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Research article

An approach to Q-neutrosophic soft rings

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Abstract: In this paper, we introduce the notion of Q-neutrosophic soft rings and discuss some of its related properties. Next, we discuss the cartesian product of Q-neutrosophic soft rings and homomorphic images and preimages of Q-neutrosophic soft rings. Moreover, Q-neutrosophic soft ideals are defined and some of their related properties are explored.

Keywords: neutrosophic soft ring; neutrosophic soft set; Q-neutrosophic soft ring; Q-neutrosophic soft set
Mathematics Subject Classification: 03G25, 08A72

1. Introduction

Neutrosophic sets (NSs) first appeared in mathematics in 1998 [31, 32] as a way to handle uncertain and indeterminate data as an extension of the concepts of the classical sets and fuzzy sets [41]. Soft sets were presented by Molodtsov [28] as another way to deal with uncertainty. NSs were further extended to neutrosophic soft sets (NSSs) [27] by joining the notions of NSs and soft sets. NSSs were further discussed in [22]. NSs and NSSs became a vital area of study, they were utilized to different branches of mathematics including graph theory and decision making [13, 15–17, 21, 23, 26, 38–40]. Q-Neutrosophic soft sets (Q-NSS) were established as a way to deal with two dimensional uncertain data as an extension of NSs, NSSs and Q–fuzzy soft sets [7]. A Q-NSS is identified via three independent membership degrees which are standard or non-standard subsets of the interval $]^{-}0, 1^{+}[$ where $^{-}0 = 0 - \epsilon, 1^{+} = 1 + \epsilon; \epsilon$ is an infinitesimal number. These memberships represent the degrees of truth, indeterminacy, and falsity; this structure makes Q-NSSs an effective common framework and empowers it to deal with two-dimensional indeterminate information. Thus, Q-NSS theory was further explored by Abu Qamar and Hassan by discussing their basic operations [1], relations [5], measures of distance, similarity and entropy [2] and also extended it further to the concept of generalized Q-neutrosophic soft expert sets [3].

Hybrid models of fuzzy sets and soft sets were extensively applied in different fields of mathematics, in particular they were extremely applied in classical algebraic structures. This was started by Rosenfeld in 1971 [30] when he established the idea of fuzzy subgroups, by applying fuzzy sets to the theory of groups. Recently, many researchers have applied different hybrid models of fuzzy sets to several algebraic structures such as groups, semirings and BCK/BCI-algebras [8–12, 19, 24, 25, 36, 37]. NSs and NSSs have received more attention in studying the algebraic structures dealing with uncertainty. Çetkin and Aygün [18] established the concept of neutrosophic subgroups. Bera and Mahapatra introduced neutrosophic soft rings [14]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. The notion of Q-fuzzy groups was discusssed in [34], neutrosophic Q-fuzzy subgroups were introduced in [35], while Q-fuzzy and anti Q-fuzzy subrings were established in [29] and Q-neutrosophic subrings were introduced in [4].

Motivated by the above discussion, in the present work, we combine the idea of Q-NSSs and ring theory to establish the concept of Q-neutrosophic soft rings (Q-NS rings) as a generalization of neutrosophic soft rings and soft rings. Some properties and basic characteristics are explored. Additionally, we define the Q-level soft set of a Q-NSSs, which is a bridge between Q-NS rings and soft rings. The concept of Q-neutrosophic soft homomorphism (Q-NS hom) is defined and homomorphic image and preimage of a Q-NS ring are investigated. Furthermore, the cartesian product of Q-NS rings is defined and some pertinent properties are examined.

2. Preliminaries

In this section, we recall some concepts relevant to this study.

Definition 2.1. [28] A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X.

Definition 2.2. [6] A soft set (F, E) over a ring R is a soft ring over R if $f(e)$ is a subring of R , $\forall e \in E$.

Definition 2.3. [20] Let (F, E) be a soft set over the ring R. Then, (F, E) is called a soft left ideal (resp. right ideal) over R if $F(e)$ is a left ideal of R for each $e \in E$ i.e.

1. $x, y \in F(e) \Rightarrow x - y \in F(e)$,
2. $x \in F(e), r \in R \Rightarrow rx \in F(e)$(resp. $xr \in F(e)$).

Definition 2.4. [20] Let (F, E) be a soft set over the ring R. Then, (F, E) is called a both sided ideal over R if $F(e)$ is a left and right ideal of R for each $e \in E$ i.e.

1. $x, y \in F(e) \Rightarrow x - y \in F(e)$,
2. $x \in F(e), r \in R \Rightarrow rx \in F(e), xr \in F(e)$.

Definition 2.5. [5] Let X be a universal set, Q be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on X with dimension $l = 1$. A pair (Γ_Q, A) is called a Q-NSS over X, where $\Gamma_Q : A \to \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

Definition 2.6. [1] The union of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Λ_Q, C) written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and for all $c \in C, (x, q) \in X \times Q$, the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$T_{\Lambda_{Q(c)}}(x,q) = \begin{cases} T_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ T_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \max\{T_{\Gamma_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_{Q(c)}}(x,q) = \begin{cases} I_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ I_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_{Q(c)}}(x,q) = \begin{cases} F_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ F_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.7. [1] The intersection of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Λ_Q, C) written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cap B$ and for all $c \in C$ and $(x,q) \in X \times Q$ the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$\begin{aligned} T_{\Lambda_{Q(c)}}(x,q) &= \min\{T_{\Gamma_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\}, \\ I_{\Lambda_{Q(c)}}(x,q) &= \max\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\}, \\ F_{\Lambda_{Q(c)}}(x,q) &= \max\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\}. \end{aligned}$$

3. Q-neutrosophic soft rings

In this section, we introduce the notion of Q-NS rings. Several basic properties and theorems related to this concept are explored.

Definition 3.1. Let (Γ_Q, A) be a Q-NSS over $(R, +, .)$. Then, (Γ_Q, A) is said to be a Q-NS ring over $(R, +, .)$ if for all $x, y \in R, q \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(x+y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x+y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$ and $F_{\Gamma_{Q(e)}}(x+y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.
2. $T_{\Gamma_{Q(e)}}(-x,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(-x,q) \leq I_{\Gamma_{Q(e)}}(x,q)$ and $F_{\Gamma_{Q(e)}}(-x,q) \leq F_{\Gamma_{Q(e)}}(x,q)$.
3. $T_{\Gamma_{Q(e)}}(x.y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x.y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$ and $F_{\Gamma_{Q(e)}}(x.y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.

Example 3.1. Let $R = (\mathbb{Z}, +, .)$ be the ring of integers and A=$\mathbb{N}$ the set of natural numbers be the parametric set. Define a Q-NSS (Γ_Q, A) as follows for $q \in Q, x \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$T_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 0 & \text{if } x \text{ is odd} \\ \frac{1}{m} & \text{if } x \text{ is even,} \end{cases}$$

$$I_{\Gamma_{Q(m)}}(x,q) = \begin{cases} \frac{1}{2m} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even,} \end{cases}$$

$$F_{\Gamma_{Q(m)}}(x,q)=\begin{cases}1-\frac{1}{m} & \text{if } x \text{ is odd}\\ 0 & \text{if } x \text{ is even.}\end{cases}$$

It is clear that $(\Gamma_Q, \mathbb{Z})$ is a Q-NS ring over R.

Theorem 3.2. *A Q-NSS (Γ_Q, A) over the ring $(R, +, .)$ is a Q-NS ring if and only if for all $x, y \in R, q \in Q$ and $e \in A$*

1. $T_{\Gamma_{Q(e)}}(x-y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}$,
   $I_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$,
   $F_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.

2. $T_{\Gamma_{Q(e)}}(x.y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}$,
   $I_{\Gamma_{Q(e)}}(x.y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$,
   $F_{\Gamma_{Q(e)}}(x.y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.

Proof. Suppose that (Γ_Q, A) is a Q-NS ring over $(R, +, .)$. Then,

$$T_{\Gamma_{Q(e)}}(x-y,q)=T_{\Gamma_{Q(e)}}(x+(-y),q)\geq\min\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(-y,q)\}\geq\min\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(y,q)\},$$
$$I_{\Gamma_{Q(e)}}(x-y,q)=I_{\Gamma_{Q(e)}}(x+(-y),q)\leq\max\{I_{\Gamma_{Q(e)}}(x,q),I_{\Gamma_{Q(e)}}(-y,q)\}\leq\max\{I_{\Gamma_{Q(e)}}(x,q),I_{\Gamma_{Q(e)}}(y,q)\},$$
$$F_{\Gamma_{Q(e)}}(x-y,q)=F_{\Gamma_{Q(e)}}(x+(-y),q)\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(-y,q)\}\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(y,q)\}.$$

Thus, conditions 1 and 2 are satisfied.

Conversely, Suppose that conditions 1 and 2 are satisfied.
For the additive identity 0_R in $(R, +, .)$,
$T_{\Gamma_{Q(e)}}(0_R,q)=T_{\Gamma_{Q(e)}}(x-x,q)\geq\min\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(x,q)\}=T_{\Gamma_{Q(e)}}(x,q)$,
$I_{\Gamma_{Q(e)}}(0_R,q)=I_{\Gamma_{Q(e)}}(x-x,q)\leq\max\{I_{\Gamma_{Q(e)}}(x,q),I_{\Gamma_{Q(e)}}(x,q)\}=I_{\Gamma_{Q(e)}}(x,q)$,
$F_{\Gamma_{Q(e)}}(0_R,q)=F_{\Gamma_{Q(e)}}(x-x,q)\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(x,q)\}=F_{\Gamma_{Q(e)}}(x,q)$.
Now,

$$\begin{aligned}T_{\Gamma_{Q(e)}}(-x,q)=T_{\Gamma_{Q(e)}}(0_R-x,q)&\geq\min\{T_{\Gamma_{Q(e)}}(0_R,q),T_{\Gamma_{Q(e)}}(x,q)\}\\&\geq\min\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(x,q)\}=T_{\Gamma_{Q(e)}}(x,q),\\I_{\Gamma_{Q(e)}}(-x,q)=I_{\Gamma_{Q(e)}}(0_R-x,q)&\leq\max\{I_{\Gamma_{Q(e)}}(0_R,q),I_{\Gamma_{Q(e)}}(x,q)\}\\&\leq\max\{I_{\Gamma_{Q(e)}}(x,q),I_{\Gamma_{Q(e)}}(x,q)\}=I_{\Gamma_{Q(e)}}(x,q),\\F_{\Gamma_{Q(e)}}(-x,q)=_{\Gamma_{Q(e)}}(0_R-x,q)&\leq\max\{F_{\Gamma_{Q(e)}}(0_R,q),F_{\Gamma_{Q(e)}}(x,q)\}\\&\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(x,q)\}=F_{\Gamma_{Q(e)}}(x,q)\}\end{aligned}$$

also,

$$\begin{aligned}T_{\Gamma_{Q(e)}}(x+y,q)=T_{\Gamma_{Q(e)}}(x-(-y),q)\geq\min\{T_{\Gamma_{Q(e)}}(x,q),T_{\Gamma_{Q(e)}}(y,q)\},\\I_{\Gamma_{Q(e)}}(x+y,q)=I_{\Gamma_{Q(e)}}(x-(-y),q)\leq\max\{I_{\Gamma_{Q(e)}}(x,q),I_{\Gamma_{Q(e)}}(y,q)\},\\F_{\Gamma_{Q(e)}}(x+y,q)=F_{\Gamma_{Q(e)}}(x-(-y),q)\leq\max\{F_{\Gamma_{Q(e)}}(x,q),F_{\Gamma_{Q(e)}}(y,q)\}.\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let* (Γ_Q, A) *and* (Ψ_Q, B) *be two Q-NS rings over* $(R, +, .)$*. Then,* $(\Gamma_Q, A) \cap (\Psi_Q, B)$ *is also a Q-NS ring over* $(R, +, .)$*.*

Proof. Let $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, A \cap B)$. Now, $\forall x, y \in R, q \in Q$ and $e \in A \cap B$,

$$\begin{aligned}
T_{\Lambda_{Q(e)}}(x-y,q) &= \min\left\{T_{\Gamma_{Q(e)}}(x-y,q), T_{\Psi_{Q(e)}}(x-y,q)\right\}\\
&\geq \min\left\{\min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, \min\{T_{\Psi_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \min\left\{\min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\}, \min\{T_{\Gamma_{Q(e)}}(y,q), T_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \min\left\{T_{\Lambda_{Q(e)}}(x,q), T_{\Lambda_{Q(e)}}(y,q)\right\},
\end{aligned}$$

also,

$$\begin{aligned}
I_{\Lambda_{Q(e)}}(x-y,q) &= \max\left\{I_{\Gamma_{Q(e)}}(x-y,q), I_{\Psi_{Q(e)}}(x-y,q)\right\}\\
&\leq \max\left\{\max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}, \max\{I_{\Psi_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \max\left\{\max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\}, \max\{I_{\Gamma_{Q(e)}}(y,q), I_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \max\left\{I_{\Lambda_{Q(e)}}(x,q), I_{\Lambda_{Q(e)}}(y,q)\right\}.
\end{aligned}$$

Similarly, $F_{\Lambda_{Q(e)}}(x-y,q) \leq \max\left\{F_{\Lambda_{Q(e)}}(x,q), F_{\Lambda_{Q(e)}}(y,q)\right\}$.

Next,

$$\begin{aligned}
T_{\Lambda_{Q(e)}}(x.y,q) &= \min\left\{T_{\Gamma_{Q(e)}}(x.y,q), T_{\Psi_{Q(e)}}(x.y,q)\right\}\\
&\geq \min\left\{\min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, \min\{T_{\Psi_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \min\left\{\min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\}, \min\{T_{\Gamma_{Q(e)}}(y,q), T_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \min\left\{T_{\Lambda_{Q(e)}}(x,q), T_{\Lambda_{Q(e)}}(y,q)\right\},
\end{aligned}$$

also,

$$\begin{aligned}
I_{\Lambda_{Q(e)}}(x.y,q) &= \max\left\{I_{\Gamma_{Q(e)}}(x.y,q), I_{\Psi_{Q(e)}}(x.y,q)\right\}\\
&\leq \max\left\{\max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}, \max\{I_{\Psi_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \max\left\{\max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\}, \max\{I_{\Gamma_{Q(e)}}(y,q), I_{\Psi_{Q(e)}}(y,q)\}\right\}\\
&= \max\left\{I_{\Lambda_{Q(e)}}(x,q), I_{\Lambda_{Q(e)}}(y,q)\right\}.
\end{aligned}$$

Similarly, we can show $F_{\Lambda_{Q(e)}}(x.y,q) \leq \max\left\{F_{\Lambda_{Q(e)}}(x,q), F_{\Lambda_{Q(e)}}(y,q)\right\}$. This completes the proof. □

Remark 3.4. For two Q-NS rings (Γ_Q, A) and (Ψ_Q, B) over $(R, +, .)$, $(\Gamma_Q, A) \cup (\Psi_Q, B)$ is not generally a Q-NS ring.

For example, let $R = (\mathbb{Z}, +, .)$, $E = 2\mathbb{Z}$. Consider two Q-NS rings (Γ_Q, E) and (Ψ_Q, E) over R as follows: for $x, m \in \mathbb{Z}$ and $q \in Q$

$$T_{\Gamma_{Q(2m)}}(x,q) = \begin{cases} 0.50 & \text{if } x = 4tm, \exists t \in \mathbb{Z},\\ 0 & \textit{otherwise}, \end{cases}$$

$$I_{\Gamma_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.25 & otherwise,\end{cases}$$

$$F_{\Gamma_{Q(2m)}}(x,q)=\begin{cases}0.40 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.10 & otherwise,\end{cases}$$

and

$$T_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0.67 & \text{if } x=8tm, \exists t\in\mathbb{Z},\\ 0 & otherwise,\end{cases}$$

$$I_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=8tm, \exists t\in\mathbb{Z},\\ 0.20 & otherwise,\end{cases}$$

$$F_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0.16 & \text{if } x=8tm, \exists t\in\mathbb{Z},\\ 0.33 & otherwise.\end{cases}$$

Let $(\Gamma_Q,A)\cup(\Psi_Q,B)=(\Lambda_Q,E)$. For $m=3, x=12, y=18$ we have,

$$T_{\Lambda_{Q(6)}}(12-18,q)=T_{\Lambda_{Q(6)}}(-6,q)=\max\left\{T_{\Gamma_{Q(6)}}(-6,q),T_{\Psi_{Q(6)}}(-6,q)\right\}=\max\{0,0\}=0$$

and

$$\begin{aligned}\min&\left\{T_{\Lambda_{Q(6)}}(12,q),T_{\Lambda_{Q(6)}}(18,q)\right\}\\ &=\min\left\{\max\{T_{\Gamma_{Q(6)}}(12,q),T_{\Psi_{Q(6)}}(12,q)\},\max\{T_{\Gamma_{Q(6)}}(18,q),T_{\Psi_{Q(6)}}(18,q)\}\right\}\\ &=\min\left\{\max\{0.50,0\},\max\{0,0.67\}\right\}\\ &=\min\left\{0.50,0.67\right\}=0.50.\end{aligned}$$

Hence, $T_{\Lambda_{Q(6)}}(12-18,q)<\min\left\{T_{\Lambda_{Q(6)}}(12,q),T_{\Lambda_{Q(6)}}(18,q)\right\}$. Thus, the union is not a Q-NS ring.

Theorem 3.5. *Let (Γ_Q,A) and (Ψ_Q,B) be two Q-NS rings over $(R,+,.)$. Then, $(\Gamma_Q,A)\wedge(\Psi_Q,B)$ is also a Q-NS ring over $(R,+,.)$.*

Proof. The proof is similar to the proof of Theorem 3.3. □

Definition 3.6. Let (Γ_Q,A) be a Q-NSS over X. Let $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma\leq 3$. Then $(\Gamma_Q,A)_{(\alpha,\beta,\gamma)}$ is a Q-level soft set of (Γ_Q,A) defined by

$$(\Gamma_Q,A)_{(\alpha,\beta,\gamma)}=\left\{x\in X,q\in Q: T_{\Gamma_{Q(e)}}(x,q)\geq\alpha, I_{\Gamma_{Q(e)}}(x,q)\leq\beta, F_{\Gamma_{Q(e)}}(x,q)\leq\gamma\right\}$$

for all $e\in A$.

The next theorem provides a bridge between Q-NS rings and soft rings.

Theorem 3.7. *Let (Γ_Q,A) be a Q-NSS over $(R,+,.)$. Then, (Γ_Q,A) is a Q-NS ring over $(R,+,.)$ if and only if for all $\alpha,\beta,\gamma\in[0,1]$ the Q-level soft set $(\Gamma_Q,A)_{(\alpha,\beta,\gamma)}\neq\phi$ is a soft ring over R.*

Proof. Let (Γ_Q, A) be a Q-NS ring over $(R, +, .)$, $x, y \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$ and $q \in Q$, for arbitrary $\alpha,\beta,\gamma \in [0, 1]$ and $e \in A$.
Then, we have $T_{\Gamma_Q(e)}(x, q) \geq \alpha, I_{\Gamma_Q(e)}(x, q) \leq \beta, F_{\Gamma_Q(e)}(x, q) \leq \gamma$. Since (Γ_Q, A) is a Q-NS ring over G, then we have

$$\begin{aligned} T_{\Gamma_Q(e)}(x-y, q) &\geq \min\left\{T_{\Gamma_Q(e)}(x, q), T_{\Gamma_Q(e)}(y, q)\right\} \geq \min\left\{\alpha, \alpha\right\} = \alpha, \\ I_{\Gamma_Q(e)}(x-y, q) &\leq \max\left\{I_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(y, q)\right\} \leq \max\left\{\beta, \beta\right\} = \beta, \\ F_{\Gamma_Q(e)}(x-y, q) &\leq \max\left\{F_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(y, q)\right\} \leq \max\left\{\gamma, \gamma\right\} = \gamma. \end{aligned}$$

Therefore, $x - y \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. Furthermore, $T_{\Gamma_Q(e)}(x.y, q) \geq \alpha, I_{\Gamma_Q(e)}(x.y, q) \leq \beta$, $F_{\Gamma_Q(e)}(x.y, q) \leq \gamma$. So, $x.y \in (\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$. Hence, $(\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$ is a subring over $(R, +, .), \forall e \in A$.

Conversely, suppose (Γ_Q, A) is not a Q-NS ring over $(R, +, .)$. Then, there exists $e \in A$ such that $\Gamma_Q(e)$ is not a Q-neutrosophic subring of R. Then, there exist $x_1, y_1 \in R$ and $q \in Q$ such that at least one of the conditions in Definition 3.1 does not hold. Without loss of generality, let us assume $T_{\Gamma_Q(e)}(x_1 - y_1, q) < \min\left\{T_{\Gamma_Q(e)}(x_1, q), T_{\Gamma_Q(e)}(y_1, q)\right\}$.
Let $T_{\Gamma_Q(e)}(x_1, q) = \alpha_1, T_{\Gamma_Q(e)}(y_1, q) = \alpha_2$ and $T_{\Gamma_Q(e)}(x_1 - y_1, q) = \alpha_3$. If we take $\alpha = \min\{\alpha_1, \alpha_2\}$, then $x_1 - y_1 \notin (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. But, since

$$T_{\Gamma_Q(e)}(x_1, q) = \alpha_1 \geq \min\{\alpha_1, \alpha_2\} = \alpha$$

and

$$T_{\Gamma_Q(e)}(y_1, q) = \alpha_2 \geq \min\{\alpha_1, \alpha_2\} = \alpha.$$

For $I_{\Gamma_Q(e)}(x_1, q) \leq \beta, I_{\Gamma_Q(e)}(y_1, q) \leq \beta, F_{\Gamma_Q(e)}(x_1, q) \leq \gamma, F_{\Gamma_Q(e)}(y_1, q) \leq \gamma$, we have $x_1, y_1 \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$.
This contradicts with the fact that $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$ is a soft ring over G.
The other cases can be obtained similarly. □

4. Cartesian product of Q-neutrosophic soft rings

In this section, we define the cartesian product of Q-NS rings and prove that it is also a Q-NS ring.

Definition 4.1. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NS rings over $(R_1, +, .)$ and $(R_2, +, .)$, respectively. Then, their cartesian product $(\Lambda_Q, A \times B) = (\Gamma_Q, A) \times (\Psi_Q, B)$, where $\Lambda_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a, b) \in A \times B$. Analytically, for $x \in R_1, y \in R_2$ and $q \in Q$

$$\Lambda_Q(a, b) = \left\{\left\langle((x, y), q), T_{\Lambda_Q(a,b)}((x, y), q), I_{\Lambda_Q(a,b)}((x, y), q), F_{\Lambda_Q(a,b)}((x, y), q)\right\rangle\right\}, \text{ where}$$

$$\begin{aligned} T_{\Lambda_Q(a,b)}((x, y), q) &= \min\{T_{\Gamma_Q(a)}(x, q), T_{\Psi_Q(b)}(y, q)\}, \\ I_{\Lambda_Q(a,b)}((x, y), q) &= \max\{I_{\Gamma_Q(a)}(x, q), I_{\Psi_Q(b)}(y, q)\}, \\ F_{\Lambda_Q(a,b)}((x, y), q) &= \max\{F_{\Gamma_Q(a)}(x, q), F_{\Psi_Q(b)}(y, q)\}. \end{aligned}$$

Theorem 4.2. *Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NS rings over $(R_1, +, .)$ and $(R_2, +, .)$, respectively. Then, their cartesian product $(\Gamma_Q, A) \times (\Psi_Q, B)$ is a Q-NS ring over $(R_1 \times R_2)$.*

Proof. Let $(\Lambda_Q, A\times B) = (\Gamma_Q, A)\times(\Psi_Q, B)$, where $\Lambda_Q(a,b) = \Gamma_Q(a)\times\Psi_Q(b)$ for $(a,b)\in A\times B$. Then, for $((x_1,y_1),q),((x_2,y_2),q)\in(R_1\times R_2)\times Q$ we have,

$$\begin{aligned}
&T_{\Lambda_Q(a,b)}\Big(((x_1,y_1)-(x_2,y_2),q)\Big)\\
&\quad= T_{\Lambda_Q(a,b)}((x_1-x_2,y_1-y_2),q)\\
&\quad= \min\{T_{\Gamma_Q(a)}((x_1-x_2),q),T_{\Psi_Q(b)}((y_1-y_2),q)\}\\
&\quad\geq \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Gamma_Q(a)}(-x_2,q)\},\min\{T_{\Psi_Q(b)}(y_1,q),T_{\Psi_Q(b)}(-y_2,q)\}\Big\}\\
&\quad\geq \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Gamma_Q(a)}(x_2,q)\},\min\{T_{\Psi_Q(b)}(y_1,q),T_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Psi_Q(b)}(y_1,q)\},\min\{T_{\Gamma_Q(a)}(x_2,q),T_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \min\Big\{T_{\Lambda_Q(a,b)}((x_1,y_1),q),T_{\Lambda_Q(a,b)}((x_2,y_2),q)\Big\}
\end{aligned}$$

also,

$$\begin{aligned}
&I_{\Lambda_Q(a,b)}\Big(((x_1,y_1)-(x_2,y_2),q)\Big)\\
&\quad= I_{\Lambda_Q(a,b)}((x_1-x_2,y_1-y_2),q)\\
&\quad= \max\{I_{\Gamma_Q(a)}((x_1-x_2),q),I_{\Psi_Q(b)}((y_1-y_2),q)\}\\
&\quad\leq \max\Big\{\max\{I_{\Gamma_Q(a)}(x_1,q),I_{\Gamma_Q(a)}(-x_2,q)\},\max\{I_{\Psi_Q(b)}(y_1,q),I_{\Psi_Q(b)}(-y_2,q)\}\Big\}\\
&\quad\leq \max\Big\{\max\{I_{\Gamma_Q(a)}(x_1,q),I_{\Gamma_Q(a)}(x_2,q)\},\max\{I_{\Psi_Q(b)}(y_1,q),I_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \max\Big\{\max\{I_{\Gamma_Q(a)}(x_1,q),I_{\Psi_Q(b)}(y_1,q)\},\max\{I_{\Gamma_Q(a)}(x_2,q),I_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \max\Big\{I_{\Lambda_Q(a,b)}((x_1,y_1),q),I_{\Lambda_Q(a,b)}((x_2,y_2),q)\Big\},
\end{aligned}$$

similarly, $F_{\Lambda_Q(a,b)}\Big(((x_1,y_1)-(x_2,y_2),q)\Big)\leq\max\Big\{F_{\Lambda_Q(a,b)}((x_1,y_1),q),F_{\Lambda_Q(a,b)}((x_2,y_2),q)\Big\}$. Next,

$$\begin{aligned}
&T_{\Lambda_Q(a,b)}\Big(((x_1,y_1).(x_2,y_2),q)\Big)\\
&\quad= T_{\Lambda_Q(a,b)}((x_1.x_2,y_1.y_2),q)\\
&\quad= \min\{T_{\Gamma_Q(a)}((x_1.x_2),q),T_{\Psi_Q(b)}((y_1.y_2),q)\}\\
&\quad\geq \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Gamma_Q(a)}(x_2,q)\},\min\{T_{\Psi_Q(b)}(y_1,q),T_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad\geq \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Gamma_Q(a)}(x_2,q)\},\min\{T_{\Psi_Q(b)}(y_1,q),T_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \min\Big\{\min\{T_{\Gamma_Q(a)}(x_1,q),T_{\Psi_Q(b)}(y_1,q)\},\min\{T_{\Gamma_Q(a)}(x_2,q),T_{\Psi_Q(b)}(y_2,q)\}\Big\}\\
&\quad= \min\Big\{T_{\Lambda_Q(a,b)}((x_1,y_1),q),T_{\Lambda_Q(a,b)}((x_2,y_2),q)\Big\},
\end{aligned}$$

$$\begin{aligned}
&I_{\Lambda_Q(a,b)}\Big(((x_1,y_1).(x_2,y_2),q)\Big)\\
&\quad= I_{\Lambda_Q(a,b)}((x_1.x_2,y_1.y_2),q)\\
&\quad= \max\{I_{\Gamma_Q(a)}((x_1.x_2),q),I_{\Psi_Q(b)}((y_1.y_2),q)\}\\
&\quad\leq \max\Big\{\max\{I_{\Gamma_Q(a)}(x_1,q),I_{\Gamma_Q(a)}(x_2,q)\},\max\{I_{\Psi_Q(b)}(y_1,q),I_{\Psi_Q(b)}(y_2,q)\}\Big\}
\end{aligned}$$

$$\leq \max\Big\{\max\{I_{\Gamma_{Q(a)}}(x_1,q), I_{\Gamma_{Q(a)}}(x_2,q)\}, \max\{I_{\Psi_{Q(b)}}(y_1,q), I_{\Psi_{Q(b)}}(y_2,q)\}\Big\}$$
$$= \max\Big\{\max\{I_{\Gamma_{Q(a)}}(x_1,q), I_{\Psi_{Q(b)}}(y_1,q)\}, \max\{I_{\Gamma_{Q(a)}}(x_2,q), I_{\Psi_{Q(b)}}(y_2,q)\}\Big\}$$
$$= \max\Big\{I_{\Lambda_{Q(a,b)}}((x_1,y_1),q), I_{\Lambda_{Q(a,b)}}((x_2,y_2),q)\Big\},$$

similarly, $F_{\Lambda_{Q(a,b)}}\Big(((x_1,y_1),q).((x_2,y_2),q)\Big) \leq \max\Big\{F_{\Lambda_{Q(a,b)}}((x_1,y_1),q), F_{\Lambda_{Q(a,b)}}((x_2,y_2),q)\Big\}$. This completes the proof. □

5. Homomorphism of Q-neutrosophic soft rings

In this section, we define the Q-neutrosophic soft function, then define the image and pre-image of a Q-NSS under a Q-neutrosophic soft function. In continuation, we introduce the notion of Q-neutrosophic soft homomorphism along with some of it's properties.

Definition 5.1. Let $g : X \times Q \to Y \times Q$ and $h : A \to B$ be two functions where $A$and B are parameter sets. Then, the pair (g, h) is called a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$.

Definition 5.2. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NSSs defined over $X \times Q$ and $Y \times Q$, respectively, and (g, h) be a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$. Then,

1. The image of (Γ_Q, A) under (g, h), denoted by $(g, h)(\Gamma_Q, A)$, is a Q-NSS over $Y \times Q$ and is defined by:
$$(g,h)(\Gamma_Q, A) = \Big(g(\Gamma_Q), h(A)\Big) = \Big\{\Big\langle b, g(\Gamma_Q)(b) : b \in h(A)\Big\rangle\Big\},$$
where for all $b \in h(A), y \in Y$ and $q \in Q$,
$$T_{g(\Gamma_Q)(b)}(y,q) = \begin{cases} \max_{g(x,q)=(y,q)} \max_{h(a)=b}[T_{\Gamma_{Q(a)}}(x,q)] & \text{if } (x,q) \in g^{-1}(y,q), \\ 0 & \textit{otherwise}, \end{cases}$$
$$I_{g(\Gamma_Q)(b)}(y,q) = \begin{cases} \min_{g(x,q)=(y,q)} \min_{h(a)=b}[I_{\Gamma_{Q(a)}}(x,q)] & \text{if } (x,q) \in g^{-1}(y,q), \\ 1 & \textit{otherwise}, \end{cases}$$
$$F_{g(\Gamma_Q)(b)}(y,q) = \begin{cases} \min_{g(x,q)=(y,q)} \min_{h(a)=b}[F_{\Gamma_{Q(a)}}(x,q)] & \text{if } (x,q) \in g^{-1}(y,q), \\ 1 & \textit{otherwise}, \end{cases}$$

2. The preimage of (Ψ_Q, B) under (g, h), denoted by $(g, h)^{-1}(\Psi_Q, B)$, is a Q-NSS over X and is defined by:
$$(g,h)^{-1}(\Psi_Q, B) = \Big(g^{-1}(\Psi_Q), h^{-1}(B)\Big) = \Big\{\Big\langle a, g^{-1}(\Psi_Q)(a) : a \in h^{-1}(B)\Big\rangle\Big\},$$
where for all $a \in h^{-1}(B), x \in X$ and $q \in Q$,
$$T_{g^{-1}(\Psi_Q)(a)}(x,q) = T_{\Psi_{Q[h(a)]}}(g(x,q)),$$
$$I_{g^{-1}(\Psi_Q)(a)}(x,q) = I_{\Psi_{Q[h(a)]}}(g(x,q)),$$
$$F_{g^{-1}(\Psi_Q)(a)}(x,q) = F_{\Psi_{Q[h(a)]}}(g(x,q)).$$

If g and h are injective (surjective), then (g, h) is injective (surjective).

Definition 5.3. Let (g, h) be a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$. If g is a homomorphism from $X \times Q$ to $Y \times Q$, then (g, h) is said to be a Q-neutrosophic soft homomorphism. If g is an isomorphism from $X \times Q$ to $Y \times Q$ and h is a one-to-one mapping from A to B, then (g, h) is said to be a Q-neutrosophic soft isomorphism.

Theorem 5.4. *Let (Γ_Q, A) be a Q-NS ring over R_1 and $(g, h) : R_1 \times Q \to R_2 \times Q$ be a Q-neutrosophic soft homomorphism. Then, $(g, h)(\Gamma_Q, A)$ is a Q-NS ring over R_2.*

Proof. Let $b \in h(A)$ and $y_1, y_2 \in R_2$. For $g^{-1}(y_1, q) = \phi$ or $g^{-1}(y_2, q) = \phi$, the proof is straight forward. So, assume there exists $x_1, x_2 \in R_1$ such that $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$\begin{aligned}
T_{g(\Gamma_Q)(b)}(y_1 - y_2, q) &= \max_{g(x,q)=(y_1-y_2,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x, q) \Big] \\
&\geq \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1 - x_2, q) \Big] \\
&\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_{Q(a)}}(x_1, q), T_{\Gamma_{Q(a)}}(-x_2, q) \Big\} \Big] \\
&\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_{Q(a)}}(x_1, q), T_{\Gamma_{Q(a)}}(x_2, q) \Big\} \Big] \\
&= \min \Big\{ \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1, q) \Big], \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_2, q) \Big] \Big\}
\end{aligned}$$

$$\begin{aligned}
T_{g(\Gamma_Q)(b)}(y_1 . y_2, q) &= \max_{g(x,q)=(y_1 . y_2,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x, q) \Big] \\
&\geq \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1 . x_2, q) \Big] \\
&\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_{Q(a)}}(x_1, q), T_{\Gamma_{Q(a)}}(x_2, q) \Big\} \Big] \\
&\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_{Q(a)}}(x_1, q), T_{\Gamma_{Q(a)}}(x_2, q) \Big\} \Big] \\
&= \min \Big\{ \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1, q) \Big], \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_2, q) \Big] \Big\}.
\end{aligned}$$

Since, the inequality is satisfied for each $x_1, x_2 \in R_1$, satisfying $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$\begin{aligned}
T_{g(\Gamma_Q)(b)}(y_1 - y_2, q) &\geq \min \Big\{ \max_{g(x_1,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1, q) \Big], \max_{g(x_2,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_2, q) \Big] \Big\} \\
&= \min \Big\{ T_{g(\Gamma_Q)(b)}(y_1, q), T_{g(\Gamma_Q)(b)}(y_2, q) \Big\}.
\end{aligned}$$

$$\begin{aligned}
T_{g(\Gamma_Q)(b)}(y_1 . y_2, q) &\geq \min \Big\{ \max_{g(x_1,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_1, q) \Big], \max_{g(x_2,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_{Q(a)}}(x_2, q) \Big] \Big\} \\
&= \min \Big\{ T_{g(\Gamma_Q)(b)}(y_1, q), T_{g(\Gamma_Q)(b)}(y_2, q) \Big\}.
\end{aligned}$$

Similarly, we show that
$I_{g(\Gamma_Q)(b)}(y_1 - y_2, q) \leq \max \Big\{ I_{g(\Gamma_Q)(b)}(y_1, q), I_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$,
$I_{g(\Gamma_Q)(b)}(y_1 . y_2, q) \leq \max \Big\{ I_{g(\Gamma_Q)(b)}(y_1, q), I_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$,

$F_{g(\Gamma_Q)(b)}(y_1 - y_2, q) \leq \max\left\{F_{g(\Gamma_Q)(b)}(y_1, q), F_{g(\Gamma_Q)(b)}(y_2, q)\right\},$
$F_{g(\Gamma_Q)(b)}(y_1.y_2, q) \leq \max\left\{F_{g(\Gamma_Q)(b)}(y_1, q), F_{g(\Gamma_Q)(b)}(y_2, q)\right\}.$

□

Theorem 5.5. *Let (Ψ_Q, B) be a Q-NS ring over R_2 and (g, h) be a Q-neutrosophic soft homomorphism from $R_1 \times Q$ to $R_2 \times Q$. Then, $(g, h)^{-1}(\Psi_Q, B)$ is a Q-NS ring over R_1.*

Proof. For $a \in h^{-1}(B)$ and $x_1, x_2 \in R_1$, we have

$$\begin{aligned}
T_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &= T_{\Psi_Q[h(a)]}(g(x_1 - x_2, q)) \\
&= T_{\Psi_Q[h(a)]}(g(x_1, q) - g(x_2, q)) \\
&\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(-g(x_2, q))\right\} \\
&\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q))\right\} \\
&= \min\left\{T_{g^{-1}(\Psi_Q)(a)}(x_1, q), T_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}
\end{aligned}$$

and

$$\begin{aligned}
T_{g^{-1}(\Psi_Q)(a)}(x_1.x_2, q) &= T_{\Psi_Q[h(a)]}(g(x_1.x_2, q)) \\
&= T_{\Psi_Q[h(a)]}(g(x_1, q).g(x_2, q)) \\
&\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q))\right\} \\
&\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q))\right\} \\
&= \min\left\{T_{g^{-1}(\Psi_Q)(a)}(x_1, q), T_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
I_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &\leq \max\left\{I_{g^{-1}(\Psi_Q)(a)}(x_1, q), I_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\
I_{g^{-1}(\Psi_Q)(a)}(x_1.x_2, q) &\leq \max\left\{I_{g^{-1}(\Psi_Q)(a)}(x_1, q), I_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\
F_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &\leq \max\left\{F_{g^{-1}(\Psi_Q)(a)}(x_1, q), F_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\
F_{g^{-1}(\Psi_Q)(a)}(x_1.x_2, q) &\leq \max\left\{F_{g^{-1}(\Psi_Q)(a)}(x_1, q), F_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}.
\end{aligned}$$

Thus, the theorem is proved. □

6. Q-neutrosophic soft ideals

In the current section, we present Q-neutrosophic soft ideals and explore some of their related properties.

Definition 6.1. A Q-NSS over (Γ_Q, A) over a ring $(R, +, .)$ is called a Q-NS left (resp. right) ideal over $(R, +, .)$ if for all $x, y \in R, q \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(x-y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$ and $F_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.
2. $T_{\Gamma_{Q(e)}}(x.y,q) \geq T_{\Gamma_{Q(e)}}(y,q), I_{\Gamma_{Q(e)}}(x.y,q) \leq I_{\Gamma_{Q(e)}}(y,q)$ and $F_{\Gamma_{Q(e)}}(x.y,q) \leq F_{\Gamma_{Q(e)}}(y,q)$ (resp. $T_{\Gamma_{Q(e)}}(x.y,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x.y,q) \leq I_{\Gamma_{Q(e)}}(x,q)$ and $F_{\Gamma_{Q(e)}}(x.y,q) \leq F_{\Gamma_{Q(e)}}(x,q)$).

Definition 6.2. A Q-NSS over (Γ_Q, A) over a ring $(R,+,.)$ is called a Q-NS ideal over $(R,+,.)$ if for all $x,y \in R, q \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(x-y,q) \geq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}$ and $F_{\Gamma_{Q(e)}}(x-y,q) \leq \max\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\}$.
2. $T_{\Gamma_{Q(e)}}(x.y,q) \geq \max\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, I_{\Gamma_{Q(e)}}(x.y,q) \leq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}$ and $F_{\Gamma_{Q(e)}}(x.y,q) \leq \min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}$.

Example 6.1. Let $R = (\mathbb{Z},+,.)$ be the ring of integers and A=$\mathbb{N}$ the set of natural numbers be the parametric set. Define a Q-NSS (Γ_Q, A) as follows for $q \in Q, x \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$T_{\Gamma_{Q(m)}}(x,q) = \begin{cases} \frac{1}{m} & \text{if } x = 2l-1, \exists l \in \mathbb{Z} \\ \frac{2}{m} & \text{if } x = 2l, \exists l \in \mathbb{Z}, \end{cases}$$

$$I_{\Gamma_{Q(m)}}(x,q) = \begin{cases} \frac{1}{m} & \text{if } x = 2l-1, \exists l \in \mathbb{Z} \\ 0 & \text{if } x = 2l-, \exists l \in \mathbb{Z}, \end{cases}$$

$$F_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 1-\frac{2}{m} & \text{if } x = 2l-1, \exists l \in \mathbb{Z} \\ 1-\frac{3}{m} & \text{if } x = 2l, \exists l \in \mathbb{Z}. \end{cases}$$

Then, by Definition 6.2 (Γ_Q, A) is a Q-NS ideal over $(\mathbb{Z},+,.)$.

Theorem 6.3. *Let* (Γ_Q, A) *and* (Ψ_Q, B) *be two Q-NS ideals over* $(R,+,.)$. *Then,* $(\Gamma_Q, A) \cap (\Psi_Q, B)$ *is also a Q-NS ideal over* $(R,+,.)$.

Proof. Let $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, A \cap B)$. Then, for $x,y \in R, q \in Q$ and $e \in A \cap B$ the first condition of Definition 6.2 is satisfied by Theorem 3.3. Now, for the second condition

$$\begin{aligned} T_{\Lambda_{Q(e)}}(x.y,q) &= \min\left\{T_{\Gamma_{Q(e)}}(x.y,q), T_{\Psi_{Q(e)}}(x.y,q)\right\} \\ &\geq \min\left\{\max\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\}, \max\{T_{\Psi_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(y,q)\}\right\} \\ &= \max\left\{\min\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\}, \min\{T_{\Gamma_{Q(e)}}(y,q), T_{\Psi_{Q(e)}}(y,q)\}\right\} \\ &= \max\left\{T_{\Lambda_{Q(e)}}(x,q), T_{\Lambda_{Q(e)}}(y,q)\right\}, \end{aligned}$$

also,

$$\begin{aligned} I_{\Lambda_{Q(e)}}(x.y,q) &= \max\left\{I_{\Gamma_{Q(e)}}(x.y,q), I_{\Psi_{Q(e)}}(x.y,q)\right\} \\ &\leq \max\left\{\min\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\}, \min\{I_{\Psi_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(y,q)\}\right\} \\ &= \min\left\{\max\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\}, \max\{I_{\Gamma_{Q(e)}}(y,q), I_{\Psi_{Q(e)}}(y,q)\}\right\} \\ &= \min\left\{I_{\Lambda_{Q(e)}}(x,q), I_{\Lambda_{Q(e)}}(y,q)\right\}. \end{aligned}$$

Similarly, we can show $F_{\Lambda_{Q(e)}}(x.y,q) \leq \min\left\{F_{\Lambda_{Q(e)}}(x,q), F_{\Lambda_{Q(e)}}(y,q)\right\}$. This completes the proof. □

Theorem 6.4. *Let* (Γ_Q, A) *and* (Ψ_Q, B) *be two Q-NS ideals over* $(R, +, .)$*. Then,* $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ *is also a Q-NS ideal over* $(R, +, .)$*.*

Proof. The proof is similar to the proof of Theorem 6.3. □

Remark 6.5. The union of two Q-NS ideals need not be a Q-NS ideal.
For example, let $R = \mathbb{Z}_6 = 0, 1, 2, 3, 4, 5$. Consider two Q-NS ideals (Γ_Q, A) and (Ψ_Q, B) over R as follows: for $x \in \mathbb{Z}, q \in Q$ and $e \in E$

$$T_{\Gamma_{Q(2m)}}(x, q) = \begin{cases} 0.30 & \text{if } x = 0 \text{ or } 3, \\ 0 & otherwise, \end{cases}$$

$$I_{\Gamma_{Q(2m)}}(x, q) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 3, \\ 0.20 & otherwise, \end{cases}$$

$$F_{\Gamma_{Q(2m)}}(x, q) = \begin{cases} 0.30 & \text{if } x = 0 \text{ or } 3, \\ 1 & otherwise, \end{cases}$$

and

$$T_{\Psi_{Q(2m)}}(x, q) = \begin{cases} 0.40 & \text{if } x = 0, 2, 4, \\ 0 & otherwise, \end{cases}$$

$$I_{\Psi_{Q(2m)}}(x, q) = \begin{cases} 0 & \text{if } x = 0, 2, 4, \\ 0.50 & otherwise, \end{cases}$$

$$F_{\Psi_{Q(2m)}}(x, q) = \begin{cases} 0.25 & \text{if } x = 0, 2, 4, \\ 1 & otherwise. \end{cases}$$

Let $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, A \cup B)$. For $x = 3, y = 2$ and $q \in Q$ we have,

$$T_{\Lambda_{Q(e)}}(3-2, q) = T_{\Lambda_{Q(e)}}(1, q) = \max\left\{T_{\Gamma_{Q(e)}}(1, q), T_{\Psi_{Q(e)}}(1, q)\right\} = \max\{0, 0\} = 0$$

and

$$\begin{aligned} \min\left\{T_{\Lambda_{Q(e)}}(3, q), T_{\Lambda_{Q(e)}}(2, q)\right\} & \\ &= \min\left\{\max\{T_{\Gamma_{Q(e)}}(3, q), T_{\Psi_{Q(e)}}(3, q)\}, \max\{T_{\Gamma_{Q(e)}}(2, q), T_{\Psi_{Q(e)}}(2, q)\}\right\} \\ &= \min\left\{\max\{0.30, 0\}, \max\{0, 0.40\}\right\} \\ &= \min\left\{0.30, 0.40\right\} = 0.30. \end{aligned}$$

Hence, $T_{\Lambda_{Q(e)}}(3-2, q) < \min\left\{T_{\Lambda_{Q(e)}}(3, q), T_{\Lambda_{Q(e)}}(2, q)\right\}$. Thus, the union is not a Q-NS ideal.

7. Conclusion

In this study, we have introduced the idea of Q-neutrosophic soft rings and discuss some of its related properties. Then, we have discussed the cartesian product of Q-neutrosophic soft rings and homomorphic image and preimage of Q-neutrosophic soft rings. Finally, we have presented Q-neutrosophic soft ideals and explored some of their related properties. The proposed notion illuminates the way to broaden the notion of Q-neutrosophic soft sets and rings by using the refined neutrosophic set [33] and different other structures.

Conflict of interest

We declare that there is no conflict of interest.

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