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Computing the Greatest X-eigenvector of Fuzzy Neutrosophic Soft Matrix

Research Article

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Abstract: A Fuzzy Neutrosophic Soft Vector(FNSV) x is said to be a Fuzzy Neutrosophic Soft Eigenvector(FNSEv) of a square max-min Fuzzy Neutrosophic Soft Matrix (FNSM) A if $A \otimes x = x$. A FNSEv x of A is called the greatest X-FNSEv of A if $x \in X = \{x : \underline{x} \leq x \leq \overline{x}\}$ and $y \leq x$ for each FNSEv $y \in X$. A max-min FNSM A is called strongly X-robust if the orbit $x, A \otimes x, A^2 \otimes x, ...$ reaches the greatest X-FNSEv with any starting FNSV of X. We suggest an $O(n^3)$ algorithm for computing the greatest X-FNSEv of A and study the strong X-robustness. The necessary and sufficient condition for strong X-robustness are introduced and an efficient algarithm for verifying these conditions is described.

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Keywords: Fuzzy Neutrosophic Soft Set (FNSS), Fuzzy Neutrosophic Soft Matrices(FNSMs), Fuzzy Neutrosophic Soft Eigenvectors(FNSEv), Interval vector, max-min Fuzzy Neutrosophic Soft Matrix(FNSM).

1. Introduction

Uncertainty forms have a very important part in our daily life. During the time we handle real life problems involving uncertainty like Medical fields, Engineering, Industry and Economics and so on. The conventional techniques may not be enough and easy, so Zadeh [34] gave the introduction of fuzzy set theory and this came out to be a gift for the study of some uncertainty types whenever old techniques did not work. Fuzzy theory and the generalizations regarding it contributed to some remarkable results in real life that involve uncertainties of certain type. Ranjit [26] has analyzed that whether the fuzzy theory is an appropriate tool for large size problem in imprecision and uncertainty in information representation and processing or not. For the motive of handling different type of uncertainties, several generalizations and modification regarding fuzzy set theory like vague sets, rough sets, soft sets, theory of Intuitionistic Fuzzy Set (IFS) and other generalization IFS is most useful. Atanassov [2, 18] developed the concept of IFS. The ideas of IFSs, were developed later in [19, 20]. In 1995, Smarandache [30] founded a theory called neutrosophic theory and neutrosophic set has capability to deal with uncertainty, imprecise, incomplete and inconsistent information which exist in real world. The theory is a powerful tool which generalizes the concept of the classical set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, interval -valued intuitionistic fuzzy set, and so on. In 1999, a Russian researcher Molodtsov [12] initiated the concept of soft set theory as a general mathematical tool for handling uncertainty and vagueness. After Molodtsov's work several researchers were studied on soft

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set theory with applications. Maji et.al, [13] initiated the concept of fuzzy soft set with some properties regarding fuzzy soft union, intersection, complement of fuzzy soft set. Moreover Maji et al. [14] extended soft sets to intuitionistic fuzzy soft sets and neutrosophic soft sets. Matrices in max-min algebra (the addition and the multiplication are formally replaced by operations of maximum and minimum) can be used in a range of practical problems related to scheduling, optimization, modeling of fuzzy discrete dynamic system, graph theory, knowledge engineering, cluster analysis, fuzzy system and also related to describing diagnosis of technical devices [33] or Medical diagnosis [28]. The research of max-min algebra can be motivated by adapting max-plus multi-processor interaction systems [3]. In these systems we have n processors which work in stages, and the algebraic model of their interactive work, entry $x_i(k)$ of a vector x(k), represents the state of processor i after some stage k, and the entry a_{ij} of a matrix A encodes the influence of the work of processor j in the previous stage on the work of processor i in the current stage. For simplicity, the system is assumed to be homogeneous, so that A does not change from stage to stage. Summing up all the influence effects multiplied by the results of previous stage, we have $x_i(k+1) = \bigoplus_j a_{ij} \otimes x_j(k)$. The summation is often interpreted as waiting till all works of the system are finished and all the necessary influence constraints are satisfied.

Thus the orbit $x, A \otimes x, ...A^k \otimes x$ where $A^k = A \otimes ... \otimes A$, (k-times) represents the evolution of such a system. Regarding the orbits, one wishes to know the set of starting vectors from which a steady state of multi-processor interaction system (an eigenvector of A; $A \otimes x = x$) can be achieved. The set of starting vectors from which a system reaches an eigenvector (the greatest eigenvector) of A after a finite number of stage, in general, contains the set of all eigenvector, but it can be an interval vector $X = [\underline{x}, \overline{x}] := \{x; \underline{x} \leq x \leq \overline{x}\}$ and also as big as the whole space. Kim and Rouch [21] introduced the concept of Fuzzy Matrix(FM). FM plays a vital role in various areas in Science and Engenering and solves the problems involving various types of uncertainties [15]. FMs deal only with membership value where as Intuitionistic Fuzzy Matrices(IFMs) deals with both membership and non-membership values. Khan et.al, [16] introduced the concept of IFMs and several interesting properties on IFMs have been obtained in [17]. Yang and Ji [32], introduced a matrix representation of fuzzy set and applied it in decision making problems. Bora et.al, [4] introduced the intuitionistic fuzzy soft matrices and applied in the application of a Medical diagnosis. Sumathi and Arokiarani [1] introduced new operation on fuzzy neutrosophic soft matrices. Dhar et.al, [6] have also defined neutrosophic fuzzy matrices and studied square neutrosophic fuzzy matrices. Uma et.al, [31] introduced two types of fuzzy neutrosophic soft matrices.

In the present paper, we consider a generalized version of the problem to compute the greatest FNSEv of A belonging to an interval FNSV X (called the greatest X-FNSEv of A), which is the main result of the paper. We show that under a certain natural condition the greatest X-FNSEv of A can be computed by an $0(n^3)$ algorithm. The next section will be occupied by some definition and notation of the max-min Fuzzy Neutosophic Soft Algebra (FNSA), leading to the discussion of the greatest X-FNSEv of the FNSM A and strong X-robustness of A. Section-6 is devoted to the main result characterizing strong X-robust FNSM with orbit of A. Let us conclude with a brief overview of the work on max-min FNSA to which this paper is related. The problem of computing the greatest eigenvector of a given max-min FNSM. The concepts of robustness (an FNSEv of A is reached with any starting vector) and strong robustness (the greatest FNSEv of A is reached with any starting vector) in max-min algebra. Some equivalent conditions and efficient algorithms for some polynomial procedures checking the weak robustness (an eigenvector is reached only if a staring vector is an eigenvector of A) in max-min algebra.

2. Preliminaries

In this section some basic definition of fuzzy neutrosophic soft set, fuzzy neutrosophic soft matrix, and fuzzy neutrosophic soft matrix Type-I.

Definition 2.1 ([30]). *A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to\ ]^-0, 1^+[$ and*

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+ \tag{1}$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (1) as $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. In short an element $\widetilde{a}$ in the neutrosophic set A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

Example 2.2. *Assume that the universe of discourse $X = \{x_1,\ x_2,\ x_3\}$ where $x_1,\ x_2$ and x_3 characterize the quality, reliability, and the price of the objects. It may be further assumed that the values of $\{x_1,\ x_2,\ x_3\}$ are in [0,1] and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that $A = \{\langle x_1,\ 0.4,\ 0.5,\ 0.3\rangle, \langle x_2,\ 0.7,\ 0.2,\ 0.4\rangle, \langle x_3, 0.8,\ 0.3,\ 0.4\rangle\}$ where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc,.*

Definition 2.3 ([12]). *Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to the fuzzy neutrosophic soft set (FNSS) over U, where F is a mapping given by $F : A \to P(U)$. Here after we simply consider A as FNSS over U instead of (F, A).*

Definition 2.4 ([1]). *Let $U = \{c_1,\ c_2,\ ...c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1,\ e_2,\ ...e_m\}$. Let $A \subset E$. A pair (F, A) be a $FNSS$ over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$ which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by $T_{R_A} : U \times E \to [0, 1]$, $I_{R_A} : U \times E \to [0, 1]$ and $F_{R_A} : U \times E \to [0, 1]$ where $T_{R_A}(u, e) \in [0, 1]$, $I_{R_A}(u, e) \in [0, 1]$ and $F_{R_A}(u, e) \in [0, 1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$. If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i, e_j),\ I_{ij}(u_i, e_j)\ , F_{ij}(u_i, e_j)]$ we define a matrix*

$$[\langle T_{ij}\ , I_{ij},\ F_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle \end{bmatrix}.$$

Which is called an $m \times n$ $FNSM$ of the FNSS (F_A, E) over U

2.1. FNSMs of Type-I

Definition 2.5 ([31]). *Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij}\rangle),\ B = \langle (b^T_{ij},\ b^I_{ij},\ b^F_{ij}\rangle) \in \mathcal{N}_{m\times n}$. The component wise addition and component wise multiplication is defined as*

$$A \oplus B = (\sup\{a^T_{ij},\ b^T_{ij}\}, \sup\{a^I_{ij},\ b^I_{ij}\}, \inf\{a^F_{ij},\ b^F_{ij}\})$$

$$A \otimes B = (\inf\{a^T_{ij},\ b^T_{ij}\}, \inf\{a^I_{ij},\ b^I_{ij}\}, \sup\{a^F_{ij},\ b^F_{ij}\})$$

Definition 2.6. *Let $A \in \mathcal{N}_{m\times n},\ B \in \mathcal{N}_{n\times p}$, the composition of A and B is defined as*

$$A \circ B = \left(\sum_{k=1}^{n}(a_{ik}^{T} \wedge\ b_{kj}^{T}),\quad \sum_{k=1}^{n}(a_{ik}^{I} \wedge\ b_{kj}^{I}),\quad \prod_{k=1}^{n}(a_{ik}^{F} \vee\ b_{kj}^{F})\right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n}(a_{ik}^{T} \wedge b_{kj}^{T}),\quad \bigvee_{k=1}^{n}(a_{ik}^{I} \wedge b_{kj}^{I}),\quad \bigwedge_{k=1}^{n}(a_{ik}^{F} \vee b_{kj}^{F})\right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$. Where $\sum(a_{ik}^{T} \wedge\ b_{kj}^{T})$ means max-min operation and $\prod_{k=1}^{n}(a_{ik}^{F} \vee\ b_{kj}^{F})$ means min-max operation.

3. Main Result

Let $(\mathcal{N}, \leq)$ be a bounded linearly ordered set with the least element in $\mathcal{N}$ denoted by $O = (0, 0, 1)$ and the greatest one by $I = (1, 1, 0)$. The set of naturals (naturals with zero) is denoted by $\mathbb{N}(\mathbb{N}_0)$. For given naturals $n, m \in \mathbb{N}$, we use the notations N and M for the set of all smaller or equal natural numbers, i.e., $N = \{1, 2, ..., n\}$ and $M = \{1, 2, ..., m\}$, respectively. The set of $n \times m$ matrices over $\mathcal{N}$ is denoted by $\mathcal{N}_{(n,m)}$, specially the set of $n \times 1$ vectors over $\mathcal{N}$ is denoted by $\mathcal{N}_{(n)}$. The max-min algebra is a triple $(\mathcal{N},\ \oplus,\ \otimes)$, where

$$a \oplus b = \max\{a, b\}$$

$$a \otimes b = \min\{a, b\}.$$

The operations $\oplus, \otimes$ are extended to the FNSM-FNSV algebra over $\mathcal{N}$ by the direct analogy to the conventional linear algebra. If each entry of a FNSM $A \in \mathcal{N}_{(n,n)}$ (a FNSV $x \in \mathcal{N}_{(n)}$) is equal to O we shall denote it as $A = O$(here O represent a zero matrix)$(x = O)$ here O represent zero vector. The greatest common divisor and the least common multiple of a set $S \subseteq \mathcal{N}$ is denoted by gcd S, and lcm S respectively. For $A, C \in \mathcal{N}_{(n,n)}$, $C \in \mathcal{N}_{(n,n)}$ we write

$$A \leq C(A < C)\ \ if\ \ \langle a_{ij}^{T}, a_{ij}^{I}, a_{ij}^{F}\rangle \leq \langle c_{ij}^{T}, c_{ij}^{I}, c_{ij}^{F}\rangle(\langle a_{ij}^{T}, a_{ij}^{I}, a_{ij}^{F}\rangle < \langle c_{ij}^{T}, c_{ij}^{I}, c_{ij}^{F}\rangle)$$

holds true for all $i, j \in N$. Similarly, for

$$\langle x^{T}, x^{I}, x^{F}\rangle = (\langle x_1^{T}, x_1^{I}, x_1^{F}\rangle, \langle x_2^{T}, x_2^{I}, x_2^{F}\rangle, ..., \langle x_n^{T}, x_n^{I}, x_n^{F}\rangle)^{t} \in \mathcal{N}(n)$$

$$\langle y_1^{T}, y_1^{I}, y_1^{F}\rangle = (\langle y_1^{T}, y_1^{I}, y_1^{F}\rangle, \langle y_2^{T}, y_2^{I}, y_2^{F}\rangle, ..., \langle y_n^{T}, y_n^{I}, y_n^{F}\rangle)^{t} \in \mathcal{N}(n)$$

we write

$$\langle x^{T}, x^{I}, x^{F}\rangle \leq \langle y^{T}, y^{I}, y^{F}\rangle(\langle x^{T}, x^{I}, x^{F}\rangle < \langle y^{T}, y^{I}, y^{F}\rangle)\ \ if\ \ \langle x_i^{T}, x_i^{I}, x_i^{F}\rangle \leq \langle y_i^{T}, y_i^{I}, y_i^{F}\rangle(\langle x_i^{T}, x_i^{I}, x_i^{F}\rangle < \langle y_i^{T}, y_i^{I}, y_i^{F}\rangle)$$

for each $i \in N$. The rth power of a FNSM A is denoted by A^r with elements $\langle a_{ij}^{T}, a_{ij}^{I}, a_{ij}^{F}\rangle^{r}$. By digraph we understand a pair $G = (V_G, E_G)$, where V_G is a non-empty finite set, called the node set, and $E_G \subseteq V_G \times V_G$, called the arc set. A digraph G' is a subdigraph of G, if $V_{G'} \subseteq V_G$ and $E_{G'} \subseteq E_G$. A path in G is the sequence of nodes $p = (v_o, v_1, ..., v_t)$ such that $(v_{k-1}, v_k) \in E_G$ for all $k = 1, 2, ..., l$ (denoted as (v_0, v_l)-path). The number $l(p) \geq O$ is called the length of p. If $v_0 = v_l$, then p is called a cycle. A cycle is elementary if all nodes except the terminal node are distinct. A digraph is called strongly connected if any two districnt nodes of G are contained in a common cycle.

By a strongly connected component $\mathcal{K}$ of $G = (N, E)$ we mean a subdigraph $\mathcal{K}$ generated by a non-empty subset $K \subseteq N$ such that any two distinct nodes $i, j \in K$ are contained in a common cycle and K is the maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$ the period of $\mathcal{K}$ is defined as per $\mathcal{K} = gcd\{l(c);\ c$ is a cycle in $\mathcal{K}, l(c) > 0\}$. If $\mathcal{K}$ is trivial, then per $\mathcal{K} = 1$. There is a well-known connection between the entries in powers of FNSMs and paths in associated digraphs: the (i, j) th entry $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle^k$ in A^k is equal to the maximum of weights of paths from $\mathcal{P}^k_{ij}$, where $\mathcal{P}^k_{ij}$ is the set of all paths of length k beginning at node i and ending at node j. If $\mathcal{P}_{ij}$ denotes the set of all paths from i to j, then $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle^* = \max\{\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle^k; k = 1, 2, ...\}$ is the maximum weight of a path from $\mathcal{P}_{ij}$ and $\langle a^T_{jj}, a^I_{jj}, a^F_{jj}\rangle^*$ is the maximum weight of a cycle containing node j. For a given FNSM $A \in \mathcal{N}(n, n)$, the number $\lambda \in \mathcal{N}$ and the n-tuple $x \in \mathcal{N}(n)$ are the so-called FNSE value of A and FNSEv of A, respectively, if

$$A \otimes x = \lambda \otimes x.$$

The FNSE space $V(A, \lambda)$ is defined as the set of all FNSEvs of A with associated FNSE value λ, i.e.,

$$V(A, \lambda) = \{\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n);\ A \otimes x = \lambda \otimes x\}.$$

In case $\lambda = I$ let us denote $V(A, I)$ by abbreviation $V(A)$. Define the greatest FNSEV $\langle x^T, x^I, x^F\rangle^*(A)$ corresponding to a FNSM A and the FNSE value I as

$$\langle x^T, x^I, x^F\rangle^*(A) = \sum_{\langle x^T, x^I, x^F\rangle \in V(A)} \langle x^T, x^I, x^F\rangle,$$

For every FNSM $A \in \mathcal{N}(n, n)$ denote

$$\langle c^T_i, c^I_i, c^F_i\rangle(A) = \bigoplus_{j \in N} \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle,$$
$$\langle c^T, c^I, c^F\rangle(A) = \bigotimes_{i \in N} \langle c^T_i, c^I_i, c^F_i\rangle(A),$$
$$\langle c^T, c^I, c^F\rangle^*(A) = (\langle c^T, c^I, c^F\rangle(A), ..., \langle c^T, c^I, c^F\rangle(A))^t \in \mathcal{N}(n).$$

4. Orbit Periodicity of Fuzzy Neutrosophic Soft Matrix

The notions of an orbit of a FNSM A generated by FNSV x and known properties of the orbit periodicity are introduced in this section.

Definition 4.1. *For any $A \in \mathcal{N}(n, n)$ and $x \in \mathcal{N}(n)$ the orbit of A generated by x is the FNSV sequence $\mathcal{O}(A, x) = (x(r); r \in \mathbb{N}_0)$ whose initial FNSV is $x(0) = x$ and successive members are defined by the formula*

$$x(r + 1) = A \otimes x(r).$$

Definition 4.2. *The sequence $\mathcal{S} = (\mathcal{S}(r);\ r \in \mathbb{N})$ is ultimataly periodic if there is a natural number p such that the following holds for some natural number $\mathcal{R} : \mathcal{S}(k + p) = \mathcal{S}(k)$ for all $k \geq \mathcal{R}$. The smallest natural number p with the above property* *is called the period of S, denoted by per (S). Both operations $\oplus$ and $\odot$ in max-min algebra are idempotent, hence no new numbers are created in the process of generating an orbit. Therefore any orbit in max-min algebra contains only a finite number of different FNSVs. Thus an orbit is always ultimately periodic. The same holds true for the power sequence $(A^k; k \in \mathbb{N})$. Hence a power sequence and an orbit $\mathcal{O}(A,x)$ are always ultimately periodic sequences. Their periods will be called the period of A and the orbit period of $\mathcal{O}(A,x)$, in notation per (A), per (A,x).*

Gavalec has proved the following theorem for per(A), when A is square matrix. This theorem can be extended to square FNSM by routine proceduers.

Theorem 4.3. *Let $A \in \mathcal{N}_{(n,n)}$ and $x \in \mathcal{N}_{(n)}$. Then $per(A) = lcm_{x \in \mathcal{N}}\ per(A,x)$. A fuzzy neutrosophic soft matrix (fuzzy neutrosophic soft vector) is called binary if $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \in \{O, I\}(\langle x_j^T, x_j^I, x_j^F\rangle \in \{O, I\})$ for each $i, j \in N$.*

Definition 4.4. *Let $A \in \mathcal{N}_{(n,n)}$ be a binary FNSM and $x \in \mathcal{N}_{(n)}$, be a binary FNSV. Then by $G(A) = (V_{G(A)}, E_{G(A)})$ we understand the digraph with $V_{G(A)} = N, E_{G(A)} = \{(i,j); \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = I\}$ and by $G(A,x)$ we understand the corresponding node-weighted digraph obtained from $G(A)$ by appending weight $\langle x_i^T, x_i^I, x_i^F\rangle$ to each node i (denoted by i^O if $\langle x_i^T, x_i^I, x_i^F\rangle = O$ and i^I if $\langle x_i^T, x_i^I, x_i^F\rangle = I$). A path in $G(A,x)$ is called an orbit path if the weight of its terminal node is I.*

Definition 4.5. *For $A \in \mathcal{N}_{(n,n)}, \langle x^T, x^I, x^F\rangle \in \mathcal{N}_{(n)}$ and $\langle h^T, h^I, h^F\rangle \in \mathcal{N}$, the threshold FNSM $A_{(h)}$ and threshold FNSV $\langle x^T, x^I, x^F\rangle_{(h)}$ corresponding to the threshold $\langle h^T, h^I, h^F\rangle$ is a binary FNSM of the same type as A and a binary FNSV of the same type as $\langle x^T, x^I, x^F\rangle$, defined as follows:*

$$\left.\begin{aligned} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle(h) &= \begin{cases} I, & if\ \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \geq \langle h^T, h^I, h^F\rangle, \\ 0, & otherwise, \end{cases} \\ \langle x_i^T, x_i^I, x_i^F\rangle(h) &= \begin{cases} I, & if\ \langle x_i^T, x_i^I, x_i^F\rangle \geq \langle h^T, h^I, h^F\rangle, \\ 0, & otherwise \end{cases} \end{aligned}\right\} \tag{2}$$

respectively. The associated digraphs $G(A_{(h)})$ and $G(A_{(h)}, \langle x^T, x^I, x^F\rangle(h))$ will be called the threshold digraphs corresponding to the threshold $\langle h^T, h^I, h^F\rangle$.

Definition 4.6. *Let $k \in N$ be a node of $G(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$. k is called removable if there is a node $j \in N$ such that (j,k)-path is an orbit path in $G(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$ with $(\langle x_j^T, x_j^I, x_j^F\rangle_{(h)}) = O$. If $(\langle x_k^T, x_k^I, x_k^F\rangle_{(h)}) = O$ then k is removable by the definition. Let $G(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$ be the threshold digraph corresponding to the threshold $\langle h^T, h^I, h^F\rangle$. Denote $\tilde{G}(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$ the digraph which arose from $G(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$ by deleting all removable nodes. A node $k \in N$ is called precyclic in $\tilde{G}(A_{(h)}, \langle x^T, x^I, x^F\rangle(h))$ if there is a path p that is finished by a cycle, i.e., $p = (k, v_1, v_2 ..., v_t ..., v_{t+s}, v_t)$. If k is lying on a cycle then k is precyclic by the definition. A characterization of $\tilde{G}(A_{(h)}, \langle x^T, x^I, x^F\rangle_{(h)})$ and a description of precyclic node is necessary for computing the greatest fuzzy neutrosophic soft eigenvector belonging to an interval fuzzy neutrosophic soft vector X.*

5. Iterval Fuzzy Neutrosophic Soft Vectors

In this section we shall deal with properties of the greatest FNSEv belonging to an interval FNSV.

Definition 5.1. *Let $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle \in \mathcal{N}(n)$, $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$. An interval FNSM X with bounds $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle$ and $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$ is defined as follows $X = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle] = \{\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n); \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \leq \langle x^T, x^I, x^F\rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle\}$. For a given $A \in \mathcal{N}_{(n,n)}$ and $X \subseteq \mathcal{N}_{(n)}$ define the greatest X-FNSEv $\langle x^T, x^I, x^F\rangle^*(A, X)$*

corresponding to a FNSM A and an intervel FNSV X as $\langle x^T, x^I, x^F\rangle^*(A,X) = \sum\limits_{x\in V(A)\cap X} \langle x^T, x^I, x^F\rangle$. *As it has been written, if $X = \mathcal{N}_{(n)}$ then $\langle x^T, x^I, x^F\rangle^*(A,X)$ exists for every FNSM A, this is in a contrast with the case that $X \subset \mathcal{N}_{(n)}$ for which $\langle x^T, x^I, x^F\rangle^*(A,X)$ exists if only if $V(A)\cap X \neq \phi$.*

Now, we define an auxiliary FNSEv $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X)$ of A belonging to X which allows us to use properties of digraphs and to characterize the structure of the greatest X-FNSEv $\langle x^T, x^I, x^F\rangle^*(A,X)$ corresponding to a FNSM A and an interval FNSV X.

Definition 5.2. *For a given $A \in \mathcal{N}(n,n)$ and $X \subseteq \mathcal{N}(n)$ define a FNSV $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X) = (\langle x_1^T, x_1^I, x_1^F\rangle^{\oplus}(A,X), ...,$ $\langle x_n^T, x_n^I, x_n^F\rangle^{\oplus}(A,X))^t$ as follows*

$$\langle x_k^T, x_k^I, x_K^F\rangle^{\oplus}(A,X) = max\{\langle h^T, h^I, h^F\rangle \in [\langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle, \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle];\ k \text{ is precyclic in } \tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)}\}.$$

Notice that if there is $k \in N$ such that $\{\langle h^T, h^I, h^F\rangle \in [\langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle, \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle];\ k$ is precyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)})\} = \phi$ then $\langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X)$ does not exist.

Example 5.3. *Let A and $X = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle]$ have the forms.*

$$A = \begin{bmatrix} \langle 0.1\ 0.1\ 0.2\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.8\ 0.7\ 0.1\rangle & \langle 0.6\ 0.5\ 0.1\rangle \\ \langle 0.0\ 0.0\ \ 1\rangle & \langle 0.4\ 0.3\ 0.2\rangle & \langle 0.6\ 0.5\ 0.1\rangle & \langle 0.2\ 0.3\ 0.4\rangle \\ \langle 0.5\ 0.4\ 0.2\rangle & \langle 0.4\ 0.3\ 0.2\rangle & \langle 0.6\ 0.5\ 0.1\rangle & \langle 0.1\ 0.1\ 0.2\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.4\ 0.3\ 0.2\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.2\ 0.1\ 0.3\rangle \end{bmatrix},\quad \underline{x} = \begin{bmatrix} \langle 0.4\ 0.3\ 0.2\rangle \\ \langle 0.4\ 0.3\ 0.2\rangle \\ \langle 0.3\ 0.2\ 0.4\rangle \\ \langle 0.4\ 0.3\ 0.2\rangle \end{bmatrix},\quad \overline{x} = \begin{bmatrix} \langle 0.8\ 0.7\ 0.1\rangle \\ \langle 0.5\ 0.4\ 0.2\rangle \\ \langle 0.6\ 0.5\ 0.1\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle \end{bmatrix}.$$

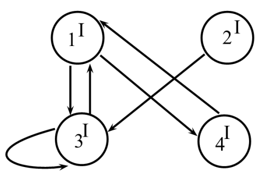

Figure 1: $G(A_{(0.5,0.4,0.2)}, X_{(0.5,0.4,0.2)}) = \widetilde{G}(A_{(0.5,0.4,0.2)}, X_{(0.5,0.4,0.2)})$

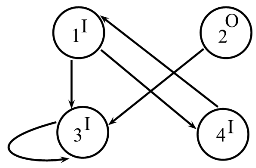

Figure 2: $G(A_{(0.6,0.5,0.1)}, X_{(0.6,0.5,0.1)})$

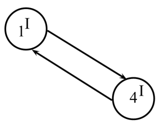

Figure 3: $\widetilde{G}(A_{(0.6,0.5,0.1)}, X_{(0.6,0.5,0.1)})$

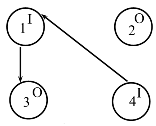

Figure 4: $G(A_{(0.7,0.6,0.1)}, X_{(0.7,0.6,0.1)})$

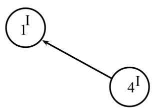

Figure 5: $\widetilde{G}(A_{(0.7,0.6,0.1)}, X_{(0.7,0.6,0.1)})$

By the definition of $\langle x^T, x^I, x^F\rangle_k^{\oplus}(A,X)$ we get that node 2 and node 3 are precyclic in $\tilde{G}(A_{\langle 0.5,0.4,0.2\rangle}, x_{\langle 0.5,0.4,0.2\rangle})$ and nodes 1, 4 are precyclic in $\tilde{G}(A_{\langle 0.6,0.5,0.1\rangle}, x_{\langle 0.6,0.5,0.1\rangle})$. The graph $\tilde{G}(A_{\langle 0.7,0.6,0.1\rangle}, x_{\langle 0.7,0.6,0.1\rangle})$ (similarly as $\tilde{G}(A_{\langle 0.8,0.7,0.1\rangle}, x_{\langle 0.8,0.7,0.1\rangle})$ and $\tilde{G}(A_{\langle 0.9,0.8,0.1\rangle}, x_{\langle 0.9,0.8,0.1\rangle})$ is acyclic and hence we get that $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X) = (\langle 0.6, 0.5, 0.1\rangle, \langle 0.5, 0.4, 0.2\rangle, \langle 0.5, 0.4, 0.2\rangle, \langle 0.6, 0.5, 0.1\rangle)^t$. From now we shall suppose that $\langle x^T, x^I, x^F\rangle_k^{\oplus}(A,X)$ exists. Then from the last definition the next lemma straightly follows.

Lemma 5.4. *Let A and X be given. Then* $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \leq \langle x^T, x^I, x^F\rangle^{\oplus}(A,X) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$.

Theorem 5.5. *Let A and X be given . Then* $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X) \in V(A)$.

Proof. Consider an arbitrary but fixed $k \in N$ and suppose that k is precyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)})$ for $\langle h^T, h^I, h^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X)$. Then there is a path $p = (k, v_1, ..., v_t, ..., v_{t+s}, v_t)$ such that $(v_t, ..., v_{t+s}, v_t)$ is a cycle and each node $v_j \in p$ is precyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)})$. Moreover

$$\langle x_{v_1}^T, x_{v_1}^I, x_{v_1}^F\rangle^{\oplus}(A,X) \geq \langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X) = \langle h^T, h^I, h^F\rangle \text{ and } \langle a_{kv_1}^T, a_{kv_1}^I, a_{kv_1}^F\rangle \geq \langle h^T, h^I, h^F\rangle.$$

Hence we get

$$\begin{aligned}(A \otimes \langle x^T, x^I, x^F\rangle^{\oplus}(A,X))_k &= \sum_{j\in N} \langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle^{\oplus}(A,X) \geq \langle a_{kv_1}^T, a_{kv_1}^I, a_{kv_1}^F\rangle \otimes \langle x_{v_1}^T, x_{v_1}^I, x_{v_1}^F\rangle^{\oplus}(A,X) \\ &\geq \langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X).\end{aligned}$$

To prove the reverse inequality suppose for a contrary that there is some index l such that $\langle a_{kl}^T, a_{kl}^I, a_{kl}^F\rangle \otimes \langle x_l^T, x_l^I, x_l^F\rangle^{\oplus}(A,X) > \langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X)$. Node l is precyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)})$ for $\langle h^T, h^I, h^F\rangle = \langle x_l^T, x_l^I, x_l^F\rangle^{\oplus}(A,X)$, i.e., there is a path $p' = (l, u_1, ..., u_t, ..., u_{t+s}, u_t)$. Then we can construct a new path $\tilde{p'} = (k, l, u_1, ...u_t, ..., u_{t+s}, u_t)$ which guarantees that k is precyclic in $\tilde{G}(A_{(h')}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h')})$ for $\langle h^T, h^I, h^F\rangle' = \langle a_{kl}^T, a_{kl}^I, a_{kl}^F\rangle \otimes \langle x_l^T, x_l^I, x_l^F\rangle^{\oplus}(A,X)$. This is a contradiction with the definition of $\langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X)$. □

Theorem 5.6. *Let A and X be given. Then* $(\forall \langle x^T, x^I, x^F\rangle \in V(A) \cap X)\ [\langle x^T, x^I, x^F\rangle \leq \langle x^T, x^I, x^F\rangle^{\oplus}(A,X)]$.

Proof. Suppose that A , X and $\langle x^T, x^I, x^F\rangle \in V(A) \cap X$ are given. Then $(A \otimes \langle x^T, x^I, x^F\rangle)_k = \langle x_k^T, x_k^I, x_k^F\rangle$ implies that there is $j_1 \in N$ such that $\langle a_{kj_1}^T, a_{kj_1}^I, a_{kj_1}^F\rangle \otimes \langle x_{j_1}^T, x_{j_1}^I, x_{j_1}^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle$ and $\langle a_{kj_1}^T, a_{kj_1}^I, a_{kj_1}^F\rangle \geq \langle x_k^T, x_k^I, x_k^F\rangle \wedge \langle x_{j_1}^T, x_{j_1}^I, x_{j_1}^F\rangle \geq \langle x_k^T, x_k^I, x_k^F\rangle$. For row index j_1 there is j_2 such that $\langle a_{j_1j_2}^T, a_{j_1j_2}^I, a_{j_1j_2}^F\rangle \otimes \langle x_{j_2}^T, x_{j_2}^I, x_{j_2}^F\rangle = \langle x_{j_1}^T, x_{j_1}^I, x_{j_1}^F\rangle$ and $\langle a_{j_1j_2}^T, a_{j_1j_2}^I, a_{j_1j_2}^F\rangle \geq \langle x_{j_1}^T, x_{j_1}^I, x_{j_1}^F\rangle \wedge \langle x_{j_2}^T, x_{j_2}^I, x_{j_2}^F\rangle \geq \langle x_{j_1}^T, x_{j_1}^I, x_{j_1}^F\rangle$. By repeating the above process at most n times we obtain a path $p = (k, j_1, ..., j_s, j_{s+1}, ..., j_s)$, i.e., k is precyclic in $\tilde{G}(A_{(x_k)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(x_k)})$. Notice that no node of p can be removable (if j_l is removable then there is a path $p' = (i_1, ...i_s = j_l)$ in $G(A_{(x_k)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(x_k)})$ with i_v^O and i_{v+1}^I and hence inequality $\langle a_{i_si_{s+1}}^T, a_{i_si_{s+1}}^I, a_{i_si_{s+1}}^F\rangle \otimes \langle x_{i_{s+1}}^T, x_{i_{s+1}}^I, x_{i_{s+1}}^F\rangle > \langle x_{i_s}^T, x_{i_s}^I, x_{i_s}^F\rangle$ contradicts the assumption that $A \otimes x = x$). The assertion follows from the fact that $\langle x_k^T, x_k^I, x_k^F\rangle^{\oplus}(A,X)$ is equal to the greatest $\langle h^T, h^I, h^F\rangle$ for which k is prpecyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle_{(h)})$. In the last three assertions we have shown that $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X)$ is an FNSEv belonging to X and fulfilling a maximality condition $(\forall \langle x^T, x^I, x^F\rangle \in V(A) \cap X)\ [\langle x^T, x^I, x^F\rangle \leq \langle x^T, x^I, x^F\rangle^{\oplus}(A,X)]$. This is a reason to formulate the next corollary. □

Corollary 5.7. *If $V(A) \cap X \neq \phi$ then* $\langle x^T, x^I, x^F\rangle^{*}(A,X) = \langle x^T, x^I, x^F\rangle^{\oplus}(A,X)$.

6. The Greatest x-fuzzy Neutrosophic Soft Eigenvector

The effective computation of $\langle x^T, x^I, x^F\rangle^{\oplus}(A,X)$ considered in this section.

Definition 6.1. *LetX be given. X is invariant under A if $\langle x^T, x^I, x^F\rangle \in X$ implies $A \otimes \langle x^T, x^I, x^F\rangle \in X$. Since A is order preserving and X is invariant under A, the following Lemma is admitable.*

Lemma 6.2. *X is invariant under A if and only if $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \leq A\otimes\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \wedge A\otimes\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$. Suppose that X is invariant under A and $\mathcal{O}(A, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle) = (\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(r); r \in \mathbb{N}_0)$ and $\mathcal{O}(A, \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle) = (\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(r); r \in \mathbb{N}_0)$ are orbits of A generated by $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$ and $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle$ respectively. Then for each $k \in \mathbb{N}_0$ we have the following.*

$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(k+1) = A^{k+1} \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle = A^k \otimes (A \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle) \leq A^k \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(k) \tag{3}$$

$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(k+1) = A^{k+1} \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle = A^k \otimes (A \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle) \geq A^k \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(k). \tag{4}$$

Lemma 6.3. *Let X be invariant under A. Then*

$$(\forall\ k \in \mathbb{N}_0)[\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n) = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n+k)], \tag{5}$$

$$(\forall\ k \in \mathcal{N}_0)[\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n) = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n+k)]. \tag{6}$$

Proof. According to (3) it is sufficient to prove that $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n+1)$. For the sake of a contradiction assume that $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n) \not\leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n+1)$, i.e., there is $i \in N$ such that $\langle \overline{x}_i^T, \overline{x}_i^I, \overline{x}_i^F\rangle(n) > \langle \overline{x}_i^T, \overline{x}_i^I, \overline{x}_i^F\rangle(n+1)$, i.e., $(A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle)_i > (A^{n+1} \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle)_i$. Then there is $s \in N$ such that for each $k \in N$ the following inequality holds true $\langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle > \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle$, or equivalently, there is a path $p \in \mathcal{P}_{is}^n$ such that next formula

$$\omega(p) \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle = \langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle > \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle \geq \omega(\tilde{p}) \otimes \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle$$

holds for each $k \in N$ and each $\tilde{p} \in \mathcal{P}_{ik}^{n+1}$. Since $l(p) = n$ then there is at least one repeated node, i.e., $p = p' \cup c$ and $l(c) \geq 1$. Now, we shall consider a path p'' such that $p'' = p' \cup c \cup c$. Let $l(p'') = v$. Then $v \geq n+1$ and we get

$$\begin{aligned}\langle \overline{x}_i^T, \overline{x}_i^I, \overline{x}_i^F\rangle(v) &\geq \langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^v \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle \geq \omega(p'') \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle = \omega(p) \otimes \langle x_s^T, x_s^I, x_s^F\rangle \\ &= \langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \overline{x}_s^T, \overline{x}_s^I, \overline{x}_s^F\rangle > \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F\rangle = \langle \overline{x}_i^T, \overline{x}_i^I, \overline{x}_i^F\rangle(n+1).\end{aligned}$$

This is a contradiction with (3) and (5) follows. To prove (6) it is enough (by (5)) to show that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n) \geq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n+1)$. For the sake of a contradiction assume that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n) \not\geq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n+1)$, i.e., there is $i \in N$ such that $\langle \underline{x}_i^T, \underline{x}_i^I, \underline{x}_i^F\rangle(n) < \langle \underline{x}_i^T, \underline{x}_i^I, \underline{x}_i^F\rangle(n+1)$ or equivalently $(A^n \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle)_i < (A^{n+1} \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle)_i$. Then there is $k \in \mathbb{N}$ such that for each $s \in \mathbb{N}$ the following inequality holds true $\langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \underline{x}_s^T, \underline{x}_s^I, \underline{x}_s^F\rangle < \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle$, or again equivalently, there is a path $p \in \mathcal{P}_{ik}^{n+1}$ such that next formula

$$\omega(p') \otimes \langle \underline{x}_s^T, \underline{x}_s^I, \underline{x}_s^F\rangle \leq \langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \underline{x}_s^T, \underline{x}_s^I, \underline{x}_s^F\rangle < \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle = \omega(p) \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle$$

holds for each $s \in N$ and each $p' \in \mathcal{P}_{is}^n$. Since $l(p) = n+1$ then there is at least one repeated node, i.e., $p = \tilde{p} \cup c$, where $\tilde{p} \in \mathcal{P}_{ik}^{l(\tilde{p})}$ with $l(\tilde{p}) \leq n$ and $l(c) \geq 1$. Now, we shall consider a path $\tilde{p}$. It is clear that $\omega(p) \leq \omega(\tilde{p})$ and hence we get

$$\begin{aligned}\langle x^T, x^I, x^F\rangle(l(\tilde{p})) &\geq \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{l(\tilde{p})} \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle \geq \omega(\tilde{p}) \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle \geq \omega(p) \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle \\ &= \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle^{n+1} \otimes \langle \underline{x}_k^T, \underline{x}_k^I, \underline{x}_k^F\rangle > \sum_{s\in N} \langle a_{is}^T, a_{is}^I, a_{is}^F\rangle^n \otimes \langle \underline{x}_s^T, \underline{x}_s^I, \underline{x}_s^F\rangle = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n).\end{aligned}$$

This is a contradiction with (4) and (6) follows. □

Corollary 6.4. *Let X be invariant under A. Then $\{\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n), \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(n)\} \subseteq V(A) \cap X$.*

Proof. The assertions follows from the facts that

$$A \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle(n) = A \otimes (A^n \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle) = A^{n+1} \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle(n+1) = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle(n),$$
$$A \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n) = A \otimes (A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle) = A^{n+1} \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n+1) = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n).$$

□

Theorem 6.5. *Let X be invariant under A. Then $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X) = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n)$.*

Proof. Suppose that X is invariant under A and $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X) = (\langle x_1^T, x_1^I, x_1^F \rangle^{\oplus}(A, X), \ldots \langle x_n^T, x_n^I, x_n^F \rangle^{\oplus}(A, X))^T$. Then by the definition of $\langle x_k^T, x_k^I, x_k^F \rangle^{\oplus}(A, X)$ there is a path $p = p' \cup c$ (p is finished by c) in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I \overline{x}^F \rangle_{(h)})$ for $\langle h^T, h^I, h^F \rangle = \langle x_k^T, x_k^I, x_k^F \rangle^{\oplus}(A, X)$ and $\omega(p) = \langle h^T, h^I, h^F \rangle$. Consider now a path $p'' = p' \cup c \cup \ldots \cup c$ with $l(p'') \geq n$ and $\omega(p) = \omega(p'')$. Then we obtain

$$\langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F \rangle(n) = \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F \rangle(l(p'')) \geq \omega(p'') = \langle x_k^T, x_k^I, x_k^F \rangle^{\oplus}(A, X).$$

Inverse inequality follows from the fact that a path $p \in \mathcal{P}_{kj}^n$ beginning in k contains a cycle and $\langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F \rangle(n) = \sum_j \langle a_{kj}^T, a_{kj}^I, a_{kj}^F \rangle^n \otimes \langle \overline{x}_j^T, \overline{x}_j^I, \overline{x}_j^F \rangle = \langle a_{kl}^T, a_{kl}^I, a_{kl}^F \rangle^n \otimes \langle \overline{x}_l^T, \overline{x}_l^I, \overline{x}_l^F \rangle = \omega(p) \otimes \langle \overline{x}_j^T, \overline{x}_j^I, \overline{x}_j^F \rangle$, i.e., k is precyclic in $\tilde{G}(A_{(h)}, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle_{(h)})$ for $\langle h^T, h^I, h^F \rangle = \langle \overline{x}_k^T, \overline{x}_k^I, \overline{x}_k^F \rangle(n)$. Hence $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X) \geq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n)$. □

6.1. Computing the Greatest X-fuzzy Neutrosophic Soft Eigenvector-General Case

In this section we compute $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X)$ when X is not invariant under A.

Let $A \in \mathcal{N}(n, n)$ and $X \subseteq \mathcal{N}(n)$ be given. Suppose that $A \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle \not\leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ and $V(A) \cap X \neq \phi$, i.e., $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X)$ exists. We look for the greatest FNSV $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \in X$ with the property that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$. For this purpose $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ will be constructed by the following algorithm.

Algorithm **Invariant upper bound**

Input. X, A.

Output. "yes" in variable gr if $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \in X$ with the property that

$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ exists;

"no" in gr otherwise.

begin

1. $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle := \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$; $M := \phi$;

2. $m := \min_{j \in N|M} \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle$; $M = \{j \in N;\ m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle\}$;

3. **While** $(\exists k \in M)(\exists j \in N)[\langle a_{kj}^T, a_{kj}^I, a_{kj}^F \rangle \otimes \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle > \langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle]$ **do**

If $\langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle \geq \langle \underline{x}_j^T, \underline{x}_j^I, \underline{x}_j^F \rangle$ **then** $\langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle := m \wedge M\ := M \cup \{j\}$ **else** gr:="no";

(comment: if the condition $A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ is not fulfilled in row k, corresponding variables $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ are modified)

4. $M = \{j \in N; m \geq \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle\}$;

(comment: M consists of row indices for which $A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ holds true)

5. **If** $N \setminus M = \phi$ then gr:=" yes" **else** go to 2;

end.

Example 6.6. *Let A and $X = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle, \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle]$ have the forms*

$$A = \begin{bmatrix} \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.5\ 0.4\ 0.3\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.8\ 0.7\ 0.1\rangle \\ \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.7\ 0.6\ 0.1\rangle & \langle 0.4\ 0.3\ 0.5\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.5\ 0.4\ 0.3\rangle & \langle 0.6\ 0.5\ 0.2\rangle \\ \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.7\ 0.6\ 0.1\rangle & \langle 0.4\ 0.3\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle\ 0\ \ \ 0\ \ \ 1\rangle \\ \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.5\ 0.4\ 0.3\rangle & \langle 0.4\ 0.3\ 0.5\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.6\ 0.5\ 0.2\rangle & \langle 0.4\ 0.3\ 0.5\rangle \\ \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.4\ 0.3\ 0.5\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.5\ 0.4\ 0.3\rangle & \langle 0.1\ 0.1\ 0.5\rangle \\ \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.5\ 0.4\ 0.3\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.1\ 0.1\ 0.5\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.2\ 0.1\ 0.5\rangle \\ \langle 0.3\ 0.2\ 0.4\rangle & \langle 0.2\ 0.1\ 0.5\rangle & \langle 0.6\ 0.5\ 0.2\rangle & \langle 0.6\ 0.5\ 0.2\rangle & \langle 0.5\ 0.1\ 0.5\rangle & \langle 0.6\ 0.5\ 0.2\rangle & \langle 0.4\ 0.3\ 0.5\rangle \end{bmatrix},$$

$$\underline{x} = \begin{bmatrix} \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \\ \langle 0.3\ 0.2\ 0.6\rangle \end{bmatrix}, \qquad \overline{x} = \begin{bmatrix} \langle 0.6\ 0.5\ 0.2\rangle \\ \langle 0.7\ 0.6\ 0.1\rangle \\ \langle 0.6\ 0.5\ 0.2\rangle \\ \langle 0.5\ 0.4\ 0.3\rangle \\ \langle 0.4\ 0.3\ 0.5\rangle \\ \langle 0.4\ 0.3\ 0.5\rangle \\ \langle 0.7\ 0.6\ 0.1\rangle \end{bmatrix}.$$

The first run of the algorithm:

By applying step 1 of the algorithm put $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F\rangle := \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle, M = \phi$ and

$$m := \min_{j\in N/M} \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle = \langle 0.4\ 0.3\ 0.5\rangle, \quad M = \{j \in N; m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{5, 6\}$$

in step 2. Since $\langle a_{63}^T, a_{63}^I, a_{63}^F\rangle \otimes \langle \tilde{x}_3^T, \tilde{x}_3^I, \tilde{x}_3^F\rangle > \langle \tilde{x}_6^T, \tilde{x}_6^I, \tilde{x}_6^F\rangle$, put $\langle \tilde{x}_3^T, \tilde{x}_3^I, \tilde{x}_3^F\rangle := m = \langle 0.4\ 0.3\ 0.5\rangle (\geq \langle \underline{x}_3^T, \underline{x}_3^I, \underline{x}_3^F\rangle)$, $M = \{j \in N;\ m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{3, 5, 6\}$ and

$$\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F\rangle = (\langle 0.6,\ 0.5,\ 0.2\rangle, \langle 0.7,\ 0.6,\ 0.1\rangle, \langle 0.4,\ 0.3,\ 0.5\rangle, \langle 0.5,\ 0.4,\ 0.3\rangle, \langle 0.4,\ 0.3,\ 0.5\rangle \langle 0.4,\ 0.3,\ 0.5\rangle, \langle 0.7,\ 0.6,\ 0.1\rangle)^t.$$

In step 4 the set $M = \{j \in N;\ m \geq \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{3, 5, 6\}$, in step 5 we obtain $N \setminus M \neq \phi$ and the algorithm goes on step 2.

The second run of the algorithm:

In step 2 we get

$$m := \min_{j\in N/M} \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle = \langle 0.5,\ 0.4,\ 0.3\rangle, \quad M = \{j \in N;\ m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{\langle 0.4,\ 0.3,\ 0.5\rangle\}.$$

Since the condition of step 3 is not fulfilled $((A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F\rangle)_4 \leq \langle \tilde{x}_4^T, \tilde{x}_4^I, \tilde{x}_4^F\rangle)$, we continue by step 4 and step 5, i.e., $M = \{j \in N; m \geq \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{3, 4, 5, 6\}$, $N \setminus M \neq \phi$ and the algorithm goes on step 2.

The third run of the algorithm:

In step 2 we get

$$m := \min_{j\in N\setminus M} \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle = \langle 0.6\ 0.5\ 0.2\rangle, \quad M = \{j \in N;\ m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = \{1\}.$$

Step 3 produces the following $\langle a_{17}^T, a_{17}^I, a_{17}^F\rangle \otimes \langle \tilde{x}_7^T, \tilde{x}_7^I, \tilde{x}_7^F\rangle > \langle \tilde{x}_1^T, \tilde{x}_1^I, \tilde{x}_1^F\rangle$, put $\langle \tilde{x}_7^T, \tilde{x}_7^I, \tilde{x}_7^F\rangle := m = \langle 0.6\ 0.5\ 0.4\rangle (\geq \langle \tilde{x}_7^T, \tilde{x}_7^I, \tilde{x}_7^F\rangle)$, $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F\rangle = (\langle 0.6,\ 0.5,\ 0.2\rangle, \langle 0.7,\ 0.6,\ 0.1\rangle, \langle 0.4,\ 0.3,\ 0.5\rangle, \langle 0.5,\ 0.4,\ 0.3\rangle \langle 0.4,\ 0.3,\ 0.5\rangle, \langle 0.4,\ 0.3,\ 0.5\rangle, \langle 0.6\ 0.5\ 0.2\rangle)^T$. Since $M = \{j \in N; m \geq \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F\rangle\} = N/\{2\}$, the algorithm again goes on step 2.

The fourth run of the algorithm:

In step 2 we get

$$m := \min_{j \in N\setminus M} \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle = \langle 0.7\ 0.6\ 0.1 \rangle, \quad M = \{j \in N; m = \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle\} = \{2\}.$$

Since the condition of step 3 is not fulfilled and $M = N$ the algorithm terminates in step 5 with the variable gr:="yes" and output $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle = (\langle 0.6,\ 0.5,\ 0.2 \rangle, \langle 0.7,\ 0.6,\ 0.1 \rangle, \langle 0.4,\ 0.3,\ 0.5 \rangle, \langle 0.5,\ 0.4,\ 0.3 \rangle, \langle 0.4,\ 0.3,\ 0.5 \rangle, \langle 0.4,\ 0.3,\ 0.5 \rangle, \langle 0.6,\ 0.5,\ 0.2 \rangle)^t$.

Theorem 6.7. *Let $A \in \mathcal{N}(n, n)$ and $X \subseteq \mathcal{N}(n)$ be given. Then the algorithm invariant upper bound is correct, its output is the greatest FNSV $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ such that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ and its computational complexity is $O(n^2)$.*

Proof. The algorithm finishes with the positive answer in step 5, where it has computed FNSV $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ such that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$. If $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ does not exists (this corresponds to else branch of the algorithm in step 3) then it is impossible to decrease $\langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle$ on the level $\langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle$ if $\langle a_{kj}^T, a_{kj}^I, a_{kj}^F \rangle \otimes \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle > \langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle$ and $\langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle < \langle \underline{x}_j^T, \underline{x}_j^I, \underline{x}_j^F \rangle$. Further, the smallest possible decrease of $\langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle$ in step 3 guarantees the maximality of $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \in X$ with property $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A \otimes \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle \leq \langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$.

For the estimation of the computational complexity observe that the algorithm checks for each coordinate $\langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle$ at most n products $\langle a_{kj}^T, a_{kj}^I, a_{kj}^F \rangle \otimes \langle \tilde{x}_j^T, \tilde{x}_j^I, \tilde{x}_j^F \rangle$ and compares it with $\langle \tilde{x}_k^T, \tilde{x}_k^I, \tilde{x}_k^F \rangle$ in step 3. The number of operations in step 3 is $O(n^2)$ and in no other step it exceeds the bound $O(n)$, hence the overall complexity is $O(n^2)$. Now, we can summarize the above results and suggest an algorithm for computing the greatest X-FNSEv of A.

Algorithm Greatest X-FNSEv

Input. X,A.

output. "yes" in variable gr if $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X) \in X$ exists; "no" in gr otherwise.

begin

1. If $A \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ then compute $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ by the algorithm Invariant upper bound; put $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle :=$ $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$;

2. Compute $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n) = A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$;

3. If $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ then $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X) := \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle(n)$ else $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X)$ does not exist;

end □

Theorem 6.8. *The algorithm Greatest X-FNSEv correctly computes $\langle x^T, x^I, x^F \rangle^{\oplus}(A, X)$ in $O(n^3)$ arithmetic operations.*

Proof. To determine a complexity of the algorithm, recall first that computing $\langle \tilde{x}^T, \tilde{x}^I, \tilde{x}^F \rangle$ by the algorithm Invariant upper bound in step 1 needs $O(n^2)$ operations. The number of operations for computing $\langle x^T, x^I, x^F \rangle(n)$ is $O(n^2) = O(n^3)$. Thus, the complexity of the whole algorithm is $O(n^3)$. □

6.2. Applications of the Greatest X-fuzzy Neutrosophic Soft Eigenvector

In this section we will analyze conditions for FNSM under which multi-processor interaction systems reach the greatest steady state with any starting FNSV belonging to an interval FNSV X. The set of starting FNSV from which a multi-processor interaction system reaches an FNSEv (the greatest FNSEv) of A after a finite number of stages, is called attraction set (strongly attraction set) of A. In general, attraction set (strongly attraction set) contains the set of all FNSEv, the set of all FNSEvs belonging to X but it can be also as big as the whole space. Let us denote the sets $attr(A)$ and $attr^*(A)$ as follows.

$$attr(A) = \{\langle x^T, x^I, x^F \rangle \in \mathcal{N}(n);\ O(A, \langle x^T, x^I, x^F \rangle) \cap V(A) \neq \phi\},$$

$$attr^*(A) = \{\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n);\ \langle x^T, x^I, x^F\rangle^*(A) \in O(A, \langle x^T, x^I, x^F\rangle)\}.$$

The set $attr(A)$ ($attr^*(A)$) allows us to characterize FNSMs for which an FNSEv (the greatest FNSEv) is reached with any starting FNSV. It is easy to see that $\langle x^T, x^I, x^F\rangle^*(A) \geq \langle c^T, c^I, c^F\rangle^*(A)$ holds true and $\langle x^T, x^I, x^F\rangle^*(A)$ can not be reached with a FNSV $\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n)$, $\langle x^T, x^I, x^F\rangle < \langle c^T, c^I, c^F\rangle^*(A)$. Let us denote the set $\{\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n); \langle x^T, x^I, x^F\rangle < \langle c^T, c^I, c^F\rangle^*(A)\}$ by $M(A)$.

Definition 6.9. *$A \in \mathcal{N}(n,n)$ is called strongly robust if $attr^*(A) = \mathcal{N}(n) \setminus M(A)$.*

Theorem 6.10. *Let $A \in \mathcal{N}(n,n)$ be a FNSM. Then A is strongly robust if and only if $\langle x^T, x^I, x^F\rangle^*(A) = \langle c^T, c^I, c^F\rangle^*(A)$ and $G(A_{(c(A))})$ is a strongly connected digraph with period equal to 1.*

Definition 6.11. *Let A, X be given. Then the strongly attraction set $attr^*(A, X)$ is defined as follows.*

$$attr^*(A, X) = \{\langle x^T, x^I, x^F\rangle \in \mathcal{N}(n);\ \langle x^T, x^I, x^F\rangle^*(A, X) \in O(A, \langle x^T, x^I, x^F\rangle)\}.$$

Definition 6.12. *Let A,X be given. A is called strongly X-robust if $X \subseteq attr^*(A, X)$.*

Lemma 6.13. *If A is strongly X-robust then $(\forall \langle x^T, x^I, x^F\rangle \in X)$ $[per(A, \langle x^T, x^I, x^F\rangle) = \langle 1,\ 1,\ 0\rangle]$.*

Proof. Suppose that A is strongly X-robust and $\langle x^T, x^I, x^F\rangle \in X$ is an arbitrary FNSV. Then there is $k \in N$ such that $\langle x^T, x^I, x^F\rangle(k) = \langle x^T, x^I, x^F\rangle^*(A, X)$ and we obtain the following.

$$\langle x^T, x^I, x^F\rangle(k) = \langle x^T, x^I, x^F\rangle^*(A, X) = A \otimes \langle x^T, x^I, x^F\rangle^*(A, X) = A \otimes \langle x^T, x^I, x^F\rangle(k) = \langle x^T, x^I, x^F\rangle(k' + 1)$$

and the assertion follows. □

Theorem 6.14. *Let $A \in \mathcal{N}(n,n)$ and X be given. Then A is strongly X-robust if and only if $A^n \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle = \langle x^T, x^I, x^F\rangle^*(A, X) = A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$.*

Proof. Suppose that $A \in \mathcal{N}(n,n)$ and X are given and $A^n \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle = \langle x^T, x^I, x^F\rangle^*(A, X) = A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle$. Then for an arbitrary FNSV $\langle x^T, x^I, x^F\rangle \in X$ we get (by monotonicity of $\otimes$) the following

$$\langle x^T, x^I, x^F\rangle^*(A, X) = A^n \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle \leq A^n \otimes \langle x^T, x^I, x^F\rangle \leq A^n \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle = \langle x^T, x^I, x^F\rangle^*(A, X).$$

The converse implication is trivial by Lemma 6.2 and Lemma 6.3. Notice that according to the last theoren the complexity of a procedure for checking strong X-robustness of a given FNSM A is $O(n^3)$ consisting of computing $\langle x^T, x^I, x^F\rangle^*(A, X)$ in $O(n^3)$ steps and each of FNSV $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(1), ..., \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n), \langle \overline{x}^T, \overline{x}^I, \overline{x}^F\rangle(1), ..., \langle \underline{x}^T, \underline{x}^I, \underline{x}^F\rangle(n)$ in $O(n^2)$ operations. Thus whole procedure has the computational complexity equal to $O(n^3) + 2nO(n^2) = O(n^3)$. □

7. Conclusion

In this work, the authors obtain a computing the greatest X-FNSEv of a FNSM in max-min algebra and study their orbit periodicity, interval vector of FNSM. Then X-FNSEv procedure for computing the greatest, general case of greatest X-FNSEv, Algorithm Invariant upper bound, Algorithm of greatest X-FNSEs, and Application of the greatest X-FNSE.

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