

Concepts of Interval-Valued Neutrosophic Graphs

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Abstract. Broumi et al. [15] proposed the concept of interval-valued neutrosophic graphs. In this research article, we first show that there are some flaws in Broumi et al. [15] 's definition, which cannot be applied in network models. We then modify the definition of an interval-valued neutrosophic graph. Further, we present some operations on interval-valued neutrosophic graphs. Moreover, we discuss the concepts of self-complementary and self weak complementary interval-valued neutrosophic complete graphs. Finally, we describe regularity of interval-valued neutrosophic graphs.

1. Introduction

In 1975, Zadeh [38] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [37] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [24]. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany on approximate reasoning [19, 20], Roy and Biswas on medical diagnosis [28], Turksen on multivalued logic [31] and Mendel on intelligent control [24]. Atanassov [13] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. The notion of intuitionistic fuzzy sets is more meaningful as well as intensive due to the presence of degree of truth, indeterminacy and falsity membership. The hesitation value of intuitionistic fuzzy set is its indeterminacy by default (and defined as 1 minus the sum of truth-membership and falsity-membership). The truth-membership degree and the falsity-membership degree are more or less independent from each other, the only requirement is that the sum of these two degrees is not greater than one. Smarandache [29, 30] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership (t), indeterminacy-membership (i) and falsity-membership (f), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-, 1^+[$ and there is no restriction on their sum. Neutrosophic set is a mathematical tool for dealing real life problems having imprecise, indeterminacy and inconsistent data. Neutrosophic set theory, as a generalization of fuzzy set theory and intuitionistic fuzzy set theory, is applied in a variety of fields, including control theory, decision making problems, topology, medicines and in many more real life problems. Wang et al. [32]

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presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0, 1]$. Wang et al. [33] presented the concept of interval-valued neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$.

Graph theory has become a powerful conceptual framework for modeling and solution of combinatorial problems that arise in various fields, including mathematics, engineering and computer science. However, in some cases, some aspects of graph theoretic concepts may be uncertain. In such cases, it is important to deal with uncertainties using the methods of fuzzy sets and logics. Kaufmann [23] gave the definition of a fuzzy graph. Fuzzy graphs were narrated by Rosenfeld [27]. After that, some remarks on fuzzy graphs were represented by Bhattacharya [14]. He showed that all the concepts on crisp graph theory do not have similarities in fuzzy graphs. Dhavaseelan et al. [18] defined strong neutrosophic graphs. Akram and Shahzadi [1] introduced the notion of neutrosophic soft graphs with applications. Akram [3] introduced the notion of single-valued neutrosophic planar graphs. Akram et al. [2] also introduced the single-valued neutrosophic hypergraphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [5]. Akram and Shahzadi [4] studied properties of single-valued neutrosophic graphs by level graphs. Akram et al. [6–12] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Recently, Broumi et al. [15, 16] proposed the concept of interval-valued neutrosophic graphs with operations. In this research article, we first show that there are some flaws in Broumi et al. [15]'s definition, which cannot be applied in network models. We then modify the definition of an interval-valued neutrosophic graph. We present some operations on interval-valued neutrosophic graphs. We discuss the concepts of self-complementary and self weak complementary interval-valued neutrosophic complete graphs. We also describe regularity of interval-valued neutrosophic graphs.

2. Interval-valued neutrosophic graphs

Definition 2.1. [33, 34] *The interval-valued neutrosophic set A in X is defined by*

$$A = \{(x, [t_A^-(x), t_A^+(x)], [i_A^-(x), i_A^+(x)], [f_A^-(x), f_A^+(x)]) : x \in X\},$$

where, $t_A^-(x)$, $t_A^+(x)$, $i_A^-(x)$, $i_A^+(x)$, $f_A^-(x)$, and $f_A^+(x)$ are neutrosophic subsets of X such that $t_A^-(x) \leq t_A^+(x)$, $i_A^-(x) \leq i_A^+(x)$ and $f_A^-(x) \leq f_A^+(x)$ for all $x \in X$.

For any two interval-valued neutrosophic sets $A = ([t_A^-(x), t_A^+(x)], [i_A^-(x), i_A^+(x)], [f_A^-(x), f_A^+(x)])$ and $B = ([t_B^-(x), t_B^+(x)], [i_B^-(x), i_B^+(x)], [f_B^-(x), f_B^+(x)])$ in X , we define:

$$A \cup B = \{(x, \max(t_A^-(x), t_B^-(x)), \max(t_A^+(x), t_B^+(x)), \max(i_A^-(x), i_B^-(x)), \max(i_A^+(x), i_B^+(x)), \min(f_A^-(x), f_B^-(x)), \min(f_A^+(x), f_B^+(x))) : x \in X\}.$$

$$A \cap B = \{(x, \min(t_A^-(x), t_B^-(x)), \min(t_A^+(x), t_B^+(x)), \min(i_A^-(x), i_B^-(x)), \min(i_A^+(x), i_B^+(x)), \max(f_A^-(x), f_B^-(x)), \max(f_A^+(x), f_B^+(x))) : x \in X\}.$$

Broumi et al. [15] defined interval-valued neutrosophic graphs as follows.

Definition 2.2. [15] *By an interval-valued neutrosophic graph with underlying set X is defined to be a pair $G = (A, B)$, where $A = ([t_A^-, t_A^+], [i_A^-, i_A^+], [f_A^-, f_A^+])$ is an interval-valued neutrosophic set on X and $B = ([t_B^-, t_B^+], [i_B^-, i_B^+], [f_B^-, f_B^+])$ is an interval-valued neutrosophic relation on E satisfying the following conditions:*

1. $X = \{x_1, x_2, x_3, \dots, x_n\}$ such that the functions $t_A^+ : X \rightarrow [0, 1]$, $t_A^- : X \rightarrow [0, 1]$, $i_A^+ : X \rightarrow [0, 1]$, $i_A^- : X \rightarrow [0, 1]$, $f_A^+ : X \rightarrow [0, 1]$ and $f_A^- : X \rightarrow [0, 1]$ denote the degree of truth-membership, degree of indeterminacy-membership and degree of falsity-membership of an element $x_i \in X$, respectively, and

$$0 \leq t_A(x_i) + i_A(x_i) + f_A(x_i) \leq 3 \quad \text{for all } x_i \in X, i = 1, 2, \dots, n.$$

2. The functions $t_B^+ : X \times X \rightarrow [0, 1]$, $t_B^- : X \times X \rightarrow [0, 1]$, $i_B^+ : X \times X \rightarrow [0, 1]$, $i_B^- : X \times X \rightarrow [0, 1]$, $f_B^+ : X \times X \rightarrow [0, 1]$, and $f_B^- : X \times X \rightarrow [0, 1]$, such that:

- (i). $t_B^+(x_i, x_j) \leq \min(t_A^+(x_i), t_A^+(x_j))$,
- (ii). $t_B^-(x_i, x_j) \leq \min(t_A^-(x_i), t_A^-(x_j))$,
- (iii). $i_B^+(x_i, x_j) \geq \max(i_A^+(x_i), i_A^+(x_j))$,
- (iv). $i_B^-(x_i, x_j) \geq \max(i_A^-(x_i), i_A^-(x_j))$,
- (v). $f_B^+(x_i, x_j) \geq \max(f_A^+(x_i), f_A^+(x_j))$,
- (vi). $f_B^-(x_i, x_j) \geq \max(f_A^-(x_i), f_A^-(x_j))$,

denote the degree of truth-membership, degree of indeterminacy-membership and degree of falsity-membership values of the edge $(x_i, x_j) \in E$, respectively, where,

$$0 \leq t_B(x_i, x_j) + i_B(x_i, x_j) + f_B(x_i, x_j) \leq 3 \quad \text{for all } (x_i, x_j) \in E, i, j = 1, 2, \dots, n.$$

We illustrate this definition by an example.

Example 2.3. We construct an interval-valued neutrosophic graph $G = (A, B)$ using Definition 2.2 as shown in the Fig. 1.

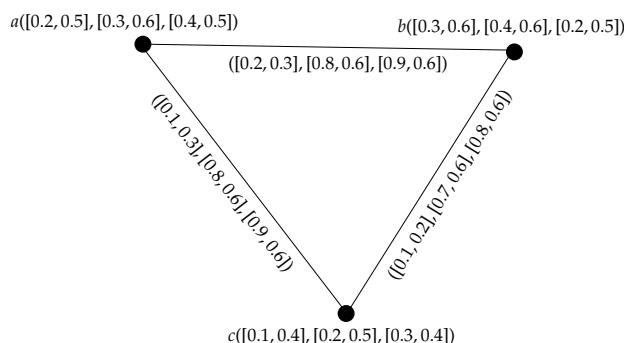


Figure 1: Interval-valued neutrosophic graph by using definition 2.2

We note that one can choose any value from $[0, 1]$ that satisfies condition 2(iii-vi) of Definition 2.2:

$$\begin{aligned} 0.8 &\geq \max(0.3, 0.4), & 0.6 &\geq \max(0.6, 0.6), \\ 0.9 &\geq \max(0.4, 0.2), & 0.6 &\geq \max(0.5, 0.5). \end{aligned}$$

We see that the degrees of indeterminacy-membership and falsity-membership of edge ab are intervals $[0.8, 0.6]$, and $[0.9, 0.6]$ which violate the property of an interval $[x, y] = \{x \mid a \leq x \leq b\}$. Hence $G = (A, B)$ is not an interval-valued neutrosophic graph.

Remark 2.4. An interval-valued neutrosophic graph is a representation of an interval-valued neutrosophic relation. In computer network model, each vertex represents a computer, and each edge between two computer represents a telephone line that operates in both directions if computer network model is simple (undirected). The intervals $[t_B^-(x_i, x_j), t_B^+(x_i, x_j)]$, $[i_B^-(x_i, x_j), i_B^+(x_i, x_j)]$ and $[f_B^-(x_i, x_j), f_B^+(x_i, x_j)]$ represent strength of relation between two computers in interval-valued neutrosophic computer network models. In applied network models, strength of a line (edge) between two computers (vertices) cannot be greater than the strength of computers (vertices) in a computer network model. Similarly, Dijkstra’s algorithm based on Broumi et al. [15]’s Definition 2.2 cannot calculate correctly shortest path between two cities in a network model. Thus, we conclude that published concepts on interval-valued neutrosophic graphs in [15–17] may be incorrect. These are factors which motivate us to modify the definition of interval-valued neutrosophic graphs.

We now define interval-valued neutrosophic graphs.

Definition 2.5. An interval-valued neutrosophic graph on a nonempty set X is a pair $G = (A, B)$, (in short, G), where A is an interval-valued neutrosophic set on X and B is an interval-valued neutrosophic relation on X such that

1. $t_B^-(xy) \leq \min(t_A^-(x), t_A^-(y)), \quad t_B^+(xy) \leq \min(t_A^+(x), t_A^+(y)),$
2. $i_B^-(xy) \leq \min(i_A^-(x), i_A^-(y)), \quad i_B^+(xy) \leq \min(i_A^+(x), i_A^+(y)),$
3. $f_B^-(xy) \leq \min(f_A^-(x), f_A^-(y)), \quad f_B^+(xy) \leq \min(f_A^+(x), f_A^+(y)), \quad \text{for all } x, y \in X.$

Note that B is called symmetric relation on A .

Example 2.6. Consider a graph G^* such that $X = \{a, b, c\}, E = \{ab, bc, ac\}$. Let A be an interval-valued neutrosophic subset of X and let B be an interval-valued neutrosophic subset of $E \subseteq X \times X$, as shown in the following Tables.

A	a	b	c
t_A^-	0.2	0.2	0.2
t_A^+	0.4	0.5	0.8
i_A^-	0.3	0.3	0.3
i_A^+	0.7	0.4	0.8
f_A^-	0.4	0.2	0.2
f_A^+	0.5	0.9	0.7

B	ab	bc	ac
t_B^-	0.1	0.1	0.1
t_B^+	0.3	0.3	0.3
i_B^-	0.2	0.2	0.2
i_B^+	0.3	0.3	0.3
f_B^-	0.2	0.2	0.2
f_B^+	0.5	0.7	0.5

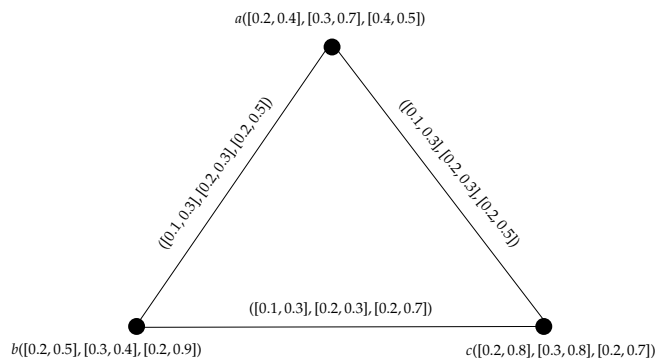


Figure 2: Interval-valued neutrosophic graph

By routine calculations, it can be observed that the graph shown in Fig.2 is an interval-valued neutrosophic graph.

Definition 2.7. The Cartesian product $G_1 \times G_2$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$ is defined as a pair $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$, such that:

1. $(t_{A_1}^- \times t_{A_2}^-)(x_1, x_2) = \min(t_{A_1}^-(x_1), t_{A_2}^-(x_2)), \quad (t_{A_1}^+ \times t_{A_2}^+)(x_1, x_2) = \min(t_{A_1}^+(x_1), t_{A_2}^+(x_2)),$
 $(i_{A_1}^- \times i_{A_2}^-)(x_1, x_2) = \min(i_{A_1}^-(x_1), i_{A_2}^-(x_2)), \quad (i_{A_1}^+ \times i_{A_2}^+)(x_1, x_2) = \min(i_{A_1}^+(x_1), i_{A_2}^+(x_2)),$
 $(f_{A_1}^- \times f_{A_2}^-)(x_1, x_2) = \min(f_{A_1}^-(x_1), f_{A_2}^-(x_2)), \quad (f_{A_1}^+ \times f_{A_2}^+)(x_1, x_2) = \min(f_{A_1}^+(x_1), f_{A_2}^+(x_2)),$
 for all $x_1, x_2 \in X$,
2. $(t_{B_1}^- \times t_{B_2}^-)((x, x_2)(x, y_2)) = \min(t_{A_1}^-(x), t_{B_2}^-(x_2y_2)), \quad (t_{B_1}^+ \times t_{B_2}^+)((x, x_2)(x, y_2)) = \min(t_{A_1}^+(x), t_{B_2}^+(x_2y_2)),$
 $(i_{B_1}^- \times i_{B_2}^-)((x, x_2)(x, y_2)) = \min(i_{A_1}^-(x), i_{B_2}^-(x_2y_2)), \quad (i_{B_1}^+ \times i_{B_2}^+)((x, x_2)(x, y_2)) = \min(i_{A_1}^+(x), i_{B_2}^+(x_2y_2)),$
 $(f_{B_1}^- \times f_{B_2}^-)((x, x_2)(x, y_2)) = \min(f_{A_1}^-(x), f_{B_2}^-(x_2y_2)), \quad (f_{B_1}^+ \times f_{B_2}^+)((x, x_2)(x, y_2)) = \min(f_{A_1}^+(x), f_{B_2}^+(x_2y_2)),$
 for all $x \in X_1$, and $x_2y_2 \in E_2$,

3. $(t_{B_1}^- \times t_{B_2}^-)((x_1, z)(y_1, z)) = \min(t_{B_1}^-(x_1 y_1), t_{A_2}^-(z)),$ $(t_{B_1}^+ \times t_{B_2}^+)((x_1, z)(y_1, z)) = \min(t_{B_1}^+(x_1 y_1), t_{A_2}^+(z)),$
 $(i_{B_1}^- \times i_{B_2}^-)((x_1, z)(y_1, z)) = \min(i_{B_1}^-(x_1 y_1), i_{A_2}^-(z)),$ $(i_{B_1}^+ \times i_{B_2}^+)((x_1, z)(y_1, z)) = \min(i_{B_1}^+(x_1 y_1), i_{A_2}^+(z)),$
 $(f_{B_1}^- \times f_{B_2}^-)((x_1, z)(y_1, z)) = \min(f_{B_1}^-(x_1 y_1), f_{A_2}^-(z)),$ $(f_{B_1}^+ \times f_{B_2}^+)((x_1, z)(y_1, z)) = \min(f_{B_1}^+(x_1 y_1), f_{A_2}^+(z)),$
 for all $z \in X_2,$ and $x_1 y_1 \in E_1.$

Example 2.8. Consider the two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2),$ as shown in Fig. 3.

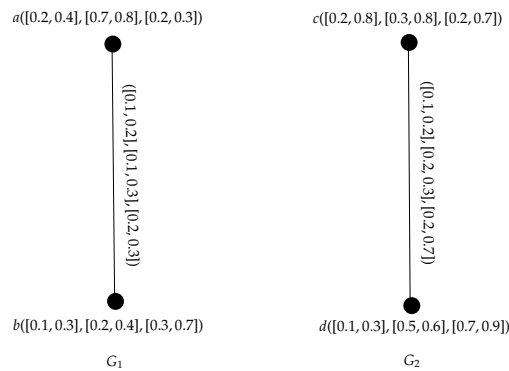


Figure 3: Interval-valued neutrosophic graphs G_1 and G_2

Then, their corresponding Cartesian product $G_1 \times G_2$ is shown in Fig. 4.

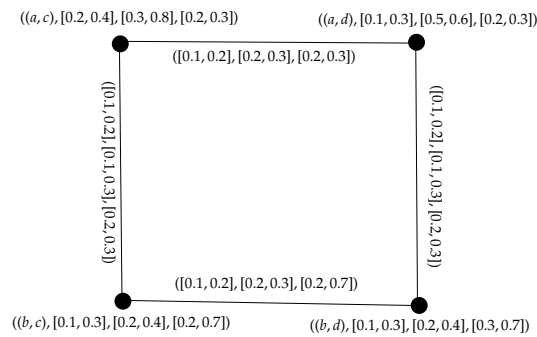


Figure 4: $G_1 \times G_2$

Proposition 2.9. The Cartesian product $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is an interval-valued neutrosophic graph.

Proof. The conditions for $A_1 \times A_2$ are obvious, therefore, we verify only conditions for $B_1 \times B_2.$

Let $x \in X_1$, and $x_2 y_2 \in E_2$. Then

$$\begin{aligned}(t_{B_1}^- \times t_{B_2}^-)((x, x_2)(x, y_2)) &= \min(t_{A_1}^-(x), t_{B_2}^-(x_2 y_2)) \\ &\leq \min(t_{A_1}^-(x), \min(t_{A_2}^-(x_2), t_{A_2}^-(y_2))) \\ &= \min(\min(t_{A_1}^-(x), t_{A_2}^-(x_2)), \min(t_{A_1}^-(x), t_{A_2}^-(y_2))) \\ &= \min((t_{A_1}^- \times t_{A_2}^-)(x, x_2), (t_{A_1}^- \times t_{A_2}^-)(x, y_2)),\end{aligned}$$

$$\begin{aligned}(t_{B_1}^+ \times t_{B_2}^+)((x, x_2)(x, y_2)) &= \min(t_{A_1}^+(x), t_{B_2}^+(x_2 y_2)) \\ &\leq \min(t_{A_1}^+(x), \min(t_{A_2}^+(x_2), t_{A_2}^+(y_2))) \\ &= \min(\min(t_{A_1}^+(x), t_{A_2}^+(x_2)), \min(t_{A_1}^+(x), t_{A_2}^+(y_2))) \\ &= \min((t_{A_1}^+ \times t_{A_2}^+)(x, x_2), (t_{A_1}^+ \times t_{A_2}^+)(x, y_2)),\end{aligned}$$

$$\begin{aligned}(i_{B_1}^- \times i_{B_2}^-)((x, x_2)(x, y_2)) &= \min(i_{A_1}^-(x), i_{B_2}^-(x_2 y_2)) \\ &\leq \min(i_{A_1}^-(x), \min(i_{A_2}^-(x_2), i_{A_2}^-(y_2))) \\ &= \min(\min(i_{A_1}^-(x), i_{A_2}^-(x_2)), \min(i_{A_1}^-(x), i_{A_2}^-(y_2))) \\ &= \min((i_{A_1}^- \times i_{A_2}^-)(x, x_2), (i_{A_1}^- \times i_{A_2}^-)(x, y_2)),\end{aligned}$$

$$\begin{aligned}(i_{B_1}^+ \times i_{B_2}^+)((x, x_2)(x, y_2)) &= \min(i_{A_1}^+(x), i_{B_2}^+(x_2 y_2)) \\ &\leq \min(i_{A_1}^+(x), \min(i_{A_2}^+(x_2), i_{A_2}^+(y_2))) \\ &= \min(\min(i_{A_1}^+(x), i_{A_2}^+(x_2)), \min(i_{A_1}^+(x), i_{A_2}^+(y_2))) \\ &= \min((i_{A_1}^+ \times i_{A_2}^+)(x, x_2), (i_{A_1}^+ \times i_{A_2}^+)(x, y_2)),\end{aligned}$$

$$\begin{aligned}(f_{B_1}^- \times f_{B_2}^-)((x, x_2)(x, y_2)) &= \min(f_{A_1}^-(x), f_{B_2}^-(x_2 y_2)) \\ &\leq \min(f_{A_1}^-(x), \min(f_{A_2}^-(x_2), f_{A_2}^-(y_2))) \\ &= \min(\min(f_{A_1}^-(x), f_{A_2}^-(x_2)), \min(f_{A_1}^-(x), f_{A_2}^-(y_2))) \\ &= \min((f_{A_1}^- \times f_{A_2}^-)(x, x_2), (f_{A_1}^- \times f_{A_2}^-)(x, y_2)),\end{aligned}$$

$$\begin{aligned}(f_{B_1}^+ \times f_{B_2}^+)((x, x_2)(x, y_2)) &= \min(f_{A_1}^+(x), f_{B_2}^+(x_2 y_2)) \\ &\leq \min(f_{A_1}^+(x), \min(f_{A_2}^+(x_2), f_{A_2}^+(y_2))) \\ &= \min(\min(f_{A_1}^+(x), f_{A_2}^+(x_2)), \min(f_{A_1}^+(x), f_{A_2}^+(y_2))) \\ &= \min((f_{A_1}^+ \times f_{A_2}^+)(x, x_2), (f_{A_1}^+ \times f_{A_2}^+)(x, y_2)).\end{aligned}$$

Similarly, for $z \in X_2$, and $x_1 y_1 \in E_1$ we have,

$$\begin{aligned}(t_{B_1}^- \times t_{B_2}^-)((x_1, z)(y_1, z)) &= \min(t_{B_1}^-(x_1 y_1), t_{A_2}^-(z)) \\ &\leq \min(\min(t_{A_1}^-(x_1), t_{A_1}^-(y_1)), t_{A_2}^-(z)) \\ &= \min(\min(t_{A_1}^-(x_1), t_{A_2}^-(z)), \min(t_{A_1}^-(y_1), t_{A_2}^-(z))) \\ &= \min((t_{A_1}^- \times t_{A_2}^-)(x_1, z), (t_{A_1}^- \times t_{A_2}^-)(y_1, z)),\end{aligned}$$

$$\begin{aligned}(t_{B_1}^+ \times t_{B_2}^+)((x_1, z)(y_1, z)) &= \min(t_{B_1}^+(x_1 y_1), t_{A_2}^+(z)) \\ &\leq \min(\min(t_{A_1}^+(x_1), t_{A_1}^+(y_1)), t_{A_2}^+(z)) \\ &= \min(\min(t_{A_1}^+(x_1), t_{A_2}^+(z)), \min(t_{A_1}^+(y_1), t_{A_2}^+(z))) \\ &= \min((t_{A_1}^+ \times t_{A_2}^+)(x_1, z), (t_{A_1}^+ \times t_{A_2}^+)(y_1, z)),\end{aligned}$$

$$\begin{aligned}
 (i_{B_1}^- \times i_{B_2}^-)((x_1, z)(y_1, z)) &= \min(i_{B_1}^-(x_1y_1), i_{A_2}^-(z)) \\
 &\leq \min(\min(i_{A_1}^-(x_1), i_{A_1}^-(y_1)), i_{A_2}^-(z)) \\
 &= \min(\min(i_{A_1}^-(x_1), i_{A_2}^-(z)), \min(i_{A_1}^-(y_1), i_{A_2}^-(z))) \\
 &= \min((i_{A_1}^- \times i_{A_2}^-)(x_1, z), (i_{A_1}^- \times i_{A_2}^-)(y_1, z)), \\
 (i_{B_1}^+ \times i_{B_2}^+)((x_1, z)(y_1, z)) &= \min(i_{B_1}^+(x_1y_1), i_{A_2}^+(z)) \\
 &\leq \min(\min(i_{A_1}^+(x_1), i_{A_1}^+(y_1)), i_{A_2}^+(z)) \\
 &= \min(\min(i_{A_1}^+(x_1), i_{A_2}^+(z)), \min(i_{A_1}^+(y_1), i_{A_2}^+(z))) \\
 &= \min((i_{A_1}^+ \times i_{A_2}^+)(x_1, z), (i_{A_1}^+ \times i_{A_2}^+)(y_1, z)), \\
 (f_{B_1}^- \times f_{B_2}^-)((x_1, z)(y_1, z)) &= \min(f_{B_1}^-(x_1y_1), f_{A_2}^-(z)) \\
 &\leq \min(\min(f_{A_1}^-(x_1), f_{A_1}^-(y_1)), f_{A_2}^-(z)) \\
 &= \min(\min(f_{A_1}^-(x_1), f_{A_2}^-(z)), \min(f_{A_1}^-(y_1), f_{A_2}^-(z))) \\
 &= \min((f_{A_1}^- \times f_{A_2}^-)(x_1, z), (f_{A_1}^- \times f_{A_2}^-)(y_1, z)), \\
 (f_{B_1}^+ \times f_{B_2}^+)((x_1, z)(y_1, z)) &= \min(f_{B_1}^+(x_1y_1), f_{A_2}^+(z)) \\
 &\leq \min(\min(f_{A_1}^+(x_1), f_{A_1}^+(y_1)), f_{A_2}^+(z)) \\
 &= \min(\min(f_{A_1}^+(x_1), f_{A_2}^+(z)), \min(f_{A_1}^+(y_1), f_{A_2}^+(z))) \\
 &= \min((f_{A_1}^+ \times f_{A_2}^+)(x_1, z), (f_{A_1}^+ \times f_{A_2}^+)(y_1, z)).
 \end{aligned}$$

This completes the proof. \square

Definition 2.10. The composition $G_1 \circ G_2$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$ is defined as a pair $G_1 \circ G_2 = (A_1 \circ A_2, B_1 \circ B_2)$, such that:

1. $(t_{A_1}^- \circ t_{A_2}^-)(x_1, x_2) = \min(t_{A_1}^-(x_1), t_{A_2}^-(x_2)),$ $(t_{A_1}^+ \circ t_{A_2}^+)(x_1, x_2) = \min(t_{A_1}^+(x_1), t_{A_2}^+(x_2)),$
 $(i_{A_1}^- \circ i_{A_2}^-)(x_1, x_2) = \min(i_{A_1}^-(x_1), i_{A_2}^-(x_2)),$ $(i_{A_1}^+ \circ i_{A_2}^+)(x_1, x_2) = \min(i_{A_1}^+(x_1), i_{A_2}^+(x_2)),$
 $(f_{A_1}^- \circ f_{A_2}^-)(x_1, x_2) = \min(f_{A_1}^-(x_1), f_{A_2}^-(x_2)),$ $(f_{A_1}^+ \circ f_{A_2}^+)(x_1, x_2) = \min(f_{A_1}^+(x_1), f_{A_2}^+(x_2)),$
for all $x_1, x_2 \in X,$
2. $(t_{B_1}^- \circ t_{B_2}^-)((x, x_2)(x, y_2)) = \min(t_{A_1}^-(x), t_{B_2}^-(x_2y_2)),$ $(t_{B_1}^+ \circ t_{B_2}^+)((x, x_2)(x, y_2)) = \min(t_{A_1}^+(x), t_{B_2}^+(x_2y_2)),$
 $(i_{B_1}^- \circ i_{B_2}^-)((x, x_2)(x, y_2)) = \min(i_{A_1}^-(x), i_{B_2}^-(x_2y_2)),$ $(i_{B_1}^+ \circ i_{B_2}^+)((x, x_2)(x, y_2)) = \min(i_{A_1}^+(x), i_{B_2}^+(x_2y_2)),$
 $(f_{B_1}^- \circ f_{B_2}^-)((x, x_2)(x, y_2)) = \min(f_{A_1}^-(x), f_{B_2}^-(x_2y_2)),$ $(f_{B_1}^+ \circ f_{B_2}^+)((x, x_2)(x, y_2)) = \min(f_{A_1}^+(x), f_{B_2}^+(x_2y_2)),$
for all $x \in X_1,$ and $x_2y_2 \in E_2,$
3. $(t_{B_1}^- \circ t_{B_2}^-)((x_1, z)(y_1, z)) = \min(t_{B_1}^-(x_1y_1), t_{A_2}^-(z)),$ $(t_{B_1}^+ \circ t_{B_2}^+)((x_1, z)(y_1, z)) = \min(t_{B_1}^+(x_1y_1), t_{A_2}^+(z)),$
 $(i_{B_1}^- \circ i_{B_2}^-)((x_1, z)(y_1, z)) = \min(i_{B_1}^-(x_1y_1), i_{A_2}^-(z)),$ $(i_{B_1}^+ \circ i_{B_2}^+)((x_1, z)(y_1, z)) = \min(i_{B_1}^+(x_1y_1), i_{A_2}^+(z)),$
 $(f_{B_1}^- \circ f_{B_2}^-)((x_1, z)(y_1, z)) = \min(f_{B_1}^-(x_1y_1), f_{A_2}^-(z)),$ $(f_{B_1}^+ \circ f_{B_2}^+)((x_1, z)(y_1, z)) = \min(f_{B_1}^+(x_1y_1), f_{A_2}^+(z)),$
for all $z \in X_2,$ and $x_1y_1 \in E_1,$
4. $(t_{B_1}^- \circ t_{B_2}^-)((x_1, x_2)(y_1, y_2)) = \min(t_{A_2}^-(x_2), t_{A_2}^-(y_2), t_{B_1}^-(x_1y_1)),$
 $(t_{B_1}^+ \circ t_{B_2}^+)((x_1, x_2)(y_1, y_2)) = \min(t_{A_2}^+(x_2), t_{A_2}^+(y_2), t_{B_1}^+(x_1y_1)),$
 $(i_{B_1}^- \circ i_{B_2}^-)((x_1, x_2)(y_1, y_2)) = \min(i_{A_2}^-(x_2), i_{A_2}^-(y_2), i_{B_1}^-(x_1y_1)),$
 $(i_{B_1}^+ \circ i_{B_2}^+)((x_1, x_2)(y_1, y_2)) = \min(i_{A_2}^+(x_2), i_{A_2}^+(y_2), i_{B_1}^+(x_1y_1)),$
 $(f_{B_1}^- \circ f_{B_2}^-)((x_1, x_2)(y_1, y_2)) = \min(f_{A_2}^-(x_2), f_{A_2}^-(y_2), f_{B_1}^-(x_1y_1)),$
 $(f_{B_1}^+ \circ f_{B_2}^+)((x_1, x_2)(y_1, y_2)) = \min(f_{A_2}^+(x_2), f_{A_2}^+(y_2), t_{B_1}^+(x_1y_1)),$ for all $x_2, y_2 \in X_2,$ $x_2 \neq y_2$ and $x_1y_1 \in E_1.$

Example 2.11. Consider the two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, as shown in Fig. 5.

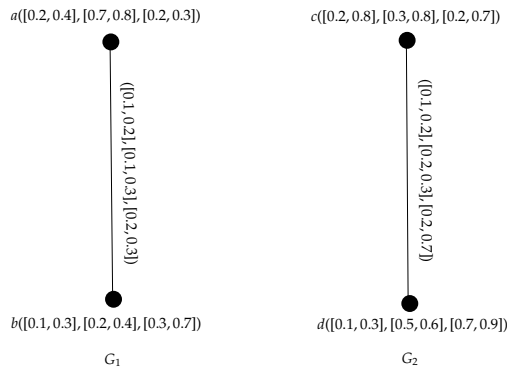


Figure 5: Interval-valued neutrosophic graphs G_1 and G_2

Then, their composition $G_1 \circ G_2$ is shown in Fig. 6.

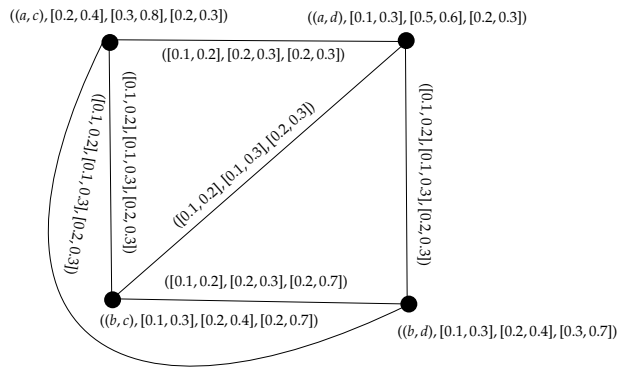


Figure 6: $G_1 \circ G_2$

Proposition 2.12. The composition $G_1 \circ G_2 = (A_1 \circ A_2, B_1 \circ B_2)$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is an interval-valued neutrosophic graph.

Proof. Similarly, as in the previous proof we verify the conditions for $B_1 \circ B_2$ only.

$$\begin{aligned}
 (t_{B_1}^- \circ t_{B_2}^-)((x, x_2)(x, y_2)) &= \min(t_{A_1}^-(x), t_{B_2}^-(x_2 y_2)) \\
 &\leq \min(t_{A_1}^-(x), \min(t_{A_2}^-(x_2), t_{A_2}^-(y_2))) \\
 &= \min(\min(t_{A_1}^-(x), t_{A_2}^-(x_2)), \min(t_{A_1}^-(x), t_{A_2}^-(y_2))) \\
 &= \min((t_{A_1}^- \circ t_{A_2}^-)(x, x_2), (t_{A_1}^- \circ t_{A_2}^-)(x, y_2)), \\
 (t_{B_1}^+ \circ t_{B_2}^+)((x, x_2)(x, y_2)) &= \min(t_{A_1}^+(x), t_{B_2}^+(x_2 y_2)) \\
 &\leq \min(t_{A_1}^+(x), \min(t_{A_2}^+(x_2), t_{A_2}^+(y_2))) \\
 &= \min(\min(t_{A_1}^+(x), t_{A_2}^+(x_2)), \min(t_{A_1}^+(x), t_{A_2}^+(y_2))) \\
 &= \min((t_{A_1}^+ \circ t_{A_2}^+)(x, x_2), (t_{A_1}^+ \circ t_{A_2}^+)(x, y_2)),
 \end{aligned}$$

$$\begin{aligned}
 (i_{B_1}^- \circ i_{B_2}^-)((x, x_2)(x, y_2)) &= \min(i_{A_1}^-(x), i_{B_2}^-(x_2 y_2)) \\
 &\leq \min(i_{A_1}^-(x), \min(i_{A_2}^-(x_2), i_{A_2}^-(y_2))) \\
 &= \min(\min(i_{A_1}^-(x), i_{A_2}^-(x_2)), \min(i_{A_1}^-(x), i_{A_2}^-(y_2))) \\
 &= \min((i_{A_1}^- \circ i_{A_2}^-)(x, x_2), (i_{A_1}^- \circ i_{A_2}^-)(x, y_2)), \\
 (i_{B_1}^+ \circ i_{B_2}^+)((x, x_2)(x, y_2)) &= \min(i_{A_1}^+(x), i_{B_2}^+(x_2 y_2)) \\
 &\leq \min(i_{A_1}^+(x), \min(i_{A_2}^+(x_2), i_{A_2}^+(y_2))) \\
 &= \min(\min(i_{A_1}^+(x), i_{A_2}^+(x_2)), \min(i_{A_1}^+(x), i_{A_2}^+(y_2))) \\
 &= \min((i_{A_1}^+ \circ i_{A_2}^+)(x, x_2), (i_{A_1}^+ \circ i_{A_2}^+)(x, y_2)),
 \end{aligned}$$

$$\begin{aligned}
 (f_{B_1}^- \circ f_{B_2}^-)((x, x_2)(x, y_2)) &= \min(f_{A_1}^-(x), f_{B_2}^-(x_2 y_2)) \\
 &\leq \min(f_{A_1}^-(x), \min(f_{A_2}^-(x_2), f_{A_2}^-(y_2))) \\
 &= \min(\min(f_{A_1}^-(x), f_{A_2}^-(x_2)), \min(f_{A_1}^-(x), f_{A_2}^-(y_2))) \\
 &= \min((f_{A_1}^- \circ f_{A_2}^-)(x, x_2), (f_{A_1}^- \circ f_{A_2}^-)(x, y_2)), \\
 (f_{B_1}^+ \circ f_{B_2}^+)((x, x_2)(x, y_2)) &= \min(f_{A_1}^+(x), f_{B_2}^+(x_2 y_2)) \\
 &\leq \min(f_{A_1}^+(x), \min(f_{A_2}^+(x_2), f_{A_2}^+(y_2))) \\
 &= \min(\min(f_{A_1}^+(x), f_{A_2}^+(x_2)), \min(f_{A_1}^+(x), f_{A_2}^+(y_2))) \\
 &= \min((f_{A_1}^+ \circ f_{A_2}^+)(x, x_2), (f_{A_1}^+ \circ f_{A_2}^+)(x, y_2)).
 \end{aligned}$$

In the case $z \in X_2$, and $x_1 y_1 \in E_1$, the proof is similar. In the case $x_2, y_2 \in X_2$, $x_2 \neq y_2$ and $x_1 y_1 \in E_1$, we have,

$$\begin{aligned}
 (t_{B_1}^- \circ t_{B_2}^-)((x_1, x_2)(y_1, y_2)) &= \min(t_{A_2}^-(x_2), t_{A_2}^-(y_2), t_{B_1}^-(x_1 y_1)) \\
 &\leq \min(t_{A_2}^-(x_2), t_{A_2}^-(y_2), \min(t_{A_1}^-(x_1), t_{A_1}^-(y_1))) \\
 &= \min(\min(t_{A_1}^-(x_1), t_{A_2}^-(x_2)), \min(t_{A_1}^-(y_1), t_{A_2}^-(y_2))) \\
 &= \min((t_{A_1}^- \circ t_{A_2}^-)((x_1, x_2), (t_{A_1}^- \circ t_{A_2}^-)((y_1, y_2)), \\
 (t_{B_1}^+ \circ t_{B_2}^+)((x_1, x_2)(y_1, y_2)) &= \min(t_{A_2}^+(x_2), t_{A_2}^+(y_2), t_{B_1}^+(x_1 y_1)) \\
 &\leq \min(t_{A_2}^+(x_2), t_{A_2}^+(y_2), \min(t_{A_1}^+(x_1), t_{A_1}^+(y_1))) \\
 &= \min(\min(t_{A_1}^+(x_1), t_{A_2}^+(x_2)), \min(t_{A_1}^+(y_1), t_{A_2}^+(y_2))) \\
 &= \min((t_{A_1}^+ \circ t_{A_2}^+)((x_1, x_2), (t_{A_1}^+ \circ t_{A_2}^+)((y_1, y_2)),
 \end{aligned}$$

$$\begin{aligned}
 (i_{B_1}^- \circ i_{B_2}^-)((x_1, x_2)(y_1, y_2)) &= \min(i_{A_2}^-(x_2), i_{A_2}^-(y_2), i_{B_1}^-(x_1 y_1)) \\
 &\leq \min(i_{A_2}^-(x_2), i_{A_2}^-(y_2), \min(i_{A_1}^-(x_1), i_{A_1}^-(y_1))) \\
 &= \min(\min(i_{A_1}^-(x_1), i_{A_2}^-(x_2)), \min(i_{A_1}^-(y_1), i_{A_2}^-(y_2))) \\
 &= \min((i_{A_1}^- \circ i_{A_2}^-)((x_1, x_2), (i_{A_1}^- \circ i_{A_2}^-)((y_1, y_2)), \\
 (i_{B_1}^+ \circ i_{B_2}^+)((x_1, x_2)(y_1, y_2)) &= \min(i_{A_2}^+(x_2), i_{A_2}^+(y_2), i_{B_1}^+(x_1 y_1)) \\
 &\leq \min(i_{A_2}^+(x_2), i_{A_2}^+(y_2), \min(i_{A_1}^+(x_1), i_{A_1}^+(y_1))) \\
 &= \min(\min(i_{A_1}^+(x_1), i_{A_2}^+(x_2)), \min(i_{A_1}^+(y_1), i_{A_2}^+(y_2))) \\
 &= \min((i_{A_1}^+ \circ i_{A_2}^+)((x_1, x_2), (i_{A_1}^+ \circ i_{A_2}^+)((y_1, y_2)),
 \end{aligned}$$

$$\begin{aligned}
 (f_{B_1}^- \circ f_{B_2}^-)((x_1, x_2)(y_1, y_2)) &= \min(f_{A_2}^-(x_2), f_{A_2}^-(y_2), f_{B_1}^-(x_1y_1)) \\
 &\leq \min(f_{A_2}^-(x_2), f_{A_2}^-(y_2), \min(f_{A_1}^-(x_1), f_{A_1}^-(y_1))) \\
 &= \min(\min(f_{A_1}^-(x_1), f_{A_2}^-(x_2)), \min(f_{A_1}^-(y_1), f_{A_2}^-(y_2))) \\
 &= \min((f_{A_1}^- \circ f_{A_2}^-)((x_1, x_2)), (f_{A_1}^- \circ f_{A_2}^-)((y_1, y_2)), \\
 (f_{B_1}^+ \circ f_{B_2}^+)((x_1, x_2)(y_1, y_2)) &= \min(f_{A_2}^+(x_2), f_{A_2}^+(y_2), f_{B_1}^+(x_1y_1)) \\
 &\leq \min(f_{A_2}^+(x_2), f_{A_2}^+(y_2), \min(f_{A_1}^+(x_1), f_{A_1}^+(y_1))) \\
 &= \min(\min(f_{A_1}^+(x_1), f_{A_2}^+(x_2)), \min(f_{A_1}^+(y_1), f_{A_2}^+(y_2))) \\
 &= \min((f_{A_1}^+ \circ f_{A_2}^+)((x_1, x_2)), (f_{A_1}^+ \circ f_{A_2}^+)((y_1, y_2)),
 \end{aligned}$$

This completes the proof. \square

Definition 2.13. The union $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ of two interval-valued neutrosophic graphs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

1. $(t_{A_1}^- \cup t_{A_2}^-)(x) = t_{A_1}^-(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(t_{A_1}^- \cup t_{A_2}^-)(x) = t_{A_2}^-(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(t_{A_1}^- \cup t_{A_2}^-)(x) = \max(t_{A_1}^-(x), t_{A_2}^-(x))$ if $x \in X_1 \cap X_2$,
 $(t_{A_1}^+ \cup t_{A_2}^+)(x) = t_{A_1}^+(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(t_{A_1}^+ \cup t_{A_2}^+)(x) = t_{A_2}^+(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(t_{A_1}^+ \cup t_{A_2}^+)(x) = \max(t_{A_1}^+(x), t_{A_2}^+(x))$ if $x \in X_1 \cap X_2$,
2. $(i_{A_1}^- \cup i_{A_2}^-)(x) = i_{A_1}^-(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(i_{A_1}^- \cup i_{A_2}^-)(x) = i_{A_2}^-(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(i_{A_1}^- \cup i_{A_2}^-)(x) = \max(i_{A_1}^-(x), i_{A_2}^-(x))$ if $x \in X_1 \cap X_2$,
 $(i_{A_1}^+ \cup i_{A_2}^+)(x) = i_{A_1}^+(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(i_{A_1}^+ \cup i_{A_2}^+)(x) = i_{A_2}^+(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(i_{A_1}^+ \cup i_{A_2}^+)(x) = \max(i_{A_1}^+(x), i_{A_2}^+(x))$ if $x \in X_1 \cap X_2$,
3. $(f_{A_1}^- \cup f_{A_2}^-)(x) = f_{A_1}^-(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(f_{A_1}^- \cup f_{A_2}^-)(x) = f_{A_2}^-(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(f_{A_1}^- \cup f_{A_2}^-)(x) = \min(f_{A_1}^-(x), f_{A_2}^-(x))$ if $x \in X_1 \cap X_2$,
 $(f_{A_1}^+ \cup f_{A_2}^+)(x) = f_{A_1}^+(x)$ if $x \in X_1$ and $x \notin X_2$,
 $(f_{A_1}^+ \cup f_{A_2}^+)(x) = f_{A_2}^+(x)$ if $x \in X_2$ and $x \notin X_1$,
 $(f_{A_1}^+ \cup f_{A_2}^+)(x) = \min(f_{A_1}^+(x), f_{A_2}^+(x))$ if $x \in X_1 \cap X_2$,
4. $(t_{B_1}^- \cup t_{B_2}^-)(xy) = t_{B_1}^-(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
 $(t_{B_1}^- \cup t_{B_2}^-)(xy) = t_{B_2}^-(xy)$ if $xy \in E_2$ and $xy \notin E_1$,
 $(t_{B_1}^- \cup t_{B_2}^-)(xy) = \max(t_{B_1}^-(xy), t_{B_2}^-(xy))$ if $xy \in E_1 \cap E_2$,
 $(t_{B_1}^+ \cup t_{B_2}^+)(xy) = t_{B_1}^+(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
 $(t_{B_1}^+ \cup t_{B_2}^+)(xy) = t_{B_2}^+(xy)$ if $xy \in E_2$ and $xy \notin E_1$,
 $(t_{B_1}^+ \cup t_{B_2}^+)(xy) = \max(t_{B_1}^+(xy), t_{B_2}^+(xy))$ if $xy \in E_1 \cap E_2$,
5. $(i_{B_1}^- \cup i_{B_2}^-)(xy) = i_{B_1}^-(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
 $(i_{B_1}^- \cup i_{B_2}^-)(xy) = i_{B_2}^-(xy)$ if $xy \in E_2$ and $xy \notin E_1$,
 $(i_{B_1}^- \cup i_{B_2}^-)(xy) = \max(i_{B_1}^-(xy), i_{B_2}^-(xy))$ if $xy \in E_1 \cap E_2$,
 $(i_{B_1}^+ \cup i_{B_2}^+)(xy) = i_{B_1}^+(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
 $(i_{B_1}^+ \cup i_{B_2}^+)(xy) = i_{B_2}^+(xy)$ if $xy \in E_2$ and $xy \notin E_1$,
 $(i_{B_1}^+ \cup i_{B_2}^+)(xy) = \max(i_{B_1}^+(xy), i_{B_2}^+(xy))$ if $xy \in E_1 \cap E_2$,
6. $(f_{B_1}^- \cup f_{B_2}^-)(xy) = f_{B_1}^-(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
 $(f_{B_1}^- \cup f_{B_2}^-)(xy) = f_{B_2}^-(xy)$ if $xy \in E_2$ and $xy \notin E_1$,
 $(f_{B_1}^- \cup f_{B_2}^-)(xy) = \min(f_{B_1}^-(xy), f_{B_2}^-(xy))$ if $xy \in E_1 \cap E_2$,

$$\begin{aligned}
 (f_{B_1}^+ \cup f_{B_2}^+)(xy) &= f_{B_1}^+(xy) && \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\
 (f_{B_1}^+ \cup f_{B_2}^+)(xy) &= f_{B_2}^+(xy) && \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\
 (f_{B_1}^+ \cup f_{B_2}^+)(xy) &= \min(f_{B_1}^+(xy), f_{B_2}^+(xy)) && \text{if } xy \in E_1 \cap E_2.
 \end{aligned}$$

Example 2.14. Consider the two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, as shown in Fig. 7.

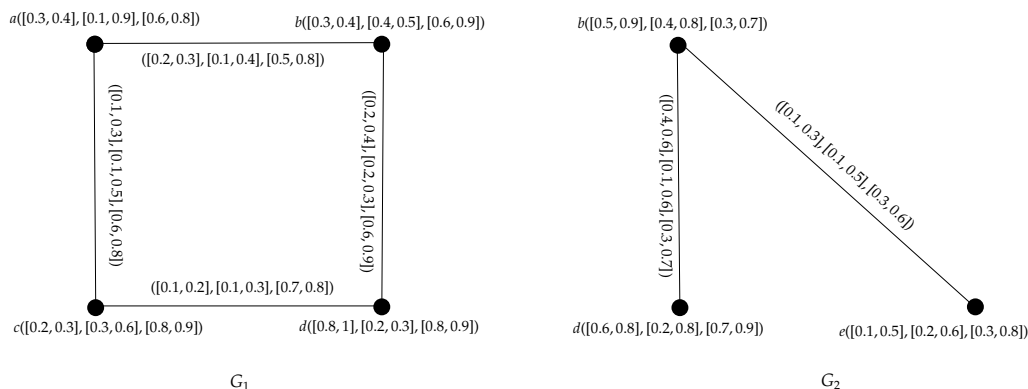


Figure 7: Interval-valued neutrosophic graphs G_1 and G_2

Then, their corresponding union $G_1 \cup G_2$ is shown in Fig. 8.

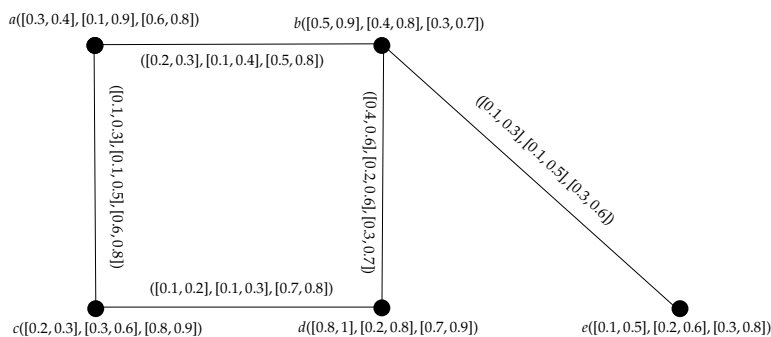


Figure 8: $G_1 \cup G_2$

Proposition 2.15. The union $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is an interval-valued neutrosophic graph.

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued neutrosophic graphs of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. We prove that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued neutrosophic graph of the graph $G_1^* \cup G_2^*$. Since all the conditions for $A_1 \cup A_2$ are automatically satisfied therefore, we verify only conditions for $B_1 \cup B_2$.

In the case, when $xy \in E_1 \cap E_2$. Then

$$\begin{aligned}(t_{B_1}^- \cup t_{B_2}^-)(xy) &= \max(t_{B_1}^-(xy), t_{B_2}^-(xy)) \\ &\leq \max(\min(t_{A_1}^-(x), t_{A_1}^-(y)), \min(t_{A_2}^-(x), t_{A_2}^-(y))) \\ &= \min(\max(t_{A_1}^-(x), t_{A_2}^-(x)), \max(t_{A_1}^-(y), t_{A_2}^-(y))) \\ &= \min((t_{A_1}^- \cup t_{A_2}^-)(x), (t_{A_1}^- \cup t_{A_2}^-)(y)),\end{aligned}$$

$$\begin{aligned}(t_{B_1}^+ \cup t_{B_2}^+)(xy) &= \max(t_{B_1}^+(xy), t_{B_2}^+(xy)) \\ &\leq \max(\min(t_{A_1}^+(x), t_{A_1}^+(y)), \min(t_{A_2}^+(x), t_{A_2}^+(y))) \\ &= \min(\max(t_{A_1}^+(x), t_{A_2}^+(x)), \max(t_{A_1}^+(y), t_{A_2}^+(y))) \\ &= \min((t_{A_1}^+ \cup t_{A_2}^+)(x), (t_{A_1}^+ \cup t_{A_2}^+)(y)),\end{aligned}$$

$$\begin{aligned}(i_{B_1}^- \cup i_{B_2}^-)(xy) &= \max(i_{B_1}^-(xy), i_{B_2}^-(xy)) \\ &\leq \max(\min(i_{A_1}^-(x), i_{A_1}^-(y)), \min(i_{A_2}^-(x), i_{A_2}^-(y))) \\ &= \min(\max(i_{A_1}^-(x), i_{A_2}^-(x)), \max(i_{A_1}^-(y), i_{A_2}^-(y))) \\ &= \min((i_{A_1}^- \cup i_{A_2}^-)(x), (i_{A_1}^- \cup i_{A_2}^-)(y)),\end{aligned}$$

$$\begin{aligned}(i_{B_1}^+ \cup i_{B_2}^+)(xy) &= \max(i_{B_1}^+(xy), i_{B_2}^+(xy)) \\ &\leq \max(\min(i_{A_1}^+(x), i_{A_1}^+(y)), \min(i_{A_2}^+(x), i_{A_2}^+(y))) \\ &= \min(\max(i_{A_1}^+(x), i_{A_2}^+(x)), \max(i_{A_1}^+(y), i_{A_2}^+(y))) \\ &= \min((i_{A_1}^+ \cup i_{A_2}^+)(x), (i_{A_1}^+ \cup i_{A_2}^+)(y)),\end{aligned}$$

$$\begin{aligned}(f_{B_1}^- \cup f_{B_2}^-)(xy) &= \min(f_{B_1}^-(xy), f_{B_2}^-(xy)) \\ &\leq \min(\min(f_{A_1}^-(x), f_{A_1}^-(y)), \min(f_{A_2}^-(x), f_{A_2}^-(y))) \\ &= \min(\min(f_{A_1}^-(x), f_{A_2}^-(x)), \min(f_{A_1}^-(y), f_{A_2}^-(y))) \\ &= \min((f_{A_1}^- \cup f_{A_2}^-)(x), (f_{A_1}^- \cup f_{A_2}^-)(y)),\end{aligned}$$

$$\begin{aligned}(f_{B_1}^+ \cup f_{B_2}^+)(xy) &= \min(f_{B_1}^+(xy), f_{B_2}^+(xy)) \\ &\leq \min(\min(f_{A_1}^+(x), f_{A_1}^+(y)), \min(f_{A_2}^+(x), f_{A_2}^+(y))) \\ &= \min(\min(f_{A_1}^+(x), f_{A_2}^+(x)), \min(f_{A_1}^+(y), f_{A_2}^+(y))) \\ &= \min((f_{A_1}^+ \cup f_{A_2}^+)(x), (f_{A_1}^+ \cup f_{A_2}^+)(y)),\end{aligned}$$

If $xy \in E_1$ and $xy \notin E_2$, then

$$(t_{B_1}^- \cup t_{B_2}^-)(xy) \leq \min((t_{A_1}^- \cup t_{A_2}^-)(x), (t_{A_1}^- \cup t_{A_2}^-)(y)),$$

$$(t_{B_1}^+ \cup t_{B_2}^+)(xy) \leq \min((t_{A_1}^+ \cup t_{A_2}^+)(x), (t_{A_1}^+ \cup t_{A_2}^+)(y)),$$

$$(i_{B_1}^- \cup i_{B_2}^-)(xy) \leq \min((i_{A_1}^- \cup i_{A_2}^-)(x), (i_{A_1}^- \cup i_{A_2}^-)(y)),$$

$$(i_{B_1}^+ \cup i_{B_2}^+)(xy) \leq \min((i_{A_1}^+ \cup i_{A_2}^+)(x), (i_{A_1}^+ \cup i_{A_2}^+)(y)),$$

$$(f_{B_1}^- \cup f_{B_2}^-)(xy) \leq \min((f_{A_1}^- \cup f_{A_2}^-)(x), (f_{A_1}^- \cup f_{A_2}^-)(y)),$$

$$(f_{B_1}^+ \cup f_{B_2}^+)(xy) \leq \min((f_{A_1}^+ \cup f_{A_2}^+)(x), (f_{A_1}^+ \cup f_{A_2}^+)(y)).$$

If $xy \in E_2$ and $xy \notin E_1$, then

$$(t_{B_1}^- \cup t_{B_2}^-)(xy) \leq \min((t_{A_1}^- \cup t_{A_2}^-)(x), (t_{A_1}^- \cup t_{A_2}^-)(y)),$$

$$(t_{B_1}^+ \cup t_{B_2}^+)(xy) \leq \min((t_{A_1}^+ \cup t_{A_2}^+)(x), (t_{A_1}^+ \cup t_{A_2}^+)(y)),$$

$$(i_{B_1}^- \cup i_{B_2}^-)(xy) \leq \min((i_{A_1}^- \cup i_{A_2}^-)(x), (i_{A_1}^- \cup i_{A_2}^-)(y)),$$

$$(i_{B_1}^+ \cup i_{B_2}^+)(xy) \leq \min((i_{A_1}^+ \cup i_{A_2}^+)(x), (i_{A_1}^+ \cup i_{A_2}^+)(y)),$$

$$(f_{B_1}^- \cup f_{B_2}^-)(xy) \leq \min((f_{A_1}^- \cup f_{A_2}^-)(x), (f_{A_1}^- \cup f_{A_2}^-)(y)),$$

$$(f_{B_1}^+ \cup f_{B_2}^+)(xy) \leq \min((f_{A_1}^+ \cup f_{A_2}^+)(x), (f_{A_1}^+ \cup f_{A_2}^+)(y)).$$

This completes the proof. \square

Definition 2.16. The join $G_1+G_2 = (A_1+A_2, B_1+B_2)$ of two interval-valued neutrosophic graphs G_1 and G_2 of the graphs G_1^* and G_2^* , where, $X_1 \cap X_2 = \emptyset$, is defined as follows:

1. $(t_{A_1}^- + t_{A_2}^-)(x) = (t_{A_1}^- \cup t_{A_2}^-)(x)$, $(t_{A_1}^+ + t_{A_2}^+)(x) = (t_{A_1}^+ \cup t_{A_2}^+)(x)$, if $x \in X_1 \cup X_2$,
2. $(i_{A_1}^- + i_{A_2}^-)(x) = (i_{A_1}^- \cup i_{A_2}^-)(x)$, $(i_{A_1}^+ + i_{A_2}^+)(x) = (i_{A_1}^+ \cup i_{A_2}^+)(x)$, if $x \in X_1 \cup X_2$,
3. $(f_{A_1}^- + f_{A_2}^-)(x) = (f_{A_1}^- \cup f_{A_2}^-)(x)$, $(f_{A_1}^+ + f_{A_2}^+)(x) = (f_{A_1}^+ \cup f_{A_2}^+)(x)$, if $x \in X_1 \cup X_2$,
4. $(t_{B_1}^- + t_{B_2}^-)(xy) = (t_{B_1}^- \cup t_{B_2}^-)(xy)$, $(t_{B_1}^+ + t_{B_2}^+)(xy) = (t_{B_1}^+ \cup t_{B_2}^+)(xy)$, if $xy \in E_1 \cap E_2$,
5. $(i_{B_1}^- + i_{B_2}^-)(xy) = (i_{B_1}^- \cup i_{B_2}^-)(xy)$, $(i_{B_1}^+ + i_{B_2}^+)(xy) = (i_{B_1}^+ \cup i_{B_2}^+)(xy)$, if $xy \in E_1 \cap E_2$,
6. $(f_{B_1}^- + f_{B_2}^-)(xy) = (f_{B_1}^- \cup f_{B_2}^-)(xy)$, $(f_{B_1}^+ + f_{B_2}^+)(xy) = (f_{B_1}^+ \cup f_{B_2}^+)(xy)$, if $xy \in E_1 \cap E_2$,
7. $(t_{B_1}^- + t_{B_2}^-)(xy) = \min(t_{A_1}^-(x), t_{A_2}^-(y))$, $(t_{B_1}^+ + t_{B_2}^+)(xy) = \min(t_{A_1}^+(x), t_{A_2}^+(y))$, if $xy \in \dot{E}$,
where \dot{E} is the set of all edges joining the vertices of X_1 and X_2 ,
8. $(i_{B_1}^- + i_{B_2}^-)(xy) = \min(i_{A_1}^-(x), i_{A_2}^-(y))$, $(i_{B_1}^+ + i_{B_2}^+)(xy) = \min(i_{A_1}^+(x), i_{A_2}^+(y))$, if $xy \in \dot{E}$,
where \dot{E} is the set of all edges joining the vertices of X_1 and X_2 ,
9. $(f_{B_1}^- + f_{B_2}^-)(xy) = \min(f_{A_1}^-(x), f_{A_2}^-(y))$, $(f_{B_1}^+ + f_{B_2}^+)(xy) = \min(f_{A_1}^+(x), f_{A_2}^+(y))$, if $xy \in \dot{E}$,
where \dot{E} is the set of all edges joining the vertices of X_1 and X_2 .

Example 2.17. Consider the two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, as shown in Fig. 9.

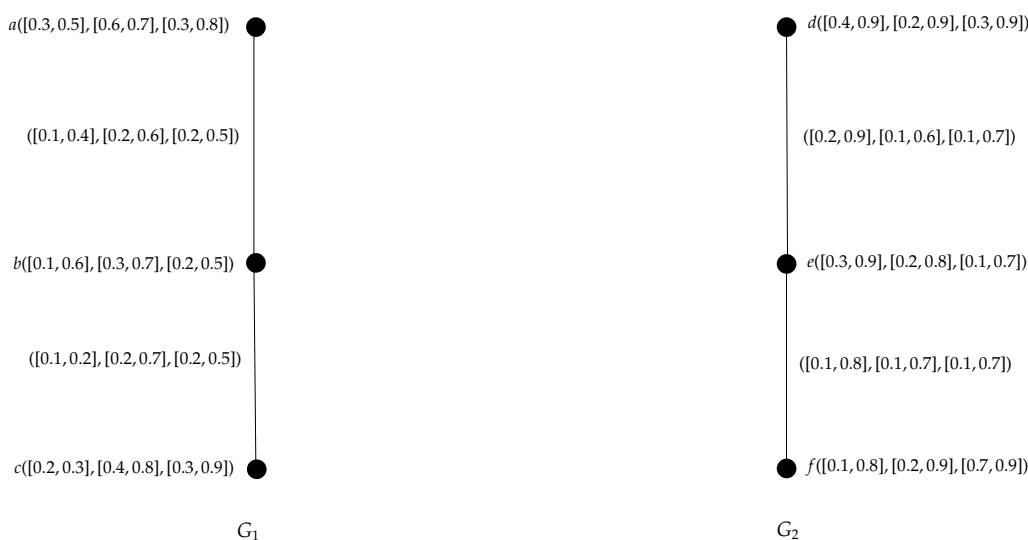


Figure 9: Interval-valued neutrosophic graphs G_1 and G_2

Then, their corresponding join G_1+G_2 is shown in Fig. 10.

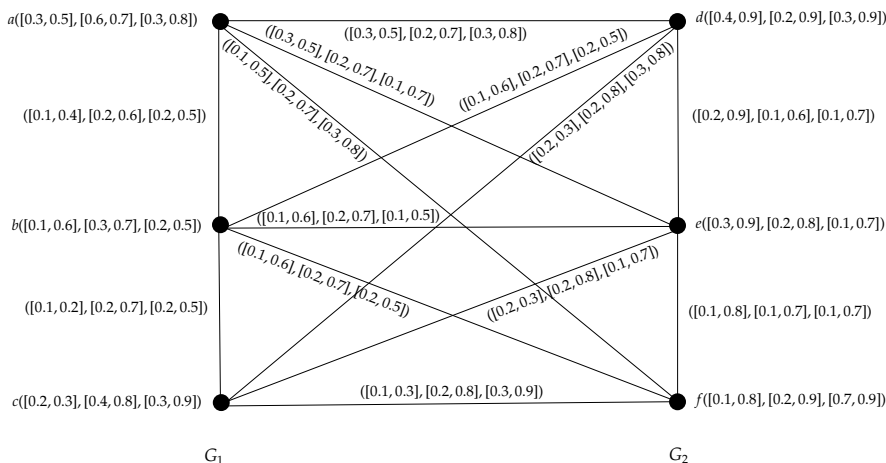


Figure 10: G_1+G_2

Proposition 2.18. The join $G_1+G_2 = (A_1+A_2, B_1+B_2)$ of two interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is an interval-valued neutrosophic graph.

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued neutrosophic graphs of the graphs $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. We prove that $G_1+G_2 = (A_1+A_2, B_1+B_2)$ is an interval-valued neutrosophic graph of the graph $G_1^*+G_2^*$. In view of Proposition 2.15 is sufficient to verify the case when $xy \in \dot{E}$. In this case we have

$$\begin{aligned} (t_{B_1}^- + t_{B_2}^-)(xy) &= \min(t_{A_1}^-(x), t_{A_2}^-(y)) \\ &\leq \min((t_{A_1}^- \cup t_{A_2}^-)(x), (t_{A_1}^- \cup t_{A_2}^-)(y)) \\ &= \min((t_{A_1}^- + t_{A_2}^-)(x), (t_{A_1}^- + t_{A_2}^-)(y)), \end{aligned}$$

$$\begin{aligned} (t_{B_1}^+ + t_{B_2}^+)(xy) &= \min(t_{A_1}^+(x), t_{A_2}^+(y)) \\ &\leq \min((t_{A_1}^+ \cup t_{A_2}^+)(x), (t_{A_1}^+ \cup t_{A_2}^+)(y)) \\ &= \min((t_{A_1}^+ + t_{A_2}^+)(x), (t_{A_1}^+ + t_{A_2}^+)(y)), \end{aligned}$$

$$\begin{aligned} (i_{B_1}^- + i_{B_2}^-)(xy) &= \min(i_{A_1}^-(x), i_{A_2}^-(y)) \\ &\leq \min((i_{A_1}^- \cup i_{A_2}^-)(x), (i_{A_1}^- \cup i_{A_2}^-)(y)) \\ &= \min((i_{A_1}^- + i_{A_2}^-)(x), (i_{A_1}^- + i_{A_2}^-)(y)), \end{aligned}$$

$$\begin{aligned} (i_{B_1}^+ + i_{B_2}^+)(xy) &= \min(i_{A_1}^+(x), i_{A_2}^+(y)) \\ &\leq \min((i_{A_1}^+ \cup i_{A_2}^+)(x), (i_{A_1}^+ \cup i_{A_2}^+)(y)) \\ &= \min((i_{A_1}^+ + i_{A_2}^+)(x), (i_{A_1}^+ + i_{A_2}^+)(y)), \end{aligned}$$

$$\begin{aligned} (f_{B_1}^- + f_{B_2}^-)(xy) &= \min(f_{A_1}^-(x), f_{A_2}^-(y)) \\ &\leq \min((f_{A_1}^- \cup f_{A_2}^-)(x), (f_{A_1}^- \cup f_{A_2}^-)(y)) \\ &= \min((f_{A_1}^- + f_{A_2}^-)(x), (f_{A_1}^- + f_{A_2}^-)(y)), \end{aligned}$$

$$\begin{aligned} (f_{B_1}^+ + f_{B_2}^+)(xy) &= \min(f_{A_1}^+(x), f_{A_2}^+(y)) \\ &\leq \min((f_{A_1}^+ \cup f_{A_2}^+)(x), (f_{A_1}^+ \cup f_{A_2}^+)(y)) \\ &= \min((f_{A_1}^+ + f_{A_2}^+)(x), (f_{A_1}^+ + f_{A_2}^+)(y)), \end{aligned}$$

This completes the proof. \square

Definition 2.19. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued neutrosophic graphs. A homomorphism $g : G_1 \rightarrow G_2$ is a mapping $g : X_1 \rightarrow X_2$ such that:

1. $t_{A_1}^-(x_1) \leq t_{A_2}^-(g(x_1)),$ $t_{A_1}^-(x_1) \leq t_{A_2}^-(g(x_1)),$
 $i_{A_1}^-(x_1) \leq i_{A_2}^-(g(x_1)),$ $i_{A_1}^-(x_1) \leq i_{A_2}^-(g(x_1)),$
 $f_{A_1}^-(x_1) \leq f_{A_2}^-(g(x_1)),$ $f_{A_1}^-(x_1) \leq f_{A_2}^-(g(x_1)),$ for all $x_1 \in X_1,$
2. $t_{B_1}^-(x_1y_1) \leq t_{B_2}^-(g(x_1)g(y_1)),$ $t_{B_1}^+(x_1y_1) \leq t_{B_2}^+(g(x_1)g(y_1)),$
 $i_{B_1}^-(x_1y_1) \leq i_{B_2}^-(g(x_1)g(y_1)),$ $i_{B_1}^+(x_1y_1) \leq i_{B_2}^+(g(x_1)g(y_1)),$
 $f_{B_1}^-(x_1y_1) \leq f_{B_2}^-(g(x_1)g(y_1)),$ $f_{B_1}^+(x_1y_1) \leq f_{B_2}^+(g(x_1)g(y_1)),$ for all $x_1y_1 \in E_1.$
 A bijective homomorphism with the property

3. $t_{A_1}^-(x_1) = t_{A_2}^-(g(x_1)),$ $t_{A_1}^-(x_1) = t_{A_2}^-(g(x_1)),$
 $i_{A_1}^-(x_1) = i_{A_2}^-(g(x_1)),$ $i_{A_1}^-(x_1) = i_{A_2}^-(g(x_1)),$
 $f_{A_1}^-(x_1) = f_{A_2}^-(g(x_1)),$ $f_{A_1}^-(x_1) = f_{A_2}^-(g(x_1)),$ for all $x_1 \in X_1,$
 is called a weak isomorphism. A weak isomorphism preserves the weights of the vertices but not necessarily the weights of the edges.
 A bijective homomorphism preserving the weights of the edges but not necessarily the weights of the vertices, i.e. a bijective homomorphism $g : G_1 \rightarrow G_2$ such that:

4. $t_{B_1}^-(x_1y_1) = t_{B_2}^-(g(x_1)g(y_1)),$ $t_{B_1}^+(x_1y_1) = t_{B_2}^+(g(x_1)g(y_1)),$
 $i_{B_1}^-(x_1y_1) = i_{B_2}^-(g(x_1)g(y_1)),$ $i_{B_1}^+(x_1y_1) = i_{B_2}^+(g(x_1)g(y_1)),$
 $f_{B_1}^-(x_1y_1) = f_{B_2}^-(g(x_1)g(y_1)),$ $f_{B_1}^+(x_1y_1) = f_{B_2}^+(g(x_1)g(y_1)),$ for all $x_1y_1 \in E_1,$ is called a weak co-isomorphism.
 A bijective mapping $g : G_1 \rightarrow G_2$ satisfying 3. and 4. is called an isomorphism.

Example 2.20. Consider interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ on $X_1 = \{a_1, b_1\}$ and $X_2 = \{a_2, b_2\}$, respectively, as shown in Fig. 11.

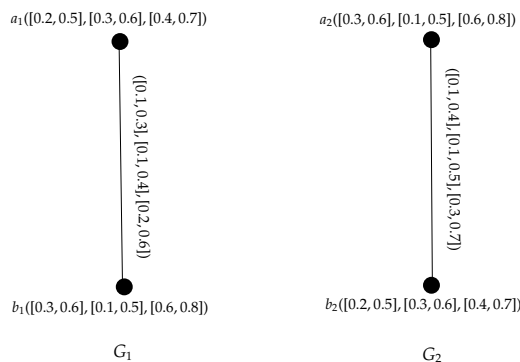


Figure 11: Interval-valued neutrosophic graphs G_1 and G_2

Then, it is easy to see that the mapping $g : X_1 \rightarrow X_2$ defined by $g(a_1) = b_2$ and $g(b_1) = a_2$ is a weak isomorphism but it is not an isomorphism.

Example 2.21. Consider interval-valued neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ on $X_1 = \{a_1, b_1\}$ and $X_2 = \{a_2, b_2\}$, respectively, as shown in Fig. 12.

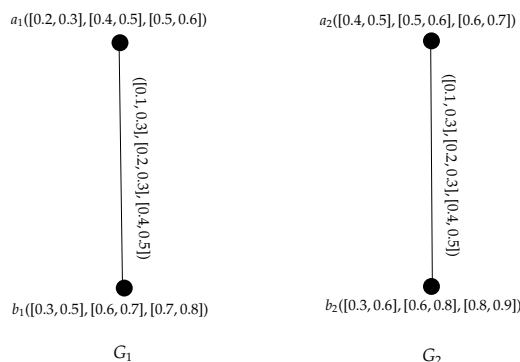


Figure 12: Interval-valued neutrosophic graphs G_1 and G_2

Then, it is easy to see that the mapping $g : X_1 \rightarrow X_2$ defined by $g(a_1b_1) = a_2b_2$ is a weak co-isomorphism but it is not an isomorphism.

Definition 2.22. An interval-valued neutrosophic graph is called complete if

1. $t_B^-(xy) = \min(t_A^-(x), t_A^-(y)),$ $t_B^+(xy) = \min(t_A^+(x), t_A^+(y)),$
2. $i_B^-(xy) = \min(i_A^-(x), i_A^-(y)),$ $i_B^+(xy) = \min(i_A^+(x), i_A^+(y)),$
3. $f_B^-(xy) = \min(f_A^-(x), f_A^-(y)),$ $f_B^+(xy) = \min(f_A^+(x), f_A^+(y)),$ for all $x, y \in X.$

Example 2.23. Consider an interval-valued neutrosophic complete graph $G = (A, B)$ on $X = \{a, b, c\}$ as shown in Fig. 13.

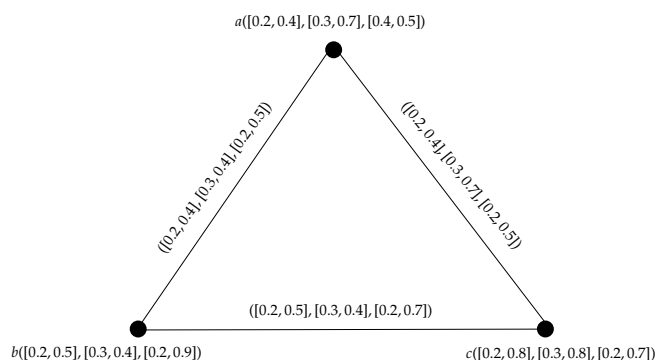


Figure 13: Interval-valued neutrosophic complete graph

Proposition 2.24. If $G = (A, B)$ is an interval-valued neutrosophic complete graph, then also $G \circ G$ is an interval-valued neutrosophic complete graph.

Definition 2.25. The complement of an interval-valued neutrosophic complete graph $G = (A, B)$ on $G^* = (X, E)$ is an interval-valued neutrosophic complete graph $\bar{G} = (\bar{A}, \bar{B})$ on $\bar{G}^* = (X, \bar{E})$, where

1. $\bar{X} = X,$
2. $\bar{t}_A^+(x_i) = t_A^+(x_i),$ $\bar{t}_A^-(x_i) = t_A^-(x_i),$ $\bar{i}_A^+(x_i) = i_A^+(x_i),$ $\bar{i}_A^-(x_i) = i_A^-(x_i),$ $\bar{f}_A^+(x_i) = f_A^+(x_i),$ $\bar{f}_A^-(x_i) = f_A^-(x_i),$ for all $x_i \in X,$

3.

$$\begin{aligned} \bar{t}_B^+(x_i, x_j) &= \begin{cases} \min[t_A^+(x_i), t_A^+(x_j)] & \text{if } t_B^+(x_i, x_j) = 0, \\ \min[t_A^+(x_i), t_A^+(x_j)] - t_B^+(x_i, x_j) & \text{if } t_B^+(x_i, x_j) > 0, \end{cases} \\ \bar{i}_B^+(x_i, x_j) &= \begin{cases} \min[i_A^+(x_i), i_A^+(x_j)] & \text{if } i_B^+(x_i, x_j) = 0, \\ \min[i_A^+(x_i), i_A^+(x_j)] - i_B^+(x_i, x_j) & \text{if } i_B^+(x_i, x_j) > 0, \end{cases} \\ \bar{f}_B^+(x_i, x_j) &= \begin{cases} \min[f_A^+(x_i), f_A^+(x_j)] & \text{if } f_B^+(x_i, x_j) = 0, \\ \min[f_A^+(x_i), f_A^+(x_j)] - f_B^+(x_i, x_j) & \text{if } f_B^+(x_i, x_j) > 0, \end{cases} \end{aligned}$$

4.

$$\begin{aligned} \bar{t}_B^-(x_i, x_j) &= \begin{cases} \min[t_A^-(x_i), t_A^-(x_j)] & \text{if } t_B^-(x_i, x_j) = 0, \\ \min[t_A^-(x_i), t_A^-(x_j)] - t_B^-(x_i, x_j) & \text{if } t_B^-(x_i, x_j) > 0, \end{cases} \\ \bar{i}_B^-(x_i, x_j) &= \begin{cases} \min[i_A^-(x_i), i_A^-(x_j)] & \text{if } i_B^-(x_i, x_j) = 0, \\ \min[i_A^-(x_i), i_A^-(x_j)] - i_B^-(x_i, x_j) & \text{if } i_B^-(x_i, x_j) > 0, \end{cases} \\ \bar{f}_B^-(x_i, x_j) &= \begin{cases} \min[f_A^-(x_i), f_A^-(x_j)] & \text{if } f_B^-(x_i, x_j) = 0, \\ \min[f_A^-(x_i), f_A^-(x_j)] - f_B^-(x_i, x_j) & \text{if } f_B^-(x_i, x_j) > 0, \end{cases} \end{aligned}$$

for all $x_i, x_j \in E$.

Definition 2.26. An interval-valued neutrosophic complete graph $G = (A, B)$ is called self-complementary if and only if $G \sim \bar{G}$.

Example 2.27. Consider a self-complementary interval-valued neutrosophic graph $G = (A, B)$ on $X = \{a, b, c\}$ as shown in Fig. 14.

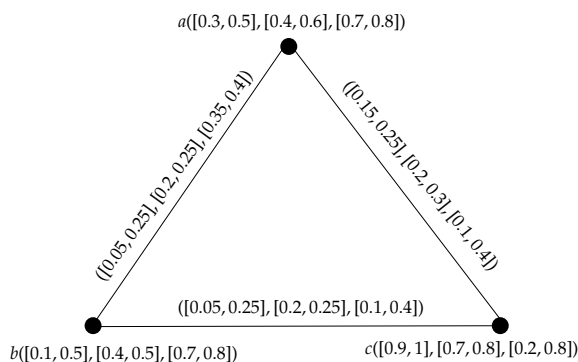


Figure 14: Self-complementary interval-valued neutrosophic graph

Proposition 2.28. In a self-complementary interval-valued neutrosophic complete graph $G = (A, B)$ we have

1. $\sum_{x \neq y} t_B^-(xy) = \frac{1}{2} \sum_{x \neq y} \min(t_A^-(x), t_A^-(y)), \quad \sum_{x \neq y} t_B^+(xy) = \frac{1}{2} \sum_{x \neq y} \min(t_A^+(x), t_A^+(y)),$
2. $\sum_{x \neq y} i_B^-(xy) = \frac{1}{2} \sum_{x \neq y} \min(i_A^-(x), i_A^-(y)), \quad \sum_{x \neq y} i_B^+(xy) = \frac{1}{2} \sum_{x \neq y} \min(i_A^+(x), i_A^+(y)),$
3. $\sum_{x \neq y} f_B^-(xy) = \frac{1}{2} \sum_{x \neq y} \min(f_A^-(x), f_A^-(y)), \quad \sum_{x \neq y} f_B^+(xy) = \frac{1}{2} \sum_{x \neq y} \min(f_A^+(x), f_A^+(y)).$

With the help of above Proposition 2.28, it can be observed that the graph shown in Fig. 14 is self-complementary.

Proposition 2.29. Let $G = (A, B)$ be an interval-valued neutrosophic complete graph such that

1. $t_B^-(xy) = \min(t_A^-(x), t_A^-(y)), \quad t_B^+(xy) = \min(t_A^+(x), t_A^+(y)),$
2. $i_B^-(xy) = \min(i_A^-(x), i_A^-(y)), \quad i_B^+(xy) = \min(i_A^+(x), i_A^+(y)),$

$$3. f_B^-(xy) = \min(f_A^-(x), f_A^-(y)), \quad f_B^+(xy) = \min(f_A^+(x), f_A^+(y)), \quad \text{for all } x, y \in X,$$

then G is self-complementary.

Proposition 2.30. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued neutrosophic complete graphs. Then $G_1 \cong G_2$ if and only if $\overline{G_1} \cong \overline{G_2}$.

Definition 2.31. An interval-valued neutrosophic graph is called strong if

1. $t_B^-(xy) = \min(t_A^-(x), t_A^-(y)), \quad t_B^+(xy) = \min(t_A^+(x), t_A^+(y)),$
2. $i_B^-(xy) = \min(i_A^-(x), i_A^-(y)), \quad i_B^+(xy) = \min(i_A^+(x), i_A^+(y)),$
3. $f_B^-(xy) = \min(f_A^-(x), f_A^-(y)), \quad f_B^+(xy) = \min(f_A^+(x), f_A^+(y)), \quad \text{for all } xy \in E.$

Example 2.32. Consider an interval-valued neutrosophic strong graph $G = (A, B)$ on $X = \{a, b, c, d\}$ as shown in Fig. 15.

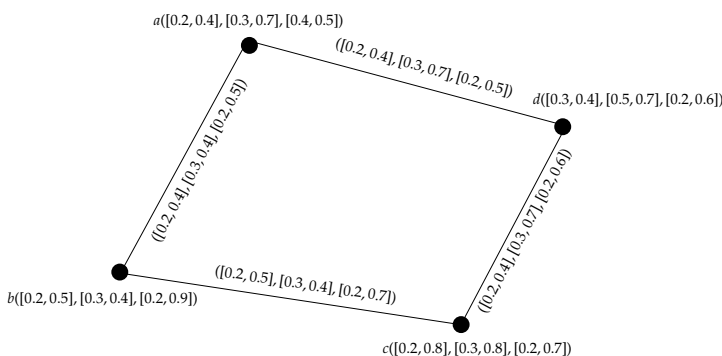


Figure 15: Interval-valued neutrosophic strong graph

Definition 2.33. Let $G = (A, B)$ be an interval-valued neutrosophic graph on G^* . If all the vertices have the same open-neighbourhood degree n , then G is called an n -regular interval-valued neutrosophic graph. The open neighbourhood degree of a vertex x in G is defined by $deg(x) = ([deg_t^-(x), deg_t^+(x)], [deg_i^-(x), deg_i^+(x)], [deg_f^-(x), deg_f^+(x)])$, where, $deg_t^-(x) = \sum_{y \in N(x)} t_A^-(y)$, $deg_t^+(x) = \sum_{y \in N(x)} t_A^+(y)$, $deg_i^-(x) = \sum_{y \in N(x)} i_A^-(y)$, $deg_i^+(x) = \sum_{y \in N(x)} i_A^+(y)$, $deg_f^-(x) = \sum_{y \in N(x)} f_A^-(y)$, and $deg_f^+(x) = \sum_{y \in N(x)} f_A^+(y)$.

Example 2.34. Consider a graph G^* such that $X = \{a, b, c\}$, $E = \{ab, bc, ac\}$. Let A be an interval-valued neutrosophic subset of X and let B be an interval-valued neutrosophic subset of $E \subseteq X \times X$, as shown in the following Tables.

A	a	b	c
t_A^-	0.2	0.2	0.2
t_A^+	0.5	0.5	0.5
i_A^-	0.3	0.3	0.3
i_A^+	0.4	0.4	0.4
f_A^-	0.2	0.2	0.2
f_A^+	0.9	0.9	0.9

B	ab	bc	ac
t_B^-	0.2	0.1	0.1
t_B^+	0.4	0.3	0.3
i_B^-	0.3	0.1	0.1
i_B^+	0.4	0.2	0.2
f_B^-	0.2	0.3	0.3
f_B^+	0.5	0.4	0.4

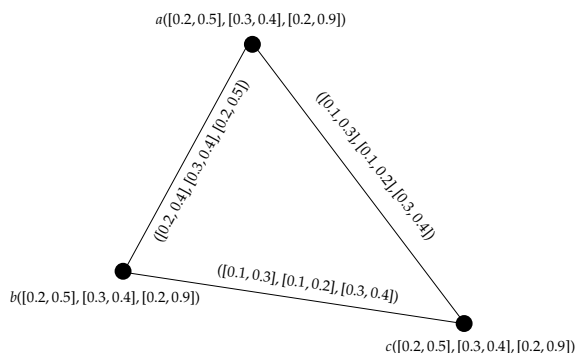


Figure 16: Regular interval-valued neutrosophic graph

By routine calculations it can be observed that the graph shown in Fig. 16 is regular interval-valued neutrosophic graph.

Following results can be proved using similar method as used in [8], hence we omit their proofs.

Theorem 2.35. Every complete interval-valued neutrosophic graph is totally regular.

Theorem 2.36. Let $G = (A, B)$ be an interval-valued neutrosophic graph of a graph G^* . Then A is a constant function if and only if the following statements are equivalent:

- (a) G is a regular interval-valued neutrosophic graph,
- (b) G is a totally regular interval-valued neutrosophic graph.

Proposition 2.37. If an interval-valued neutrosophic graph G is regular and totally regular, then A is a constant function.

Theorem 2.38. Let G be an interval-valued neutrosophic graph where a crisp graph G^* is an odd cycle. Then G is a regular interval-valued neutrosophic graph if and only if B is a constant function.

3. Conclusions

An interval-valued neutrosophic set is an extension of an interval-valued fuzzy set. The models which are based on interval-valued neutrosophic sets are more appropriate and well-suited as compare to traditional models. In this paper we have discussed some operations on interval-valued neutrosophic graphs. We are extending our research work to (1) Irregular Interval-valued neutrosophic graphs, (2) Interval-valued neutrosophic directed hypergraphs, (3) Rough neutrosophic graphs, (4) Interval-valued neutrosophic competition graphs, and (5) Application of Interval-valued neutrosophic graphs in decision support systems.

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