Behaviour of ring ideal in neutrosophic and soft sense

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Abstract . This article enriches the idea of neutrosophic soft ideal (NSI). The notion of neutrosophic soft prime ideal (NSPI) is also introduced here. The characteristics of both NSI and NSPI are investigated. Their relations are drawn with the concept of ideal and prime ideal in crisp sense. Any neutrosophic soft set (Nss) can be made into NSI or NSPI using the respective cut set under a situation. The homomorphic characters of ideal and prime ideal in this new class are also drawn critically.

Keywords : Neutrosophic soft ideal (NSI); Neutrosophic soft prime ideal (NSPI); Homomorphic image.

1 Introduction

In today's world, the most of our routine activities are full of uncertainty and ambiguity. Whenever solving any problem arisen in decision making, political affairs, medicine, management, industrial and many other different real worlds, analysts suffer from a major confusion instead of directly moving towards a positive decision. The situation can be nicely conducted by practice of Neutrosophic set (N_S) theory introduced by Smarandache [7,8]. This theory represents an object by an additional value namely indeterministic function beside another two characters seen in Attanasov's theory [16]. So, Attanasov's theory can not be a proper choice in uncertain situation. Hence, the N_S theory is more reliable to an analyst, since an object is estimated here by three independent characters namely true value, indeterminate value and false value. The analysis of uncertain fact is possible in a more convenient way on the availability of adequate parameters. The soft set theory innovated by Molodtsov [5] brought that opportunity to practice the different theories in uncertain atmosphere.

Researchers are trying to extend the various mathematical structures over fuzzy set, intuitionistic fuzzy set, soft set from the very beginning. Some attempts [1,2,3,4,6,11,12,21,32,33,45] allied to group and ring theory are pointed out. Maji [22] took a successful effort to combine the neutrosophic logic with soft set theory and thus the Nss theory was brought forth. Later, modifying the different operations of Nss theory using t-norm and s-norm, Deli and Broumi [13] gave this Nss theory a new look. Doing the habit of this modified formation, Bera and Mahapatra [36] began to study the notion of NSI. From initiation, the authors are making attempt to unite with the neutrosophic logic in different mathematical areas and in many real sectors. These [9,10,14,15, 17-20, 23-31, 34-44] are some accomplishments.

The present study investigates the characteristics of NSI. Section 2 states some necessary definitions to carry on the main result. In Section 3, the structural characteristics of NSIs are investigated. Section 4 introduces and develops the concept of NSPI. Section 5 describes the nature of homomorphic image of NSI and the conclusion is given in Section 6.

2 Preliminaries

We shall remember some definitions here to make out the main thought.

2.1 Definition [38]

1. A continuous t-norm $\triangle$ maps $[0,1]\times[0,1]\to[0,1]$ and satisfies the followings.
(i) $\triangle$ is continuous and associative.
(ii) $m\triangle q = q\triangle m,\ \forall m,q\in[0,1]$.
(iii) $m\triangle 1 = 1\triangle m = m,\ \forall m\in[0,1]$.
(iv) $m\triangle q\le n\triangle s$ if $m\le n,\ q\le s$ with $m,q,n,s\in[0,1]$.
$m\triangle q = mq, m\triangle q=\min\{m,q\}, m\triangle q=\max\{m+q-1,0\}$ are some necessary continuous t-norms.
2. A continuous t - conorm (s - norm) $\bigtriangledown$ maps $[0,1]\times[0,1]\to[0,1]$ and obeys the followings.
(i) $\bigtriangledown$ is continuous and associative.
(ii) $w\bigtriangledown p = p\bigtriangledown w,\ \forall w,p\in[0,1]$.
(iii) $w\bigtriangledown 0 = 0\bigtriangledown w = w,\ \forall w\in[0,1]$.
(iv) $w\bigtriangledown p\le v\bigtriangledown q$ if $w\le v,\ p\le q$ with $w,v,p,q\in[0,1]$.
$w\bigtriangledown p = w+p-wp, w\bigtriangledown p=\max\{w,p\}, w\bigtriangledown p=\min\{w+p,1\}$ are some useful continuous s-norms.

2.2 Definition [7]

An element u of a universal set X is described under an $N_S H$ by three characters $viz.$ truth-membership T_H, indeterminacy-membership I_H and falsity-membership F_H such that $T_H(u), I_H(u), F_H(u)\in]^-0,1^+[$ and $^-0\le \sup T_H(u)+\sup I_H(u)+\sup F_H(u)\le 3^+$. For $1^+ = 1+\epsilon$, 1 is the standard part and ϵ is the non-standard part and so on for $^-0$ also. The non-standard subsets of $]^-0,1^+[$ is practiced in philosophical ground but in real atmosphere, only the standard subsets of $]^-0,1^+[$ i.e., $[0,1]$ is used. Thus the $N_S H$ is put as : $\{<u,(T_H(u),I_H(u),F_H(u))>: u\in X\}$.

2.3 Definition [5]

Suppose X be the universe of discourse and E be a parametric set. Then for $B\subseteq E$ and $\wp(X)$ being the set of all subsets of X, a soft set is narrated by a pair (G,B) when G maps $B\to\wp(X)$.

2.4 Definition [22]

Suppose X be the universe of discourse and E be a parametric set. Then for $B\subseteq E$ and $N_S(X)$ being the set of all N_Ss over X, an Nss is narrated by a pair (G,B) when G maps $B\to N_S(X)$.
The Nss theory appeared in a new look by Deli and Broumi [13] as follows.

2.5 Definition [13]

Suppose X be the universe of discourse and E being a parametric set describes the elements of X. An Nss D over (X,E) is put as : $\{(b,h_D(b)) : b\in E\}$ where h_D maps $E\to N_S(X)$ given by $h_D(b)=\{<u,(T_{h_D(b)}(u),I_{h_D(b)}(u),F_{h_D(b)}(u))>: u\in X\}$. $T_{h_D(b)}, I_{h_D(b)}, F_{h_D(b)}\in[0,1]$ are three characters of $h_D(b)$ as mentioned in Definition [7] and they are connected by the relation $0\le T_{h_D(b)}(u)+I_{h_D(b)}(u)+F_{h_D(b)}(u)\le 3$.

2.5.1 Definition [13]

Over (X, E), suppose P, Q be two Nss. $\forall b \in E$ and $\forall u \in X$, if $T_{h_{P(b)}}(u) \leq T_{h_{Q(b)}}(u)$, $I_{h_{P(b)}}(u) \geq I_{h_{Q(b)}}(u)$, $F_{h_{P(b)}}(u) \geq F_{h_{Q(b)}}(u)$, then P is called a neutrosophic soft subset of Q (denoted as $P \subseteq Q$)

2.6 Proposition [34]

A neutrosophic soft group (NSG) D is an Nss on (V, o), a classical group, obeying the inequalities mentioned below with respect to $m \bigtriangleup q = \min\{m, q\}$ and $p \bigtriangledown n = \max\{p, n\}$.

$T_{h_{D(b)}}(uov^{-1}) \geq T_{h_{D(b)}}(u) \bigtriangleup T_{h_{D(b)}}(v)$, $I_{h_{D(b)}}(uov^{-1}) \leq I_{h_{D(b)}}(u) \bigtriangledown I_{h_{D(b)}}(v)$ and
$F_{h_{D(b)}}(uov^{-1}) \leq F_{h_{D(b)}}(u) \bigtriangledown F_{h_{D(b)}}(v)$, $\forall u, v \in V, \forall b \in E$.

2.7 Definition [36]

1. For a neutrosophic soft ring (NSR) D on a ring $(S, +, \cdot)$ in crisp sense if each $h_D(b)$ is a neutrosophic left ideal for $b \in E$, then D is called a neutrosophic soft left ideal (NSLI) i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_{D(b)}}(x.y) \geq T_{h_{D(b)}}(y)$, $I_{h_{D(b)}}(x.y) \leq I_{h_{D(b)}}(y)$, $F_{h_{D(b)}}(x.y) \leq F_{h_{D(b)}}(y)$; for $x, y \in S$.

2. For an NSR D on $(S, +, \cdot)$ if each $h_D(b)$ is a neutrosophic right ideal for $b \in E$, then D is called a neutrosophic soft right ideal (NSRI) i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_{D(b)}}(x.y) \geq T_{h_{D(b)}}(x)$, $I_{h_{D(b)}}(u.v) \leq I_{h_{D(b)}}(x)$, $F_{h_{D(b)}}(x.y) \leq F_{h_{D(b)}}(x)$; for $x, y \in S$.

3. For an NSR D on $(S, +, \cdot)$ if each $h_D(b)$ is an NSLI as well as NSRI for $b \in E$, then D is called an NSI i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_{D(b)}}(x.y) \geq \max\{T_{h_{D(b)}}(x), T_{h_{D(b)}}(y)\}$, $I_{h_{D(b)}}(x.y) \leq \min\{I_{h_{D(b)}}(x), I_{h_{D(b)}}(y)\}$ and
$F_{h_{D(b)}}(x.y) \leq \min\{F_{h_{D(b)}}(x), F_{h_{D(b)}}(y)\}$; for $x, y \in S$.

2.8 Definition [35]

1. Let M be an N_S on the universe of discourse X. Then $M_{(\sigma,\eta,\delta)}$ is called (σ, η, δ)-cut of M and is described as a set $\{u \in X : T_M(u) \geq \sigma, I_M(u) \leq \eta, F_M(u) \leq \delta\}$ where $\sigma, \eta, \delta \in [0, 1]$ and $0 \leq \sigma + \eta + \delta \leq 3$. This $M_{(\sigma,\eta,\delta)}$ is called (σ, η, δ)-level set or (σ, η, δ)-cut set of the N_S M and clearly, $M_{(\sigma,\eta,\delta)} \subset X$.

2. Let D be an Nss on (X, E). Then the soft set $D_{(\sigma,\eta,\delta)} = \{(b, [h_D(b)]_{(\sigma,\eta,\delta)}) : b \in E\}$ is called (σ, η, δ)-level soft set or (σ, η, δ)-cut soft set for $\sigma, \eta, \delta \in [0, 1]$ with $0 \leq \sigma + \eta + \delta \leq 3$. Here each $[h_D(b)]_{(\sigma,\eta,\delta)}$ is an (σ, η, δ)-level set of the N_S $h_D(b)$ over X.

In the main results, we shall restrict ourselves by the t-norm as $m \bigtriangleup q = \min\{m, q\}$ and s-norm as $p \bigtriangledown n = \max\{p, n\}$ and shall take $b \in E$, a parametric set, as an arbitrary parameter.

3 Neutrosophic soft ideal

Some features of NSI are studied by developing a number of theorems here.

3.1 Proposition

Let K be an NSLI (NSRI) on (S,E). If 0_S is the additive identity of the ring S, then
(i) $T_{h_K(b)}(u) \leq T_{h_K(b)}(0_S),\ I_{h_K(b)}(u) \geq I_{h_K(b)}(0_S),\ F_{h_K(b)}(u) \geq F_{h_K(b)}(0_S),\ \forall u \in R$ and $\forall b \in E$.
(ii) $K_{(\sigma,\eta,\delta)}$ is a left (right) ideal for $0 \leq \sigma \leq T_{h_K(b)}(0_S),\ \ I_{h_K(b)}(0_S) \leq \eta \leq 1,\ \ F_{h_K(b)}(0_S) \leq \delta \leq 1$.

Proof. (i) Here, for every $b \in E$, $h_K(b)$ is a neutrosophic subgroup of $(S,+)$. Then $\forall u \in S$ and $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(0_S) &= T_{h_K(b)}(u-u) \geq T_{h_K(b)}(u) \vartriangle T_{h_K(b)}(u) = T_{h_K(b)}(u),\\ I_{h_K(b)}(0_S) &= I_{h_K(b)}(u-u) \leq I_{h_K(b)}(u) \triangledown I_{h_K(b)}(u) = I_{h_K(b)}(u),\\ F_{h_K(b)}(0_S) &= F_{h_K(b)}(u-u) \leq F_{h_K(b)}(u) \triangledown F_{h_K(b)}(u) = F_{h_K(b)}(u); \end{aligned}$$

(ii) Let $u, v \in K_{(\sigma,\eta,\delta)}$ and $r \in S$. Then,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq T_{h_K(b)}(u) \vartriangle T_{h_K(b)}(v) \geq \sigma \vartriangle \sigma = \sigma,\\ I_{h_K(b)}(u-v) &\leq I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v) \leq \eta \triangledown \eta = \eta,\\ F_{h_K(b)}(u-v) &\leq F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v) \leq \delta \triangledown \delta = \delta; \end{aligned}$$

and $T_{h_K(b)}(ru) \geq T_{h_K(b)}(u) \geq \sigma,\ I_{h_K(b)}(ru) \leq I_{h_K(b)}(u) \leq \eta,\ F_{h_K(b)}(ru) \leq F_{h_K(b)}(u) \leq \delta$.
Hence $u - v,\ ru \in K_{(\sigma,\eta,\delta)}$ and so $K_{(\sigma,\eta,\delta)}$ is a left ideal of S. Similarly, one right ideal of S is $K_{(\sigma,\eta,\delta)}$ also.

3.2 Theorem

(i) Q be a non-empty ideal of crisp ring S if and only if $\exists$ an NSI K on (S,E) where $h_K : E \longrightarrow N_S(S)$ is given as, $\forall b \in E$,

$$T_{h_K(b)}(u) = \begin{cases} p_1 & \text{if } u \in Q \\ s_1\ (< p_1) & \text{if } u \notin Q. \end{cases} \quad I_{h_K(b)}(u) = \begin{cases} p_2 & \text{if } u \in Q \\ s_2\ (> p_2) & \text{if } u \notin Q. \end{cases} \quad F_{h_K(b)}(u) = \begin{cases} p_3 & \text{if } u \in Q \\ s_3\ (> p_3) & \text{if } u \notin Q. \end{cases}$$

Briefly stated $$h_K(b)(u) = \begin{cases} (p_1,p_2,p_3) & \text{when } u \in Q \\ (s_1,s_2,s_3) & \text{when } u \notin Q. \end{cases}$$

where $s_1 < p_1, s_2 > p_2, s_3 > p_3$ and $p_i, s_i \in [0,1]$ for all $i = 1,2,3$.
(ii) Specifically, Q is a non empty ideal of a crisp ring S iff it's characteristic function λ_Q is an NSI on (S,E) where $\lambda_Q : E \longrightarrow N_S(S)$ is given as, $\forall b \in E$,

$$T_{\lambda_Q(b)}(u) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q. \end{cases} \quad I_{\lambda_Q(b)}(u) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q. \end{cases} \quad F_{\lambda_Q(b)}(u) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q. \end{cases}$$

*Proof.***(i)** First let Q be a non empty ideal of S in crisp sense and consider an Nss K on (S,E). We now take the following cases.
Case 1 : When $u, v \in Q$, then $u - v \in Q$, an ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= p_1 = p_1 \vartriangle p_1 = T_{h_K(b)}(u) \vartriangle T_{h_K(b)}(v)\\ I_{h_K(b)}(u-v) &= p_2 = p_2 \triangledown p_2 = I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v)\\ F_{h_K(b)}(u-v) &= p_3 = p_3 \triangledown p_3 = F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v) \end{aligned}$$

Case 2 : If $u \in Q$ but $v \notin Q$, then $u - v \notin Q$. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= s_1 = p_1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= s_2 = p_2 \bigtriangledown s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= s_3 = p_3 \bigtriangledown s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 3 : If $u, v \notin Q$, then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq s_1 = s_1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &\leq s_2 = s_2 \bigtriangledown s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &\leq s_3 = s_3 \bigtriangledown s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Thus in any case $\forall u, v \in R$ and $\forall b \in E$,

$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v)$, $I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v)$ and $F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v)$.

We shall now test the 2nd condition of the Definition [2.7].

Case 1 : When $u \in Q$ then $uv,\ vu \in Q$, an ideal on S, for $v \in S$. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(uv) &= T_{h_K(b)}(vu) = p_1 = T_{h_K(b)}(u), \\ I_{h_K(b)}(uv) &= I_{h_K(b)}(vu) = p_2 = I_{h_K(b)}(u), \\ F_{h_K(b)}(uv) &= F_{h_K(b)}(vu) = p_3 = F_{h_K(b)}(u); \end{aligned}$$

Case 2 : If $u \notin Q$ then either $uv \in Q$ or $uv \notin Q$ and so, $\forall b \in E$,

$$\begin{aligned} &T_{h_K(b)}(uv) \geq s_1 = T_{h_K(b)}(u),\ T_{h_K(b)}(vu) \geq s_1 = T_{h_K(b)}(u), \\ &I_{h_K(b)}(uv) \leq s_2 = I_{h_K(b)}(u),\ I_{h_K(b)}(vu) \leq s_2 = I_{h_K(b)}(u), \\ &F_{h_K(b)}(uv) \leq s_3 = F_{h_K(b)}(u),\ F_{h_K(b)}(vu) \leq s_3 = F_{h_K(b)}(u); \end{aligned}$$

This shows that K is NSLI and also NSRI on (S, E). Thus K is an NSI on (S, E).

Reversely, suppose K be an NSI on (S, E) in the specified form. We are to show $Q(\neq \phi)$ is a crisp ideal of S. Let $u, v \in Q$ and $a \in S$. Then $T_{h_K(b)}(u) = T_{h_K(b)}(v) = p_1$, $I_{h_K(b)}(u) = I_{h_K(b)}(v) = p_2$, $F_{h_K(b)}(u) = F_{h_K(b)}(v) = p_3$. Now,

$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) = p_1$, $I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) = p_2$ and $F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) = p_3$.

Further, as K is an NSI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{h_K(b)}(u-v) \leq T_{h_K(b)}(0_S) \leq p_1$, $I_{h_K(b)}(u-v) \geq I_{h_K(b)}(0_S) \geq p_2$, $F_{h_K(b)}(u-v) \geq F_{h_K(b)}(0_S) \geq p_3$.

This implies $T_{h_K(b)}(u-v) = p_1$, $I_{h_K(b)}(u-v) = p_2$, $F_{h_K(b)}(u-v) = p_3$ and so by construction of K, $u-v \in Q$.

Next, K is an NSLI over (S, E) and so,

$T_{h_K(b)}(au) \geq T_{h_K(b)}(u) = p_1$, $I_{h_K(b)}(au) \leq I_{h_K(b)}(u) = p_2$, $F_{h_K(b)}(au) \leq F_{h_K(b)}(u) = p_3$.

Again K is an NSLI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{h_K(b)}(au) \leq T_{h_K(b)}(0_S) \leq p_1$, $I_{h_K(b)}(au) \geq I_{h_K(b)}(0_S) \geq p_2$, $F_{h_K(b)}(au) \geq F_{h_K(b)}(0_S) \geq p_3$.

This shows $T_{h_K(b)}(au) = p_1, I_{h_K(b)}(au) = p_2, F_{h_K(b)}(au) = p_3$. So, $au \in Q$ by structure of K. In a same corner, $ua \in Q$. Therefore, Q is a crisp ideal of S.

(ii) First suppose Q be a non empty crisp ideal of S and on (S,E), λ_Q be an Nss. Following cases are needed to discuss.

Case 1 : When $u,v \in Q$, then $u-v \in Q$, an ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &= 1 = 1 \triangle 1 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &= 0 = 0 \bigtriangledown 0 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &= 0 = 0 \bigtriangledown 0 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Case 2 : If $u \in Q$ but $v \notin Q$, then $u-v \notin Q$. Then $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &= 0 = 1 \triangle 0 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &= 1 = 0 \bigtriangledown 1 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &= 1 = 0 \bigtriangledown 1 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Case 3 : If $u,v \notin Q$, then $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &\geq 0 = 0 \triangle 0 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &\leq 1 = 1 \bigtriangledown 1 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &\leq 1 = 1 \bigtriangledown 1 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Thus in any case $\forall u,v \in S$ and $\forall b \in E$,

$T_{\lambda_{Q(b)}}(u-v) \geq T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v)$, $I_{\lambda_{Q(b)}}(u-v) \leq I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v)$ and $F_{\lambda_{Q(b)}}(u-v) \leq F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v)$.

We shall now test the 2nd condition of Definition [2.7].

Case 1 : When $u \in Q$ then $uv,\ vu \in Q$, an ideal of S, for $v \in S$. So, $\forall b \in E$,

$T_{\lambda_{Q(b)}}(uv) = T_{\lambda_{Q(b)}}(vu) = 1 = T_{\lambda_{Q(b)}}(u)$, $I_{\lambda_{Q(b)}}(uv) = I_{\lambda_{Q(b)}}(vu) = 0 = I_{\lambda_{Q(b)}}(u)$ and $F_{\lambda_{Q(b)}}(uv) = F_{\lambda_{Q(b)}}(vu) = 0 = F_{\lambda_{Q(b)}}(u)$.

Case 2 : If $u \notin Q$ then either $uv \in Q$ or $uv \notin Q$ and so $\forall b \in E$,

$$\begin{aligned} &T_{\lambda_{Q(b)}}(uv) \geq 0 = T_{\lambda_{Q(b)}}(u),\ T_{\lambda_{Q(b)}}(vu) \geq 0 = T_{\lambda_{Q(b)}}(u), \\ &I_{\lambda_{Q(b)}}(uv) \leq 1 = I_{\lambda_{Q(b)}}(u),\ \ I_{\lambda_{Q(b)}}(vu) \leq 1 = I_{\lambda_{Q(b)}}(u), \\ &F_{\lambda_{Q(b)}}(uv) \leq 1 = F_{\lambda_{Q(b)}}(u),\ F_{\lambda_{Q(b)}}(vu) \leq 1 = F_{\lambda_{Q(b)}}(u); \end{aligned}$$

This shows that λ_Q is NSLI and NSRI on (S,E). Thus λ_Q is NSI on (S,E).

Reversely, let λ_Q be an NSI over (S,E) in the prescribed form. We shall have to show $Q(\neq \phi)$ is a crisp ideal of S. Let $u,v \in Q$ and $a \in S$. Then $T_{\lambda_{Q(b)}}(u) = T_{\lambda_{Q(b)}}(v) = 1$, $I_{\lambda_{Q(b)}}(u) = I_{\lambda_{Q(b)}}(v) = 0$, $F_{\lambda_{Q(b)}}(u) = F_{\lambda_{Q(b)}}(v) = 0$. Now,

$T_{\lambda_{Q(b)}}(u-v) \geq T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) = 1$, $I_{\lambda_{Q(b)}}(u-v) \leq I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) = 0$ and $F_{\lambda_{Q(b)}}(u-v) \leq F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) = 0$.

Further, as λ_Q is an NSI over (S,E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{\lambda_{Q(b)}}(u-v) \leq T_{\lambda_{Q(b)}}(0_S) \leq 1$, $I_{\lambda_{Q(b)}}(u-v) \geq I_{\lambda_{Q(b)}}(0_S) \geq 0$, $F_{\lambda_{Q(b)}}(u-v) \geq F_{\lambda_{Q(b)}}(0_S) \geq 0$.

This implies $T_{\lambda_{Q(b)}}(u-v) = 1$, $I_{\lambda_{Q(b)}}(u-v) = 0$, $F_{\lambda_{Q(b)}}(u-v) = 0$ and so by construction of λ_Q, $u-v \in Q$.

Next, λ_Q is an NSLI over (S,E) and so,

$T_{\lambda_{Q(b)}}(au) \geq T_{\lambda_{Q(b)}}(u) = 1$, $I_{\lambda_{Q(b)}}(au) \leq I_{\lambda_{Q(b)}}(u) = 0$, $F_{\lambda_{Q(b)}}(au) \leq F_{\lambda_{Q(b)}}(u) = 0$.

Again λ_Q is an NSLI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{\lambda_{Q(b)}}(au) \leq T_{\lambda_{Q(b)}}(0_S) \leq 1,\ I_{\lambda_{Q(b)}}(au) \geq I_{\lambda_{Q(b)}}(0_S) \geq 0,\ F_{\lambda_{Q(b)}}(au) \geq F_{\lambda_{Q(b)}}(0_S) \geq 0.$

This shows $T_{\lambda_{Q(b)}}(au) = 1, I_{\lambda_{Q(b)}}(au) = 0, F_{\lambda_{Q(b)}}(au) = 0$. So, $au \in Q$ by structure of λ_Q. By same logic, $ua \in Q$. Thus, Q is a crisp ideal of S.

3.3 Theorem

Consider an NSLI (NSRI) Q over (S, E). Then, $Q_0 = \{u \in S : T_{h_{Q(b)}}(u) = T_{h_{Q(b)}}(0_S),\ I_{h_{Q(b)}}(u) = I_{h_{Q(b)}}(0_S), F_{h_{Q(b)}}(u) = F_{h_{Q(b)}}(0_S)\}$ is a crisp left (right) ideal of S for $b \in E$.

Proof. Following the reverse part of Theorem [3.2], it will be as usual.

3.4 Theorem

Q, an Nss on (S, E), is an NSLI (NSRI) iff $\widehat{Q} = \{u \in S : T_{h_{Q(b)}}(u) = 1,\ I_{h_{Q(b)}}(u) = 0,\ F_{h_{Q(b)}}(u) = 0\}$ with $0_S \in \widehat{Q}$ is a crisp left (right) ideal of S.

Proof. We can put Q, an Nss on (S, E), as given below, $\forall b \in E$,

$$h_Q(b)(u) = \begin{cases} (1, 0, 0) & \text{when } u \in \widehat{Q} \\ (s_1, s_2, s_3) & \text{when } u \notin \widehat{Q}. \end{cases}$$

where $0 \leq s_1 < 1,\ 0 < s_2 \leq 1,\ 0 < s_3 \leq 1$. Assume $\widehat{Q}$ be a crisp left ideal of S for Q being an Nss on (S, E). We shall now take the cases stated below.

Case 1 : When $u, v \in \widehat{Q}$, then $u - v \in \widehat{Q}$, a crisp left ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{h_{Q(b)}}(u - v) &= 1 = 1 \vartriangle 1 = T_{h_{Q(b)}}(u) \vartriangle T_{h_{Q(b)}}(v) \\ I_{h_{Q(b)}}(u - v) &= 0 = 0 \triangledown 0 = I_{h_{Q(b)}}(u) \triangledown I_{h_{Q(b)}}(v) \\ F_{h_{Q(b)}}(u - v) &= 0 = 0 \triangledown 0 = F_{h_{Q(b)}}(u) \triangledown F_{h_{Q(b)}}(v) \end{aligned}$$

Case 2 : If $u \in \widehat{Q}$ but $v \notin \widehat{Q}$, then $u - v \notin \widehat{Q}$. Then $\forall b \in E$,

$$\begin{aligned} T_{h_{Q(b)}}(u - v) &= s_1 = 1 \vartriangle s_1 = T_{h_{Q(b)}}(u) \vartriangle T_{h_{Q(b)}}(v) \\ I_{h_{Q(b)}}(u - v) &= s_2 = 0 \triangledown s_2 = I_{h_{Q(b)}}(u) \triangledown I_{h_{Q(b)}}(v) \\ F_{h_{Q(b)}}(u - v) &= s_3 = 0 \triangledown s_3 = F_{h_{Q(b)}}(u) \triangledown F_{h_{Q(b)}}(v) \end{aligned}$$

Case 3 : If $u, v \notin \widehat{Q}$, then $\forall b \in E$,

$$\begin{aligned} T_{h_{Q(b)}}(u - v) &\geq s_1 = s_1 \vartriangle s_1 = T_{h_{Q(b)}}(u) \vartriangle T_{h_{Q(b)}}(v) \\ I_{h_{Q(b)}}(u - v) &\leq s_2 = s_2 \triangledown s_2 = I_{h_{Q(b)}}(u) \triangledown I_{h_{Q(b)}}(v) \\ F_{h_{Q(b)}}(u - v) &\leq s_3 = s_3 \triangledown s_3 = F_{h_{Q(b)}}(u) \triangledown F_{h_{Q(b)}}(v) \end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_{Q(b)}}(u - v) \geq T_{h_{Q(b)}}(u) \vartriangle T_{h_{Q(b)}}(v),\quad I_{h_{Q(b)}}(u - v) \leq I_{h_{Q(b)}}(u) \triangledown I_{h_{Q(b)}}(v)$ and $F_{h_{Q(b)}}(u - v) \leq F_{h_{Q(b)}}(u) \triangledown F_{h_{Q(b)}}(v)$.

We are to test now the 2nd condition of Definition [2.7].

Case 1 : If $u \in \widehat{Q}$ then $vu \in \widehat{Q}$, a crisp left ideal on S, for $v \in S$. So, $\forall b \in E$,

$T_{h_Q(b)}(vu) = 1 = T_{h_Q(b)}(u),\ I_{h_Q(b)}(vu) = 0 = I_{h_Q(b)}(u),\ F_{h_Q(b)}(vu) = 0 = F_{h_Q(b)}(u).$

Case 2 : If $u \notin \widehat{Q}$ then either $vu \in \widehat{Q}$ or $vu \notin \widehat{Q}$ for $v \in R$ and so $\forall b \in E$,

$T_{h_Q(b)}(vu) \geq s_1 = T_{h_Q(b)}(u),\ I_{h_Q(b)}(vu) \leq s_2 = I_{h_Q(b)}(u),\ F_{h_Q(b)}(vu) \leq s_3 = F_{h_Q(b)}(u).$

This shows that Q is an NSLI over (S, E).

Conversely, let Q be an NSLI on (S, E) in the assumed structure. Let $u, v \in \widehat{Q}$ and $a \in S$. Then $T_{h_Q(b)}(u) = T_{h_Q(b)}(v) = 1,\ I_{h_Q(b)}(u) = I_{h_Q(b)}(v) = 0,\ F_{h_Q(b)}(u) = F_{h_Q(b)}(v) = 0$. Now,

$T_{h_Q(b)}(u - v) \geq T_{h_Q(b)}(u) \triangle T_{h_Q(b)}(v) = 1,\ \ I_{h_Q(b)}(u - v) \leq I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v) = 0$ and
$F_{h_Q(b)}(u - v) \leq F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v) = 0.$

Further, as Q is an NSLI over (R, E) and as either $0_S \in \widehat{Q}$ or $0_S \notin \widehat{Q}$,

$T_{h_Q(b)}(u - v) \leq T_{h_Q(b)}(0_S) \leq 1,\ I_{h_Q(b)}(u - v) \geq I_{h_Q(b)}(0_S) \geq 0,\ F_{h_Q(b)}(u - v) \geq F_{h_Q(b)}(0_S) \geq 0.$

This implies $T_{h_Q(b)}(u - v) = 1,\ I_{h_Q(b)}(u - v) = 0, F_{h_Q(b)}(u - v) = 0$ and so by construction of Q, $u - v \in \widehat{Q}$.
Next, Q is an NSLI over (R, E) and so,

$T_{h_Q(b)}(au) \geq T_{h_Q(b)}(u) = 1,\ I_{h_Q(b)}(au) \leq I_{h_Q(b)}(u) = 0,\ F_{h_Q(b)}(au) \leq F_{h_Q(b)}(u) = 0.$

Again Q is an NSLI over (R, E) and as either $0_R \in \widehat{Q}$ or $0_R \notin \widehat{Q}$,

$T_{h_Q(b)}(au) \leq T_{h_Q(b)}(0_R) \leq 1,\ I_{h_Q(b)}(au) \geq I_{h_Q(b)}(0_R) \geq 0,\ F_{h_Q(b)}(au) \geq F_{h_Q(b)}(0_R) \geq 0.$

This shows $T_{h_Q(b)}(au) = 1, I_{h_Q(b)}(au) = 0, F_{h_Q(b)}(au) = 0$ i.e., $au \in \widehat{Q}$. Therefore, $\widehat{Q}$ is a crisp left ideal of S and so is $\widehat{Q}$ over S similarly.

3.5 Theorem

Let K be an Nss over (S, E). Then K is an NSLI (NSRI) iff each nonempty cut set $[h_K(b)]_{(\delta,\eta,\sigma)}$ of the N_S $h_K(b)$ is a crisp left (right) ideal of S for $\delta \in Im\, T_{h_K(b)},\ \eta \in Im\, I_{h_K(b)},\ \sigma \in Im\, F_{h_K(b)}$.

Proof. Let K be an NSLI (NSRI) over (S, E) and $u, v \in [h_K(b)]_{(\delta,\eta,\sigma)}, r \in S$. Then,

$$T_{h_K(b)}(u - v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \geq \delta \triangle \delta = \delta$$
$$I_{h_K(b)}(u - v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \leq \eta \bigtriangledown \eta = \eta$$
$$F_{h_K(b)}(u - v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \leq \sigma \bigtriangledown \sigma = \sigma \text{ and}$$

$T_{h_K(b)}(ru) \geq T_{h_K(b)}(u) \geq \delta,\ I_{h_K(b)}(ru) \leq I_{h_K(b)}(u) \leq \eta,\ \ F_{h_K(b)}(ru) \leq F_{h_K(b)}(u) \leq \sigma.$

Hence $u - v,\ ru \in [h_K(b)]_{(\delta,\eta,\sigma)}$ and so $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp left ideal of S. By same way, $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a right ideal of S.

Reversely, assume $[h_K(b)]_{(\delta,\eta,\sigma)}$ be a crisp left (right) ideal of S and $u, v \in S$. If possible, let

$T_{h_K(b)}(u - v) < T_{h_K(b)}(u) \triangle T_{h_K(b)}(v),\ I_{h_K(b)}(u - v) > I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v)$ and
$F_{h_K(b)}(u - v) > F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v).$

If $T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) = s$ (say), then $T_{h_K(b)}(u) \geq s$ and $T_{h_K(b)}(v) \geq s$. As cut set is a crisp left ideal, so $T_{h_K(b)}(u - v) \geq s$ is natural. It shows a contradiction for $T_{h_K(b)}(u - v) < s$. Hence $T_{h_K(b)}(u - v) \geq T_{h_K(b)}(u) \triangle T_{h_K(x)}(v)$. Other two can be shown as usual.

For $r \in S$, let, $T_{h_K(b)}(ru) < T_{h_K(b)}(u),\ \ I_{h_K(b)}(ru) > I_{h_K(b)}(u)$ and $F_{h_K(b)}(ru) > F_{h_K(b)}(u)$.

If $T_{h_K(b)}(u) = t$, then $T_{h_K(b)}(ru) < t$. As cut set is a crisp left ideal, then $T_{h_K(x)}(ru) \geq t$ is obvious. It is against our assumption. So, $T_{h_K(x)}(ru) \geq T_{h_K(x)}(u)$. Other two can be set naturally. Thus K is an NSLI on

(S, E). K can also be shown an NSRI over (S, E) by same path and thus the theorem is ended.

4 Neutrosophic soft prime ideal

This section defines and illustrates NSPI along with the development of some theorems.

4.1 Definition

A constant Nss K on (S, E) is one whose $h_K(b)$ is constant $\forall b \in E$. It means, for every $b \in E$, the triplet $(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u))$ always gives same value $\forall u \in S$.
If for every $b \in E$, the triplet $(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u))$ is at least of two different kinds $\forall u \in S$, then K is called a nonconstant Nss.

4.2 Definition

Let C, D be two Nss on (S, E). Then CoD $(= P$, say) is also an Nss on (S, E). $\forall b \in E$ and $\forall u \in S$, it is defined as :

$$T_{h_P(b)}(u) = \begin{cases} \max_{u=vz}[T_{h_C(x)}(v) \bigtriangleup T_{h_D(x)}(z)] \\ 0 \quad \text{if } u \text{ is not put as } u = vz. \end{cases}$$

$$I_{h_P(b)}(u) = \begin{cases} \min_{u=vz}[I_{h_C(x)}(v) \bigtriangledown I_{h_D(x)}(z)] \\ 1 \quad \text{if } u \text{ is not put as } u = vz. \end{cases}$$

$$F_{h_P(b)}(u) = \begin{cases} \min_{u=vz}[F_{h_C(x)}(v) \bigtriangledown F_{h_D(x)}(z)] \\ 1 \quad \text{if } x \text{ is not put as } u = vz. \end{cases}$$

4.3 Definition

An NSI K over (S, E) is called an NSPI when (i) K is not constant NSI, (ii) for any two NSIs C, D over (S, E), $CoD \subseteq K$ implies either $C \subseteq K$ or $D \subseteq K$.

4.3.1 *Example*

Consider the integer set Z and the parametric set $E = \{b_1, b_2, b_3\}$. Take a division Z into $3Z$ and $Z - 3Z$. Consider an Nss K on (Z, E) given below.

Table 1 : Tabular form of Nss K

	$h_K(b_1)$	$h_K(b_2)$	$h_K(b_3)$
$3Z$	$(0.9, 0.4, 0.1)$	$(0.4, 0.3, 0.4)$	$(0.8, 0.7, 0.3)$
$Z - 3Z$	$(0.6, 0.7, 0.5)$	$(0.1, 0.6, 0.5)$	$(0.2, 0.9, 0.4)$

Now the following several cases are taken into consideration.
Case 1 : If $u, v \in 3Z$ then $u - v, uv \in 3Z$.
Case 2 : If $u, v \in Z - 3Z$ then $u - v \in 3Z$ or $Z - 3Z, uv \in Z - 3Z$.
Case 3 : If $u \in 3Z, v \in Z - 3Z$ then $u - v \in Z - 3Z$ and $uv \in 3Z$.

Obviously, K is an NSI on (Z, E). To make out that, consider Case 3 with respect to the parameter b_1. Other two are as usual.

$$\begin{cases} T_{h_K(b_1)}(u-v) = 0.6 = \min\{0.9, 0.6\} = T_{h_K(b_1)}(u) \bigtriangleup T_{h_K(b_1)}(v) \\ I_{h_K(b_1)}(u-v) = 0.7 = \max\{0.4, 0.7\} = I_{h_K(b_1)}(u) \bigtriangledown I_{h_K(b_1)}(v) \\ F_{h_K(b_1)}(u-v) = 0.5 = \max\{0.1, 0.5\} = F_{h_K(b_1)}(u) \bigtriangledown F_{h_K(b_1)}(v). \end{cases}$$

$$\begin{cases} T_{h_K(b_1)}(uv) = 0.9 = \max\{0.9, 0.6\} = \max\{T_{h_K(b_1)}(u), T_{h_K(b_1)}(v)\} \\ I_{h_K(b_1)}(uv) = 0.4 = \min\{0.4, 0.7\} = \min\{I_{h_K(b_1)}(u), I_{h_K(b_1)}(v)\} \\ F_{h_K(b_1)}(uv) = 0.1 = \min\{0.1, 0.5\} = \min\{F_{h_K(b_1)}(u), F_{h_K(b_1)}(v)\}. \end{cases}$$

To prove K as NSPI, we now let another two NSIs C (by Table 2) and D (by Table 3) on (Z, E). Table 4 refers the operation CoD.

Table 2 : Table for NSI C

	$h_C(b_1)$	$h_C(b_2)$	$h_C(b_3)$
$3Z$	$(0.3, 0.4, 0.6)$	$(0.7, 0.2, 0.5)$	$(0.6, 0.5, 0.1)$
$Z - 3Z$	$(0.1, 0.5, 0.8)$	$(0.1, 0.6, 0.7)$	$(0.3, 0.8, 0.2)$

Table 3 : Table for NSI D

	$h_D(b_1)$	$h_D(b_2)$	$h_D(b_3)$
$3Z$	$(0.6, 0.4, 0.5)$	$(0.3, 0.5, 0.6)$	$(0.4, 0.8, 0.4)$
$Z - 3Z$	$(0.2, 0.8, 0.9)$	$(0.1, 0.7, 0.8)$	$(0.1, 1.0, 0.5)$

Table 4 : Table for $CoD = Q$(say)

	$h_Q(b_1)$	$h_Q(b_2)$	$h_Q(b_3)$
$3Z$	$(0.3, 0.4, 0.6)$	$(0.3, 0.5, 0.6)$	$(0.4, 0.8, 0.4)$
$Z - 3Z$	$(0.1, 0.8, 0.9)$	$(0.1, 0.7, 0.8)$	$(0.1, 1.0, 0.5)$

The discussion of $h_Q(b_1)$ is provided to convince the Table 4.
When $uv \in 3Z$, then either $u, v \in 3Z$ or $u \in 3Z, v \in Z - 3Z$ or $u \in Z - 3Z, v \in 3Z$.
When $uv \in Z - 3Z$, then $u, v \in Z - 3Z$ only. Now for $w = uv \in 3Z$,

$$T_{h_Q(b_1)}(w) = \max_w\{T_{h_C(b_1)}(u) \bigtriangleup T_{h_D(b_1)}(v)\} = \max\{0.3 \bigtriangleup 0.6, 0.3 \bigtriangleup 0.2, 0.1 \bigtriangleup 0.6\} = 0.3$$
$$I_{h_Q(b_1)}(w) = \min_w\{I_{h_C(b_1)}(u) \bigtriangledown I_{h_D(b_1)}(v)\} = \min\{0.4 \bigtriangledown 0.4, 0.4 \bigtriangledown 0.8, 0.5 \bigtriangledown 0.4\} = 0.4$$
$$F_{h_Q(b_1)}(w) = \min_w\{F_{h_C(b_1)}(u) \bigtriangledown F_{h_D(b_1)}(v)\} = \min\{0.6 \bigtriangledown 0.5, 0.6 \bigtriangledown 0.9, 0.8 \bigtriangledown 0.5\} = 0.6$$

Next for $u = uv \in Z - 3Z$,

$$T_{h_Q(b_1)}(u) = \max_u\{T_{h_C(b_1)}(u) \bigtriangleup T_{h_D(b_1)}(v)\} = \max\{0.1 \bigtriangleup 0.2\} = 0.1$$
$$I_{h_Q(b_1)}(u) = \min_u\{I_{h_C(b_1)}(u) \bigtriangledown I_{h_D(b_1)}(v)\} = \min\{0.5 \bigtriangledown 0.8\} = 0.8$$
$$F_{h_Q(b_1)}(u) = \min_u\{F_{h_C(b_1)}(u) \bigtriangledown F_{h_D(b_1)}(v)\} = \min\{0.8 \bigtriangledown 0.9\} = 0.9$$

Table 1, Table 3, Table 4 execute that $D \subset K$ and $CoD \subset K$. Therefore, K is an NSPI on (Z, E).

4.4 Theorem

Consider an NSPI K on (S, E). Then $\forall b \in E$, $h_K(b)$ exactly attains two distinct values on S i.e., $|h_K(b)| = 2$.

Proof. As K is non-constant, hence $|h_K(b)| \geq 2, \forall b \in E$. Let $|h_K(b)| > 2$. Take $x = glb\{T_{h_K(b)}(u)\}$, $y = lub\{I_{h_K(b)}(u)\}, z = lub\{F_{h_K(b)}(u)\}$. Then $\exists\, s_1, p_1, s_2, p_2, s_3, p_3$ such that $x \leq s_1 < p_1 < T_{h_K(b)}(0_S)$, $y \geq s_2 > p_2 > I_{h_K(b)}(0_S)$, $z \geq s_3 > p_3 > F_{h_K(b)}(0_S)$. Define two Nss C, D on (S, E) as :

$T_{h_C(b)}(u) = \frac{1}{2}(s_1 + p_1),\ I_{h_C(b)}(u) = \frac{1}{2}(s_2 + p_2),\ \ F_{h_C(b)}(u) = \frac{1}{2}(s_3 + p_3),\ \forall u \in S$ and

$T_{h_D(b)}(u) = x,\ I_{h_D(b)}(u) = y,\ F_{h_D(b)}(u) = z$ if $u \notin K_{(p_1,p_2,p_3)}$,

$T_{h_D(b)}(u) = T_{h_K(b)}(0_S),\ I_{h_D(b)}(u) = I_{h_K(b)}(0_S),\ F_{h_D(b)}(u) = F_{h_K(b)}(0_S)$ if $u \in K_{(p_1,p_2,p_3)}$.

Clearly, C is an NSI on (S, E). We are to prove that D is an NSI over (S, E). Since K is an NSI on (S, E) then $K_{(p_1,p_2,p_3)}$ is a crisp ideal of S. Let $u, v \in S$. Following facts are considered.

Case 1 : When $u, v \in K_{(p_1,p_2,p_3)}$ then $u - v \in K_{(p_1,p_2,p_3)}$. So,

$$T_{h_D(b)}(u - v) = T_{h_K(b)}(0_S) = T_{h_K(b)}(0_S) \bigtriangleup T_{h_K(b)}(0_S) = T_{h_D(b)}(u) \bigtriangleup T_{h_D(b)}(v)$$
$$I_{h_D(b)}(u - v) = I_{h_K(b)}(0_S) = I_{h_K(b)}(0_S) \bigtriangledown I_{h_K(b)}(0_S) = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$$
$$F_{h_D(b)}(u - v) = F_{h_K(b)}(0_S) = F_{h_K(b)}(0_S) \bigtriangledown F_{h_K(b)}(0_S) = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)$$

Case 2 : When $u \in K_{(p_1,p_2,p_3)}$, $v \notin K_{(p_1,p_2,p_3)}$ then $u - v \notin K_{(p_1,p_2,p_3)}$ and so,

$$T_{h_D(b)}(u - y) = x = T_{h_K(b)}(0_S) \bigtriangleup x = T_{h_D(b)}(u) \bigtriangleup T_{h_D(b)}(v)$$
$$I_{h_D(b)}(u - v) = y = I_{h_K(b)}(0_S) \bigtriangledown y = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$$
$$F_{h_D(b)}(u - v) = z = F_{h_K(b)}(0_S) \bigtriangledown z = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)$$

Case 3 : When $u, v \notin K_{(p_1,p_2,p_3)}$ then,

$$T_{h_D(b)}(u - v) \geq x = x \bigtriangleup x = T_{h_D(b)}(u) \bigtriangleup T_{h_D(b)}(v)$$
$$I_{h_D(b)}(u - v) \leq y = y \bigtriangledown y = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$$
$$F_{h_D(b)}(u - v) \leq z = z \bigtriangledown z = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_D(b)}(u - v) \geq T_{h_D(b)}(u) \bigtriangleup T_{h_D(b)}(v)$, $\ I_{h_D(b)}(u - v) \leq I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$ and $F_{h_D(b)}(u - v) \leq F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)$.

We are to test the 2nd condition of Definition [2.7].

Case 1 : When $u \in K_{(p_1,p_2,p_3)}$ then $uv,\ vu \in K_{(p_1,p_2,p_3)}$, a crisp ideal of S, for $u, v \in S$. So,

$$\begin{aligned} T_{h_D(b)}(uv) &= T_{h_D(b)}(vu) = T_{h_K(b)}(0_S) = T_{h_D(b)}(u) \\ I_{h_D(b)}(uv) &= I_{h_D(b)}(vu) = I_{h_K(b)}(0_S) = I_{h_D(b)}(u) \\ F_{h_D(b)}(uv) &= F_{h_D(b)}(vu) = F_{h_K(b)}(0_S) = F_{h_D(b)}(u) \end{aligned}$$

Case 2 : If $u \notin K_{(p_1,p_2,p_3)}$ then,

$$T_{h_D(b)}(uv) \geq x = T_{h_D(b)}(u),\ \ T_{h_D(b)}(vu) \geq x = T_{h_D(b)}(u)$$
$$I_{h_D(b)}(uv) \leq y = I_{h_D(b)}(u),\ \ I_{h_D(b)}(vu) \leq y = I_{h_D(b)}(u)$$
$$F_{h_D(b)}(uv) \leq z = F_{h_D(b)}(u),\ \ F_{h_D(b)}(vu) \leq z = F_{h_D(b)}(u)$$

This shows that D is both NSLI and NSRI over (S, E). So, D is an NSI on (S, E). We claim $CoD \subseteq K$. We require following cases to analyse.

Case 1 : Tell $P = CoD$. For $u = 0_S$,

$$\begin{aligned}
T_{h_{P(b)}}(u) &= \max_{u=vw}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(w)] \le \frac{1}{2}(s_1 + p_1) \triangle T_{h_{K(b)}}(0_S) \\
&< T_{h_{K(b)}}(0_S) \triangle T_{h_{K(b)}}(0_S) \text{ [as } s_1 < p_1 < T_{h_{K(b)}}(0_S)] = T_{h_{K(b)}}(0_S) \\
I_{h_{P(b)}}(u) &= \min_{u=vw}[I_{h_{C(b)}}(v) \triangledown I_{h_{D(b)}}(w)] \ge \frac{1}{2}(s_2 + p_2) \triangledown I_{h_{K(b)}}(0_S) \\
&> I_{h_{K(b)}}(0_S) \triangledown I_{h_{K(b)}}(0_S) \text{ [as } s_2 > p_2 > I_{h_{K(b)}}(0_S)] = I_{h_{K(b)}}(0_S) \\
F_{h_{P(b)}}(u) &= \min_{u=vw}[F_{h_{C(b)}}(v) \triangledown F_{h_{D(b)}}(w)] \ge \frac{1}{2}(s_3 + p_3) \triangledown F_{h_{K(b)}}(0_S) \\
&> F_{h_{K(b)}}(0_S) \triangledown F_{h_{K(b)}}(0_S) \text{ [as } s_3 > p_3 > F_{h_{K(b)}}(0_S)] = F_{h_{K(b)}}(0_S)
\end{aligned}$$

Case 2 : For $u \neq 0_S$ but $u \in K_{(p_1,p_2,p_3)}$,

$$\begin{aligned}
T_{h_{P(b)}}(u) &= \max_{u=vw}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(w)] \le \frac{1}{2}(s_1 + p_1) \triangle T_{h_{K(b)}}(0_S) \\
&= \frac{1}{2}(s_1 + p_1) \quad \text{[as } s_1 < p_1 < T_{h_{K(b)}}(0_S)] \\
&< p_1 \quad \text{[as } s_1 < p_1] \le T_{h_{K(b)}}(u) \\
I_{h_{P(b)}}(u) &= \min_{u=vw}[I_{h_{C(b)}}(v) \triangledown I_{h_{D(b)}}(w)] \ge \frac{1}{2}(s_2 + p_2) \triangledown I_{h_{K(b)}}(0_S) \\
&= \frac{1}{2}(s_2 + p_2) \quad \text{[as } s_2 > p_2 > I_{h_{K(b)}}(0_S)] \\
&> p_2 \quad \text{[as } t_2 > m_2] \ge I_{h_{K(b)}}(u) \\
F_{h_{P(b)}}(u) &= \min_{u=vw}[F_{h_{C(b)}}(v) \triangledown F_{h_{D(b)}}(w)] \ge \frac{1}{2}(s_3 + p_3) \triangledown F_{h_{K(b)}}(0_S) \\
&= \frac{1}{2}(s_3 + p_3) \quad \text{[as } s_3 > p_3 > F_{h_{K(b)}}(0_S)] \\
&> p_3 \quad \text{[as } s_3 > p_3] \ge F_{h_{K(b)}}(u)
\end{aligned}$$

Case 3 : When $0_S \neq u \notin K_{(p_1,p_2,p_3)}$, for $v, w \in S$ such that $u = vw$, $v \notin K_{(p_1,p_2,p_3)}$ and $w \notin K_{(p_1,p_2,p_3)}$,

$$\begin{aligned}
T_{h_{P(b)}}(u) &= \max_{u=vw}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(w)] = \frac{1}{2}(s_1 + p_1) \triangle x = x \quad \text{[as } x \le s_1 < p_1] \le T_{h_{K(b)}}(u) \\
I_{h_{P(b)}}(u) &= \min_{u=vw}[I_{h_{C(b)}}(v) \triangledown I_{h_{D(b)}}(w)] = \frac{1}{2}(s_2 + p_2) \triangledown y = y \quad \text{[as } y \ge s_2 > p_2] \ge I_{h_{K(b)}}(u) \\
F_{h_{P(b)}}(u) &= \min_{u=vw}[F_{h_{C(b)}}(v) \triangledown F_{h_{D(b)}}(w)] = \frac{1}{2}(s_3 + p_3) \triangledown z = z \quad \text{[as } z \ge s_3 > p_3] \ge F_{h_{K(b)}}(u)
\end{aligned}$$

Therefore, $CoD \subseteq K$. Lastly, let $v \in S$ such that $T_{h_{K(b)}}(v) = s_1$, $I_{h_{K(b)}}(v) = s_2$, $F_{h_{K(b)}}(v) = s_3$. Then, $T_{h_{C(b)}}(v) = \frac{1}{2}(s_1 + p_1) > T_{h_{K(b)}}(v)$. Then $C \nsubseteq K$. Again assume $w \in S$ for which $T_{h_{K(b)}}(w) = p_1$, $I_{h_{K(b)}}(w) = p_2$, $F_{h_{K(b)}}(w) = p_3$ i.e., $w \in K_{(p_1,p_2,p_3)}$. Then $T_{h_{D(b)}}(w) = T_{h_{K(b)}}(0_S) > p_1 = T_{h_{K(b)}}(w)$ imply $D \nsubseteq K$. Hence, neither $C \nsubseteq K$ nor $D \nsubseteq K$ if $CoD \subseteq K$. Therefore, K is not an NSPI on (S, E) and it is against the hypothesis. So, $h_K(b)$ exactly attains two distinct values on S for $b \in E$ i.e., $|h_K(b)| = 2$.

4.5 Theorem

If K is an NSPI on (S, E), then $T_{h_K(b)}(0_S) = 1,\ I_{h_K(b)}(0_S) = 0,\ F_{h_K(b)}(0_S) = 0,\ \forall b \in E.$

Proof. For K being an NSPI on (S, E), $|h_K(b)| = 2, \forall b \in E$. Assume $T_{h_K(b)}(0_S) < 1, I_{h_K(b)}(0_S) > 0, F_{h_K(b)}(0_S) > 0$. For K being nonconstant, $\exists\, u \in S$ for which $T_{h_K(b)}(u) < T_{h_K(b)}(0_S), I_{h_K(b)}(u) > I_{h_K(b)}(0_S), F_{h_K(b)}(u) > F_{h_K(b)}(0_S)$. Let $T_{h_K(b)}(u) = p_1, T_{h_K(b)}(0_S) = m_1, I_{h_K(b)}(u) = p_2, I_{h_K(b)}(0_S) = m_2, F_{h_K(b)}(u) = p_3, F_{h_K(b)}(0_S) = m_3$. Take s_1, s_2, s_3 for that $p_1 < m_1 < s_1 \leq 1,\ p_2 > m_2 > s_2 \geq 0,\ p_3 > m_3 > s_3 \geq 0$. We assume two Nss C, D on (S, E) so that,

$T_{h_C(b)}(u) = \frac{1}{2}(p_1 + m_1),\ I_{h_C(b)}(u) = \frac{1}{2}(p_2 + m_2),\ F_{h_C(b)}(u) = \frac{1}{2}(p_3 + m_3),\ \forall u \in S$ and

$T_{h_D(b)}(u) = p_1,\ I_{h_D(b)}(u) = p_2,\ F_{h_D(b)}(u) = p_3$ for $u \notin K_0$,

$T_{h_D(b)}(u) = s_1,\ I_{h_D(b)}(u) = s_2,\ F_{h_D(b)}(u) = s_3$ if $u \in K_0$

where $K_0 = \{u \in S : T_{h_K(b)}(u) = T_{h_K(b)}(0_S), I_{h_K(b)}(u) = I_{h_K(b)}(0_S), F_{h_K(b)}(u) = F_{h_K(b)}(0_S)\}$.
Clearly, C is an NSI on (S, E). D is an NSI on (S, E) for K_0 being an ideal of S. We are now to show that $CoD \subseteq K$. Following facts are needed to consider.

Case 1 : Take $Q = CoD$. For $u = 0_S$,

$$\begin{aligned}
T_{h_Q(b)}(u) &= \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \max[\frac{1}{2}(p_1 + m_1) \triangle p_1, \frac{1}{2}(p_1 + m_1) \triangle s_1] \\
&= \max[p_1, \frac{1}{2}(p_1 + m_1)] = \frac{1}{2}(p_1 + m_1) < m_1 = T_{h_K(b)}(0_S) \\
I_{h_Q(b)}(u) &= \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) > m_2 = I_{h_K(b)}(0_S) \\
F_{h_Q(b)}(u) &= \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) > m_3 = F_{h_K(b)}(0_S)
\end{aligned}$$

Case 2 : When $0_S \neq u = vw \in K_0$ for $v, w \in K_0 \subset S$,

$$T_{h_Q(b)}(u) = \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \frac{1}{2}(p_1 + m_1) \triangle s_1 = \frac{1}{2}(p_1 + m_1) < m_1 = T_{h_K(b)}(0_S) = T_{h_K(b)}(u)$$
$$I_{h_Q(b)}(u) = \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) \triangle s_2 = \frac{1}{2}(p_2 + m_2) > m_2 = I_{h_K(b)}(0_S) = I_{h_K(b)}(u)$$
$$F_{h_Q(b)}(u) = \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) \triangle s_3 = \frac{1}{2}(p_3 + m_3) > m_3 = F_{h_K(b)}(0_S) = F_{h_K(b)}(u)$$

Case 3 : When $0_S \neq u = vw \notin K_0$ for $v, w \in S - K_0$,

$$T_{h_Q(b)}(u) = \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \frac{1}{2}(p_1 + m_1) \triangle p_1 = p_1 = T_{h_K(b)}(u)$$
$$I_{h_Q(b)}(u) = \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) \bigtriangledown p_2 = p_2 = I_{h_K(b)}(u)$$
$$F_{h_Q(b)}(u) = \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) \bigtriangledown p_3 = p_3 = F_{h_K(b)}(u)$$

So including all, $CoD \subseteq K$. As $T_{h_K(b)}(0_S) = m_1 < s_1 = T_{h_D(b)}(0_S)$, so $D \not\subseteq K$. Further $\exists\, u \in S$ so that $T_{h_K(b)}(u) = p_1 < \frac{1}{2}(p_1 + m_1) = T_{h_C(b)}(u)$ impliy $C \not\subseteq K$. This means that K is not an NSPI which is against the hypothesis. Therefore $T_{h_K(b)}(0_S) = 1,\ I_{h_K(b)}(0_S) = 0,\ F_{h_K(b)}(0_S) = 0,\ \forall b \in E$.

4.6 Theorem

For an Nss K on (S,E), let $|h_K(b)| = 2$ and $T_{h_K(b)}(0_S) = 1$, $I_{h_K(b)}(0_S) = 0$, $F_{h_K(b)}(0_S) = 0, \forall b \in E$. If $K_0 = \{u \in S : T_{h_K(b)}(u) = T_{h_K(b)}(0_S), I_{h_K(b)}(u) = I_{h_K(b)}(0_S), F_{h_K(b)}(u) = F_{h_K(b)}(0_S)\}$ is a prime ideal on S, then K is an NSPI on (S,E).

Proof. By hypothesis, $\exists$ one $u \in S$ with $s_1 = T_{h_K(b)}(u) < 1, s_2 = I_{h_K(b)}(u) > 0, s_3 = F_{h_K(b)}(u) > 0$. The facts stated below are taken.

Case 1 : When $u, v \in K_0$, then $u - v \in K_0$, an ideal. So $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= T_{h_K(b)}(0) = 1 = 1 \triangle 1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= I_{h_K(b)}(0) = 0 = 0 \bigtriangledown 0 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= I_{h_K(b)}(0) = 0 = 0 \bigtriangledown 0 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 2 : If $u \in K_0$ but $v \notin K_0$, then $u - v \notin K_0$. Then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= s_1 = 1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= s_2 = 0 \bigtriangledown s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= s_3 = 0 \bigtriangledown s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 3 : If $u, v \notin K_0$, then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &\leq s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &\leq s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,
$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v),\ I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v)$ and
$F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v)$.

To verify the final item, we consider the following cases.

Case 1 : When $u \in K_0$ then $uv,\ vu \in K_0$, an ideal over S, for $v \in s$. So $\forall b \in E$,
$T_{h_K(b)}(uv) = T_{h_K(b)}(vu) = 1 = T_{h_K(b)}(u),\ \ I_{h_K(b)}(uv) = I_{h_K(b)}(vu) = 0 = I_{h_K(b)}(u)$,
$F_{h_K(b)}(uv) = F_{h_K(b)}(vu) = 0 = F_{h_K(b)}(u)$.

Case 2 : If $u \notin K_0$ then,

$$\begin{aligned} &T_{h_K(b)}(uv) \geq s_1 = T_{h_K(b)}(u),\ T_{h_K(b)}(vu) \geq s_1 = T_{h_K(b)}(u) \\ &I_{h_K(b)}(uv) \leq s_2 = I_{h_K(b)}(u),\ I_{h_K(b)}(vu) \leq s_2 = I_{h_K(b)}(u) \\ &F_{h_K(b)}(uv) \leq s_3 = F_{h_K(b)}(u),\ F_{h_K(b)}(vu) \leq s_3 = F_{h_K(b)}(u) \end{aligned}$$

This shows that K is NSI over (S,E). Let $CoD \subseteq K$ but $C \nsubseteq K$, $D \nsubseteq K$ for C, D being two NSIs on (S,E). So, $\forall u, v \in S$ and $\forall b \in E$,
$T_{h_C(b)}(u) > T_{h_K(b)}(u),\ I_{h_C(b)}(u) < I_{h_K(b)}(u),\ F_{h_C(b)}(u) < F_{h_K(b)}(u)$ and
$T_{h_D(b)}(v) > T_{h_K(b)}(v),\ I_{h_D(b)}(v) < I_{h_K(b)}(v),\ F_{h_D(b)}(v) < F_{h_K(b)}(v)$.

Clearly, these $u, v \notin K_0$ otherwise $T_{h_C(b)}(u) > T_{h_K(b)}(u) = T_{h_K(b)}(0_S) = 1$ and $T_{h_D(b)}(u) > T_{h_K(b)}(u) = T_{h_K(b)}(0_S) = 1$ which are impossible. Then $rv,\ urv \notin K_0$, a prime ideal of S, for $r \in S$. Thus,

$T_{h_{K(b)}}(urv) = s_1 = T_{h_{K(b)}}(u) = T_{h_{K(b)}}(v),\ I_{h_{K(b)}}(urv) = s_2 = I_{h_{K(b)}}(u) = I_{h_{K(b)}}(v)$ and
$F_{h_{K(b)}}(urv) = s_3 = F_{h_{K(b)}}(u) = F_{h_{K(b)}}(v)$.

Now, if $Q = CoD$ then $\forall b \in E$ and $\forall w \in S$,

$$\begin{aligned} T_{h_{Q(b)}}(w) &= \max_{w=yz}[T_{h_{C(b)}}(y) \triangle T_{h_{D(b)}}(z)] \geq T_{h_{C(b)}}(u) \triangle T_{h_{D(b)}}(rv) \geq T_{h_{C(b)}}(u) \triangle T_{h_{D(b)}}(v) \\ &> T_{h_{K(b)}}(u) \triangle T_{h_{K(b)}}(v) = s_1 \triangle s_1 = T_{h_{K(b)}}(w) \end{aligned}$$

Hence $CoD \not\subseteq K$. Then either $C \subseteq K$ or $D \subseteq K$ implies K is an NSPI on (S, E).

4.7 Theorem

For an NSPI K on (S, E), $K_0 = \{u \in R : T_{h_{K(b)}}(u) = T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) = I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) = F_{h_{K(b)}}(0_S)\}$ is a crisp prime ideal of S.

Proof. Here, K_0 is a crisp ideal of S by Theorem [3.3]. To prove K_0 being prime, let A, B be two crisp ideals of K_0 with $AB \subseteq K_0$. Assume C, D as two Nss on (S, E) as given below, $\forall b \in E$,

$$h_C(b) = \begin{cases} (T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(0_S)) & \text{if } u \in A \\ (0, 1, 1) & \text{if } u \notin A. \end{cases}$$
$$h_D(b) = \begin{cases} (T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(0_S)) & \text{if } u \in B \\ (0, 1, 1) & \text{if } u \notin B. \end{cases}$$

Clearly C, D are two NSIs on (R, E) by Theorem [3.2]. We are to prove $CoD \subseteq K$. Consider the following facts.

Case 1 : If $Q = CoD$ and $u \in K_0$,

$$T_{h_{Q(b)}}(u) = \max_{u=vz}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(z)] \leq T_{h_{K(b)}}(0_S) \triangle T_{h_{K(b)}}(0_S) = T_{h_{K(b)}}(0_S) = T_{h_{K(b)}}(u)$$
$$I_{h_{Q(b)}}(u) = \min_{u=vz}[I_{h_{C(b)}}(v) \bigtriangledown I_{h_{D(b)}}(z)] \geq I_{h_{K(b)}}(0_S) \bigtriangledown I_{h_{K(b)}}(0_S) = I_{h_{K(b)}}(0_S) = I_{h_{K(b)}}(u)$$
$$F_{h_{Q(b)}}(u) = \min_{u=vz}[F_{h_{C(b)}}(v) \bigtriangledown F_{h_{D(b)}}(z)] \geq F_{h_{K(b)}}(0_S) \bigtriangledown F_{h_{K(b)}}(0_S) = F_{h_{K(b)}}(0_S) = F_{h_{K(b)}}(u)$$

Case 2 : If $u \notin K_0$ then for $v, z \in R$ such that $u = vz$, $v \notin K_0$ and $z \notin K_0$. Now,

$$T_{h_{Q(b)}}(u) = \max_{u=vz}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(z)] = 0 \leq T_{h_{K(b)}}(u)$$
$$I_{h_{Q(b)}}(u) = \min_{u=vz}[I_{h_{C(b)}}(v) \bigtriangledown I_{h_{D(b)}}(z)] = 1 \geq I_{h_{K(b)}}(u)$$
$$F_{h_{Q(b)}}(u) = \min_{u=vz}[F_{h_{C(b)}}(v) \bigtriangledown F_{h_{D(b)}}(z)] = 1 \geq F_{h_{K(b)}}(u)$$

Thus in either case $CoD \subseteq K$. Then either $C \subseteq K$ or $D \subseteq K$, an NSPI over (S, E). Suppose $C \subseteq K$ but $A \not\subseteq K_0$. Then $\exists u \in A$ such that $u \notin K_0$ i.e., $T_{h_{K(b)}}(u) \neq T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) \neq I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) \neq F_{h_{K(b)}}(0_S)$, $\forall x \in E$. This implies $T_{h_{K(b)}}(u) < T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) > I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) > F_{h_{K(b)}}(0_S)$ by Proposition [3.1](i). Thus $T_{h_{C(b)}}(u) = T_{h_{K(b)}}(0_S) > T_{h_{K(b)}}(u)$, $I_{h_{C(b)}}(u) = I_{h_{K(b)}}(0_S) < I_{h_{K(b)}}(u)$, $F_{h_{C(b)}}(u) = F_{h_{K(b)}}(0_S) < F_{h_{K(b)}}(u)$ which is against the assumption $C \subseteq K$. So, $A \subseteq K_0$. Identically, $D \subseteq K \Rightarrow B \subseteq K_0$. Hence $AB \subseteq K_0 \Rightarrow$ either $A \subseteq K_0$ or $B \subseteq K_0$ implies K_0 is a prime ideal.

4.8 Theorem

(i) Q is a non empty crisp prime ideal of S if and only if $\exists$ an NSPI M on (S,E) where $h_M : E \longrightarrow N_S(S)$ is put as, $\forall b \in E$,

$$h_M(b) = \begin{cases} (1,0,0) & \text{when } u \in Q \\ (p_1,p_2,p_3) & \text{when } u \notin Q. \end{cases}$$

with $0 \leq p_1, p_2, p_3 \leq 1$.

(ii) Particularly, Q is a non empty crisp prime ideal of S if and only if it's characteristic function λ_Q is an NSPI on (S,E) when $\lambda_Q : E \longrightarrow N_S(S)$ is put as, $\forall b \in E$,

$$\lambda_Q(b)(u) = \begin{cases} (1,0,0) & \text{when } u \in Q \\ (0,1,1) & \text{when } u \notin Q. \end{cases}$$

Proof. **(i)** If Q be a crisp prime ideal, then M is an NSI on (S,E) by Theorem [3.2]. Consider two NSIs C, D on (S,E) with $CoD \subseteq M$ but $C \nsubseteq M$ and $D \nsubseteq M$. For $u, v \in S$ and $b \in E$,

$T_{h_C(b)}(u) > T_{h_M(b)}(u),\ I_{h_C(b)}(u) < I_{h_M(b)}(u),\ F_{h_C(b)}(u) < F_{h_M(b)}(u)$ and
$T_{h_D(b)}(v) > T_{h_M(b)}(v),\ I_{h_D(b)}(v) < I_{h_M(b)}(v),\ F_{h_D(b)}(v) < F_{h_M(b)}(v)$.

Obviously $u, v \notin Q$ otherwise $T_{h_C(b)}(u) > 1, I_{h_C(b)}(u) < 0,\ F_{h_C(b)}(u) < 0$ and $T_{h_D(b)}(v) > 1, I_{h_D(b)}(v) < 0, F_{h_D(b)}(v) < 0$ which are impossible. Then $z = uv \notin Q$ i.e., $T_{h_M(b)}(z) = p_1,\ I_{h_M(b)}(z) = p_2,\ F_{h_M(b)}(z) = p_3$. Now since $CoD \subseteq M$, then

$$p_1 = T_{h_M(b)}(z) \geq T_{h_{CoD}(b)}(z) = \max_{z=uv}[T_{h_C(b)}(u) \triangle T_{h_D(b)}(v)] > T_{h_M(b)}(u) \triangle T_{h_M(b)}(v) = p_1 \triangle p_1 = p_1$$

So $p_1 > p_1$ makes a contradiction and thus $C \nsubseteq M$ and $D \nsubseteq M$ are false. Hence $CoD \subseteq M$ implies either $C \subseteq M$ or $D \subseteq M$ i.e., M is an NSPI on (S,E).
The 'only if' part can be drawn from Theorem [4.7] by taking $T_{h_M(b)}(0_S) = 1,\ I_{h_M(b)}(0_S) = 0,\ F_{h_M(b)}(0_S) = 0$.
(ii) Following the sense of 1st part, it can be easily proved.

4.9 Theorem

An Nss K on (S,E) with $|h_K(b)| = 2,\ \forall b \in E$ is an NSPI over (S,E) if and only if $\widehat{K} = \{u \in S : T_{h_K(b)}(u) = 1,\ I_{h_K(b)}(u) = 0,\ F_{h_K(b)}(u) = 0,\ \forall b \in E\}$ with $0_S \in \widehat{K}$ is a crisp prime ideal of S.

Proof. Combining Theorem [4.7] and Theorem [4.8], it can be proved.

4.10 Theorem

An Nss K on (S,E) is an NSPI iff each nonempty cut set $[h_K(b)]_{(\delta,\eta,\sigma)}$ of $h_K(b)$, an N_S, is a crisp prime ideal of S when $\delta \in Im\,T_{h_K(b)},\ \eta \in Im\,I_{h_K(b)},\ \sigma \in Im\,F_{h_K(b)},\ \forall b \in E$.

Proof. Let K be an NSPI over (S,E). Then, by Theorem [3.5], $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp ideal of S. Consider another two crisp ideals A, B of S so as $AB \subseteq [h_K(b)]_{(\delta,\eta,\sigma)}$. On (S,E), define two Nss C, D as :

$$h_C(b) = \begin{cases} (\delta,0,0) & \text{if } u \in A \\ (0,\eta,\sigma) & \text{otherwise}. \end{cases} \quad h_D(b) = \begin{cases} (\delta,0,0) & \text{if } u \in B \\ (0,\eta,\sigma) & \text{otherwise}. \end{cases}$$

Then C, D are two NSIs over (R,E) and $CoD \subseteq K$. Since K is an NSPI over (R,E) then either $C \subseteq K$ or $D \subseteq K$. Now if possible, suppose $A \nsubseteq [h_K(b)]_{(\delta,\eta,\sigma)}$. Then $\exists\, u \in A$ such that $u \notin [h_K(b)]_{(\delta,\eta,\sigma)}$ i.e.,

$T_{h_K(b)}(u) < \delta,\ I_{h_K(b)}(u) > \eta,\ F_{h_K(b)}(u) > \sigma$. Now for $u \in A$,

$T_{h_C(b)}(u) = \delta > T_{h_K(b)}(u),\ I_{h_C(b)}(u) = 0 \leq \eta < I_{h_K(b)}(u),\ \ F_{h_C(b)}(u) = 0 \leq \sigma < F_{h_K(b)}(u).$

This shows $C \nsubseteq K$. Also $D \nsubseteq K$ similarly. These are against the situation. Therefore $A \subseteq [h_K(b)]_{(\delta,\eta,\sigma)}$ means $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp prime ideal of S.

Reversely, we need to clear that K is an NSPI over (S, E) if $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp prime ideal of S. Take two NSIs C, D on (S, E) so as $CoD \subseteq K$. Let $C \nsubseteq K,\ D \nsubseteq K$. Then $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_C(b)}(u) > T_{h_K(b)}(u),\ I_{h_C(b)}(u) < I_{h_K(b)}(u),\ F_{h_C(b)}(u) < F_{h_K(b)}(u)$ and
$T_{h_D(b)}(v) > T_{h_K(b)}(v),\ I_{h_D(b)}(v) < I_{h_K(b)}(v),\ F_{h_D(b)}(v) < F_{h_K(b)}(v).$

Clearly $T_{h_K(b)}(u) \neq 1,\ I_{h_K(b)}(u) \neq 0,\ F_{h_K(b)}(u) \neq 0$ and $T_{h_K(b)}(v) \neq 1,\ I_{h_K(b)}(v) \neq 0,\ F_{h_K(b)}(v) \neq 0$. Let $T_{h_K(b)}(u) = T_{h_K(b)}(v) = p,\ I_{h_K(b)}(u) = I_{h_K(b)}(v) = q,\ F_{h_K(b)}(u) = F_{h_K(b)}(v) = r$. Then $T_{h_C(b)}(u) > p,\ I_{h_C(b)}(u) < q,\ F_{h_C(b)}(u) < r$ and $T_{h_D(b)}(v) > p,\ I_{h_D(b)}(v) < q,\ F_{h_D(b)}(v) < r$ i.e., $u \in [h_C(b)]_{(p,q,r)}$ and $v \in [h_D(b)]_{(p,q,r)}$. Now since $CoD \subseteq K$,

$$T_{h_K(b)}(z) \geq \max_{z=uv}[T_{h_C(b)}(u) \triangle T_{h_D(b)}(v)] > T_{h_C(b)}(u) \triangle T_{h_D(b)}(v) > p$$
$$I_{h_K(b)}(z) \leq \min_{z=uv}[I_{h_C(b)}(u) \bigtriangledown I_{h_D(b)}(v)] < I_{h_C(b)}(u) \bigtriangledown I_{h_D(b)}(v) < q$$
$$F_{h_K(b)}(z) \leq \min_{z=uv}[F_{h_C(b)}(u) \bigtriangledown F_{h_D(b)}(v)] < F_{h_C(b)}(u) \bigtriangledown F_{h_D(b)}(v) < r$$

Thus $z = uv \in [h_K(b)]_{(p,q,r)}$ i.e., $[h_C(b)]_{(p,q,r)}[h_D(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$, a crisp prime ideal of S. Then either $[h_C(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$ or $[h_D(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$. If $[h_C(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$, then $u \in [h_C(b)]_{(p,q,r)}$ implies $u \in [h_K(b)]_{(p,q,r)}$. This means $T_{h_C(b)}(u) \geq p \Rightarrow T_{h_K(b)}(u) \geq p,\ I_{h_C(b)}(u) \leq q \Rightarrow I_{h_K(b)}(u) \leq q,\ F_{h_C(b)}(u) \leq r \Rightarrow F_{h_K(b)}(u) \leq r$ i.e., $T_{h_K(b)}(u) \geq T_{h_C(b)}(u),\ I_{h_K(b)}(u) \leq I_{h_C(b)}(u),\ F_{h_K(b)}(u) \leq F_{h_C(b)}(u)$. It is against the assumption. Therefore, $C \subseteq K$ or $D \subseteq K$ and the proof is reached.

5 Homomorphic image of NSI and NSPI

The homomorphic image of NSI and NSPI are analysed here. We let R_1, R_2 as two crisp rings and $\pi : R_1 \longrightarrow R_2$ being a ring homomorphism throughout this section.

5.1 Definition

If C, D be two Nss on $(R_1, E), (R_2, E)$ respectively, then $\pi(C), \pi^{-1}(D)$ are also Nss over $(R_2, E), (R_1, E)$ respectively and these are described as :

(i) $\pi(C)(v) = \{(T_{h_{\pi(C)}(b)}(v), I_{h_{\pi(C)}(b)}(v), F_{h_{\pi(C)}(b)}(v)) : b \in E\},\ \forall v \in R_2$ where

$$T_{h_{\pi(C)}(b)}(v) = \begin{cases} \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 0 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$
$$I_{h_{\pi(C)}(b)}(v) = \begin{cases} \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 1 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$
$$F_{h_{\pi(C)}(b)}(v) = \begin{cases} \min\{F_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 1 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$

(ii) $\pi^{-1}(D)(u) = \{(T_{h_{\pi^{-1}(D)}(b)}(u), I_{h_{\pi^{-1}(D)}(b)}(u), F_{h_{\pi^{-1}(D)}(b)}(u)) : b \in E\},\ \forall u \in R_1$ where

$T_{h_{\pi^{-1}(D)}(b)}(u) = T_{h_D(b)}[\pi(u)],\ \ I_{h_{\pi^{-1}(D)}(b)}(u) = I_{h_D(b)}[\pi(u)]$ and $F_{h_{\pi^{-1}(D)}(b)}(u) = F_{h_D(b)}[\pi(u)]$.

5.2 Proposition

Let C and D be two NSLIs (NSRIs) on (R_1, E) and (R_2, E) respectively. Then,
(i) $\pi(C)$ is an NSLIs (NSRIs) over (R_2, E) if π is epimorphism.
(ii) $\pi^{-1}(D)$ is an NSLIs (NSRIs) over (R_1, E).

Proof. (i) Let $v_1, v_2, s \in R_2$. If $\pi^{-1}(v_1) = \phi$ or $\pi^{-1}(v_2) = \phi$, the proof is usual. So, let $\exists\, u_1, u_2, r \in R_1$ so as $\pi(u_1) = v_1, \pi(u_2) = v_2, \pi(r) = s$. Now,

$$T_{h_{\pi(C)}(b)}(v_1 - v_2) = \max_{\pi(u)=v_1-v_2}\{T_{h_C(b)}(b)\} \geq T_{h_C(b)}(u_1 - u_2) \geq T_{h_C(b)}(u_1) \bigtriangleup T_{h_C(b)}(u_2),$$
$$T_{h_{\pi(C)}(b)}(sv_1) = \max_{\pi(u)=sv_1}\{T_{h_C(b)}(u)\} \geq T_{h_C(b)}(ru_1) \geq T_{h_C(b)}(u_1)$$

As all the inequalities are carried $\forall u_1, u_2, r \in R_1$ obeying $\pi(u_1) = v_1, \pi(u_2) = v_2, \pi(r) = s$ hence,

$$T_{h_{\pi(C)}(b)}(v_1 - v_2) \geq (\max_{\pi(u_1)=v_1}\{T_{h_C(b)}(u_1)\}) \bigtriangleup (\max_{\pi(u_2)=v_2}\{T_{h_C(b)}(u_2)\}) = T_{h_{\pi(C)}(b)}(v_1) \bigtriangleup T_{h_{\pi(C)}(b)}(v_2),$$
$$T_{h_{\pi(C)}(b)}(sv_1) \geq \max_{\pi(u_1)=v_1}\{T_{h_C(b)}(u_1)\} = T_{h_{\pi(C)}(b)}(v_1). \text{ Next,}$$
$$I_{h_{\pi(C)}(b)}(v_1 - v_2) = \min_{\pi(u)=v_1-v_2}\{I_{h_C(b)}(u)\} \leq I_{h_C(b)}(u_1 - u_2) \leq I_{h_C(b)}(u_1) \bigtriangledown I_{h_C(b)}(u_2),$$
$$I_{h_{\pi(C)}(b)}(sv_1) = \min_{\pi(u)=sv_1}\{I_{h_C(b)}(u)\} \leq I_{h_C(b)}(ru_1) \leq I_{h_C(b)}(u_1).$$

As all the inequalities are carried $\forall u_1, u_2, r \in R_1$ obeying $\pi(u_1) = y_1, \pi(u_2) = v_2, \pi(r) = s$ hence,

$$I_{h_{\pi(C)}(b)}(v_1 - v_2) \leq (\min_{\pi(u_1)=v_1}\{I_{h_C(b)}(u_1)\}) \bigtriangledown (\min_{\pi(u_2)=v_2}\{I_{h_C(b)}(u_2)\}) = I_{h_{\pi(C)}(b)}(v_1) \bigtriangledown I_{h_{\pi(C)}(b)}(v_2),$$
$$I_{h_{\pi(C)}(b)}(sv_1) \leq \min_{\pi(u_1)=v_1}\{I_{h_C(b)}(u_1)\} = I_{h_{\pi(C)}(b)}(v_1).$$

Similarly, we can show that
$F_{h_{\pi(C)}(b)}(v_1 - v_2) \leq F_{h_{\pi(C)}(b)}(v_1) \bigtriangledown F_{h_{\pi(C)}(b)}(v_2), \quad F_{h_{\pi(C)}(b)}(sv_1) \leq F_{h_{\pi(C)}(b)}(v_1).$
This brings the 1st result.
(ii) For $u_1, u_2 \in R_1$, we have,

$$\begin{aligned}
T_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) &= T_{h_D(b)}[\pi(u_1 - u_2)] = T_{h_D(b)}[\pi(u_1) - \pi(u_2)] \\
&\geq T_{h_D(b)}[\pi(u_1)] \bigtriangleup T_{h_D(b)}[\pi(u_2)] = T_{h_{\pi^{-1}(D)}(b)}(u_1) \bigtriangleup T_{h_{\pi^{-1}(D)}(b)}(u_2), \\
T_{h_{\pi^{-1}(D)}(b)}(ru_1) &= T_{h_D(b)}[\pi(ru_1)] = T_{h_D(b)}[\pi(r)\pi(u_1)] = T_{h_D(b)}[s\pi(u_1)] \\
&\geq T_{h_D(b)}[\pi(u_1)] = T_{h_{\pi^{-1}(D)}(b)}(u_1), \\
I_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) &= I_{h_D(b)}[\pi(u_1 - u_2)] = I_{h_D(b)}[\pi(u_1) - \pi(u_2)] \\
&\leq I_{h_D(b)}[\pi(u_1)] \bigtriangledown I_{h_D(b)}[\pi(u_2)] = I_{h_{\pi^{-1}(D)}(b)}(u_1) \bigtriangledown I_{h_{\pi^{-1}(D)}(b)}(u_2), \\
I_{h_{\pi^{-1}(D)}(b)}(ru_1) &= I_{h_D(b)}[\pi(ru_1)] = I_{h_D(b)}[\pi(r)\pi(u_1)] = I_{h_D(b)}[s\pi(u_1)] \\
&\leq I_{h_D(b)}[\pi(u_1)] = I_{h_{\pi^{-1}(D)}(b)}(u_1).
\end{aligned}$$

In a similar fashion,
$F_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) \leq F_{h_{\pi^{-1}(D)}(b)}(u_1) \bigtriangledown F_{h_{\pi^{-1}(D)}(b)}(u_2), \quad F_{h_{\pi^{-1}(D)}(b)}(ru_1) \leq F_{h_{\pi^{-1}(D)}(b)}(u_1).$
This brings the 2nd result.

5.3 Proposition

Take two NSLIs (NSRIs) C, D over (R_1, E) and (R_2, E), respectively. If $0_1, 0_2$ are the additive identities of R_1, R_2 respectively, then (i) $\pi(C)(0_2) = C(0_1)$ (ii) $\pi^{-1}(D)(0_1) = D(0_2)$

Proof. (i) Here $\pi(C)(0_2) = \{(T_{h_{\pi(C)}(b)}(0_2), I_{h_{\pi(C)}(b)}(0_2), F_{h_{\pi(C)}(b)}(0_2)) : b \in E\}$ and
$C(0_1) = \{(T_{h_C(b)}(0_1), I_{h_C(b)}(0_1), F_{h_C(b)}(0_1)) : b \in E\}$; Now,
$T_{h_{\pi(C)}(b)}(0_2) = \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \geq T_{h_C(b)}(0_1)$ [as $\pi(0_1) = 0_2$]
Since C is an NSLIs over (R_1, E), so $\forall u \in R$ and $\forall b \in E$,
$T_{h_C(b)}(u) \leq T_{h_C(b)}(0_1) \Rightarrow \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \leq T_{h_C(b)}(0_1) \Rightarrow T_{h_{\pi(C)}(b)}(0_2) \leq T_{h_C(b)}(0_1)$
Thus $T_{h_{\pi(C)}(b)}(0_2) = T_{h_C(b)}(0_1)$. Next,
$I_{h_{\pi(C)}(b)}(0_2) = \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \leq I_{h_C(b)}(0_1)$ [as $\pi(0_1) = 0_2$]
Since C is an NSLIs over (R_1, E), so $\forall u \in R$ and $\forall b \in E$,
$I_{h_C(b)}(u) \geq I_{h_C(b)}(0_1) \Rightarrow \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \geq I_{h_C(b)}(0_1) \Rightarrow I_{h_{\pi(C)}(b)}(0_2) \geq I_{h_C(b)}(0_1)$.
Thus $I_{h_{\pi(C)}(b)}(0_2) = I_{h_C(b)}(0_1)$. Similarly, $F_{h_{\pi(C)}(b)}(0_2) = F_{h_C(b)}(0_1)$ and this follows the 1st result.
(ii) Here, we have
$T_{h_{\pi^{-1}(D)}(b)}(0_1) = T_{h_D(b)}[\pi(0_1)] = T_{h_D(b)}(0_2)$, $I_{h_{\pi^{-1}(D)}(b)}(0_1) = I_{h_D(b)}[\pi(0_1)] = I_{h_D(b)}(0_2)$ and
$F_{h_{\pi^{-1}(D)}(b)}(0_1) = F_{h_D(b)}[\pi(0_1)] = F_{h_D(b)}(0_2)$. This follows the 2nd result.

5.4 Definition

Consider two nonempty sets X, E and a lattice $[0, 1]$. Then $K = \{(T_{h_K(b)}, I_{h_K(b)}, F_{h_K(b)})|b \in E\} : X \longrightarrow [0, 1] \times [0, 1] \times [0, 1]$ attains the sup property when $T_{h_K(b)}(X) = \{T_{h_K(b)}(x) : x \in X\}$ (the image of $T_{h_K(b)}$) admits a maximal element and each of $I_{h_K(b)}(X) = \{I_{h_K(b)}(x) : x \in X\}$, $F_{h_K(b)}(X) = \{F_{h_K(b)}(x) : x \in X\}$ (the image of $I_{h_K(b)}, F_{h_K(b)}$ respectively) admits a minimal element $\forall b \in E$.

5.5 Proposition

For two NSLIs (NSRIs) K, L on (R_1, E) and (R_2, E), respectively, followings hold.
(i) $\pi(K_0) \subseteq (\pi(K))_0$ (Theorem [3.3] describes K_0).
(ii) $\pi(K_0) = (\pi(K))_0$ when K attains sup property.
(iii) $\pi^{-1}(L_0) = (\pi^{-1}(L))_0$.

Proof. (i) If $v \in \pi(K_0)$ signifies $v = \pi(u)$ for $u \in K_0 \subset R_1$ so as $T_{h_K(b)}(u) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(u) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(u) = F_{h_K(b)}(0_1)$. Now,

$$\begin{aligned} T_{h_{\pi(K)}(b)}(v) &= \max\{T_{h_K(b)}(u) : u \in \pi^{-1}(v)\} = \max\{T_{h_K(b)}(0_1)\} = T_{h_K(b)}(0_1) = T_{h_{\pi(K)}(b)}(0_2) \\ I_{h_{\pi(K)}(b)}(v) &= \min\{I_{h_K(b)}(u) : u \in \pi^{-1}(v)\} = \min\{I_{h_K(b)}(0_1)\} = I_{h_K(b)}(0_1) = I_{h_{\pi(K)}(b)}(0_2) \end{aligned}$$

Similarly, $F_{h_{\pi(K)}(b)}(v) = F_{h_{\pi(K)}(b)}(0_2)$. It signifies $v \in (\pi(K))_0$ when $v \in \pi(K_0)$ i.e., $\pi(K_0) \subseteq (\pi(K))_0$.
(ii)Take $u \in R_1$ so as $v = \pi(u) \in (\pi(K))_0 \subset R_2$. Then $\forall b \in E$,
$T_{h_{\pi(K)}(b)}(0_2) = T_{h_{\pi(K)}(b)}(v) \Rightarrow T_{h_K(b)}(0_1) = \max\{T_{h_K(b)}(t) : t \in \pi^{-1}(v)\} = T_{h_K(b)}(t)$
for $t \in R_1$ so as $t \in \pi^{-1}(v)$. Further,
$I_{h_{\pi(K)}(b)}(0_2) = I_{h_{\pi(K)}(b)}(v) \Rightarrow I_{h_K(b)}(0_1) = \min\{I_{h_K(b)}(t) : t \in \pi^{-1}(v)\} = I_{h_K(b)}(t)$
for $t \in R_1$ so as $t \in \pi^{-1}(v)$.

Identical picture is drawn for F and thus $t \in K_0$ i.e., $\pi(t) \in \pi(K_0) \Rightarrow v = \pi(u) \in \pi(K_0)$. Therefore $(\pi(K))_0 \subseteq \pi(K_0)$. Then $\pi(K_0) = (\pi(K))_0$ using (i).

$$\begin{aligned}
\text{(iii)} \quad & u \in \pi^{-1}(L_0) \subset R_1 \\
\Leftrightarrow \quad & T_{h_L(b)}[\pi(u)] = T_{h_L(b)}(0_2) = T_{h_L(b)}[\pi(0_1)], I_{h_L(b)}[\pi(u)] = I_{h_L(b)}(0_2) = I_{h_L(b)}[\pi(0_1)] \text{ and} \\
& F_{h_L(b)}[\pi(u)] = F_{h_L(b)}(0_2) = F_{h_L(b)}[\pi(0_1)]; \\
\Leftrightarrow \quad & T_{h_{\pi^{-1}(L)}(b)}(u) = T_{h_{\pi^{-1}(L)}(b)}(0_1), I_{h_{\pi^{-1}(L)}(b)}(u) = I_{h_{\pi^{-1}(L)}(b)}(0_1), F_{h_{\pi^{-1}(L)}(b)}(u) = F_{h_{\pi^{-1}(L)}(b)}(0_1); \\
\Leftrightarrow \quad & u \in (\pi^{-1}(L))_0
\end{aligned}$$

Therefore, $\pi^{-1}(L_0) = (\pi^{-1}(L))_0$.

5.6 Definition

Take a classical function $\pi : R_1 \longrightarrow R_2$ and an Nss $K(u) = \{(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u)) : b \in E\}$, $u \in R_1$. Then K is said to be π- invariant if $\pi(u) = \pi(v) \Rightarrow K(u) = K(v)$ for $u, v \in R_1$. $K(u) = K(v)$ hold if $T_{h_K(b)}(u) = T_{h_K(b)}(v), I_{h_K(b)}(u) = I_{h_K(b)}(v), F_{h_K(b)}(u) = F_{h_K(b)}(v)$, $\forall b \in E$.

5.7 Theorem

Let $\pi : R_1 \longrightarrow R_2$ be an epimorphism and K be a π- invariant NSI on (R_1, E). Then the followings hold.
(i) If K attains sup property, then $(\pi(K))_0$ is a crisp prime ideal of R_2 when K_0 is a prime ideal of R_1.
(ii) If $K(R_1)$ is finite and K_0 is prime ideal of R_1, then $\pi(K_0)$ is so of R_2 and $\pi(K_0) = (\pi(K))_0$.
(iii) If K is an NSPI over (R_1, E), then $\pi(K)$ is also an NSPI over (R_2, E).

Proof. (i) By Theorem [5.5], $\pi(K_0) = (\pi(K))_0$ obviously. Let $y, z \in R_2$ such that $yz \in \pi(K_0) = (\pi(K))_0$. Then there exists $u, v \in R_1$ so as $\pi(u) = y, \pi(v) = z$ and $\pi(uv) = \pi(u)\pi(v) = yz \in (\pi(K))_0$. Then $\forall b \in E$,

$$\begin{aligned}
& T_{h_{\pi(K)}(b)}[\pi(uv)] = T_{h_{\pi(K)}(b)}(0_2) \Rightarrow \max\{T_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = T_{h_K(b)}(0_1), \\
& I_{h_{\pi(K)}(b)}[\pi(uv)] = I_{h_{\pi(K)}(b)}(0_2) \Rightarrow \min\{I_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = I_{h_K(b)}(0_1), \\
& F_{h_{\pi(K)}(b)}[\pi(uv)] = F_{h_{\pi(K)}(b)}(0_2) \Rightarrow \min\{F_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = F_{h_K(b)}(0_1).
\end{aligned}$$

For $w \in \pi^{-1}(yz)$ i.e., for $\pi(w) = yz = \pi(uv)$, sup property tells,
$T_{h_K(b)}(w) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(w) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(w) = F_{h_K(b)}(0_1)$.
But as K is π-invariant, so $K(w) = K(uv)$. Then $\forall b \in E$,
$T_{h_K(b)}(uv) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(uv) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(uv) = F_{h_K(b)}(0_1)$.
Therefore, $uv \in K_0$. As K_0 is a crisp prime ideal of R_1, so $u \in K_0$ or $v \in K_0$. It refers $\pi(u) \in \pi(K_0)$ or $\pi(v) \in \pi(K_0)$. This furnishes the proof.

(ii) Combining the 1st part and Theorem [5.5], the proof is onward.

(iii) By Proposition [5.2](i), $\pi(K)$ is an NSI over (R_2, E). Since K is an NSPI over (R, E), then $|h_K(b)| = 2$, $[h_K(b)](0_1) = (1, 0, 0)$, $\forall b \in E$ and using Theorems [4.4, 4.5, 4.7], K_0 is a prime ideal. But $[h_{\pi(K)}(b)](0_2) = [h_K(b)](0_1) = (1, 0, 0)$, $\forall b \in E$ and by 1st part, $(\pi(K))_0$ is a prime ideal of R_2. As $|h_K(b)| = 2$, $\exists\, u \in R_1$ so as $[h_K(b)](u) = (p_1, p_2, p_3)$ for $b \in E$. Then,

$$\begin{aligned} T_{h_{\pi(K)}(b)}(\pi(u)) &= \max\{T_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_1 \\ I_{h_{\pi(K)}(b)}(\pi(u)) &= \min\{I_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_2 \\ F_{h_{\pi(K)}(b)}(\pi(u)) &= \min\{F_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_3 \end{aligned}$$

So, $[h_{\pi(K)}(b)](\pi(u)) = (p_1, p_2, p_3) = [h_K(b)](u)$ for $b \in E$. Then $[h_K(b)](R_1) = [h_{\pi(K)}(b)](R_2)$ as π is epimorphism and u is arbitrary. Now consider two NSIs L, M over (R_2, E) such that $LoM \subseteq \pi(K)$ but $L \not\subseteq \pi(K)$ and $M \not\subseteq \pi(K)$. Then for all $y, z \in R_2$,

$T_{h_L(b)}(y) > T_{h_{\pi(K)}(b)}(y),\ I_{h_L(b)}(y) < I_{h_{\pi(K)}(b)}(y),\ F_{h_L(b)}(y) < F_{h_{\pi(K)}(b)}(y)$ and
$T_{h_M(b)}(z) > T_{h_{\pi(K)}(b)}(z),\ I_{h_M(b)}(z) < I_{h_{\pi(K)}(b)}(z),\ F_{h_M(b)}(z) < F_{h_{\pi(K)}(b)}(z)$.

For $y, z \in R_2 - (\pi(K))_0$, consider $T_{h_{\pi(K)}(b)}(y) = T_{h_{\pi(K)}(b)}(z) = p_1$, $I_{h_{\pi(K)}(b)}(y) = I_{h_{\pi(K)}(b)}(z) = p_2$ and $F_{h_{\pi(K)}(b)}(y) = F_{h_{\pi(K)}(b)}(z) = p_3$. Then,

$T_{h_L(b)}(y) > p_1,\ I_{h_L(b)}(y) < p_2,\ F_{h_L(b)}(y) < p_3$ and $T_{h_M(b)}(z) > p_1,\ I_{h_M(b)}(z) < p_2,\ F_{h_M(b)}(z) < p_3$.

Clearly, $yz \notin (\pi(K))_0$ as $y, z \notin (\pi(K))_0$, a prime ideal of R_2.

Then, $T_{h_{\pi(K)}(b)}(yz) = p_1,\ I_{h_{\pi(K)}(b)}(yz) = p_2,\ F_{h_{\pi(K)}(b)}(yz) = p_3$.

Now, $p_1 = T_{h_{\pi(K)}(b)}(yz) \geq T_{h_{LoM}(b)}(yz) = T_{h_L(b)}(y) \triangle T_{h_M(b)}(z) > p_1 \triangle p_1 = p_1$

The opposition $p_1 > p_1$ ensures $L \subseteq \pi(K), M \subseteq \pi(K)$ and this furnishes the 1st part.

5.8 Theorem

Let Q be an NSI over (R_2, E) and π is onto homomorphism. Then,
(i) $(\pi^{-1}(Q))_0$ is a crisp prime ideal on R_1 when Q_0 is so over R_2.
(ii) $\pi^{-1}(Q)$ is NSPI on (R_1, E) when Q is an NSPI over (R_2, E).

Proof. (i) We have by Theorem [5.5], $\pi^{-1}(Q_0) = (\pi^{-1}(Q))_0$. Let $u, v \in R_1$ so as $uv \in \pi^{-1}(Q_0)$. Then $\pi(uv) = \pi(u)\pi(v) \in Q_0$. Again $\pi(u) \in Q_0$ or $\pi(v) \in Q_0$ as Q_0 is a prime ideal.

$\pi(u) \in Q_0 \Rightarrow T_{h_Q(b)}[\pi(u)] = T_{h_Q(b)}(0_2) \Rightarrow T_{h_{\pi^{-1}(Q)}(b)}(u) = T_{h_{\pi^{-1}(Q)}(b)}(0_1) \Rightarrow u \in (\pi^{-1}(Q))_0$.

Identically, $v \in (\pi^{-1}(Q))_0$ when $\pi(v) \in Q_0$. Therefore, $uv \in (\pi^{-1}(Q))_0$ refers $u \in (\pi^{-1}(Q))_0$ or $v \in (\pi^{-1}(Q))_0$. Hence, the 1st part follows.

(ii) By Theorem [5.2], $\pi^{-1}(Q)$ is an NSI over (R_1, E) and by Theorem [5.3], $\pi^{-1}(Q)(0_1) = Q(0_2)$. Also since Q is an NSPI over (R_2, E), then $|h_Q(b)| = 2$, $[h_Q(b)](0_2) = (1, 0, 0)$ and Q_0 is a crisp prime ideal of R_2 respectively by Theorem [4.4], Theorem [4.5] and Theorem [4.7]. Then, by 1st result, $(\pi^{-1}(Q))_0$ is a crisp prime ideal of R_1 and $[h_{\pi^{-1}(Q)}(b)](0_1) = (1, 0, 0)$. Construct $[h_Q(b)](R_2) = \{(1, 0, 0) \cup (q_1, q_2, q_3)\}$ for a fixed $b \in E$ with $(1, 0, 0) \neq (q_1, q_2, q_3)$. Let $[h_Q(b)](v) = (q_1, q_2, q_3)$ for $v \in R_2$. Then $\exists\, u \in R_1$ for which $\pi(u) = v$ and $[h_{\pi^{-1}(Q)}(b)](u) = [h_Q(b)](v) = (q_1, q_2, q_3)$. Therefore, $[\pi^{-1}(Q)](R_1) = Q(R_2)$ as $b \in E$ is arbitrary and π is epimorphism.
For two NSIs A, B on (R_1, E), let $AoB \subseteq \pi^{-1}(Q)$ with $A \not\subseteq \pi^{-1}(Q)$ and $B \not\subseteq \pi^{-1}(Q)$. Then $\forall\, u, v \in R_1$,

$T_{h_A(b)}(u) > T_{h_{\pi^{-1}(Q)}(b)}(u),\ I_{h_A(b)}(u) < I_{h_{\pi^{-1}(Q)}(b)}(u),\ F_{h_A(b)}(u) < F_{h_{\pi^{-1}(Q)}(b)}(u)$ and
$T_{h_B(b)}(v) > T_{h_{\pi^{-1}(Q)}(b)}(v),\ I_{h_B(b)}(v) < I_{h_{\pi^{-1}(Q)}(b)}(v),\ F_{h_B(b)}(v) < F_{h_{\pi^{-1}(Q)}(b)}(v)$.

For $u, v \in R_1 - (\pi^{-1}(Q))_0$, let $T_{h_{\pi^{-1}(Q)}(b)}(u) = T_{h_{\pi^{-1}(Q)}(b)}(v) = q_1$, $I_{h_{\pi^{-1}(Q)}(b)}(u) = I_{h_{\pi^{-1}(Q)}(b)}(v) = q_2$ and $F_{h_{\pi^{-1}(Q)}(b)}(u) = F_{h_{\pi^{-1}(Q)}(b)}(v) = q_3$. Then,

$T_{h_A(b)}(u) > q_1,\ I_{h_A(b)}(u) < q_2,\ F_{h_A(b)}(u) < q_3$ and $T_{h_B(b)}(v) > q_1,\ I_{h_B(b)}(v) < q_2,\ F_{h_B(b)}(v) < q_3$.

It indicates $uv \notin (\pi^{-1}(Q))_0$ as $u, v \notin (\pi^{-1}(Q))_0$, a prime ideal of R_1.
Then, $T_{h_{\pi^{-1}(Q)}(b)}(uv) = q_1$, $I_{h_{\pi^{-1}(Q)}(b)}(uv) = q_2$, $F_{h_{\pi^{-1}(Q)}(b)}(uv) = q_3$ and
so, $q_1 = T_{h_{\pi^{-1}(Q)}(b)}(uv) \geq T_{h_{AoB}(b)}(uv) = T_{h_A(b)}(u) \triangle T_{h_B(b)}(v) > q_1 \triangle q_1 = q_1$
The opposition $q_1 > q_1$ ensures $A \subseteq \pi^{-1}(Q), B \subseteq \pi^{-1}(Q)$ and this leads the 2nd part.

6 Conclusion

This effort is made to extend the notion of ideal and prime ideal of a classical ring in the parlance of N_S theory and soft set theory. Their structural behaviours are innovated by developing a number of properties and theorems. Using neutrosophic cut set, it is shown how an Nss will be an NSI or NSPI. The nature of homomorphic image of NSI and NSPI are also studied in different aspect. This theoretical attempt will help to cultivate the N_S theory in several mode in future, we think.

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