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Bipolar fuzzy soft mappings with application to bipolar disorders

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Bipolar disorder is a neurological disorder that consists of two main factors, i.e. mania and depression. There are two main drawbacks in clinical diagnosis of the bipolar disorder. First, bipolar disorder is mostly wrongly diagnosed as unipolar depression in clinical diagnosis. This is, because in clinical diagnosis, the first factor is often neglected due to its approach toward positivity. Consequently, the element of bipolarity vanishes and the disease becomes worse. Second, the types of bipolar disorder are mostly misdiagnosed due to similar symptoms. To overcome these problems, the bipolar fuzzy soft set (BFS-set) and bipolar fuzzy soft mappings (BFS-mappings) are useful to tackle bipolarity and to construct a strong mathematical modeling process to diagnose this disease correctly. This technique is extensive but simple as compared to existing medical diagnosis methods. A chart (relation between different types and symptoms of bipolar disorder) is provided which contains different ranges over the interval $[-1, 1]$. A process of BFS-mappings is also provided to obtain correct diagnosis and to suggest the best treatment. Lastly, a generalized BFS-mapping is introduced which is helpful to keep patient's improvement record. The case study indicates the reliability, efficiency and capability of the achieved theoretical results. Further, it reveals that the connection of soft set with bipolar fuzzy set is fruitful to construct a connection between symptoms which minimize the complexity of the case study.

Keywords: BFS-set; BFS-mappings; operations and properties of BFS-mappings; Bipolarity; Bipolar I disorder; Bipolar II disorder; Cyclothymia.

Mathematics Subject Classification 2010: 03B52, 03E72

1. Introduction

In daily life problems, we encounter many circumstances, which involve ambiguities and uncertainties due to insufficient knowledge, meager information, incompatible data, inconsistent and rare information. To overcome such kind of problems, Zadeh

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[42] brought out the idea of fuzzy sets in 1965. Fuzzy set theory has been auspiciously adapted in the field of medical sciences [10, 16, 24]. The theory has been extended with many hybrid structures. The researchers have been published a good number of research papers on extensions of fuzzy sets. As an extension of fuzzy theory, many set theories have been developed, including, interval valued set theory [43], intuitionistic fuzzy set theory [5], bipolar fuzzy set theory [44], neutrosophic set theory [35], m-polar fuzzy set theory [7], etc. All these theories have been developed according to the necessity of handling specific type of data and its useability in different suitable domains. Riaz *et al.* worked on different extensions of fuzzy set theory and their applications [25–30, 32, 34].

In 1999, Molodstov [21] brought out new abstraction, namely, soft set (a mathematical tool which handle ambiguities and imprecisions in parametric manners). The role of parameters is an essential part in analyzing and scrutinizing information or making a decision. Recently, the theory has been affirmed to be an appropriate parametrization mechanism. In short, theory of soft set fascinatingly overcame many problems raised on implementing the existing theories. The theory got great attention of the scientists and researchers because of its miscellaneous applications. In last few years, the domain of soft set has been popular among researchers. Maji *et al.* [18, 19] introduced some useful basic operations on soft sets as well as its implementation in decision support system. Ali *et al.* [1, 2] presented some new operation on soft set and algebraic structures of soft sets.

For data analyzing of many types, bipolarity of knowledge is a vital part to be considered while developing a mathematical framework for most of the situations. Bipolarity indicates the positive and negative aspects of a particular problem. The concept behind the bipolarity is that a huge range of human decisions analysis is involved in bipolar subjective thoughts. For illustration, happiness and grief, sweetness and sourness, effects and side effects are two different aspects of decision analysis. The equilibrium and mutual coexistence of these two aspects are treated as a key for balanced social environment. The fuzzy set and soft set theories are not enough to handle such type of bipolarity; for instance, a medicine which is not effective may not be having any side effect. Zhang [45] introduced the extension of fuzzy set with bipolarity, called, bipolar-valued fuzzy sets. Bipolar fuzzy set is suitable for information which involves property as well as its counter property. Lee [17] discussed some basic operations of bipolar-valued fuzzy set. Aslam *et al.* [4] brought out the abstraction of bipolar fuzzy soft set (BFS-set). They discussed the operations of BFS-sets and a problem of decision-making. In short, bipolar fuzzy set has been used in many mathematical domains [11, 12, 23, 37–41]. The concept of mapping is a useful tool to be considered while developing models for many problems. A mapping is an association between two or more domains under some specific rules. Different fields including, mathematics, computer sciences, chemistry, psychology and logics adapted concept of mapping under their specific criteria. In 2010, Majumdar and Samanta [20] introduced mappings on soft sets and their implementation in medical diagnosis. Kharal and Ahmad presented the idea of soft mappings [15] and fuzzy soft mappings [14]. Gunduz and Bayramov [9] also discussed continuous, open and closed fuzzy soft mappings. Bashir and Saleh [6] introduced mappings on intuitionistic fuzzy soft classes. Karataş and Akdağ [13] discussed continuous, open and closed intuitionistic fuzzy mappings.

According to world mental health survey (WHOs), Bipolar disorders have been classified as the disease with second major cause of incapability to perform daily task [3]. Since there does not exist any biological marker for this disorder, the character of clinical diagnosis of bipolar disorder is still indispensable. Unfortunately, statistics shows that only 20% of patients with bipolar disorder having a depressive episode are diagnozed with bipolar disorder within the first year of seeking treatment, since they look similar to the ones with unipolar depression [8]. People with bipolar disorder go through unexpected changes in mood. These changes include feeling very cheerful and hopeful which is represented by "highs" and then feeling very cheerless and hopeless which is represented by "lows". They often swing back to normal mood. The feelings of highs are known as mania, on the other hand, feelings of lows are known as depression. The secondary level of mania periods are called hypomania episodes and severe feelings of depression are called major depression. Bipolar disorder occurs in the ratio of 1 in every 100 adults. Bipolar disorder is common at the age of early 20s and rarely usual after the age of 40. It effects both male and female equally. The symptoms can vary from person to person, depending on the person's history of illness. The symptoms also vary from time to time. The bipolarity can exist in other diseases as well, but in bipolar disorder it is usual rather than anomaly. The chief motivation of this paper is to develop a mathematical model for stable bipolarity in bipolar disorder by considering domains of bipolarity, fuzziness and softness. Since there exist many models in fuzzy system which have been efficiently applied to medical diagnosis, among which bipolar fuzzy soft set is more suitable to diagnose bipolar disorder. There is no doubt that the methods presented by [20] and [15] are simple and useful, but they are not comprehensive enough to diagnoze such type of complicated diseases, resulting inaccuracy in the real world clinical diagnosis. The area of the brain linked with bipolar disorder can be seen in Fig. 1 Thus, we propose a novel BFS-mapping method to bipolar disorder diagnosis. By considering the features of bipolar disorder diagnosis, a comprehensive BFS-mapping method to bipolar disorder diagnosis is presented. The proposed method is dependent on different types of BFS-mappings and their relevant properties. The proposed method is more efficient as compared to the existing methods of medical diagnosis. The advantages of this method are given as follows: First, a chart of conditions and their intensity is provided, which is helpful to determine the severeness of disease. Second, the soft set is involved which is efficient enough to present the parameters or attribute to whole problem nicely. Third, BFS-mappings and BFS max–min composition are used to create relations between two different phenomena. Fourth, a range is provided for final diagnosis of three types of bipolar disorder and equilibrium state of mind. Fifth, the best treatment is also discussed in the method. Last, the generalized BFS-mapping is

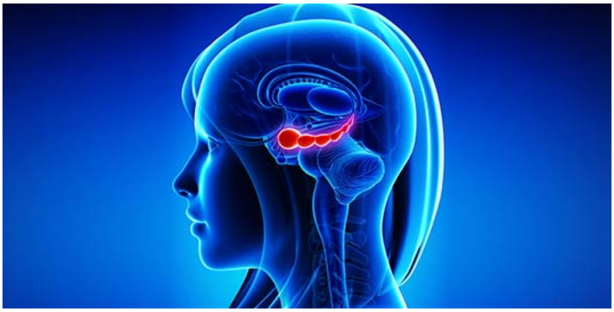


Fig. 1. Area of brain linked to bipolar disorder.
Source: University of Texas Health Science Center at Houston.

provided which is helpful to keep a record of improvement of each patient. The motivational work of this paper is listed as follows: (1) In theory, different types of mapping their operations and properties of a bipolar fuzzy soft set are discussed. (2) In methodology, a hybridization of fuzzy decision-making method to bipolar disorder diagnosis is proposed, which is helpful to diagnose types of bipolar disorder, best treatment and improvement record. The remainder of this paper is structured as follows. In Sec. 2, some basic definitions are presented. In Sec. 3, by studying the features of bipolar disorder diagnosis, the motivation to introduce the new method is presented. Second, some novel concepts and theories about bipolar fuzzy soft information are proposed. Different types of mappings their operations and properties are presented. Last, a hybrid method to bipolar disorder diagnosis is proposed based on BFS-mappings. In Sec. 4, the computative case is discussed. In Sec. 5, comparison and discussion are included to highlight the useability, reliability and necessity of the theoretical results obtained. The paper is concluded in Sec. 6.

2. Preliminaries

In this section, we review some basic definitions.

Definition 2.1 ([42]). Consider the universal set V and membership function $f : V \to [0,1]$. Then, V_f is a fuzzy set on V if each element $\xi \in V$ is associated with degree of membership, which is a real number in $[0,1]$ and it is denoted by f_ξ.

Definition 2.2 ([21]). Consider the universal set V and a set of decision variables $\mathcal{D}$. Let $\mathcal{A}_1 \subseteq \mathcal{D}$ and $\mathcal{K} : \mathcal{A}_1 \to \mathcal{P}(V)$ be the set-valued function, where $\mathcal{P}(V)$ is the power set of V. Then, $\mathcal{K}_{\mathcal{A}_1}$ or $(\mathcal{K}, \mathcal{A}_1)$ denotes a soft set on V.

Definition 2.3 ([45]). A bipolar fuzzy set on V is of the form

$$\mathcal{K} = \{(\xi, \delta^+_{\mathcal{K}}(\xi), \delta^-_{\mathcal{K}}(\xi)) : \text{for all } \xi \in V\},$$

where $\delta^+_{\mathcal{K}}(\xi)$ denotes the positive membership ranges over $[0,1]$ and $\delta^-_{\mathcal{K}}(\xi)$ denotes the negative membership ranges over $[-1,0]$.

Definition 2.4 ([4]). Let $\mathcal{A}_1 \subset \mathcal{D}$ and define a mapping $\mathcal{K} : \mathcal{A}_1 \to \mathcal{BF}(V)$, where $\mathcal{BF}(V)$ represents the family of all bipolar fuzzy subsets of V. Then, $\mathcal{K}_{\mathcal{A}_1}$ or $(\mathcal{K}, \mathcal{A}_1)$ is called a bipolar fuzzy set (BFS-set) on V. A BFS-set can be defined as

$$\mathcal{K}_{\mathcal{A}_1} = \{\mathcal{K}_p = (\xi, \delta_p^+(\xi), \delta_p^-(\xi)) : \text{ for all } \xi \in V \text{ and } p \in \mathcal{A}_1\}.$$

Example 2.5. Let $V = \{\xi_1, \xi_2, \xi_3\}$ be the set of three companies of home appliances and $\mathcal{D} = \{p_1, p_2, p_3\}$ be the set of decision variables related to their productivity, where

p_1: represents durability.
p_2: represents expensive.
p_3: represents economical.

Suppose that $\mathcal{A}_1 = \{p_1, p_3\} \subset \mathcal{D}$, now a BFS-set $\mathcal{K}_{\mathcal{A}_1}$ can be written as follows:

$$\mathcal{K}_{\mathcal{A}_1} = \begin{Bmatrix} \mathcal{K}_{p_1} = \{(\xi_1, 0.12, -0.53), (\xi_2, 0.31, -0.62), (\xi_3, 0.41, -0.23)\}, \\ \mathcal{K}_{p_3} = \{(\xi_1, 0.82, -0.11), (\xi_2, 0.33, -0.61), (\xi_3, 0.42, -0.32)\}. \end{Bmatrix}$$

Definition 2.6 ([4]). A BFS-set $\mathcal{K}_{\mathcal{D}}$ is called a null BFS-set, if $\delta_p^+(\xi) = 0$ and $\delta_p^-(\xi) = 0$, for each $p \in \mathcal{D}$ and $\xi \in V$ and we write it as $\phi_{\mathcal{D}}$.

Definition 2.7 ([4]). A BFS-set $\mathcal{K}_{\mathcal{D}}$ is called an absolute BFS-set, if $\delta_p^+(\xi) = 1$ and $\delta_p^-(\xi) = -1$, for each $p \in \mathcal{D}$ and $\xi \in V$ and we write it as $V_{\mathcal{D}}$.

Definition 2.8 ([4]). The complement of a BFS-set $\mathcal{K}_{\mathcal{A}_1}$ is represented by $(\mathcal{K}_{\mathcal{A}_1})^c$ and is defined by $(\mathcal{K}_{\mathcal{A}_1})^c = \{\mathcal{K}_p = (\xi, 1 - \delta_p^+(\xi), -1 - \delta_p^-(\xi)) : \xi \in V, p \in \mathcal{A}_1\}$.

Definition 2.9 ([4]). Consider two BFS-sets $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ on V. Then, $\mathcal{K}_{\mathcal{A}_1}$ is a BFS-subset of $\mathcal{K}_{\mathcal{A}_2}$, if

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$.
(ii) $\delta_{\mathcal{A}_1}^+(\xi) \leq \delta_{\mathcal{A}_2}^+(\xi)$, $\delta_{\mathcal{A}_1}^-(\xi) \geq \delta_{\mathcal{A}_2}^-(\xi)$.

Then we can write it as $\mathcal{K}_{\mathcal{A}_1} \subseteq \mathcal{K}_{\mathcal{A}_2}$.

Definition 2.10 ([4]). Let $\mathcal{K}_{\mathcal{A}_1}^1$ and $\mathcal{K}_{\mathcal{A}_2}^2 \,\tilde{\in}\, \mathcal{BF}(V_{\mathcal{D}})$. Then, intersection of $\mathcal{K}_{\mathcal{A}_1}^1$ and $\mathcal{K}_{\mathcal{A}_2}^2$ is a BFS-set $\mathcal{K}_{\mathcal{A}}$, where $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$, $\mathcal{K} : \mathcal{A} \to \mathcal{BF}(V)$ is a mapping defined by $\mathcal{K}_p = \mathcal{K}_p^1 \,\tilde{\cap}\, \mathcal{K}_p^2 \ \forall\, p \in \mathcal{A}$ and it is written as $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}_1}^1 \,\tilde{\cap}\, \mathcal{K}_{\mathcal{A}_2}^2$.

Definition 2.11 ([4]). Let $\mathcal{K}_{\mathcal{A}_1}^1$ and $\mathcal{K}_{\mathcal{A}_2}^2 \,\tilde{\in}\, \mathcal{BF}(V_{\mathcal{D}})$. The union of $\mathcal{K}_{\mathcal{A}_1}^1$ and $\mathcal{K}_{\mathcal{A}_2}^2$ is a BFS-set $\mathcal{K}_{\mathcal{A}}$, where $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{K} : \mathcal{A} \to \mathcal{BF}(V)$ is a mapping defined as

$$\begin{aligned} \mathcal{K}_p &= \mathcal{K}_p^1 \quad \text{if } p \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ &= \mathcal{K}_p^2 \quad \text{if } p \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ &= \mathcal{K}_p^1 \,\tilde{\cup}\, \mathcal{K}_p^2 \quad \text{if } p \in \mathcal{A}_1 \cap \mathcal{A}_2, \end{aligned}$$

and it is written as $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}_1}^1 \,\tilde{\cup}\, \mathcal{K}_{\mathcal{A}_2}^2$.

3. Propounded Technique

In this section, first, by considering the features of the bipolar disorder diagnosis, the motivation of the presented technique is proposed. Second, relevant theories of method to bipolar disorder diagnosis, including different types of mappings, useful operations and properties under the bipolar fuzzy soft environment are propounded. Last, depending on the analysis, a BFS-mapping method to bipolar disorder diagnosis, best treatment and improvement record is presented.

3.1. *Investigation of bipolar disorder diagnosis problem*

In clinical psychopharmacology and biomedical engineering, the modeling of mathematics and diagnostic study of psychological disorders have great importance [36]. To find an appropriate mathematical framework, the features of the bipolar disorder diagnosis are studied, and a proper mathematical framework to tackle it is addressed as follows: As said earlier, bipolarity is the main feature of bipolar disorder diagnosis. Similarly, vagueness or fuzziness of bipolar disorder diagnosis is another feature. Thus, it is essential to find a proper fuzzy framework, which can tackle uncertainties as well as bipolarity in diagnosis of bipolar disorder. The following table provides information on which set is more suitable to capture bipolarity in bipolar disorder diagnosis process. If we look at the bipolarity from semantics point of view, we obtain Table 1 which is a semantical comparison of different set theories. Although in pattern, bipolar fuzzy set is equivalent to neutrosophic set [22]. Whereas, semantically, bipolar fuzzy set [46] represents equilibrium and neutrosophic set which provided a usual impartiality. No doubt the neutrosophic sets have been auspiciously applied to medical diagnosis. By this comparison, we conclude that bipolar fuzzy soft set is definitely a suitable combination to set a framework for bipolar diagnostic.

Zhang [44] introduced an order relation defined as $\mathcal{K}_1 \leq \mathcal{K}_2 \Leftrightarrow \delta_1^+ \leq \delta_2^+$ and $\delta_1^- \geq \delta_2^+$. The reason for the success of traditional Chinese medicine is equilibrium

Table 1. Semantic comparison of different theories.

Set theories	Advantages	Semantic drawbacks
Fuzzy sets	Provided information about specific property	Does not consider counter property simultaneously
Interval-valued sets	Truth based	Does not contain bipolar stability
Intuitionistic fuzzy sets	Perfectly identify uncertainty in single property	Does not inform about counter property
Neutrosophic sets	indeterminacy of specific property	Does not represent equilibrium
M-polar fuzzy set	mathematically 2-polar is equivalent to bipolar	but not suitable in human thinking containing bipolarity
Bipolar fuzzy set	Represents bipolarity	Does not create connections among symptoms

or stability principle. In Chinese medicine, symptoms are considered to be the loss of stability of two sides. Thus, we use Zhang's equilibrium concept and order relation in our paper. Because bipolar disorders are imbalance of mood and disturbed harmony of two different sides of a stability, we use multiple symptoms and their connections as parameters of soft set to diagnose correctly. There are three main types of bipolar disorders.

Bipolar I Disorder

In this type, a person was had one or more episodes of high or mania, which has a duration of one week or more. A person may encounter with only mania episodes, but mostly some people also encounter with episodes of depression as well.

Bipolar II Disorder

In this type, a person has had one or more periods of major depression as well as one or more episodes of hypomania (less severe form of mania). In this type, a person never encounters with a mania episode. This type is not light form of type I, but is diagnosed separately. Note that mania episodes of type I can be severe and serious which can cause a serious loss, whereas long-term depression periods of type II can also be dangerous and may lead to suicidal attempt.

Cyclothymia

In this type, swings of mood as described in previous types are not as much severe as in types I and II, but can last for a long time, which can cause for developing it into type I or type II. A person may have less severe episodes of hypomania and some periods of depression (less severe than major depression). The comparison of normal brain with disordered brains can be seen in Fig. 2.

3.2. *Proposed theories about bipolar fuzzy soft set*

In this section, the BFS- mappings and inverse BFS-mappings are defined. Different types of BFS-mappings, including, BFS-surjective, BFS-injective and BFS-bijective mappings are also presented in this section. Some basic operations and properties of

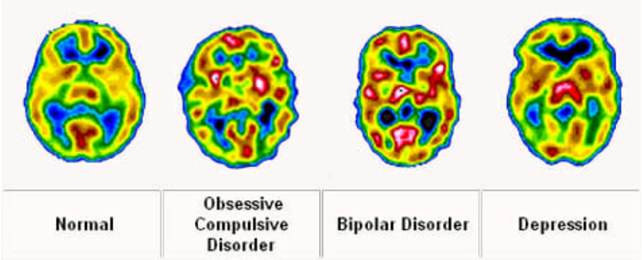


Fig. 2. Comparison of normal brain with bipolar disorder brain.
Source: Brain Matters Imaging Centers (2007).

BFS-mappings are included. BFS max–min composition is defined in this section. These BFS-mappings are useful throughout this technique.

Definition 3.1. Consider a universal set V and a set of decision variables $\mathcal{D}$, then $\mathcal{BF}(V_{\mathcal{D}})$ represents the family of all BFS-sets on V with decision variables from $\mathcal{D}$ and it is called class of BFS-sets.

Definition 3.2. Consider two BFS-classes $\mathcal{BF}(V_{\mathcal{D}})$ and $\mathcal{BF}(W_{\mathcal{D}'})$ on V and W with corresponding set of decision variables $\mathcal{D}$ and $\mathcal{D}'$, respectively. Suppose that $\mathfrak{u} : V \to W$ and $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ be two mappings. Then, we define a mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ as, if $\mathcal{K}_{\mathcal{A}}$ is a BFS-set in $\mathcal{BF}(V_{\mathcal{D}})$ for $\mathcal{A} \subseteq \mathcal{D}$ then we have a BFS-set $\mathfrak{f}(\mathcal{K}_{\mathcal{A}})$ in $\mathcal{BF}(W_{\mathcal{D}'})$, having form

$$\mathfrak{f}(\mathcal{K}_{\mathcal{A}}) = \{\mathcal{K}_{p'} = (\varsigma, \delta^{(+)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}})}(p')(\varsigma), \delta^{(-)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}})}(p')(\varsigma)) : \text{for } \varsigma \in W \text{ and } p' \in \mathfrak{g}(\mathcal{D}) \subseteq \mathcal{D}'\},$$

where

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}})}(p')(\varsigma) = \begin{cases} \max\limits_{\xi \in \mathfrak{u}^{-1}(\varsigma)} \left(\max\limits_{p \in \mathfrak{g}^{-1}(p') \cap \mathcal{A}} \delta^{(+)}_{\mathcal{K}_p} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap \mathcal{A} \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}})}(p')(\varsigma) = \begin{cases} \min\limits_{\xi \in \mathfrak{u}^{-1}(\varsigma)} \left(\min\limits_{p \in \mathfrak{g}^{-1}(p') \cap \mathcal{A}} \delta^{(-)}_{\mathcal{K}_p} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap \mathcal{A} \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

On combining (1) and (2), we get

$$\mathfrak{f}(\mathcal{K}_{\mathcal{A}})(p')(\varsigma) = \begin{cases} \widetilde{\bigcup\limits_{\xi \in \mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup\limits_{p \in \mathfrak{g}^{-1}(p') \cap \mathcal{A}}} \mathcal{K}_p \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \ \mathfrak{g}^{-1}(p') \cap \mathcal{A} \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

Then, $\mathfrak{f}(\mathcal{K}_{\mathcal{A}})$ is called BFS-image of BFS-set $\mathcal{K}_{\mathcal{A}}$ under $\mathfrak{f}$, which can be obtained by using (3).

Definition 3.3. Suppose that $\mathfrak{u} : V \to W$ and $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ be two mappings. Then, we define a mapping $\mathfrak{f} : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ as follows: if $\mathcal{K}_{\mathcal{A}'}$ is a BFS-set in $\mathcal{BF}(W_{\mathcal{D}'})$ for $\mathcal{A}' \subseteq \mathcal{D}'$, then we have a BFS-set $\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})$ in $\mathcal{BF}(V_{\mathcal{D}})$, having form

$$\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'}) = \{\mathcal{K}_p = (\xi, \delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})}(p)(\xi), \delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})}(p)(\xi)) : \\ \text{for } \xi \in V \text{ and } p \in \mathfrak{g}^{-1}(\mathcal{A}') \subseteq \mathcal{D}\},$$

where

$$\delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})}(p)(\xi) = \begin{cases} \delta^{(+)}_{\mathcal{K}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}', \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

$$\delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})}(p)(\xi) = \begin{cases} \delta^{(-)}_{\mathcal{K}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}', \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

On combining (4) and (5), we have

$$\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})(p)(\xi) = \begin{cases} \mathcal{K}_{(\mathfrak{g}(p))}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}', \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Then, $\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})$ is called BFS inverse image of $\mathcal{K}_{\mathcal{A}'}$, which can be obtained by using (6).

Example 3.4. Consider two universal sets $V = \{\xi_1, \xi_2, \xi_3\}$ and $W = \{\varsigma_1, \varsigma_2, \varsigma_3\}$. Let $\mathcal{D} = \{p_1, p_2, p_3\}$ and $\mathcal{D}' = \{p'_1, p'_2, p'_3\}$ be corresponding sets of decision variables, respectively. Let us consider two classes of BFS-sets $\mathcal{BF}(V_{\mathcal{D}})$ and $\mathcal{BF}(W_{\mathcal{D}'})$. Suppose that the mappings $\mathfrak{u} : V \to W$ and $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ are defined as

$$\mathfrak{u}(\xi_1) = \varsigma_1, \quad \mathfrak{u}(\xi_2) = \varsigma_2, \quad \mathfrak{u}(\xi_3) = \varsigma_3,$$
$$\mathfrak{g}(p_1) = p'_1, \quad \mathfrak{g}(p_2) = p'_2, \quad \mathfrak{g}(p_3) = p'_2.$$

Let $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{A}'}$ be two BFS-sets in $\mathcal{BF}(V_{\mathcal{D}})$ and $\mathcal{BF}(W_{\mathcal{D}'})$, respectively.

$$\mathcal{K}_{\mathcal{A}} = \begin{Bmatrix} \mathcal{K}_{p_1} = \{(\xi_1, 0.33, -0.54), (\xi_2, 0.53, -0.64), (\xi_3, 0.63, -0.44)\}, \\ \mathcal{K}_{p_2} = \{(\xi_1, 0.53, -0.44), (\xi_2, 0.63, -0.54), (\xi_3, 0.73, -0.54)\}, \\ \mathcal{K}_{p_3} = \{(\xi_1, 0.63, -0.24), (\xi_2, 0.23, -0.14), (\xi_3, 0.43, -0.24)\}. \end{Bmatrix}$$

$$\mathcal{K}_{\mathcal{A}'} = \begin{Bmatrix} \mathcal{K}_{p'_1} = \{(\varsigma_1, 0.51, -0.33), (\varsigma_2, 0.32, -0.21), (\varsigma_3, 0.22, -0.31)\}, \\ \mathcal{K}_{p'_2} = \{(\varsigma_1, 0.32, -0.43), (\varsigma_2, 0.51, -0.33), (\varsigma_3, 0.71, -0.51)\}, \\ \mathcal{K}_{p'_3} = \{(\varsigma_1, 0.31, -0.53), (\varsigma_2, 0.21, -0.43), (\varsigma_3, 0.71, -0.41)\}. \end{Bmatrix}$$

Now, we obtain the image of $\mathcal{K}_{\mathcal{A}}$ under the mapping $\mathfrak{f} : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ as follows:

$$\begin{aligned} \mathfrak{f}(\mathcal{K}_{\mathcal{A}})(p'{}_1)(\varsigma_1) &= \widetilde{\bigcup_{\xi \in \mathfrak{u}^{-1}(\varsigma_1)}} \left(\widetilde{\bigcup_{p \in \mathfrak{g}^{-1}(p'_1) \cap \mathcal{A}}} \mathcal{K}_p \right)(\xi) \\ &= \widetilde{\bigcup_{\xi \in \{\xi_1\}}} \left(\widetilde{\bigcup_{p \in \{p_1\}}} \mathcal{K}_p \right)(\xi) \end{aligned}$$

$$= \widetilde{\bigcup_{\xi\in\{\xi_1\}}} (\{(\xi_1, 0.33, -0.54), (\xi_2, 0.53, -0.64), (\xi_3, 0.63, -0.44)\})(\xi)$$

$$= \widetilde{\bigcup}(0.33, -0.54) = (0.33, -0.54);$$

similarly,

$$\mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_1)(\varsigma_2) = (0.53, -0.64), \quad \mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_1)(\varsigma_3) = (0.63, -0.44);$$

by similar calculation we obtain

$$\begin{aligned}
\mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_2)(\varsigma_1) &= \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma_1)}} \left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p'_2)\cap\mathcal{A}}} \mathcal{K}_p \right)(\xi) \\
&= \widetilde{\bigcup_{\xi\in\{\xi_1\}}} \left(\widetilde{\bigcup_{p\in\{p_2,p_3\}}} \mathcal{K}_p \right)(\xi) \\
&= \widetilde{\bigcup_{\xi\in\{\xi_1\}}} (\{(\xi_1, 0.63, -0.24), (\xi_2, 0.63, -0.14), (\xi_3, 0.73, -0.24)\})(\xi) \\
&= \widetilde{\bigcup}(0.63, -0.24) = (0.63, -0.24);
\end{aligned}$$

similarly,

$$\mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_2)(\varsigma_2) = (0.63, -0.14), \mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_2)(\varsigma_3) = (0.73, -0.24),$$

and

$$\mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_3)(\varsigma_1) = (0, 0), \mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_3)(\varsigma_2) = (0, 0), \mathfrak{f}(\mathcal{K}_\mathcal{A})({p'}_3)(\varsigma_3) = (0, 0).$$

Hence, we obtain the BFS-image of $\mathcal{K}_\mathcal{A}$ under the given mapping as

$$\mathfrak{f}(\mathcal{K}_\mathcal{A}) = \left\{ \begin{matrix} \mathcal{K}_{p'_1} = \{(\varsigma_1, 0.33, -0.54), (\varsigma_2, 0.53, -0.64), (\varsigma_3, 0.63, -0.44)\}, \\ \mathcal{K}_{p'_2} = \{(\varsigma_1, 0.63, -0.24), (\varsigma_2, 0.63, -0.14), (\varsigma_3, 0.73, -0.24)\}, \\ \mathcal{K}_{p'_3} = \{(\varsigma_1, 0, 0), (\varsigma_2, 0, 0), (\varsigma_3, 0, 0)\}. \end{matrix} \right\}$$

Now we calculate the image of $\mathcal{K}_{\mathcal{A}'}$ under inverse mapping as follows:

$$\begin{aligned}
\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})(p_1)(\xi_1) &= \mathcal{K}_{(\mathfrak{g}(p_1))}(\mathfrak{u}(\xi_1)) \\
&= \mathcal{K}_{(p'_1)}(\varsigma_1) \\
&= \{(\varsigma_1, 0.51, -0.33), (\varsigma_2, 0.32, -0.21), (\varsigma_3, 0.22, -0.31)\}(\varsigma_1) \\
&= (0.51, -0.33), \\
\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})(p_1)(\xi_2) &= \mathcal{K}_{(\mathfrak{g}(p_1))}(\mathfrak{u}(\xi_2)) \\
&= \mathcal{K}_{(p'_1)}(\varsigma_2)
\end{aligned}$$

$$
\begin{aligned}
&= \{(\varsigma_1, 0.51, -0.33), (\varsigma_2, 0.32, -0.21), (\varsigma_3, 0.22, -0.31)\}(\varsigma_2) \\
&= (0.32, -0.21), \\
\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'})(p_1)(\xi_3) &= \mathcal{K}_{(\mathfrak{g}(p_1))}(\mathfrak{u}(\xi_3)) \\
&= \mathcal{K}_{(p_1')}(\varsigma_3) \\
&= \{(\varsigma_1, 0.51, -0.33), (\varsigma_2, 0.32, -0.21), (\varsigma_3, 0.22, -0.31)\}(\varsigma_3) \\
&= (0.22, -0.31).
\end{aligned}
$$

Similar calculation gives

$$
\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}'}) = \left\{\begin{aligned}
\mathcal{K}_{p_1} &= \{(\xi_1, 0.51, -0.33), (\xi_2, 0.32, -0.21), (\xi_3, 0.22, -0.31)\}, \\
\mathcal{K}_{p_2} &= \{(\xi_1, 0.32, -0.43), (\xi_2, 0.51, -0.33), (\xi_3, 0.71, -0.51)\}, \\
\mathcal{K}_{p_3} &= \{(\xi_1, 0.32, -0.43), (\xi_2, 0.51, -0.33), (\xi_3, 0.71, -0.51)\}.
\end{aligned}\right\}
$$

Definition 3.5.

(i) A BFS-mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g})$ is said to be injective if both $\mathfrak{u}$ and $\mathfrak{g}$ are injective mappings.

(ii) A BFS-mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g})$ is said to be surjective if both $\mathfrak{u}$ and $\mathfrak{g}$ are surjective mappings.

(iii) A BFS-mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g})$ is said to be bijective if both $\mathfrak{u}$ and $\mathfrak{g}$ are bijective (injective as well as surjective) mappings.

Example 3.6. Let $V = \{\xi_1, \xi_2, \xi_3\}$, $W = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ and $\mathcal{D} = \{p_1, p_2, p_3\}$ $\mathcal{D}' = \{p_1', p_2', p_3'\}$. Let us consider two classes of BFS-sets $\mathcal{BF}(V_{\mathcal{D}})$ and $\mathcal{BF}(W_{\mathcal{D}'})$. Suppose that the mappings $\mathfrak{u} : V \to W$ and $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ are defined as

$$
\mathfrak{u}(\xi_1) = \varsigma_3, \quad \mathfrak{u}(\xi_2) = \varsigma_1, \quad \mathfrak{u}(\xi_3) = \varsigma_2,
$$

$$
\mathfrak{g}(p_1) = p_1', \quad \mathfrak{g}(p_2) = p_3', \quad \mathfrak{g}(p_3) = p_2'.
$$

Let $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{A}'}$ be two BFS-sets in $\mathcal{BF}(V_{\mathcal{D}})$ and $\mathcal{BF}(W_{\mathcal{D}'})$, respectively.

$$
\mathcal{K}_{\mathcal{A}} = \left\{\begin{aligned}
\mathcal{K}_{p_1} &= \{(\xi_1, 0.33, -0.54), \quad (\xi_2, 0.53, -0.64), \quad (\xi_3, 0.63, -0.44)\}, \\
\mathcal{K}_{p_2} &= \{(\xi_1, 0.53, -0.44), \quad (\xi_2, 0.63, -0.54), \quad (\xi_3, 0.73, -0.54)\}, \\
\mathcal{K}_{p_3} &= \{(\xi_1, 0.63, -0.24), \quad (\xi_2, 0.23, -0.14), \quad (\xi_3, 0.43, -0.24)\}.
\end{aligned}\right\}
$$

$$
\mathcal{K}_{\mathcal{A}'} = \left\{\begin{aligned}
\mathcal{K}_{p_1'} &= \{(\varsigma_1, 0.51, -0.33), (\varsigma_2, 0.32, -0.21), (\varsigma_3, 0.22, -0.31)\}, \\
\mathcal{K}_{p_2'} &= \{(\varsigma_1, 0.32, -0.43), (\varsigma_2, 0.51, -0.33), (\varsigma_3, 0.71, -0.51)\}, \\
\mathcal{K}_{p_3'} &= \{(\varsigma_1, 0.31, -0.53), (\varsigma_2, 0.21, -0.43), (\varsigma_3, 0.71, -0.41)\}.
\end{aligned}\right\}
$$

Then, it can be seen that $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ is a bijective mapping.

Definition 3.7. Suppose that $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ be a BFS-mapping. Let $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ be two BFS-sets in $\mathcal{BF}(V_{\mathcal{D}})$, then BFS-union and

BFS-intersection of BFS-images $\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})$ and $\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})$ in $\mathcal{BF}(W_{\mathcal{D}'})$, for $p' \in \mathcal{D}'$ and $\varsigma \in W$ defined as follows:

$$(\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2}))(p')(\varsigma) = \mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})(p')(\varsigma)\,\tilde{\cup}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma),$$

and

$$(\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})\,\tilde{\cap}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2}))(p')(\varsigma) = \mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})(p')(\varsigma)\,\tilde{\cap}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma).$$

Definition 3.8. Suppose that $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$ be a BFS-mapping. Let $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ be two BFS-sets in $\mathcal{BF}(W_{\mathcal{D}'})$, then BFS-union and BFS-intersection of BFS-images $\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_1})$ and $\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2})$ in $\mathcal{BF}(V_{\mathcal{D}})$, for $p \in \mathcal{D}$ and $\xi \in V$ defined as follows:

$$(\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2}))(p)(\xi) = \mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_1})(p)(\xi)\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2})(p)(\xi),$$

and

$$(\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_1})\,\tilde{\cap}\,\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2}))(p)(\xi) = \mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_1})(p)(\xi)\,\tilde{\cap}\,\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2})(p)(\xi).$$

Theorem 3.9. *Suppose that* $\mathcal{K}_{\mathcal{A}}, \mathcal{K}^1_{\mathcal{A}_1}$ *and* $\mathcal{K}^2_{\mathcal{A}_2} \,\tilde{\in}\, \mathcal{BF}(V_{\mathcal{D}})$ *and consider a BFS-mapping* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$, *then*

(i) $\mathfrak{f}(\phi_{\mathcal{D}}) = \phi_{\mathcal{D}}$.
(ii) $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2}) = \mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})$.
(iii) $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cap}\,\mathcal{K}^2_{\mathcal{A}_2})\,\tilde{\subseteq}\,\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cap}\,\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})$.
(iv) $\mathcal{K}_{\mathcal{A}}\,\tilde{\subseteq}\,\mathfrak{f}^{-1}(\mathfrak{f}(\mathcal{K}_{\mathcal{A}}))$. *The equality does hold if* $\mathfrak{u}$ *is injective mapping.*
(v) *If* $\mathcal{K}_{\mathcal{A}_1}\,\tilde{\subseteq}\,\mathcal{K}_{\mathcal{A}_2} \Rightarrow \mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})\,\tilde{\subseteq}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})$.

Proof. (i) Obvious

(ii) Suppose that for $p' \in \mathfrak{g}(\mathcal{D}) \subseteq \mathcal{D}'$ and $\varsigma \in W$, we have to show that $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2})(p')(\varsigma) = \mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})(p')(\varsigma)\,\tilde{\cup}\,\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})(p')(\varsigma)$. Consider the left hand side first, let $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2})(p')(\varsigma) = \mathfrak{f}(\mathcal{H}_{\mathcal{A}_1 \cup \mathcal{A}_2})(p')(\varsigma)$, by definition of BFS-mapping, we get

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})}(p')(\varsigma) = \begin{cases} \max\limits_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\max\limits_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(+)}_{\mathcal{H}_p} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap (\mathcal{A}_1 \cup \mathcal{A}_2) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})}(p')(\varsigma) = \begin{cases} \min\limits_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\min\limits_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(-)}_{\mathcal{H}_p} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap (\mathcal{A}_1 \cup \mathcal{A}_2) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})(p')(\varsigma) = \begin{cases} \tilde{\bigcup}_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\tilde{\bigcup}_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \mathcal{H}_p\right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma)\neq\emptyset, \\ & \mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)\neq\emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{H}_p = \left\{ \begin{array}{ll} \mathcal{K}_p^1, & \text{if } p\in\mathcal{A}_1\backslash\mathcal{A}_2 \\ \mathcal{K}_p^2, & \text{if } p\in\mathcal{A}_2\backslash\mathcal{A}_1 \\ \mathcal{K}_p^1\,\tilde{\cup}\,\mathcal{K}_p^2, & \text{if } p\in\mathcal{A}_1\cap\mathcal{A}_2 \end{array} \right\},$$

for some $p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)$. The trivial case is obvious, we only consider non trivial case.

$$\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})(p')(\varsigma) = \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup} \left\{ \begin{array}{ll} \mathcal{K}_p^1(\xi), & \text{if } p\in\mathcal{A}_1\backslash\mathcal{A}_2\cap\mathfrak{g}^{-1}(p') \\ \mathcal{K}_p^2(\xi), & \text{if } p\in\mathcal{A}_2\backslash\mathcal{A}_1\cap\mathfrak{g}^{-1}(p') \\ (\mathcal{K}_p^1\,\tilde{\cup}\,\mathcal{K}_p^2)(\xi), & \text{if } p\in\mathcal{A}_1\cap\mathcal{A}_2\cap\mathfrak{g}^{-1}(p') \end{array} \right\} \right). \tag{7}$$

Now for right-hand side, by definition of union of BFS-mappings, we have

$$\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma) = \mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})(p')(\varsigma)\,\tilde{\cup}\,\mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma),$$

$$\begin{aligned} &\delta^{(+)}_{\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})}(p')(\varsigma)\,\tilde{\cup}\,\delta^{(+)}_{\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})}(p')(\varsigma) \\ &\quad= \left(\max_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \max_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(+)}_{\mathcal{K}^1_p}(\xi)\right) \\ &\qquad \max\left(\max_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \max_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(+)}_{\mathcal{K}^2_p}(\xi)\right) \\ &\quad= \max_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \max_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \left(\max(\delta^{(+)}_{\mathcal{K}^1_p}, \delta^{(+)}_{\mathcal{K}^2_p})\right)(\xi) \\ &\quad= \max_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\max \left\{ \begin{array}{ll} \delta^{(+)}_{\mathcal{K}^1_p}(\xi), & \text{if } p\in\mathcal{A}_1\backslash\mathcal{A}_2\cap\mathfrak{g}^{-1}(p') \\ \delta^{(+)}_{\mathcal{K}^2_p}(\xi), & \text{if } p\in\mathcal{A}_2\backslash\mathcal{A}_1\cap\mathfrak{g}^{-1}(p') \\ \max(\delta^{(+)}_{\mathcal{K}^1_p}, \delta^{(+)}_{\mathcal{K}^2_p})(\xi), & \text{if } p\in\mathcal{A}_1\cap\mathcal{A}_2\cap\mathfrak{g}^{-1}(p') \end{array} \right\} \right). \end{aligned} \tag{8}$$

$$\begin{aligned} &\delta^{(-)}_{\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})}(p')(\varsigma)\,\tilde{\cup}\,\delta^{(-)}_{\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})}(p')(\varsigma) \\ &\quad= \left(\min_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \min_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(-)}_{\mathcal{K}^1_p}(\xi)\right) \\ &\qquad \min\left(\min_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \min_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \delta^{(-)}_{\mathcal{K}^2_p}(\xi)\right) \end{aligned}$$

$$= \min_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \min_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \left(\min(\delta^{(-)}_{\mathcal{K}^1_p}, \delta^{(-)}_{\mathcal{K}^2_p})\right)(\xi)$$

$$= \min_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\min \begin{Bmatrix} \delta^{(-)}_{\mathcal{K}^1_p}(\xi), & \text{if } p \in \mathcal{A}_1\backslash\mathcal{A}_2 \cap \mathfrak{g}^{-1}(p') \\ \delta^{(-)}_{\mathcal{K}^2_p}(\xi), & \text{if } p \in \mathcal{A}_2\backslash\mathcal{A}_1 \cap \mathfrak{g}^{-1}(p') \\ \min(\delta^{(-)}_{\mathcal{K}^1_p}, \delta^{(-)}_{\mathcal{K}^2_p})(\xi), & \text{if } p \in \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathfrak{g}^{-1}(p') \end{Bmatrix} \right), \quad (9)$$

from (8) and (9), we get

$$\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})(p')(\varsigma) \,\tilde{\cup}\, \mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma) = \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup} \begin{Bmatrix} \mathcal{K}^1_p(\xi), & \text{if } p \in \mathcal{A}_1\backslash\mathcal{A}_2 \cap \mathfrak{g}^{-1}(p') \\ \mathcal{K}^2_p(\xi), & \text{if } p \in \mathcal{A}_2\backslash\mathcal{A}_1 \cap \mathfrak{g}^{-1}(p') \\ (\mathcal{K}^1_p \,\tilde{\cup}\, \mathcal{K}^2_p)(\xi), & \text{if } p \in \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathfrak{g}^{-1}(p') \end{Bmatrix} \right), \quad (10)$$

from (7) and (10) it is evident that $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1} \,\tilde{\cup}\, \mathcal{K}^2_{\mathcal{A}_2}) = \mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1}) \,\tilde{\cup}\, \mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})$.

(iii) Suppose that for $p' \in \mathfrak{g}(\mathcal{D}) \subseteq \mathcal{D}'$ and $\varsigma \in W$ and by definition of intersection of BFS-mappings, we have $\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1} \,\tilde{\cup}\, \mathcal{K}^2_{\mathcal{A}_2})(p')(\varsigma) = \mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p')(\varsigma)$. Now by the definition of BFS-mappings, we get

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})}(p')(\varsigma) = \begin{cases} \max_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\max_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)} \delta^{(+)}_{\mathcal{H}_p}\right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap (\mathcal{A}_1 \cap \mathcal{A}_2) \neq \emptyset, \end{cases}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})}(p')(\varsigma) = \begin{cases} \min_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\min_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)} \delta^{(-)}_{\mathcal{H}_p}\right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap (\mathcal{A}_1 \cap \mathcal{A}_2) \neq \emptyset, \end{cases}$$

$$\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p')(\varsigma) = \begin{cases} \tilde{\bigcup}_{\xi\in\mathfrak{u}^{-1}(\varsigma)} \left(\tilde{\bigcup}_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cup\mathcal{A}_2)} \mathcal{H}_p \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma) \neq \emptyset, \\ & \mathfrak{g}^{-1}(p') \cap (\mathcal{A}_1 \cup \mathcal{A}_2) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{H}_p = (\mathcal{K}^1_{\mathcal{A}_1} \,\tilde{\cap}\, \mathcal{K}^2_{\mathcal{A}_2})$$

$$\mathfrak{f}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p')(\varsigma) = \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)}} (\mathcal{K}^1_{\mathcal{A}_1} \,\tilde{\cup}\, \mathcal{K}^2_{\mathcal{A}_2}) \right)(\xi)$$

$$= \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)}} (\mathcal{K}^1_{\mathcal{A}_1}(\xi) \,\tilde{\cap}\, \mathcal{K}^2_{\mathcal{A}_2}(\xi)) \right)$$

$$\tilde{\subseteq} \left(\widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}} \left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)}} \mathcal{K}^1_{\mathcal{A}_1}(\xi) \right) \right)$$

$$\tilde{\cap}\left(\widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}}\left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap(\mathcal{A}_1\cap\mathcal{A}_2)}}\mathcal{K}^2_{\mathcal{A}_2}(\xi)\right)\right)$$
$$= \mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})(p')(\varsigma)\,\tilde{\cap}\,\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2})(p')(\varsigma)$$
$$= (\mathfrak{f}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cap}\,\mathfrak{f}(\mathcal{K}^2_{\mathcal{A}_2}))(p')(\varsigma).$$

(iv) Straightforward.

(v) Suppose that for $p' \in \mathfrak{g}(\mathcal{D}) \subseteq \mathcal{D}'$ and $\varsigma \in W$ and by the definition of BFS-mappings, we get

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})}(p')(\varsigma) = \begin{cases} \max\limits_{\xi\in\mathfrak{u}^{-1}(\varsigma)}\left(\max\limits_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{A}_1}\delta^{(+)}_{\mathcal{K}_{(p)}}\right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma)\neq\emptyset, \\ & \mathfrak{g}^{-1}(p')\cap\mathcal{A}_1\neq\emptyset, \end{cases} \tag{11}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})}(p')(\varsigma) = \begin{cases} \min\limits_{\xi\in\mathfrak{u}^{-1}(\varsigma)}\left(\min\limits_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{A}_1}\delta^{(-)}_{\mathcal{K}_{(p)}}\right)(\xi), & \text{if } \mathfrak{u}^{-1}(\varsigma)\neq\emptyset, \\ & \mathfrak{g}^{-1}(p')\cap\mathcal{A}_1\neq\emptyset, \end{cases} \tag{12}$$

on combining (11) and (12), we obtain

$$\mathfrak{f}(\mathcal{K}_{\mathcal{A}_1})(p')(\varsigma) = \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}}\left(\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{A}_1}}\mathcal{K}_{\mathcal{A}_1}\right)(\xi)$$
$$= \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}}\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{A}_1}}\mathcal{K}_{\mathcal{A}_1}(\xi)$$
$$\subseteq \widetilde{\bigcup_{\xi\in\mathfrak{u}^{-1}(\varsigma)}}\widetilde{\bigcup_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{A}_2}}\mathcal{K}_{\mathcal{A}_2}(\xi)$$
$$= \mathfrak{f}(\mathcal{K}_{\mathcal{A}_2})(p')(\varsigma).$$
□

Theorem 3.10. *Suppose that* $\mathcal{K}_{\mathcal{A}}, \mathcal{K}^1_{\mathcal{A}_1}$ *and* $\mathcal{K}^2_{\mathcal{A}_2}\tilde{\in}\mathcal{BF}(W_{\mathcal{D}'})$ *and consider a BFS-mapping* $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(W_{\mathcal{D}'})$, *then*

(i) $\mathfrak{f}^{-1}(\phi_{\mathcal{D}}) = \phi_{\mathcal{D}}$.
(ii) $\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2}) = \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})$.
(iii) $\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2}) = \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})$.
(iv) $\mathfrak{f}(\mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}}))\,\tilde{\subseteq}\,\mathcal{K}_{\mathcal{A}}$. *The equality does hold if* $\mathfrak{u}$ *and* $\mathfrak{g}$ *are surjective mappings.*
(v) *If* $\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\subseteq}\,\mathcal{K}^2_{\mathcal{A}_2} \Rightarrow \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\subseteq}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})$

Proof. (i) It is obvious.

(ii) Suppose that for $p \in \mathcal{D}$ and $\xi \in V$, $\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2})(p)(\xi) = \mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})(p)(\xi) = \mathcal{H}(\mathfrak{g}(p))(\mathfrak{u}(\varsigma))$, where $(\mathfrak{g}(p) \in (\mathcal{A}_1 \cup \mathcal{A}_2), \mathfrak{u}(\varsigma) \in W)$, taking right hand-side and by the definition of inverse BFS-mapping, we get

$$\delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})}(p)(\xi) = \begin{cases} \delta^{(+)}_{\mathcal{H}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cup \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

$$\delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})}(p)(\xi) = \begin{cases} \delta^{(-)}_{\mathcal{H}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cup \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

$$\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})(p)(\xi) = \begin{cases} \mathcal{H}_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cup \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where

$$\mathcal{H}_{\mathfrak{g}(p)} = \left\{ \begin{array}{ll} \mathcal{K}^1_{\mathfrak{g}(p)}, & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ \mathcal{K}^2_{\mathfrak{g}(p)}, & \text{if } \mathfrak{g}(p) \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ (\mathcal{K}^1_{\mathfrak{g}(p)} \tilde{\cup} \mathcal{K}^2_{\mathfrak{g}(p)}), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \cap \mathcal{A}_2 \end{array} \right\}.$$

considering only non trivial case, we have

$$\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cup\mathcal{A}_2})(p)(\xi) = \left\{ \begin{array}{ll} \mathcal{K}^1_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ \mathcal{K}^2_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ (\mathcal{K}^1_{\mathfrak{g}(p)} \tilde{\cup} \mathcal{K}^2_{\mathfrak{g}(p)})(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \cap \mathcal{A}_2 \end{array} \right\}, \tag{16}$$

Now, by using definition of union of inverse BFS-mappings, we have

$$(\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}) \tilde{\cup} \mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2}))(p)(\xi) = \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})(p)(\xi) \tilde{\cup} \mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})(p)(\xi),$$

$$\begin{aligned} \delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})}(p)(\xi) \tilde{\cup} \delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})}(p)(\xi) &= \max\left(\delta^{(+)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), \delta^{(+)}_{\mathcal{K}^2_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)) \right) \\ &= \left\{ \begin{array}{ll} \delta^{(+)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ \delta^{(+)}_{\mathcal{K}^2_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ \max(\delta^{(+)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}, \delta^{(+)}_{\mathcal{K}^2_{\mathfrak{g}(p)}})(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \cap \mathcal{A}_2 \end{array} \right\}, \end{aligned} \tag{17}$$

$$\begin{aligned} \delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})}(p)(\xi) \tilde{\cup} \delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})}(p)(\xi) &= \min\left(\delta^{(-)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), \delta^{(-)}_{\mathcal{K}^2_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)) \right) \\ &= \left\{ \begin{array}{ll} \delta^{(-)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ \delta^{(-)}_{\mathcal{K}^2_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ \min(\delta^{(-)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}, \delta^{(+)}_{\mathcal{K}^2_{\mathfrak{g}(p)}})(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \cap \mathcal{A}_2 \end{array} \right\}, \end{aligned} \tag{18}$$

on combining (17) and (18), we obtain

$$\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})(p)(\xi)\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})(p)(\xi) = \left\{\begin{array}{ll} \mathcal{K}^1_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \backslash \mathcal{A}_2 \\ \mathcal{K}^2_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_2 \backslash \mathcal{A}_1 \\ (\mathcal{K}^1_{\mathfrak{g}(p)}\,\tilde{\cup}\,\mathcal{K}^2_{\mathfrak{g}(p)})(\mathfrak{u}(\xi)), & \text{if } \mathfrak{g}(p) \in \mathcal{A}_1 \cap \mathcal{A}_2 \end{array}\right\}, \tag{19}$$

from (16) and (19) it is evident that $\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cup}\,\mathcal{K}^2_{\mathcal{A}_2}) = \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cup}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})$.

(iii) Suppose that for $p \in \mathcal{D}$ and $\xi \in V$ and by definition of intersection of inverse BFS-mappings, we have $\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1}\,\tilde{\cap}\,\mathcal{K}^2_{\mathcal{A}_2})(p)(\xi) = \mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p)(\xi)$. Now, by the definition of inverse BFS-mappings, we obtain

$$\delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})}(p)(\xi) = \begin{cases} \delta^{(+)}_{\mathcal{H}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cap \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})}(p)(\xi) = \begin{cases} \delta^{(-)}_{\mathcal{H}_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cap \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p)(\xi) = \begin{cases} \mathcal{H}_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in (\mathcal{A}_1 \cap \mathcal{A}_2), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \mathcal{H}_{\mathfrak{g}(p)} &= (\mathcal{K}^1_{\mathfrak{g}(p)}\,\tilde{\cap}\,\mathcal{K}^2_{\mathfrak{g}(p)}), \\ \mathfrak{f}^{-1}(\mathcal{H}_{\mathcal{A}_1\cap\mathcal{A}_2})(p)(\xi) &= \mathcal{H}_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)) \\ &= (\mathcal{K}^1_{\mathfrak{g}(p)}\,\tilde{\cap}\,\mathcal{K}^2_{\mathfrak{g}(p)})(\mathfrak{u}(\xi)) \\ &= \mathcal{K}^1_{\mathfrak{g}(p)}(\mathfrak{u}(\xi))\,\tilde{\cap}\,\mathcal{K}^2_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)) \\ &= \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})(p)(\xi)\,\tilde{\cap}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2})(p)(\xi) \\ &= (\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})\,\tilde{\cap}\,\mathfrak{f}^{-1}(\mathcal{K}^2_{\mathcal{A}_2}))(p)(\xi). \end{aligned}$$

(iv) Straightforward.

(v) Suppose that for $p \in \mathcal{D}$ and $\xi \in V$ and by the definition of inverse BFS-mappings, we get

$$\delta^{(+)}_{\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})}(p)(\xi) = \begin{cases} \delta^{(+)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}_1, \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

$$\delta^{(-)}_{\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})}(p)(\xi) = \begin{cases} \delta^{(-)}_{\mathcal{K}^1_{\mathfrak{g}(p)}}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}_1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

On combining (20) and (21), we have

$$\mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})(p)(\xi) = \begin{cases} \mathcal{K}^1_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)), & \text{for } \mathfrak{g}(p) \in \mathcal{A}_1, \\ 0 & \text{otherwise}, \end{cases} \tag{22}$$

$$\begin{aligned} \mathfrak{f}^{-1}(\mathcal{K}^1_{\mathcal{A}_1})(p)(\xi) &= (\mathcal{K}^1_{\mathfrak{g}(p)})(\mathfrak{u}(\xi)) \\ &= \mathcal{K}^1_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)) \\ &\subseteq \mathcal{K}^2_{\mathfrak{g}(p)}(\mathfrak{u}(\xi)) \\ &= \mathfrak{f}^{-1}(\mathcal{K}_{\mathcal{A}_2})(p)(\xi). \end{aligned}$$ □

Definition 3.11. A BFS-relation $V \times W$ can be defined as $\mathcal{R} = \{(\xi,\varsigma), \delta^{(+)}_{\mathcal{R}}(\xi,\varsigma), \delta^{(-)}_{\mathcal{R}}(\xi,\varsigma), (\xi,\varsigma) \in V \times W\}$, where $\delta^{(+)}_{\mathcal{R}} : V \times W \to [0,1]$, $\delta^{(-)}_{\mathcal{R}} : V \times W \to [-1,0]$ are called positive and negative membership functions. The set of all BFS-relations is denoted by BFS-$(V \times W)$.

Definition 3.12. Let $\mathcal{A}_1 \in$ BFS-$(V \times W)$ and $\mathcal{A}_2 \in$ BFS-$(W \times Z)$, a BFS max–min composition $\mathcal{A}_1 \circ \mathcal{A}_2$ can be defined as $\mathcal{A}_1 \circ \mathcal{A}_2 = \{\{(\xi,\rho), \delta^{(+)}_{\mathcal{A}_1\circ\mathcal{A}_2}(\xi,\rho), \delta^{(-)}_{\mathcal{A}_1\circ\mathcal{A}_2}(\xi,\rho)\} : \xi \in V, \rho \in Z\}\}$, where

$$\delta^{(+)}_{\mathcal{A}_1\circ\mathcal{A}_2}(\xi,\rho) = \max_{\varsigma\in W}\{\min[\delta^{(+)}_{\mathcal{A}_2}(\xi,\varsigma), \delta^{(+)}_{\mathcal{A}_1}(\varsigma,\rho)]\},$$

$$\delta^{(-)}_{\mathcal{A}_1\circ\mathcal{A}_2}(\xi,\rho) = \min_{\varsigma\in W}\{\max[\delta^{(-)}_{\mathcal{A}_2}(\xi,\varsigma), \delta^{(-)}_{\mathcal{A}_1}(\varsigma,\rho)]\}.$$

3.3. *Methodology*

In this section, we discuss our proposed method of BFS-mapping to diagnose bipolar disorder, best treatment and progress of treatment episodes. For bipolar disorder diagnosis

Pre-Step: Bipolar disorder diagnosis has two difficulties: first diagnosing bipolar disorder correctly and then diagnosing its types correctly as well. It is very hard to differentiate among its types. Because the symptoms for mania and hypomania are similar, on the other hand, symptoms for depression and major depression are similar. As we see that it involves uncertainties and vague information, so bipolar fuzzy soft set is suitable to handle such kind of data. We associate $[0,1]$ interval of BFS-set with feelings of "highs" and $[-1,0]$ interval with feelings of "lows" in bipolar disorder (see Fig. 4). We further associate decision variables of BFS-set with symptoms of bipolar disorder. We also use BFS-mappings to diagnose the type of bipolar disorder (see Fig. 5). We further see that the BFS-mappings work perfectly to know the progress in the tenure of treatment (see Fig. 6). We make a chart for above three types and assign different ranges of membership degrees to mania, hypomania, depression and major depression from their associated interval. For mania 0.80 to 1 (severe mania), the value 0.80 to 0.89 is for one week continuously manic conditions, 0.90 to 0.99 is for two to three weeks manic conditions and 1 for above three weeks manic conditions. In case of hypomania (moderate mania or less severe than mania), the value 0.50 to 0.59 is for first week, 0.60 to 0.69 for second and third week and 0.70 to 0.79 for the hypomanic conditions which lasted for more than three weeks. The range 0.20–0.40 (mild mania) is less severe form of hypomania and set according to the symptoms which lasted for one year and above. Similarly, for major depression (extreme depression) conditions of one week, we set -0.80 to -0.89, -0.90 to -0.99 for two to three weeks and -1 for above three weeks continuous episodes of major depression. As in bipolar I disorder, feelings of depression or major depression may or may not occur therefore we set the range -1 to 0 (severe to nil depression). In cyclothymic, the person only feels the mild or less severe hypomania and depressive (low to moderate depression or less severe than extreme depression) for long time so we set the range accordingly. The graph of main conditions relevant to different types of bipolar disorder can be seen in Fig. 3.

Now, we set a range of specific values to obtain the final results.

Step 1. Identify and frame the bipolar disorder problem. Let $V = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$ be the set of patient and $\mathcal{D} = \{p_1, p_2, \ldots, p_o\}$ be the set of related symptoms of bipolar disorder. A doctor makes t number of weeks diagnosis charts (represented by BFS-sets) to obtain a final correct decision. The BFS-set provided by doctor at εth time can be written as $\mathcal{K}^{\varepsilon}_{\mathcal{D}} = (\delta^{\varepsilon+}_{\ell k}, \delta^{\varepsilon-}_{\ell k})$, where $\delta^{\varepsilon+}_{\ell k}$ and $\delta^{\varepsilon-}_{\ell k}$ are fuzzy membership degrees evaluation of mania and depression for kth symptom and ℓth patient, respectively, $\varepsilon = 1, 2, \ldots, n$, $\ell = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, o$. Take BFS-union of ε number of week charts to obtain final BFS-set to examine disorder further.

Step 2. Consider a set $\mathcal{D}' = \{p'_1, p'_2, \ldots, p'_{o'}\}$ of connected symptoms (primary symptoms containing relevant basic symptoms). Construct a BFS-set (doctor's

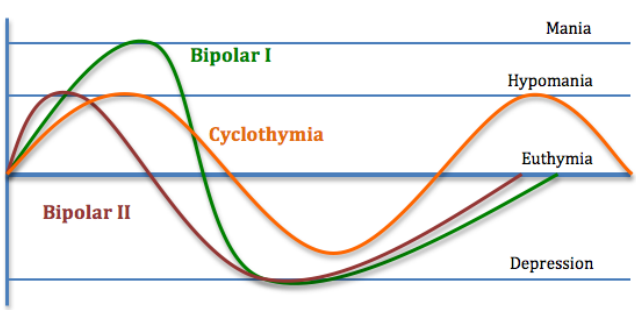


Fig. 3. Graph of types of bipolar disorder with symptoms level.
Source: https://meandcyclothymia.wordpress.com/2015/01/19/opening-up-to-cyclothymia/.

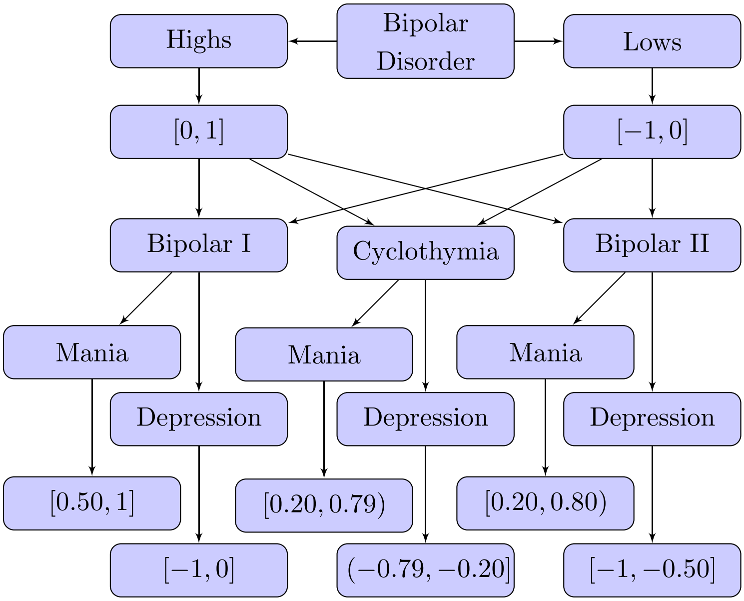


Fig. 4. Flow chart of the different ranges assigned to conditions.

assigning weights by keeping in view the patient's ε number of week evaluation of basic symptoms) based on primary symptoms and patients.

Step 3. Define mapping which is basically a relation between final set obtained in **Step 1** and BFS-set obtained in **Step 2**, let $\mathfrak{u} : V \to V$, $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ defined as follows: $\mathfrak{u}(\xi_\ell) = \xi_\ell$ and $\mathfrak{g}(p_k) = p'_{k'}$ (depends on interrelation between basic and primary symptoms), consider the mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_\mathcal{D}) \to \mathcal{BF}(V_{\mathcal{D}'})$ defined by

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{K}_\mathcal{D})}(p')(\xi) = \mid \delta^{i+}_{p'_k} \mid \begin{cases} \max\limits_{\xi\in\mathfrak{u}^{-1}(\xi)} \left(\max\limits_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{D}} \delta^{(+)}_{\mathcal{K}_{(p)}} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\xi) \neq \emptyset,\ \ \mathfrak{g}^{-1}(p') \cap \mathcal{D} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{K}_\mathcal{D})}(p')(\xi) = \mid \delta^{i-}_{p'_k} \mid \begin{cases} \min\limits_{\xi\in\mathfrak{u}^{-1}(\xi)} \left(\min\limits_{p\in\mathfrak{g}^{-1}(p')\cap\mathcal{D}} \delta^{(-)}_{\mathcal{K}_{(p)}} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\xi) \neq \emptyset,\ \ \mathfrak{g}^{-1}(p') \cap \mathcal{D} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta^{i+}_{p'_k}$ and $\delta^{i-}_{p'_k}$ are associated weights from $\mathcal{K}_{\mathcal{D}'}$. Obtain the image of $\mathcal{K}^{\varepsilon}_{\mathcal{D}}$ under $\mathfrak{f}$.

Step 4. Compare the value of obtained set with Table 2 and prepare a Table of pre-diagnosis which helps to check accuracy of final results.

Step 5. Calculate $\mathcal{S} = \frac{\delta^{i+}_{p_j} + \delta^{i-}_{p_j}}{m}$ where m is number of connected symptoms, conclude the result by using Table 3.

For best treatment, do the following steps:

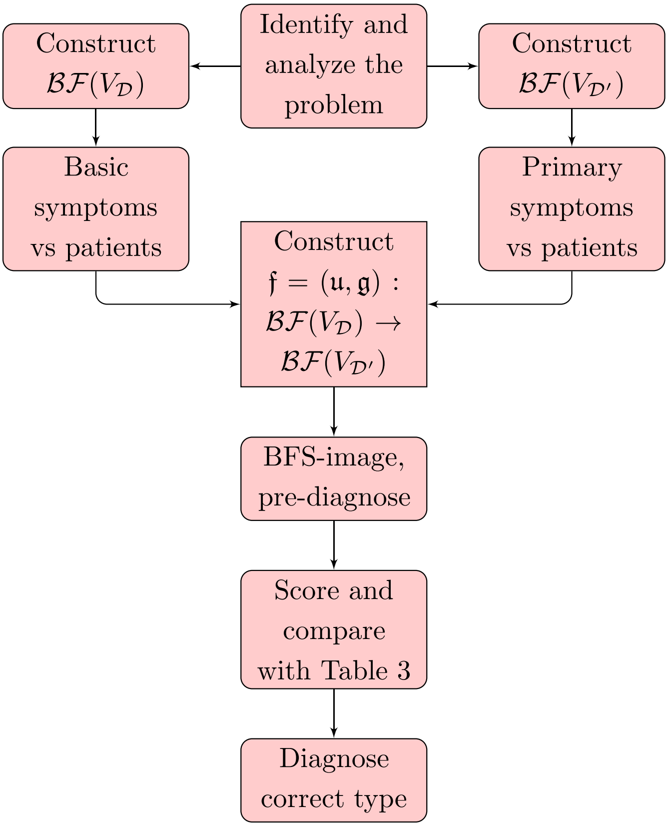


Fig. 5. Flow chart of the diagnosis of bipolar disorder.

Step 6. Consider a set of $\mathcal{D}' = \{p'_1, p'_2, \ldots, p'_{o'}\}$ of connected symptoms and $\mathcal{D}'' = \{p''_1, p''_2, \ldots, p''_{o''}\}$ of possible treatments and construct $\mathcal{K}_{\mathcal{D}''}$

Step 7. Take $\mathcal{K}_{\mathcal{D}''}$, $\mathcal{K}'_{\mathcal{D}'}$, perform BFS max–min composition and obtain $\mathcal{K}_{\mathcal{D}'''}$.

Step 8. Choose the treatment with more benefits and less side effects.

To track the improvement record, do the following steps:

Step 9. Define a generalized mapping $\mathfrak{f}' = (\mathfrak{u}', \mathfrak{g}') : V^{n-1}_{\mathcal{D}''} \to V^1_{\mathcal{D}''}$, where $\mathfrak{u}' : V^{n-1} \to V^1$ and $\mathfrak{g}' : (\mathcal{D}'')^{n-1} \to \mathcal{D}''$ defined as follows:

$$\mathfrak{u}'(\xi_1) = \xi_1, \mathfrak{u}'(\xi_2) = \xi_2, \mathfrak{u}'(\xi_3) = \xi_3, \mathfrak{g}'(p''_1) = p''_1, \mathfrak{g}'(p''_2) = p''_2, \mathfrak{g}'(p''_3) = p''_3,$$

$$V^n_{\mathcal{D}''} = \mathfrak{f}'(V^{n-1}_{\mathcal{D}''})(p'')(\xi) = 1/n \begin{cases} \widetilde{\bigcup}_{\alpha \in \mathfrak{u}'^{-1}(\xi)} \left(\widetilde{\bigcup}_{\beta \in \mathfrak{g}'^{-1}(p'') \cap \mathcal{D}''} V^{n-1}_{\mathcal{D}''} \right)(\alpha), & \text{if } \mathfrak{u}'^{-1}(\xi), \\ & \mathfrak{g}'^{-1}(p'') \cap \mathcal{D}'' \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

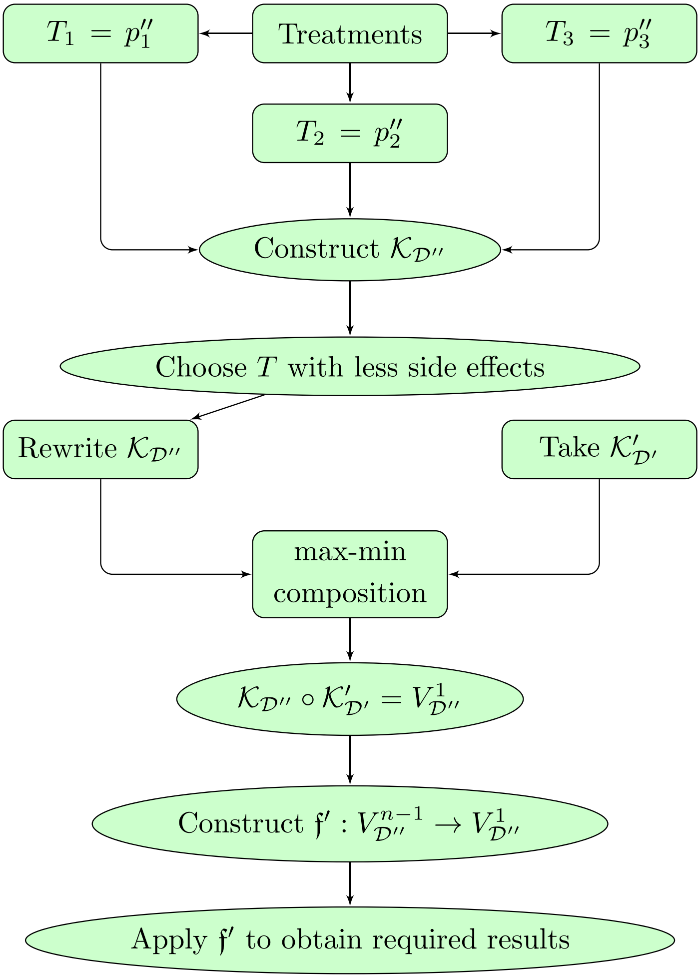


Fig. 6. Flow chart of the treatment and improvement.

Table 2. Conditions and their intensity based on weekly observation to diagnose bipolar disorder.

Conditions	≤1 week	2–3 weeks	>3 weeks
Severe Mania (SM)	0.80 to 0.89	0.90 to 0.99	1
Hypomania (HM)	0.50 to 0.59	0.60–0.69	0.70 to 0.79
Mild Mania (MM)	0.20 to 0.29	0.30 to 0.39	0.40 to 0.49
Extreme Depression (ED)	−0.80 to −0.89	−0.90 to −0.99	−1
Moderate Depression (MD)	−0.50 to −0.59	−0.60 to −0.69	−0.70 to −0.079
Low Depression (LD)	−0.20 to −0.29	−0.30 to −0.39	−0.40 to −0.49

Table 3. Final diagnosis chart.

Types of disorder	Different ranges of $[-1, 1]$
Bipolar I Disorder	$[0.5, 1]$
Bipolar II Disorder	$[-0.5, -1]$
Cyclothymic	$(-0.5, -0.2] \cup [0.2, 0.5)$
No Bipolar Disorder	$(-0.2, 0.2)$

where $n = 2, 3, \ldots$ is number of treatment episodes, $p'' \in \mathfrak{g}(\mathcal{D}') \subseteq \mathcal{D}''$ and $\xi \in V^1$. $\alpha \in V^{n-1}$, $\beta \in (\mathcal{D}'')^{n-1}$.

Step 10. Repeat application in **Step 9** until the required results are achieved.

4. Application of BFS-Mappings to Bipolar Disorders

In this section, we apply our method to a case under consideration.

Case Study

Suppose that a psychiatrist wants to diagnose the bipolar disorder and its types in his four patients. Mostly, type of the bipolar disorder cannot be diagnosed correctly due to its symptoms matching with other diseases. Therefore, after a complete physical exam, the psychiatrist rules out all the impossible factors. He considers all possible factors, such as complete and careful history of medical, genetics, family history, neurological (episodes of high stress, such as the death of a sibling or other traumatic event, structure and functioning of brain and anxiety disorder), environmental, physical illnesses, which are along with bipolar disorder (including thyroid disease, heart disease, migraine headaches, diabetes and obesity) and substance abuse (drug or alcohol), etc. Let $V = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the set of four patients and $\mathcal{D} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the set of symptoms of their illness which a psychiatrist conducts after a complete mental evaluation of the patients by analyzing the daily prepared mood charts after some visits and discussion with family members and friends of the patients. Here,

- p_1 represents feelings of euphoric, exultant and sometimes dejected, sorrowful,
- p_2 represents full energized meanwhile decreased levels of activity,
- p_3 represents disturbed sleep due to racing thoughts, unusual active as well as eat too little or too much,
- p_4 represents full activated mood on the other hand disturbed mood,
- p_5 represents most talkative and touchy as well as can not enjoy anything,
- p_6 represents Think about risky things and also sleep too little or too much, think about death or suicide.

For a perfect and right diagnosis, construct charts (which are BFS-sets) of three consecutive weeks $\mathcal{K}^{\varepsilon}_{\mathcal{D}} \in \mathcal{BF}(V_{\mathcal{D}})$ and assign membership degrees to the symptoms of the patients by keeping in view all possible factors. First week chart is as follows:

$$\mathcal{K}^1_{\mathcal{D}} = \begin{Bmatrix} \mathcal{K}^1_{p_1} = \{(\xi_1, 0.87, -0.61), (\xi_2, 0.52, -0.51), (\xi_3, 0.89, -0.10), (\xi_4, 0.40, -0.30)\}, \\ \mathcal{K}^1_{p_2} = \{(\xi_1, 0.26, -0.76), (\xi_2, 0.63, -0.76), (\xi_3, 0.83, -0.15), (\xi_4, 0.60, -0.00)\}, \\ \mathcal{K}^1_{p_3} = \{(\xi_1, 0.15, -0.55), (\xi_2, 0.50, -0.78), (\xi_3, 0.86, -0.23), (\xi_4, 0.60, -0.30)\}, \\ \mathcal{K}^1_{p_4} = \{(\xi_1, 0.14, -0.43), (\xi_2, 0.81, -0.87), (\xi_3, 0.31, -0.73), (\xi_4, 0.50, -0.50)\}, \\ \mathcal{K}^1_{p_5} = \{(\xi_1, 0.23, -0.86), (\xi_2, 0.40, -0.91), (\xi_3, 0.40, -0.24), (\xi_4, 0.20, -0.10)\}, \\ \mathcal{K}^1_{p_6} = \{(\xi_1, 0.15, -0.83), (\xi_2, 0.36, -0.81), (\xi_3, 0.31, -0.66), (\xi_4, 0.20, -0.40)\}. \end{Bmatrix}$$

Second week chart is

$$\mathcal{K}^2_{\mathcal{D}} = \begin{Bmatrix} \mathcal{K}^2_{p_1} = \{(\xi_1, 0.71, -0.81), (\xi_2, 0.27, -0.71), (\xi_3, 0.39, -0.20), (\xi_4, 0.60, -0.40)\}, \\ \mathcal{K}^2_{p_2} = \{(\xi_1, 0.23, -0.86), (\xi_2, 0.53, -0.76), (\xi_3, 0.88, -0.15), (\xi_4, 0.20, -0.60)\}, \\ \mathcal{K}^2_{p_3} = \{(\xi_1, 0.25, -0.55), (\xi_2, 0.00, -0.78), (\xi_3, 0.23, -0.13), (\xi_4, 0.00, -0.30)\}, \\ \mathcal{K}^2_{p_4} = \{(\xi_1, 0.14, -0.43), (\xi_2, 0.10, -0.27), (\xi_3, 0.21, -0.43), (\xi_4, 0.10, -0.40)\}, \\ \mathcal{K}^2_{p_5} = \{(\xi_1, 0.23, -0.56), (\xi_2, 0.10, -0.11), (\xi_3, 0.60, -0.44), (\xi_4, 0.40, -0.10)\}, \\ \mathcal{K}^2_{p_6} = \{(\xi_1, 0.15, -0.73), (\xi_2, 0.66, -0.91), (\xi_3, 0.21, -0.56), (\xi_4, 0.10, -0.70)\}. \end{Bmatrix}$$

Third and forth week charts are

$$\mathcal{K}^3_{\mathcal{D}} = \begin{Bmatrix} \mathcal{K}^3_{p_1} = \{(\xi_1, 0.21, -0.61), (\xi_2, 0.52, -0.51), (\xi_3, 0.20, -0.20), (\xi_4, 0.20, -0.10)\}, \\ \mathcal{K}^3_{p_2} = \{(\xi_1, 0.12, -0.76), (\xi_2, 0.63, -0.76), (\xi_3, 0.13, -0.25), (\xi_4, 0.10, -0.30)\}, \\ \mathcal{K}^3_{p_3} = \{(\xi_1, 0.15, -0.85), (\xi_2, 0.20, -0.78), (\xi_3, 0.20, -0.33), (\xi_4, 0.00, -0.50)\}, \\ \mathcal{K}^3_{p_4} = \{(\xi_1, 0.14, -0.93), (\xi_2, 0.00, -0.87), (\xi_3, 0.87, -0.13), (\xi_4, 0.20, -0.30)\}, \\ \mathcal{K}^3_{p_5} = \{(\xi_1, 0.23, -0.76), (\xi_2, 0.49, -0.91), (\xi_3, 0.80, -0.24), (\xi_4, 0.70, -0.00)\}, \\ \mathcal{K}^3_{p_6} = \{(\xi_1, 0.25, -0.53), (\xi_2, 0.16, -0.81), (\xi_3, 0.81, -0.56), (\xi_4, 0.10, -0.30)\}. \end{Bmatrix}$$

Now, by taking BFS-union of above sets, we obtain

$$\tilde{\cup}_{\varepsilon}\mathcal{K}^{\varepsilon}_{\mathcal{D}} = \begin{Bmatrix} \mathcal{K}^{\varepsilon}_{p_1} = \{(\xi_1, 0.87, -0.81), (\xi_2, 0.52, -0.71), (\xi_3, 0.89, -0.20), (\xi_4, 0.60, -0.40)\}, \\ \mathcal{K}^{\varepsilon}_{p_2} = \{(\xi_1, 0.26, -0.86), (\xi_2, 0.63, -0.76), (\xi_3, 0.88, -0.25), (\xi_4, 0.60, -0.60)\}, \\ \mathcal{K}^{\varepsilon}_{p_3} = \{(\xi_1, 0.25, -0.85), (\xi_2, 0.50, -0.78), (\xi_3, 0.86, -0.33), (\xi_4, 0.60, -0.50)\}, \\ \mathcal{K}^{\varepsilon}_{p_4} = \{(\xi_1, 0.14, -0.93), (\xi_2, 0.81, -0.87), (\xi_3, 0.87, -0.73), (\xi_4, 0.50, -0.50)\}, \\ \mathcal{K}^{\varepsilon}_{p_5} = \{(\xi_1, 0.23, -0.86), (\xi_2, 0.49, -0.91), (\xi_3, 0.80, -0.44), (\xi_4, 0.70, -0.10)\}, \\ \mathcal{K}^{\varepsilon}_{p_6} = \{(\xi_1, 0.25, -0.93), (\xi_2, 0.66, -0.91), (\xi_3, 0.81, -0.66), (\xi_4, 0.20, -0.70)\}. \end{Bmatrix}$$

Let $\mathcal{D}' = \{p'_1, p'_2, p'_3\}$ be the possible set of connected symptoms of bipolar disorder, where

p'_1 represents Mood symptoms,
p'_2 represents Behavioral disorders,
p'_3 represents Thought disorders,

Construct class of BFS-set $\mathcal{BF}(V_{\mathcal{D}'})$ on the same universal set V, which has weights given by doctor for connected symptoms and patients collected data

$$\mathcal{K}_{\mathcal{D}'} = \begin{Bmatrix} \mathcal{K}_{p'_1} = \{(\xi_1, 0.60, -0.81), (\xi_2, 0.52, -0.87), (\xi_3, 0.81, -0.10), (\xi_4, 0.11, -0.54)\}, \\ \mathcal{K}_{p'_2} = \{(\xi_1, 0.12, -0.86), (\xi_2, 0.63, -0.87), (\xi_3, 0.95, -0.25), (\xi_4, 0.41, -0.21)\}, \\ \mathcal{K}_{p'_3} = \{(\xi_1, 0.15, -0.93), (\xi_2, 0.70, -0.81), (\xi_3, 0.89, -0.30), (\xi_4, 0.54, -0.50)\}. \end{Bmatrix}$$

Defining mappings $\mathfrak{u} : V \to V$, $\mathfrak{g} : \mathcal{D} \to \mathcal{D}'$ as follows:

$$\mathfrak{u}(\xi_1) = \xi_1, \mathfrak{u}(\xi_2) = \xi_2, \mathfrak{u}(\xi_3) = \xi_3,$$

$$\mathfrak{g}(p_1) = p'_1, \mathfrak{g}(p_4) = p'_1, \mathfrak{g}(p_2) = p'_2, \mathfrak{g}(p_5) = p'_2, \mathfrak{g}(p_3) = p'_3, \mathfrak{g}(p_6) = p'_3,$$

the mapping $\mathfrak{f} = (\mathfrak{u}, \mathfrak{g}) : \mathcal{BF}(V_{\mathcal{D}}) \to \mathcal{BF}(V_{\mathcal{D}'})$ defined by

$$\delta^{(+)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{D}})}(p')(\xi) = |\ \delta^{i+}_{p'_j}\ | \begin{cases} \max\limits_{\xi \in \mathfrak{u}^{-1}(\xi)} \left(\max\limits_{p \in \mathfrak{g}^{-1}(p') \cap \mathcal{D}} \delta^{(+)}_{\mathcal{K}(p)} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\xi) \neq \emptyset, \quad \mathfrak{g}^{-1}(p') \cap \mathcal{D} \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

$$\delta^{(-)}_{\mathfrak{f}(\mathcal{K}_{\mathcal{D}})}(p')(\xi) = |\ \delta^{i-}_{p'_j}\ | \begin{cases} \min\limits_{\xi \in \mathfrak{u}^{-1}(\xi)} \left(\min\limits_{p \in \mathfrak{g}^{-1}(p') \cap \mathcal{D}} \delta^{(-)}_{\mathcal{K}(p)} \right)(\xi), & \text{if } \mathfrak{u}^{-1}(\xi) \neq \emptyset, \quad \mathfrak{g}^{-1}(p') \cap \mathcal{D} \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

the image of $\mathcal{K}^{\varepsilon}_{\mathcal{D}}$ under $\mathfrak{f}$ is given by

$$\mathcal{K}'_{\mathcal{D}'} = \begin{Bmatrix} \mathcal{K}'_{p'_1} = \{(\xi_1, 0.52, -0.75), (\xi_2, 0.42, -0.75), (\xi_3, 0.72, -0.07), (\xi_4, 0.06, -0.27)\}, \\ \mathcal{K}'_{p'_2} = \{(\xi_1, 0.03, -0.73), (\xi_2, 0.39, -0.79), (\xi_3, 0.83, -0.08), (\xi_4, 0.28, -0.12)\}, \\ \mathcal{K}'_{p'_3} = \{(\xi_1, 0.03, -0.86), (\xi_2, 0.46, -0.73), (\xi_3, 0.76, -0.19), (\xi_4, 0.32, -0.35)\} \end{Bmatrix}.$$

On comparing the values of above set with Table 2, we get a chart (Table 4) of pre-diagnosis, which we compare later with our results to check accuracy.

Calculating the score function, for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$

$$\mathcal{S} = \frac{\delta^{i+}_{p_j} + \delta^{i-}_{p_j}}{m},$$

where m is number of connected symptoms, here $m = 3$. Thus, we obtain $\xi_1 = -0.58$, $\xi_2 = -0.33$, $\xi_3 = 0.65$ and $\xi_4 = -0.02$. On comparing the score values with Table 3 we get, ξ_1 is diagnosed with bipolar II disorder, ξ_2 with cyclothymic disorder, ξ_3 is diagnosed with bipolar I disorder and ξ_4 is free from bipolar disorder.

Table 4. Pre-diagnosis chart.

Pre-diagnosis	ξ_1	ξ_2	ξ_3	ξ_4
$\mathcal{K}'_{p'_1}$	(HM, MD)	(MM, MD)	(HM, N)	(N, LD)
$\mathcal{K}'_{p'_2}$	(N, MD)	(MM, MD)	(SM, N)	(MM, N)
$\mathcal{K}'_{p'_3}$	(N, ED)	(MM, MD)	(HM, N)	(MM, LD)

After diagnosis, the psychiatrist suggests best medication along with psychotherapy. He constructs another BFS-set with suggesting treatment and types of disease.

Let $W = \{p'_1, p'_2, p'_3\}$ be the set of primary symptoms of bipolar disorder and $\mathcal{D}'' = \{p''_1, p''_2, p''_3\}$ be the set of recommended treatments, where

p''_1 represents treatment with high potency medication and electroconvulsive therapy,

p''_2 represents treatment with moderate potency medication and some psychotherapies,

p''_3 represents treatment with cognitive behavior therapy and some mild medications.

Now suppose that $\mathcal{K}_{\mathcal{D}''} \in \mathcal{BF}(W_{\mathcal{D}''})$, where

$$\mathcal{K}_{\mathcal{D}''} = \begin{Bmatrix} \mathcal{K}_{p''_1} = \{(p'_1, 0.90, -0.50), (p'_2, 0.50, -0.90), (p'_3, 0.40, -0.95)\}, \\ \mathcal{K}_{p''_2} = \{(p'_1, 0.70, -0.50), (p'_2, 0.90, -0.80), (p'_3, 0.60, -0.50)\}, \\ \mathcal{K}_{p''_3} = \{(p'_1, 0.50, -0.25), (p'_2, 0.60, -0.40), (p'_3, 0.90, -0.20)\}, \end{Bmatrix}$$

the membership degrees are assigned according to history of medical of each patient. Here, positive membership degrees show that effectiveness of treatments for each type on the other hand negative membership degrees indicate the side effects of medications and less effectiveness of the treatment for each type. Now, by applying BFS max–min composition between $\mathcal{K}_{\mathcal{D}''}$ and $\mathcal{K}'_{\mathcal{D}'}$,

$\mathcal{K}_{\mathcal{D}''}$	p'_1	p'_2	p'_3
$\mathcal{K}_{p''_1}$	(0.90, −0.50)	(0.50, −0.90)	(0.40, −0.95)
$\mathcal{K}_{p''_2}$	(0.70, −0.50)	(0.90, −0.80)	(0.60, −0.50)
$\mathcal{K}_{p''_3}$	(0.50, −0.25)	(0.60, −0.40)	(0.90, −0.20),

now, the $\mathcal{K}'_{\mathcal{D}'}$ can be written as

$\mathcal{K}'_{\mathcal{D}'}$	ξ_1	ξ_2	ξ_3
p'_1	(0.52, −0.75)	(0.42, −0.75)	(0.72, −0.07)
p'_2	(0.03, −0.73)	(0.39, −0.79)	(0.83, −0.08)
p'_3	(0.03, −0.86)	(0.46, −0.73)	(0.76, −0.19),

we obtain a BFS-set, which is relation between patients and suggested treatment according to their disease.

$\mathcal{K}_{\mathcal{D}''} \circ \mathcal{K}_{\mathcal{D}'}$	ξ_1	ξ_2	ξ_3
$\mathcal{K}_{p''_1}$	(0.52, −0.86)	(0.42, −0.79)	(0.72, −0.19)
$\mathcal{K}_{p''_2}$	(0.52,−0.73)	(0.46, −0.79)	(0.83, −0.19)
$\mathcal{K}_{p''_3}$	(0.50,−0.40)	(0.46, −0.40)	(0.76, −0.19)

There is no doubt that a doctor wants a treatment with less side effects and more benefits. So, according to suitability of the treatment for each patient, we get p''_3 for ξ_1, p''_3 for ξ_2 and p''_2 for ξ_3.

The psychiatrist sets the episodes of the treatment for each patient according to the intensity of the symptoms. The episodes of the treatment may be repeated until the disease is cured well. The progress after each episode of the treatment can be seen by applying the self-mapping given in what follows. Let $\mathfrak{f}' = (\mathfrak{u}', \mathfrak{g}') : V^{n-1}_{\mathcal{D}''} \to V^1_{\mathcal{D}''}$, where $\mathfrak{u}' : V^{n-1} \to V^1$ and $\mathfrak{g}' : (\mathcal{D}'')^{n-1} \to \mathcal{D}''$ defined as follows:

$$\mathfrak{u}'(\xi_1) = \xi_1, \mathfrak{u}'(\xi_2) = \xi_2, \mathfrak{u}'(\xi_3) = \xi_3,$$

$$\mathfrak{g}'(p''_1) = p''_1, \mathfrak{g}'(p''_2) = p''_2, \mathfrak{g}'(p''_3) = p''_3,$$

then it is given that

$$V^1_{\mathcal{D}''} = \begin{Bmatrix} \mathcal{K}_{p''_1} = \{(\xi_1, 0.52, -0.75), (\xi_2, 0.42, -0.75), (\xi_3, 0.72, -0.07)\}, \\ \mathcal{K}_{p''_2} = \{(\xi_1, 0.03, -0.73), (\xi_2, 0.39, -0.79), (\xi_3, 0.83, -0.08)\}, \\ \mathcal{K}_{p''_3} = \{(\xi_1, 0.03, -0.86), (\xi_2, 0.46, -0.73), (\xi_3, 0.76, -0.19)\}. \end{Bmatrix}$$

$$\begin{aligned} V^n_{\mathcal{D}''} &= \mathfrak{f}'(V^{n-1}_{\mathcal{D}''})(p'')(\xi) \\ &= 1/n \begin{cases} \widetilde{\bigcup}_{\alpha \in \mathfrak{u}'^{-1}(\xi)} \left(\widetilde{\bigcup}_{\beta \in \mathfrak{g}'^{-1}(p'') \cap \mathcal{D}''} V^{n-1}_{\mathcal{D}''} \right)(\alpha), & \text{if } \mathfrak{u}'^{-1}(\xi), \mathfrak{g}'^{-1}(p'') \cap \mathcal{D}'' \neq \emptyset, \\ 0 & \text{otherwise ,} \end{cases} \end{aligned}$$

where $n = 2, 3, \ldots$ is number of treatment episodes, $p'' \in \mathfrak{g}(\mathcal{D}') \subseteq \mathcal{D}''$ and $\xi \in V^1$. $\alpha \in V^{n-1}$, $\beta \in (\mathcal{D}'')^{n-1}$. At the first episode of the treatment, when $n = 1$ it is given that

$$V^1_{\mathcal{D}''} = \begin{Bmatrix} \mathcal{K}_{p''_1} = \{(\xi_1, 0.52, -0.75), (\xi_2, 0.42, -0.75), (\xi_3, 0.72, -0.07)\}, \\ \mathcal{K}_{p''_2} = \{(\xi_1, 0.03, -0.73), (\xi_2, 0.39, -0.79), (\xi_3, 0.83, -0.08)\}, \\ \mathcal{K}_{p''_3} = \{(\xi_1, 0.03, -0.86), (\xi_2, 0.46, -0.73), (\xi_3, 0.76, -0.19)\}. \end{Bmatrix}$$

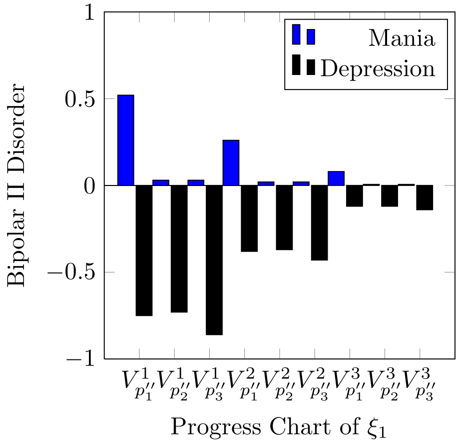
Now, apply second episode of the treatment and take $n = 2$, by applying $\mathfrak{f}'$ on the consequent BFS-set, another BFS-set can be obtained as

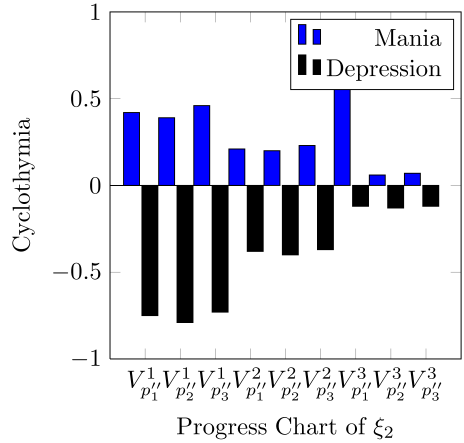
$$V^2_{\mathcal{D}''} = \begin{Bmatrix} \mathcal{K}_{p''_1} = \{(\xi_1, 0.26, -0.38), (\xi_2, 0.21, -0.38), (\xi_3, 0.36, -0.04)\}, \\ \mathcal{K}_{p''_2} = \{(\xi_1, 0.02, -0.37), (\xi_2, 0.20, -0.40), (\xi_3, 0.42, -0.04)\}, \\ \mathcal{K}_{p''_3} = \{(\xi_1, 0.02, -0.43), (\xi_2, 0.23, -0.37), (\xi_3, 0.38, -0.10)\}. \end{Bmatrix}$$

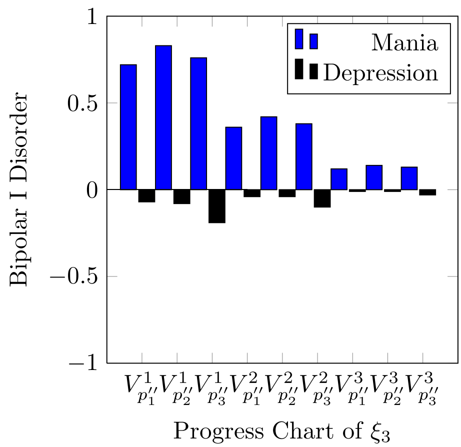
Similarly, when $n = 3$, another application of $\mathfrak{f}'$ to the resultant set gives

$$V^3_{\mathcal{D}''} = \begin{Bmatrix} \mathcal{K}_{p''_1} = \{(\xi_1, 0.08, -0.12), (\xi_2, 0.7, -0.12), (\xi_3, 0.12, -0.01)\}, \\ \mathcal{K}_{p''_2} = \{(\xi_1, 0.006, -0.12), (\xi_2, 0.06, -0.13), (\xi_3, 0.14, -0.01)\}, \\ \mathcal{K}_{p''_3} = \{(\xi_1, 0.006, -0.14), (\xi_2, 0.07, -0.12), (\xi_3, 0.13, -0.03)\}, \end{Bmatrix}$$

if we apply again score function, then we will observe that patients are entering in normal domain. If a patient does not get an improvement by an episode of treatment, then we apply inverse BFS-mapping (by doing this we get the same set because the mappings are injective and by theorem (3.9) part(iv) equality hold) to restrict the values in previous phase of treatment and we do this until we get an improvement. The duration of a single episode of a treatment can be recommended according to the severeness of the disease. The effectiveness of the method can be seen throughout the process. Since psychological treatments take a longer time to recover, so this method is also helpful to keep record and history of each patient. The membership degrees are helpful to know the severeness of the causes and right diagnose of the disease. In particular, when we have uncertain bipolar information or data.







5. Comparison and Discussion

In this section, we discuss the comparison analysis of our method and its accuracy. First, we examine that a multiple weeks chart provided a great fuzzy environment evaluation of patients which is necessary to diagnose perfectly because psychological disorder cannot be judged in a single visit. The BFS-union also provided maximum degrees of patient's basic symptoms which is necessary to examine full manic episode and extreme depression episode if occurs. Second, we see that the connection between primary and basic symptoms and weights provided by doctor (according to severeness of basic symptoms) are very important, if we consider only basic symptoms, then we cannot correctly conclude. In this case, patient 4 also has bipolar disorder which has negative results in our case. Third, the pre-diagnosis chart makes sure the accuracy of the method, we can see patient 1 encounters with one episode of hypomania, 2 episodes of moderate depression and 1 episode of extreme depression which are indicating about bipolar II disorder according to definition. We get the same result after calculating and comparing the score function. A similar evaluation can be made for remaining patients. This evaluation makes sure the method's reliability. In second phase, we see the treatment and its related degrees play an important role to finalize the most accurate treatment for a patient according to their medical history. In third phase of the method, a generalized mapping is helpful to check the improvement of treatment throughout the tenure, if a patient cannot get the convergence state in next episode of treatment, then inverse BFS-mapping shifts it automatically in the previous episode. The method is useful to treat a large number of patients. In particular, this mathematical framework is reliable and capable to handle bipolar disorder efficiently.

6. Conclusion

In this study, we proposed a technique to diagnose a bipolar disorder and its types perfectly. The symptoms and their connections play an importants role to less the complexity of diagnosis its types. In this regard, we define mappings on BFS-set and some useful basic operations and properties of BFS-mappings. We propose an extensive model to correctly diagnose bipolar disorder in the environment of BFS-mappings. The method is helpful to suggest a perfect treatment as well as improvements in treatment episodes. We introduce mappings on bipolar fuzzy soft set, we define concepts of BFS-mappings. We find the image of a BFS-set under defined BFS-mapping. We discuss certain properties of BFS-mappings. Finally, we present an application of BFS-mappings in the area of neurological disorders. We apply the concept of BFS-mapping to diagnose the bipolar disorder and its type. The process is also helpful to track the progress of the treatment episodes. In short, the process is reliable and suitable to treat such kind of medical issues in which uncertain bipolar type information is involved. In future, we extend our work to discuss disorder under the environment of pythagorean fuzzy set and multiple symptoms disorder using m-polar fuzzy information.

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