Bipolar Neutrosophic Graph Structures

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Abstract

In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

Key-words: Graph structure, Bipolar single-valued neutrosophic graph structure, Operations.

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1 Introduction

Fuzzy graph theory has a number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1973, Kauffmann [13] illustrated the notion of fuzzy graphs based on Zadeh's fuzzy relations [24]. Rosenfeld [16] discussed several basic graph-theoretic concepts, including bridges, cut-nodes, connectedness, trees and cycles. Bhattacharya [7] gave some remarks on fuzzy graphs. Later, Bhattacharya [7] gave some remarks on fuzzy graphs. 1994, Mordeson and Chang-Shyh [14] defined some operations on fuzzy graphs. The complement of fuzzy graph was defined in [14]. Further, this concept was discussed by Sunitha and Vijayakumar [20]. Akram described bipolar fuzzy graphs in 2011 [1]. Akram and Shahzadi [4] described the concept of neutrosophic soft graphs with applications. Dinesh and Ramakrishnan [12] introduced the concept of the fuzzy graph structure and investigated some related properties. Akram and Akmal [3] proposed the notion of bipolar fuzzy graph structures. On the other hand, Dhavaseelan et al. [10] defined strong neutrosophic graphs. Broumi et al. [8] portrayed bipolar single-valued neutrosophic graphs. Akram and Shahzadi [4] introduced the notion of neutrosophic soft graphs with applications. Akram [2] introduced the notion of single-valued neutrosophic planar graphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [5]. Single-valued neutrosophic minimum spanning tree and its clustering method were studied by Ye [22]. In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

2 Bipolar Single-Valued Neutrosophic Graph Structures

Smarandache [19] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set has three constituents: truth-membership, indeterminacy-membership and falsity-membership, in which each membership value is a real standard or non-standard subset of the unit interval $]0^-, 1^+[$. In real-life problems, neutrosophic sets can be applied more appropriately by using the single-valued neutrosophic sets defined by Smarandache [19] and Wang et al [21].

Definition 2.1. [19] A *neutrosophic set* N on a non-empty set V is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where, $T_N, I_N, F_N : V \to]0^-, 1^+[$ and there is no restriction on the sum of $T_N(v)$, $I_N(v)$ and $F_N(v)$ for all $v \in V$.

Definition 2.2. [21] A *single-valued neutrosophic set* N on a non-empty set V is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where, $T_N, I_N, F_N : V \to [0, 1]$ and sum of $T_N(v)$, $I_N(v)$ and $F_N(v)$ is confined between 0 and 3 for all $v \in V$.

Deli et al. [9] defined bipolar neutrosophic sets a generalization of bipolar fuzzy sets. They also studied some operations and applications in decision making problems.

Definition 2.3. [9] A *bipolar single-valued neutrosophic set* on a non-empty set V is an object of the form

$$B = \{(v, T_B^P(v), I_B^P(v), F_B^P(v), T_B^N(v), I_B^N(v), F_B^N(v)) : v \in V\}$$

where, $T_B^P, I_B^P, F_B^P : V \to [0, 1]$ and $T_B^N, I_B^N, F_B^N : V \to [-1, 0]$. The positive values $T_B^P(v), I_B^P(v), F_B^P(v)$ denote the truth, indeterminacy and falsity membership values of an element $v \in V$, whereas negative values $T_B^N(v), I_B^N(v), F_B^N(v)$ indicates the implicit counter property of truth, indeterminacy and falsity membership values of an element $v \in V$.

Definition 2.4. A *bipolar single-valued neutrosophic graph* on a non-empty set V is a pair $G = (B, R)$, where B is a bipolar single-valued neutrosophic set on V and R is a bipolar single-valued neutrosophic relation in V such that

$$T_R^P(bd) \le T_B^P(b) \wedge T_B^P(d), \quad I_R^P(bd) \le I_B^P(b) \wedge I_B^P(d), \quad F_R^P(bd) \le F_B^P(b) \vee F_B^P(d),$$
$$T_R^N(bd) \ge T_B^N(b) \vee T_B^N(d), \quad I_R^N(bd) \ge I_B^N(b) \vee I_B^N(d), \quad F_R^N(bd) \ge F_B^N(b) \wedge F_B^N(d) \quad \text{for all } b, d \in V.$$

We now define bipolar single-valued neutrosophic graph structure.

Definition 2.5. $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is called *bipolar single-valued neutrosophic graph structure(BSVNGS)* of graph structure $\check{G}_s = (V, V_1, V_2, \ldots, V_m)$ if $B =< b, T^P(b), I^P(b), F^P(b), T^N(b), I^N(b), F^N(b) >$ and $B_k =< (b, d), T_k^P(b, d), I_k^P(b, d), F_k^P(b, d), T_k^N(b, d), I_k^N(b, d), F_k^N(b, d) >$ are bipolar single-valued neutrosophic(BSVN) sets on V and V_k, respectively, such that

$$T_k^P(b, d) \le \min\{T^P(b), T^P(d)\},\ I_k^P(b, d) \le \min\{I^P(b), I^P(d)\},\ F_k^P(b, d) \le \max\{F^P(b), F^P(d)\},$$
$$T_k^N(b, d) \ge \max\{T^N(b), T^N(d)\},\ I_k^N(b, d) \ge \max\{I^N(b), I^N(d)\},\ F_k^N(b, d) \ge \min\{F^N(b), F^N(d)\}.$$

$\forall b, d \in V$. Note that $0 \le T_k^P(b, d) + I_k^P(b, d) + F_k^P(b, d) \le 3$, $-3 \le T_k^N(b, d) + I_k^N(b, d) + F_k^N(b, d) \le 0$ $\forall (b, d) \in V_k$.

Example 2.6. Consider graph structure(GSR) $\check{G}_s = (V, V_1, V_2)$ such that $V = \{b_1, b_2, b_3, b_4\}$, $V_1 = \{b_1b_3, b_1b_2, b_3b_4\}$, $V_2 = \{b_1b_4, b_2b_3\}$. By defining bipolar single-valued neutrosophic sets B, B_1 and B_2 on V, V_1 and V_2, respectively, we can draw a bipolar SVNGS as depicted in Fig. 2.1.

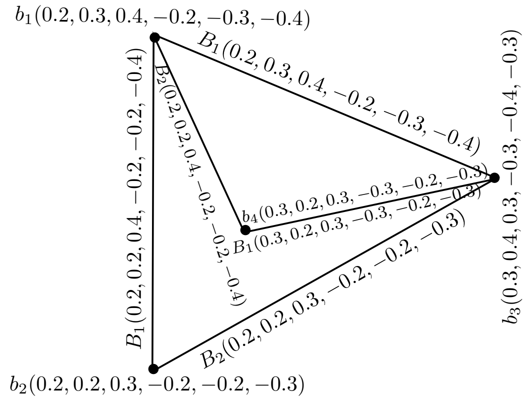


Figure 2.1: A bipolar single-valued neutrosophic graph structure

Definition 2.7. Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS of GSR $\check{G}_s$. If $\check{H}_{bn} = (B', B'_1, B'_2, \ldots, B'_m)$ is a BSVNGS of $\check{G}_s$ such that

$$T'^P(b) \leq T^P(k), I'^P(b) \leq I^P(b), F'^P(b) \geq F^P(b),\ T'^N(b) \geq T^P(k), I'^N(b) \geq I^P(b), F'^N(b) \leq F^N(b)\ ,$$

$$\begin{gathered} T'^P_k(b,d) \leq T^P_k(b,d),\ I'^P_k(b,d) \leq I^P_k(b,d),\ F'^P_k(b,d) \geq F^P_k(b,d), \\ T'^N_k(b,d) \geq T^N_k(b,d),\ I'^N_k(b,d) \geq I^N_k(b,d),\ F'^N_k(b,d) \leq F^N_k(b,d), \\ \forall\ b \in V \text{ and } (b,d) \in V_k,\ k = 1, 2, \ldots, m. \end{gathered}$$

Then $\check{H}_{bn}$ is named as a *bipolar single-valued neutrosophic(BSVN) subgraph structure* of BSVNGS $\check{G}_{bn}$.

Example 2.8. Consider a BSVNGS $\check{H}_{bn} = (B', B'_1, B'_2)$ of GSR $\check{G}_s = (V, V_1, V_2)$ as depicted in Fig. 2.2. Routine calculations indicate that $\check{H}_{bn}$ is BSVN subgraph-structure of BSVNGS $\check{G}_{bn}$.

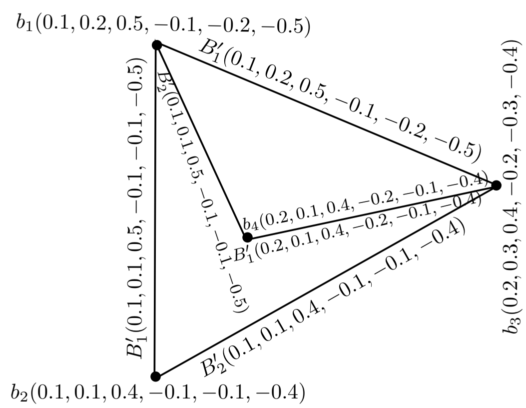


Figure 2.2: A BSVN subgraph structure

Definition 2.9. A BSVNGS $\check{H}_{bn} = (B', B'_1, B'_2, \ldots, B'_m)$ is called a *BSVN induced subgraph-structure* of BSVNGS $\check{G}_{bn}$ by $Q \subseteq V$ if

$$T'^P(b) = T^P(b), I'^P(b) = I^P(b), F'^P(b) = F^P(b),\ T'^N(b) = T^N(b), I'^N(b) = I^N(b), F'^N(b) = F^N(b),$$

$$\begin{gathered} T'^P_k(b,d) = T^P_k(b,d),\ I'^P_k(b,d) = I^P_k(b,d),\ F'^P_k(b,d) = F^P_k(b,d),\ T'^N_k(b,d) = T^N_k(b,d), \\ I'^N_k(b,d) = I^N_k(b,d),\ F'^N_k(b,d) = F^N_k(b,d),\ \forall b, d \in Q,\ k = 1, 2, \ldots, m. \end{gathered}$$

Example 2.10. A BSVNGS depicted in Fig. 2.3 is a BSVN induced subgraph-structure of BSVNGS represented in Fig. 2.1.

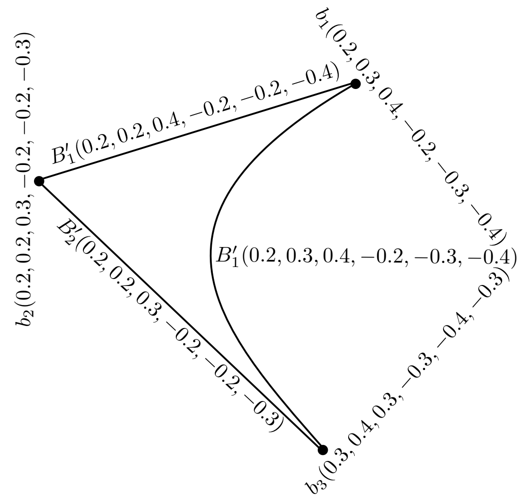


Figure 2.3: A BSVN induced subgraph-structure

Definition 2.11. A BSVNGS $\check{H}_{bn} = (B', B'_1, B'_2, \ldots, B'_m)$ is called *BSVN spanning subgraph-structure* of BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ if $B' = B$ and

$$T'^P_k(b, d) \leq T^P_k(b, d),\ I'^P_k(b, d) \leq I^P_k(b, d),\ F'^P_k(b, d) \geq F^P_k(b, d),\ T'^N_k(b, d) \geq T^N_k(b, d),$$
$$I'^N_k(b, d) \geq I^N_k(b, d),\ F'^N_k(b, d) \leq F^N_k(b, d),\ k = 1, 2, \ldots, m.$$

Example 2.12. A BSVNGS represented in Fig. 2.4 is a BSVN spanning subgraph-structure of BSVNGS represented in Fig. 2.1.

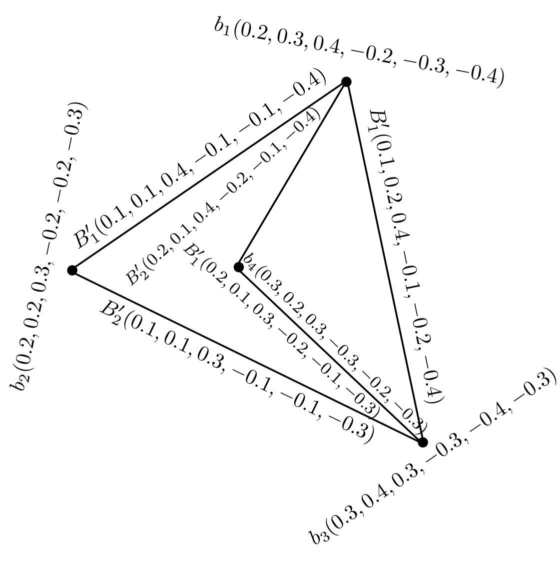


Figure 2.4: A BSVN spanning subgraph-structure

Definition 2.13. Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS. Then $bd \in B_k$ is called a *BSVN* B_k*-edge* or shortly B_k*-edge*, if
$T_k^P(b,d) > 0$ or $I_k^P(b,d) > 0$ or $F_k^P(b,d) > 0$ or $T_k^N(b,d) < 0$ or $I_k^N(b,d) < 0$ or $F_k^N(b,d) < 0$ or all these conditions are satisfied. Consequently, support of B_k is;

$$supp(B_k) = \{bd \in B_k : T_k^P(b,d) > 0\} \cup \{bd \in B_k : I_k^P(b,d) > 0\} \cup \{bd \in B_k : F_k^P(b,d) > 0\} \cup$$
$$\{bd \in B_k : T_k^N(b,d) < 0\} \cup \{bd \in B_k : I_k^N(b,d) < 0\} \cup \{bd \in B_k : F_k^N(b,d) < 0\},\ k = 1, 2, \ldots, m.$$

Definition 2.14. B_k*-path* in BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is a sequence $b_1, b_2, \ldots, b_m$ of distinct nodes(vertices) (except $b_m = b_1$) in V, such that $b_{k-1}b_k$ is a BSVN B_k-edge $\forall k = 2, \ldots, m$.

Definition 2.15. A BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is B_k*-strong* for any $k \in \{1, 2, \ldots, m\}$ if

$$T_k^P(b,d) = \min\{T^P(b), T^P(d)\},\ I_k^P(b,d) = \min\{I^P(b), I^P(d)\},\ F_k^P(b,d) = \max\{F^P(b), F^P(d)\},$$
$$T_k^N(b,d) = \max\{T^N(b), T^N(d)\},\ I_k^N(b,d) = \max\{I^N(b), I^N(d)\},\ F_k^N(b,d) = \min\{F^N(b), F^N(d)\},$$

$\forall\ bd \in supp(B_k)$. If $\check{G}_{bn}$ is B_k-strong $\forall\ k \in \{1, 2, \ldots, m\}$, then $\check{G}_{bn}$ is called *strong BSVNGS.*

Example 2.16. Consider BSVNGS $\check{G}_{bn} = (B, B_1, B_2, B_3)$ as depicted in Fig. 2.5. Then $\check{G}_{bn}$ is strong BSVNGS, since it is B_1-, B_2- and B_3-*strong*.

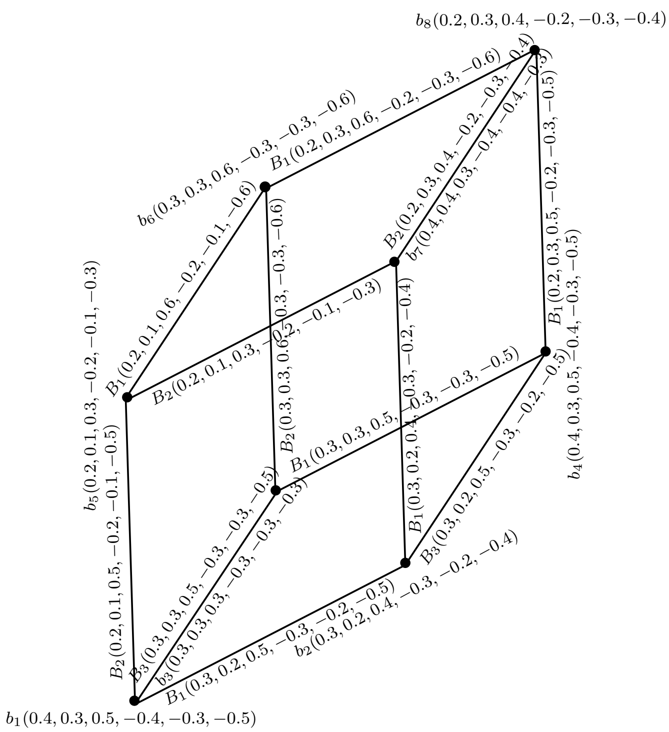


Figure 2.5: A Strong BSVNGS

Definition 2.17. A BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is called complete BSVNGS , if

1. $\check{G}_{bn}$ is strong BSVNGS.
2. $supp(B_k) \neq \emptyset$, for all k = 1, 2, . . . , m.
3. For all $b, d \in V$, bd is a $B_k - edge$ for some k.

Example 2.18. Let $\check{G}_{bn} = (B, B_1, B_2)$ be BSVNGS of GSR $\check{G} = (V, V_1, V_2)$, such that $V = \{b_1, b_2, b_3, b_4\}$, $V_1 = \{b_1b_2, b_3b_4\}$, $V_2 = \{b_1b_3, b_2b_3, b_1b_4, b_2b_4\}$. Through direct calculations, it may be easily shown that $\check{G}_{bn}$ is strong BSVNGS.

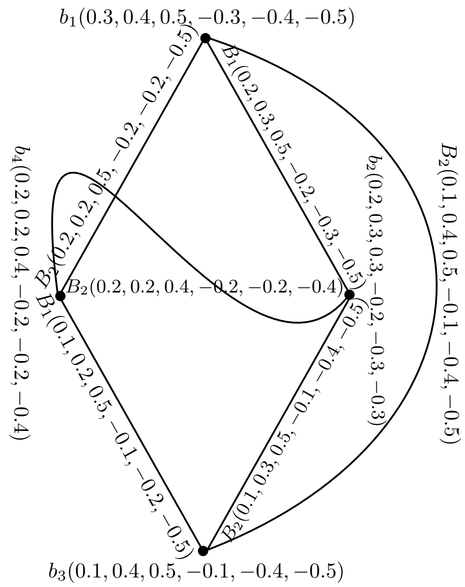


Figure 2.6: A complete BSVNGS

Moreover, $supp(B_1) \neq \emptyset$, $supp(B_2) \neq \emptyset$, and each pair b_kb_l of nodes in V, is either a B_1-edge or B_2-edge. Hence $\check{G}_{bn}$ is complete BSVNGS, that is, B_1B_2-complete BSVNGS.

Definition 2.19. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be two BSVNGSs. *Lexicographic product* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1m} \bullet B_{2m}),$$

is defined as:

$$\text{(i)} \begin{cases} T^P_{(B_1 \bullet B_2)}(bd) = (T^P_{B_1} \bullet T^P_{B_2})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_1 \bullet B_2)}(bd) = (I^P_{B_1} \bullet I^P_{B_2})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_1 \bullet B_2)}(bd) = (F^P_{B_1} \bullet F^P_{B_2})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$$

$$\text{(ii)} \begin{cases} T^N_{(B_1 \bullet B_2)}(bd) = (T^N_{B_1} \bullet T^N_{B_2})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_1 \bullet B_2)}(bd) = (I^N_{B_1} \bullet I^N_{B_2})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_1 \bullet B_2)}(bd) = (F^N_{B_1} \bullet F^N_{B_2})(bd) = F^N_{B_1}(b) \wedge F^N_{B_2}(d) \end{cases}$$

for all $(bd) \in V_1 \times V_2$,

$$\text{(iii)} \begin{cases} T^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}} \bullet T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}} \bullet I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}} \bullet F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$$

(iv) $\begin{cases} T^N_{(B_{1k}\bullet B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}} \bullet T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\bullet B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}} \bullet I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\bullet B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}} \bullet F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$

for all $b \in V_1$, $(d_1d_2) \in V_{2k}$,

(v) $\begin{cases} T^P_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} \bullet T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} \bullet I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} \bullet F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$

(vi) $\begin{cases} T^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} \bullet T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} \bullet I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} \bullet F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$

for all $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$.

Example 2.20. Consider $\check{G}_{b1} = (B_1, B_{11}, B_{12})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22})$ are two BSVNGSs of GSRs $\check{G}_{s1} = (V_1, V_{11}, V_{12})$ and $\check{G}_{s2} = (V_2, V_{21}, V_{22})$, respectively, as depicted in Fig. 2.7, where $V_{11} = \{b_1b_2\}$, $V_{12} = \{b_3b_4\}$, $V_{21} = \{d_1d_2\}$, $V_{22} = \{d_2d_3\}$.

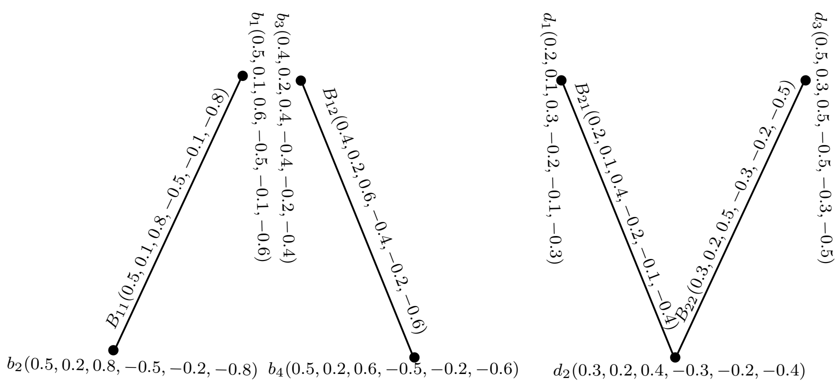


Figure 2.7: Two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$

Lexicographic product of BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 2.7 is defined as $\check{G}_{b1} \bullet \check{G}_{b2} = \{B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}\}$ and is depicted in Fig. 2.8.

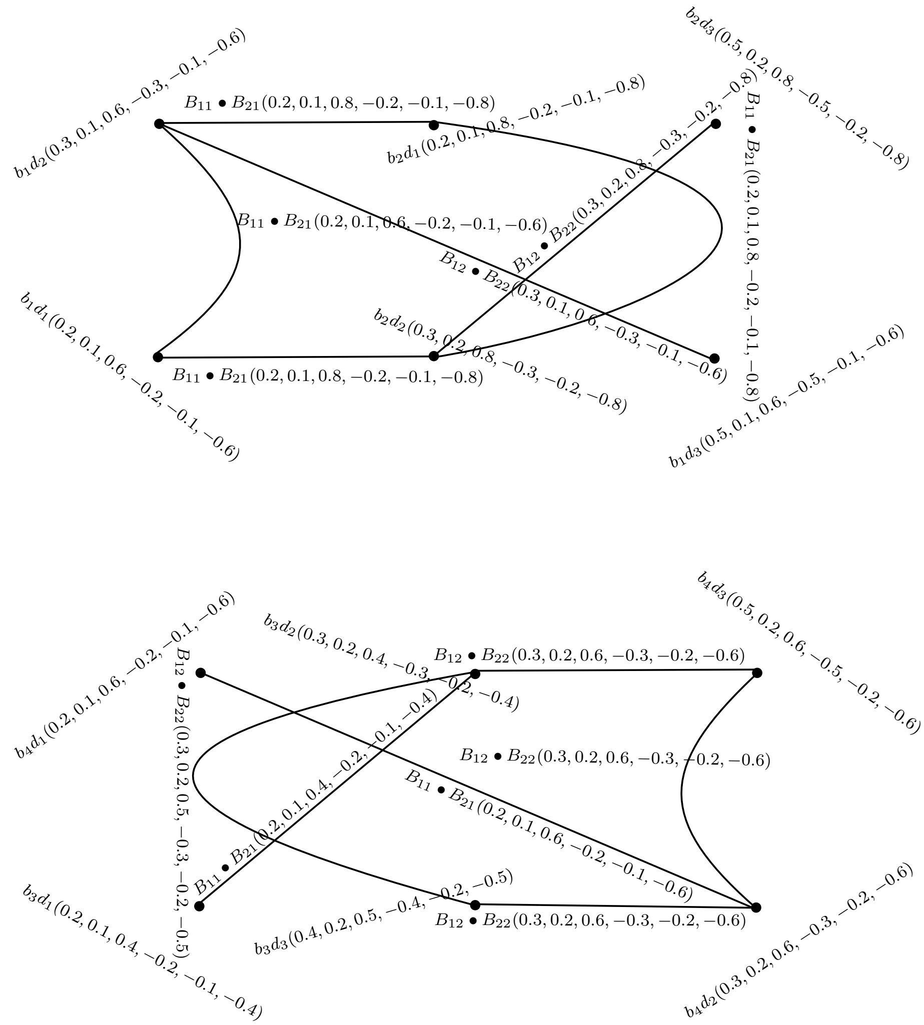


Figure 2.8: $\check{G}_{b1} \bullet \check{G}_{b2}$

Theorem 2.21. *Lexicographic product* $\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1m} \bullet B_{2m})$ *of two BSVNSGSs of GSRs* $\check{G_{s1}}$ *and* $\check{G_{s2}}$ *is a BSVNGS of* $\check{G_{s1}} \bullet \check{G_{s2}}$.

Proof. Consider two cases:

Case 1. For $b \in V_1$, $d_1d_2 \in V_{2k}$

$$\begin{aligned} T^P_{(B_{1k} \bullet B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ &\leq T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1 \bullet B_2)}(bd_1) \wedge T^P_{(B_1 \bullet B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ &\geq T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1\bullet B_2)}(bd_1) \vee T^N_{(B_1\bullet B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} I^P_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ &\leq I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)] \\ &= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)] \\ &= I^P_{(B_1\bullet B_2)}(bd_1) \wedge I^P_{(B_1\bullet B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} I^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ &\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)] \\ &= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)] \\ &= I^N_{(B_1\bullet B_2)}(bd_1) \vee I^N_{(B_1\bullet B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \\ &\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)] \\ &= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)] \\ &= F^P_{(B_1\bullet B_2)}(bd_1) \vee F^P_{(B_1\bullet B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \\ &\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)] \\ &= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)] \\ &= F^N_{(B_1\bullet B_2)}(bd_1) \wedge F^N_{(B_1\bullet B_2)}(bd_2), \end{aligned}$$

for $bd_1, bd_2 \in V_1 \bullet V_2$.

Case 2. For $b_1b_2 \in V_{1k}$, $d_1d_2 \in V_{2k}$

$$\begin{aligned} T^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ &\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2] \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b_1) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b_2) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1\bullet B_2)}(b_1d_1) \wedge T^P_{(B_1\bullet B_2)}(b_2d_2), \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ &\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2] \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b_1) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b_2) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1\bullet B_2)}(b_1d_1) \vee T^N_{(B_1\bullet B_2)}(b_2d_2), \end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2)\wedge I^P_{B_{2k}}(d_1d_2)\\
&\le [I^P_{B_1}(b_1)\wedge I^P_{B_1}(b_2]\wedge[I^P_{B_2}(d_1)\wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b_1)\wedge I^P_{B_2}(d_1)]\wedge[I^P_{B_1}(b_2)\wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\bullet B_2)}(b_1d_1)\wedge I^P_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2)\vee I^N_{B_{2k}}(d_1d_2)\\
&\ge [I^N_{B_1}(b_1)\vee I^N_{B_1}(b_2]\vee[I^N_{B_2}(d_1)\vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b_1)\vee I^N_{B_2}(d_1)]\vee[I^N_{B_1}(b_2)\vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\bullet B_2)}(b_1d_1)\vee I^N_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2)\vee F^P_{B_{2k}}(d_1d_2)\\
&\le [F^P_{B_1}(b_1)\vee F^P_{B_1}(b_2]\vee[F^P_{B_2}(d_1)\vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b_1)\vee F^P_{B_2}(d_1)]\vee[F^P_{B_1}(b_2)\vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\bullet B_2)}(b_1d_1)\vee F^P_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2)\wedge F^N_{B_{2k}}(d_1d_2)\\
&\ge [F^N_{B_1}(b_1)\wedge F^N_{B_1}(b_2]\wedge[F^N_{B_2}(d_1)\wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b_1)\wedge F^N_{B_2}(d_1)]\wedge[F^N_{B_1}(b_2)\wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\bullet B_2)}(b_1d_1)\wedge F^N_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$b_1d_1, b_2d_2 \in V_1 \bullet V_2$ and $h \in \{1, 2, \ldots, m\}$. This completes the proof.

□

Definition 2.22. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be two BSVNGSs. *Strong product* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} \boxtimes \check{G}_{b2} = (B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1m} \boxtimes B_{2m}),$$

is defined as:

$$\text{(i)}\ \begin{cases} T^P_{(B_1\boxtimes B_2)}(bd) = (T^P_{B_1}\boxtimes T^P_{B_2})(bd) = T^P_{B_1}(b)\wedge T^P_{B_2}(d)\\ I^P_{(B_1\boxtimes B_2)}(bd) = (I^P_{B_1}\boxtimes I^P_{B_2})(bd) = I^P_{B_1}(b)\wedge I^P_{B_2}(d)\\ F^P_{(B_1\boxtimes B_2)}(bd) = (F^P_{B_1}\boxtimes F^P_{B_2})(bd) = F^P_{B_1}(b)\vee F^P_{B_2}(d)\end{cases}$$

$$\text{(ii)}\ \begin{cases} T^N_{(B_1\boxtimes B_2)}(bd) = (T^N_{B_1}\boxtimes T^N_{B_2})(bd) = T^N_{B_1}(b)\vee T^N_{B_2}(d)\\ I^N_{(B_1\boxtimes B_2)}(bd) = (I^N_{B_1}\boxtimes I^N_{B_2})(bd) = I^N_{B_1}(b)\vee I^N_{B_2}(d)\\ F^N_{(B_1\boxtimes B_2)}(bd) = (F^N_{B_1}\boxtimes F^N_{B_2})(bd) = F^N_{B_1}(b)\wedge F^P_{B_2}(d)\end{cases}$$

for all $(bd) \in V_1 \times V_2$,

$$\text{(iii)}\ \begin{cases} T^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}}\boxtimes T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b)\wedge T^P_{B_{2k}}(d_1d_2)\\ I^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}}\boxtimes I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b)\wedge I^P_{B_{2k}}(d_1d_2)\\ F^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}}\boxtimes F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b)\vee F^P_{B_{2k}}(d_1d_2)\end{cases}$$

(iv) $\begin{cases} T^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}} \boxtimes T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}} \boxtimes I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}} \boxtimes F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$

for all $b \in V_1$, $(d_1d_2) \in V_{2k}$,

(v) $\begin{cases} T^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (T^P_{B_{1k}} \boxtimes T^P_{B_{2k}})(b_1d)(b_2d) = T^P_{B_2}(d) \wedge T^P_{B_{1k}}(b_1b_2) \\ I^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (I^P_{B_{1k}} \boxtimes I^P_{B_{2k}})(b_1d)(b_2d) = I^P_{B_2}(d) \wedge I^P_{B_{2k}}(b_1b_2) \\ F^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (F^P_{B_{1k}} \boxtimes F^P_{B_{2k}})(b_1d)(b_2d) = F^P_{B_2}(d) \vee F^P_{B_{2k}}(b_1b_2) \end{cases}$

(vi) $\begin{cases} T^N_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (T^N_{B_{1k}} \boxtimes T^N_{B_{2k}})(b_1d)(b_2d) = T^N_{B_2}(d) \vee T^N_{B_{1k}}(b_1b_2) \\ I^N_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (I^N_{B_{1k}} \boxtimes I^N_{B_{2k}})(b_1d)(b_2d) = I^N_{B_2}(d) \vee I^N_{B_{2k}}(b_1b_2) \\ F^N_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (F^N_{B_{1k}} \boxtimes F^N_{B_{2k}})(b_1d)(b_2d) = F^N_{B_2}(d) \wedge F^N_{B_{2k}}(b_1b_2) \end{cases}$

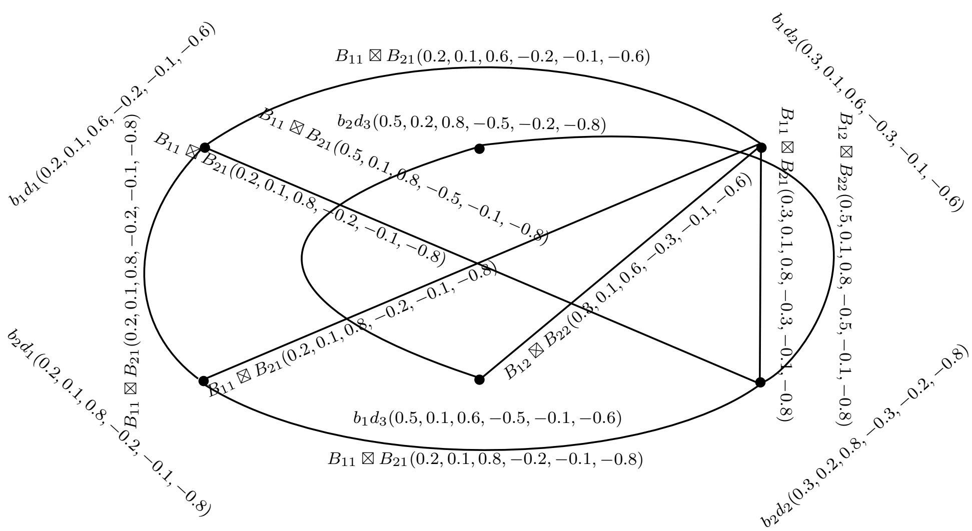
for all $d \in V_2$, $(b_1b_2) \in V_{1k}$.

(vii) $\begin{cases} T^P_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} \boxtimes T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} \boxtimes I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} \boxtimes F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$

(viii) $\begin{cases} T^N_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} \boxtimes T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} \boxtimes I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} \boxtimes F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$

for all $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$.

Example 2.23. Strong product of BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 2.7 is defined as $\check{G}_{b1} \boxtimes \check{G}_{b2} = \{B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}\}$ and is depicted in Fig. 2.9.



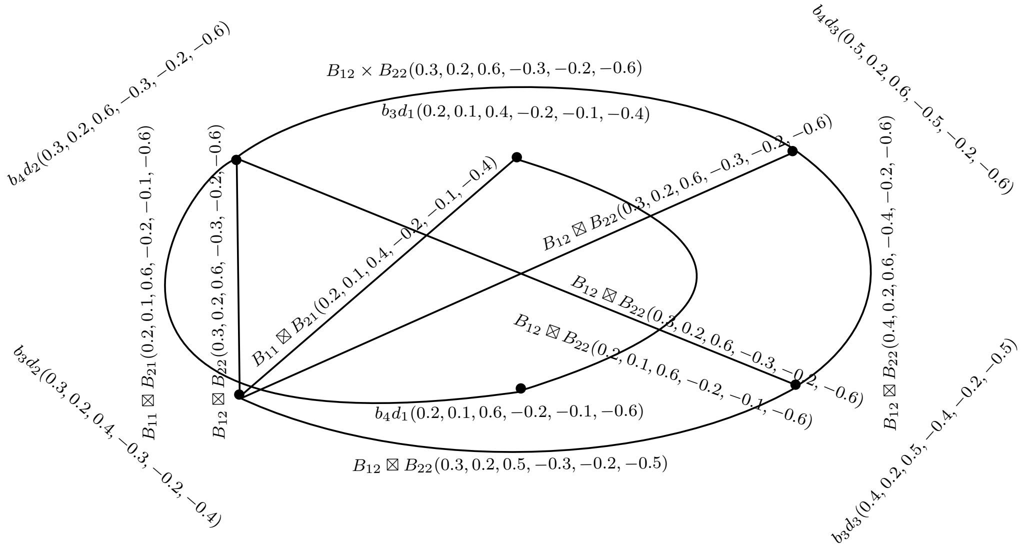


Figure 2.9: $\check{G}_{b1} \boxtimes \check{G}_{b2}$

Theorem 2.24. *Strong product $\check{G}_{b1} \boxtimes \check{G}_{b2} = (B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1m} \boxtimes B_{2m})$ of two BSVNGSs of GSRs $\check{G_{s1}}$ and $\check{G_{s2}}$ is a BSVNGS of $\check{G_{s1}} \boxtimes \check{G_{s2}}$.*

Proof. Consider three cases:

Case 1. For $b \in V_1$, $d_1d_2 \in V_{2k}$

$$\begin{aligned} T^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ &\le T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)] \\ &= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)] \\ &= T^P_{(B_1\boxtimes B_2)}(bd_1) \wedge T^P_{(B_1\boxtimes B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ &\ge T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)] \\ &= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)] \\ &= T^N_{(B_1\boxtimes B_2)}(bd_1) \vee T^N_{(B_1\boxtimes B_2)}(bd_2), \end{aligned}$$

$$\begin{aligned} I^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ &\le I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)] \\ &= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)] \\ &= I^P_{(B_1\boxtimes B_2)}(bd_1) \wedge I^P_{(B_1\boxtimes B_2)}(bd_2), \end{aligned}$$

$$
\begin{aligned}
I^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2)\\
&\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\boxtimes B_2)}(bd_1) \vee I^N_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}
$$

$$
\begin{aligned}
F^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2)\\
&\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\boxtimes B_2)}(bd_1) \vee F^P_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}
$$

$$
\begin{aligned}
F^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\boxtimes B_2)}(bd_1) \wedge F^N_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}
$$

for $bd_1, bd_2 \in V_1 \boxtimes V_2$.

Case 2. For $b \in V_2$, $d_1d_2 \in V_{1k}$

$$
\begin{aligned}
T^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= T^P_{B_2}(b) \wedge T^P_{B_{1k}}(d_1d_2)\\
&\leq T^P_{B_2}(b) \wedge [T^P_{B_1}(d_1) \wedge T^P_{B_1}(d_2)]\\
&= [T^P_{B_2}(b) \wedge T^P_{B_1}(d_1)] \wedge [T^P_{B_2}(b) \wedge T^P_{B_1}(d_2)]\\
&= T^P_{(B_1\boxtimes B_2)}(d_1b) \wedge T^P_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

$$
\begin{aligned}
T^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= T^N_{B_2}(b) \vee T^N_{B_{1k}}(d_1d_2)\\
&\geq T^N_{B_2}(b) \vee [T^N_{B_1}(d_1) \vee T^P_{B_1}(d_2)]\\
&= [T^N_{B_2}(b) \vee T^N_{B_1}(d_1)] \vee [T^N_{B_2}(b) \vee T^N_{B_1}(d_2)]\\
&= T^N_{(B_1\boxtimes B_2)}(d_1b) \vee T^N_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

$$
\begin{aligned}
I^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= I^P_{B_2}(b) \wedge I^P_{B_{1k}}(d_1d_2)\\
&\leq I^P_{B_2}(b) \wedge [I^P_{B_1}(d_1) \wedge I^P_{B_1}(d_2)]\\
&= [I^P_{B_2}(b) \wedge I^P_{B_1}(d_1)] \wedge [I^P_{B_2}(b) \wedge I^P_{B_1}(d_2)]\\
&= I^P_{(B_1\boxtimes B_2)}(d_1b) \wedge I^P_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

$$
\begin{aligned}
I^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= I^N_{B_2}(b) \vee I^N_{B_{1k}}(d_1d_2)\\
&\geq I^N_{B_2}(b) \vee [I^N_{B_1}(d_1) \vee I^N_{B_1}(d_2)]\\
&= [I^N_{B_2}(b) \vee I^N_{B_1}(d_1)] \vee [I^N_{B_2}(b) \vee I^N_{B_1}(d_2)]\\
&= I^N_{(B_1\boxtimes B_2)}(d_1b) \vee I^N_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

$$
\begin{aligned}
F^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= F^P_{B_2}(b)\vee F^P_{B_{1k}}(d_1d_2)\\
&\le F^P_{B_2}(b)\vee [F^P_{B_1}(d_1)\vee F^P_{B_1}(d_2)]\\
&= [F^P_{B_2}(b)\vee F^P_{B_1}(d_1)]\vee [F^P_{B_2}(b)\vee F^P_{B_1}(d_2)]\\
&= F^P_{(B_1\boxtimes B_2)}(d_1b)\vee F^P_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

$$
\begin{aligned}
F^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= F^N_{B_2}(b)\wedge F^N_{B_{1k}}(d_1d_2)\\
&\ge F^N_{B_2}(b)\wedge [F^N_{B_1}(d_1)\wedge F^N_{B_1}(d_2)]\\
&= [F^N_{B_2}(b)\wedge F^N_{B_1}(d_1)]\wedge [F^N_{B_2}(b)\wedge F^N_{B_1}(d_2)]\\
&= F^N_{(B_1\boxtimes B_2)}(d_1b)\wedge F^N_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}
$$

for $d_1b, d_2b \in V_1\boxtimes V_2$.

Case 3. For $b_1b_2\in V_{1k}$, $d_1d_2\in V_{2k}$

$$
\begin{aligned}
T^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2)\wedge T^P_{B_{2k}}(d_1d_2)\\
&\le [T^P_{B_1}(b_1)\wedge T^P_{B_1}(b_2]\wedge [T^P_{B_2}(d_1)\wedge T^P_{B_2}(d_2)]\\
&= [T^P_{B_1}(b_1)\wedge T^P_{B_2}(d_1)]\wedge [T^P_{B_1}(b_2)\wedge T^P_{B_2}(d_2)]\\
&= T^P_{(B_1\boxtimes B_2)}(b_1d_1)\wedge T^P_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
T^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2)\vee T^N_{B_{2k}}(d_1d_2)\\
&\ge [T^N_{B_1}(b_1)\vee T^N_{B_1}(b_2]\vee [T^N_{B_2}(d_1)\vee T^N_{B_2}(d_2)]\\
&= [T^N_{B_1}(b_1)\vee T^N_{B_2}(d_1)]\vee [T^N_{B_1}(b_2)\vee T^N_{B_2}(d_2)]\\
&= T^N_{(B_1\boxtimes B_2)}(b_1d_1)\vee T^N_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
I^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2)\wedge I^P_{B_{2k}}(d_1d_2)\\
&\le [I^P_{B_1}(b_1)\wedge I^P_{B_1}(b_2]\wedge [I^P_{B_2}(d_1)\wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b_1)\wedge I^P_{B_2}(d_1)]\wedge [I^P_{B_1}(b_2)\wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\boxtimes B_2)}(b_1d_1)\wedge I^P_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
I^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2)\vee I^N_{B_{2k}}(d_1d_2)\\
&\ge [I^N_{B_1}(b_1)\vee I^N_{B_1}(b_2]\vee [I^N_{B_2}(d_1)\vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b_1)\vee I^N_{B_2}(d_1)]\vee [I^N_{B_1}(b_2)\vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\boxtimes B_2)}(b_1d_1)\vee I^N_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
F^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2)\vee F^P_{B_{2k}}(d_1d_2)\\
&\le [F^P_{B_1}(b_1)\vee F^P_{B_1}(b_2]\vee [F^P_{B_2}(d_1)\vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b_1)\vee F^P_{B_2}(d_1)]\vee [F^P_{B_1}(b_2)\vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\boxtimes B_2)}(b_1d_1)\vee F^P_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
F^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2)\wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq [F^N_{B_1}(b_1)\wedge F^N_{B_1}(b_2]\wedge[F^N_{B_2}(d_1)\wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b_1)\wedge F^N_{B_2}(d_1)]\wedge[F^N_{B_1}(b_2)\wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\boxtimes B_2)}(b_1d_1)\wedge F^N_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}
$$

$b_1d_1, b_2d_2 \in V_1 \boxtimes V_2$.

All cases hold $\forall\ k \in \{1, 2, \ldots, m\}$. □

Definition 2.25. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be BSVNGSs. *Union* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} \cup \check{G}_{b2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1m} \cup B_{2m}),$$

is defined as:

$$
\text{(i)}\ \begin{cases}
T^P_{(B_1\cup B_2)}(b) = (T^P_{B_1}\cup T^P_{B_2})(b) = T^P_{B_1}(b)\vee T^P_{B_2}(b)\\
I^P_{(B_1\cup B_2)}(b) = (I^P_{B_1}\cup I^P_{B_2})(b) = (I^P_{B_1}(b)+I^P_{B_2}(b))/2\\
F^P_{(B_1\cup B_2)}(b) = (F^P_{B_1}\cup F^P_{B_2})(b) = F^P_{B_1}(b)\wedge F^P_{B_2}(b)
\end{cases}
$$

$$
\text{(ii)}\ \begin{cases}
T^N_{(B_1\cup B_2)}(b) = (T^N_{B_1}\cup T^N_{B_2})(b) = T^N_{B_1}(b)\wedge T^N_{B_2}(b)\\
I^N_{(B_1\cup B_2)}(b) = (I^N_{B_1}\cup I^N_{B_2})(b) = (I^N_{B_1}(b)+I^N_{B_2}(b))/2\\
F^N_{(B_1\cup B_2)}(b) = (F^N_{B_1}\cup F^N_{B_2})(b) = F^N_{B_1}(b)\vee F^N_{B_2}(b)
\end{cases}
$$

for all $b \in V_1 \cup V_2$,

$$
\text{(iii)}\ \begin{cases}
T^P_{(B_{1k}\cup B_{2k})}(bd) = (T^P_{B_{1k}}\cup T^P_{B_{2k}})(bd) = T^P_{B_{1k}}(bd)\vee T^P_{B_{2k}}(bd)\\
I^P_{(B_{1k}\cup B_{2k})}(bd) = (I^P_{B_{1k}}\cup I^P_{B_{2k}})(bd) = (I^P_{B_{1k}}(bd)+I^P_{B_{2k}}(bd))/2\\
F^P_{(B_{1k}\cup B_{2k})}(bd) = (F^P_{B_{1k}}\cup F^P_{B_{2k}})(bd) = F^P_{B_{1k}}(bd)\wedge F^P_{B_{2k}}(bd)
\end{cases}
$$

$$
\text{(iv)}\ \begin{cases}
T^N_{(B_{1k}\cup B_{2k})}(bd) = (T^N_{B_{1k}}\cup T^N_{B_{2k}})(bd) = T^N_{B_{1k}}(bd)\wedge T^N_{B_{2k}}(bd)\\
I^N_{(B_{1k}\cup B_{2k})}(bd) = (I^N_{B_{1k}}\cup I^N_{B_{2k}})(bd) = (I^N_{B_{1k}}(bd)+I^N_{B_{2k}}(bd))/2\\
F^N_{(B_{1k}\cup B_{2k})}(bd) = (F^N_{B_{1k}}\cup F^N_{B_{2k}})(bd) = F^N_{B_{1k}}(bd)\vee F^N_{B_{2k}}(bd)
\end{cases}
$$

for all $(bd) \in V_{1k} \cup V_{2k}$.

Example 2.26. Union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 2.7 is defined as $\check{G}_{b1} \cup \check{G}_{b2} = \{B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}\}$ and is depicted in Fig. 2.10.

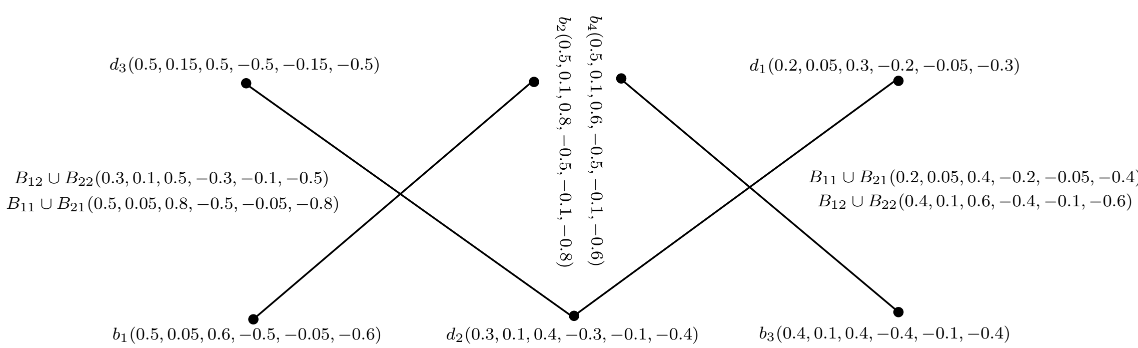


Figure 2.10: $\check{G}_{b1} \cup \check{G}_{b2}$

Theorem 2.27. *Union $\check{G}_{b1} \cup \check{G}_{b2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1m} \cup B_{2m})$ of two BSVNGSs of the GSRs $\check{G}_1$ and $\check{G}_2$ is BSVNGS of $\check{G}_1 \cup \check{G}_2$.*

Proof. Let $b_1 b_2 \in V_{1k} \cup V_{2k}$. Two cases arise:

Case 1. For $b_1, b_2 \in V_1$, by definition 2.25, $T^P_{B_2}(b_1) = T^P_{B_2}(b_2) = T^P_{B_{2k}}(b_1 b_2) = 0$, $I^P_{B_2}(b_1) = I^P_{B_2}(b_2) = I^P_{B_{2k}}(b_1 b_2) = 0$, $F^P_{B_2}(b_1) = F^P_{B_2}(b_2) = F^P_{B_{2k}}(b_1 b_2) = 1$,
$T^N_{B_2}(b_1) = T^N_{B_2}(b_2) = T^N_{B_{2k}}(b_1 b_2) = 0$, $I^N_{B_2}(b_1) = I^N_{B_2}(b_2) = I^N_{B_{2k}}(b_1 b_2) = 0$, $F^N_{B_2}(b_1) = F^N_{B_2}(b_2) = F^N_{B_{2k}}(b_1 b_2) = -1$, so

$$\begin{aligned}
T^P_{(B_{1k} \cup B_{2k})}(b_1 b_2) &= T^P_{B_{1k}}(b_1 b_2) \vee T^P_{B_{2k}}(b_1 b_2) \\
&= T^P_{B_{1k}}(b_1 b_2) \vee 0 \\
&\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2)] \vee 0 \\
&= [T^P_{B_1}(b_1) \vee 0] \wedge [T^P_{B_1}(b_2) \vee 0] \\
&= [T^P_{B_1}(b_1) \vee T^P_{B_2}(b_1)] \wedge [T^P_{B_1}(b_2) \vee T^P_{B_2}(b_2)] \\
&= T^P_{(B_1 \cup B_2)}(b_1) \wedge T^P_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k} \cup B_{2k})}(b_1 b_2) &= T^N_{B_{1k}}(b_1 b_2) \wedge T^N_{B_{2k}}(b_1 b_2) \\
&= T^N_{B_{1k}}(b_1 b_2) \wedge 0 \\
&\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2)] \wedge 0 \\
&= [T^N_{B_1}(b_1) \wedge 0] \vee [T^N_{B_1}(b_2) \wedge 0] \\
&= [T^N_{B_1}(b_1) \wedge T^N_{B_2}(b_1)] \vee [T^N_{B_1}(b_2) \wedge T^N_{B_2}(b_2)] \\
&= T^N_{(B_1 \cup B_2)}(b_1) \vee T^N_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k} \cup B_{2k})}(b_1 b_2) &= F^P_{B_{1k}}(b_1 b_2) \wedge F^P_{B_{2k}}(b_1 b_2) \\
&= F^P_{B_{1k}}(b_1 b_2) \wedge 1 \\
&\leq [F^P_{B_1}(b_1) \vee F^P_{B_1}(b_2)] \wedge 1 \\
&= [F^P_{B_1}(b_1) \wedge 1] \vee [F^P_{B_1}(b_2) \wedge 1] \\
&= [F^P_{B_1}(b_1) \wedge F^P_{B_2}(b_1)] \vee [F^P_{B_1}(b_2) \wedge F^P_{B_2}(b_2)] \\
&= F^P_{(B_1 \cup B_2)}(b_1) \vee F^P_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k} \cup B_{2k})}(b_1 b_2) &= F^N_{B_{1k}}(b_1 b_2) \vee F^N_{B_{2k}}(b_1 b_2) \\
&= F^N_{B_{1k}}(b_1 b_2) \vee -1 \\
&\geq [F^N_{B_1}(b_1) \wedge F^N_{B_1}(b_2)] \vee -1 \\
&= [F^N_{B_1}(b_1) \vee -1] \wedge [F^N_{B_1}(b_2) \vee -1] \\
&= [F^N_{B_1}(b_1) \vee F^N_{B_2}(b_1)] \wedge [F^N_{B_1}(b_2) \vee F^N_{B_2}(b_2)] \\
&= F^N_{(B_1 \cup B_2)}(b_1) \wedge F^N_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$
\begin{aligned}
I^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^P_{B_{1k}}(b_1b_2) + I^P_{B_{2k}}(b_1b_2)}{2} \\
&= \frac{I^P_{B_{1k}}(b_1b_2) + 0}{2} \\
&\leq \frac{[I^P_{B_1}(b_1) \wedge I^P_{B_1}(b_2)] + 0}{2} \\
&= [\frac{I^P_{B_1}(b_1)}{2} + 0] \wedge [\frac{I^P_{B_1}(b_2)}{2} + 0] \\
&= \frac{[I^P_{B_1}(b_1) + I^P_{B_2}(b_1)]}{2} \wedge \frac{[I^P_{B_1}(b_2) + I^P_{B_2}(b_2)]}{2} \\
&= I^P_{(B_1\cup B_2)}(b_1) \wedge I^P_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
I^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^N_{B_{1k}}(b_1b_2) + I^N_{B_{2k}}(b_1b_2)}{2} \\
&= \frac{I^N_{B_{1k}}(b_1b_2) + 0}{2} \\
&\geq \frac{[I^N_{B_1}(b_1) \vee I^N_{B_1}(b_2)] + 0}{2} \\
&= [\frac{I^N_{B_1}(b_1)}{2} + 0] \vee [\frac{I^N_{B_1}(b_2)}{2} + 0] \\
&= \frac{[I^N_{B_1}(b_1) + I^N_{B_2}(b_1)]}{2} \vee \frac{[I^N_{B_1}(b_2) + I^N_{B_2}(b_2)]}{2} \\
&= I^N_{(B_1\cup B_2)}(b_1) \vee I^N_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

for $b_1, b_2 \in V_1 \cup V_2$.

Case 2. For $b_1, b_2 \in V_2$, by definition 2.25, $T^P_{B_1}(b_1) = T^P_{B_1}(b_2) = T^P_{B_{1k}}(b_1b_2) = 0$, $I^P_{B_1}(b_1) = I^P_{B_1}(b_2) = I^P_{B_{1k}}(b_1b_2) = 0$, $F^P_{B_1}(b_1) = F^P_{B_2}(b_2) = F^P_{B_{1k}}(b_1b_2) = 1$,
$T^N_{B_1}(b_1) = T^N_{B_1}(b_2) = T^N_{B_{1k}}(b_1b_2) = 0$, $I^N_{B_1}(b_1) = I^N_{B_1}(b_2) = I^N_{B_{1k}}(b_1b_2) = 0$, $F^N_{B_1}(b_1) = F^N_{B_2}(b_2) = F^N_{B_{1k}}(b_1b_2) = -1$, so

$$
\begin{aligned}
T^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= T^P_{B_{1k}}(b_1b_2) \vee T^P_{B_{2k}}(b_1b_2) \\
&= T^P_{B_{2k}}(b_1b_2) \vee 0 \\
&\leq [T^P_{B_2}(b_1) \wedge T^P_{B_2}(b_2)] \vee 0 \\
&= [T^P_{B_2}(b_1) \vee 0] \wedge [T^P_{B_2}(b_2) \vee 0] \\
&= [T^P_{B_2}(b_1) \vee T^P_{B_1}(b_1)] \wedge [T^P_{B_2}(b_2) \vee T^P_{B_1}(b_2)] \\
&= T^P_{(B_1\cup B_2)}(b_1) \wedge T^P_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
T^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= T^N_{B_{1k}}(b_1b_2) \wedge T^N_{B_{2k}}(b_1b_2) \\
&= T^N_{B_{2k}}(b_1b_2) \wedge 0 \\
&\geq [T^N_{B_2}(b_1) \vee T^N_{B_2}(b_2)] \wedge 0 \\
&= [T^N_{B_2}(b_1) \wedge 0] \vee [T^N_{B_2}(b_2) \wedge 0] \\
&= [T^N_{B_2}(b_1) \wedge T^N_{B_1}(b_1)] \vee [T^N_{B_2}(b_2) \wedge T^N_{B_1}(b_2)] \\
&= T^N_{(B_1\cup B_2)}(b_1) \vee T^N_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
F^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^P_{B_{1k}}(b_1b_2) \wedge F^P_{B_{2k}}(b_1b_2)\\
&= F^P_{B_{2k}}(b_1b_2) \wedge (1)\\
&\leq [F^P_{B_2}(b_1) \vee F^P_{B_2}(b_2)] \wedge (1)\\
&= [F^P_{B_2}(b_1) \wedge (1)] \vee [F^P_{B_2}(b_2) \wedge (1)]\\
&= [F^P_{B_2}(b_1) \wedge F^P_{B_1}(b_1)] \vee [F^P_{B_2}(b_2) \wedge F^P_{B_1}(b_2)]\\
&= F^P_{(B_1\cup B_2)}(b_1) \vee F^P_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
F^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^N_{B_{1k}}(b_1b_2) \vee F^N_{B_{2k}}(b_1b_2)\\
&= F^N_{B_{2k}}(b_1b_2) \vee (-1)\\
&\geq [F^N_{B_2}(b_1) \wedge F^N_{B_2}(b_2)] \vee (-1)\\
&= [F^N_{B_2}(b_1) \vee (-1)] \wedge [F^N_{B_2}(b_2) \vee (-1)]\\
&= [F^N_{B_2}(b_1) \vee F^N_{B_1}(b_1)] \wedge [F^N_{B_2}(b_2) \vee F^N_{B_1}(b_2)]\\
&= F^N_{(B_1\cup B_2)}(b_1) \wedge F^N_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
I^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^P_{B_{1k}}(b_1b_2) + I^P_{B_{2k}}(b_1b_2)}{2}\\
&= \frac{I^P_{B_{2k}}(b_1b_2) + 0}{2}\\
&\leq \frac{[I^P_{B_2}(b_1) \wedge I^P_{B_2}(b_2)] + 0}{2}\\
&= [\frac{I^P_{B_2}(b_1)}{2} + 0] \wedge [\frac{I^P_{B_2}(b_2)}{2} + 0]\\
&= \frac{[I^P_{B_2}(b_1) + I^P_{B_1}(b_1)]}{2} \wedge \frac{[I^P_{B_2}(b_2) + I^P_{B_1}(b_2)]}{2}\\
&= I^P_{(B_1\cup B_2)}(b_1) \wedge I^P_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

$$
\begin{aligned}
I^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^N_{B_{1k}}(b_1b_2) + I^N_{B_{2k}}(b_1b_2)}{2}\\
&= \frac{I^N_{B_{2k}}(b_1b_2) + 0}{2}\\
&\geq \frac{[I^N_{B_2}(b_1) \vee I^N_{B_2}(b_2)] + 0}{2}\\
&= [\frac{I^N_{B_2}(b_1)}{2} + 0] \vee [\frac{I^N_{B_2}(b_2)}{2} + 0]\\
&= \frac{[I^N_{B_2}(b_1) + I^N_{B_1}(b_1)]}{2} \vee \frac{[I^N_{B_2}(b_2) + I^N_{B_1}(b_2)]}{2}\\
&= I^N_{(B_1\cup B_2)}(b_1) \vee I^N_{(B_1\cup B_2)}(b_2),
\end{aligned}
$$

for $b_1, b_2 \in V_1 \cup V_2$.

Both cases hold $\forall\ k \in \{1, 2, \ldots, m\}$. This completes the proof. □

Theorem 2.28. *Let $\check{G}_s = (V_1 \cup V_2, V_{11} \cup V_{21}, V_{12} \cup V_{22}, \ldots, V_{1m} \cup V_{2m})$ be union of GSRs $\check{G}_{s1} = (V_1, V_{11}, V_{12}, \ldots, V_{1m})$ and $\check{G}_{s2} = (V_2, V_{21}, V_{22}, \ldots, V_{2m})$. Then every BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ of $\check{G}_s$ is union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ of GSRs $\check{G}_{s1}$ and $\check{G}_{s2}$, respectively.*

Proof. Firstly, we define B_1, B_2, B_{1k} and B_{2k} for $k \in \{1, 2, \ldots, m\}$ as:
$T^P_{B_1}(b) = T^P_B(b), I^P_{B_1}(b) = I^P_B(b), F^P_{B_1}(b) = F^P_B(b)$,
$T^N_{B_1}(b) = T^N_B(b), I^N_{B_1}(b) = I^N_B(b), F^N_{B_1}(b) = F^N_B(b)$, if $b \in V_1$.

$T^P_{B_2}(b) = T^P_B(b), I^P_{B_2}(b) = I^P_B(b), F^P_{B_2}(b) = F^P_B(b)$,
$T^N_{B_2}(b) = T^N_B(b), I^N_{B_2}(b) = I^N_B(b), F^N_{B_2}(b) = F^N_B(b)$, if $b \in V_2$.

$T^P_{B_{1k}}(b_1b_2) = T^P_{B_k}(b_1b_2), I^P_{B_{1k}}(b_1b_2) = I^P_{B_k}(b_1b_2), F^P_{B_{1k}}(b_1b_2) = F^P_{B_k}(b_1b_2), T^N_{B_{1k}}(b_1b_2) = T^N_{B_k}(b_1b_2), I^N_{B_{1k}}(b_1b_2) = I^N_{B_k}(b_1b_2), F^N_{B_{1k}}(b_1b_2) = F^N_{B_k}(b_1b_2)$, if $b_1b_2 \in V_{1k}$.
$T^P_{B_{2k}}(b_1b_2) = T^P_{B_k}(b_1b_2), I^P_{B_{2k}}(b_1b_2) = I^P_{B_k}(b_1b_2), F^P_{B_{2k}}(b_1b_2) = F^P_{B_k}(b_1b_2), T^N_{B_{2k}}(b_1b_2) = T^N_{B_k}(b_1b_2), I^N_{B_{2k}}(b_1b_2) = I^N_{B_k}(b_1b_2), F^N_{B_{2k}}(b_1b_2) = F^N_{B_k}(b_1b_2)$, if $b_1b_2 \in V_{2k}$.

Then $B = B_1 \cup B_2$ and $B_k = B_{1k} \cup B_{2k}$, $k \in \{1, 2, \ldots, m\}$. Now for $b_1b_2 \in V_{tk}$, $t = 1, 2$, $k \in \{1, 2, \ldots, m\}$:
$T^P_{B_{tk}}(b_1b_2) = T^P_{B_k}(b_1b_2) \leq T^P_B(b_1) \wedge T^P_B(b_2) = T^P_{B_t}(b_1) \wedge T^P_{B_t}(b_2)$, $I^P_{B_{tk}}(b_1b_2) = I^P_{B_k}(b_1b_2) \leq I^P_B(b_1) \wedge I^P_B(b_2) = I^P_{B_t}(b_1) \wedge I^P_{B_t}(b_2)$, $F^P_{B_{tk}}(b_1b_2) = F^P_{B_k}(b_1b_2) \leq F^P_B(b_1) \vee F^P_B(b_2) = F^P_{B_t}(b_1) \vee F^P_{B_t}(b_2)$,

$T^N_{B_{tk}}(b_1b_2) = T^N_{B_k}(b_1b_2) \geq T^N_B(b_1) \vee T^N_B(b_2) = T^N_{B_t}(b_1) \vee T^N_{B_t}(b_2)$, $I^N_{B_{tk}}(b_1b_2) = I^N_{B_k}(b_1b_2) \geq I^N_B(b_1) \vee I^N_B(b_2) = I^N_{B_t}(b_1) \vee I^N_{B_t}(b_2)$, $F^N_{B_{tk}}(b_1b_2) = F^N_{B_k}(b_1b_2) \geq F^N_B(b_1) \wedge F^N_B(b_2) = F^N_{B_t}(b_1) \wedge F^N_{B_t}(b_2)$, i.e.,
$\check{G}_{bl} = (B_l, B_{l1}, B_{l2}, \ldots, B_{lm})$ is a BSVNGS of $\check{G}_t$, $t = 1, 2$. Thus $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$, a BSVNGS of $\check{G}_s = \check{G}_{s1} \cup \check{G}_{s2}$, is the union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$. □

Definition 2.29. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be BSVNGSs and let $V_1 \cap V_2 = \emptyset$. *Join* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1m} + B_{2m}),$$

is defined as:

(i) $\begin{cases} T^P_{(B_1+B_2)}(b) = T^P_{(B_1\cup B_2)}(b) \\ I^P_{(B_1+B_2)}(b) = I^P_{(B_1\cup B_2)}(b) \\ F^P_{(B_1+B_2)}(b) = F^P_{(B_1\cup B_2)}(b) \end{cases}$

(ii) $\begin{cases} T^N_{(B_1+B_2)}(b) = T^N_{(B_1\cup B_2)}(b) \\ I^N_{(B_1+B_2)}(b) = I^N_{(B_1\cup B_2)}(b) \\ F^N_{(B_1+B_2)}(b) = F^N_{(B_1\cup B_2)}(b) \end{cases}$
for all $b \in V_1 \cup V_2$,

(iii) $\begin{cases} T^P_{(B_{1k}+B_{2k})}(bd) = T^P_{(B_{1k}\cup B_{2k})}(bd) \\ I^P_{(B_{1k}+B_{2k})}(bd) = I^P_{(B_{1k}\cup B_{2k})}(bd) \\ F^P_{(B_{1k}+B_{2k})}(bd) = F^P_{(B_{1k}\cup B_{2k})}(bd) \end{cases}$

(iv) $\begin{cases} T^N_{(B_{1k}+B_{2k})}(bd) = T^N_{(B_{1k}\cup B_{2k})}(bd) \\ I^N_{(B_{1k}+B_{2k})}(bd) = I^N_{(B_{1k}\cup B_{2k})}(bd) \\ F^N_{(B_{1k}+B_{2k})}(bd) = F^N_{(B_{1k}\cup B_{2k})}(bd) \end{cases}$
for all $(bd) \in V_{1k} \cup V_{2k}$,

(v) $\begin{cases} T^P_{(B_{1k}+B_{2k})}(bd) = (T^P_{B_{1k}} + T^P_{B_{2k}})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_{1k}+B_{2k})}(bd) = (I^P_{B_{1k}} + I^P_{B_{2k}})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_{1k}+B_{2k})}(bd) = (F^P_{B_{1k}} + F^P_{B_{2k}})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$

(vi) $\begin{cases} T^N_{(B_{1k}+B_{2k})}(bd) = (T^N_{B_{1k}} + T^N_{B_{2k}})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_{1k}+B_{2k})}(bd) = (I^N_{B_{1k}} + I^N_{B_{2k}})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_{1k}+B_{2k})}(bd) = (F^N_{B_{1k}} + F^N_{B_{2k}})(bd) = F^N_{B_1}(b) \wedge F^N_{B_2}(d) \end{cases}$

for all $b \in V_1$, $d \in V_2$.

Example 2.30. Join of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 2.7 is defined as $\check{G}_{b1} + \check{G}_{b2} = \{B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}\}$ and is depicted in Fig. 2.11.

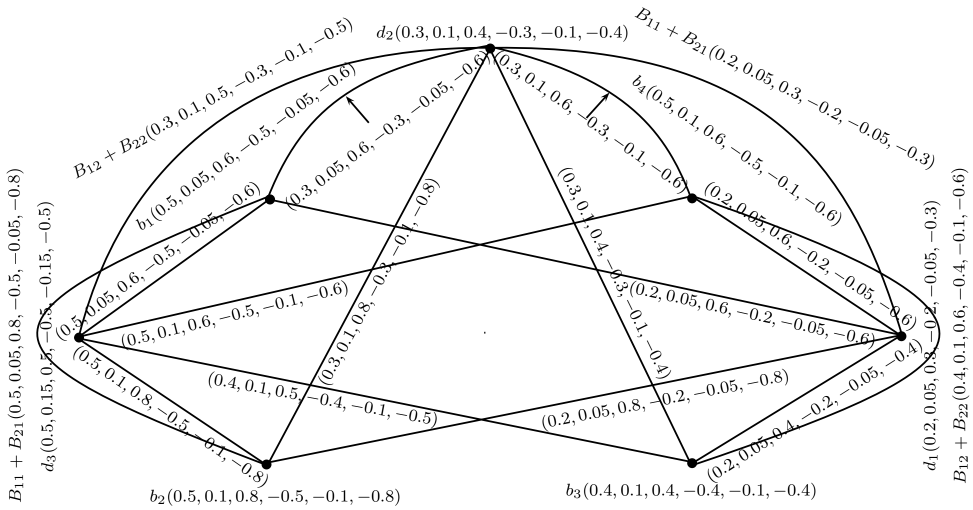


Figure 2.11: $\check{G}_{b1} + \check{G}_{b2}$

Theorem 2.31. *Join* $\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1m} + B_{2m})$ *of two BSVNGSs of the GSRs* $\check{G}_1$ *and* $\check{G}_2$ *is BSVNGS of* $\check{G}_1 + \check{G}_2$.

3 Conclusions

Bipolar fuzzy graph theory has numerous applications in various fields of science and technology including, artificial intelligence, operations research and decision making. A bipolar neutrosophic graph constitutes a generalization of the notion bipolar fuzzy graph. In this research paper, We have introduced the idea of bipolar single-valued neutrosophic graph structure and discussed many relevant notions. We also discussed a worthwhile application of bipolar single-valued neutrosophic graph structure in decision-making. In future, we aim to generalize our notions to (1) BSVN hypergraph structures, (2) BSVN vague hypergraph structures, (3) BSVN interval-valued hypergraph structures, and(4) BSVN rough hypergraph structures.

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