A Study of Systems of Neutrosophic Linear Equations

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Abstract

Operations research methods are among the modern scientific methods that have occupied a prominent place among the mathematical methods used in planning and managing various economic and military activities. They have been able to help specialists in developing ideal plans in terms of costs, production, storage, or investment of human energies. One of its most important methods is the method Linear programming, which was built based on the sets of linear equations that represent the constraints for any linear model. Based on the methods for solving the systems of linear equations, researchers were able to prepare algorithms for solving linear models, such as the direct Simplex algorithm and its modifications. After the emergence of neutrosophic science, we found that research methods had to be reformulated. Operations using the concepts of this science, and as a basis and foundation for neutrosophic linear programming. In this research, we will reformulate the systems of linear equations and some methods for solving them using the concepts of neutrosophic to be a basis for any study presented in the field of neutrosophic linear programming.

Keywords: Operations research; linear programming; systems of linear equations; neutrosophic science; systems of neutrosophic linear equations; methods for solving neutrosophic linear equations.

1. Introduction:

Through our endeavor to provide everything new and keep pace with scientific development and given the great importance of the linear programming method as one of the methods of operations research, we found it necessary to reformulate the systems of linear equations and some methods for solving them using the concepts of neutrosophic, the science in which research and studies that use its concepts topped the research list in most publishing houses. The reason for this is the accuracy of the results obtained in studies that use the concepts of this science, some research [1-13].

Operations research methods are among the modern scientific methods that have occupied a prominent place among the mathematical methods used in planning and managing various economic and military activities. They have been able to help specialists in developing ideal plans in terms of costs, production, storage, or investment of human energies. One of its most important methods is the method Linear programming, which relies on mathematical models to express any practical system from real life or a proposed idea. Mathematical models consist of an objective function and constraints. Classical operations research methods provided many algorithms that help to find the optimal solution for these models. The construction of these algorithms was based on sentences Linear equations. Based on the methods used to solve these sentences, such as the Jardin-Diver method and the Simplex method, the direct Simplex algorithm and its modifications were developed to solve linear models, and in view of the importance of the linear programming method and to keep pace with everything new in the field of scientific research and after the emergence of neutrosophic science, the science that caused a great revolution. In all fields of science, we found that it is necessary to reformulate the methods of operations research using the concepts of this science, the science that takes into account all the changes that may occur to the issue under study through the indeterminacy of the issue's data. Therefore, we present in this research a study of sentences of linear equations and some methods for solving them using the concepts of neutrosophic science to be a basis for any study presented in the field of neutrosophic linear programming .we will take the matrix of constants on the right side of these equations as neutrosophic values. Of the form $Nb_j = b_j + \delta_j$ where δ_j is indeterminacy and can take one of the forms$\delta_j \in \{\mu_{i1}, \mu_{i2}\}$ or $\delta_j \in [\mu_{i1}, \mu_{i2}]$, , also a_{ij} on the right side we will take it Neutrosophic values

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i.e., $Na_{ij} = a_{ij} + \gamma_{ij}$ where $\gamma_{ij}$is indeterminacy and can take In one of the forms $\gamma_{ij} \in [\varphi_{ij1}, \varphi_{ij2}]$, or $\gamma_{ij} \in \{\varphi_{ij1}, \varphi_{ij2}\}$, which helps us obtain more accurate solution results that take into account all conditions.

2. Discussion:

Based on the study mentioned in the classical references on the systems of linear equations and methods for solving them in some references [14-17].

We present the following study according to the concepts of neutrosophic science:

The systems of linear equations in which the number of equations equals m and the number of variables in them equals n are given according to classical logic in the following general form:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

It is written in the following matrix form:

$$A.X = B$$

Where:

$$A = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \dots \dots \dots & \dots \dots \dots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Where a_{ij} and b_i are real numbers for all values of $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$

We distinguished three cases of the systems of linear equations:

The first case: The number of equations is equal to the number of variables, i.e., $m = n$

The second case: The number of equations is greater than the number of variables, i.e., $m > n$

Third case: The number of equations is less than the number of variables, i.e., $m < n$

Below we will present the systems of linear equations using the concepts of neutrosophic science, where we will take the real numbers a_{ij} and b_i as neutrosophic numbers, that is, of the form Nb_i and Na_{ij}, indefinite values. Perfectly determined, they can be any neighborhood of the real numbers a_{ij} and b_i, written in one of the forms. next:

$Na_{ij} = a_{ij} + \varepsilon_{ij}$ and $Nb_i = b_i + \mu_i$ where $\varepsilon_{ij} \in [\lambda_{1ij}, \lambda_{2ij}]$ or $\varepsilon_{ij} \in \{\lambda_{1ij}, \lambda_{2ij}\}$ or otherwise, then the systems of neutrosophic linear equations is written in the following form:

The systems of neutrosophic linear equations in which the number of equations equals m and the number of variables in them equals $n$are given in the following general form:

$$\begin{gathered} Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n = Nb_1 \\ Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n = Nb_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ Na_{mn}x_1 + Na_{mn}x_2 + \cdots + Na_{mn}x_m = Nb_m \end{gathered}$$

In the following matrix form**:**

$$NA.X = NB$$

Where:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2n} \\ \dots \dots \dots & \dots \dots \dots \\ Na_{m1} & Na_{m2} \dots Na_{mn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ \vdots \\ Nb_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

To find the general solution to the previous systems of equations, we examine them according to the three aforementioned cases:

The first case: The number of equations is equal to the number of variables, i.e., $m = n$.

We write the systems of equations as follows:

$$\begin{gathered} Na_{11}x_1 + Na_{12}x_2 + \cdots + Na_{1n}x_n = Nb_1 \\ Na_{21}x_1 + Na_{22}x_2 + \cdots + Na_{2n}x_n = Nb_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \end{gathered}$$

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$$Na_{n1}x_1 + Na_{n2}x_2 + \cdots + Na_{nn}x_n = Nb_n$$

In matrix form:

$$NA\,.X = NB$$

Where:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \ldots Na_{1n} \\ Na_{21} & Na_{22} \ \ldots Na_{2n} \\ \ldots \ldots \ldots & \ldots \ldots \ldots \ldots \\ Na_{n1} & Na_{n2} \ldots Na_{nn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix}$$

The proverbs matrix is a square matrix whose determinant is $\Delta_N = |NA|$

Here we distinguish two cases:

1- $\Delta_N = 0$. This case results in two cases:

a. If $\Delta_N = 0$ and $\Delta_{N_{x_j}} \neq 0$ where $\Delta_{N_{x_j}}$ is the determinant resulting from the determinant of the matrix of proverbs Δ_N after replacing the column containing the proverbs of the unknown x_j with the column of constants (the values on the side The second of the equations) then the systems have no solution.

b. If $\Delta_N = 0$ and $\Delta_{N_{x_j}} = 0$, this means that the systems of equations are not linearly independent, meaning that some of them are linearly related to each other. To address this case, we delete one of the two linearly related equations, thus the number of equations decreases and becomes $m'$where $m' = m - 1$ and $m' < n$, which is identical to the second case that will be dealt with later.

c. If $\Delta_N \neq 0$, that is, the systems of equations are linearly independent and the systems have a single solution, which can be found in several ways. In this research, we study Gaussian- Jordan method, which is the basis for the direct simplex algorithm that we use to obtain the optimal solution for linear models.

Gaussian- Jordan method for solving systems of neutrosophic linear equations in which $m = n$:

To clarify the mathematical basis of this method, we write the equations in the following matrix form:

$$\begin{bmatrix} Na_{11} & Na_{12} \ldots Na_{1n} \\ Na_{21} & Na_{22} \ \ldots Na_{2n} \\ \ldots \ldots \ldots & \ldots \ldots \ldots \ldots \\ Na_{n1} & Na_{n2} \ldots Na_{nn} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \quad (1)$$

Or in the following abbreviated form:

$$NA\,.X = NB \quad (2)$$

Since $\Delta_N = |NA| \neq 0$, this means that the matrix NA has an inverse that is NA^{-1}. We multiply both sides of equation (2) by $NA^{-1}$and we find:

$$NA^{-1}.(NA\,.X) = NA^{-1}.NB$$

Hence, we get:

$$I.\mathrm{X} = NB'$$

Which is written in the following detailed form:

$$\begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ 0 & 1 & 0 \ldots 0 \\ \ldots & \ldots \ldots & \ldots \ldots \\ 0 & 0 & 0 \ldots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix} \quad (3)$$

This process is the basis of the Gaussian- Jordan method for solving a system of linear equations. In order to convert Figure (1) to Figure (2), we follow the following steps:

1- We express Figure (1) in the following table:

Table No. (1): Table of equations

Variables / Equations	x_1	x_2	....	x_n	NB
1	Na_{11}	Na_{12}	....	Na_{1n}	Nb_1
2	Na_{21}	Na_{22}	...	Na_{2n}	Nb_2
....	...	...	...	...	...
n	Na_{n1}	Na_{n2}	...	Na_{nn}	Nb_n

2- We convert the matrix NA to the unit matrix I by processing the lines of the table so that we make all non-diagonal elements in all its rows equal to zero and the diagonal elements equal to one. When we want to remove the variable x_s from the equation t, we follow these steps:

a- We divide all the elements of row t in which we want to make x_s equal to one by Na_{ts}, so x_s becomes equal to one and the other expressions change.

b- We set all elements of the column with x_s (except row t) equal to zero.

c- We calculate the rest of the elements of the new table from the following two relationships:

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$$\left.\begin{aligned} Na'_{ij} &= \left(Na_{ij} - Na_{is}\frac{Na_{tj}}{Na_{ts}}\right) = \frac{Na_{ij}Na_{ts} - Na_{is}Na_{tj}}{Na_{ts}} \\ Nb'_{i} &= \left(Nb_{i} - Na_{is}\frac{Nb_{t}}{Na_{ts}}\right) = \frac{Nb_{i}Na_{ts} - Na_{is}Nb_{t}}{Na_{ts}} \end{aligned}\right] \quad (4)$$

The element Na_{ts} is called the pivot element

From the previous processing we get the following table:

Table No. (2) Final solution table

Variables / Equations	Nx_1	Nx_2	....	Nx_n	NB'
1	1	0	....	0	Nb'_1
2	0	1	...	0	Nb'_2
....	...	...	...	...	...
n	0	0	...	1	Nb'_n

Through the table, the systems of linear equations are written in the following matrix form:

$$I.NX = NB'$$

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \dots \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} Nx_1 \\ Nx_2 \\ .. \\ Nx_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_n \end{bmatrix}$$

$$\Rightarrow Nx_1 = Nb'_1 \,, Nx_2 = Nb'_2 \,, \dots , Nx_n = Nb'_n$$

The second case:

The number of equations is greater than the number of variables, i.e., $m > n$:

In this case, we form a new system from the set of equations in which the number of equations is equal to the number of variables by excluding a number of equations of $\mathrm{m} - \mathrm{n}$. Then we solve the new systems as we did in the first case and replace the resulting solution in the equations that were excluded to ensure that they are satisfied.

The third case: The number of variables is greater than the number of equations, i.e., $m < n$:

In this case, we are faced with a set of equations of the following form:

$$\left.\begin{aligned} Na_{11}x_1 + Na_{12}x_2 + \dots + Na_{1\mathrm{m}}x_m + \dots + Na_{1\mathrm{n}}x_n &= Nb_1 \\ Na_{21}x_1 + Na_{22}x_2 + \dots + Na_{2\mathrm{m}}x_m + \dots + Na_{2\mathrm{n}}x_n &= Nb_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots & \dots \\ Na_{m1}x_1 + Na_{m2}x_2 + \dots + Na_{\mathrm{mm}}x_m + \dots + Na_{mn}x_n &= Nb_m \end{aligned}\right] \quad (5)$$

Which is written in the following matrix form:

$$NA = \begin{bmatrix} Na_{11} & Na_{12} \dots Na_{1m} \dots \dots Na_{1n} \\ Na_{21} & Na_{22} \dots Na_{2m} \dots \dots Na_{1n} \\ \dots \dots \dots \dots \dots & \dots \dots \dots \dots \dots \dots \dots \\ Na_{m1} & Na_{m2} \dots Na_{mm} \dots \dots Na_{mn} \end{bmatrix} \quad NB = \begin{bmatrix} Nb_1 \\ Nb_2 \\ .. \\ Nb_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \\ .. \\ .. \\ x_n \end{bmatrix}$$

In the following brief form:

$$NA_{(m.n)} \,.\, X_{(n.1)} = NB_{(m.1)} \quad (6)$$

1- We partition the matrix $NA_{(m.n)}$ into two matrices:
   a- A square matrix of rank $(m.m)$ and we denote it as $NC_{(m.m)}$.
   b- And a rectangular matrix of rank $(m.n-m)$ and we denote it $ND_{(m.n-m)}$
2- We partition the column matrix $X_{(n.1)}$ into two matrices $X'_{(m.1)}$ and $X''_{(n-m.1)}$.

Then the systems of equations (5) are written in the following matrix form:

$$\left[NC_{(m.m)}, ND_{(m.n-m)}\right] \cdot \begin{bmatrix} X'_{(m.1)} \\ X''_{(n-m.1)} \end{bmatrix} = NB_{(m.1)} \quad (7)$$

$$NC_{(m.m)} . X'_{(m.1)} + ND_{(m.n-m)} . X''_{(n-m.1)} = NB_{(m.1)}$$

From them we find that:

$$NC_{(m.m)} . X'_{(m.1)} = NB_{(m.1)} - ND_{(m.n-m)} . X''_{(n-m.1)} \quad (8)$$

Assuming that $|NC| \neq 0$, we multiply both sides in relation (8) by NC^{-1} and we find:

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$$NC^{-1}.NC.X' = NC^{-1}.(NB - ND.X^{"})$$
$$I.X' = NC^{-1}.NB - NC^{-1}.ND.X^{"}) \quad (9)$$

Assuming that $NC^{-1}.NB = NB'$ and $NC^{-1}.ND = ND'_{(m.n-m)}$ we find that:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_m \end{bmatrix} - \begin{bmatrix} Nd'_{11} & Nd'_{12} \dots Nd'_{1(n-m)} \\ Nd'_{21} & Nd'_{22} \dots Nd'_{1(n-m)} \\ \dots & \dots \\ Nd'_{m1} & Nd'_{m2} \dots Nd'_{m(n-m)} \end{bmatrix} . \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \dots \\ x_n \end{bmatrix} \quad (10)$$

It is transformed into a set of linear equations as follows:

$$Nx_1 = Nb'_1 - (Nd'_{11}x_{m+1} + Nd'_{12}x_{m+2} + \cdots + Nd'_{1(n-m)}x_n)$$
$$Nx_2 = Nb'_2 - (Nd'_{21}x_{m+1} + Nd'_{22}x_{m+2} + \cdots + Nd'_{2(n-m)}x_n)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$Nx_m = Nb'_m - (Nd'_{m1}x_{m+1} + Nd'_{m2}x_{m+2} + \cdots + Nd'_{m(n-m)}x_n)$$

This means that we were able to calculate m in terms of $(n-m)$in terms of $(n-m)$, $x_{m+1}, x_{m+2}, \dots, x_n$,we note that the values of the variables $x_1, x_2, \dots, x_m$, it relates to the values taken by the variables $x_{m+1}, x_{m+2}, \dots, x_n$, or in other words, what we give to the variables $x_{m+1}, x_{m+2}, \dots, x_n$, and that for every sentence of Values such as $\beta_{m+1}, \beta_{m+2}, \dots, \beta_n$for these variables we get a set of values for the variables $x_1, x_2, \dots, x_m$ is:

$$Nx_1 = Nb'_1 - (Nd'_{11}\beta_{m+1} + Nd'_{12}\beta_{m+2} + \cdots + Nd'_{1(n-m)}\beta_n)$$
$$Nx_2 = Nb'_2 - (Nd'_{21}\beta_{m+1} + Nd'_{22}\beta_{m+2} + \cdots + Nd'_{2(n-m)}\beta_n)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$Nx_m = Nb'_m - (Nd'_{m1}\beta_{m+1} + Nd'_{m2}\beta_{m+2} + \cdots + Nd'_{m(n-m)}\beta_n)$$

Thus, we obtain a solution that includes all the variables of sentence (5)
The solution is arranged as follows:

$$(\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_n)$$

But since the variables $x_{m+1}, x_{m+2}, \dots, x_n$can take an infinite number of qualitative values (even if they are restricted by certain conditions), we obtain an infinite number of corresponding values for the variables $x_1, x_2, \dots, x_m$.
Therefore, if $|NC| \neq 0$, then the set of equations (5) has an infinite number of acceptable solutions of the form:

$$(x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}, \dots, x_n)$$

Thus, we obtain a solution that includes all variables of the sentence, which is the ordered solution:

$$(\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_n)$$

3. Basic solutions

Since sentence (5) has an infinite number of acceptable solutions, we will try to limit ourselves to a limited number of them by setting the variables $x_{m+1}, x_{m+2}, \dots, x_n$ equal to zero. Then sentence (9) takes the following form:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ .. \\ Nb'_m \end{bmatrix} \quad (11)$$

From it we get:

$$x_1 = Nb'_1 \ \ , x_2 = Nb'_2 \ , \dots, x_m = Nb'_m$$

So, the complete solution is:

$$(Nb'_1, Nb'_2, \dots, Nb'_m, 0,0, \dots ,0)$$

We call this solution the basic solution because it is attributed to the rule with single normal vectors in the space R^m as follows:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix} \quad \dots \quad \dots \quad e_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ .. \\ m \end{bmatrix}$$

The set of vectors $e_1, e_2, \dots, e_m$ form a rule because they are linearly independent, and the vector NB' can be expressed in terms of it using the factorials $x_1, x_2, \dots, x_m$ as follows:

$$NB' = e_1x_1 + e_2x_2 + \cdots + e_mx_m$$

We call the variables$x_1, x_2, \dots, x_m$, basic variables and we call other variables $x_{m+1}, x_{m+2}, \dots, x_n$ free or non-basic variables because they take qualitative values.

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The process of choosing the variables $x_1, x_2, \dots, x_m$ to be basic variables is a random process, as we can form other basic solutions, knowing that the possibilities available to obtain basic solutions are:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

It is a finite number of infinite acceptable solutions.

Example 1:

Find the joint solution of the following two linear equations:

$$2x_1 + 7x_2 + 3x_3 + 2x_4 = [2,5]$$
$$3x_1 + 9x_2 + 4x_3 + x_4 = [3,7]$$
$$x_1 + 5x_2 + 3x_3 + 4x_4 = [4,8]$$

In the set of equations, the number of variables is $n = 4$ and the number of equations is $m = 3$. Therefore, the number of basic variables is equal to 3 and the number of non-basic free variables is $n - m = 1$. The number of possible solutions is calculated from the relationship:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

i.e.,

$$C_4^3 = \frac{4!}{3!\,(4-3)!} = 4$$

Write as follows:

$$(x_1, x_2, x_3, 0) \quad, (x_1, x_2, 0, x_4)\,, (x_1, 0, x_3, x_4), (0, x_2, x_3, x_4)$$

To obtain these solutions, we write the systems of equations in the following form:

$$2x_1 + 7x_2 + 3x_3 = [2,5] - 2x_4$$
$$3x_1 + 9x_2 + 4x_3 + x_4 = [3,7] - x_4$$
$$x_1 + 5x_2 + 3x_3 = [4,8] - 4x_4$$

The previous sentence is written in the following matrix form:

$$\begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4] \quad (*)$$

$$C = \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} \qquad X' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad NB = \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} \qquad D = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad X'' = [x_4]$$

We calculate the determinant $|C|$.

We find:

$$|C| = \begin{vmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{vmatrix} = -3 \neq 0$$

To find the solutions, we find the reciprocal of the matrix, $C = \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix}$

We find:

$$C^{-1} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix}$$

We compensate in the relationship:

$$NC^{-1}.NC.X' = NC^{-1}.(NB - ND.X'')$$

We get:

$$\begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \left(\begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4] \right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Nx_1 \\ Nx_2 \\ Nx_3 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} [2,5] \\ [3,7] \\ [4,8] \end{bmatrix} - \begin{bmatrix} \frac{-7}{3} & 2 & \frac{-1}{3} \\ \frac{5}{3} & -1 & \frac{-1}{3} \\ -2 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} . [x_4]$$

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It is transformed into the following systems of equations:

$$Nx_1 = \begin{bmatrix} \left[0, \frac{-1}{3}\right] \\ -\left[1, \frac{4}{3}\right] \\ [3,5] \end{bmatrix} - \begin{bmatrix} \frac{-16}{3} \\ 1 \\ 1 \end{bmatrix} . [x_4]$$

Setting the free variable x_4 equal to zero we get:

$$Nx_1 = \begin{bmatrix} \left[0, \frac{-1}{3}\right] \\ -\left[1, \frac{4}{3}\right] \\ [3,5] \end{bmatrix}$$

i.e.,

$$x_1 = \left[0, \frac{-1}{3}\right], x_2 = -\left[1, \frac{4}{3}\right], x_3 = [3,5]$$

Thus, we obtain the first neutrosophic basic solution, which is:

$$(x_1, x_2, x_3, 0) = \left(\left[0, \frac{-1}{3}\right], -\left[1, \frac{4}{3}\right], [3,5], 0\right)$$

In the same way we obtain other basic solutions.

4. Dissolved basic solutions:

The base solution is a degenerate and invalid solution if in the final result we obtain a value of zero for the variables that we chose as the base.

Gaussian- Jordan method for solving a set of linear equations in which $m < n$:

Based on the previous mathematical principles, the basic steps of the Gaussian- Jordan method are as follows:

1- We write the systems of equations (5) in the following matrix form:

$$I.X' + NC^{-1}.D.X^{"} = NC^{-1}.NB = NB'$$

$$[I\ , NC^{-1}.ND] . \begin{bmatrix} X' \\ X^{"} \end{bmatrix} = NB' \qquad (12)$$

Which is written in the following detailed form:

$$\begin{bmatrix} 1 & 0 & 0 \dots 0 & Nd'_{11} & Nd'_{12} \dots Nd'_{1(n-m)} \\ 0 & 1 & 0 \dots 0 & Nd'_{21} & Nd'_{22} \dots Nd'_{1(n-m)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 & Nd'_{m1} & Nd'_{m2} \dots Nd'_{m(n-m)} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} Nb'_1 \\ Nb'_2 \\ \vdots \\ Nb'_m \end{bmatrix} \qquad (13)$$

The transition from Figure (5) to Figure (12) is done by following the same steps that we mentioned in the previous paragraph, but this method does not give us a basic solution unless we set all the free variables equal to zero. If we do that, we only get the first solution, and to get all the solutions, we do the following steps:

a. We organize the following table:

Table No. (3) The first table for the Gaussian- Jordan method

Variables / Equations	x_1	x_2	....	x_m	x_{m+1}	x_{m+2}	....	x_n	NB
1	a_{11}	a_{12}	....	a_{1m}	a_{1m+1}	a_{1m+2}	....	a_{1n}	Nb'_1
2	a_{21}	a_{22}	...	a_{2m}	a_{2m+1}	a_{2m+2}	....	a_{2n}	Nb'_2
....	...	...	...	...	...	...	...	...	...
m	a_{m1}	a_{m2}	...	a_{mm}	a_{mm+1}	a_{mm+2}	...	a_{mn}	Nb'_m

b. We find the identity matrix $I_{m\times m}$ by processing the rows of the previous table in the same way that was explained in the previous paragraph. This is done by specifying the variables that will be entered into the base and let them be $x_1, x_2, \dots, x_m$. As a result of this processing, we get the following table:

Table No. (4): Table of the first basic solution

Variables / Equations	x_1	x_2	....	x_m	x_{m+1}	x_{m+2}	....	x_n	NB'
1	1	0	....	0	Nd'_{11}	Nd'_{12}	....	Nd'_{1n-m}	Nb'_1
2	0	1	...	0	Nd'_{21}	Nd'_{22}	....	Nd'_{2n-m}	Nb'_2
....	...	...	...	...	...	...	...	...	...
m	0	0	...	1	Nd'_{m1}	Nd'_{22}	...	Nd'_{mn-m}	Nb'_2

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c. Setting all the free variables in Table (4) equal to zero, we obtain the following first basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_m, 0,0, \dots ,0)$$

d. To obtain a second basic solution, we replace one of the basic variables, say x_m, with one of the non-basic variables x_{m+1}, by selecting the appropriate pivot element, and here it is Nq'_{m1}. We work to delete x_{m+1} from all equations except equation m .We set the coefficient of this variable in this equation equal to one. We perform the appropriate calculations through the two relations (4). We solve the following second basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_{m-1}, 0, Nb'_{m+1}, 0, \dots ,0)$$

To obtain other basic solutions, we repeat what was stated in step (d).

Non-negative basic solutions:

If all or some of the variables are conditioned to be non-negative, then some of the basic solutions are unacceptable because they violate the condition. In such a case, we have to look for positive basic solutions from among the basic solutions.

Due to the difficulty of applying the method mentioned in the example, especially in the case that includes a large number of variables, the Gaussian- Jordan method was developed so that positive solutions are directly obtained.

The new method was called the simplex method, which is carried out according to the following steps:

The simplex method for finding non-negative basic solutions to a system of linear equations in which $m < n$:

In the systems of equations (5):

1- We make all elements of the constant's column NB on the second side of the equations non-negative, by multiplying the equation whose second side is negative by (-1)
2- We put the coefficients of the new systems in a table.
3- We form a rule consisting of m variables by choosing the variable that we want to enter into the rule, for example, x_s, and then we calculate the index.

$$\theta = Min\left[\frac{Nb_i}{Na_{is}}\right] = \frac{Nb_t}{Na_{ts}} > 0; \quad Na_{is} > 0, Nb_i > 0$$

We call the element Na_{ts} the pivot element, we delete the variable x_s from all equations according to the Gaussian- Jordan method, except for the equation t, in which its coefficient is equal to one. We repeat the previous step until we form a base consisting of m variables.

4- Setting the non-basic variables equal to zero we obtain the following non-negative basic solution:

$$(Nb'_1, Nb'_2, \dots, Nb'_{m-1}, 0, Nb'_{m+1}, 0, \dots ,0)$$

5- To obtain other new non-negative basic solutions, we choose one of the variables to be a basic variable, then we determine the pivot element and repeat the work we did for the previously mentioned variable x_s. We obtain a new non-negative basic solution, and so we continue working until we obtain all non-negative basic solutions.

We explain the above through the following example:

Example 2:

$$x_1 - 3x_4 + 2x_5 = -[1,3]$$
$$x_2 + 2x_4 - 3x_5 = [2,8]$$

We multiply the first equation by (-1) until the condition $Nb_i > 0$ is met, we obtain the following new systems:

$$-x_1 - 3x_3 - 2x_5 = [1,3]$$
$$x_2 + 2x_4 - 3x_5 = [2,8]$$

The stopping criterion is when we do not find a free column that we did not use for switching that contains a positive element (at least one), meaning that all the elements of the free columns that were not used during the swap are negative values.

In the systems of equations, the number of variables is $n = 5$ and the number of equations is $m = 2$ Therefore, the number of basic variables is equal to 2 and the number of non-basic free variables is $n - m = 3$. The number of possible solutions is calculated from the relationship:

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

i.e.,

$$C_5^2 = \frac{5!}{2!\,(5-2)!} = 10$$

Write as follows:

$$(x_1, x_2, 0,0,0)\ , (x_1, 0, x_3, 0,0), (x_3, 0,0, x_4, 0), (x_1, 0,0,0, x_5), (0, x_2, x_3, 0,0)$$
$$(0, x_2, 0, x_4, 0), (0, x_2, 0,0, x_5), (0,0, x_3, x_4, 0), (0,0, x_3, 0, x_5), (0,0,0, x_4, x_5)$$

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To obtain these solutions, we organize the following table:

Table No. (5): The first table for the simplex method

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB
1	-1	0	0	3	-2	[1,3]
2	0	1	0	2	-3	[2,8]

To find a basic solution to the set of equations, we choose a variable, for example x_4, to be a basic variable, and to determine the appropriate anchor element, we calculate the index:

$$\theta = Min\left[\frac{Nb_i}{Na_{is}}\right] = Min\left[\frac{[1,3]}{3}, \frac{[2,8]}{2}\right] = \frac{[1,3]}{3}$$

That is, the fulcrum is $a_{14} = 3$. By performing the necessary calculations to delete the variable x_4 from the two equations, we obtain the following table:

Table No. (6) The second table for the simplex method

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB'
x_4	$-\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	$\left[\frac{1}{3}, 1\right]$
2	$\frac{2}{3}$	1	0	0	$\frac{5}{3}$	$\left[\frac{4}{3}, 6\right]$

We choose another variable to be a basic variable. We note that the variable x_2 is ready to be a basic variable, and thus we get the following table:

Table No. (7): Final solution table

Variables / Equations	x_1	x_2	x_3	x_4	x_5	NB'
x_4	$-\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	$\left[\frac{1}{3}, 1\right]$
x_2	$\frac{2}{3}$	1	0	0	$\frac{5}{3}$	$\left[\frac{4}{3}, 6\right]$

Thus, we obtain a base consisting of the variables x_2 , x_4. We set the free variables equal to zero, and we obtain the following non-negative neutrosophic basic solution:

$$\left(0, \left[\frac{4}{3}, 6\right], 0, \left[\frac{1}{3}, 1\right], 0\right)$$

To obtain other solutions, we repeat what we did to determine the previous solution.

5. Conclusion and results

As a basis for neutrosophic linear programming, we presented in this research a study of the sets of neutrosophic linear equations, and the Gaussian- Jordan method, which is considered the mathematical basis for the simplex method used to find positive basic solutions, which can be used when there are restrictions on some or all of the variables to be positive values, which in turn was the basis for the method. Direct simplex used to find the optimal solution for linear models. Through the examples that we presented on the systems of neutrosophic equations, we arrived at basic neutrosophic solutions that express unspecified values. Such sentences can be used in cases where the data provided to the systems that operate according to these systems of equations is subject to change. Here we can benefit from the margin of freedom offered by neutrosophic values.

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