

Contemporary Concepts of Neutrosophic Fuzzy Soft *BCK*-submodules

R. S. Alghamdi¹ and Noura Omair Alshehri²

¹Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia; ralgamdi0068@stu.kau.edu.sa, rsaalghamdi@uj.edu.sa

²Department of Mathematics, Faculty of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia; noal-shehri@uj.edu.sa

Abstract

In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft *BCK*-submodules and discuss the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft *BCK*-submodules. Finally, we investigate some results.

Keywords: *BCK*-algebras, *BCK*-modules, soft sets, fuzzy soft sets, neutrosophic sets, neutrosophic soft sets, neutrosophic fuzzy soft *BCK*-submodules, neutrosophic fuzzy soft translations, neutrosophic fuzzy soft multiplications and neutrosophic fuzzy soft extensions.

1 Introduction

Fuzzy set theory which was developed by Zadeh [23] is an appropriate theory for dealing with vagueness. It is considered as the one of theories can be handled with uncertainties. Combining fuzzy set models with other mathematical models has attracted the attention of many researchers. Interval-valued fuzzy sets [24], hesitant fuzzy sets [21], intuitionistic fuzzy sets [3, 4], Intuitionistic Fuzzy *BCK*-submodules [5] and $(\epsilon, \epsilon \vee q)$ -fuzzy *BCK*-submodules [2] are some of the researches that have dealt this subject.

Neutrosophic algebraic structure is a very recent study. It was applied in many fields in order to solve problems related to uncertainties and indeterminacy where they happen to be one of the major factors in almost all real-world problems. Neutrosophic set is a generalization of the fuzzy set especially of intuitionistic fuzzy set. The intuitionistic fuzzy set has the degree of non-membership

as introduced by K. Atanassov [3]. Smarandache in 1998 [19] has introduced the degree of indeterminacy as an independent component and defined the neutrosophic set on three components: truth, indeterminacy and falsity.

The concept of *BCK*-algebra was first initiated by Imai and Iseki [8]. In 1994, the notion of *BCK*-modules was introduced by H. Abujable, M. Aslam and A. Thaheem as an action of *BCK*-algebras on abelian group [1]. *BCK*-modules theory then was developed by Z. perveen, M. Aslam and A. Thaheem [18]. Bakhshi [6] presented the concept of fuzzy *BCK*-submodules and investigated their properties. Recently, H. Bashir and Z. Zahid applied the theory of soft sets on *BCK*-modules in [12].

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft *BCK*-submodules and discuss the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft *BCK*-submodules. Finally, we investigate some results.

2 Preliminaries

In this section, some preliminaries from the soft set theory, neutrosophic soft sets, *BCK*-algebras and *BCK*-modules are induced.

Definition 2.1.[17] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a nonempty subset of E . A pair $F_A = (F, A)$ is called a soft set over U , where $A \subseteq E$ and $F : A \rightarrow P(U)$ is a set-valued mapping, called the approximate function of the soft set (F, A) . It is easy to represent a soft set (F, A) by a set of ordered pairs as follows:

$$(F, A) = \{(x, F(x)) : x \in A\}$$

Definition 2.2.[20] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in U\}$ where $T_A : X \rightarrow]^{-}0, 1^{+}[$ is a truth membership function, $I_A : U \rightarrow]^{-}0, 1^{+}[$ is an indeterminate membership function, and $F_A : X \rightarrow]^{-}0, 1^{+}[$ is a false membership function and $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^{-}0, 1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3.[13] Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $P(U)$ denotes the set of all neutrosophic sets of U . The collection (F, A) is termed to be the neutrosophic soft set (NSS) over U , where F is a mapping given by $F : A \rightarrow P(U)$.

Definition 2.4.[8, 9] An algebra $(X, *, 0)$ of type $(2, 0)$ is called *BCK*-algebra if it satisfying the following axioms:

$$(BCK-1) ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCK-2) (x * (x * y)) * y = 0,$$

$$(BCK-3) x * x = 0,$$

$$(BCK-4) 0 * x = 0,$$

$$(BCK-5) x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y, \text{ for all } x, y, z \in X.$$

A partial ordering “ \leq ” is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A *BCK*-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$ and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2.5.[1] Let X be a *BCK*-algebra. Then by a left X -module (abbreviated X -module), we mean an abelian group M with an operation $X \times M \rightarrow M$ with $(x, m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

$$(i) (x \wedge y)m = x(y m),$$

$$(ii) x(m + n) = xm + xn,$$

$$(iii) 0m = 0.$$

If X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

A mapping $\mu : X \rightarrow [0, 1]$ is called a fuzzy set in a *BCK*-algebra X . For any fuzzy set μ in X and any $t \in [0, 1]$, we define set $U(\mu; t) = \mu^t = \{x \in X | \mu(x) \geq t\}$, which is called upper t -level cut of μ .

Definition 2.6.[6] A fuzzy subset μ of M is said to be a fuzzy *BCK*-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold:

$$(FBCKM1) \mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\},$$

$$(FBCKM2) \mu(-m) = \mu(m),$$

$$(FBCKM3) \mu(xm) \geq \mu(m).$$

Definition 2.7.[6] Let M, N be modules over a *BCK*-algebra X . A mapping $f : M \rightarrow N$ is called *BCK*-module homomorphism if

$$(1) f(m_1 + m_2) = f(m_1) + f(m_2),$$

$$(2) f(xm) = xf(m) \text{ for all } m, m_1, m_2 \in M \text{ and } x \in X.$$

A *BCK*-module homomorphism is said to be monomorphism (epimorphism) if it is one to one (onto). If it is both one to one and onto, then we say that it is an isomorphism.

Definition 2.8.[12] Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \rightarrow N, g : A \rightarrow B$ be two functions. Then we say that (f, g) is a soft *BCK*-homomorphism if the following conditions are satisfied:

$$(1) f \text{ is a homomorphism from } M \text{ onto } N,$$

- (2) g is a mapping from A onto B , and
- (3) $f(F(x)) = G(g(x))$ for all $x \in A$.

3 Neutrosophic fuzzy soft BCK -submodules

Definition 3.1. A neutrosophic fuzzy soft set (F, A) over a BCK -module M is said to be a neutrosophic fuzzy soft BCK -submodule over M if for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$ the following axioms hold :

$$\begin{aligned}
 \text{(NFSS1)} \quad & T_{F(\varepsilon)}(m_1 + m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}, \\
 & I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}, \\
 & F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}, \\
 \text{(NFSS2)} \quad & T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}(m), \\
 & I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m), \\
 & F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m), \\
 \text{(NFSS3)} \quad & T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), \\
 & I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), \\
 & F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m).
 \end{aligned}$$

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a set along with a binary operation $*$ defined in Table 1, then $(X, *, 0)$ forms a commutative BCK -algebra which is not bounded (see [16]). Let $M = \{0, a, b, c\}$ be a subset of X along with an operation $+$ defined by Table 2. Then $(M, +)$ forms a commutative group. Table 3 explains the action of X on M under the operation $xm = x \wedge m$ for all $x \in X$ and $m \in M$. Consequently, M forms an X -module (see [11]).

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	d
d	d	d	d	d	0

Table 1

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 2

\wedge	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c
d	0	0	0	0

Table 3

Let $A = \{0, a\}$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 4

Consequently, a routine exercise of calculations show that (F, A) forms a neutrosophic fuzzy soft BCK -submodule over M .

(F, A)	0	a	b	c
$T_{F(0)}$	0.9	0.7	0.8	0.7
$I_{F(0)}$	0.8	0.5	0.6	0.5
$F_{F(0)}$	0.1	0.1	0.1	0.1
$T_{F(a)}$	0.5	0.2	0.3	0.2
$I_{F(a)}$	0.3	0.1	0.3	0.1
$F_{F(a)}$	0.1	0.5	0.4	0.5

Table 4

For the sake of simplicity, we shall use the symbols $NFS(M)$ and $NFSS(M)$ for the set of all neutrosophic fuzzy soft sets over M and the set of all neutrosophic fuzzy soft BCK -submodules over M , respectively.

Theorem 3.3. A neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if

(i) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), \quad F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m),$

(ii) $T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

for all $m, m_1, m_2 \in M, x \in X$ and $\varepsilon \in A$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK -submodule over M then by the definition(3.1) condition (i) is hold.

(ii) $T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 - m_2) = I_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 - m_2) = F_{F(\varepsilon)}(m_1 + (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\} = \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

Conversely suppose (F, A) satisfies the conditions (i),(ii). Then we have by (i)

$$T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}((-1)m) \geq T_{F(\varepsilon)}(m),$$

and

$$T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}((-1)(-1)m) \geq T_{F(\varepsilon)}(-m).$$

Thus, $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}(-m)$. Similarly for $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$.

$T_{F(\varepsilon)}(m_1 + m_2) = T_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\},$

$I_{F(\varepsilon)}(m_1 + m_2) = I_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$

$F_{F(\varepsilon)}(m_1 + m_2) = F_{F(\varepsilon)}(m_1 - (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\} = \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$

Hence (F, A) is a neutrosophic fuzzy soft BCK -submodule over M .

Theorem 3.4. A neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if for all $m, m_1, m_2 \in M$, $x, y \in X$ and $\varepsilon \in A$ the following statements hold:

- (i) $T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$,
- (ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
 $I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,
 $F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Proof. Let $(F, A) \in NFSS(M)$ then by theorem (3.3) and since $0m = 0$ for all $m \in M$, we have

- (i) $T_{F(\varepsilon)}(0) = T_{F(\varepsilon)}(0m) \geq T_{F(\varepsilon)}(m)$,
 $I_{F(\varepsilon)}(0) = I_{F(\varepsilon)}(0m) \geq I_{F(\varepsilon)}(m)$, and
 $F_{F(\varepsilon)}(0) = F_{F(\varepsilon)}(0m) \leq F_{F(\varepsilon)}(m)$.
- (ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(xm_1), T_{F(\varepsilon)}(ym_2)\}$
 $\geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$.

Similarly for

$$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$$

and

$$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies (i),(ii), then we have

$$T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m) \text{ and } F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m).$$

Then

$$T_{F(\varepsilon)}(xm) = T_{F(\varepsilon)}(x(m - 0)) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(0)\} = T_{F(\varepsilon)}(m).$$

Similarly for

$$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) \text{ and } F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m).$$

Also,

$$T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(1m_1 - 1m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\} \text{ and } F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK -submodule over M .

Definition 3.5. Let (F, A) be a neutrosophic fuzzy soft set over a BCK -module M and $\alpha \in [0, \perp]$ such that $\perp = 1 - \sup \{F_{F(\varepsilon)}(m) : m \in M, \varepsilon \in A\}$. Then $\tilde{T}_\alpha [(F, A)] = (G, A_\alpha^T)$ is called a neutrosophic fuzzy soft α -translation of (F, A) if it satisfies:

$$G(\varepsilon) = \left((T_{F(\varepsilon)})_\alpha^T(m), (I_{F(\varepsilon)})_\alpha^T(m), (F_{F(\varepsilon)})_\alpha^T(m) \right),$$

for all $\varepsilon \in A, m \in M$ where:

$$\begin{aligned} (T_{F(\varepsilon)})_\alpha^T(m) &= T_{F(\varepsilon)}(m) + \alpha, \\ (I_{F(\varepsilon)})_\alpha^T(m) &= I_{F(\varepsilon)}(m), \\ (F_{F(\varepsilon)})_\alpha^T(m) &= F_{F(\varepsilon)}(m) - \alpha. \end{aligned}$$

Theorem 3.6. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK -submodule over M if and only if the α -translation neutrosophic fuzzy soft set $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M for all $\alpha \in [0, \perp]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK -submodule over M and $\alpha \in [0, \perp]$, then by Theorem (3.3)

$$\begin{aligned} (T_{F(\varepsilon)})_\alpha^T(xm) &= T_{F(\varepsilon)}(xm) + \alpha \geq T_{F(\varepsilon)}(m) + \alpha = (T_{F(\varepsilon)})_\alpha^T(m), \\ (F_{F(\varepsilon)})_\alpha^T(xm) &= F_{F(\varepsilon)}(xm) - \alpha \leq F_{F(\varepsilon)}(m) - \alpha = (F_{F(\varepsilon)})_\alpha^T(m), \end{aligned}$$

for all $m \in M, x \in X$. Also, for all $m_1, m_2 \in M$ we have

$$\begin{aligned} (T_{F(\varepsilon)})_\alpha^T(m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2) + \alpha \\ &\geq \min \{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\} + \alpha \\ &= \min \{T_{F(\varepsilon)}(m_1) + \alpha, T_{F(\varepsilon)}(m_2) + \alpha\} \\ &= \min \left\{ (T_{F(\varepsilon)})_\alpha^T(m_1), (T_{F(\varepsilon)})_\alpha^T(m_2) \right\}, \end{aligned}$$

and

$$\begin{aligned} (F_{F(\varepsilon)})_\alpha^T(m_1 - m_2) &= F_{F(\varepsilon)}(m_1 - m_2) - \alpha \\ &\leq \max \{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\} - \alpha \\ &= \max \{F_{F(\varepsilon)}(m_1) - \alpha, F_{F(\varepsilon)}(m_2) - \alpha\} \\ &= \max \left\{ (F_{F(\varepsilon)})_\alpha^T(m_1), (F_{F(\varepsilon)})_\alpha^T(m_2) \right\}. \end{aligned}$$

Hence $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M .

Conversely, assume that $\tilde{T}_\alpha [(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M for some $\alpha \in [0, \perp]$. Then for all $m \in M, x \in X$

$$\begin{aligned} T_{F(\varepsilon)}(xm) + \alpha &= (T_{F(\varepsilon)})_\alpha^T(xm) \geq (T_{F(\varepsilon)})_\alpha^T(m) = T_{F(\varepsilon)}(m) + \alpha \\ \implies T_{F(\varepsilon)}(xm) &\geq T_{F(\varepsilon)}(m). \end{aligned}$$

Also,

$$\begin{aligned} F_{F(\varepsilon)}(xm) - \alpha &= (F_{F(\varepsilon)})_{\alpha}^T(xm) \leq (F_{F(\varepsilon)})_{\alpha}^T(m) = F_{F(\varepsilon)}(m) - \alpha \\ &\implies F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m). \end{aligned}$$

Now let $m_1, m_2 \in M$, then

$$\begin{aligned} T_{F(\varepsilon)}(m_1 - m_2) + \alpha &= (T_{F(\varepsilon)})_{\alpha}^T(m_1 - m_2) \\ &\geq \min \left\{ (T_{F(\varepsilon)})_{\alpha}^T(m_1), (T_{F(\varepsilon)})_{\alpha}^T(m_2) \right\} \\ &= \min \left\{ T_{F(\varepsilon)}(m_1) + \alpha, T_{F(\varepsilon)}(m_2) + \alpha \right\} \\ &= \min \left\{ T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2) \right\} + \alpha \\ &\implies T_{F(\varepsilon)}(m_1 - m_2) \geq \min \left\{ T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2) \right\}, \end{aligned}$$

and

$$\begin{aligned} F_{F(\varepsilon)}(m_1 - m_2) - \alpha &= (F_{F(\varepsilon)})_{\alpha}^T(m_1 - m_2) \\ &\leq \max \left\{ (F_{F(\varepsilon)})_{\alpha}^T(m_1), (F_{F(\varepsilon)})_{\alpha}^T(m_2) \right\} \\ &= \max \left\{ F_{F(\varepsilon)}(m_1) - \alpha, F_{F(\varepsilon)}(m_2) - \alpha \right\} \\ &= \max \left\{ F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2) \right\} - \alpha \\ &\implies F_{F(\varepsilon)}(m_1 - m_2) \leq \max \left\{ F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2) \right\}. \end{aligned}$$

Hence by Theorem (3.3), (F, A) is a neutrosophic fuzzy soft BCK -submodule over M .

Definition 3.7. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK -module M . If $A \subset B$ and $T_{F(\varepsilon)}(m) \leq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \leq I_{G(\varepsilon)}(m)$, $F_{F(\varepsilon)}(m) \geq F_{G(\varepsilon)}(m)$, $\forall \varepsilon \in A$ and $m \in M$. Then we say that (G, B) is a neutrosophic fuzzy soft extension of (F, A) .

Definition 3.8. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK -module M . Then (G, B) is a neutrosophic fuzzy soft s -extension of (F, A) if the following assertions hold:

- (i) (G, B) is a neutrosophic fuzzy soft extension of (F, A) .
- (ii) If (F, A) is a neutrosophic fuzzy soft BCK -submodule over M , then so (G, B) .

Theorem 3.9. Let (F, A) be a neutrosophic fuzzy soft BCK -submodule over M and $\alpha \in [0, \perp]$. Then the neutrosophic fuzzy soft α -translation $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft s -extension of (F, A) .

Proof. Since $\tilde{T}_{\alpha}[(F, A)]$ is an α -translation, we know that $(T_{F(\varepsilon)})_{\alpha}^T(m) \geq T_{F(\varepsilon)}(m)$, $(I_{F(\varepsilon)})_{\alpha}^T(m) = I_{F(\varepsilon)}(m)$ and $(F_{F(\varepsilon)})_{\alpha}^T(m) \leq F_{F(\varepsilon)}(m)$ for all $m \in M, \varepsilon \in A$. Hence $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft extension of (F, A) . According to Theorem (3.6), $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft s -extension of (F, A) .

The converse of Theorem (3.9) is not true in general as seen in the following example:

Example 3.10. Let $X = \{0, a, b, c\}$ along with a binary operation $*$ defined in Table 5, then $(X, *, 0)$ forms a bounded implicative *BCK*-algebra (see [16]). Let $M = \{0, a\}$ be a subset of X with a binary operation $+$ defined by $x + y = (x * y) \vee (y * x)$. Then M is a commutative group as shown in table 6. Define scalar multiplication $(X, M) \rightarrow M$ by $xm = x \wedge m$ for all $x \in X$ and $m \in M$ that is given in Table 7. Consequently, M forms an X -module (see [11]).

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 5

+	0	a
0	0	a
a	a	0

Table 6

\wedge	0	a
0	0	0
a	0	a
b	0	0
c	0	a

Table 7

Let $A = M$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 8.

(F, A)	0	a
$T_{F(0)}$	0.9	0.5
$I_{F(0)}$	0.8	0.6
$F_{F(0)}$	0.1	0.3
$T_{F(a)}$	0.3	0.3
$I_{F(a)}$	0.2	0.2
$F_{F(a)}$	0.3	0.5

Table 8

Then (F, A) is a neutrosophic fuzzy soft *BCK*-submodule over M . Let (G, B) be a neutrosophic fuzzy soft set over M given by Table 9.

Then (G, B) is also a neutrosophic fuzzy soft *BCK*-submodule over M . Since $T_{F(\varepsilon)}(m) \geq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \geq I_{G(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(m) \leq F_{G(\varepsilon)}(m)$ for all $m \in M$ and $\varepsilon \in A \subset B$, hence (F, A) is a neutrosophic fuzzy soft s -extension of (G, B) , but since $I_{F(0)}(0) = 0.8 \neq I_{G(0)}(0) = 0.7$ then (F, A) is not a neutrosophic fuzzy soft α -translation of (G, B) for all $\alpha \in [0, \perp]$.

(G, B)	0	a
$T_{G(0)}$	0.5	0.3
$I_{G(0)}$	0.7	0.6
$F_{G(0)}$	0.1	0.4
$T_{G(a)}$	0.2	0.2
$I_{G(a)}$	0.1	0.1
$F_{G(a)}$	0.4	0.5

Table 9

Definition 3.11. Let (F, A) be a neutrosophic fuzzy soft set over a BCK -module M and $v \in [0, 1]$. A neutrosophic fuzzy soft v -multiplication of (F, A) denoted by $\tilde{M}_v [(F, A)] = (G, m_v(A))$ is defined as:

$$G(\varepsilon) = (m_v(T_{F(\varepsilon)})(m), m_v(I_{F(\varepsilon)})(m), m_v(F_{F(\varepsilon)})(m)),$$

where

$$m_v(T_{F(\varepsilon)})(m) = T_{F(\varepsilon)}(m) \cdot v,$$

$$m_v(I_{F(\varepsilon)})(m) = I_{F(\varepsilon)}(m),$$

$$m_v(F_{F(\varepsilon)})(m) = F_{F(\varepsilon)}(m) \cdot v,$$

for all $\varepsilon \in A$ and $m \in M$.

Theorem.3.12. If $(F, A) \in NFSS(M)$, then the neutrosophic fuzzy soft v -multiplication $\tilde{M}_v [(F, A)] \in NFSS(M)$ for all $v \in [0, 1]$.

Proof. Assume that (F, A) is a neutrosophic fuzzy soft BCK -submodule over M and let $m, m_1, m_2 \in M, x \in X$ and $\varepsilon \in A$. Then

$$m_v(T_{F(\varepsilon)})(xm) = T_{F(\varepsilon)}(xm) \cdot v \geq T_{F(\varepsilon)}(m) \cdot v = m_v(T_{F(\varepsilon)})(m),$$

$$m_v(I_{F(\varepsilon)})(xm) = I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) = m_v(I_{F(\varepsilon)})(m),$$

$$m_v(F_{F(\varepsilon)})(xm) = F_{F(\varepsilon)}(xm) \cdot v \leq F_{F(\varepsilon)}(m) \cdot v = m_v(F_{F(\varepsilon)})(m).$$

Moreover,

$$\begin{aligned} m_v(T_{F(\varepsilon)})(m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2) \cdot v \\ &\geq \min \{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\} \cdot v \\ &= \min \{T_{F(\varepsilon)}(m_1) \cdot v, T_{F(\varepsilon)}(m_2) \cdot v\} \\ &= \min \{m_v(T_{F(\varepsilon)})(m_1), m_v(T_{F(\varepsilon)})(m_2)\}, \end{aligned}$$

$$\begin{aligned} m_v (I_{F(\varepsilon)}) (m_1 - m_2) &= I_{F(\varepsilon)} (m_1 - m_2) \\ &\geq \min \{ I_{F(\varepsilon)} (m_1), I_{F(\varepsilon)} (m_2) \} \\ &= \min \{ m_v (I_{F(\varepsilon)}) (m_1), m_v (I_{F(\varepsilon)}) (m_2) \}, \end{aligned}$$

$$\begin{aligned} m_v (F_{F(\varepsilon)}) (m_1 - m_2) &= F_{F(\varepsilon)} (m_1 - m_2) \cdot v \\ &\leq \max \{ F_{F(\varepsilon)} (m_1), F_{F(\varepsilon)} (m_2) \} \cdot v \\ &= \max \{ F_{F(\varepsilon)} (m_1) \cdot v, F_{F(\varepsilon)} (m_2) \cdot v \} \\ &= \max \{ m_v (F_{F(\varepsilon)}) (m_1), m_v (F_{F(\varepsilon)}) (m_2) \}. \end{aligned}$$

Therefore by Theorem (3.3), $\tilde{M}_v [(F, A)]$ is a neutrosophic fuzzy soft *BCK*-submodule over M .

The converse of Theorem (3.12) is not true in general as seen in the following example:

Example 3.13. Consider a *BCK*-algebra $X = \{0, a, b, c\}$ and X -module $M = \{0, a\}$ that are defined in Example 3.10. Table 10 defines a neutrosophic fuzzy soft set (F, A) over M

(F, A)	0	a
$T_{F(0)}$	0.3	0.4
$I_{F(0)}$	0.7	0.5
$F_{F(0)}$	0.1	0.5
$T_{F(a)}$	0.1	0.1
$I_{F(a)}$	0.1	0.1
$F_{F(a)}$	0.5	0.6

Table 10

If we take $v = 0$, then the v -multiplication is a neutrosophic fuzzy soft *BCK*-submodule over M since

$$\begin{aligned} m_0 (T_{F(\varepsilon)}) (xm) &= 0 = m_0 (T_{F(\varepsilon)}) (m), \\ m_0 (I_{F(\varepsilon)}) (xm) &\geq m_0 (I_{F(\varepsilon)}) (m), \\ m_0 (F_{F(\varepsilon)}) (xm) &= 0 = m_0 (F_{F(\varepsilon)}) (m), \end{aligned}$$

and

$$\begin{aligned} m_0 (T_{F(\varepsilon)}) (m_1 - m_2) &= 0 = \min \{ m_0 (T_{F(\varepsilon)}) (m_1), m_0 (T_{F(\varepsilon)}) (m_2) \}, \\ m_0 (I_{F(\varepsilon)}) (m_1 - m_2) &\geq \min \{ m_0 (I_{F(\varepsilon)}) (m_1), m_0 (I_{F(\varepsilon)}) (m_2) \}, \\ m_0 (F_{F(\varepsilon)}) (m_1 - m_2) &= 0 = \min \{ m_0 (F_{F(\varepsilon)}) (m_1), m_0 (F_{F(\varepsilon)}) (m_2) \}, \end{aligned}$$

for all $m, m_1, m_2 \in M$ and $x \in X$. But if we take $m_1 = 0, m_2 = a$ and $\varepsilon = 0$ then

$$T_{F(0)}(0 + a) = T_{F(0)}(a) = 0.4 \not\geq \min \{T_{F(0)}(0), T_{F(0)}(a)\} = 0.3.$$

Hence (F, A) is not a neutrosophic fuzzy soft BCK -submodule over M .

Theorem.3.14. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK -submodule over M if and only if the v -multiplication neutrosophic fuzzy set $\tilde{M}_v[(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M for all $v \in (0, 1]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK -submodule over M then by Theorem (3.12) $\tilde{M}_v[(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M for all $v \in (0, 1]$.

Now let $v \in (0, 1]$ be such that $\tilde{M}_v[(F, A)]$ is a neutrosophic fuzzy soft BCK -submodule over M and let $m, m_1, m_2 \in M, x \in X$ and $\varepsilon \in A$. Then

$$\begin{aligned} T_{F(\varepsilon)}(xm) \cdot v &= m_v(T_{F(\varepsilon)}(xm)) \geq m_v(T_{F(\varepsilon)}(m)) = T_{F(\varepsilon)}(m) \cdot v, \\ I_{F(\varepsilon)}(xm) &= m_v(I_{F(\varepsilon)}(xm)) \geq m_v(I_{F(\varepsilon)}(m)) = I_{F(\varepsilon)}(m), \\ F_{F(\varepsilon)}(xm) \cdot v &= m_v(F_{F(\varepsilon)}(xm)) \leq m_v(F_{F(\varepsilon)}(m)) = F_{F(\varepsilon)}(m) \cdot v, \end{aligned}$$

and since $v \neq 0$, then $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$. Now

$$\begin{aligned} T_{F(\varepsilon)}(m_1 - m_2) \cdot v &= m_v(T_{F(\varepsilon)}(m_1 - m_2)) \\ &\geq \min \{m_v(T_{F(\varepsilon)}(m_1)), m_v(T_{F(\varepsilon)}(m_2))\} \\ &= \min \{T_{F(\varepsilon)}(m_1) \cdot v, T_{F(\varepsilon)}(m_2) \cdot v\} \\ &= \min \{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\} \cdot v, \end{aligned}$$

which means that

$$T_{F(\varepsilon)}(m_1 - m_2) \geq \min \{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly,

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max \{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK -submodule over M .

4 Isomorphism Theorem Of Neutrosophic Fuzzy Soft BCK -submodules

Definition 4.1. Let M and N be two BCK -modules over a BCK -algebra X . Let $f : M \rightarrow N$ be a BCK -submodule homomorphism and let $(F, A), (G, B)$ be two neutrosophic fuzzy soft BCK -submodule over M and N respectively. Then the image of (F, A) is a neutrosophic fuzzy soft set over N defined as follows for all $x \in M, y \in N$ and $\varepsilon \in A$.

$$f(F(\varepsilon))(x) = (T_{f(F)}(y), I_{f(F)}(y), F_{f(F)}(y)) = (f(T_F)(y), f(I_F)(y), f(F_F)(y)),$$

where

$$\begin{aligned}
 f(T_F)(y) &= \begin{cases} \sup T_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}, \\
 f(I_F)(y) &= \begin{cases} \sup I_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}, \\
 f(F_F)(y) &= \begin{cases} \inf F_F(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases},
 \end{aligned}$$

and the preimage of (G, B) is a neutrosophic fuzzy soft set over M defined as

$$f^{-1}(G(\delta))(y) = (T_{f^{-1}(G)}(x), I_{f^{-1}(G)}(x), F_{f^{-1}(G)}(x)) = (T_G(f(x)), I_G(f(x)), F_G(f(x))),$$

where $\delta \in B$.

Theorem 4.2. Let $(X, *, 0)$ be a *BCK*-algebra, M and N are modules of X . A mapping $f : M \rightarrow N$ is a *BCK*-submodule homomorphism and $(F, A) \in NFSS(N)$, then the inverse image $(f^{-1}(F), A) \in NFSS(M)$.

Proof. Since (F, A) is a neutrosophic fuzzy soft *BCK*-submodule over N . Let $m \in M, \varepsilon \in A$ then by Theorem (3.4)

$$\begin{aligned}
 T_{f^{-1}(F)}(0) &= T_{F(\varepsilon)}(f(0)) = T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(f(m)) = T_{f^{-1}(F)}(m), \\
 I_{f^{-1}(F)}(0) &= I_{F(\varepsilon)}(f(0)) = I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(f(m)) = I_{f^{-1}(F)}(m), \\
 F_{f^{-1}(F)}(0) &= F_{F(\varepsilon)}(f(0)) = F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(f(m)) = F_{f^{-1}(F)}(m).
 \end{aligned}$$

Now let $m_1, m_2 \in M, x, y \in X$, and $\varepsilon \in A$, then

$$\begin{aligned}
 T_{f^{-1}(F)}(xm_1 - ym_2) &= T_{F(\varepsilon)}(f(xm - ym_2)) \\
 &= T_{F(\varepsilon)}(xf(m_1) - yf(m_2)) \\
 &\geq \min \{T_{F(\varepsilon)}(f(m_1)), T_{F(\varepsilon)}(f(m_2))\} \\
 &= \min \{T_{f^{-1}(F)}(m_1), T_{f^{-1}(F)}(m_2)\}.
 \end{aligned}$$

Similarly for

$$I_{f^{-1}(F)}(xm_1 - ym_2) \geq \min \{I_{f^{-1}(F)}(m_1), I_{f^{-1}(F)}(m_2)\},$$

and

$$F_{f^{-1}(F)}(xm_1 - ym_2) \leq \max \{F_{f^{-1}(F)}(m_1), F_{f^{-1}(F)}(m_2)\}.$$

Hence $(f^{-1}(F), A)$ is a neutrosophic fuzzy soft *BCK*-submodule over M .

Theorem.4.3. Let $(X, *, 0)$ be a *BCK*-algebra, M and N are modules of X . A mapping $f : M \rightarrow N$ is a *BCK*-submodule epimorphism. If (F, A) is a neutrosophic fuzzy soft set over N such that $(f^{-1}(F), A) \in NFSS(M)$, then $(F, A) \in NFSS(N)$.

Proof. Assume that $(f^{-1}(F), A)$ is a neutrosophic fuzzy soft BCK -submodule over M . Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then for all $\varepsilon \in A$

$$\begin{aligned} T_{F(\varepsilon)}(n) &= T_{F(\varepsilon)}(f(m)) = T_{f^{-1}(F)}(m) \leq T_{f^{-1}(F)}(0) = T_{F(\varepsilon)}(f(0)) = T_{F(\varepsilon)}(0), \\ I_{F(\varepsilon)}(n) &= I_{F(\varepsilon)}(f(m)) = I_{f^{-1}(F)}(m) \leq I_{f^{-1}(F)}(0) = I_{F(\varepsilon)}(f(0)) = I_{F(\varepsilon)}(0), \\ F_{F(\varepsilon)}(n) &= F_{F(\varepsilon)}(f(m)) = F_{f^{-1}(F)}(m) \geq F_{f^{-1}(F)}(0) = F_{F(\varepsilon)}(f(0)) = F_{F(\varepsilon)}(0). \end{aligned}$$

Let $m, \hat{m} \in M, n, \hat{n} \in N$ such that $f(m) = n$ and $f(\hat{m}) = \hat{n}$ and $x, y \in X$ then

$$\begin{aligned} T_{F(\varepsilon)}(xn - y\hat{n}) &= T_{F(\varepsilon)}(xf(m) - yf(\hat{m})) \\ &= T_{F(\varepsilon)}(f(xm - y\hat{m})) \\ &= T_{f^{-1}(F)}(xm - y\hat{m}) \\ &\geq \min \{T_{f^{-1}(F)}(m), T_{f^{-1}(F)}(\hat{m})\} \\ &= \min \{T_{F(\varepsilon)}(f(m)), T_{F(\varepsilon)}(f(\hat{m}))\} \\ &= \min \{T_{F(\varepsilon)}(n), T_{F(\varepsilon)}(\hat{n})\}. \end{aligned}$$

Similarly for

$$I_{F(\varepsilon)}(xn - y\hat{n}) \geq \min \{I_{F(\varepsilon)}(n), I_{F(\varepsilon)}(\hat{n})\},$$

and

$$F_{F(\varepsilon)}(xn - y\hat{n}) \leq \max \{F_{F(\varepsilon)}(n), F_{F(\varepsilon)}(\hat{n})\}.$$

Hence according to Theorem (3.4), (F, A) is a neutrosophic fuzzy soft BCK -submodule over N .

Theorem.4.4. Let $(X, *, 0)$ be a BCK -algebra, M and N are modules of X . A mapping $f : M \rightarrow N$ is a BCK -submodule epimorphism and let (F, A) be a neutrosophic fuzzy soft BCK -submodule over M . Then the homomorphic image $(f(F), A)$ is a neutrosophic fuzzy soft BCK -submodule over N .

Proof. Assume that (F, A) is a neutrosophic fuzzy soft BCK -submodule over M . Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then

$$\begin{aligned} T_{f(F)}(n) &= f(T_F)(n) = \sup T_F(m) \leq \sup T(0) = f(T_F)(0) = T_{f(F)}(0), \\ I_{f(F)}(n) &= f(I_F)(n) = \sup I_F(m) \leq \sup I(0) = f(I_F)(0) = I_{f(F)}(0), \\ F_{f(F)}(n) &= f(F_F)(n) = \inf F_F(m) \geq \inf F(0) = f(F_F)(0) = F_{f(F)}(0). \end{aligned}$$

Let $m_1, m_2 \in M, n_1, n_2 \in N$ such that $f(m_1) = n_1$ and $f(m_2) = n_2$ and $x, y \in X$ then

$$\begin{aligned} T_{f(F)}(xn_1 - yn_2) &= f(T_F)(xn_1 - yn_2) \\ &= \sup T_F(xm_1 - ym_2) \\ &\geq \sup\{\min\{T_F(m_1), T_F(m_2)\}\} \\ &= \min\{\sup T_F(m_1), \sup T_F(m_2)\} \\ &= \min\{f(T_F)(n_1), f(T_F)(n_2)\} \\ &= \min\{T_{f(F)}(n_1), T_{f(F)}(n_2)\}. \end{aligned}$$

Similarly for

$$I_{f(F)}(xn_1 - yn_2) \geq \min\{I_{f(F)}(n_1), I_{f(F)}(n_2)\},$$

and

$$F_{f(F)}(xn_1 - yn_2) \leq \max\{F_{f(F)}(n_1), F_{f(F)}(n_2)\}.$$

Hence by Theorem (3.4), $(f(F), A)$ is a neutrosophic fuzzy soft BCK -submodule over N .

Corollary 4.5. Let $f : M \rightarrow N$ be a homomorphism of BCK -submodules and (F, A) is a neutrosophic fuzzy soft set over N . If (F, A) is a neutrosophic fuzzy soft BCK -submodule, then so is $(f^{-1}(F), A_\alpha^T)$ for any α -translation $\tilde{T}_\alpha[(F, A)]$ of (F, A) with $\alpha \in [0, \perp]$.

Proof. Directly by Theorem(3.6) and Theorem(4.2).

Joining Theorems (3.6), (4.3) and (4.4) we have the following corollaries:

Corollary 4.6. Let $f : M \rightarrow N$ be an epimorphism of BCK -submodules and (F, A) is a neutrosophic fuzzy soft set over N . If the inverse image of a neutrosophic fuzzy soft α -translation of (F, A) is a neutrosophic fuzzy soft BCK -submodule for some $\alpha \in [0, \perp]$, then so is (F, A) .

Corollary 4.7. Let $f : M \rightarrow N$ be an epimorphism of BCK -submodules and (F, A) is a neutrosophic fuzzy soft BCK -submodule over M , then the homomorphic image of a neutrosophic fuzzy soft α -translation of (F, A) is a neutrosophic fuzzy soft BCK -submodule over N for any $\alpha \in [0, \perp]$.

Using Theorems (3,14), (4.2), (4.3) and (4.4), we deduce the following results:

Corollary 4.8. Let $f : M \rightarrow N$ be a homomorphism of BCK -submodules and (F, A) is a neutrosophic fuzzy soft BCK -submodule over N , then the inverse image of a neutrosophic fuzzy soft v -multiplication of (F, A) is a neutrosophic fuzzy soft BCK -submodule over M for any v -multiplication of (F, A) with $v \in [0, 1]$.

Corollary 4.9. Let $f : M \rightarrow N$ be an epimorphism of BCK -submodules. If the inverse image of a neutrosophic fuzzy soft v -multiplication of (F, A) is a neutrosophic fuzzy soft BCK -submodule over M for some $v \in (0, 1]$, then (F, A) is a neutrosophic fuzzy soft BCK -submodule over N .

Corollary 4.10. Let $f : M \rightarrow N$ be an epimorphism of BCK -submodules and (F, A) is a neutrosophic fuzzy soft BCK -submodule over M , then the homomorphic image of a neutrosophic

fuzzy soft v -multiplication of (F, A) is a neutrosophic fuzzy soft BCK -submodule over N for any $v \in (0, 1]$.

5 Conclusion

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK -submodules were introduced and the relation between them were discussed. Also, the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK -submodules was defined. Finally, some results were investigated.

6 Compliance with Ethical Standards

Conflict of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical Approval: This artical does not contain any studies with human participants or animals performed by any of the authors.

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