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Cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data

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Abstract
In the present study, we discuss the concept of internal cubic bipolar fuzzy (ICBF) sets and external cubic bipolar fuzzy (ECBF) sets. We also discuss some properties of ICBF-sets and ECBF-sets under both $\mathcal{P}$-Order and $\mathcal{R}$-Order. We present examples and counterexamples to support our concepts. Furthermore, we see the importance of ICBF-sets and EBCF-sets in multiple attribute decision making. We proposed two cubic bipolar fuzzy ordered weighted geometric aggregation operators, including, $\mathcal{P}$-CBFOWG operator and $\mathcal{R}$-CBFOWG operator to aggregate cubic bipolar fuzzy information with both perspectives, i.e., ICBF data and ECBF data. Finally, we present a multiple attribute decision making problem to examine the useability and capability of these operators and a comparison between ICBF information and ECBF information.

Keywords Internal cubic bipolar fuzzy sets · External cubic bipolar fuzzy sets · Properties of ICBF-sets and ECBF-sets · $\mathcal{P}$-CBFOWG and $\mathcal{R}$-CBFOWG operators · Multiple attribute decision making

Mathematics Subject Classification 90B50 · 03E72

1 Introduction

In daily life, we encounter with many decision making circumstances, which involve ambiguities and uncertainties due to insufficient knowledge, meager information, incompatible data, inconsistent and rare information. To overcome such kind of problems Zadeh (1965) brought out the idea of fuzzy sets. The theory of fuzzy set is universality of classical set theory.

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Zadeh's theory has been successfully applied in decision analysis to handle uncertain and ambiguous data from many decades. Fuzzy sets have been auspiciously adapted in domains of computer sciences (Frank and Seliger 1997). They used fuzzy logic and neural network to diagnose faults. Madore (2004) used fuzzy spaces in physics and worked and introduced concept of fuzziness in quantum mechanics. Innocent and Jhon (2004) constructed different useful method to medical diagnosis. Recently, Fuzzy sets have been used in many fields of life including, commercial appliances (air conditioner, washing machine and heating ventilation, etc.), forecasting system of weather, system of traffic monitoring (in Japan fuzzy controller use to run the train all day long).

The researchers have been published a good number of research papers on extensions of fuzzy sets. Many set theories have been developed, including, interval valued set theory (Zadeh 1965), intuitionistic fuzzy set theory (Atanassov 1986), bipolar fuzzy set theory (Zhang 2017), neutrosophic set theory (Smarandache 2010), soft set theory (Molodtsov 1999). All these theories have been developed according to necessity of handling specific type of data and its useability in different suitable domains.

For data analyzing of many types, bipolarity of knowledge is a vital part to be considered while developing a mathematical framework for most of the situations. Bipolarity indicates the positive and negative aspects of a particular problem. The concept behind the bipolarity is that a huge range of human decisions analysis is involved bipolar subjective thoughts. For illustration, happiness and grief, sweetness and sourness, effects and side effects are two different aspects of decision analysis. The equilibrium and mutual coexistence of these two aspects are treated as a key for balanced social environment. Zhang (1998), introduced the extension of fuzzy set with bipolarity, called, bipolar-valued fuzzy sets. Bipolar fuzzy set is suitable for information which involve property as well as its counter property. Lee (2000) discussed some basic operations of bipolar-valued fuzzy set. Lee and Cios (2004) proposed a comparison of intuitionistic fuzzy set, interval-valued set and bipolar fuzzy set. In bipolar-valued fuzzy set interval of membership value is $[-1, 1]$. The bipolar fuzzy set involves positive and negative memberships. The elements with 0 membership indicate that they are not satisfying the specific property, the interval $(0, 1]$ indicates elements satisfying property with different degrees of membership, whereas $[-1, 0)$ shows that elements satisfying implicit counter property. Bipolar fuzzy sets have been also used in many decision analysis which involve bipolar type information (Aslam et al. 2014; Tahir et al. 2016). Jun et al. (2012) introduced the abstraction of cubic set (extension of interval-valued fuzzy set and fuzzy set) and its operations. They presented idea of internal and external cubic sets and their characteristics. The concept has been extended with many other theories. Cubic set has been also successfully used in decision making processes due to its efficiency of containing enough information of a particular type of data. Decision making has a vital role in our daily life problems. We encounter with many difficulties to handle a problem which contains huge amount of data. In this regard, different techniques with aggregation operators have been introduced to handle such type of data. Many authors have been auspiciously adopted these capable techniques to handle ambiguities in decision analysis (Ghodsypour and 'Brien 1998; Gülçin and Çifçi 2012; Hwang and Yoon 1981).

The motivation of this paper is to develop a strong mathematical model for bipolar type decision analysis. From latest research surveys of hybrid fuzzy set models, most of the researchers in fuzzy-set-inspired models focused on real numbers between 0 and 1. But in most of the real life problems, we encounter with the negative part of a particular decision. For example, a medicine which is not effective may not be has any side effect. So the bipolarity is an important aspect of human decisions. In human decisions the second important part is ranking and rating of different alternatives obtained after particular evaluation. A verity of bipolar fuzzy decision making with different technique is available in literature. This range of applications of the theories can be used to deal with bipolar vagueness and uncertainty, which introduced the simple bipolar fuzzy characterizations of the universe of options that depend on a limited number of grades. So the concept of simple bipolar fuzzy set is insufficient to provide the information about the occurrence of ratings or grades with accuracy because information is limited, and it is also unable to describe the occurrence of uncertainty and vagueness very well specially, when sensitive cases are involved in decision making problems. Similarly, the concept of interval valued bipolar fuzzy sets (IVBFSs) is also insufficient to provide the information about the opinion of experts depending upon the properties of alternatives. For this purpose, we introduced the novel model with applications called cubic bipolar fuzzy sets as the generalization of bipolar fuzzy sets. So, by the hybridization of two well-known concepts called bipolar fuzzy sets and IVBFSs, we introduce a new hybrid model called cubic bipolar fuzzy sets (CBFSs). This model provides more accuracy and flexibility as compared to previously existing approaches, because it contains more information and it is more comprehensive and reasonable. Proposed model provides complete information about occurrence of ratings, uncertainty and bipolarity. It is also valuable because we extend the method of aggregating operators using geometric operations under cubic bipolar fuzzy data for MAGDM and compile the final decision using these extended operators method because of the complex structure of proposed model. In this paper, firstly, we discuss ICBF-sets and ECBF-sets and their relative characteristics under both orders, secondly, we introduce two cubic bipolar fuzzy ordered weighted geometric aggregation operators to aggregate the ICBF and ECBF data, then we apply both concepts to a multiple-criteria decision making problem to handle this specific type of information. The paper is planed in following sequence: in Sect. 2 we discuss some preliminaries , in Sect. 3 we introduce ICBF-sets and ECBF-sets and their properties for $\mathcal{P}$-Order and $\mathcal{R}$-Order. In Sect. 4 we introduce cubic bipolar fuzzy ordered weighted geometric aggregation operators under both order. In Sect. 5 we see the useability and application of these operators. In Sect. 6 we finally concluded our study.

2 Preliminaries

In the present section, we make another look on some useful basic definitions. Throughout this paper V is a universal set.

Definition 2.1 (Zadeh 1965) Let f be a membership function defined as $f : V \to [0, 1]$. Then V_f is a fuzzy set defined on V if each element $\& \in V$ is associated with membership's degree, which is a real number in $[0, 1]$ and it is denoted by $f(\varsigma)$.

Definition 2.2 (Zadeh 1971) Let $\mathcal{I} = [0, 1]$ be closed interval and $\ddot{\jmath} = [\jmath_\ell, \jmath_u]$ ba a closed subinterval of $\mathcal{I}$, where $0 \leq \jmath_\ell \leq \jmath_u \leq 1$. Let $[\mathcal{J}]$ be the set of all subintervals. The interval-valued fuzzy set (IVFS) defined on V is a function $f : V \to [\mathcal{J}]$. The cumulation of all IVFSs is denoted by $[\mathcal{J}]^V$ and $\ddot{\jmath}(\varsigma) = [\jmath_\ell(\varsigma), \jmath_u(\varsigma)]$,for each $\ddot{\jmath} \in [\mathcal{J}]^V$ and $\varsigma \in V$, is called membership's degree of ς to $\ddot{\jmath}$, where $\jmath_\ell(\varsigma)$ and $\jmath_u(\varsigma)$ are called lower or inferior fuzzy sets and upper or superior fuzzy sets, respectively.

Definition 2.3 (Jun et al. 2012) A cubic set on V can be defined as $\mathcal{K} = \{\langle \varsigma, \ddot{\jmath}(\varsigma), V_f(\varsigma)\rangle \mid \varsigma \in V\}$, where $\ddot{\jmath}$ is interval-valued fuzzy set on V and V_f is a fuzzy set on V.

Definition 2.4 (Zhang 1998) A bipolar fuzzy set on V is of the form $\mathcal{K} = \{(\varsigma,\ \delta^P_{\mathcal{K}}(\varsigma),\ \delta^N_{\mathcal{K}}(\varsigma)\,) :$ for all $\varsigma \in V\}$, where $\delta^P_{\mathcal{K}}(\varsigma)$ denotes the positive memberships ranges over $[0, 1]$ and $\delta^N_{\mathcal{K}}(\varsigma)$

denotes the negative memberships ranges over $[-1, 0]$. A bipolar fuzzy element (BFE) is simply written as $\mathcal{K} = (\delta^N_{\mathcal{K}}, \delta^P_{\mathcal{K}})$.

2.1 Cubic bipolar fuzzy sets

The present subsection is devoted for the study of cubic bipolar fuzzy sets (CBFSs). We review some definition and operations of CBFSs with examples.

Definition 2.5 (Riaz and Tehrim 2019b) Let $[\mathcal{J}]$ be the cumulation of all closed subintervals of $[-1, 1]$. An interval-valued bipolar fuzzy set (IVBFS) is of the form $\mathcal{K} = \{\langle \varsigma, \{\mathcal{M}^P_{\mathcal{K}}(\varsigma), \mathcal{M}^N_{\mathcal{K}}(\varsigma)\}\rangle\}$ where $\mathcal{M}^P_{\mathcal{K}}(\varsigma) : V \to [0, 1]$ and $\mathcal{M}^N_{\mathcal{K}}(\varsigma) : V \to [-1, 0]$ are interval-valued positive and negative membership degrees of $\varsigma \in V$. An interval-valued bipolar fuzzy element (IVBFE) can be written as $\mathcal{K} = \{\mathcal{M}^P_{\mathcal{K}}, \mathcal{M}^N_{\mathcal{K}}\} = \{[\delta^P_{\mathcal{K}\ell}, \delta^P_{\mathcal{K}u}], [\delta^N_{\mathcal{K}\ell}, \delta^N_{\mathcal{K}u}]\}$, where $\delta^P_{\mathcal{K}\ell}, \delta^P_{\mathcal{K}u}, \delta^N_{\mathcal{K}\ell}$ and $\delta^N_{\mathcal{K}u}$ are called upper and lower limits of the IVBFEs $\mathcal{M}^P$ and $\mathcal{M}^N$ respectively.

Definition 2.6 (Riaz and Tehrim 2019b) Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be IVBFEs, and $\varrho\rangle 0$, then operations on IVBFEs are defined below

(i) $\mathcal{K}_1 \oplus \mathcal{K}_2 = \{[\delta^P_{\mathcal{K}\ell_1} + \delta^P_{\mathcal{K}\ell_2} - \delta^P_{\mathcal{K}\ell_1}\delta^P_{\mathcal{K}\ell_2}, \delta^P_{\mathcal{K}u_1} + \delta^P_{\mathcal{K}u_2} - \delta^P_{\mathcal{K}u_1}\delta^P_{\mathcal{K}u_2}], [-(-\delta^N_{\mathcal{K}\ell_1} \cdot -\delta^N_{\mathcal{K}\ell_2}), -(-\delta^N_{\mathcal{K}u_1} \cdot -\delta^N_{\mathcal{K}u_2})]\}$

(ii) $\mathcal{K}_1 \otimes \mathcal{K}_2 = \{[\delta^P_{\mathcal{K}\ell_1} \cdot \delta^P_{\mathcal{K}\ell_2}, \delta^P_{\mathcal{K}u_1} \cdot \delta^P_{\mathcal{K}u_2}], [-(-\delta^N_{\mathcal{K}\ell_1} - \delta^N_{\mathcal{K}\ell_2} - (\delta^N_{\mathcal{K}\ell_1} \cdot \delta^N_{\mathcal{K}\ell_2}), -(-\delta^N_{\mathcal{K}u_1} - \delta^N_{\mathcal{K}u_2} - (\delta^N_{\mathcal{K}u_1} \cdot \delta^N_{\mathcal{K}u_2})]\}$

(iii) $\varrho\mathcal{K}_1 = [(1 - (1 - \delta^P_{\mathcal{K}\ell_1})^{\varrho}, (1 - (1 - \delta^P_{\mathcal{K}u_1})^{\varrho}], [-((-\delta^N_{\mathcal{K}\ell_1})^{\varrho}), -((-\delta^N_{\mathcal{K}u_1})^{\varrho})]$

(iv) $\mathcal{K}_1{}^{\varrho} = [(\delta^P_{\mathcal{K}\ell_1})^{\varrho}, (\delta^P_{\mathcal{K}u_1})^{\varrho}], [-(1 - (1 - (-\delta^N_{\mathcal{K}\ell_1}))^{\varrho}), -(1 - (1 - (-\delta^N_{\mathcal{K}u_1}))^{\varrho})]$

(v) $\mathcal{K}_1{}^{c} = [1 - \delta^P_{\mathcal{K}u_1}, 1 - \delta^P_{\mathcal{K}\ell_1}], [-1 - \delta^N_{\mathcal{K}u_1}, -1 - \delta^N_{\mathcal{K}\ell_1}]$

(vi) $\mathcal{K}_1 \cup \mathcal{K}_2 = [\max\{\delta^P_{\mathcal{K}\ell_1}, \delta^P_{\mathcal{K}\ell_2}\}, \max\{\delta^P_{\mathcal{K}u_1}, \delta^P_{\mathcal{K}u_2}\}], [\min\{\delta^N_{\mathcal{K}\ell_1}, \delta^N_{\mathcal{K}\ell_2}\}, \min\{\delta^N_{\mathcal{K}u_1}, \delta^N_{\mathcal{K}u_2}\}]$

(vii) $\mathcal{K}_1 \cap \mathcal{K}_2 = [\min\{\delta^P_{\mathcal{K}\ell_1}, \delta^P_{\mathcal{K}\ell_2}\}, \min\{\delta^P_{\mathcal{K}u_1}, \delta^P_{\mathcal{K}u_2}\}], [\max\{\delta^N_{\mathcal{K}\ell_1}, \delta^N_{\mathcal{K}\ell_2}\}, \max\{\delta^N_{\mathcal{K}u_1}, \delta^N_{\mathcal{K}u_2}\}]$

Definition 2.7 (Riaz and Tehrim 2019b) A set having form $\mathcal{K} = \{\langle \varsigma, \{\mathcal{M}^P_{\mathcal{K}}(\varsigma), \mathcal{M}^N_{\mathcal{K}}(\varsigma)\}, \mathcal{N}_{\mathcal{K}}(\varsigma)\rangle : \varsigma \in V\}$ is called cubic bipolar fuzzy set (CBFS), where $\mathcal{M} = \{\mathcal{M}^P_{\mathcal{K}}(\varsigma), \mathcal{M}^N_{\mathcal{K}}(\varsigma)\}$ is called interval-valued bipolar fuzzy set (IVBFS) and $\mathcal{N} = \mathcal{N}_{\mathcal{K}}(\varsigma)$ is bipolar fuzzy set (BFS). Consider the Interval $\mathcal{I} = [-1, 1]$. Suppose that $[\mathcal{I}]$ and $[\mathcal{J}]$ be the collection of all sub-intervals of $[0, 1]$ and $[-1, 0]$ respectively. Then we obtain the mappings $\mathcal{M}^P_{\mathcal{K}}(\varsigma) \to [\mathcal{I}] \mid \mathcal{M}^P_{\mathcal{K}}(\varsigma) = [\delta^P_{\mathcal{K}\ell}(\varsigma), \delta^P_{\mathcal{K}u}(\varsigma)]$ and $\mathcal{M}^N_{\mathcal{K}}(\varsigma) \to [\mathcal{J}] \mid \mathcal{M}^N_{\mathcal{K}}(\varsigma) = [\delta^N_{\mathcal{K}\ell}(\varsigma), \delta^N_{\mathcal{K}u}(\varsigma)]$, similarly we get $\mathcal{N}_{\mathcal{K}}(\varsigma) \to \mathcal{I} \mid \mathcal{N}_{\mathcal{K}}(\varsigma) = [\rho^P_{\mathcal{K}}(\varsigma), \rho^N_{\mathcal{K}}(\varsigma)]$, so $\mathcal{K}$ becomes

$$\mathcal{K} = \langle \mathcal{M}, \mathcal{N}\rangle = \{\langle \varsigma, \{[\delta^P_{\mathcal{K}\ell}(\varsigma), \delta^P_{\mathcal{K}u}(\varsigma)], [\delta^N_{\mathcal{K}\ell}(\varsigma), \delta^N_{\mathcal{K}u}(\varsigma)]\}, \{\rho^P_{\mathcal{K}}(\varsigma), \rho^N_{\mathcal{K}}(\varsigma)\}\rangle : \varsigma \in V\}$$

A cubic bipolar fuzzy element (CBFE) can be denoted by $\mathcal{K} = \{\langle\{[\delta^P_{\mathcal{K}\ell}, \delta^P_{\mathcal{K}u}], [\delta^N_{\mathcal{K}\ell}, \delta^N_{\mathcal{K}u}]\}, \{\rho^P_{\mathcal{K}}, \rho^N_{\mathcal{K}}\}\rangle\}$.

Example 2.8 (Riaz and Tehrim 2019b) Let $V = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ be a universal set then a CBFS on V is given as follow:

$$\begin{aligned}\ddot{\mathcal{K}} &= \{\langle \varsigma_1, \{[0.44, 0.56], [-0.73, -0.61]\}, \{0.21, -0.33\}\rangle, \langle \varsigma_2, \{[0.73, 0.81],\\ &\quad [-0.61, -0.46]\}, \{0.19, -0.23\}\rangle, \langle \varsigma_3, \{[0.51, 0.62], [-0.41, -0.23]\}, \{0.31, -0.57\}\rangle\}.\end{aligned}$$

Definition 2.9 (Riaz and Tehrim 2019b) (*Equality*)
Two cubic bipolar fuzzy sets (CBFSs) $\mathcal{K}_1$ and $\mathcal{K}_2$ on V are said to be equal if

(i) $[\delta^P_{\mathcal{K}\ell_1}(\varsigma), \delta^P_{\mathcal{K}u_1}(\varsigma)] = [\delta^P_{\mathcal{K}\ell_2}(\varsigma), \delta^P_{\mathcal{K}u_2}(\varsigma)]$ and $[\delta^N_{\mathcal{K}\ell_1}(\varsigma), \delta^N_{\mathcal{K}u_1}(\varsigma)] = [\delta^N_{\mathcal{K}\ell_2}(\varsigma), \delta^N_{\mathcal{K}u_2}(\varsigma)]$

(ii) $\{\rho^P_{\mathcal{K}_1}(\varsigma) = \rho^P_{\mathcal{K}_2}(\varsigma)\}$ and $\{\rho^N_{\mathcal{K}_1}(\varsigma) = \rho^N_{\mathcal{K}_2}(\varsigma)\}$.

Definition 2.10 (Riaz and Tehrim 2019b) ($\mathcal{P}$-Order)
Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two CBFSs on V. Then $\mathcal{K}_1 \subseteq_{\mathcal{P}} \mathcal{K}_2$, if

(i) $[\delta^P_{\mathcal{K}\ell_1}(\varsigma), \delta^P_{\mathcal{K}u_1}(\varsigma)] \leq [\delta^P_{\mathcal{K}\ell_2}(\varsigma), \delta^P_{\mathcal{K}u_2}(\varsigma)]$ and $[\delta^N_{\mathcal{K}\ell_1}(\varsigma), \delta^N_{\mathcal{K}u_1}(\varsigma)] \geq [\delta^N_{\mathcal{K}\ell_2}(\varsigma), \delta^N_{\mathcal{K}u_2}(\varsigma)]$

(ii) $\{\rho^P_{\mathcal{K}_1}(\varsigma) \leq \rho^P_{\mathcal{K}_2}(\varsigma)\}$ and $\{\rho^N_{\mathcal{K}_1}(\varsigma) \geq \rho^N_{\mathcal{K}_2}(\varsigma)\}$.

Definition 2.11 (Riaz and Tehrim 2019b) ($\mathcal{R}$-Order)
Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two CBFSs on V. Then $\mathcal{K}_1 \subseteq_{\mathcal{R}} \mathcal{K}_2$, if

(i) $[\delta^P_{\mathcal{K}\ell_1}(\varsigma), \delta^P_{\mathcal{K}u_1}(\varsigma)] \leq [\delta^P_{\mathcal{K}\ell_2}(\varsigma), \delta^P_{\mathcal{K}u_2}(\varsigma)]$ and $[\delta^N_{\mathcal{K}\ell_1}(\varsigma), \delta^N_{\mathcal{K}u_1}(\varsigma)] \geq [\delta^N_{\mathcal{K}\ell_2}(\varsigma), \delta^N_{\mathcal{K}u_2}(\varsigma)]$

(ii) $\{\rho^P_{\mathcal{K}_1}(\varsigma) \geq \rho^P_{\mathcal{K}_2}(\varsigma)\}$ and $\{\rho^N_{\mathcal{K}_1}(\varsigma) \leq \rho^N_{\mathcal{K}_2}(\varsigma)\}$.

Definition 2.12 (Riaz and Tehrim 2019b) The complement of $\mathcal{K}$ can be defined as $\mathcal{K}^c = \{\langle \varsigma, \{[1-\delta^P_{\mathcal{K}u}(\varsigma), 1-\delta^P_{\mathcal{K}\ell}(\varsigma)], [-1-\delta^N_{\mathcal{K}u}(\varsigma), -1-\delta^N_{\mathcal{K}\ell}(\varsigma)]\}, \{1-\rho^P_{\mathcal{K}}(\varsigma), -1-\rho^N_{\mathcal{K}}(\varsigma)\}\rangle : \varsigma \in V\}$.

Example 2.13 (Riaz and Tehrim 2019b) ($\mathcal{P}$-Order)
Consider $\ddot{\mathcal{K}_1} = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\ddot{\mathcal{K}_2} = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ two CBFSs on V where,

$$\begin{aligned}
\ddot{\mathcal{K}_1} &= \{\langle \varsigma_1, \{[0.23, 0.61], [-0.63, -0.45]\}, \{0.71, -0.68\}\rangle,\\
&\quad \langle \varsigma_2, \{[0.46, 0.71], [-0.73, -0.61]\}, \{0.81, -0.71\}\rangle,\\
&\quad \langle \varsigma_3, \{[0.24, 0.62], [-0.65, -0.47]\}, \{0.69, -0.72\}\rangle\},\\
\ddot{\mathcal{K}_2} &= \{< \varsigma_1, \{[0.33, 0.73], [-0.83, -0.51]\}, \{0.81, -0.71\}\rangle,\\
&\quad \langle \varsigma_2, \{[0.53, 0.82], [-0.80, -0.70]\}, \{0.90, -0.80\}\rangle,\\
&\quad \langle \varsigma_3, \{[0.33, 0.73], [-0.83, -0.71]\}, \{0.78, -0.81\}\rangle\},
\end{aligned}$$

It is clear $\ddot{\mathcal{K}_1} \subseteq_{\mathcal{P}} \ddot{\mathcal{K}_2}$.

Example 2.14 (Riaz and Tehrim 2019b) ($\mathcal{R}$-Order)
Consider $\ddot{\mathcal{K}_1} = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\ddot{\mathcal{K}_2} = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ two CBFSs on V where,

$$\begin{aligned}
\ddot{\mathcal{K}_1} &= \{\langle \varsigma_1, \{[0.23, 0.61], [-0.63, -0.45]\}, \{0.90, -0.80\}\rangle,\\
&\quad \langle \varsigma_2, \{[0.46, 0.71], [-0.73, -0.61]\}, \{0.81, -0.71\}\rangle,\\
&\quad \langle \varsigma_3, \{[0.24, 0.62], [-0.65, -0.47]\}, \{0.69, -0.72\}\rangle\},\\
\ddot{\mathcal{K}_2} &= \{< \varsigma_1, \{[0.33, 0.73], [-0.83, -0.51]\}, \{0.81, -0.71\}\rangle,\\
&\quad \langle \varsigma_2, \{[0.53, 0.82], [-0.80, -0.70]\}, \{0.71, -0.68\}\rangle,\\
&\quad \langle \varsigma_3, \{[0.33, 0.73], [-0.83, -0.71]\}, \{0.60, -0.63\}\rangle\},
\end{aligned}$$

It is clear $\ddot{\mathcal{K}_1} \subseteq_{\mathcal{R}} \ddot{\mathcal{K}_2}$.

Definition 2.15 (Riaz and Tehrim 2019b) ($\mathcal{P}$-Union)
Let $\mathcal{K}_\varepsilon$ be a collection of CBFSs on V, then we define $\mathcal{P}$-Union as $\cup^n_{\varepsilon=1}\mathcal{K}_\varepsilon = \{\langle \max_{\varepsilon\in\Omega}[\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)], \min_{\varepsilon\in\Omega}[\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)]\}, \max_{\varepsilon\in\Omega}(\rho^P_{\mathcal{K}_\varepsilon}(\varsigma)), \min_{\varepsilon\in\Omega}(\rho^N_{\mathcal{K}_\varepsilon}(\varsigma))\}\rangle$. We write it $\cup_{\mathcal{P}}$.

Definition 2.16 (Riaz and Tehrim 2019b) ($\mathcal{P}$-Intersection)
Let $\mathcal{K}_\varepsilon$ be a collection of CBFSs on V, then we define $\mathcal{P}$-Intersection as $\cup_{\varepsilon=1}^{n}\mathcal{K}_\varepsilon = \{\langle \min_{\varepsilon\in\Omega}[\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)], \max_{\varepsilon\in\Omega}[\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)]\}, \min_{\varepsilon\in\Omega}(\rho^P_{\mathcal{K}_\varepsilon}(\varsigma)), \max_{\varepsilon\in\Omega} (\rho^N_{\mathcal{K}_\varepsilon}(\varsigma))\}\rangle$. We write it $\cap_{\mathcal{P}}$.

Definition 2.17 (Riaz and Tehrim 2019b) ($\mathcal{R}$-Union)
Let $\mathcal{K}_\varepsilon$ be a collection of CBFSs on V, then we define $\mathcal{R}$-Union as $\cup_{\varepsilon=1}^{n}\mathcal{K}_\varepsilon = \{\langle \max_{\varepsilon\in\Omega}[\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)], \min_{\varepsilon\in\Omega}[\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)]\}, \{\min_{\varepsilon\in\Omega}(\rho^P_{\mathcal{K}_\varepsilon}(\varsigma)), \max_{\varepsilon\in\Omega} (\rho^N_{\mathcal{K}_\varepsilon}(\varsigma))\}\rangle$. We write it $\cup_{\mathcal{R}}$.

Definition 2.18 (Riaz and Tehrim 2019b) ($\mathcal{R}$-Intersection)
Let $\mathcal{K}_\varepsilon$ be a collection of CBFSs on V, then we define $\mathcal{R}$-Intersection as $\cup_{\varepsilon=1}^{n}\mathcal{K}_\varepsilon = \{\langle \min_{\varepsilon\in\Omega}[\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)], \max_{\varepsilon\in\Omega}[\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)]\}, \{\max_{\varepsilon\in\Omega}(\rho^P_{\mathcal{K}_\varepsilon}(\varsigma)), \min_{\varepsilon\in\Omega} (\rho^N_{\mathcal{K}_\varepsilon}(\varsigma))\}\rangle$. We write it $\cap_{\mathcal{R}}$.

Example 2.19 (Riaz and Tehrim 2019b) Consider $\ddot{\mathcal{K}}_1 = \langle \mathcal{M}_1, \mathcal{N}_1\rangle$ and $\ddot{\mathcal{K}}_2 = \langle \mathcal{M}_2, \mathcal{N}_2\rangle$ two CBFSs on V, where $\ddot{\mathcal{K}}_1 = \{\langle \varsigma_1, \{[0.22, 0.34], [-0.48, -0.36]\}, \{0.45, -0.67\}\rangle, \langle \varsigma_2, \{[0.37, 0.46], [-0.67, -0.59]\}, \{0.81, -0.72\}\rangle\}$, $\ddot{\mathcal{K}}_2 = \{\langle \varsigma_1, \{[0.35, 0.43], [-0.63, -0.51]\}, \{0.56, -0.67\}\rangle, \langle \varsigma_2, \{[0.38, 0.52], [-0.70, -0.54]\}, \{0.67, -0.43\}\rangle\}$, then

$$
\begin{aligned}
\ddot{\mathcal{K}}_1 \cup_{\mathcal{P}} \ddot{\mathcal{K}}_2 &= \{\langle \varsigma_1, \{[0.35, 0.43], [-0.63, -0.51]\}, \{0.56, -0.67\}\rangle,\\
&\qquad \langle \varsigma_2, \{[0.38, 0.52], [-0.70, -0.59]\}, \{0.81, -0.72\}\rangle\}\\
\ddot{\mathcal{K}}_1 \cap_{\mathcal{P}} \ddot{\mathcal{K}}_2 &= \{\langle \varsigma_1, \{[0.22, 0.34], [-0.48, -0.36]\}, \{0.45, -0.61\}\rangle,\\
&\qquad \langle \varsigma_2, \{[0.37, 0.46], [-0.67, -0.54]\}, \{0.67, -0.43\}\rangle\}\\
\ddot{\mathcal{K}}_1 \cup_{\mathcal{R}} \ddot{\mathcal{K}}_2 &= \{\langle \varsigma_1, \{[0.35, 0.43], [-0.63, -0.51]\}, \{0.45, -0.61\}\rangle,\\
&\qquad \langle \varsigma_2, \{[0.38, 0.52], [-0.70, -0.59]\}, \{0.67, -0.43\}\rangle\}\\
\ddot{\mathcal{K}}_1 \cap_{\mathcal{R}} \ddot{\mathcal{K}}_2 &= \{\langle \varsigma_1, \{[0.35, 0.43], [-0.63, -0.51]\}, \{0.56, -0.67\}\rangle,\\
&\qquad \langle \varsigma_2, \{[0.38, 0.52], [-0.70, -0.59]\}, \{0.81, -0.72\}\rangle\}
\end{aligned}
$$

Definition 2.20 (Riaz and Tehrim 2019b) Let $\mathcal{K}_1, \mathcal{K}_2$ be two CBFSs on V and $\varrho\rangle 0$ be a real number. Then we define following algebraic operations for $\mathcal{P}$-Order.

$$
\begin{aligned}
\mathcal{K}_1 \oplus_{\mathcal{P}} \mathcal{K}_2 =& \Bigg\{\Bigg\{\Bigg[\delta^P_{\mathcal{K}\ell_1}(\varsigma) + \delta^P_{\mathcal{K}\ell_2}(\varsigma) - \left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\delta^P_{\mathcal{K}\ell_2}(\varsigma)\right), \delta^P_{\mathcal{K}u_1}(\varsigma) + \delta^P_{\mathcal{K}u_2}(\varsigma)\\
& - \left(\delta^P_{\mathcal{K}u_1}(\varsigma)\delta^P_{\mathcal{K}u_2}(\varsigma)\right)\Bigg],\\
& \Bigg[-\left(\delta^N_{\mathcal{K}\ell_1}(\varsigma)\delta^N_{\mathcal{K}\ell_2}(\varsigma)\right), -\left(\delta^N_{\mathcal{K}u_1}(\varsigma)\delta^N_{\mathcal{K}u_2}(\varsigma)\right)\Bigg]\Bigg\}, \Bigg\{\rho^P_{\mathcal{K}_1}(\varsigma) + \rho^P_{\mathcal{K}_2}(\varsigma)\\
& - \left(\rho^P_{\mathcal{K}_1}(\varsigma)\rho^P_{\mathcal{K}_2}(\varsigma)\right), -\left(\rho^N_{\mathcal{K}_1}(\varsigma)\rho^N_{\mathcal{K}_2}(\varsigma)\right)\Bigg\}\Bigg\}\\
\mathcal{K}_1 \otimes_{\mathcal{P}} \mathcal{K}_2 =& \Bigg\{\Bigg\{\Bigg[\left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\delta^P_{\mathcal{K}\ell_2}(\varsigma)\right), \left(\delta^P_{\mathcal{K}u_1}(\varsigma)\delta^P_{\mathcal{K}u_2}(\varsigma)\right)\Bigg], \Bigg[-\Bigg(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)\\
& -\delta^N_{\mathcal{K}\ell_2}(\varsigma) - \left(\delta^N_{\mathcal{K}\ell_1}(\varsigma)\delta^N_{\mathcal{K}\ell_2}(\varsigma)\right)\Bigg),\\
& -\left(-\delta^N_{\mathcal{K}u_1}(\varsigma) - \delta^N_{\mathcal{K}u_2}(\varsigma) - \left(\delta^N_{\mathcal{K}u_1}(\varsigma)\delta^N_{\mathcal{K}u_2}(\varsigma)\right)\right)\Bigg]\Bigg\}, \Bigg\{\left(\rho^P_{\mathcal{K}_1}(\varsigma)\rho^P_{\mathcal{K}_2}(\varsigma)\right),
\end{aligned}
$$

$$-\left(-\rho^N_{\mathcal{K}_1}(\varsigma)-\rho^N_{\mathcal{K}_2}(\varsigma)-\left(\rho^N_{\mathcal{K}_1}(\varsigma)\rho^N_{\mathcal{K}_2}(\varsigma)\right)\right)\Big\}\Big\}$$

$$\mathcal{K}_1^{\varrho}=\Big\{\Big\{\Big[\left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho},\left(\delta^P_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\Big],\Big[-\left(1-\left(1-\left(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)\right)\right)^{\varrho}\right),$$
$$-\left(1-\left(1-\left(-\delta^N_{\mathcal{K}u_1}(\varsigma)\right)\right)^{\varrho}\right)\Big]\Big\},$$
$$\Big\{\left(\rho^P_{\mathcal{K}_1}(\varsigma)\right)^{\varrho},-\left(1-\left(1-\left(-\rho^N_{\mathcal{K}_1}(\varsigma)\right)\right)^{\varrho}\right)\Big\}\Big\}$$

$$\varrho\mathcal{K}_1=\Big\{\Big\{\Big[1-\left(1-\delta^P_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho},1-\left(1-\delta^P_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\Big],\Big[-\left(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho},$$
$$-\left(-\delta^N_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\Big]\Big\},\Big\{1-\left(1-\rho^P_{\mathcal{K}_1}(\varsigma)\right)^{\varrho},-\left(-\rho^N_{\mathcal{K}_1}(\varsigma)\right)^{\varrho}\Big\}\Big\}$$

Example 2.21 (Riaz and Tehrim 2019b) Let $\ddot{\mathcal{K}}_1=\{\langle\varsigma_1,\{[0.44,0.56],[-0.73,-0.61]\},\{0.21,-0.33\}\rangle\}$ and $\ddot{\mathcal{K}}_2=\{\langle\varsigma_1,\{[0.73,0.81],[-0.61,-0.46]\},\{0.54,-0.62\}\rangle\}$ and $\varrho=2$, then

$$\ddot{\mathcal{K}}_1\oplus_{\mathcal{P}}\ddot{\mathcal{K}}_2=\Big\{\Big\langle\Big\{\Big[0.44+0.73-(0.32),0.56+0.81-(0.45)\Big],$$
$$\Big[-(0.44),-(0.28)\Big]\Big\},\Big\{0.21+0.54-(0.11),-(0.20)\Big\}\Big\rangle\Big\}$$
$$\ddot{\mathcal{K}}_1\oplus_{\mathcal{P}}\ddot{\mathcal{K}}_2=\{\langle\varsigma_1,\{[0.85,0.92],[-0.44,-0.28]\},\{0.64,-0.20\}\rangle\}$$
$$\ddot{\mathcal{K}}_1\otimes_{\mathcal{P}}\ddot{\mathcal{K}}_2=\Big\{\Big\langle\Big\{\Big[(0.32),(0.45)\Big],\Big[-\left(0.73+0.61-(0.44)\right),$$
$$-\left(0.61+0.46-(0.28)\right)\Big]\Big\},\Big\{(0.096),-\left(0.33+0.62-(0.20)\right)\Big\}\Big\rangle\Big\}$$
$$\ddot{\mathcal{K}}_1\otimes_{\mathcal{P}}\ddot{\mathcal{K}}_2=\{\langle\varsigma_1,\{[0.32,0.45],[-0.90,-0.79]\},\{0.096,-0.75\}\rangle\}$$
$$\ddot{\mathcal{K}}_1^2=\Big\{\Big\langle\Big\{\Big[(0.44)^2,(0.56)^2\Big],\Big[-\left(1-(1-0.73)^2\right),$$
$$-\left(1-(1-0.61)^2\right)\Big]\Big\}\Big\{(0.21)^2,-\left(1-(1-0.33)^2\right)\Big\}\Big\rangle\Big\}$$
$$\ddot{\mathcal{K}}_1^2=\{\langle\varsigma_1,\{[0.19,0.31],[-0.92,-0.84]\},\{0.044,-0.55\}\rangle\}$$
$$2\ddot{\mathcal{K}}_1=\Big\{\Big\langle\Big\{\Big[1-(1-0.44)^2,1-(1-0.56)^2\Big],\Big[-(0.73)^2,-(0.61)^2\Big]\Big\},$$
$$\Big\{1-(1-0.21)^2,-(0.33)^2\Big\}\Big\rangle\Big\}$$
$$2\ddot{\mathcal{K}}_1=\{\langle\varsigma_1,\{[0.68,0.80],[-0.53,-0.37]\},\{0.37,-0.10\}\rangle\}$$

Definition 2.22 (Riaz and Tehrim 2019b) Let $\mathcal{K}_1,\mathcal{K}_2$ be two CBFSs on V and $\varrho\rangle 0$ be a real number. Then we define following algebraic operations for $\mathcal{R}$-Order.

$$\mathcal{K}_1\oplus_{\mathcal{R}}\mathcal{K}_2=\Big\{\Big\{\Big[\delta^P_{\mathcal{K}\ell_1}(\varsigma)+\delta^P_{\mathcal{K}\ell_2}(\varsigma)-\left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\delta^P_{\mathcal{K}\ell_2}(\varsigma)\right),\delta^P_{\mathcal{K}u_1}(\varsigma)$$
$$+\delta^P_{\mathcal{K}u_2}(\varsigma)-\left(\delta^P_{\mathcal{K}u_1}(\varsigma)\delta^P_{\mathcal{K}u_2}(\varsigma)\right)\Big],$$

$$\left[-\left(\delta^N_{\mathcal{K}\ell_1}(\varsigma)\delta^N_{\mathcal{K}\ell_2}(\varsigma)\right), -\left(\delta^N_{\mathcal{K}u_1}(\varsigma)\delta^N_{\mathcal{K}u_2}(\varsigma)\right)\right]\left\{\left(\rho^P_{\mathcal{K}_1}(\varsigma)\rho^P_{\mathcal{K}_2}(\varsigma)\right),\right.$$
$$\left.\left.-\left(-\rho^N_{\mathcal{K}_1}(\varsigma)-\rho^N_{\mathcal{K}_2}(\varsigma)-\left(\rho^N_{\mathcal{K}_1}(\varsigma)\rho^N_{\mathcal{K}_2}(\varsigma)\right)\right)\right\}\right\}$$

$$\mathcal{K}_1 \otimes_{\mathcal{R}} \mathcal{K}_2 = \left\{\left\{\left[\left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\delta^P_{\mathcal{K}\ell_2}(\varsigma)\right), \left(\delta^P_{\mathcal{K}u_1}(\varsigma)\delta^P_{\mathcal{K}u_2}(\varsigma)\right)\right],\right.\right.$$
$$\left[-\left(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)-\delta^N_{\mathcal{K}\ell_2}(\varsigma)-\left(\delta^N_{\mathcal{K}\ell_1}(\varsigma)\delta^N_{\mathcal{K}\ell_2}(\varsigma)\right)\right),\right.$$
$$\left.\left.-\left(-\delta^N_{\mathcal{K}u_1}(\varsigma)-\delta^N_{\mathcal{K}u_2}(\varsigma)-\left(\delta^N_{\mathcal{K}u_1}(\varsigma)\delta^N_{\mathcal{K}u_2}(\varsigma)\right)\right)\right]\right\}, \Big\},$$
$$\left\{\rho^P_{\mathcal{K}_1}(\varsigma)+\rho^P_{\mathcal{K}_2}(\varsigma)-\left(\rho^P_{\mathcal{K}_1}(\varsigma)\rho^P_{\mathcal{K}_2}(\varsigma)\right),\right.$$
$$\left.\left.-\left(\rho^N_{\mathcal{K}_1}(\varsigma)\rho^N_{\mathcal{K}_2}(\varsigma)\right)\right\}\right\}$$

$$\mathcal{K}_1^{\varrho} = \left\{\left\{\left[\left(\delta^P_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho}, \left(\delta^P_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\right], \left[-\left(1-\left(1-\left(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)\right)\right)^{\varrho}\right),\right.\right.\right.$$
$$\left.\left.-\left(1-\left(1-\left(-\delta^N_{\mathcal{K}u_1}(\varsigma)\right)\right)^{\varrho}\right)\right]\right\},$$
$$\left.\left\{1-\left(1-\rho^P_{\mathcal{K}_1}(\varsigma)\right)^{\varrho}, -\left(-\rho^N_{\mathcal{K}_1}(\varsigma)\right)^{\varrho}\right\}\right\}$$

$$\varrho\mathcal{K}_1 = \left\{\left\{\left[1-\left(1-\delta^P_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho}, 1-\left(1-\delta^P_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\right],\right.\right.$$
$$\left[-\left(-\delta^N_{\mathcal{K}\ell_1}(\varsigma)\right)^{\varrho}, -\left(-\delta^N_{\mathcal{K}u_1}(\varsigma)\right)^{\varrho}\right]\left\{\left(\rho^P_{\mathcal{K}_1}(\varsigma)\right)^{\varrho},\right.$$
$$\left.\left.-\left(1-\left(1-\left(-\rho^N_{\mathcal{K}_1}(\varsigma)\right)\right)^{\varrho}\right)\right\}\right\}$$

Example 2.23 (Riaz and Tehrim 2019b) Consider $\ddot{\mathcal{K}}_1 = \{\langle\varsigma_1, \{[0.44, 0.56], [-0.73, -0.61]\}, \{0.21, -0.33\}\rangle\}$ and $\ddot{\mathcal{K}}_2 = \{\langle\varsigma_1, \{[0.73, 0.81], [-0.61, -0.46]\}, \{0.54, -0.62\}\rangle\}$ and $\varrho = 2$, then

$$\begin{aligned}
\ddot{\mathcal{K}}_1 \oplus_{\mathcal{R}} \ddot{\mathcal{K}}_2 &= \{\langle\varsigma_1, \{[0.85, 0.92], [-0.44, -0.28]\}, \{0.096, -0.75\}\rangle\} \\
\ddot{\mathcal{K}}_1 \otimes_{\mathcal{R}} \ddot{\mathcal{K}}_2 &= \{\langle\varsigma_1, \{[0.32, 0.45], [-0.90, -0.79]\}, \{0.64, -0.20\}\rangle\} \\
\ddot{\mathcal{K}}_1^2 &= \{\langle\varsigma_1, \{[0.19, 0.31], [-0.92, -0.84]\}, \{0.37, -0.10\}\rangle\} \\
2\ddot{\mathcal{K}}_1 &= \{\langle\varsigma_1, \{[0.68, 0.80], [-0.53, -0.37]\}, \{0.044, -0.55\}\rangle\}
\end{aligned}$$

Lemma 2.24 (Riaz and Tehrim 2019b) *Let $\mathcal{K}_\varepsilon$ be a family of cubic bipolar fuzzy sets and $\varrho\rangle 0$ be any real number, then for any $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{K}_\varepsilon$ the $\mathcal{K}_1 \oplus \mathcal{K}_2$, $\mathcal{K}_1 \otimes \mathcal{K}_2$, $\mathcal{K}_1^{\varrho}$ and $\varrho\mathcal{K}_1$ are also cubic bipolar fuzzy sets under $\mathcal{P}$-Order and $\mathcal{R}$-Order.*

Theorem 2.25 (Riaz and Tehrim 2019b) *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two CBFSs and $\varrho_1, \varrho_2, \varrho_3\rangle 0$ be real numbers then the following properties are hold under $\mathcal{P}$-Order and $\mathcal{R}$-Order.*

(i) $\varrho(\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) = \varrho\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \varrho\mathcal{K}_2$,
(ii) $(\mathcal{K}_1^{\varrho} \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2^{\varrho}) = (\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)^{\varrho}$

(iii) $\mathcal{K}_1^{\varrho_1} \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_1^{\varrho_2} = \mathcal{K}_1{}^{\varrho_1+\varrho_2}$,
(iv) $(\varrho_1\varrho_2)\mathcal{K}_1 = \varrho_1(\varrho_2\mathcal{K}_1)$
(v) $(\mathcal{K}_1{}^{\varrho_1\varrho_2}) = (\mathcal{K}_1{}^{\varrho_1})^{\varrho_2}$,
(vi) $\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2 = \mathcal{K}_2 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_1$
(vii) $\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2 = \mathcal{K}_2 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_1$,
(viii) $\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2 = \mathcal{K}_2 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_1$
(ix) $\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2 = \mathcal{K}_2 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_1$,
(x) $\{\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2\}^c = \mathcal{K}_1{}^c \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2{}^c$
(xi) $\{\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2\}^c = \mathcal{K}_1{}^c \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2{}^c$,
(xii) $\{\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2\}^c = \mathcal{K}_1{}^c \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2{}^c$
(xiii) $\{\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2\}^c = \mathcal{K}_1{}^c \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2{}^c$.

2.2 Ordered weighted aggregation operators

An aggregation operator is some sort of function that used to combine information or data. It is used to combine n number of data, for instance, n numerical values in a single outcome. The aggregation operators can be obtained by different mathematical operations for example arithmetic operations and geometric operations. A weight factor is also used in aggregation operator which allow us to weight the different data according to their relevance. There exist a large number of different aggregation operators that differ on the assumptions on the data (data types) and about the type of information that we can incorporate in the model. In the present study we use ordered weighted aggregation operator which were first introduced by Yager (1988) under averaging operations.

Definition 2.26 (Yager 1988) Let $\ddot{\mathcal{K}}_\varepsilon$, for $\{\varepsilon = 1, 2, \ldots, n\}$, be the collection of data and $\Upsilon = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n\}^T$ be the weighted vector, where ordered weighted averaging aggregation operator is a mapping $\mathfrak{A} : \ddot{\mathcal{K}}^n \to \ddot{\mathcal{K}}$ which can be defined as follows

$$\mathfrak{A}(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) = \left\{\overset{n}{\underset{\varepsilon=1}{\oplus}}\right\}_{\mathcal{P}} (\mathfrak{w}_\varepsilon(\ddot{\mathcal{K}}_\varepsilon))$$

Definition 2.27 (Yager 1988) (Properties)
The weighted averaging aggregation operator defined on $\ddot{\mathcal{K}}_\varepsilon$ satisfy the following properties.

(i) (Idempotent) For $\ddot{\mathcal{K}}_\varepsilon = \ddot{\mathcal{K}}$, we have $CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) = \ddot{\mathcal{K}}$
(ii) (Bounded) $\cap CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) \leq CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) \leq \cup CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n)$
(iii) (Commutative) If $(\ddot{\mathcal{K}}'_1, \ddot{\mathcal{K}}_2{}', \ldots, \ddot{\mathcal{K}}'_n)$ be different alteration of $(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n)$, then $CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) = CBFG(\ddot{\mathcal{K}}'_1, \ddot{\mathcal{K}}_2{}', \ldots, \ddot{\mathcal{K}}'_n)$
(iv) (Monotonic) Let $\ddot{\mathcal{K}}_\varepsilon$ and $\ddot{\mathcal{K}}'_\varepsilon$ then for $\ddot{\mathcal{K}}_\varepsilon \leq \ddot{\mathcal{K}}'_\varepsilon$, we get $CBFG(\ddot{\mathcal{K}}_1, \ddot{\mathcal{K}}_2, \ldots, \ddot{\mathcal{K}}_n) \leq CBFG(\ddot{\mathcal{K}}'_1, \ddot{\mathcal{K}}_2{}', \ldots, \ddot{\mathcal{K}}'_n)$.

The researchers developed a number of aggregation operators according to necessity of data. Here, we develop some ordered weighted geometric aggregation operators for cubic bipolar fuzzy information.

3 Internal and external CBFSs

Definition 3.1 A CBFS $\mathcal{K} = \langle \mathcal{M}, \mathcal{N} \rangle$ is said to be an internal cubic bipolar fuzzy set (ICBFS) if for all indexing set $\varepsilon \in \Omega$ and $\varsigma \in V$ the following conditions are satisfied $\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq$

$\rho^P_{\mathcal{K}_\varepsilon} \leq \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)$, $\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq \rho^N_{\mathcal{K}_\varepsilon} \leq \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)$ where $\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^{(P)}_{\mathcal{K}u_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma) \in \mathcal{M}$, $\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon} \in \mathcal{N}$.

Definition 3.2 A CBFS $\mathcal{K} = \langle \mathcal{M}, \mathcal{N} \rangle$ is called an external cubic bipolar fuzzy set (ECBFS) if for all indexing set $\varepsilon \in \Omega$ and $\varsigma \in V$ the following conditions are satisfied $\rho^P_{\mathcal{K}_\varepsilon} \notin \big((\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$, $\rho^N_{\mathcal{K}_\varepsilon} \notin \big(\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ where $\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma)$, $\delta^N_{\mathcal{K}u_\varepsilon}(\varsigma) \in \mathcal{M}$, $\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon} \in \mathcal{N}$.

Example 3.3 The given below set is an ICBFS

$$\mathcal{K} = \left\{ \begin{array}{l} \langle \varsigma_1, \{[0.32, 0.45], [-0.51, -0.42]\}, \{0.38, -0.48\}\rangle, \\ \langle \varsigma_2, \{[0.41, 0.55], [-0.56, -0.47]\}, \{0.47, -0.50\}\rangle, \\ \langle \varsigma_3, \{[0.63, 0.73], [-0.83, -0.71]\}, \{0.68, -0.79\}\rangle \end{array} \right\}$$

Example 3.4 The given below set is an ECBFS

$$\mathcal{K} = \left\{ \begin{array}{l} \langle \varsigma_1, \{[0.23, 0.33], [-0.51, -0.41]\}, \{0.41, -0.25\}\rangle, \\ \langle \varsigma_2, \{[0.43, 0.56], [-0.83, -0.71]\}, \{0.71, -0.26\}\rangle, \\ \langle \varsigma_3, \{[0.15, 0.21], [-0.49, -0.39]\}, \{0.33, -0.51\}\rangle \end{array} \right\}$$

Theorem 3.5 *Let $\mathcal{K} = \langle \mathcal{M}, \mathcal{N} \rangle$ be a CBFS on V, which is not ECBFS then $\exists\ \varsigma \in V$ such that $\rho^P_{\mathcal{K}_\varepsilon} \in \big(\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ and $\rho^N_{\mathcal{K}_\varepsilon} \in \big(\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$.*

Proof Obvious. □

Theorem 3.6 *Let $\mathcal{K} = \langle \mathcal{M}, \mathcal{N} \rangle$ be a CBFS on V. If $\mathcal{K}$ is an ICBFS as well as ECBFS then $\rho^P_{\mathcal{K}_\varepsilon} \in \big(\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma) \cup \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ and $\rho^N_{\mathcal{K}_\varepsilon} \in \big(\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma) \cup \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ for all $\varsigma \in V$.*

Proof By definition of ICBFS and ECBFS, we have $\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq \rho^P_{\mathcal{K}_\varepsilon} \leq \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)$ and $\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq \rho^N_{\mathcal{K}_\varepsilon} \leq \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)$, $\rho^P_{\mathcal{K}_\varepsilon} \notin \big(\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ and $\rho^N_{\mathcal{K}_\varepsilon} \notin \big(\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ which shows that $\rho^P_{\mathcal{K}_\varepsilon} = \delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma)$ or $\rho^P_{\mathcal{K}_\varepsilon} = \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)$. Similarly, $\rho^N_{\mathcal{K}_\varepsilon} = \delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma)$ or $\rho^N_{\mathcal{K}_\varepsilon} = \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)$. Thus $\rho^P_{\mathcal{K}_\varepsilon} \in \big(\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma) \cup \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ *and* $\rho^N_{\mathcal{K}_\varepsilon} \in \big(\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma) \cup \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)\big)$ for all $\varsigma \in V$. □

Theorem 3.7 *Let $\mathcal{K} = \langle \mathcal{M}, \mathcal{N} \rangle$ be a CBFS, which is both ICBFS as well as ECBFS then $\mathcal{K}^c = \langle \mathcal{M}, \mathcal{N} \rangle$ is also an ICBFS and ECBFS.*

Proof By definition of ICBFS and ECBFS, we have $\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq \rho^P_{\mathcal{K}_\varepsilon} \leq \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma)$, $\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma) \leq \rho^N_{\mathcal{K}_\varepsilon} \leq \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma)$ and $\rho^P_{\mathcal{K}_\varepsilon} \notin (\delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma))$, $\rho^N_{\mathcal{K}_\varepsilon} \notin (\delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma), \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma))$ now,

$$1 - \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma) \leq 1 - \rho^P_{\mathcal{K}_\varepsilon} \leq 1 - \delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma), \ -1 - \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma) \leq -1 - \rho^N_{\mathcal{K}_\varepsilon} \leq -1 - \delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma)$$
$$1 - \rho^P_{\mathcal{K}_\varepsilon} \notin (1 - \delta^P_{\mathcal{K}u_\varepsilon}(\varsigma), 1 - \delta^P_{\mathcal{K}\ell_\varepsilon}(\varsigma)), \ -1 - \rho^N_{\mathcal{K}_\varepsilon} \notin (-1 - \delta^N_{\mathcal{K}u_\varepsilon}(\varsigma), -1 - \delta^N_{\mathcal{K}\ell_\varepsilon}(\varsigma))$$

By definition of compliment of CBFS, $\mathcal{K}^c$ is also ICBFS and ECBFS for all $\varsigma \in V$. □

Remark If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then

(i) $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ may not to be an ICBFS.
(ii) $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ may not to be an ICBFS.
(iii) $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ may not to be an ECBFS.
(iv) $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ may not to be an ECBFS.
(v) $\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2$ and $\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2$ may not to be ICBFSs.

The supporting counter examples for above remark are given below.

Example 3.8 If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ may not to be an ICBFS.

$\mathcal{K}_1 = \{\langle \varsigma_1, \{[0.23, 0.33], [-0.51, -0.41]\}, \{0.41, -0.25\} \rangle\}$,
$\mathcal{K}_2 = \{\langle \varsigma_1, \{[0.21, 0.31], [-0.50, -0.40]\}, \{0.40, -0.24\} \rangle\}$ then
$\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.23, 0.33], [-0.51, -0.41]\}, \{0.41, -0.25\} \rangle\}$ is not an ICBFS.

Example 3.9 If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ may not to be an ICBFS.

$\mathcal{K}_1 = \{\langle \varsigma_1, \{[0.23, 0.33], [-0.51, -0.41]\}, \{0.41, -0.25\} \rangle\}$,
$\mathcal{K}_2 = \{\langle \varsigma_1, \{[0.21, 0.31], [-0.50, -0.40]\}, \{0.40, -0.24\} \rangle\}$ then
$\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.21, 0.31], [-0.50, -0.40]\}, \{0.40, -0.24\} \rangle\}$ is not an ICBFS.

Example 3.10 If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ may not to be an ECBFS.

$\mathcal{K}_1 = \{\langle \varsigma_1, \{[0.42, 0.81], [-0.56, -0.21]\}, \{0.39, -0.70\} \rangle\}$,
$\mathcal{K}_2 = \{\langle \varsigma_1, \{[0.51, 0.61], [-0.46, -0.31]\}, \{0.93, -0.25\} \rangle\}$ then
$\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.51, 0.81], [-0.56, -0.31]\}, \{0.93, -0.70\} \rangle\}$ is not an ECBFS.

Example 3.11 If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ may not to be an ECBFS.

$\mathcal{K}_1 = \{\langle \varsigma_1, \{[0.42, 0.81], [-0.56, -0.26]\}, \{0.39, -0.70\} \rangle\}$,
$\mathcal{K}_2 = \{\langle \varsigma_1, \{[0.51, 0.61], [-0.46, -0.46]\}, \{0.93, -0.25\} \rangle\}$ then
$\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.42, 0.61], [-0.46, -0.26]\}, \{0.39, -0.25\} \rangle\}$ is not an ECBFS.

Example 3.12 If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ECBFSs on V. Then $\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2$ and $\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2$ may not to be ICBFSs.

$\mathcal{K}_1 = \{\langle \varsigma_1, \{[0.21, 0.58], [-0.73, -0.61]\}, \{0.62, -0.21\} \rangle\}$,
$\mathcal{K}_2 = \{\langle \varsigma_1, \{[0.52, 0.31], [-0.83, -0.41]\}, \{0.72, -0.11\} \rangle\}$ then

$$\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.52, 0.58], [-0.83, -0.61]\}, \{0.62, -0.11\} \rangle\}$$
$$\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2 = \{\langle \varsigma_1, \{[0.21, 0.31], [-0.73, -0.41]\}, \{0.72, -0.21\} \rangle\}$$

are not ICBFSs.

Theorem 3.13 *If $\mathcal{K}_1 = \langle \mathcal{M}_1, \mathcal{N}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ be two ICBFSs on V. Then $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ and $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ are also ICBFSs on V.*

Proof By definition of ICBFS, for $\mathcal{K}_1$, we have $\delta^P_{\mathcal{K}\ell_1}(\varsigma) \le \rho^P_{\mathcal{K}_1} \le \delta^P_{\mathcal{K}u_1}(\varsigma)$, $\delta^N_{\mathcal{K}\ell_1}(\varsigma) \le \rho^N_{\mathcal{K}_1} \le \delta^N_{\mathcal{K}u_1}(\varsigma)$ similarly for $\mathcal{K}_2$, we get $\delta^P_{\mathcal{K}\ell_2}(\varsigma) \le \rho^P_{\mathcal{K}_2} \le \delta^P_{\mathcal{K}u_2}(\varsigma)$, $\delta^N_{\mathcal{K}\ell_2}(\varsigma) \le \rho^N_{\mathcal{K}_2} \le \delta^N_{\mathcal{K}u_2}(\varsigma)$ on taking $\mathcal{P}$-Union,

$$\max[\delta^P_{\mathcal{K}\ell_1}(\varsigma), \delta^P_{\mathcal{K}\ell_2}(\varsigma)] \le \max[\rho^P_{\mathcal{K}_1}, \rho^P_{\mathcal{K}_2}] \le \max[\delta^P_{\mathcal{K}u_1}(\varsigma), \delta^P_{\mathcal{K}u_2}(\varsigma)],$$
$$\min[\delta^N_{\mathcal{K}\ell_1}(\varsigma), \delta^N_{\mathcal{K}\ell_2}(\varsigma)] \le \min[\rho^N_{\mathcal{K}_1}, \rho^N_{\mathcal{K}_2}] \le \min[\delta^N_{\mathcal{K}u_1}(\varsigma), \delta^N_{\mathcal{K}u_2}(\varsigma)]$$

similarly $\mathcal{P}$-Intersection,

$$\min[\delta^P_{\mathcal{K}\ell_1}(\varsigma), \delta^P_{\mathcal{K}\ell_2}(\varsigma)] \leq \min[\rho^P_{\mathcal{K}_1}, \rho^P_{\mathcal{K}_2}] \leq \min[\delta^P_{\mathcal{K}u_1}(\varsigma), \delta^P_{\mathcal{K}u_2}(\varsigma)],$$
$$\max[\delta^N_{\mathcal{K}\ell_1}(\varsigma), \delta^N_{\mathcal{K}\ell_2}(\varsigma)] \leq \max[\rho^N_{\mathcal{K}_1}, \rho^N_{\mathcal{K}_2}] \leq \max[\delta^N_{\mathcal{K}u_1}(\varsigma), \delta^N_{\mathcal{K}u_2}(\varsigma)]$$

So, $\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2$ and $\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2$ are ICBFSs on V. □

Example 3.14 Consider $\mathcal{K}_1 = \{\langle\varsigma_1, \{[0.32, 0.60], [-0.50, -0.42]\}, \{0.38, -0.48\}\rangle\}$, and $\mathcal{K}_2 = \{\langle\varsigma_2, \{[0.41, 0.55], [-0.56, -0.41]\}, \{0.47, -0.50\}\rangle\}$, then

$$\mathcal{K}_1 \cup_{\mathcal{P}} \mathcal{K}_2 = \{\langle\varsigma_1, \{[0.41, 0.60], [-0.56, -0.42]\}, \{0.47, -0.50\}\rangle\}$$
$$\mathcal{K}_1 \cap_{\mathcal{P}} \mathcal{K}_2 = \{\langle\varsigma_1, \{[0.32, 0.55], [-0.50, -0.41]\}, \{0.38, -0.48\}\rangle\}$$

are ICBFSs on V.

The following counter example shows that the $\mathcal{R}$-Union and $\mathcal{R}$-Intersection of two ICBFSs may not be an ICBFS.

Example 3.15 If $\mathcal{K}_1 = \langle\mathcal{M}_1, \mathcal{N}_1\rangle$ and $\mathcal{K}_2 = \langle\mathcal{M}_2, \mathcal{N}_2\rangle$ be two ICBFSs on V. Then $\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2$ and $\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2$ may not to be ICBFSs.
$\mathcal{K}_1 = \{\langle\varsigma_1, \{[0.23, 0.37], [-0.73, -0.61]\}, \{0.30, -0.70\}\rangle\}$,
$\mathcal{K}_2 = \{\langle\varsigma_1, \{[0.51, 0.59], [-0.62, -0.54]\}, \{0.55, -0.59\}\rangle\}$ then

$$\mathcal{K}_1 \cup_{\mathcal{R}} \mathcal{K}_2 = \{\langle\varsigma_1, \{[0.51, 0.59], [-0.73, -0.61]\}, \{0.30, -0.59\}\rangle\}$$
$$\mathcal{K}_1 \cap_{\mathcal{R}} \mathcal{K}_2 = \{\langle\varsigma_1, \{[0.23, 0.37], [-0.62, -0.54]\}, \{0.55, -0.70\}\rangle\}$$

are not ICBFSs.

The following properties holds for all CBFSs, including, ICBFSs, and ECBFSs.

Theorem 3.16 *Let $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ be three CBFSs, then the following properties are hold under $\mathcal{P}$-Order as well as $\mathcal{R}$-Order.*

(i) $\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_1 = \mathcal{K}_1$
(ii) $\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_1 = \mathcal{K}_1$
(iii) $(\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_3 = \mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_3)$
(iv) $(\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_3 = \mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_3)$
(v) $(\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_3 = \mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_3)$
(vi) $(\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_3 = \mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_3)$
(vii) $\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cap_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$
(viii) $\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cup_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$
(ix) $\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cup_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$
(x) $\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cap_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \oplus_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$
(xi) $\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cup_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cup_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$
(xii) $\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} (\mathcal{K}_2 \cap_{\mathcal{P},\mathcal{R}} \mathcal{K}_3) = (\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2) \cap_{\mathcal{P},\mathcal{R}} (\mathcal{K}_1 \otimes_{\mathcal{P},\mathcal{R}} \mathcal{K}_2)$

Proof The proofs can be obtained by applying definitions. □

4 Cubic bipolar fuzzy ordered weighted geometric aggregation operators for CBFSs

In this section we define two aggregation operators for CBFSs. We define cubic bipolar fuzzy ordered weighted geometric aggregation operators under the operations of $\mathcal{P}$-Order and $\mathcal{R}$-Order. We discuss some certain features of both operators. We also examine capability and reliability of these operators in aggregating the ICBF and ECBF data.

4.1 Score and accuracy functions of CBFSs

To compare the cubic bipolar fuzzy elements(CBFEs) we use the score function. After applying proposed algorithm and CBFOWG-operators, we obtain different values for different alternatives using score function if two or more values are same then we use accuracy function for those particular alternatives. The score function is defined as

Definition 4.1 (Riaz and Tehrim 2019a) The score function for CBFEs under $\mathcal{P}$-Order can be computed as

$$\mathcal{S}_{\mathcal{P}}(\mathcal{K}) = \frac{[\delta^P_{\mathcal{K}\ell}(\varsigma) + \delta^P_{\mathcal{K}u}(\varsigma)] + [\delta^N_{\mathcal{K}\ell}(\varsigma) + \delta^N_{\mathcal{K}u}(\varsigma)] - \rho^P_{\mathcal{K}}(\varsigma) - \rho^N_{\mathcal{K}}(\varsigma)}{6}$$

where $\mathcal{S}_{\mathcal{P}}(\mathcal{K}) \in [-1, 1]$. If $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_1)\rangle\mathcal{S}_{\mathcal{P}}(\mathcal{K}_2)$, then $\mathcal{K}_1 \succ \mathcal{K}_2$. However, If $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_1) = \mathcal{S}_{\mathcal{P}}(\mathcal{K}_2)$, then $\mathcal{A}(\mathcal{K}_1) = \mathcal{A}(\mathcal{K}_2)$ implies $\mathcal{K}_1 = \mathcal{K}_2$, where $\mathcal{A}(\mathcal{K})$ is defined in definition 4.2. The score function for CBFEs under $\mathcal{R}$-Order can be computed as

$$\mathcal{S}_{\mathcal{R}}(\mathcal{K}) = \frac{[\delta^P_{\mathcal{K}\ell}(\varsigma) + \delta^P_{\mathcal{K}u}(\varsigma)] + [\delta^N_{\mathcal{K}\ell}(\varsigma) + \delta^N_{\mathcal{K}u}(\varsigma)] + \rho^P_{\mathcal{K}}(\varsigma) + \rho^N_{\mathcal{K}}(\varsigma)}{6}$$

where $\mathcal{S}_{\mathcal{R}}(\mathcal{K}) \in [-1, 1]$. If $\mathcal{S}_{\mathcal{R}}(\mathcal{K}_1)\rangle\mathcal{S}_{\mathcal{R}}(\mathcal{K}_2)$, then $\mathcal{K}_1 \succ \mathcal{K}_2$. However, If $\mathcal{S}_{\mathcal{R}}(\mathcal{K}_1) = \mathcal{S}_{\mathcal{R}}(\mathcal{K}_2)$, then $\mathcal{A}(\mathcal{K}_1) = \mathcal{A}(\mathcal{K}_2)$ implies $\mathcal{K}_1 = \mathcal{K}_2$. Where $\mathcal{A}(\mathcal{K})$ is defined in definition 4.2.

Here "$\succ$" denote the symbol of ranking or preference between alternatives.

Remark If $\mathcal{K}_1 \succ \mathcal{K}_2$ then $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_1)\rangle\mathcal{S}_{\mathcal{P}}(\mathcal{K}_2)$ or $\mathcal{S}_{\mathcal{R}}(\mathcal{K}_1)\rangle\mathcal{S}_{\mathcal{R}}(\mathcal{K}_2)$ may or may not hold.

Definition 4.2 (Riaz and Tehrim 2019a) The accuracy function for CBFEs can be computed as

$$\mathcal{A}(\mathcal{K}) = \frac{[\delta^P_{\mathcal{K}\ell}(\varsigma) + \delta^P_{\mathcal{K}u}(\varsigma)] - [\delta^N_{\mathcal{K}\ell}(\varsigma) + \delta^N_{\mathcal{K}u}(\varsigma)] + \rho^P_{\mathcal{K}}(\varsigma) - \rho^N_{\mathcal{K}}(\varsigma)}{6}$$

where $\mathcal{A}(\mathcal{K}) \in [0, 1]$. If $\mathcal{A}(\mathcal{K}_1)\rangle\mathcal{A}(\mathcal{K}_2)$, then $\mathcal{K}_1 \succ \mathcal{K}_2$. If $\mathcal{A}(\mathcal{K}_1) = \mathcal{A}(\mathcal{K}_2)$, then $\mathcal{K}_1 = \mathcal{K}_2$.

Example 4.3 Let us consider some CBFEs

$$\mathfrak{B} = \begin{bmatrix} \varsigma_1 = \{[0.45, 0.57], [-0.83, -0.53], \{0.52, -0.78\}\} \\ \varsigma_2 = \{[0.34, 0.50], [-0.65, -0.49], \{0.39, -0.61\}\} \end{bmatrix}$$

then $\mathcal{S}_{\mathcal{P}}(\varsigma_1) = -0.013$,and $\mathcal{S}_{\mathcal{P}}(\varsigma_2) = -0.08$ which shows that $\varsigma_1 \succ \varsigma_2$ using $\mathcal{S}_{\mathcal{P}}(\mathcal{K})$. Similarly, $\mathcal{S}_{\mathcal{R}}(\varsigma_1) = -0.01$,and $\mathcal{S}_{\mathcal{R}}(\varsigma_2) = -0.08$ which shows that $\varsigma_1 \succ \varsigma_2$ using $\mathcal{S}_{\mathcal{R}}(\mathcal{K})$

4.2 $\mathcal{P}$-order cubic bipolar fuzzy ordered weighted geometric($\mathcal{P}$-CBFWG) operator

Definition 4.4 Suppose $\mathcal{K}_\varepsilon = \{\langle\{[\delta^P_{\mathcal{K}\ell_\varepsilon}, \delta^P_{\mathcal{K}u_\varepsilon}], [\delta^N_{\mathcal{K}\ell_\varepsilon}, \delta^N_{\mathcal{K}u_\varepsilon}]\}, \{\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon}\}\rangle\}$, for $\{\varepsilon = 1, 2, \ldots, n\}$, be a family of CBFEs. Then consider a weighted vector $\Upsilon = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_3\}^T$, where $\Sigma^n_{\varepsilon=1}\mathfrak{w}_\varepsilon = 1$ and $0 \le \mathfrak{w}_\varepsilon \le 1$. Let o_ε be an alteration which shows the greatest $\mathcal{K}_\varepsilon$ then cubic bipolar fuzzy ordered weighted geometric(CBFOWG) aggregation operator is a mapping $\mathfrak{A} : \mathcal{K}^n \to \mathcal{K}$ which can be computed under $\mathcal{P}$-Order as follows

$$\mathcal{P}\text{-}CBFOWG(\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n) = \left\{\bigotimes_{\varepsilon=1}^{n}\right\}_{\mathcal{P}} (\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon}))$$
$$= \left\{\left\{\left[\prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right], \left[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^N_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\right.\right.\right.$$
$$\left.\left.-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^N_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right]\right\}, \left\{\prod_{\varepsilon=1}^{n}\left(\rho^P_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^N_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}\right\}$$

Example 4.5 Let us consider CBFEs $\mathcal{K}_1 = \{[0.42, 0.53], [-0.45, -0.34], \{0.45, -0.39\}\}$, $\mathcal{K}_2 = \{[0.66, 0.73], [-0.53, -0.40], \{0.68, -0.45\}\}$,$\mathcal{K}_3 = \{[0.45, 0.67], [-0.62, -0.34], \{0.48, -0.51\}\}$ and$\mathcal{K}_4 = \{[0.23, 0.61], [-0.71, -0.60], \{0.34, -0.63\}\}$ with weight vector $\Upsilon = \{0.1, 0.2, 0.3, 0.4\}^T$. Now we rearrange the CBFEs using score function under $\mathcal{P}$-Order. By applying score function, we obtain $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_1) = 0.016$, $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_2) = 0.0383$, $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_3) = 0.0316$, $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_4) = -0.030$. This shows that $\mathcal{S}_{\mathcal{P}}(\mathcal{K}_2)\rangle\mathcal{S}_{\mathcal{P}}(\mathcal{K}_3)\rangle\mathcal{S}_{\mathcal{P}}(\mathcal{K}_1)\rangle\mathcal{S}_{\mathcal{P}}(\mathcal{K}_4)$ this implies that $\mathcal{K}_2 \succ \mathcal{K}_3 \succ \mathcal{K}_1 \succ \mathcal{K}_4$. Now we obtain the following ordered sequence of CBFEs $\mathcal{K}_{o1} = \{[0.66, 0.73], [-0.53, -0.40], \{0.68, -0.45\}\}$, $\mathcal{K}_{o2} = \{[0.45, 0.67], [-0.62, -0.34], \{0.48, -0.51\}\}$, $\mathcal{K}_{o3} = \{[0.42, 0.53], [-0.45, -0.34], \{0.45, -0.39\}\}$ and $\mathcal{K}_{o4} = \{[0.23, 0.61], [-0.71, -0.60], \{0.34, -0.63\}\}$ $\Pi^4_{\varepsilon=1}\left(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = \left((0.66)^{0.1} \times (0.45)^{0.2} \times (0.42)^{0.3} \times (0.23)^{0.4}\right) = 0.350$

$$\prod_{\varepsilon=1}^{4}\left(\delta^P_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = \left((0.73)^{0.1} \times (0.67)^{0.2} \times (0.53)^{0.3} \times (0.61)^{0.4}\right) = 0.606$$
$$\prod_{\varepsilon=1}^{4}\left(1-\delta^N_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = (1-0.53)^{0.1} \times (1-0.62)^{0.2} \times (1-0.45)^{0.3} \times (1-0.71)^{0.4} = -0.610$$
$$\prod_{\varepsilon=1}^{4}\left(1-\delta^N_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = (1-0.40)^{0.1} \times (1-0.34)^{0.2} \times (1-0.34)^{0.3} \times (1-0.60)^{0.4} = -0.464$$
$$\prod_{\varepsilon=1}^{4}\left(\rho^P_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = \left((0.68)^{0.1} \times (0.48)^{0.2} \times (0.45)^{0.3} \times (0.34)^{0.4}\right) = 0.424$$
$$\prod_{\varepsilon=1}^{4}\left(1-\rho^N_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} = (1-0.45)^{0.1} \times (1-0.51)^{0.2} \times (1-0.39)^{0.3} \times (1-0.63)^{0.4} = -0.526$$
$$\mathcal{P}\text{-CBFOWG}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4) = \{[0.350, 0.606], [-0.610, -0.464], \{0.424, -0.526\}\}$$

Theorem 4.6 *Suppose that* $\mathcal{K}_\varepsilon = \{\langle\{[\delta^P_{\mathcal{K}\ell_\varepsilon}, \delta^P_{\mathcal{K}u_\varepsilon}], [\delta^N_{\mathcal{K}\ell_\varepsilon}, \delta^N_{\mathcal{K}u_\varepsilon}]\}, \{\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon}\}\rangle\}$ *be a collection of CBFEs. Let* $\Upsilon = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n\}$ *be weight vector and* $0 \leq \mathfrak{w}_\varepsilon \leq 1 \mid \Sigma^n_{\varepsilon=1} = 1$, *then* $\mathcal{P}$*-CBFOWG operator is Idempotent, bounded and monotonic.*

Proof **Idempotent:**
For $\mathcal{K}_\varepsilon = \mathcal{K}$, we will prove that $\mathcal{P}$-CBFOWG$(\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n) = \mathcal{K}$. By definition of $\mathcal{P}$-CBFOWG, we have

$$\mathcal{P}\text{-CBFOWG}(\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n) = \left\{\bigotimes_{\varepsilon=1}^{n}\right\}_{\mathcal{P}} (\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon}))$$
$$= \left\{\left\{\left[\prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right], \left[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^N_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\right.\right.\right.$$
$$\left.\left.-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^N_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right]\right\},$$
$$\left.\left\{\prod_{\varepsilon=1}^{n}\left(\rho^P_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^N_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}\right\}$$
$$= \left\{\left\{\left[\prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}\ell o}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\delta^P_{\mathcal{K}uo}\right)^{\mathfrak{w}_\varepsilon}\right], \left[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^N_{\mathcal{K}\ell o})\right)^{\mathfrak{w}_\varepsilon}\right),\right.\right.\right.$$

$$
\begin{aligned}
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}uo})\right)^{\mathfrak{w}_\varepsilon}\right)\Big]\Big\},\\
&\Big\{\prod_{\varepsilon=1}^{n}\left(\rho^{P}_{\mathcal{K}o}\right)^{\mathfrak{w}_\varepsilon}, -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^{N}_{\mathcal{K}o})\right)^{\mathfrak{w}_\varepsilon}\right)\Big\}\Big\}\\
&=\Big\{\Big\{\Big[\left(\delta^{P}_{\mathcal{K}\ell_o}\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}, \left(\delta^{P}_{\mathcal{K}u_o}\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}\Big], \Big[-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}\ell_o})\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}\right),\\
&-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_o})\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}\right)\Big]\Big\},\\
&\Big\{\left(\rho^{P}_{\mathcal{K}_o}\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}, -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_o})\right)^{\sum_{\varepsilon=1}^{n}\mathfrak{w}_\varepsilon}\right)\Big\}\Big\}\\
&=\mathcal{K}
\end{aligned}
$$

Bounded:
$\cap_{\mathcal{P}}\mathcal{P}$-CBFOWG$(\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_n) \le \mathcal{P}-CBFOWG(\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_n) \le \cup_{\mathcal{P}}\mathcal{P}-CBFOWG(\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_n)$, that is every CBFE is bounded between the operation $\cap_{\mathcal{P}}$ and $\cup_{\mathcal{P}}$. Consider

$$
\begin{aligned}
&\mathcal{P}\text{-CBFOWG}(\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_n)\\
&=\Big\{\Big\{\Big[\prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\Big], \Big[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\\
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big]\Big\},\\
&\Big\{\prod_{\varepsilon=1}^{n}\left(\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big\}\Big\}\\
&\cap_{\mathcal{P}}\mathcal{P}\text{-CBFOWG}(\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_n)\\
&=\Big\{\Big\{\Big[\prod_{\varepsilon=1}^{n}\left(\min\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\min\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\Big], \Big[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\max(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\\
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\max(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big]\Big\}, \Big\{\prod_{\varepsilon=1}^{n}\left(\min\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon},\\
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\max(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big\}\Big\}
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\cup_{\mathcal{P}}\mathcal{P}-CBFOWG(\mathcal{K}_1,\mathcal{K}_2,v,\mathcal{K}_n)\\
&=\Big\{\Big\{\Big[\prod_{\varepsilon=1}^{n}\left(\max\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\left(\max\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\Big], \Big[-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\\
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big]\Big\}, \Big\{\prod_{\varepsilon=1}^{n}\left(\max\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon},\\
&-\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\Big\}\Big\}
\end{aligned}
$$

we have,

$$
\begin{aligned}
&\left\{ \min(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}) \leq \delta^{P}_{\mathcal{K}\ell_{o\varepsilon}} \leq \max(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}})\right\} \\
&\quad \Rightarrow \left\{ \prod_{\varepsilon=1}^{n}\left(\min(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\max(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \right\}
\end{aligned}
$$

A similar computation gives,

$$
\begin{aligned}
&\left\{ \prod_{\varepsilon=1}^{n}\left(\min(\delta^{P}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\max(\delta^{P}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \right\} \\
&\left\{ -\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \right.\\
&\left. \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-\max(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\} \\
&\left\{ -\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \leq -\left(\prod_{\varepsilon=1}^{n}\left(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right) \right.\\
&\left. \leq -\left(\prod_{\varepsilon=1}^{n}\left(1-\max(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\} \\
&\left\{ \prod_{\varepsilon=1}^{n}\left(\min(\rho^{P}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\max(\rho^{P}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{ -\left(1-\prod_{\varepsilon=1}^{n}\left(1-\min(-\rho^{N}_{o\mathcal{K}_\varepsilon})\right)^{\mathfrak{w}_\varepsilon}\right) \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \right.\\
&\quad \left. \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-\max(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}
\end{aligned}
$$

If the CBFEs satisfied property of idempotent then the equality does holds.

Monotonic:
Consider $\mathcal{K}_\varepsilon$ and $\mathcal{K}'_\varepsilon$ two collections of CBFEs then for $\mathcal{K}_\varepsilon \leq \mathcal{K}'_\varepsilon$, we get $\mathcal{P}$-CBFOWG$(\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n) \leq \mathcal{P}$-CBFOWG$(\mathcal{K}'_1, \mathcal{K}_2', \ldots, \mathcal{K}'_n)$. We have, $\mathcal{K}_\varepsilon \leq \mathcal{K}'_\varepsilon$

$$
\begin{aligned}
&\Rightarrow \left\{(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}) \leq (\delta^{P}_{\mathcal{K}'\ell_{o\varepsilon}})\right\} \Rightarrow \left\{ \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left((\delta^{P}_{\mathcal{K}'\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right\} \\
&\Rightarrow \left\{ \prod_{\varepsilon=1}^{n}\left((\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left((\delta^{P}_{\mathcal{K}'\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right\}
\end{aligned}
$$

A similar computation gives,

$$
\begin{aligned}
&\left\{ \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \leq \prod_{\varepsilon=1}^{n}\left(\delta^{P}_{\mathcal{K}'u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right\} \\
&\left\{ -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}'\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\} \\
&\left\{ -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \leq -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\delta^{N}_{\mathcal{K}'u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}
\end{aligned}
$$

$$\left\{\prod_{\varepsilon=1}^{n}\left(\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon} \le \prod_{\varepsilon=1}^{n}\left(\rho^{P}_{\mathcal{K'}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right\} \text{ and } \left\{-\left(\prod_{\varepsilon=1}^{n}\left(1-(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right) \le -\left(1-\prod_{\varepsilon=1}^{n}\left(1-(-\rho^{N}_{\mathcal{K'}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}$$

Thus $\mathcal{P}$-CBFOWG is monotonic. □

Theorem 4.7 *Let $\mathcal{K}_\varepsilon = \{\langle\{[\delta^{P}_{\mathcal{K}\ell_\varepsilon}, \delta^{P}_{\mathcal{K}u_\varepsilon}], [\delta^{N}_{\mathcal{K}\ell_\varepsilon}, \delta^{N}_{\mathcal{K}u_\varepsilon}]\}, \{\rho^{P}_{\mathcal{K}_\varepsilon}, \rho^{N}_{\mathcal{K}_\varepsilon}\}\rangle\}$, for $\{\varepsilon = 1, 2, \ldots, n\}$, be the collection of CBFEs. Then their aggregating value computed by $\mathcal{P}$-CBFOWG is again a CBFE.*

Proof Let $\mathcal{K}_{o\varepsilon}$ be the collection of CBFEs after applying $\mathcal{S}_{\mathcal{P}}$, where o is alteration of alternatives shows the greatest $\mathcal{K}_\varepsilon$. We prove this by mathematical induction, suppose that for $n = 2$

$$\mathfrak{w}_1\mathcal{K}_{o1} = \left\{\left\{\left[\left(\delta^{P}_{\mathcal{K}\ell_{o1}}\right)^{\mathfrak{w}_1}, \left(\delta^{P}_{\mathcal{K}u_{o1}}\right)^{\mathfrak{w}_1}\right], \left[-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o1}})\right)^{\mathfrak{w}_1}\right),\right.\right.\right.$$
$$\left.\left.\left. -\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_{o1}})\right)^{\mathfrak{w}_1}\right)\right]\right\}, \left\{\left(\rho^{P}_{\mathcal{K}_{o1}}\right)^{\mathfrak{w}_1}, -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_{o1}})\right)^{\mathfrak{w}_1}\right)\right\}\right\}$$
$$\mathfrak{w}_2\mathcal{K}_{o2} = \left\{\left\{\left[\left(\delta^{P}_{\mathcal{K}\ell_{o2}}\right)^{\mathfrak{w}_2}, \left(\delta^{P}_{\mathcal{K}u_{o2}}\right)^{\mathfrak{w}_2}\right], \left[-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o2}})\right)^{\mathfrak{w}_2}\right),\right.\right.\right.$$
$$\left.\left.\left. -\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_{o2}})\right)^{\mathfrak{w}_2}\right)\right]\right\}, \left\{\left(\rho^{P}_{\mathcal{K}_{o2}}\right)^{\mathfrak{w}_2}, -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_{o2}})\right)^{\mathfrak{w}_2}\right)\right\}\right\}$$
$$\mathfrak{w}_1\mathcal{K}_{o1} \otimes_{\mathcal{P}} \mathfrak{w}_2\mathcal{K}_{o2} = \left\{\left\{\left[\left(\delta^{P}_{\mathcal{K}\ell_{o1}}\right)^{\mathfrak{w}_1}, \left(\delta^{P}_{\mathcal{K}u_{o1}}\right)^{\mathfrak{w}_1}\right], \left[-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o1}})\right)^{\mathfrak{w}_1}\right),\right.\right.\right.$$
$$\left.\left.\left. -\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_{o1}})\right)^{\mathfrak{w}_1}\right)\right]\right\}, \left\{\left(\rho^{P}_{\mathcal{K}_{o1}}\right)^{\mathfrak{w}_1}, -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_{o1}})\right)^{\mathfrak{w}_1}\right)\right\}\right\} \otimes_{\mathcal{P}}$$
$$\left\{\left\{\left[\left(\delta^{P}_{\mathcal{K}\ell_{o2}}\right)^{\mathfrak{w}_2}, \left(\delta^{P}_{\mathcal{K}u_{o2}}\right)^{\mathfrak{w}_2}\right], \left[-\left(1-\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o2}})\right)^{\mathfrak{w}_2}\right), -\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_{o2}})\right)^{\mathfrak{w}_2}\right)\right]\right\},\right.$$
$$\left.\left\{\left(\rho^{P}_{\mathcal{K}_{o2}}\right)^{\mathfrak{w}_2}, -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_{o2}})\right)^{\mathfrak{w}_2}\right)\right\}\right\}$$
$$\mathfrak{w}_1\mathcal{K}_{o1} \otimes_{\mathcal{P}} \mathfrak{w}_2\mathcal{K}_{o2} = \left\{\left\{\left[\left(\delta^{P}_{\mathcal{K}\ell_{o1}}\right)^{\mathfrak{w}_1}\left(\delta^{P}_{\mathcal{K}\ell_{o2}}\right)^{\mathfrak{w}_2}, \left(\delta^{P}_{\mathcal{K}u_{o1}}\right)^{\mathfrak{w}_1}\left(\delta^{P}_{\mathcal{K}u_{o2}}\right)^{\mathfrak{w}_2}\right],\right.\right.$$
$$\left[-\left(1-\left(-1(-\delta^{N}_{\mathcal{K}\ell_{o1}})\right)^{\mathfrak{w}_1}\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o2}})\right)^{\mathfrak{w}_2}\right), -\left(1-\left(1-(-\delta^{N}_{\mathcal{K}u_{o1}})\right)^{\mathfrak{w}_1}\right.\right.$$
$$\left.\left.\left.\left(1-(-\delta^{N}_{\mathcal{K}u_{o2}})\right)^{\mathfrak{w}_2}\right)\right]\right\}, \left\{\left(\rho^{P}_{\mathcal{K}_{o1}}\right)^{\mathfrak{w}_1}\left(\rho^{P}_{\mathcal{K}_{o2}}\right)^{\mathfrak{w}_2},\right.$$
$$\left.\left. -\left(1-\left(1-(-\rho^{N}_{\mathcal{K}_{o1}})\right)^{\mathfrak{w}_1}\left(1-(-\rho^{N}_{\mathcal{K}_{o2}})\right)^{\mathfrak{w}_2}\right)\right\}\right\}$$

suppose the statement is true for $n = z$

$$\left\{\bigotimes_{\varepsilon=1}^{z}\right\}_{\mathcal{P}}(\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon}))$$
$$= \left\{\left\{\left[\prod_{\varepsilon=1}^{z}\left(\delta^{P}_{\mathcal{K}\ell_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{z}\left(\delta^{P}_{\mathcal{K}u_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}\right], \left[-\left(1-\prod_{\varepsilon=1}^{z}\left(1-(-\delta^{N}_{\mathcal{K}\ell_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right),\right.\right.\right.$$
$$\left.\left.\left. -\left(1-\prod_{\varepsilon=1}^{z}\left(1-(-\delta^{N}_{\mathcal{K}u_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right]\right\}, \left\{\prod_{\varepsilon=1}^{z}\left(\rho^{P}_{\mathcal{K}_{o\varepsilon}}\right)^{\mathfrak{w}_\varepsilon}, -\left(1-\prod_{\varepsilon=1}^{z}\left(1-(-\rho^{N}_{\mathcal{K}_{o\varepsilon}})\right)^{\mathfrak{w}_\varepsilon}\right)\right\}\right\}$$

now we will show that the statement is true for $n = z + 1$.

$$\left\{\bigotimes_{\varepsilon=1}^{z}\right\}_{\mathcal{P}} (\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon})) \otimes_{\mathcal{P}} \mathfrak{w}_{z+1}(\mathcal{K}_{oz+1})$$
$$= \Bigg\{\Bigg\{\Bigg[\prod_{\varepsilon=1}^{z}\big(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{z}\big(\delta^P_{\mathcal{K}u_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}\Bigg], \Bigg[-\Bigg(1-\prod_{\varepsilon=1}^{z}\big(1-(-\delta^N_{\mathcal{K}\ell_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg),$$
$$-\Bigg(1-\prod_{\varepsilon=1}^{z}\big(1-(-\delta^N_{\mathcal{K}u_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg]\Bigg\},$$
$$\Bigg\{\prod_{\varepsilon=1}^{z}\big(\rho^P_{\mathcal{K}_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, -\Bigg(1-\prod_{\varepsilon=1}^{z}\big(1-(-\rho^N_{\mathcal{K}_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg\}\Bigg\} \otimes_{\mathcal{P}}$$
$$\Bigg\{\Bigg\{\Bigg[\big(\delta^P_{\mathcal{K}\ell_{oz+1}}\big)^{\mathfrak{w}_{z+1}}, \big(\delta^P_{\mathcal{K}u_{oz+1}}\big)^{\mathfrak{w}_{z+1}}\Bigg],$$
$$\Bigg[-\Bigg(1-\big(1-(-\delta^N_{\mathcal{K}\ell_{oz+1}})\big)^{\mathfrak{w}_{z+1}}\Bigg), -\Bigg(1-\big(1-(-\delta^N_{\mathcal{K}u_{oz+1}})\big)^{\mathfrak{w}_{z+1}}\Bigg)\Bigg]\Bigg\}, \Bigg\{\big(\rho^P_{\mathcal{K}_{oz+1}}\big)^{\mathfrak{w}_{z+1}},$$
$$-\Bigg(1-\big(1-(-\rho^N_{\mathcal{K}_{oz+1}})\big)^{\mathfrak{w}_{z+1}}\Bigg)\Bigg\}\Bigg\}$$

thus after some simplification, we get the final expression

$$\left\{\bigotimes_{\varepsilon=1}^{z+1}\right\}_{\mathcal{P}} (\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon}))$$
$$= \Bigg\{\Bigg\{\Bigg[\prod_{\varepsilon=1}^{z+1}\big(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{z+1}\big(\delta^P_{\mathcal{K}u_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}\Bigg], \Bigg[-\Bigg(1-\prod_{\varepsilon=1}^{z+1}\big(1-(-\delta^N_{\mathcal{K}\ell_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg),$$
$$-\Bigg(1-\prod_{\varepsilon=1}^{z+1}\big(1-(-\delta^N_{\mathcal{K}u_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg]\Bigg\},$$
$$\Bigg\{\prod_{\varepsilon=1}^{z+1}\big(\rho^P_{\mathcal{K}_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, -\Bigg(1-\prod_{\varepsilon=1}^{z+1}\big(1-(-\rho^N_{\mathcal{K}_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg\}\Bigg\}$$

which is again a cubic bipolar fuzzy element. Hence the statement is true for all n. □

Now we define cubic bipolar fuzzy ordered weighted geometric aggregating operator under $\mathcal{R}$-Order, the CBFOWG operator under $\mathcal{R}$-Order share similar properties as in $\mathcal{P}$-Order.

4.3 $\mathcal{R}$-order cubic bipolar fuzzy ordered weighted geometric($\mathcal{R}$-CBFOWG) operator

Definition 4.8 Suppose $\mathcal{K}_\varepsilon = \{\langle\{[\delta^P_{\mathcal{K}\ell_\varepsilon}, \delta^P_{\mathcal{K}u_\varepsilon}], [\delta^N_{\mathcal{K}\ell_\varepsilon}, \delta^N_{\mathcal{K}u_\varepsilon}]\}, \{\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon}\}\rangle\}$, for $\{\varepsilon = 1, 2, \ldots, n\}$, be a collection of CBFEs. Let o is the alteration of alternatives. Consider a weight vector $\Upsilon = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_3\}^T$, where $\Sigma^n_{\varepsilon=1}\mathfrak{w}_\varepsilon = 1$ and $0 \le \mathfrak{w}_\varepsilon \le 1$, then CBFOWG is a mapping $\mathfrak{A} : \mathcal{K}^n \to \mathcal{K}$ which can be computed under $\mathcal{R}$-Order as follows

$$\mathcal{R}\text{-CBFOWG}(\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n) = \left\{\bigotimes_{\varepsilon=1}^{n}\right\}_{\mathcal{R}} (\mathfrak{w}_\varepsilon(\mathcal{K}_{o\varepsilon}))$$
$$= \Bigg\{\Bigg\{\Bigg[\prod_{\varepsilon=1}^{n}\big(\delta^P_{\mathcal{K}\ell_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, \prod_{\varepsilon=1}^{n}\big(\delta^P_{\mathcal{K}u_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}\Bigg], \Bigg[-\Bigg(1-\prod_{\varepsilon=1}^{n}\big(1-(-\delta^N_{\mathcal{K}\ell_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg),$$
$$-\Bigg(1-\prod_{\varepsilon=1}^{n}\big(1-(-\delta^N_{\mathcal{K}u_{o\varepsilon}})\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg]\Bigg\}, \Bigg\{1-\prod_{\varepsilon=1}^{n}\big(1-\rho^P_{\mathcal{K}_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}, -\Bigg(\prod_{\varepsilon=1}^{n}\big(-\rho^N_{\mathcal{K}_{o\varepsilon}}\big)^{\mathfrak{w}_\varepsilon}\Bigg)\Bigg\}\Bigg\}$$

Theorem 4.9 (Properties)
Let $\mathcal{K}_\varepsilon = \{\langle\{[\delta^P_{\mathcal{K}\ell_\varepsilon}, \delta^P_{\mathcal{K}u_\varepsilon}], [\delta^N_{\mathcal{K}\ell_\varepsilon}, \delta^N_{\mathcal{K}u_\varepsilon}]\}, \{\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon}\}\rangle\}$ *be the collection of CBFEs then $\mathcal{R}$-CBOFWG operator is idempotent, bounded, commutative and monotonic.*

Proof The proofs are similar as for $\mathcal{P}$-Order. □

Lemma 4.10 *Let* $\mathcal{K}_\varepsilon = \{\langle\{[\delta^P_{\mathcal{K}\ell_\varepsilon}, \delta^P_{\mathcal{K}u_\varepsilon}], [\delta^N_{\mathcal{K}\ell_\varepsilon}, \delta^N_{\mathcal{K}u_\varepsilon}]\}, \{\rho^P_{\mathcal{K}_\varepsilon}, \rho^N_{\mathcal{K}_\varepsilon}\}\rangle\}$, *for* $\{\varepsilon = 1, 2, \ldots, n\}$ *be the collection of CBFEs, then their aggregating value calculated by $\mathcal{R}$-CBFWG operator is a CBFE.*

5 Application of $\mathcal{P}$-CBFOWG and $\mathcal{R}$-CBFOWG operators to multiple attribute decision analysis using ICBFSs and ECBFSs

The multiple attribute decision analysis is a process in which a collection of different attributes or parameters are involved. All these attributes may be conflicting with each other. For example, a person want a highly efficient car in low cost, then how can be it possible to manage high quality in low price? So these are conflicting attributes. This type of decision making is more difficult when the problems involve more uncertainties and ambiguities. The role of bipolarity is also important in decision making process, which involve uncertain bipolar type information. The bipolar fuzzy sets have been successfully used to handle uncertain data which can distinguish the positive aspect as well as negative aspect of a particular problem. The cubic set is also an important and successful tool to get proper results by the mean of interval valued fuzzy set. Here we use a combination of both extensions to get proper results through this hybrid of bipolar fuzzy set.

5.1 Proposed technique

In this subsection, we discuss the process of cubic bipolar fuzzy ordered weighted geometric aggregation operators to solve a multiple attribute decision making problem under the environment ICBF and ECBF information. We see the behavior of ICBF and ECBF information in decision analysis. In this terminology the following steps are included.

Algorithm:

Step 1: Let $\mathfrak{X} = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_m\}$ be the set of alternatives $\mathfrak{Y} = \{\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_k\}$ be the set of attributes or parameters and $\dot{\Upsilon} = \{\dot{\mathfrak{w}}_1, \dot{\mathfrak{w}}_2, \ldots, \dot{\mathfrak{w}}_n\}^T$ be the weight under the condition that $0 \leq \mathfrak{w} \leq 1$ and $\sum\limits_{\varepsilon=1}^{n} \mathfrak{w}_\varepsilon = 1$.

Step 2: We further assume that $\mathfrak{M} = \left[\mathfrak{b}_{\iota\jmath}\right]_{m\times k}$, for $\iota = \{1, 2, \ldots m\}$ and $\jmath = \{1, 2, \ldots k\}$, be the decision matrix of ICBF or ECBF data provided by experts, where each $\mathfrak{b}_{\iota\jmath} = \{\{[\delta_{\ell_{\iota\jmath}}, \delta_{u_{\iota\jmath}}]^P, [\delta_{\ell_{\iota\jmath}}, \delta_{u_{\iota\jmath}}]^N\}, \{\rho^P_{\ell_{\iota\jmath}}, \rho^N_{u_{\iota\jmath}}\}\}$ is ICBF-element(ICBFE) or ECBF-element(ECBFE). These matrices are specified by the set of alternatives with respect to the set of attributes.

Step 3: Compute $\mathfrak{M}' = \left[\mathfrak{b}'_{\iota\jmath}\right]_{m\times k}$, for $\iota = \{1, 2, \ldots m\}$ and $\jmath = \{1, 2, \ldots k\}$ the matrix of ordered alternatives by calculating $\mathcal{S}_\mathcal{P}$ or $\mathcal{S}_\mathcal{R}$

Step 4: Apply $\mathcal{P}$-CBFOWG or $\mathcal{R}$-CBFOWG operator to $\mathfrak{M}' = \left[\mathfrak{b}'_{\iota\jmath}\right]_{m\times k}$ and obtain aggregating matrix $\mathfrak{M}'' = \left[\mathfrak{b}''_{\iota\jmath}\right]_{m\times k}$

Step 5: Calculate

$$\mathcal{S}_\mathcal{P}(\mathfrak{b}''_\iota) = \frac{[\delta_{\ell_\iota} + \delta_{u_\iota}]^P + [\delta_{\ell_\iota} + \delta_{u_\iota}]^N - \rho^P_{\ell_\iota} - \rho^N_{u_\iota}}{6}$$

or

$$\mathcal{S}_{\mathcal{R}}(\mathfrak{b}''_{\iota}) = \frac{[\delta_{\ell_\iota} + \delta_{u_\iota}]^P + [\delta_{\ell_\iota} + \delta_{u_\iota}]^N + \rho^P_{\ell_\iota} + \rho^N_{u_\iota}}{6}$$

Step 6: Rank the alternatives.

5.2 Computative example of $\mathcal{P}$-CBFOWG and $\mathcal{R}$-CBFWG operators in multiple attribute decision making problem

The multiple attribute decision analysis unambiguously solves numerous incompatible points of comparison in decision making in the terms such as government, business, medicine, physics, information technology, etc. These incompatible points of comparison are classical in measuring options, for example, cost is main attribute and standard is another attribute incompatible with cost. In this case a healthy and beneficial decision is more difficult. In literature, there are many decision making problems solved by different techniques and approaches. In this subsection we present a different, but useful method to tackle a problem of decision analysis. In this regard, we consider ICBFEs and ECBFEs and use CBFOWG operators of both orders to examine a real life problem. The computative example is illustrated here.

5.3 Numerical example

Suppose that the government want to assign a contract of construction to the best construction company on three attributes. The are three construction companies as three alternatives including, ς_1 , ς_2 = and ς_3 =. The three attributes are $\dot{y}_1$ = necessity of peoples, $\dot{y}_2$ = 5-year status of all three alternatives and $\dot{y}_3$ = budget limit to examine all three alternatives. The decision maker(DM) assume ICBFEs and ECBFEs because is more helpful in examine the decision with different perspectives. So, ICBFEs and ECBFEs are suitable to handle such kind of data. We associate positive membership degree with positive status of a particular alternative, whereas negative membership degrees associate with negative status of that alternative with respect to each attribute. The decision maker construct decision matrix first with ICBFEs and then with ECBFEs for alternatives ς_ι with respect to attributes $\dot{y}_\jmath$, where $\iota = 1, 2, 3$ and $\jmath = 1, 2, 3$ (Table 1).

By calculating $\mathcal{S}_{\mathcal{P}}$ we obtain the matrix of ordered alternatives (Table 2).

Table 1 $\mathfrak{M} = \left[\mathfrak{b}_{\iota\jmath}\right]_{m\times k}$ decision matrix of ICBFSs provided by DM

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.45, 0.59], [−0.92, −0.42], {0.53, −0.88}}	{[0.63, 0.73], [−0.83, −0.71], {0.68, −0.79}}	{[0.23, 0.34], [−0.37, −0.22], {0.31, −0.32}}
ς_2	{[0.28, 0.53], [−0.78, −0.60], {0.35, −0.75}}	{[0.41, 0.49], [−0.52, −0.44], {0.45, −0.49}}	{[0.39, 0.48], [−0.54, −0.39], {0.42, −0.49}}
ς_3	{[0.34, 0.48], [−0.84, −0.72], {0.41, −0.77}}	{[0.52, 0.60], [−0.63, −0.55], {0.56, −0.60}}	{[0.46, 0.57], [−0.60, −0.48], {0.22, −0.11}}

Table 2 $\mathfrak{M}' = \left[\mathfrak{b}'_{\imath\jmath}\right]_{m\times k}$ decision matrix with ordered alternatives of ICBFSs using $\mathcal{S}_{\mathcal{P}}$

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.45, 0.59], [−0.92, −0.42], {0.53, −0.88}}	{[0.23, 0.34], [−0.37, −0.22], {0.31, −0.32}}	{[0.63, 0.73], [−0.83, −0.71], {0.68, −0.79}}
ς_2	{[0.39, 0.48], [−0.54, −0.39], {0.42, −0.49}}	{[0.41, 0.49], [−0.52, −0.44], {0.45, −0.49}}	{[0.28, 0.53], [−0.78, −0.60], {0.35, −0.75}}
ς_3	{[0.52, 0.60], [−0.63, −0.55], {0.56, −0.60}}	{[0.34, 0.48], [−0.84, −0.72], {0.41, −0.77}}	{[0.46, 0.57], [−0.60, −0.48], {0.22, −0.11}}

Table 3 $\mathfrak{M}' = \left[\mathfrak{b}'_{\imath\jmath}\right]_{m\times k}$ decision matrix with ordered alternatives of ICBFSs using $\mathcal{S}_{\mathcal{R}}$

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.23, 0.34], [−0.37, −0.22], {0.31, −0.32}}	{[0.63, 0.73], [−0.83, −0.71], {0.68, −0.79}}	{[0.45, 0.59], [−0.92, −0.42], {0.53, −0.88}}
ς_2	{[0.41, 0.49], [−0.52, −0.44], {0.45, −0.49}}	{[0.39, 0.48], [−0.54, −0.39], {0.42, −0.49}}	{[0.28, 0.53], [−0.78, −0.60], {0.35, −0.75}}
ς_3	{[0.46, 0.57], [−0.60, −0.48], {0.22, −0.11}}	{[0.52, 0.60], [−0.63, −0.55], {0.56, −0.60}}	{[0.34, 0.48], [−0.84, −0.72], {0.41, −0.77}}

Now compute $\mathcal{P}$-CBFOWG to aggregate $\mathfrak{M}' = \left[\mathfrak{b}'_{\imath\jmath}\right]_{m\times k}$, with weight vector $\Upsilon = \{0.4, 0.2, 0.4\}$

$$\mathfrak{M}'' = \begin{bmatrix} \varsigma_1 = \{[0.450, 0.575], [-0.836, -0.533], \{0.526, -0.787\}\} \\ \varsigma_2 = \{[0.345, 0.501], [-0.654, -0.493], \{0.395, -0.616\}\} \\ \varsigma_3 = \{[0.454, 0.562], [-0.677, -0.566], \{0.362, -0.506\}\} \end{bmatrix}$$

by calculating $\mathcal{S}_{\mathcal{P}}$ we obtain the ranking $\varsigma_1 \succ \varsigma_3 \succ \varsigma_2$. So ς_1 is best choice.
Now we consider decision matrix of ICBFEs and perform same process under $\mathcal{R}$-Order. By calculating $\mathcal{S}_{\mathcal{R}}$ we obtain the matrix of ordered alternatives (Table 3).

Now compute $\mathcal{R}$-CBFOWG to aggregate $\mathfrak{M}' = \left[\mathfrak{b}'_{\imath\jmath}\right]_{m\times k}$, with weight vector $\Upsilon = \{0.4, 0.2, 0.4\}$

$$\mathfrak{M}'' = \begin{bmatrix} \varsigma_1 = \{[0.367, 0.493], [-0.787, -0.431], \{0.492, -0.574\}\} \\ \varsigma_2 = \{[0.348, 0.503], [-0.651, -0.502], \{0.405, -0.580\}\} \\ \varsigma_3 = \{[0.417, 0.537], [-0.727, -0.605], \{0.377, -0.336\}\} \end{bmatrix}$$

by calculating $\mathcal{S}_{\mathcal{R}}$ we obtain the ranking $\varsigma_3 \succ \varsigma_1 \succ \varsigma_2$. So ς_3 is best choice. If decision maker consider decision matrix with ECBFEs (Table 4). By calculating $\mathcal{S}_{\mathcal{P}}$ we obtain the matrix of ordered alternatives (Table 5).

Table 4 $\mathfrak{M} = \left[\mathfrak{b}_{\iota\jmath}\right]_{m\times k}$ decision matrix with ECBFEs provided by DM

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.54, 0.67], [−0.12, −0.10], {0.23, −0.56}}	{[0.76, 0.87], [−0.13, −0.12], {0.60, −0.64}}	{[0.81, 0.92], [−0.19, −0.18], {0.70, −0.73}}
ς_2	{[0.64, 0.77], [−0.13, −0.11], {0.32, −0.37}}	{[0.80, 0.92], [−0.19, −0.18], {0.24, −0.29}}	{[0.76, 0.82], [−0.21, −0.18], {0.23, −0.27}}
ς_3	{[0.74, 0.87], [−0.23, −0.20], {0.33, −0.40}}	{[0.66, 0.78], [−0.31, −0.26], {0.34, −0.43}}	{[0.73, 0.77], [−0.30, −0.29], {0.26, −0.44}}

Table 5 $\mathfrak{M}' = \left[\mathfrak{b}'_{\iota\jmath}\right]_{m\times k}$ decision matrix with ordered alternatives of ECBFEs using $\mathcal{S}_{\mathcal{P}}$

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.76, 0.87], [−0.13, −0.12], {0.60, −0.64}}	{[0.81, 0.92], [−0.19, −0.18], {0.70, −0.73}}	{[0.54, 0.67], [−0.12, −0.10], {0.23, −0.56}}
ς_2	{[0.80, 0.92], [−0.19, −0.18], {0.24, −0.29}}	{[0.76, 0.82], [−0.21, −0.18], {0.23, −0.27}}	{[0.64, 0.77], [−0.13, −0.11], {0.32, −0.37}}
ς_3	{[0.74, 0.87], [−0.23, −0.20], {0.33, −0.40}}	{[0.73, 0.77], [−0.30, −0.29], {0.26, −0.44}}	{[0.66, 0.78], [−0.31, −0.26], {0.34, −0.43}}

Table 6 $\mathfrak{M} = \left[\mathfrak{b}_{\iota\jmath}\right]_{m\times k}$ decision matrix with ordered alternatives of ECBFEs $\mathcal{S}_{\mathcal{R}}$

$\mathfrak{X}$ vs $\mathfrak{Y}$	$\dot{y}_1$	$\dot{y}_2$	$\dot{y}_3$
ς_1	{[0.76, 0.87], [−0.13, −0.12], {0.60, −0.64}}	{[0.81, 0.92], [−0.19, −0.18], {0.70, −0.73}}	{[0.54, 0.67], [−0.12, −0.10], {0.23, −0.56}}
ς_2	{[0.80, 0.92], [−0.19, −0.18], {0.24, −0.29}}	{[0.76, 0.82], [−0.21, −0.18], {0.23, −0.27}}	{[0.64, 0.77], [−0.13, −0.11], {0.32, −0.37}}
ς_3	{[0.74, 0.87], [−0.23, −0.20], {0.33, −0.40}}	{[0.66, 0.78], [−0.31, −0.26], {0.34, −0.43}}	{[0.73, 0.77], [−0.30, −0.29], {0.26, −0.44}}

now compute $\mathcal{P}$-CBFOWG to aggregate $\mathfrak{M}' = \left[\mathfrak{b}'_{\iota\jmath}\right]_{m\times k}$, with weight vector $\Upsilon = \{0.4, 0.2, 0.4\}$

$$\mathfrak{M}'' = \begin{bmatrix} \varsigma_1 = \{[0.671, 0.792], [-0.138, -0.124], \{0.421, -0.631\}\} \\ \varsigma_2 = \{[0.724, 0.837], [-0.170, -0.152], \{0.266, -0.319\}\} \\ \varsigma_3 = \{[0.704, 0.812], [-0.276, -0.242], \{0.318, -0.420\}\} \end{bmatrix}$$

by calculating $\mathcal{S}_{\mathcal{P}}$ we obtain the ranking $\varsigma_1 \succ \varsigma_2 \succ \varsigma_3$. So ς_1 is best choice. Now we consider decision matrix with ECBFEs and operator with $\mathcal{R}$-Order (Table 6).

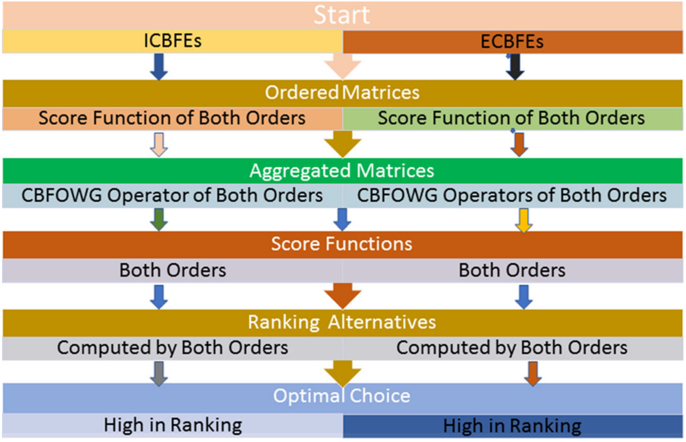

Fig. 1 Flow chart of whole process

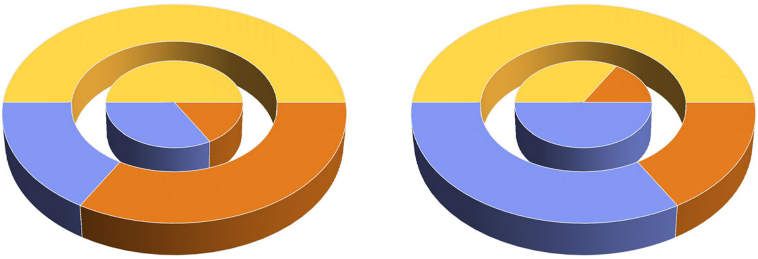

Fig. 2 Yellow $= \varsigma_1$, Brown $= \varsigma_2$,Blue $= \varsigma_3$, the internal pie graph is representing ranking of alternatives using ICBFEs w.r.t $\mathcal{P}$-Order (Fig. 1) and $\mathcal{R}$-Order (Fig. 2) and the external pie graph is representing ranking of alternatives using ECBFEs w.r.t $\mathcal{P}$-Order (Fig. 1) and $\mathcal{R}$-Order (Fig. 2)

Now compute $\mathcal{R}$-CBFOWG to aggregate $\mathfrak{M}' = \left[\mathfrak{b}'_{\imath\jmath}\right]_{m\times k}$, with weight vector $\Upsilon = \{0.4, 0.2, 0.4\}$

$$\mathfrak{M}'' = \begin{bmatrix} \varsigma_1 = \{[0.671, 0.792], [-0.138, -0.124], \{0.509, -0.622\}\} \\ \varsigma_2 = \{[0.724, 0.837], [-0.170, -0.152], \{0.271, -0.315\}\} \\ \varsigma_3 = \{[0.719, 0.810], [-0.274, -0.249], \{0.304, -0.421\}\} \end{bmatrix}$$

by calculating $\mathcal{S}_{\mathcal{R}}$ we obtain the ranking $\varsigma_1 \succ \varsigma_3 \succ \varsigma_2$. So ς_1 is best choice.

6 Conclusion

In this paper, we present the concept of internal cubic bipolar fuzzy(ICBF) sets and external cubic bipolar fuzzy(ECBF) sets. We proposed two cubic bipolar fuzzy ordered weighted geometric aggregation operators, including, $\mathcal{P}$-CBFOWG operator and $\mathcal{R}$-CBFOWG operator to aggregate cubic bipolar fuzzy information with both perspectives, i.e., ICBF data and ECBF data. Finally, we present a multiple attribute decision making problem to examine the useability and capability of these operators and a comparison between ICBF information and ECBF information. We see the final results are more reliable after applying different geometric operators with different perspectives. We examine that in all four processes ς_1 is most reliable alternative. We conclude that the techniques is useful in bipolar type decision making due to its internal and external point of view.

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