu [24] proposed Frank triangular norms for triangular interval type-2 fuzzy set and applied it in a decision-making process.

Ye [25] improved the correlation coefficient of SVNSs, examined their properties and extended the concept to interval neutrosophic sets (INSs). They also applied the proposed concepts in decision-making problems. Ye [26] generalized Jaccard, Dice and cosine similarity measures in vector space and presented three vector similarity measures between simplified NSs (SNSs). They also used it in a decision-making problem. Ye [27] proposed the concept of SNS, its operational laws and two aggregation operators, namely, simplified neutrosophic weighted (SNW) arithmetic average operator and SNW geometric averaging operator and applied them in a decision-making problem.

Singh et al. [28] used comparative analysis of neural network and fuzzy algorithm in implementing ACC. Shafeek [29] designed an autopilot to control the header of an aircraft using PD-like T1 and T2 fuzzy logic controllers. Wen et al. [30] proposed an intelligent signal controller using T2FL and NSGAII. Lafta and Hassan [31] introduced mobile automation control using fuzzy logic. Broumi and Smarandache [32] proposed interval-valued neutrosophic soft rough sets. Smarandache [33] explained very clearly about neutrosophic theory symbolically. Ye [34] proposed the ranking method on possibility degree for INNs from the probability aspect. Poyen et al. [35] designed a dynamic traffic signal system based on density where the signal timing changes automatically on sensing traffic. Sharma and Sahu [36] reviewed fuzzy logic-based traffic signal control.

Singh et al. [37] analyzed an uncertainty for the provided many-valued context. Ye [38] proposed new exponential laws of INSs, interval neutrosophic weighted exponential aggregation operator and its dual operator and applied them in a decision-making problem for global supplier selection. Ye [39] introduced a credibility induced INWA averaging operator and a credibility induced INWG averaging operator and examined their properties. They also presented a measure of projection between INNs and its ranking method and applied it in a decision-making problem. Bouyahia et al. [40] used fuzzy switching linear models to present real-time traffic smoothing from GPS spare measures. Chen and Ye [41] derived the mathematical properties of Dombi triangular norms based on a SVNS and applied them in a decision- making method.

Singh [42] discovered some of the important hidden patterns in the interval-valued neutrosophic context. Ye [43] presented the concepts of neutrosophic linear equations (NLEs) and, neutrosophic matrix (NM), and proposed NM operations for the first time. They also introduced some solving methods on NMs. Laxmi et al. [44] proposed an intelligent system for traffic control to enable emergency vehicles to pass without any disruptions. Noormohammadpour and Raghavendra [45] have brought out the important characteristic of traffic control in data centers. Shi and Ye [46] derived Dombi aggregation operators of neutrosophic cubic sets and applied them in a decision-making process. Liu and Wang [47] proposed interval-valued intuitionist fuzzy Schweizer–Sklar power aggregation operators and applied them in a decision-making problem for supplier selection.

Broumi et al. [48] discussed the lack of knowledge partially for [0, 1] using IVNSs. Nagarajan et al. [49] applied triangular norms under a type-2 fuzzy environment for edge detection on a DICOM image. Mayouf et al. [50] developed an accident management system applicable for cellular technology in public transportation. Sumia and Ranga [51] proposed a new intelligent traffic management system (TMS). Ankam et al. [52] designed a new TMS for the benefit of vehicle owners to carry the documents such as license and insurance during investigation by the authorities.

This review is the motivation of the present study, as the use of aggregation operators for controlling traffic flow has not yet been studied.

3. Fundamental perception

In this section, basic concepts of a traffic control system, role of fuzzy logic, output methods from fuzzy linguistic terms and structure of the fuzzy control system have been given for better understanding.

3.1. Traffic signal control [33]

This is a pretimed or induced or flexible control and is described as follows.

Pretimed Controllers - Such controllers decide the signal timings in advance, which are collected from earlier pattern of traffic, and repeat the same.

Actuated Controllers – These will identify the moving and interrupted traffic on each lane towards cross-roads and estimate the duration of the signal phase.

Adaptive Controllers – These identify the entire cross-roads and modify the signal phase and response timings to real-time traffic.

3.2. Levels of signal control [33]

The fuzziness of signal control can be classified into three levels, namely, input, control and output levels, and are described as follows.

Input Level – Here, a partial picture of the succeeding traffic environment will be drawn using measurements.

Control Level – At this level, there will be various possibilities and it is difficult to decide the right or the best possibility because the relationship of source and reaction of the signal control cannot be explained.

Output Level – Here, the exact criteria of the control are not known, such as extension gap.

3.3. Fuzzy logic in traffic control system [22]

As fuzzy logic is theoretically easy to understand, adaptable, lenient with uncertain data, can design nonlinear functions of inconsistent complexity, can be built with the knowledge of the experts, and flexible with traditional control approach, a fuzzy logic-based control system has been a successful pursuit to implement intelligence in a traffic control system.

A nonlinear mapping of an input data set to scalar output is called a fuzzy logic system. It consists of four parts i.e., fuzzifier, fuzzy rules, inference engine and defuzzifier. The fuzzy system converts the input to the output. Here, the linguistic values are divided into fuzzy sets, e.g., traffic flow can be defined as low, high and medium. The degree of addiction to every fuzzy set is shown by membership functions. The input value of the fuzzy system may exist in more than one fuzzy set. The corresponding numeric values to fuzzy set are called fuzzification and the reverse is called defuzzification.

Fuzzy IF-THEN rules are the main logic of the inference system and involve vague reasoning. Fuzzy rules are well defined using an expert's knowledge, and hence a mathematical model is not necessary for the objects and the system is very flexible. The parameters of the membership functions and its values, operators, fuzzy rules, defuzzification and other parameters can be modified according to the desired result.

Interval type-2 fuzzy logic systems are used to recognize control laws to minimize control errors. The output of this system, called the control signal, is supposed to be monotonic with respect to the error and/or variation of the error called inputs of the system.

The fuzzy control is found to be preferable in complicated problems with multi-objective decisions. Various traffic flows contest for the same time and space and various preferences are frequently set to different flows or vehicle groups.

3.4. Defuzzification methods [19]

The following methods are often applied in a control system to get the precise output from the fuzzy inputs: Karnik Mendel, Du Ying, Begian Melek Mendel, Wu Tan and Nie Tan methods.

3.5. Role of membership functions [21]

Application of membership function is an essential role in the stage of fuzzification and defuzzification of the fuzzy logic system to calculate the nonfuzzy input values to fuzzy linguistic terms and for the converse. It is used to measure the linguistic term. An amazing characteristic of the fuzzy logic lying in the fuzzification of the numerical value is that it need not be fuzzified using only one membership function, and hence the value can be described by different sets at a particular time.

3.6. Algorithm of fuzzy logic [5]

The following figure represents the algorithm of traffic control system using fuzzy logic.

3.7. Triangular norms considered

In this paper, Schweizer and Sklar (SS) triangular norms have been considered and defined as follows. [38]

$$\textbf{T}-\textbf{norm}:\ TN(p,q)=p\otimes q=1-[(1-p)^{\varphi}+(1-q)^{\varphi}-(1-p)^{\varphi}(1-q)^{\varphi}]^{\frac{1}{\varphi}},\ p,q\in[0,1]^2 \tag{1}$$

$$\textbf{T}-\textbf{conorm}:\ TCN(p,q)=p\oplus q=(p^{\varphi}+q^{\varphi}-p^{\varphi}q^{\varphi})^{\frac{1}{\varphi}},\ p,q\in[0,1]^2 \tag{2}$$

where$\varphi>0$is the parameter.

3.8. Triangular interval type-2 weighted arithmetic/geometric operator [24]

Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}]), i = 1, 2, ..., n$ be a set of TIT2FNs of the triangular interval type-2 fuzzy Set X. Let $TIT2WA_{\varpi}/TIT2WG_{\varpi}$: $\Omega^n \to \Omega$ if,

$$TIT2WA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \varpi_1 \cdot \overline{F_1} \otimes \varpi_2 \cdot \overline{F_2} \otimes .....\otimes \varpi_n \cdot \overline{F_n} \tag{3}$$

$$TIT2WG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \overline{F_1}^{\varpi_1} \otimes \overline{F_2}^{\varpi_2} \otimes .....\otimes \overline{F_n}^{\varpi_n} \tag{4}$$

then the function TIT2WA/TIT2WG are called triangular interval type-2 weighted arithmetic and geometric operators respectively, and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of $\overline{F_i} = , i = 1, 2, ..., n$, $\varpi_i \geq 0$ and $\sum_{i=1}^{n} \varpi_i = 1$. If $\varpi = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ then the TIT2WG operator is reduced to a triangular interval type-2 geometric averaging operator of dimension n.

3.9. Score function of TIT2FN [24]

To rank two TIT2FNs the following score function is used, defined by

$$SF(\overline{F}) = \left(\frac{\underline{l}_F + \overline{r_F}}{2} + 1\right) \times \frac{\underline{l}_F + \overline{l_F} + \underline{r}_F + \overline{r_F} + 4c_F}{8} \tag{5}$$

4. Proposed operational laws

The following operational laws can be defined using triangular interval type-2 fuzzy numbers for SS triangular norms. Consider three triangular interval type-2 fuzzy numbers $\overline{F}$, $\overline{F_1}$, $\overline{F_2}$ and $\varphi > 0$.

Addition

$$\overline{F_1} \oplus \overline{F_2} = \left\langle \left[(sum(\underline{l}_{F_i})^{\varphi} - prod(\underline{l}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, (sum(\overline{l_{F_i}})^{\varphi} - prod(\overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}} \right], (sum(c_{F_i})^{\varphi} - prod(c_{F_i})^{\varphi})^{\frac{1}{\varphi}}, \left[(sum(\underline{r}_{F_i})^{\varphi} - prod(\underline{r}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, (sum(\overline{r_{F_i}})^{\varphi} - prod(\overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}} \right] \right\rangle \tag{6}$$

where $\sum_{i=1}^{2} =, \prod_{i=1}^{2} = \wp$

Numerical Example:

If $\overline{F_1} = \langle [0.4, 0.5], 0.6, [0.7, 0.8] \rangle$ and $\overline{F_2} = \langle [0.5, 0.6], 0.7, [0.8, 0.9] \rangle$ be any two TIT2FNs and if $\varphi = 2$ then

$$\overline{F_1} \oplus \overline{F_2} = \langle [((0.4)^2 + (0.5)^2 - (0.4)^2.(0.5)^2)^{1/2}, ((0.5)^2 + (0.6)^2 - (0.5)^2.(0.6)^2)^{1/2}], ((0.6)^2 + (0.7)^2 - (0.6)^2.(0.7)^2)^{1/2}, [((0.7)^2 + (0.8)^2 - (0.7)^2.(0.8)^2)^{1/2}, ((0.8)^2 + (0.9)^2 - (0.8)^2.(0.9)^2)^{1/2}] \rangle = \langle [0.6096, 0.7211], 0.8207, [0.9035, 0.9652] \rangle$$

Multiplication

$$\overline{F_1} \otimes \overline{F_2} = \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi} - \wp(1 - \underline{l}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}} \right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{1}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}} \right] \right\rangle \tag{7}$$

, where $\sum_{i=1}^{2} =, \prod_{i=1}^{2} = \wp$

Numerical Example:

$$\overline{F_1} \otimes \overline{F_2} = \langle [1 - ((1 - 0.4)^2 + (1 - 0.5)^2 - (1 - 0.4)^2.(1 - 0.5)^2)^{1/2}, 1 - ((1 - 0.5)^2 + (1 - 0.6)^2 - (1 - 0.5)^2.(1 - 0.6)^2)^{1/2}] 1 - ((1 - 0.6)^2 + (1 - 0.7)^2 - (1 - 0.6)^2.(1 - 0.7)^2)^{1/2}, [1 - ((1 - 0.7)^2 + (1 - 0.8)^2 - (1 - 0.7)^2.(1 - 0.8)^2)^{1/2}, 1 - ((1 - 0.8)^2 + (1 - 0.9)^2 - (1 - 0.8)^2.(1 - 0.9)^2)^{1/2}] \rangle$$

$$= \langle [0.2789, 0.3917], 0.5146, [0.6445, 0.777] \rangle = \text{TIT2FN}$$

Multiplication by an ordinary number

$$\nu.\overline{F_1} = \left\langle \left[((\underline{l}_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, ((\overline{l_{F_1}})^{\varphi})^{\frac{\nu}{\varphi}} \right], ((c_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, \left[((\underline{r}_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, ((\overline{r_{F_1}})^{\varphi})^{\frac{\nu}{\varphi}} \right] \right\rangle \tag{8}$$

Here, ν is an ordinary number.

Numerical Example:

Consider $\overline{F_1} = \langle [0.4, 0.5], 0.6, [0.7, 0.8] \rangle$ and $\nu = 0.3$

$$0.3.\overline{F_1} = \left\langle \left[((0.4)^2)^{\frac{0.3}{2}}, ((0.5)^2)^{\frac{0.3}{2}} \right], ((0.6)^2)^{\frac{0.3}{2}}, \left[((0.7)^2)^{\frac{0.3}{2}}, ((0.8)^2)^{\frac{0.3}{2}} \right] \right\rangle$$

$$= \langle [0.2789, 0.3917], 0.5146, [0.6445, 0.777] \rangle = \text{TIT2FN}$$

Power Operation

$$\overline{F_1}^{\nu} = \left\langle \left[1 - ((1 - \underline{l}_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, 1 - ((1 - \overline{l_{F_1}})^{\varphi})^{\frac{\nu}{\varphi}} \right], 1 - ((1 - c_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_1})^{\varphi})^{\frac{\nu}{\varphi}}, 1 - ((1 - \overline{r_{F_1}})^{\varphi})^{\frac{\nu}{\varphi}} \right] \right\rangle \tag{9}$$

Numerical Example: Consider $\nu = 0.3$ and $\overline{F_1}$

$$\overline{F_1}^{0.3} = \left\langle \left[1 - ((1 - 0.4)^2)^{\frac{0.3}{2}}, 1 - ((1 - 0.5)^2)^{\frac{0.3}{2}} \right], 1 - ((1 - 0.6)^2)^{\frac{0.3}{2}}, \left[1 - ((1 - 0.7)^2)^{\frac{0.3}{2}}, 1 - ((1 - 0.8)^2)^{\frac{0.3}{2}} \right] \right\rangle$$

$$= \langle [0.1421, 0.1877], 0.2403, [0.3032, 0.383] \rangle = \text{TIT2FN}$$

4.1. Proposed theorems using TIT2SSWG operator

Here, the SS operator under triangular interval type-2 setting has been developed and proposed as a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on SS triangular norms.

4.1.1. Theorem

Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}])$, $i = 1, 2, ..., n$ be a set of TIT2FNs; then their aggregated value using TIT2SSWG operator is still a TIT2FN, $0 \leq \underline{l}_{F_i} \leq \overline{l_{F_i}} \leq c_{F_i} \leq \underline{r}_{F_i} \leq \overline{r_{F_i}} \leq 1$, $i = 1, 2, ..., n$ and

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi} - \wp(1 - \underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle \tag{10}$$

where $\sum_{i=1}^{n} =, \prod_{i=1}^{n} = \wp$

Proof. :

By mathematical induction method, we prove this theorem.

For $n = 2$

Consider the power operation

$$\overline{F}^{v_1} = \left\langle \left[1-((1-\underline{l}_F)^{\varphi})^{\frac{v1}{\varphi}}, 1-((1-\overline{l_F})^{\varphi})^{\frac{v1}{\varphi}}\right], 1 - ((1-c_F)^{\varphi})^{\frac{v1}{\varphi}}, \left[1-((1-\underline{r}_F)^{\varphi})^{\frac{v1}{\varphi}}, 1-((1-\overline{r_F})^{\varphi})^{\frac{v1}{\varphi}}\right]\right\rangle$$

$$\overline{F}^{v_2} = \left\langle \left[1-((1-\underline{l}_F)^{\varphi})^{\frac{v2}{\varphi}}, 1-((1-\overline{l_F})^{\varphi})^{\frac{v2}{\varphi}}\right], 1 - ((1-c_F)^{\varphi})^{\frac{v2}{\varphi}}, \left[1-((1-\underline{r}_F)^{\varphi})^{\frac{v2}{\varphi}}, 1-((1-\overline{r_F})^{\varphi})^{\frac{v2}{\varphi}}\right]\right\rangle$$

$$TIT2WG_{\varpi}(\overline{F_1}, \overline{F_2}) = \overline{F_1}^{\varpi_1} \otimes \overline{F_1}^{\varpi_2}$$
$$= \left\langle \left[1-((1-\underline{l}_{F_i})^{\varphi} - \wp(1-\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{F_i}})^{\varphi} - \wp(1-\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1-((1-c_{F_i})^{\varphi} - \wp(1-c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi} - \wp(1-\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1-\overline{r_{F_i}})^{\varphi} - \wp(1-\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{2} =, \prod_{i=1}^{2} = \wp$

For $n = k$

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_k}) = \overline{F_1}^{\varpi_1} \otimes \overline{F_2}^{\varpi_2} \otimes ...\otimes \overline{F_k}^{\varpi_k}$$
$$= \left\langle \left[1-((\underline{l}_{F_i})^{\varphi} - \wp(\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1-((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1-((1-c_{F_i})^{\varphi} - \wp(1-c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi} - \wp(1-\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1-((1-\overline{r_{F_i}})^{\varphi} - \wp(1-\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{k}$ =and $\prod_{i=1}^{k} = \wp$

For $n = k + 1$

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_{k+1}}) = (\overline{F_1}^{\varpi_1} \otimes \overline{F_2}^{\varpi_2} \otimes ...\otimes \overline{F_k}^{\varpi_k}) \otimes \overline{F_{k+1}}^{\varpi_{k+1}}$$
$$= \left\langle \left[1-((\underline{l}_{F_i})^{\varphi} - \wp(\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1-((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1-((1-c_{F_i})^{\varphi} - \wp(1-c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi} - \wp(1-\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1-\overline{r_{F_i}})^{\varphi} - \wp(1-\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$
$$\otimes \left\langle \left[1-((1-\underline{l}_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, 1-((1-\overline{l_{S_{k+1}}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}\right], 1-((1-c_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, \left[1-((1-\underline{r}_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, 1-((1-\overline{r_{F_{k+1}}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}\right]\right\rangle$$
$$= \left\langle \left[1-((1-\underline{l}_{F_i})^{\varphi} - \wp(1-\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{F_i}})^{\varphi} - \wp(1-\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1-((1-c_{F_i})^{\varphi} - \wp(1-c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi} - \wp(1-\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1-((1-\overline{r_{F_i}})^{\varphi} - \wp(1-\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{k+1}$ =and $\prod_{i=1}^{k+1} = \wp$

Hence, the result is true for all values of n.

Numerical Example:

For $n = 2$ the calculation has been given and the computation is similar for all the values of n. In addition, consider the weight vector $\varpi_1 = 0.55$and$\varpi_2 = 0.45$. Without loss of generality, take $\varphi = 2$throughout the paper.

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}) = \overline{F_1}^{0.55} \otimes \overline{F_2}^{0.45}$$
$$= \left\langle \left[1-((1-0.4)^2 + (1-0.5)^2 - (1-0.4)^2.\ (1-0.5)^2)^{\frac{0.55+0.45}{2}}, 1 - ((1-0.5)^2 + (1-0.6)^2 - (1-0.5)^2.\ (1-0.6)^2)^{\frac{0.55+0.45}{2}}\right], 1-((1-0.6)^2 + (1-0.7)^2 - (1-0.6)^2.\ (1-0.7)^2)^{\frac{0.55+0.45}{2}}, \left[1-((1-0.7)^2 + (1-0.8)^2 - (1-0.7)^2.\ (1-0.8)^2)^{\frac{0.55+0.45}{2}}, 1 - ((1-0.8)^2 + (1-0.9)^2 - (1-0.8)^2.\ (1-0.9)^2)^{\frac{0.55+0.45}{2}}\right]\right\rangle$$
$$= \langle[0.48, 0.63], 0.7644, [0.8736, 0.9504]\rangle = \text{TIT2FN}$$

4.1.2. Theorem (Idempotency)

Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}]),\ i = 1, 2, ..., n$ be a set of TIT2FNs, $0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1,\ i = 1, 2, ..., n.$

If all $\overline{F_i},\ i = 1, 2, ..., n$ are equal, i.e., $\overline{F_i} = \overline{F}$ then $TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \overline{F}$ (11)

Proof. :

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n})$$
$$= \left\langle \left[1-((1-\underline{l}_{F_i})^{\varphi} - \wp(1-\underline{l}_{S_t})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{S_t}})^{\varphi} - \wp(1-\overline{l_{S_t}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1-((1-m_{S_t})^{\varphi} - \wp(1-m_{S_t})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi} - \wp(1-\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1-((1-\overline{r_{F_i}})^{\varphi} - \wp(1-\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$
$$= \left\langle \left[1-((1-\underline{l}_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}, 1-((1-\overline{l_{F_i}})^{\varphi})^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}\right], (1-(1-c_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}, 1-((1-\overline{r_{F_i}})^{\varphi})^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}\right]\right\rangle$$
$$= \left\langle \left[1-((1-\underline{l}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1-((1-\overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right], 1-((1-m_{S_t})^{\varphi})^{\frac{1}{\varphi}}, \left[1-((1-\underline{r}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1-((1-\overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right]\right\rangle$$
$$= \langle[(1-\underline{l}_{F_i})^{\varphi}, (1-\overline{l_{F_i}})^{\varphi}],\ (1-c_{F_i})^{\varphi},\ [(1-\underline{r}_{F_i})^{\varphi}, (1-\overline{r_{F_i}})^{\varphi}]\rangle$$
$$= \langle[(1-\underline{l}_{F_i}), (1-\overline{l_{F_i}})],\ (1-c_{F_i}),\ [(1-\underline{r}_{F_i}), (1-\overline{r_{F_i}})]\rangle$$
$$= \langle[\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}]\rangle$$
$$= \langle[\underline{l}_F, \overline{l_F}], c_F, [\underline{r}_F, \overline{r_F}]\rangle$$
$$= \overline{F},\ \text{where } \sum_{i=1}^{n} = \text{ and } \prod_{i=1}^{n} = \wp$$

4.1.3. Theorem

Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}]),\ i = 1, 2, ..., n$ be a set of TIT2FNs, $0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1,\ i = 1, 2, ..., n.$ if $\overline{S}_{n+1} = ([\underline{l}_{S_{n+1}}, \overline{l_{S_{n+1}}}], m_{S_{n+1}}, [\underline{r}_{S_{n+1}}, \overline{r_{S_{n+1}}}])$ is also a TIT2FN on X then,

$$TIT2SSWG_{\varpi}(\overline{F_1} \otimes \overline{S_{n+1}}, \overline{F_2} \otimes \overline{F_{n+1}}, ...\overline{F_n} \otimes \overline{F_{n+1}}) = TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \otimes \overline{F_{n+1}} \quad (12)$$

Proof. :

$$\text{Since}, \overline{F_i} \otimes \overline{F_{n+1}} = \left\langle \left[1-(*(1-\underline{l_{Ft}})^{\varphi} - \wp^*(1-\underline{l_{Ft}})^{\varphi})^{\frac{1}{\varphi}}, 1 - (*(1-\overline{l_{Fi}})^{\varphi} - \wp^*(1-\overline{l_{Ft}})^{\varphi})^{\frac{1}{\varphi}} \right], \right.$$

$$1-(*(1-c_{Ft})^{\varphi} - \wp^*(1-c_{Ft})^{\varphi})^{\frac{1}{\varphi}},$$

$$\left. \left[1-(*(1-\underline{r_{Ft}})^{\varphi} - \wp^*(1-\underline{r_{Ft}})^{\varphi})^{\frac{1}{\varphi}}, 1-(*(1-\overline{r_{Ft}})^{\varphi} - \wp^*(1-\overline{r_{Ft}})^{\varphi})^{\frac{1}{\varphi}} \right] \right\rangle$$

where $* = \sum_{t=i,n+1}$, $\wp^* = \prod_{t=i,n+1}$

Consider LHS,

$$TIT2SSWG_{\varpi}(\overline{F_1} \otimes \overline{S_{n+1}}, \overline{F_2} \otimes \overline{F_{n+1}}, \ldots, \overline{F_n} \otimes \overline{F_{n+1}})$$

$$= \left\langle \left[1-((*(1-\underline{l_{Fi}})^{\varphi}) - \wp(\wp^*(1-\underline{l_{Fi}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}, 1-((*(1-\overline{l_{Fi}}))^{\xi} - \wp(\wp^*(1-\overline{l_{Fi}}))^{\varphi})^{\frac{\varpi_i}{\varphi}} \right], \right.$$

$$1-((*(1-c_{Fi})^{\varphi}) - \wp(\wp^*(1-c_{Fi})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$

$$\left. \left[1-((*(1-\underline{r_{Fi}})^{\varphi}) - \wp(\wp^*(1-\underline{r_{Fi}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}, 1-((*(1-\overline{r_{Fi}}))^{\xi} - \wp(\wp^*(1-\overline{r_{Fi}}))^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle$$

$$= \left\langle \left[1-\left((1-\underline{l_{Fi}})^{\varphi} + \left(1-\underline{l_{Fn+1}}\right)^{\varphi} - \wp(1-\underline{l_{Fi}})^{\varphi}\left(1-\underline{l_{Fn+1}}\right)\right)^{\varphi} \right)^{\frac{\varpi_i}{\varphi}}, 1 \right.$$

$$\left. - ((1-\overline{l_{Fi}})^{\varphi} + (1-\overline{l_{Fn+1}})^{\varphi} - \wp(1-\overline{l_{Fi}})^{\varphi}(1-\overline{l_{Fn+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right],$$

$$1-((1-c_{Fi})^{\varphi} + (1-c_{Fn+1})^{\varphi} - \wp(1-c_{Fi})^{\varphi}(1-c_{Fn+1})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left((1-\underline{r_{Fi}})^{\varphi} + \left(1-\underline{r_{Fn+1}}\right)^{\varphi} - \wp(1-\underline{r_{Ft}})^{\varphi}\left(1-\underline{r_{Fn+1}}\right)\right)^{\varphi} \right)^{\frac{\varpi_i}{\varphi}},$$

$$\left. 1-((1-\overline{r_{Fi}})^{\varphi} + (1-\overline{r_{Fn+1}})^{\varphi} - \wp(1-\overline{r_{Fi}})^{\varphi}(1-\overline{r_{Fn+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle$$

$$= \left\langle \left[1-\left(\left((1-\underline{l_{Fi}})^{\varphi} + \left(1-\underline{l_{Fn+1}}\right)^{\varphi} \right) - \wp\left((1-\underline{l_{Fi}})^{\varphi} + \left(1-\underline{l_{Fn+1}}\right)^{\varphi} \right) \right)^{\frac{\varpi_i}{\varphi}}, \right. \right.$$

$$\left. 1-(((1-\overline{l_{Fi}})^{\varphi} + (1-\overline{l_{Fn+1}})^{\varphi}) - \wp((1-\overline{l_{Fi}})^{\varphi} + (1-\overline{l_{Fn+1}})^{\varphi}))^{\frac{\varpi_i}{\varphi}} \right],$$

$$1-(((1-c_{Fi})^{\varphi} + (1-c_{Fn+1})^{\varphi}) - \wp((1-c_{Fi})^{\varphi} + (1-c_{Fn+1})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left(\left((1-\underline{r_{Fi}})^{\varphi} + \left(1-\underline{r_{Fn+1}}\right)^{\varphi} \right) - \wp\left((1-\underline{r_{Fi}})^{\varphi} + \left(1-\underline{r_{Fn+1}}\right)^{\varphi} \right) \right)^{\frac{\varpi_i}{\varphi}}, \right.$$

$$\left. \left. 1-(((1-\overline{r_{Fi}})^{\varphi} + (1-\overline{r_{Fn+1}})^{\varphi}) - \wp((1-\overline{r_{Fi}})^{\varphi} + (1-\overline{r_{Fn+1}})^{\varphi}))^{\frac{\varpi_i}{\varphi}} \right] \right\rangle \tag{13}$$

$$\text{RHS} = TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}) \otimes \overline{F_{n+1}}$$

$$= \left\langle \left[1-((1-\underline{l_{Fi}})^{\varphi} - \wp(1-\underline{l_{Fi}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((1-\overline{l_{Fi}})^{\varphi} - \wp(1-\overline{l_{Fi}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right], \right.$$

$$1-((1-c_{Fi})^{\varphi} - \wp(1-c_{Fi})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$

$$\left. \left[1-((1-\underline{r_{Fi}})^{\varphi} - \wp(1-\underline{r_{Fi}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((1-\overline{r_{Fi}})^{\varphi} - \wp(1-\overline{r_{Fi}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle \oplus$$

$$\left\langle \left[1-\left(\left(1-\underline{l_{Fn+1}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}, 1-((1-\overline{l_{Fn+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right], (1-(1-c_{Fn+1})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \right.$$

$$\left. \left[1-\left(\left(1-\underline{r_{Fn+1}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}, 1-((1-\overline{r_{Fn+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle$$

$$= \left\langle \left[1-\left(\left((1-\underline{l_{Fi}})^{\varphi} + \left(1-\underline{l_{Fn+1}}\right)^{\varphi} \right) - \wp\left((1-\underline{l_{Fi}})^{\varphi} + \left(1-\underline{l_{Fn+1}}\right)^{\varphi} \right) \right)^{\frac{\varpi_i}{\varphi}}, \right. \right.$$

$$\left. 1-(((1-\overline{l_{Fi}})^{\varphi} + (1-\overline{l_{Fn+1}})^{\varphi}) - \wp((1-\overline{l_{Fi}})^{\varphi} + (1-\overline{l_{Fn+1}})^{\varphi}))^{\frac{\varpi_i}{\xi}} \right],$$

$$1-(((1-c_{Fi})^{\varphi} + (1-c_{Fn+1})^{\varphi}) - \wp((1-c_{Fi})^{\varphi} + (1-c_{Fn+1})^{\varphi}))^{\frac{\varpi_i}{\varphi}}$$

$$\left[1-\left(\left((1-\underline{r_{Fi}})^{\varphi} + \left(1-\underline{r_{Fn+1}}\right)^{\varphi} \right) - \wp\left((1-\underline{r_{Fi}})^{\varphi} + \left(1-\underline{r_{Fn+1}}\right)^{\varphi} \right) \right)^{\frac{\varpi_i}{\varphi}}, \right.$$

$$\left. \left. 1-(((1-\overline{r_{Fi}})^{\varphi} + (1-\overline{r_{Fn+1}})^{\varphi}) - \wp((1-\overline{r_{Fi}})^{\varphi} + (1-\overline{r_{Fn+1}})^{\varphi}))^{\frac{\varpi_i}{\varphi}} \right] \right\rangle \tag{14}$$

From (13) and (14), the theorem holds.

4.1.4. Theorem

Let $\overline{F_i} = ([\underline{l_{Fi}}, \overline{l_{Fi}}], c_{Fi}, [\underline{r_{Fi}}, \overline{r_{Fi}}])$, $i = 1, 2, \ldots, n$ be a set of TIT2FNs, $0 \le \underline{l_{Fi}} \le \overline{l_{Fi}} \le c_{Fi} \le \underline{r_{Fi}} \le \overline{r_{Fi}} \le 1$, $i = 1, 2, \ldots, n$. If $k > 0$, $\overline{F_{n+1}} = \left(\left[\underline{l_{Fn+1}}, \overline{l_{Fn+1}}\right], c_{Fn+1}, \left[\underline{r_{Fn+1}}, \overline{r_{Fn+1}}\right]\right)$ is a TIT2FN on X then,

$$TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, \ldots, \overline{F_n}^k) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}))^k \tag{15}$$

Proof. :

$$k.\overline{S_i} = \left\langle \left[1-((1-\underline{l_{Fi}})^{\varphi})^{\frac{k}{\varphi}}, 1-(\wp(1-\overline{l_{Fi}})^{\varphi})^{\frac{k}{\varphi}} \right], 1 - ((1-c_{Fi})^{\varphi})^{\frac{k}{\varphi}}, \left[1-((1-\underline{r_{Fi}})^{\varphi})^{\frac{k}{\varphi}}, 1-(\wp(1-\overline{r_{Fi}})^{\varphi})^{\frac{k}{\varphi}} \right] \right\rangle$$

$$\text{LHS} = TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, \ldots, \overline{F_n}^k)$$

$$= \left\langle \left[1-([((1-\underline{l_{Fi}})^{\varphi}) - \wp(\wp(1-\underline{l_{Fi}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}, 1 - ([((1-\overline{l_{Fi}})^{\xi}) - \wp(\wp(1-\overline{l_{Fi}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}} \right], \right.$$

$$1-[([((1-c_{Fi})^{\varphi})])^k - ([\wp(\wp(1-c_{Fi})^{\varphi})])^k]^{\frac{\varpi_i}{\varphi}},$$

$$\left. \left[1-([((1-\underline{r_{Fi}})^{\varphi}) - \wp(\wp(1-\underline{r_{Fi}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}, 1 - ([((1-\overline{r_{Fi}})^{\xi}) - \wp(\wp(1-\overline{r_{Fi}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}} \right] \right\rangle$$

$$= \left\langle \left[1-((1-\underline{l_{Fi}})^{\varphi} - \wp(1-\underline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{Fi}})^{\varphi} - \wp(1-\overline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right], \right.$$

$$1-((1-c_{Fi})^{\varphi} - \wp(1-c_{Fi})^{\varphi})^{\frac{k\varpi_i}{\varphi}},$$

$$\left. \left[1-((1-\underline{r_{Fi}})^{\varphi} - \wp(1-\underline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{r_{Fi}})^{\varphi} - \wp(1-\overline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right] \right\rangle \tag{16}$$

$$\text{RHS} = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}))^k$$

$$= \left\langle \left[1-((1-\underline{l_{Fi}})^{\varphi} - \wp(1-\underline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{Fi}})^{\varphi} - \wp(1-\overline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right], \right.$$

$$1-((1-c_{Fi})^{\varphi} - \wp(1-c_{Fi})^{\varphi})^{\frac{k\varpi_i}{\varphi}},$$

$$\left. \left[1-((1-\underline{r_{Fi}})^{\varphi} - \wp(1-\underline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{r_{Fi}})^{\varphi} - \wp(1-\overline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right] \right\rangle$$

$$= \left\langle \left[1-((1-\underline{l_{Fi}})^{\varphi} - \wp(1-\underline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{l_{Fi}})^{\varphi} - \wp(1-\overline{l_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right], \right.$$

$$1-((1-c_{Fi})^{\varphi} - \wp(1-c_{Fi})^{\varphi})^{\frac{k\varpi_i}{\varphi}},$$

$$\left. \left[1-((1-\underline{r_{Fi}})^{\varphi} - \wp(1-\underline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1-\overline{r_{Fi}})^{\varphi} - \wp(1-\overline{r_{Fi}})^{\varphi})^{\frac{k\varpi_i}{\varphi}} \right] \right\rangle \tag{17}$$

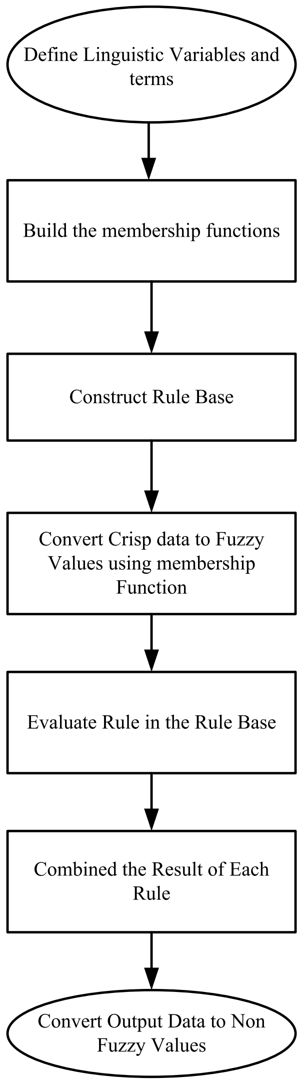

Fig. 1. Traffic control system using fuzzy logic.

From (16) and (17),

$$TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, ..., \overline{F_n}^k) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}))^k$$

4.1.5. Theorem (Stability)

Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}])$, $i = 1, 2, ..., n$ be a set of TIT2FNs, $0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1$, If $\overline{F_{n+1}} = ([\underline{l}_{F_{n+1}}, \overline{l_{F_{n+1}}}], c_{F_{n+1}}, [\underline{r}_{F_{n+1}}, \overline{r_{F_{n+1}}}])$ is also a TIT2FN on X. If $k > 0$ then,

$$TIT2SSWG_{\varpi}(\overline{F_1}^k \otimes \overline{F_{n+1}}, \overline{F_2}^k \otimes \overline{F_{n+1}}, ..., \overline{F_n}^k \otimes \overline{F_{n+1}}) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}))^k \otimes \overline{F_{n+1}}. \quad (18)$$

Proof. :

From theorems 4.1.4. and 4.1.5. $TIT2SSWG_{\varpi}(\overline{F_1}^k \otimes \overline{F_{n+1}}, \overline{F_2}^k \otimes \overline{F_{n+1}}, ..., \overline{F_n}^k \otimes \overline{F_{n+1}})$ is true.

$$= (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}))^k \otimes \overline{F_{n+1}}$$

4.2. Proposed theorems using TIT2SSWA operator

In this section, the same theorems are listed out and the proof is similar

4.2.1. Let $\overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}])$, $i = 1, 2, ..., n$ be a set of TIT2FNs; then their aggregated value using TIT2SSWA operator is still a TIT2FN, $0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1$, $i = 1, 2, ..., n$ and

$$TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \left\langle \left[((\underline{l}_{F_i})^{\varphi} - \wp(\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right], ((c_{F_i})^{\varphi} - \wp(c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[((1 - \underline{r}_{F_i})^{\varphi} - \wp(\underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((\overline{r_{F_i}})^{\varphi} - \wp(\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}} \right] \right\rangle \quad (19)$$

where$\sum_{i=1}^{n} =$, $\prod_{i=1}^{n} = \wp$

If all $\overline{F_i}$, $i = 1, 2, ..., n$ are equal, i.e., $\overline{F_i} = \overline{F}$ then $TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) = \overline{F}$. (20)

$$TIT2SSWA_{\varpi}(\overline{F_1} \oplus \overline{F_{n+1}}, \overline{F_2} \oplus \overline{F_{n+1}}, ..., \overline{F_n} \oplus \overline{F_{n+1}}) = TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \oplus \overline{F_{n+1}} \quad (21)$$

$$TIT2SSWA_{\varpi}(k \cdot \overline{F_1}, k \cdot \overline{F_2}, ..., k \cdot \overline{F_n}) = k \cdot TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \quad (22)$$

$$TIT2SSWA_{\varpi}(k \cdot \overline{F_1} \oplus \overline{F_{n+1}}, k \cdot \overline{F_2} \oplus \overline{F_{n+1}}, ..., k \cdot \overline{F_n} \oplus \overline{F_{n+1}}) = k \cdot TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \oplus \overline{F_{n+1}} \quad (23)$$

4.3. Proposed method for traffic flow control using TIT2SSWA operator

The traffic flow of the junction is considered during rush hour on a working day inFig. 2. The arrow marks represent the direction of the flow in each direction. The average number of vehicles per hour coming in and departing at each intersecting point are taken as triangular interval type-2 fuzzy numbers instead of crisp numbers. The aim of this work is to identify the junction that has a higher number of vehicles (traffic) that need to be cleared first using the score value of the IT2FNs, and the result can be concluded based on the greater score value.

Using Eq. (19),

$$TIT2SSWA_{\varpi}(Z_1, Z_2) = (0.45)Z_1 \oplus (0.55)Z_2$$

$$= \left\langle \left[((0.6)^2 + (0.2)^2 - (0.6)^2.\ (0.2)^2)^{\frac{0.45+0.55}{2}}, ((0.7)^2 + (0.3)^2 - (0.7)^2.\ (0.3)^2)^{\frac{0.45+0.55}{2}} \right], ((0.8)^2 + (0.4)^2 - (0.8)^2.\ (0.4)^2)^{\frac{0.45+0.55}{2}}, \left[((0.9)^2 + (0.5)^2 - (0.9)^2.\ (0.5)^2)^{\frac{0.45+0.55}{2}}, ((1)^2 + (0.6)^2 - (1)^2.\ (0.6)^2)^{\frac{0.45+0.55}{2}} \right] \right\rangle$$

$$= \langle [0.62, 0.73], 0.84, [0.92, 1] \rangle$$

Similarly,

$$TIT2SSWA_{\varpi}(Z_2, Z_3) = (0.45)Z_2 \oplus (0.55)Z_3 = \langle [0.56, 0.68], 0.78, [0.87, 0.9] \rangle$$

$$TIT2SSWA_{\varpi}(Z_3, Z_4) = (0.45)Z_3 \oplus (0.55)Z_4 = \langle [0.41, 0.53], 0.65, [0.75, 0.85] \rangle$$

$$TIT2SSWA_{\varpi}(Z_4, Z_1) = (0.45)Z_4 \oplus (0.55)Z_1 = \langle [0.56, 0.68], 0.79, [0.88, 0.95] \rangle$$

Finding the score values (SVs)

Using Eq. (5)

$$SV(Z_1, Z_2) = \left(\frac{0.62 + 1}{2} + 1 \right) \times \frac{0.62 + 0.73 + 0.92 + 1 + 4(0.84)}{8} = 1.5$$

Similarly,

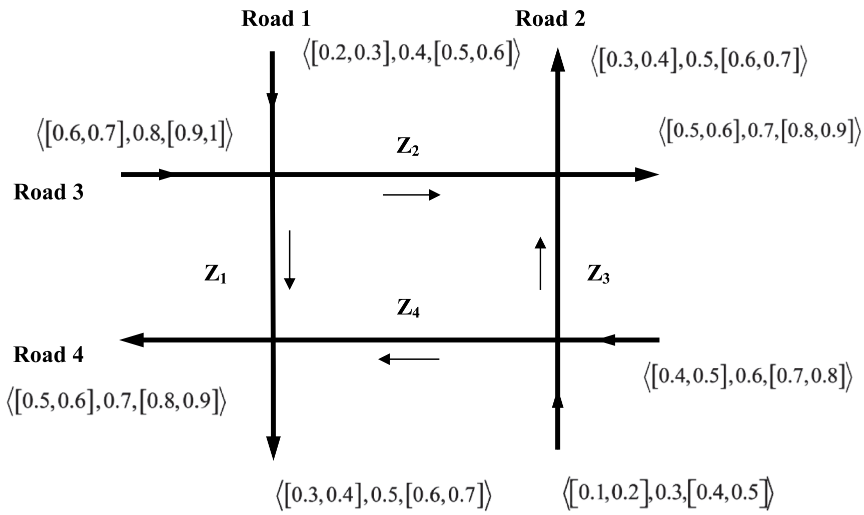

Fig. 2. Traffic flow on the road with four junctions using triangular interval type-2 fuzzy numbers.

$SV(Z_2, Z_3) = 1.33,\ SV(Z_3, Z_4) = 1.05,\ SV(Z_4, Z_1) = 1.37$

From the score values, the junction between $Z_1$and Z_2 has a higher value, and therefore it is recommended that this junction has more traffic and may be cleared first.

5. Neutrosophic perspective

The concept of interval type-2 fuzzy sets can be extended to interval neutrosophic sets. As fuzzy sets handle only truth and false membership grades whereas neutrosophic sets handle not only truth and false membership grades but also indeterminacy grade, extension of the above results would provide an efficient way of handling uncertainties existing in the real-world problems.

The above theorems have been extended to an interval neutrosophic setting. The following are the basic concepts related to interval neutrosophic sets.

5.1. Basic concepts

In this section, essential definitions of interval neutrosophic set and numbers are given, and based on these definitions, Schweizer and Sklar operations of INNs have been proposed.

5.1.1. Interval Neutrosophic Set (INS) [46]

Let U be a nonempty set. An interval neutrosophic set $B$is defined as follows.

$B = \{x, \langle T(x),\ I(x),\ F(x)\rangle | x \in B\}$, where the intervals

$T(x) = [T^L(x), T^U(x)] \subseteq [0, 1]$, $I(x) = [I^L(x), I^U(x)] \subseteq [0, 1]$, $F(x) = [F^L(x), F^U(x)] \subseteq [0, 1]$ for $x \in U$are the grades of the truth-membership, indeterminacy-membership and false-membership respectively.

5.1.2. Interval neutrosophic numbers [32]

Let $X = \{x_1, x_2, ..., x_n\}$be an INS, where $x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$for $j = 1, 2, 3, ..., n$ is a collection of INNs and$T_j^L, T_j^U, I_j^L, I_j^U, F_j^L, F_j^U \in (0, 1)$, $\upsilon > 0$and$\delta > 0$. Then, the SS T-norm and T-conorm operations on INNs, are defined as follows.

5.2. Proposed schweizer and sklar operations of interval neutrosophic numbers

Addition:

$$x_1 \oplus x_2 = \{[((T_1^L)^\delta + (T_2^L)^\delta - (T_1^L)^\delta (T_2^L)^\delta)^{1/\delta}, ((T_1^U)^\delta + (T_2^U)^\delta - (T_1^U)^\delta (T_2^U)^\delta)^{1/\delta}],$$

$$[1 - ((1 - I_1^L)^\delta + (1 - I_2^L)^\delta - (1 - I_1^L)^\delta (1 - I_2^L)^\delta)^{1/\delta}, 1 - ((1 - I_1^U)^\delta + (1 - I_2^U)^\delta - (1 - I_1^U)^\delta (1 - I_2^U)^\delta)^{1/\delta}],$$
$$[1 - ((1 - F_1^L)^\delta + (1 - F_2^L)^\delta - (1 - F_1^L)^\delta (1 - I_2^L)^\delta)^{1/\delta}, 1 - ((1 - F_1^U)^\delta + (1 - F_2^U)^\delta - (1 - F_1^U)^\delta (1 - F_2^U)^\delta)^{1/\delta}]\} \quad (24)$$

Numerical Example:

If $x_1 = \langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2] \rangle$ and $x_2 = \langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4] \rangle$are the two INNs and $\delta = 2$then

$$x_1 \oplus x_2 = \{[((0.7)^2 + (0.4)^2 - (0.7)^2.\ (0.4)^2)^{1/2}, ((0.8)^2 + (0.5)^2 - (0.8)^2.\ (0.5)^2)^{1/2}],$$

$$[1 - ((1 - 0)^2 + (1 - 0.2)^2 - (1 - 0)^2.\ (1 - 0.2)^2)^{1/2}, 1 - ((1 - 0.1)^2 + (1 - 0.3)^2 - (1 - 0.1)^2.\ (1 - 0.3)^2)^{1/2}],$$
$$[1 - ((1 - 0.1)^2 + (1 - 0.3)^2 - (1 - 0.1)^2.\ (1 - 0.3)^2)^{1/2}, 1 - ((1 - 0.2)^2 + (1 - 0.4)^2 - (1 - 0.2)^2.\ (1 - 0.4)^2)^{1/2}]\}$$

$x_1 \oplus x_2 = \{[0.76, 0.85], [0.11, 0.13], [0.05, 0.12]\}$= INN

Multiplication:

$$x_1 \otimes x_2 = \{[1 - ((1 - T_1^L)^\delta + (1 - T_2^L)^\delta - (1 - T_1^L)^\delta (1 - T_2^L)^\delta)^{1/\delta}$$
$$1 - ((1 - T_1^U)^\delta + (1 - T_2^U)^\delta - (1 - T_1^U)^\delta (1 - T_2^U)^\delta)^{1/\delta}],$$
$$[((I_1^L)^\delta + (I_2^L)^\delta - (I_1^L)^\delta (I_2^L)^\delta)^{1/\delta}, ((I_1^U)^\delta + (I_2^U)^\delta - (I_1^U)^\delta (I_2^U)^\delta)^{1/\delta}],$$
$$[((F_1^L)^\delta + (F_2^L)^\delta - (F_1^L)^\delta (I_2^L)^\delta)^{1/\delta}, ((F_1^U)^\delta + (F_2^U)^\delta - (F_1^U)^\delta (F_2^U)^\delta)^{1/\delta}]\}. \quad (25)$$

Numerical Example:

$$x_1 \otimes x_2 = \{[1 - ((1 - 0.7)^2 + (1 - 0.4)^2 - (1 - 0.7)^2.\ (1 - 0.4)^2)^{1/2},$$
$$1 - ((1 - 0.8)^2 + (1 - 0.5)^2 - (1 - 0.8)^2.\ (1 - 0.5)^2)^{1/2}],$$
$$[((0)^2 + (0.2)^2 - (0)^2.\ (0.2)^2)^{1/2}, ((0.1)^2 + (0.3)^2 - (0.1)^2.\ (0.3)^2)^{1/2}],$$
$$[((0.1)^2 + (0.3)^2 - (0.1)^2.\ (0.3)^2)^{1/2}, ((0.2)^2 + (0.4)^2 - (0.2)^2.\ (0.4)^2)^{1/2}]\}$$

$x_1 \otimes x_2 = \{[0.35, 0.47], [0.2, 0.31], [0.3148, 0.44]\}$ = INN

Multiplication by an ordinary Numbers:

$$g.\ x_1 = \{[(g(T_1^L)^\delta)^{1/\delta}, (g(T_1^U)^\delta)^{1/\delta}], [1 - (g(1 - I_1^L)^\delta)^{1/\delta}, 1 - (g(1 - I_1^U)^\delta)^{1/\delta}],$$
$$[1 - (g(1 - F_1^L)^\delta)^{1/\delta}, 1 - (g(1 - F_1^U)^\delta)^{1/\delta}]\} \quad (26)$$

Numerical Example: Consider $g = 0.2$

$$(0.2).\ x_1 = \{[(0.2(0.7)^2)^{1/2}, (0.2(0.8)^2)^{1/2}], [1-(0.2(1-0.0)^2)^{1/2}, 1-(0.2(1-0.1)^2)^{1/2}], [1-(0.2(1-0.1)^2)^{1/2}, 1-(0.2(1-0.2)^2)^{1/2}]\} = \{[0.3130, 357], [0.55, 0.5976], [0.5976, 0.6422]\} = \text{INN}$$

Power Operation:

$$x_1{}^g = \{[1-(g(1-T_1^L)^\delta)^{1/\delta}, 1-(g(1-T_1^U)^\delta)^{1/\delta}], [(g(I_1^L)^\delta)^{1/\delta}, (g(I_1^U)^\delta)^{1/\delta}], [(g(F_1^L)^\delta)^{1/\delta}, (g(F_1^U)^\delta)^{1/\delta}]\} \tag{27}$$

Numerical Example: Consider $g = 0.2$

$$x_1{}^{0.2} = \{[1-(0.2(1-0.7)^2)^{1/2}, 1-(0.2(1-0.8)^2)^{1/2}], [(0.2(0.0)^2)^{1/2}, (0.2(0.1)^2)^{1/2}], [(0.2(0.1)^2)^{1/2}, (0.2(0.2)^2)^{1/2}]\} = \{[0.8658, 0.9106], [0.0, 0.0447], [0.0447, 0.0894]\} = \text{INN}$$

5.2.1. Proposed score function

For ranking INNs, a new score function is proposed in this section, defined by

$$SF(\overline{F}) = \frac{1}{2}[(T_F^L + T_F^U) - (I_F^L I_F^U) + (I_F^U - 1)^2 + F_F^U] \tag{28}$$

5.3. Proposed theorems using INSSWG operator

The following theorems are proved using the proposed aggregation operator

5.3.1. Theorem

Let $x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, \ldots, n$ be a collection of INNs and their weight vector is $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$, $\upsilon_i \in [0, 1]$ and $\sum_{j=1}^n \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operator is still an INN, i.e.,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_n) = \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})], \left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right], [((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\} \tag{29}$$

where $\sum_{i=1}^n =, \prod_{i=1}^n = \wp$.

Proof. :

Mathematical induction is used to prove this theorem.

When $n = 2$, using SS triangular norms, we get,

$$INSSWG_\upsilon(x_1, x_2) = x_1{}^{\upsilon_1} \otimes x_2{}^{\upsilon_2}$$
$$= \{[1-(\upsilon_1(1-T_1^L)^\delta + \upsilon_2(1-T_2^L)^\delta - \upsilon_1(1-T_1^L)^\delta \upsilon_2(1-T_2^L)^\delta)^{1/\delta}, 1-(\upsilon_1(1-T_1^U)^\delta + \upsilon_2(1-T_2^U)^\delta - \upsilon_1(1-T_1^U)^\delta \upsilon_2(1-T_2^U)^\delta)^{1/\delta}] [(\upsilon_1(I_1^L)^\delta + \upsilon_2(I_2^L)^\delta - \upsilon_1(I_1^L)^\delta \upsilon_2(I_2^L)^\delta)^{1/\delta}, (\upsilon_1(I_1^U)^\delta + \upsilon_2(I_2^U)^\delta - \upsilon_1(I_1^U)^\delta \upsilon_2(I_2^U)^\delta)^{1/\delta}], [(\upsilon_1(F_1^L)^\delta + \upsilon_2(F_2^L)^\delta - \upsilon_1(F_1^L)^\delta \upsilon_2(I_2^L)^\delta)^{1/\delta}, (\upsilon_1(F_1^U)^\delta + \upsilon_2(F_2^U)^\delta - \upsilon_1(F_1^U)^\delta \upsilon_2(F_2^U)^\delta)^{1/\delta}]\}$$
$$= \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})], \left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right], [((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\}$$

where $\sum_{j=1}^2 =, \prod_{j=1}^2 = \wp$.

When $n = k$,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_k) = x_1{}^{\upsilon_1} \otimes x_2{}^{\upsilon_2} \otimes \ldots \otimes x_k{}^{\upsilon_k}$$
$$= \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})], \left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right], [((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\}$$

where $\sum_{j=1}^k =, \prod_{j=1}^k = \wp$.

When $n = k + 1$,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_k, x_{k+1}) = (x_1{}^{\upsilon_1} \otimes x_2{}^{\upsilon_2} \otimes \ldots \otimes x_k{}^{\upsilon_k}) \otimes x_{k+1}{}^{\upsilon_{k+1}}$$
$$= \langle \{[1-(((\upsilon_i(1-T_i^L)^\delta) - \wp(\upsilon_i(1-T_i^L)^\delta))^{1/\delta}, 1-((\upsilon_i(1-T_i^U)^\delta) - \wp(\upsilon_i(1-T_i^U)^\delta))^{1/\delta})], \left[\left(\left(\upsilon_{i_{(I_i^L)^\delta}}\right) - \wp(\upsilon_i(I_i^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{i_{(I_i^U)^\delta}}\right) - \wp(\upsilon_i(I_i^U)^\delta)\right)^{1/\delta}\right], [((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\} \rangle \otimes \upsilon_{k+1} x_{k+1}$$
$$= \left\{\left[1 - \left(\sum_{j=1}^{k+1}(\upsilon_j(1-T_j^L)^\delta) - \prod_{j=1}^{k+1}(\upsilon_j(1-T_j^L)^\delta)\right)^{1/\delta}, 1 - \left(\sum_{j=1}^{k+1}(\upsilon_j(1-T_j^U)^\delta) - \prod_{j=1}^{k+1}(\upsilon_j(1-T_j^U)^\delta)\right)^{1/\delta}\right], \left[\left(\sum_{j=1}^{k+1}\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \prod_{j=1}^{k+1}(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\sum_{j=1}^{k+1}\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \prod_{j=1}^{k+1}(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right], \left[\left(\sum_{j=1}^{k+1}(\upsilon_j(F_j^L)^\delta) - \prod_{j=1}^{k+1}(\upsilon_j(F_j^L)^\delta)\right)^{1/\delta}, \left(\sum_{j=1}^{k+1}(\upsilon_j(F_j^U)^\delta) - \prod_{j=1}^{k+1}(\upsilon_j(F_j^U)^\delta)\right)^{1/\delta}\right]\right\}$$

Hence, the theorem is true for all values of n.

Numerical Example: For $n = 2$

Consider the same x_1 and x_2, and consider the weight vectors $\upsilon_1 = 0.55$ and $\upsilon_1 = 0.45$

$INSSWG_\upsilon(x_1, x_2) = x_1^{0.55} \otimes x_2^{0.45}$
$= \{[1 - (0.55(1-0.7)^2 + 0.45(1-0.4)^2 - (0.55(1-0.7)^2)(0.45(1-0.4)^2))^{1/2},$
$1 - (0.55(1-0.8)^2 + 0.45(1-0.5)^2 - (0.55(1-0.8)^2)(0.45(1-0.5)^2))^{1/2}]$
$[(0.55(0)^2 + 0.45(0.1)^2 - (0.55(0)^2)(0.45(0.1)^2))^{1/2},$
$(0.55(0.1)^2 + 0.45(0.3)^2 - (0.55(0.1)^2)(0.45(0.3)^2))^{1/2}]$
$[(0.55(0.1)^2 + 0.45(0.3)^2 - (0.55(0.1)^2)(0.45(0.3)^2))^{1/2},$
$(0.55(0.2)^2 + 0.45(0.4)^2 - (0.55(0.2)^2)(0.45(0.4)^2))^{1/2}]\}$
$= \{[0.5489, 0.6367], [0.0671, 0.2140], [0.2140, 0.3040]\}.$

5.3.2. Theorem

If $\upsilon = (1/n, 1/n, \ldots, 1/n)$then,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_n) = \left\{\left[1 - \left(\left(\left(\tfrac{1}{n}(1-T_j^L)^\delta\right) - \wp\left(\tfrac{1}{n}(1-T_j^L)^\delta\right)\right)\right)^{1/\delta}, 1 - \left(\left(\tfrac{1}{n}(1-T_j^U)^\delta\right) - \wp\left(\tfrac{1}{n}(1-T_j^U)^\delta\right)\right)^{1/\delta}\right], \left[\left(\left(\tfrac{1}{n}(I_j^L)^\delta\right) - \wp\left(\tfrac{1}{n}(I_j^L)^\delta\right)\right)^{1/\delta}, \left(\left(\tfrac{1}{n}(I_j^U)^\delta\right) - \wp\left(\tfrac{1}{n}(I_j^U)^\delta\right)\right)^{1/\delta}\right], \left[\left(\left(\tfrac{1}{n}(F_j^L)^\delta\right) - \wp\left(\tfrac{1}{n}(F_j^L)^\delta\right)\right)^{1/\delta}, \left(\left(\tfrac{1}{n}(F_j^U)^\delta\right) - \wp\left(\tfrac{1}{n}(F_j^U)^\delta\right)\right)^{1/\delta}\right]\right\} \quad (30)$$

where $\sum_{i=1}^{n} =, \prod_{i=1}^{n} = \wp$. Therefore, INSSWG operator reduces into an interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) operator when the weight vector$\upsilon = (1/n, 1/n, \ldots, 1/n)$.

5.3.3. Theorem (Idempotency)

Let$x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$,
$j = 1, 2, 3, \ldots, n$ be a collection of INNs and if $x_j = x$, then

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_n) = x. \quad (31)$$

Proof. :

$INSSWG_\upsilon(x_1, x_2, \ldots, x_n)$
$= \{[1 - (((\upsilon_i(1-T_i^L)^\delta) - \wp(\upsilon_i(1-T_i^L)^\delta))^{1/\delta}, 1 - ((\upsilon_i(1-T_i^U)^\delta) - \wp(\upsilon_i(1-T_i^U)^\delta))^{1/\delta})],$
$\left[\left(\left(\upsilon_{i_{(I_i^L)^\delta}}\right) - \wp(\upsilon_i(I_i^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{i_{(I_i^U)^\delta}}\right) - \wp(\upsilon_i(I_i^U)^\delta)\right)^{1/\delta}\right],$
$[((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\}$

$= \{[1 - ((1-T^L)^\delta)^{1/\delta}, 1 - ((1-T^U)^\delta)^{1/\delta}],$
$\left[\left(_{(F^L)^\delta}\right)^{1/\delta}, \left(_{(F^U)^\delta}\right)^{1/\delta}\right]\}\left[\left(_{(I^L)^\delta}\right)^{1/\delta}, \left(_{(I^U)^\delta}\right)^{1/\delta}\right],$

$= \{[1 - ((1-T^L)), 1 - ((1-T^U))], [(_{(I^L)}), (_{(I^U)})], [(_{(F^L)}), (_{(F^U)})]\}$
$= \{[1 - T^L, 1 - T^U], [I^L, I^U], [F^L, F^U]\}$
$= \{[T^L, T^U], [I^L, I^U], [F^L, F^U]\}$
$= x$

Hence, the theorem is proved.
Numerical computation can be performed as in theorem 5.3.1.

5.3.4. Theorem (Boundedness)

Let x_j, $j = 1, 2, \ldots, n$be a collection of INNs and let

$x^- = \left\langle \left\{\left[\min_j\left(T_j^L\right), \min_j\left(T_j^U\right)\right], \left[\max_j\left(I_j^L\right), \max_j\left(I_j^U\right)\right], \left[\max_j\left(F_j^L\right), \max_j\left(F_j^U\right)\right]\right\}\right\rangle$ and

$x^+ = \left\langle \left\{\left[\max_j\left(T_j^L\right), \max_j\left(T_j^U\right)\right], \left[\min_j\left(I_j^L\right), \min_j\left(I_j^U\right)\right], \left[\min_j\left(F_j^L\right), \min_j\left(F_j^U\right)\right]\right\}\right\rangle$. then,

$$x^- \le INSSWG_\upsilon(x_1, x_2, \ldots, x_n) \le x^+ \quad (32)$$

Proof. :

Since, $\min_j(T_j^L) \le T_j^L \le \max_j(T_j^L)$, $\min_j(T_j^U) \le T_j^U \le \max_j(T_j^U)$
$\min_j(I_j^L) \le I_j^L \le \max_j(I_j^L)$, $\min_j(I_j^U) \le I_j^U \le \max_j(I_j^U)$
$\min_j(F_j^L) \le F_j^L \le \max_j(F_j^L)$, $\min_j(F_j^U) \le F_j^U \le \max_j(F_j^U)$

the following inequalities are holding good.

$1 - ((\upsilon_j \min(1-T_j^L)^\delta) - \wp(\upsilon_j \min(1-T_j^L)^\delta))^{1/\delta}$
$\le 1 - ((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}$
$\le 1 - ((\upsilon_j \max(1-T_j^L)^\delta) - \wp(\upsilon_j \max(1-T_j^L)^\delta))^{1/\delta}$
$1 - ((\upsilon_j \min(1-T_j^U)^\delta) - \wp(\upsilon_j \min(1-T_j^U)^\delta))^{1/\delta}$
$\le 1 - ((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta}$
$\le 1 - ((\upsilon_j \max(1-T_j^U)^\delta) - \wp(\upsilon_j \max(1-T_j^U)^\delta))^{1/\delta}$,
$((\upsilon_j \min(I_j^L)^\delta) - \wp(\upsilon_j \min(I_j^L)^\delta))^{1/\delta} \le ((\upsilon_j(I_j^L)^\delta) - \wp(\upsilon_j(I_j^L)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(I_j^L)^\delta) - \wp(\upsilon_j \max(I_j^L)^\delta))^{1/\delta}$,
$((\upsilon_j \min(I_j^U)^\delta) - \wp(\upsilon_j \min(I_j^U)^\delta))^{1/\delta} \le ((\upsilon_j(I_j^U)^\delta) - \wp(\upsilon_j(I_j^U)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(I_j^U)^\delta) - \wp(\upsilon_j \max(I_j^U)^\delta))^{1/\delta}$,
$((\upsilon_j \min(F_j^L)^\delta) - \wp(\upsilon_j \min(F_j^L)^\delta))^{1/\delta} \le ((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(F_j^L)^\delta) - \wp(\upsilon_j \max(F_j^L)^\delta))^{1/\delta}$,
$((\upsilon_j \min(F_j^U)^\delta) - \wp(\upsilon_j \min(F_j^U)^\delta))^{1/\delta} \le ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(F_j^U)^\delta) - \wp(\upsilon_j \max(F_j^U)^\delta))^{1/\delta}$

Therefore, $x^- \le INSSWG_\upsilon(x_1, x_2, \ldots, x_n) \le x^+$. Hence, the result.Numerical computation can be performed as in theorem 5.3.1.

5.3.5. Theorem (Stability)

Let$x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, \ldots, n$be a collection of INNs and their weight vector is$\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$, $\upsilon_i \in [0, 1]$ and$\sum_{j=1}^{n} \upsilon_j = 1$. If $x_{n+1} = \langle [T_{n+1}^L, T_{n+1}^U], [I_{n+1}^L, I_{n+1}^U], [F_{n+1}^L, F_{n+1}^U] \rangle$ is also an INN and $k > 0$ then

$$INSSWG_\upsilon(x_1{}^k \otimes x_{n+1}, x_2{}^k \otimes x_{n+1}, \ldots, x_n{}^k \otimes x_{n+1}) = (INSSWG_\upsilon(x_1, x_2, \ldots, x_n))^k \otimes x_{n+1} \quad (33)$$

Proof. :

Based on the operational laws and above results, the following results are true for INNs.

$$INSSWG_\upsilon(x_1 \otimes x_{n+1}, x_2 \otimes x_{n+1}, \ldots, x_n \otimes x_{n+1}) = INSSWG_\upsilon(x_1, x_2, \ldots, x_n) \otimes x_{n+1} \quad (34)$$

$$INSSWG_\upsilon(x_1{}^k, x_2{}^k, \ldots, x_n{}^k) = (INSSWG_\upsilon(x_1, x_2, \ldots, x_n))^k \quad (35)$$

From (34) and (35), it is obvious that,

$$INSSWG_\upsilon(x_1{}^k \otimes x_{n+1}, x_2{}^k \otimes x_{n+1}, \ldots, x_n{}^k \otimes x_{n+1}) = (INSSWG_\upsilon(x_1, x_2, \ldots, x_n))^k \otimes x_{n+1}$$

Numerical computation can be performed as in theorem 5.3.1.

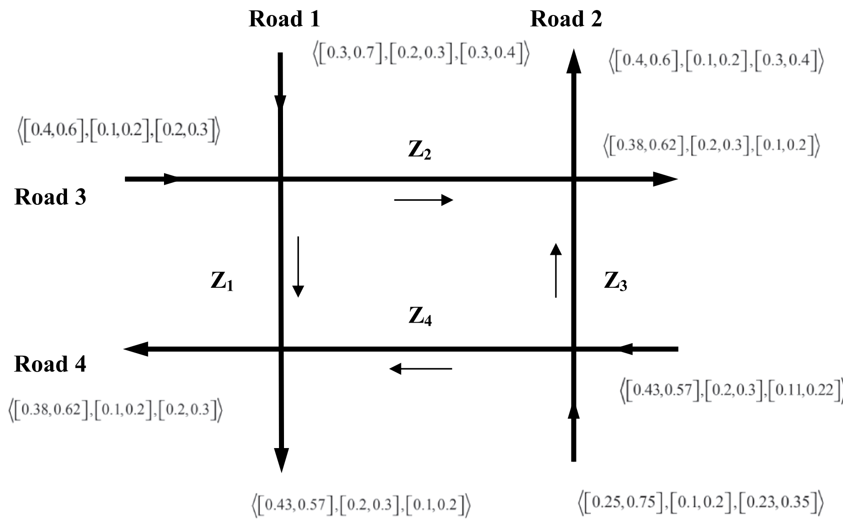

Fig. 3. Traffic flow on the road with four junctions using interval neutrosophic numbers.

5.4. Proposed theorems using INSSWA operator

Here, statements of the above theorems are given, and the proof is similar.

5.4.1. Theorem

Let $x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, ..., n$ be a collection of INNs and their weight vector is $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$, $\upsilon_i \in [0, 1]$ and $\sum_{j=1}^{n} \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted averaging (INSSWA) operator is still an INN, i.e.,

$$INSSWA_{\upsilon}(x_1, x_2, ...x_n)$$
$$= \{[(((\upsilon_j (T_j^L)^{\delta}) - \wp(\upsilon_j (T_j^L)^{\delta}))^{1/\delta}, ((\upsilon_j (T_j^U)^{\delta}) - \wp(\upsilon_j (T_j^U)^{\delta}))^{1/\delta})],$$
$$\left[1 - \left(\left(\upsilon_{j_{(1-I_j^L)^{\delta}}}\right) - \wp(\upsilon_j (1 - I_j^L)^{\delta})\right)^{1/\delta}, 1 - ((\upsilon_j (1 - I_j^U)^{\delta}) - \wp(\upsilon_j (1 - I_j^U)^{\delta}))^{1/\delta}\right],$$
$$[1 - ((\upsilon_j (1 - F_j^L)^{\delta}) - \wp(\upsilon_j (1 - F_j^L)^{\delta}))^{1/\delta}, 1 - ((\upsilon_j (1 - F_j^U)^{\delta}) - \wp(\upsilon_j (1 - F_j^U)^{\delta}))^{1/\delta}]\} \quad (36)$$

If all x_i, $i = 1, 2, ..., n$ are equal, i.e., $x_i = x$ then $INSSWA_{\varpi}(x_1, x_2, ..., x_n) = x.$. (37)

$$INSSWA_{\upsilon}(x_1 \oplus x_{n+1}, x_2 \oplus x_{n+1}, ...x_n \oplus x_{n+1}) = INSSWA_{\upsilon}(x_1, x_2, ...x_n) \oplus x_{n+1} \quad (38)$$

$$INSSWA_{\varpi}(k \cdot x_1, k \cdot x_2, ..., k \cdot x_n) = k \cdot INSSWA_{\varpi}(x_1, x_2, ..., x_n) \quad (39)$$

$$TIT2SSWA_{\varpi}(k \cdot \overline{F_1} \oplus \overline{F_{n+1}}, k \cdot \overline{F_2} \oplus \overline{F_{n+1}}, ..., k \cdot \overline{F_n} \oplus \overline{F_{n+1}}) = k \cdot TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \oplus \overline{F_{n+1}} \quad (40)$$

5.5. Proposed method for traffic flow control using INSSWA operator

For the same experiment as in the previous case, the average number of vehicles per hour coming in and departing at each intersecting point is taken as INNs instead of crisp numbers as in Fig. 3. The aim of this work is to identify the junction that has more vehicles (traffic), which need to be cleared first using the score value of the INN and the higher score value represents the junction that has more traffic.

Using Eq. (36),

$$INSSWA_{\varpi}(Z_1, Z_2) = (0.45)Z_1 \oplus (0.55)Z_2$$
$$= \langle [(0.45(0.3)^2 + 0.55(0.4)^2 - (0.45(0.3)^2)(0.55(0.4)^2))^{1/2},$$
$$(0.45(0.7)^2 + 0.55(0.6)^2 - (0.45(0.7)^2)(0.55(0.6)^2))^{1/2}],$$
$$[1 - (0.45(1 - 0.2)^2 + 0.55(1 - 0.1)^2 - (0.45(1 - 0.2)^2)(0.55(1 - 0.1)^2))^{1/2},$$
$$1 - (0.45(1 - 0.3)^2 + 0.55(1 - 0.2)^2 - (0.45(1 - 0.3)^2)(0.55(1 - 0.2)^2))^{1/2}],$$
$$[1 - (0.45(1 - 0.3)^2 + 0.55(1 - 0.2)^2 - (0.45(1 - 0.3)^2)(0.55(1 - 0.2)^2))^{1/2},$$
$$1 - (0.45(1 - 0.4)^2 + 0.55(1 - 0.3)^2 - (0.45(1 - 0.4)^2)(0.55(1 - 0.3)^2))^{1/2}] \rangle$$
$$= \langle [0.35, 0.61], [0.22, 0.29], [0.29, 0.37] \rangle$$

Similarly,

$$INSSWA_{\varpi}(Z_2, Z_3) = (0.45)Z_2 \oplus (0.55)Z_3 = \langle [0.38, 0.58], [0.22, 0.30], [0.26, 0.34] \rangle$$
$$INSSWA_{\varpi}(Z_3, Z_4) = (0.45)Z_3 \oplus (0.55)Z_4 = \langle [0.35, 0.62], [0.23, 0.31], [0.23, 0.32] \rangle$$
$$INSSWA_{\varpi}(Z_4, Z_1) = (0.45)Z_4 \oplus (0.55)Z_1 = \langle [0.4, 0.57], [0.23, 0.31], [0.22, 0.29] \rangle$$

Finding the score values (SVs)

Using Eq. (28),

$$SV(Z_1, Z_2) = \frac{1}{2}[(0.35 + 0.61) - (0.22 \times 0.29) + (0.29 - 1)^2 + 0.37] = 0.89$$

Similarly, $SV(Z_2, Z_3) = 0.85$, $SV(Z_3, Z_4) = 0.84$, $SV(Z_4, Z_1) = 0.83$

Based on the score values, the junction between Z_1 and Z_2 has higher value, and therefore it is recommended that this junction may be cleared first as it has more traffic.

6. Traffic flow using proposed operators

The proposed operators under interval type-2 fuzzy environment and interval neutrosophic environment are listed in Table 1. Controlling traffic flow has been handled using TIT2SSWA and INSSWA operators. There is a similar procedure for the geometric case.

In Table 1, junction (Z_1, Z_2) has the higher score value, as determined using both the proposed methods, and therefore the traffic may be cleared in that junction first.

7. Qualitative comparison of traffic control management using crisp sets, fuzzy sets, type-2 fuzzy sets, neutrosophic set and interval neutrosophic sets

In this section, a comparative analysis has been done with

Table 1
Aggregated traffic flow and score value.

Junction	TIT2SSWA	SV	INSSWA	SV
(Z_1, Z_2)	⟨[0.62, 0.73], 0.84, [0.92, 1]⟩	**1.5**	⟨[0.35, 0.61], [0.22, 0.29], [0.29, 0.37]⟩	**0.89**
(Z_2, Z_3)	⟨[0.56, 0.68], 0.78, [0.87, 0.9]⟩	**1.33**	⟨[0.38, 0.58], [0.22, 0.30], [0.26, 0.34]⟩	**0.85**
(Z_3, Z_4)	⟨[0.41, 0.53], 0.65, [0.75, 0.85]⟩	**1.05**	⟨[0.35, 0.62], [0.23, 0.31], [0.23, 0.32]⟩	**0.84**
(Z_4, Z_1)	⟨[0.56, 0.68], 0.79, [0.88, 0.95]⟩	**1.37**	⟨[0.4, 0.57], [0.23, 0.31], [0.22, 0.29]⟩	**0.83**

advantages and limitations of different types of sets such as crisp, fuzzy, type-2 fuzzy, neutrosophic and interval valued neutrosophic sets in traffic control management. This analysis will be helpful in understanding the role of all types of sets mentioned and will provide the motivation for conducting research on these areas and applying them in real-world problems according to the capacity of the type of sets. From the analysis, it is found that interval-based fuzzy and neutrosophic sets can handle more uncertainties than the single-valued type of sets. This point will give a different perspective to new researchers.

Traffic control management	Advantages	Limitations
Using crisp sets	• Fixed time period for all traffic densities • Achieved to characterize the real situation appropriately	• Cannot act while there is a fluctuation in traffic density • Unable to react immediately to unpredictable changes such as a driver's behavior • Unable to handle rapid momentous changes that disturb the continuity of the traffic
Using fuzzy sets	• Various time durations can be considered according to the traffic density • Follow a rule-based approach that accepts uncertainties • Able to model the reasoning of an experienced human being • Adaptive and intelligent • Able to apply and handle real-life rules identical to human thinking • Admits fuzzy terms and conditions • Has the best security • Makes it simpler to convert knowledge beyond the domain	• Adaptiveness is missing while computing the connectedness of the interval-based input • Cannot be used to show uncertainty as it applies crisp and accurate functions • Cannot handle uncertainties such as stability, flexibility and on-line planning completely as consequences can be uncertain
Using type-2 fuzzy sets	• Rule-based approach that accepts uncertainties completely • Adaptiveness (Fixed type-1 fuzzy sets are used to calculate the bounds of the type-reduced interval change as input changes) • Novelty (the upper and lower membership functions may be used concurrently in calculating every bound of the type-reduced interval)	• Computational complexity is high as the membership functions are themselves fuzzy
Neutrosophic set	• Deals not only with uncertainty but also indeterminacy owing to unpredictable environmental disturbances	• Unable to round up and down errors of calculations
Interval neutrosophic set	• Deals with more uncertainties and indeterminacy • Flexible and adaptable • Able to address issues with a set of numbers in the real unit interval, not just a particular number • Able to round up and down errors of calculations	• Unable to deal with criterion incomplete weight information

8. Conclusion

Controlling and clearing traffic is an essential daily traffic management task. In this paper, operational laws, and aggregation operators have been proposed under triangular interval type-2 fuzzy and interval neutrosophic environments. The validity of the proposed concepts has been verified using a numerical example. Furthermore, a novel traffic flow control method using the proposed operators is proposed. An improved score function is also proposed. Using TIT2SSWA and INSSWA operators, the traffic flow is analyzed with the score values using the score functions and the same can be derived using TIT2SSWG and INSSWG operators. The junction identified as having more traffic is the same for both the methods applied.

Notes

Compliance with ethical standards

Conflicts of interest

The authors declare that they have no conflicts of interest.

Ethical approval

The article does not involve any studies with human participants or animals. All activities have been performed by one or more of the authors.

Informed consent

Informed consent was obtained from all individual participants included in the study.

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