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Double-Valued Neutrosophic Sets, their Minimum Spanning Trees, and Clustering Algorithm

Abstract: Neutrosophy (neutrosophic logic) is used to represent uncertain, indeterminate, and inconsistent information available in the real world. This article proposes a method to provide more sensitivity and precision to indeterminacy, by classifying the indeterminate concept/value into two based on membership: one as indeterminacy leaning towards truth membership and the other as indeterminacy leaning towards false membership. This paper introduces a modified form of a neutrosophic set, called Double-Valued Neutrosophic Set (DVNS), which has these two distinct indeterminate values. Its related properties and axioms are defined and illustrated in this paper. An important role is played by clustering in several fields of research in the form of data mining, pattern recognition, and machine learning. DVNS is better equipped at dealing with indeterminate and inconsistent information, with more accuracy, than the Single-Valued Neutrosophic Set, which fuzzy sets and intuitionistic fuzzy sets are incapable of. A generalised distance measure between DVNSs and the related distance matrix is defined, based on which a clustering algorithm is constructed. This article proposes a Double-Valued Neutrosophic Minimum Spanning Tree (DVN-MST) clustering algorithm, to cluster the data represented by double-valued neutrosophic information. Illustrative examples are given to demonstrate the applications and effectiveness of this clustering algorithm. A comparative study of the DVN-MST clustering algorithm with other clustering algorithms like Single-Valued Neutrosophic Minimum Spanning Tree, Intuitionistic Fuzzy Minimum Spanning Tree, and Fuzzy Minimum Spanning Tree is carried out.

Keywords: Neutrosophic set, Double-Valued Neutrosophic Set (DVNS), Minimum Spanning Tree (MST), Clustering algorithm, Double-Valued Neutrosophic Minimum Spanning Tree (DVN-MST) clustering algorithm.

JEL Classification: 62H30, 05C90, 03B52, 03B60.

1 Introduction

Clustering plays a vital role in data mining, pattern recognition, and machine learning in several scientific and engineering fields. Clustering categorises the entities into groups (clusters); that is, it gathers similar samples into the same class and gathers the dissimilar ones into different classes. Clustering analysis has traditionally been kind of hard, which strictly allocates an object to a specific or particular class. It cannot be directly ascertained which class they should belong to, as a great deal of objects have no rigid restrictions. Therefore, it is required to divide them softly.

The fuzzy set theory introduced by Zadeh [29] provides a constructive analytic tool for such "soft" divisions of sets. The concept of fuzzy division was put forth by Ruspini [16], which marked the start of fuzzy clustering analysis. Zadeh's fuzzy set theory [29] was extended to the intuitionistic fuzzy set (A-IFS), in which each element is assigned a membership degree and a non-membership degree by Atanassov [1]. A-IFS seems to be more suitable in dealing with data that has fuzziness and uncertainty than the fuzzy set. A-IFS

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was further generalised into the notion of interval-valued intuitionistic fuzzy set (IVIFS) by Atanassov and Gargov [2].

Because of the fuzziness and uncertainty of the real world, the attributes of the samples are often represented with A-IFSs; there is a need to cluster intuitionistic fuzzy data. Zahn [30] proposed the clustering algorithm using the minimum spanning tree (MST) of the graph. He had defined several criteria of edge inconsistency for detecting clusters of various shapes, and proposed a clustering algorithm using MST. Then, Chen et al. [4] put forward a maximal tree clustering method of the fuzzy graph by constructing the fuzzy similarity relation matrix and used the threshold of fuzzy similarity relation matrix to cut the maximum spanning tree, and obtained the classification.

Dong et al. [6] had given a hierarchical clustering algorithm based on fuzzy graph connectedness. Three MST algorithms were introduced and applied to clustering gene expression data by Xu et al. [24]. Two intuitionistic fuzzy MST (IF-MST) clustering algorithms to deal with intuitionistic fuzzy information was proposed and then extended to clustering interval-valued intuitionistic fuzzy information by Zhao et al. [33]. Furthermore, Zhang and Xu [31] introduced an MST algorithm-based clustering method under a hesitant fuzzy environment. The graph theory-based clustering algorithm [3, 4, 6, 30, 31] is an active research area.

To represent uncertain, imprecise, incomplete, inconsistent, and indeterminate information that are present in the real world, the concept of a neutrosophic set from the philosophical point of view was proposed by Smarandache [19]. The neutrosophic set is a prevailing framework that generalises the concept of the classic set, fuzzy set, intuitionistic fuzzy set (IFS), interval-valued fuzzy set, IVIFS, paraconsistent set, paradoxist set, and tautological set. Truth membership, indeterminacy membership, and falsity membership are represented independently in the neutrosophic set. However, the neutrosophic set generalises the abovementioned sets from the philosophical point of view, and its functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$; that is, $T_A(x): X \rightarrow]^-0, 1^+[$, $I_A(x):X \rightarrow]^-0, 1^+[$, and $F_A(x):X \rightarrow]^-0, 1^+[$, with the condition $^-0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$. It is difficult to apply the neutrosophic set in this form in real scientific and engineering areas.

To overcome this difficulty, Wang et al. [22] introduced a Single-Valued Neutrosophic Set (SVNS), which is an instance of a neutrosophic set. SVNS can deal with indeterminate and inconsistent information, which fuzzy sets and IFSs are incapable of. Ye [25–27] presented the correlation coefficient of SVNS and its crossentropy measure, and applied them to single-valued neutrosophic decision-making problems. Recently, Ye [28] proposed a Single-Valued Neutrosophic Minimum Spanning Tree (SVN-MST) clustering algorithm to deal with the data represented by SVNSs. Owing to the fuzziness, uncertainty, and indeterminate nature of many practical problems in the real world, neutrosophy has found application in many fields including social network analysis [17], image processing [5, 18, 32], and socio-economic problems [20, 21]. Liu et al. have applied neutrosophy to group decision problems and multiple attribute decision-making problems [8–14], etc.

To provide more accuracy and precision to indeterminacy in this paper, the indeterminacy value present in the neutrosophic set has been classified into two based on membership: one as indeterminacy leaning towards truth membership and the other as indeterminacy leaning towards false membership. When the indeterminacy I can be identified as indeterminacy that is more of the truth value than the false value but cannot be classified as truth, it is considered to be indeterminacy leaning towards truth (I_T). When the indeterminacy can be identified to be indeterminacy that is more of the false value than the truth value but cannot be classified as false, it is considered to be indeterminacy leaning towards false (I_F). The single value of indeterminacy has vagueness, for one is not certain of whether the indeterminacy is favouring truth or false membership. When the indeterminacy is favouring or leaning towards truth and when it is favouring or leaning towards false is captured, the results or outcome will certainly be better than when a single value is used. Indeterminacy leaning towards truth and indeterminacy leaning towards falsity makes the indeterminacy involved in the scenario to be more accurate and precise. This modified neutrosophic set is defined as a Double-Valued Neutrosophic Set (DVNS).

Consider the scenario where the expert's opinion is requested about a particular statement; he/she may state that the possibility in which the statement is true is 0.6 and in which the statement is false is

0.5, the degree in which he/she is not sure but thinks it is true is 0.2, and the degree in which he/she is not sure but thinks it is false is 0.1. Using a dual-valued neutrosophic notation, it can be expressed as *x* (0.6, 0.2, 0.1, 0.5). Assume another example: suppose there are 10 voters during a voting process; two people vote yes, two people vote no, three people are for yes but still undecided, and two people are favouring towards a no but still undecided. Using a dual-valued neutrosophic notation, it can be expressed as *x* (0.2, 0.3, 0.3, 0.2). However, these expressions are beyond the scope of representation using the existing SVNS. Therefore, the notion of a dual-valued neutrosophic set is more general and it overcomes the aforementioned issues.

This paper is organised into eight sections. Section 1 is introductory in nature. The basic concepts related to this paper are recalled in Section 2. Section 3 introduces DVNSs, their set operators, and discusses their properties. The distance measure of DVNS is defined, and its properties are discussed in Section 4. Section 5 proposes the Double-Valued Neutrosophic Minimum Spanning Tree (DVN-MST) clustering algorithm. Illustrative examples of the DVN-MST clustering algorithm and comparison of the DVN-MST clustering algorithm with other clustering algorithms such as SVN-MST, IF-MST, and FMST is carried out in Section 7. Conclusions and future work are provided in Section 8.

2 Preliminaries/Basic Concepts

2.1 Neutrosophy and SVNS

Neutrosophy is a branch of philosophy introduced by Smarandache [19], which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It considers a proposition, concept, theory, event, or entity, "*A*," in relation to its opposite, "Anti-*A*," and that which is not *A*, "Non-*A*," and that which is neither "*A*" nor "Anti-*A*," denoted by "Neut-*A*." Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.

The concept of a neutrosophic set from a philosophical point of view, introduced by Smarandache [19], is as follows.

Definition 1 [19]: Let *X* be a space of points (objects), with a generic element in *X* denoted by *x*. A neutrosophic set *A* in *X* is characterised by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$; that is, $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$, and $F_A(x): X \to]^-0, 1^+[$, with the condition $^-0 \le sup T_A(x) + sup I_A(x) + sup F_A(x) \le 3^+$.

This definition of a neutrosophic set is difficult to apply in the real world in scientific and engineering fields. Therefore, the concept of SVNS, which is an instance of a neutrosophic set, was introduced by Wang et al. [22].

Definition 2 [22]: Let *X* be a space of points (objects) with generic elements in *X* denoted by *x*. An SVNS *A* in *X* is characterised by truth membership function $T_A(x)$, indeterminacy membership function $I_A(x)$, and falsity membership function $F_A(x)$. For each point *x* in *X*, there are $T_A(x), I_A(x), F_A(x) \in [0, 1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$. Therefore, an SVNS *A* can be represented by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in X\}.$$

The following expressions are defined in Ref. [22] for SVNSs *A*, *B*:
-   $A \in B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for any *x* in *X*.
-   $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
-   $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle | x \in X\}$.

2.2 Graphs and MSTs

Graphs and their MSTs have been extensively studied and developed; they are recalled with examples.

A graph G is a pair of sets $G = (V, E)$, where V is the set of vertices (or nodes) and E is the set of edges. Graphs are either undirected or directed. Each edge in the undirected graph is an unordered pair v_i, v_j, whereas each edge in the directed graph is an ordered pair v_i, v_j, where the vertices v_i and v_j are the end points of an edge. A path is a sequence of edges and vertices that can be traversed between two different vertices.

A connected acyclic graph that contains all nodes of G is called a spanning tree of the graph, which is a set of edges connecting pairs of vertices such that
1. There are no closed loops.
2. Each vertex is visited by at least one edge.
3. A tree is connected.

An MST is the spanning tree with the minimum length. Algorithms to obtain the MST of a graph have been proposed in Refs. [7, 15], and they operate iteratively. At any stage, the edge belongs to one of two sets, i.e. set A, which contains those edges that belong to the MST and set B, which contains those that do not belong to the MST.

Kruskal's algorithm [7] assigns iteratively to set A the shortest edge in the set B, which does not form a closed loop with any of the edges in A. Initially, A is empty, and the iteration stops when A contains $(n-1)$ edges. Prim's algorithm [15] begins with any one of the given nodes and initially assigns to A the shortest edge starting from this node; it continues to add the shortest edge from B, which connects at least one edge from A without forming a closed loop with the edge in A. The iteration stops when A has $(n-1)$ edges.

Usually, clustering data sets are represented as weighted graphs, where nodes are represented as entities to be clustered and edges are represented as a distance measure (or called dissimilarity measure) between those entities. A fuzzy relation R over $V \times V$ has the membership function $\mu_R(v_1, v_2)$, where $(v_1, v_2) \in V \times V$; it takes different values from 0 to 1, and such a graph is called a fuzzy graph. When R is an intuitionistic fuzzy relation over $V \times V$, then its related graph is called an intuitionistic fuzzy graph [33]. When R is a neutrosophic relation over $V \times V$, its related graph is called a single-valued neutrosophic graph [28].

3 DVNSs and their Properties

Indeterminacy deals with uncertainty that is faced in every sphere of life by everyone. It makes research/science more realistic and sensitive by introducing the indeterminate aspect of life as a concept. There are times in the real world where the indeterminacy I can be identified to be indeterminacy that has more of the truth value than the false value but cannot be classified as truth. Similarly, in some cases, the indeterminacy can be identified to be indeterminacy that has more of the false value than the truth value but cannot be classified as false. To provide more sensitivity to indeterminacy, this kind of indeterminacy is classified into two. When the indeterminacy I can be identified as indeterminacy that is more of the truth value than the false value but cannot be classified as truth, it is considered to be indeterminacy leaning towards truth (I_T). Whereas in case the indeterminacy can be identified to be indeterminacy that is more of the false value than the truth value but cannot be classified as false, it is considered to be indeterminacy leaning towards false (I_F). Indeterminacy leaning towards truth and indeterminacy leaning towards falsity makes the indeterminacy involved in the scenario to be more accurate and precise. It provides a better and detailed view of the existing indeterminacy.

The definition of DVNS is as follows:

Definition 3: Let X be a space of points (objects) with generic elements in X denoted by x. A DVNS A in X is characterised by truth membership function $T_A(x)$, indeterminacy leaning towards truth membership function $I_{TA}(x)$, indeterminacy leaning towards falsity membership function $I_{FA}(x)$, and falsity membership function $F_A(x)$. For each generic element $x \in X$, there are

$$T_A(x), I_{TA}(x), I_{FA}(x), F_A(x) \in [0, 1],$$

and

$$0 \le T_A(x) + I_{TA}(x) + I_{FA}(x) + F_A(x) \le 4.$$

Therefore, a DVNS A can be represented by

$$A = \{\langle x, T_A(x), I_{TA}(x), I_{FA}(x), F_A(x) \rangle \mid x \in X\}.$$

A DVNS A is represented as

$$A = \int_X \{\langle T(x), I_T(x), I_F(x), F(x) \rangle / dx, \, x \in X\} \tag{1}$$

when X is continuous. It is represented as

$$A = \sum_{i=1}^{n} \{\langle T(x_i), I_T(x_i), I_F(x_i), F(x_i) \rangle \mid x_i, \, x_i \in X\} \tag{2}$$

when X is discrete.

To illustrate the applications of DVNS in the real world, consider parameters that are commonly used to define the quality of service of semantic web services, like capability, trustworthiness, and price, for illustrative purposes. The evaluation of the quality of service of semantic web services [23] is used to illustrate set-theoretic operation on DVNSs.

Example 1: Let $X = [x_1, x_2, x_3]$ where x_1 is capability, x_2 is trustworthiness, and x_3 is price. The values of x_1, x_2, and x_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts; their option could be a degree of "good service," indeterminacy leaning towards a degree of "good service," indeterminacy leaning towards a degree of "poor service," and a degree of "poor service." A is a DVNS of X defined by

$$A = \langle 0.3, 0.4, 0.2, 0.5 \rangle / x_1 + \langle 0.5, 0.1, 0.3, 0.3 \rangle / x_2 + \langle 0.7, 0.2, 0.1, 0.2 \rangle / x_3.$$

B is a DVNS of X defined by

$$B = \langle 0.6, 0.1, 0.3, 0.2 \rangle / x_1 + \langle 0.2, 0.1, 0.2, 0.4 \rangle / x_2 + \langle 0.4, 0.1, 0.1, 0.3 \rangle / x_3.$$

Definition 4: The complement of a DVNS A denoted by $c(A)$ is defined as
1. $T_{c(A)}(x) = F_A(x),$
2. $I_{Tc(A)}(x) = 1 - I_{TA}(x),$
3. $I_{Fc(A)}(x) = 1 - I_{FA(A)}(x),$
4. $F_{c(A)}(x) = T_A(x),$

for all x in X.

Example 2: Consider the DVNS A defined in Example 1. Then, the complement of A is

$$c(A) = \langle 0.5, 0.6, 0.8, 0.3 \rangle / x_1 + \langle 0.3, 0.9, 0.7, 0.5 \rangle / x_2 + \langle 0.2, 0.8, 0.9, 0.7 \rangle / x_3.$$

Definition 5: A DVNS A is contained in DVNS B, $A \subseteq B$, if and only if
1. $T_A(x) \le T_B(x),$
2. $I_{TA}(x) \le I_{TB}(x),$
3. $I_{FA}(x) \le I_{FB}(x),$
4. $F_A(x) \ge F_B(x),$

for all x in X.

Note that by the definition of containment relation, X is a partially ordered set and not a totally ordered set.

For example, let A and B be the DVNSs as defined in Example 1, then A is not contained in B and B is not contained in A.

Definition 6: Two DVNSs A and B are equal, denoted as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Theorem 1: $A \subseteq B$ *if and only if* $c(B) \subseteq c(A)$.
   Proof.

$$A \subseteq B \Leftrightarrow T_A \leq T_B, I_{TA} \leq I_{TB}, I_{FA} \leq I_{FB}, F_B \geq F_A$$
$$\Leftrightarrow F_A \leq F_B, 1 - I_{TB} \leq 1 - I_{TA}, 1 - I_{FB} \leq 1 - I_{FA}, T_B \geq T_A$$
$$\Leftrightarrow c(B) \subseteq c(A).$$

$\square$

Definition 7: The union of two DVNSs A and B is a DVNS C, denoted as $C = A \cup B$, whose truth membership, indeterminacy leaning towards truth membership, indeterminacy leaning towards falsity membership, and falsity membership functions are related to those of A and B by the following:
1. $T_C(x) = max(T_A(x), T_B(x))$,
2. $I_{TC}(x) = max(I_{TA}(x), I_{TB}(x))$,
3. $I_{FC}(x) = max(I_{FA}(x), I_{FB}(x))$,
4. $F_C(x) = min(F_A(x), F_B(x))$,

for all x in X.

Example 3: Consider the DVNSs A and B defined in Example 1. Then,

$$A \cup B = \langle 0.6, 0.4, 0.3, 0.2 \rangle / x_1 + \langle 0.5, 0.1, 0.3, 0.3 \rangle / x_2 + \langle 0.7, 0.2, 0.1, 0.2 \rangle / x_3.$$

Theorem 2. $A \cup B$ *is the smallest DVNS containing both A and B.*
   Proof. It is direct from the definition of the union operator. $\square$

Definition 8: The intersection of two DVNSs A and B is a DVNS C, denoted as $C = A \cap B$, whose truth membership, indeterminacy leaning towards truth membership, indeterminacy leaning towards falsity membership, and falsity membership functions are related to those of A and B by the following:
1. $T_C(x) = min(T_A(x), T_B(x))$,
2. $I_{TC}(x) = min(I_{TA}(x), I_{TB}(x))$,
3. $I_{FC}(x) = min(I_{FA}(x), I_{FB}(x))$,
4. $F_C(x) = max(F_A(x), F_B(x))$,

for all $x \in X$.

Example 4: Consider the DVNSs A and B as defined in Example 1. Then,

$$A \cap B = \langle 0.3, 0.1, 0.2, 0.5 \rangle / x_1 + \langle 0.2, 0.1, 0.2, 0.4 \rangle / x_2 + \langle 0.4, 0.1, 0.1, 0.3 \rangle / x_3.$$

Theorem 3: $A \cap B$ *is the largest DVNS contained in both A and B.*
   Proof. It is straightforward from the definition of intersection operator. $\square$

Definition 9: The difference of two DVNSs D, written as $D = A \backslash B$, whose truth membership, indeterminacy leaning towards truth membership, indeterminacy leaning towards falsity membership, and falsity membership functions are related to those of A and B by

1. $T_D(x) = min(T_A(x), F_B(x))$,
2. $I_{TD}(x) = min(I_{TA}(x), 1 - I_{TB}(x))$,
3. $I_{FD}(x) = min(I_{FA}(x), 1 - I_{FB}(x))$,
4. $F_D(x) = min(F_A(x), T_B(x))$,

for all x in X.

Example 5: Consider A and B the DVNSs defined in Example 1. Then,

$$A \setminus B = \langle 0.2,\ 0.4,\ 0.2,\ 0.5 \rangle / x_1 + \langle 0.4,\ 0.1,\ 0.3,\ 0.2 \rangle / x_2 + \langle 0.3,\ 0.2,\ 0.1,\ 0.2 \rangle / x_3.$$

Three operators, called truth favourite (Δ), falsity favourite (∇), and indeterminacy neutral (∇), are defined over DVNSs. Two operators, truth favourite (Δ) and falsity favourite (∇), are defined to remove the indeterminacy in the DVNSs and transform it into IFSs or paraconsistent sets. Similarly, a DVNS can be transformed into an SVNS by applying the indeterminacy neutral (∇) operator that combines the indeterminacy values of the DVNS. These three operators are unique on DVNSs.

Definition 10: The truth favourite of a DVNS A is written as $B = \Delta A$, whose truth membership and falsity membership functions are related to those of A by
1. $T_B(x) = min(T_A(x) + I_{TA}(x), 1)$,
2. $I_{TB}(x) = 0$,
3. $I_{FB}(x) = 0$,
4. $F_B(x) = F_A(x)$,

for all x in X.

Example 6: The truth favourite of the DVNS A given in Example 1 is $B = \Delta A$, where

$$B = \langle 0.7,\ 0,\ 0,\ 0.5 \rangle / x_1 + \langle 0.6,\ 0,\ 0,\ 0.2 \rangle / x_2 + \langle 0.9,\ 0,\ 0,\ 0.2 \rangle / x_3.$$

Definition 11: The falsity favourite of a DVNS A, written as $B = \nabla A$, whose truth membership and falsity membership functions are related to those of A by
1. $T_B(x) = T_A(x)$,
2. $I_{TB}(x) = 0$,
3. $I_{FB}(x) = 0$,
4. $F_B(x) = min(F_A(x) + I_{FA}(x), 1)$,

for all x in X.

Example 7: Let A be the DVNS defined in Example 1. Then, B is the falsity favourite of the DVNS $B = \nabla A$:

$$B = \langle 0.3,\ 0,\ 0,\ 0.7 \rangle / x_1 + \langle 0.5,\ 0,\ 0,\ 0.6 \rangle / x_2 + \langle 0.7,\ 0,\ 0,\ 0.3 \rangle / x_3.$$

Definition 12: The indeterminacy neutral of a DVNS A, written as $B = \nabla A$, whose truth membership, indeterminate membership, and falsity membership functions are related to those of A by
1. $T_B(x) = T_A(x)$,
2. $I_{TB}(x) = min(I_{TA}(x) + I_{TB}(x), 1)$,
3. $I_{FB}(x) = 0$,
4. $F_B(x) = F_A(x)$,

for all x in X.

Example 8: Consider the DVNS A defined in Example 1. Then, the indeterminacy neutral of the DVNS $B = \nabla A$, is

$$B = \langle 0.3,\ 0.6,\ 0,\ 0.5 \rangle / x_1 + \langle 0.5,\ 0.4,\ 0,\ 0.3 \rangle / x_2 + \langle 0.7,\ 0.3,\ 0,\ 0.2 \rangle / x_3.$$

Proposition 1: The following set theoretic operators are defined over DVNSs A, B, and C.
1. (Property 1) (commutativity):

$$A \cup B = B \cup A,\ \ A \cap B = B \cap A,\ \ A \times B = B \times A.$$

2. (Property 2) (associativity):

$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cup C,$$
$$A \times (B \times C) = (A \times B) \times C.$$

3. (Property 3) (distributivity):

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. (Property 4) (idempotency):

$$A \cup A = A,\ \ A \cap A = A,$$
$$\Delta \Delta A = \Delta A,\ \ \nabla \nabla A = \nabla A.$$

5. (Property 5)

$$A \cap \phi = \phi,\ A \cap X = X,$$

where $T\phi = I\phi = 0$, $F\phi = 1$ and $T_X = I_{TX} = I_{FX} = 1$, $F_X = 0$.

6. (Property 6)

$$A \cup \phi = A,\ A \cap X = A,$$

where $T\phi = I_I\phi = I_F\phi = 0$, $F\phi = 1$ and $T_X = I_{TX} = I_{FX} = 1$, $F_X = 0$.

7. (Property 7) (absorption):

$$A \cup (A \cap B) = A,\ A \cap (A \cup B) = A.$$

8. (Property 8) (De Morgan's laws):

$$c(A \cup B) = c(A) \cap c(B),\ c(A \cap B) = c(A) \cup c(B).$$

9. (Property 9) (involution):

$$c(c(A)) = A.$$

The definition of complement, union, and intersection of DVNSs and the DVNS itself satisfy most properties of the classical set, fuzzy set, IFS, and SNVS. Similar to the fuzzy set, IFS, and SNVS, it does not satisfy the principle of middle exclude.

4 Distance Measures of DVNS

The distance measures over DVNSs is defined in the following and the related algorithm for determining the distance is given.

Consider two DVNSs A and B in a universe of discourse, $X = x_1, x_2, \ldots, x_n$, which are denoted by

$$A = \{\langle x_i,\ T_A(x_i),\ I_{TA}(x_i),\ I_{FA}(x_i),\ F_A(x_i) \rangle | x_i \in X\},\ \text{and}$$
$$B = \{\langle x_i,\ T_B(x_i),\ I_{TB}(x_i),\ I_{FB}(x_i),\ F_A(x_i) \rangle | x_i \in X\},$$

where $T_A(x_i)$, $I_{TA}(x_i)$, $I_{FA}(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_{TB}(x_i)$, $I_{FB}(x_i)$, $F_B(x_i) \in [0, 1]$ for every $x_i \in X$. Let $w_i (i = 1, 2, \ldots, n)$ be the weight of an element $x_i (i = 1, 2, \ldots, n)$, with $w_i \geq 0 (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$.

Then, the generalised double-valued neutrosophic weighted distance is defined as follows:

$$d_\lambda(A, B) = \left\{ \frac{1}{4} \sum_{i=1}^{n} w_i [|T_A(x_i) - T_B(x_i)|^\lambda + |I_{TA}(x_i) - I_{TB}(x_i)|^\lambda + |I_{FA}(x_i) - I_{FB}(x_i)|^\lambda + |F_A(x_i) - F_B(x_i)|^\lambda] \right\}^{1/\lambda}, \quad (3)$$

where $\lambda > 0$.

Equation (3) reduces to the double-valued neutrosophic weighted Hamming distance and the double-valued neutrosophic weighted Euclidean distance, when $\lambda = 1, 2$, respectively. The double-valued neutrosophic weighted Hamming distance is given as

$$d_\lambda(A, B) = \frac{1}{4} \sum_{i=1}^{n} w_i [|T_A(x_i) - T_B(x_i)| + |I_{TA}(x_i) - I_{TB}(x_i)| + |I_{FA}(x_i) - I_{FB}(x_i)| + |F_A(x_i) - F_B(x_i)|], \quad (4)$$

where $\lambda = 1$ in Eq. (3).

The double-valued neutrosophic weighted Euclidean distance is given as

$$d_\lambda(A, B) = \left\{ \frac{1}{4} \sum_{i=1}^{n} w_i [|T_A(x_i) - T_B(x_i)|^2 + |I_{TA}(x_i) - I_{TB}(x_i)|^2 + |I_{FA}(x_i) - I_{FB}(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2] \right\}^{1/2}, \quad (5)$$

where $\lambda = 2$ in Eq. (3).

The algorithm to obtain the generalised double-valued neutrosophic weighted distance $d_\lambda(A, B)$ is given in Algorithm 1.

The following proposition is given for the distance measure:

Proposition 2: The generalised double-valued neutrosophic weighted distance $d_\lambda(A, B)$ for $\lambda > 0$ satisfies the following properties:
1. (Property 1) $d_\lambda(A, B) \geq 0$
2. (Property 2) $d_\lambda(A, B) = 0$ if and only if $A = B$
3. (Property 3) $d_\lambda(A, B) = d_\lambda(B, A)$

Algorithm 1: Generalised Double-Valued Neutrosophic Weighted Distance $d_\lambda (A; B)$.

Input: $X = x_1, x_2, \ldots, x_n$, DVNS A, B where
   $A = \{\langle x_i, T_A(x), I_{TA}(x), I_{FA}(x), F_A(x)\rangle \mid x_i \in X\}$,
   $B = \{\langle x_i, T_B(x), I_{TB}(x), I_{FB}(x), F_A(x)\rangle \mid x_i \in X\}$,
   $w_i (i = 1, 2, \ldots, n)$
Output: $d_\lambda(A, B)$
begin
  $d_\lambda \leftarrow 0$
  For $i = 1$ to n **do**
    $d_\lambda \leftarrow d_\lambda + w_i [|T_A(x_i) - T_B(x_i)|^\lambda + |I_{TA}(x_i) - I_{TB}(x_i)|^\lambda$
        $+ |I_{FA}(x_i) - I_{FB}(x_i)|^\lambda + |F_A(x_i) - F_B(x_i)|^\lambda]$
  end
  $d_\lambda \leftarrow d_\lambda / 4$
  $d_\lambda \leftarrow d_\lambda^{\left(\frac{1}{\lambda}\right)}$
end

---

4. (Property 4) If $A \subseteq B \subseteq C$, C is a DVNS in X, then $d_\lambda(A, C) \geq d_\lambda(A, B)$ and $d_\lambda(A, C) \geq d_\lambda(B, C)$.

It can be easily seen that $d_\lambda(A, B)$ satisfies (Property 1) to (Property 3); hence, only (Property 4) is proved. Let $A \subseteq B \subseteq C$, then $T_A(x_i) \le T_B(x_i) \le T_C(x_i)$, $I_A(x_i) \le I_B(x_i) \le I_C(x_i)$, and $F_A(x_i) \ge F_B(x_i) \ge F_C(x_i)$ for every $x_i \in X$. Then, the following relations are obtained:

$$|T_A(x_i) - T_B(x_i)|^\lambda \le |T_A(x_i) - T_C(x_i)|^\lambda,$$

$$|T_B(x_i) - T_C(x_i)|^\lambda \le |T_A(x_i) - T_C(x_i)|^\lambda,$$

$$|I_{TA}(x_i) - I_{TB}(x_i)|^\lambda \le |I_{TA}(x_i) - I_{TC}(x_i)|^\lambda,$$

$$|I_{TB}(x_i) - I_{TC}(x_i)|^\lambda \le |I_{TA}(x_i) - I_{TC}(x_i)|^\lambda,$$

$$|I_{FA}(x_i) - I_{FB}(x_i)|^\lambda \le |I_{FA}(x_i) - I_{FC}(x_i)|^\lambda,$$

$$|I_{FB}(x_i) - I_{FC}(x_i)|^\lambda \le |I_{FA}(x_i) - I_{FC}(x_i)|^\lambda,$$

$$|F_A(x_i) - F_B(x_i)|^\lambda \le |F_A(x_i) - F_C(x_i)|^\lambda, \text{ and}$$

$$|F_B(x_i) - F_C(x_i)|^\lambda \le |F_A(x_i) - F_C(x_i)|^\lambda.$$

Hence,

$$|T_A(x_i) - T_B(x_i)|^\lambda + |I_{TA}(x_i) - I_{TB}(x_i)|^\lambda + |I_{FA}(x_i) - I_{FB}(x_i)|^\lambda + |F_A(x_i) - F_B(x_i)|^\lambda \le |T_A(x_i) - T_C(x_i)|^\lambda +$$

$$\le |I_{TA}(x_i) - I_{TC}(x_i)|^\lambda + \le |I_{FA}(x_i) - I_{FC}(x_i)|^\lambda + |F_A(x_i) - F_C(x_i)|^\lambda.$$

$$|T_B(x_i) - T_C(x_i)|^\lambda + |I_{TB}(x_i) - I_{TC}(x_i)|^\lambda + |I_{FB}(x_i) - I_{FC}(x_i)|^\lambda + |F_B(x_i) - F_C(x_i)|^\lambda \le |T_A(x_i) - T_C(x_i)|^\lambda$$

$$+ |I_{TA}(x_i) - I_{TC}(x_i)|^\lambda + |I_{FA}(x_i) - I_{FC}(x_i)|^\lambda + |F_A(x_i) - F_C(x_i)|^\lambda.$$

Combining the above inequalities with the distance formula given in Eq. (3), the following is obtained:

$$d_\lambda(A, B) \ge d_\lambda(A, C) \text{ and } d_\lambda(B, C) \ge d_\lambda(A, C).$$

Algorithm 2: Double-Valued Neutrosophic Weighted Distance Matrix D.

Input: DVNS $A_1, \dots, A_m$,
Output: Distance matrix D with elements d_{ij}
begin
  for $i = 1$ to m **do**
    for $j = 1$ to m **do**
      if $i = j$ **then**
        $d_{ij} \leftarrow 0$
      else
        $d_{ij} \leftarrow \{d_\lambda(A_i, A_j)\}$
      end
    end
  end
end

---

for $\lambda > 0$. Thus, (Property 4) is obtained.

The double-valued neutrosophic distance matrix D is defined in the following.

Definition 13: Let $A_j (j = 1, 2, \dots, m)$ be a collection of m DVNSs; then, $D = (d_{ij})_{m \times m}$ is called a double-valued neutrosophic distance matrix, where $d_{ij} = d_\lambda(A_i, A_j)$ is the generalised double-distance-valued neutrosophic between A_i and A_j, and its properties are as follows:

1. $0 \le d_{ij} \le 1$ for all $i, j = 1, 2, \dots, m$;
2. $d_{ij} = 0$ if and only if $A_i = A_j$;
3. $d_{ij} = d_{ji}$ for all $i, j = 1, 2, \dots, m$.

The algorithm to calculate the double-valued neutrosophic weighted distance matrix D is given in Algorithm 2.

The following section provides the DVN-MST clustering algorithm.

5 DVN-MST Clustering Algorithms

In this section, a DVN-MST clustering algorithm is proposed as a generalisation of the IF-MST and SVN-MST clustering algorithms.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an attribution space and the weight vector of an element $x_i(i = 1, 2, \ldots, n)$ be $w = \{w_1, w_2, \ldots, w_n\}$, with $w_i \geq 0(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Consider that $A_j(j = 1, 2, \ldots, m)$ is a collection of m DVNSs, which has m samples that need to be clustered. Then, they are represented in the following form: $A_j = \{\langle x_i, T_{A_j}(x_j), I_{TA_j}(x_j), I_{FA_j}(x_j), F_{A_j}(x_j)\rangle | x_j \in X\}$. Algorithm 3 provides the DVN-MST clustering algorithm. The description of the algorithm is as follows:

Algorithm 3: DVN-MST Clustering Algorithm.

Input: Distance matrix $D = (d_{ij})_{m \times m}$
Output: MST S and clusters
begin
  Step 1: Calculate distance matrix D of $A_1, \ldots, A_m$
  $D(A_1, \ldots, A_m)$ //`Distance matrix D is from Algorithm 2`
  Step 2: Create graph $G(V, E)$
  for $i = 1$ to m **do**
    for $j = 1$ to m **do**
    If $i \mathrel{!=} j$ **then**
      Draw the edge between A_i and A_j with weight d_{ij} based
      on $D = (d_{ij})_{m \times m}$
    end
    end
  end
  Step 3: Compute the MST of the double-valued neutrosophic graph
  $G(V, E)$: //`Using Kruskal's algorithm`
  Sort all the edges in non-decreasing order of their weight in E.
  while *No. of edges in subgraph S of $G < (V − 1)$* **do**
    Select the smallest edge (v_i, v_j).
    Delete (v_i, v_j) from E
    If (v_i, v_j) forms a cycle with the spanning tree S **then**
      Discard the edge v_i, v_j
    else
      Include the edge v_i, v_j in S
    end
  end
  S is the MST of the double-valued neutrosophic graph $G(V, E)$.
  Step 4: Perform clustering
  for $i = 1$ to m **do**
    for $j = 1$ to m **do**
    If $d_{ij} \geq r$ **then** // `r is the threshold`
      Disconnect edge
    else
      Edge is not disconnected
    end
    end
  end
  Results in clusters automatically; it is tabulated
end

Step 1: Calculate the distance matrix $D = d_{ij} = d_\lambda(A_i, A_j)$ by Algorithm 2 (take $\lambda = 2$). The double-valued neutrosophic distance matrix $D = (d_{ij})_{m \times m}$ obtained is

$$D = \begin{bmatrix} 0 & d_{12} & \dots & d_{1m} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \dots & 0 \end{bmatrix}.$$

Step 2: The double-valued neutrosophic graph $G(V, E)$, where every edge between A_i and $A_j(i, j = 1, 2, \ldots, m)$ is assigned the double-valued neutrosophic weighted distance d_{ij}, is an element of the double-valued neutrosophic distance matrix $D = (d_{ij})_{m \times m}$, which represents the dissimilarity degree between the samples A_i and A_j. The double-valued neutrosophic graph $G(V, E)$ is represented as a graph.

Step 3: Construct the MST of the double-valued neutrosophic graph $G(V, E)$.

1. The sorted list of distances of edges of $G(V, E)$ in increasing order by weights is constructed.
2. Keep an empty subgraph S of $G(V, E)$ and select the edge e with the smallest weight to add in S, where the end points of e are disconnected.
3. The smallest edge e is added to S and deleted from the sorted list.
4. The next smallest edge is selected and if no cycle is formed in S, it is added to S and deleted from the list.
5. Repeat process 4 until the subgraph S has $(m - 1)$ edges.

Thus, the MST of the double-valued neutrosophic graph $G(V, E)$ is obtained and illustrated as a graph.

Step 4: Select a threshold r and disconnect all the edges of the MST with weights greater than r to obtain a certain number of clusters; list it as a table. The clustering results induced by the subtrees do not depend on some particular MST [30, 31].

6 Illustrative Examples

Two descriptive examples are presented and utilised to demonstrate the real-world applications and the effectiveness of the proposed DVN-MST clustering algorithm. This example is adopted from Ref. [28].

Example 9: A car market is going to classify eight different cars of $A_j(j = 1, 2, \ldots, 8)$. For each car, the six evaluation factors (attributes) are as follows: x_1, fuel consumption; x_2, coefficient of friction; x_3, price; x_4, comfortable degree; x_5, design; x_6, security coefficient. The characteristics of each car under the six attributes are represented by the form of DVNSs, and then the double-valued neutrosophic data are as follows:

$$A_1 = \{\langle x_1, 0.3, 0.2, 0.1, 0.5 \rangle, \langle x_2, 0.6, 0.3, 0.4, 0.1 \rangle, \langle x_3, 0.4, 0.3, 0.2, 0.3 \rangle,$$
$$\langle x_4, 0.8, 0.1, 0.2, 0.1 \rangle, \langle x_5, 0.1, 0.3, 0.4, 0.6 \rangle, \langle x_6, 0.5, 0.2, 0.1, 0.4 \rangle\},$$

$$A_2 = \{\langle x_1, 0.6, 0.3, 0.2, 0.3 \rangle, \langle x_2, 0.5, 0.4, 0.1, 0.2 \rangle, \langle x_3, 0.6, 0.2, 0.2, 0.1 \rangle,$$
$$\langle x_4, 0.7, 0.2, 0.3, 0.1 \rangle, \langle x_5, 0.3, 0.1, 0.4, 0.6 \rangle, \langle x_6, 0.4, 0.3, 0.2, 0.3 \rangle\},$$

$$A_3 = \{\langle x_1, 0.4, 0.2, 0.3, 0.4 \rangle, \langle x_2, 0.8, 0.2, 0.3, 0.1 \rangle, \langle x_3, 0.5, 0.3, 0.2, 0.1 \rangle,$$
$$\langle x_4, 0.6, 0.1, 0.2, 0.2 \rangle, \langle x_5, 0.4, 0.1, 0.2, 0.5 \rangle, \langle x_6, 0.3, 0.2, 0.1, 0.2 \rangle\},$$

$$A_4 = \{\langle x_1, 0.2, 0.4, 0.3, 0.4 \rangle, \langle x_2, 0.4, 0.5, 0.2, 0.1 \rangle, \langle x_3, 0.9, 0.2, 0.3, 0.0 \rangle,$$
$$\langle x_4, 0.8, 0.2, 0.2, 0.1 \rangle, \langle x_5, 0.2, 0.3, 0.4, 0.5 \rangle, \langle x_6, 0.7, 0.3, 0.2, 0.1 \rangle\},$$

$$A_5 = \{\langle x_1, 0.2, 0.3, 0.2, 0.3 \rangle, \langle x_2, 0.3, 0.2, 0.1, 0.6 \rangle, \langle x_3, 0.5, 0.1, 0.2, 0.4 \rangle,$$
$$\langle x_4, 0.7, 0.1, 0.3, 0.1 \rangle, \langle x_5, 0.4, 0.2, 0.2, 0.4 \rangle, \langle x_6, 0.3, 0.2, 0.1, 0.6 \rangle\},$$

$$A_6 = \{\langle x_1, 0.3, 0.2, 0.1, 0.4 \rangle, \langle x_2, 0.2, 0.1, 0.4, 0.7 \rangle, \langle x_3, 0.4, 0.2, 0.1, 0.5 \rangle,$$
$$\langle x_4, 0.8, 0.0, 0.1, 0.1 \rangle, \langle x_5, 0.4, 0.3, 0.2, 0.5 \rangle, \langle x_6, 0.2, 0.1, 0.2, 0.7 \rangle\},$$

$$A_7 = \{\langle x_1, 0.4, 0.4, 0.1, 0.3\rangle, \langle x_2, 0.5, 0.3, 0.2, 0.1\rangle, \langle x_3, 0.6, 0.1, 0.2, 0.2\rangle,$$
$$\langle x_4, 0.2, 0.3, 0.1, 0.7\rangle, \langle x_5, 0.3, 0.1, 0.2, 0.5\rangle, \langle x_6, 0.7, 0.2, 0.2, 0.1\rangle\}$$
$$A_8 = \{\langle x_1, 0.4, 0.1, 0.1, 0.2\rangle, \langle x_2, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_3, 0.8, 0.2, 0.2, 0.1\rangle,$$
$$\langle x_4, 0.7, 0.2, 0.3, 0.1\rangle, \langle x_5, 0.1, 0.1, 0.2, 0.8\rangle, \langle x_6, 0.2, 0.1, 0.1, 0.8\rangle\}.$$

Let the weight vector of the attribute $x_i(i=1, 2, \ldots, 6)$ be $w=(0.16, 0.12, 0.25, 0.2, 0.15, 0.12)^T$; then, the DVN-MST clustering algorithm given in Algorithm 3 is used to group the eight different cars of $A_j(j=1, 2, \ldots, 8)$.

Step 1: Calculate the distance matrix $D=d_{ij}=d_\lambda(A_i, A_j)$ by Algorithm 2 (take $\lambda=2$). The double-valued neutrosophic distance matrix $D=(d_{ij})_{m\times m}$ is obtained as follows:

$$D = \begin{bmatrix}
0 & 0.14465 & 0.13775 & 0.18934 & 0.17292 & 0.18221 & 0.24145 & 0.18768 \\
0.14465 & 0 & 0.12207 & 0.15108 & 0.1683 & 0.2213 & 0.20887 & 0.15652 \\
0.13775 & 0.12207 & 0 & 0.18993 & 0.18486 & 0.22243 & 0.19856 & 0.18083 \\
0.18934 & 0.15108 & 0.18993 & 0 & 0.22672 & 0.28346 & 0.23484 & 0.21915 \\
0.17292 & 0.1683 & 0.18486 & 0.22672 & 0 & 0.11247 & 0.2511 & 0.19602 \\
0.18221 & 0.2213 & 0.22243 & 0.28346 & 0.11247 & 0 & 0.29774 & 0.23383 \\
0.24145 & 0.20887 & 0.19856 & 0.23484 & 0.2511 & 0.29774 & 0 & 0.265 \\
0.18768 & 0.15652 & 0.18083 & 0.21915 & 0.19602 & 0.23383 & 0.265 & 0
\end{bmatrix}.$$

Step 2: The double-valued neutrosophic graph $G(V, E)$, where every edge between A_i and $A_j(i, j=1, 2, \ldots, 8)$ is assigned the double-valued neutrosophic weighted distance d_{ij}, is an element of the double-valued neutrosophic distance matrix $D=(d_{ij})_{m\times m}$, which represents the dissimilarity degree between the samples A_i and A_j. The double-valued neutrosophic graph $G(V, E)$ is shown in Figure 1.

Step 3: Construct the MST of the double-valued neutrosophic graph $G(V, E)$.

1. The sorted list of distances of edges of G in increasing order by weights is $d_{56} \le d_{32} \le d_{31} \le d_{12} \le d_{42} \le d_{82} \le d_{52} \le d_{51} \le d_{38} \le d_{16} \le d_{35} \le d_{81} \le d_{41} \le d_{34} \le d_{85} \le d_{73} \le d_{72} \le d_{84} \le d_{26}$.
2. Keep an empty subgraph S of G and add the edge e with the smallest weight to S, where the end points of e are disconnected.
3. The edge between A_5 and A_6, $d_{56}=0.11247$, is the smallest; it is added to S and deleted from the sorted list.
4. The next smallest edge is selected from G and if no cycle is formed in S, it is added to S and deleted from the list.
5. Repeat process 4 until the subgraph S has $(7-1)$ edges or spans eight nodes.

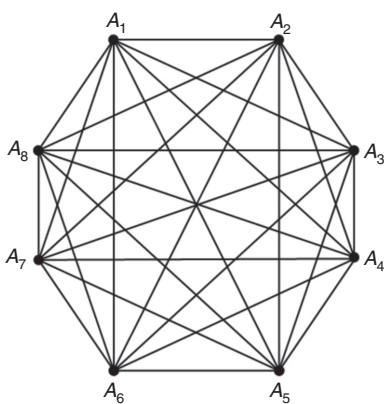Thus, the MST of the double-valued neutrosophic graph $G(V, E)$ is obtained, as illustrated in Figure 2.

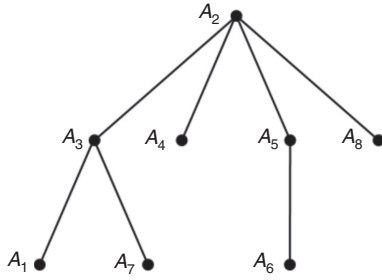**Figure 1:** Double-Valued Neutrosophic Graph *G*.

Figure 2: DVN-MST S of Graph G.

Table 1: Clustering Results of the Eight Different Cars Using the DVN-MST Clustering Algorithm.

Threshold r	Corresponding clustering result
$r = d_{73} = 0.19856$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$
$r = d_{52} = 0.1683$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_8\}, \{A_7\}$
$r = d_{82} = 0.15652$	$\{A_1, A_2, A_3, A_4, A_8,\}, \{A_6, A_5,\}, \{A_7\}$
$r = d_{42} = 0.151108$	$\{A_1, A_2, A_3, A_4\}, \{A_6, A_5\}, \{A_8\}, \{A_7\}$
$r = d_{31} = 0.13775$	$\{A_1, A_2, A_3\}, \{A_4\}, \{A_6, A_5\}, \{A_8\}, \{A_7\}$
$r = d_{32} = 0.12207$	$\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_6, A_5\}, \{A_7\}, \{A_8\}$
$r = d_{56} = 0.11247$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_6, A_5\}, \{A_7\}, \{A_8\}$
$r = 0$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}$

Step 4: Select a threshold r and disconnect all the edges of the MST with weights greater than r to obtain a certain number of subtrees (clusters), as listed in Table 1.

To compare the DVN-MST clustering algorithm with the SVN-MST clustering algorithm, IF-MST clustering algorithm, and fuzzy MST clustering algorithm, the following example discussed in Refs. [28, 30, 31] is introduced for comparative convenience.

Example 10: For the completion of an operational mission, the six sets of operational plans are made (adapted from Refs. [28, 30, 31]). To group these operational plans with respect to their comprehensive function, a military committee has been set up to provide assessment information on them. The attributes that are considered here in assessment of the six operational plans, $A_j (j = 1, 2, …, 6)$, are based on the effectiveness of (i) operational organisation (x_1) and (ii) operational command (x_2). The weight vector of the attributes $x_i (i = 1, 2)$ is $w = (0.45, 0.55)^T$. The military committee evaluates the performance of the six operational plans $A_j (j = 1, 2, …, 6)$ with respect to the attributes $x_i (i = 1, 2)$ and provides the DVNSs as follows:

$$A_1 = \{\langle x_1, 0.7, 0.2, 0.05, 0.15\rangle, \langle x_2, 0.6, 0.3, 0.4, 0.2\rangle\},$$
$$A_2 = \{\langle x_1, 0.4, 0.3, 0.04, 0.35, \rangle, \langle x_2, 0.8, 0.1, 0.3, 0.1\rangle\},$$
$$A_3 = \{\langle x_1, 0.55, 0.2, 0.02, 0.25, \rangle, \langle x_2, 0.7, 0.1, 0.05, 0.15\rangle\},$$
$$A_4 = \{\langle x_1, 0.44, 0.2, 0.3, 0.35, \rangle, \langle x_2, 0.6, 0.2, 0.1, 0.2\rangle\},$$
$$A_5 = \{\langle x_1, 0.5, 0.15, 0.2, 0.35, \rangle, \langle x_2, 0.75, 0.1, 0.1, 0.2\rangle\},$$
$$A_6 = \{\langle x_1, 0.55, 0.2, 0.3, 0.25, \rangle, \langle x_2, 0.57, 0.2, 0.3, 0.15\rangle\}.$$

Then, the DVN-MST clustering algorithm is utilised to group these operational plans $A_j (j = 1, 2, …, 6)$.

Step 1: Calculate the distance matrix $D = d_{ij} = d_\lambda (A_i, A_j)$ by Algorithm 3 (take $\lambda = 2$). The double-valued neutrosophic distance matrix $D = (d_{ij})_{m \times m}$ is obtained as follows:

$$
D = \begin{bmatrix}
0 & 0.17179 & 0.16679 & 0.18135 & 0.18106 & 0.11792 \\
0.17179 & 0 & 0.12304 & 0.15053 & 0.11728 & 0.14621 \\
0.16679 & 0.12304 & 0 & 0.12142 & 0.079773 & 0.1453 \\
0.18135 & 0.15053 & 0.12142 & 0 & 0.079246 & 0.091944 \\
0.18106 & 0.11728 & 0.079773 & 0.079246 & 0 & 0.12036 \\
0.11792 & 0.14621 & 0.1453 & 0.091944 & 0.12036 & 0
\end{bmatrix}.
$$

Step 2: The double-valued neutrosophic graph $G(V, E)$ where every edge between A_i and $A_j (i, j = 1, 2, \ldots,$ 8) is assigned the double-valued neutrosophic weighted distance d_{ij} coming from an element of the double-valued neutrosophic distance matrix $D = (d_{ij})_{m \times m}$, which represents the dissimilarity degree between the samples A_i and A_j. Then, the double-valued neutrosophic graph $G(V, E)$ is shown in Figure 3.

Step 3: Construct the MST of the double-valued neutrosophic graph $G(V, E)$.

1. The sorted list of distances of edges of G in increasing order by weights is $d_{54} \le d_{35} \le d_{46} \le d_{52} \le d_{61} \le d_{56} \le d_{43} \le d_{32} \le d_{63} \le d_{26} \le d_{42} \le d_{13} \le d_{12} \le d_{51} \le d_{41}$

2. Keep an empty subgraph S of G and add the edge e with the smallest weight to S, where the end points of e are disconnected.

3. The edge between A_5 and A_4, $d_{54} = 0.079246$, is the smallest; it is added to S and deleted from the sorted list.

4. The next smallest edge is selected from G and if no cycle formed in S, it is added to S and deleted from the list.

5. Repeat process 4 until the subgraph S spans six nodes or has $(5 - 1)$ edges.

The MST S of the double-valued neutrosophic graph $G(V, E)$ is obtained, as shown in Figure 4.

Step 4: Select a threshold r and disconnect all the edges of the MST with weights greater than r to obtain clusters, as listed in Table 2.
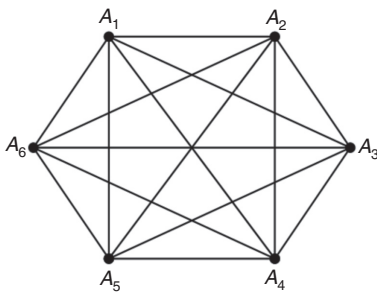

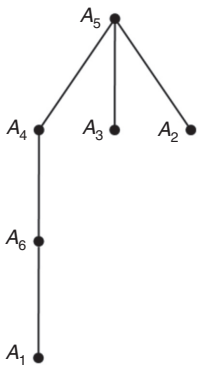**Figure 3:** Double-Valued Neutrosophic Graph G for Operational Plans.

Figure 4: DVN-MST S for Operational Plans.

Table 2: Clustering Results of the Six Sets of Operational Plans Using DVN-MST.

Threshold r	Corresponding clustering result
$r = d_{61} = 0.11792$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
$r = d_{52} = 0.11728$	$\{A_2, A_3, A_4, A_5, A_6\}, \{A_1\}$
$r = d_{64} = 0.091944$	$\{A_3, A_4, A_5, A_6\}, \{A_2\} \{A_1\}$
$r = d_{53} = 0.079773$	$\{A_3, A_4, A_5\}, \{A_6\}, \{A_2\}, \{A_1\}$
$r = d_{54} = 0.079246$	$\{A_3\}, \{A_4, A_5\}, \{A_6\}, \{A_2\}, \{A_1\}$
$r = 0$	$\{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_2\}, \{A_1\}$

7 Comparison of DVN-MST Clustering Algorithm with Other Clustering Algorithms

For comparative purposes, the DVN-MST clustering algorithm, SVN-MST clustering algorithm, IF-MST clustering algorithm, and fuzzy clustering algorithm are applied to the problem given in Example 10. In Example 10, the DVN-MST clustering algorithm is applied to the military committee evaluation of the performance of the six operational plans; its related clustering results are given in Table 2.

7.1 Results of the SVN-MST Clustering Algorithm

To perform a comparison of the DVN-MST clustering algorithm with the SVN-MST clustering algorithm that was proposed in Ref. [28], assume that the indeterminacy is not classified into two but is represented as a single-valued neutrosophic set; then, the information will be the single-valued neutrosophic data (adopted in Ref. [28]). The accuracy of the indeterminacy is lost in this SVNS representation.

Example 11: The military committee evaluates the performance of the six operational plans $A_j(j=1, 2, \ldots, 6)$ with respect to the attributes $x_i(i=1, 2)$ and gives the SVNSs as follows:

$$A_1 = \{\langle x_1, 0.7, 0.2, 0.15 \rangle, \langle x_2, 0.6, 0.3, 0.2 \rangle\},$$
$$A_2 = \{\langle x_1, 0.4, 0.3, 0.35, \rangle, \langle x_2, 0.8, 0.1, 0.1 \rangle\},$$
$$A_3 = \{\langle x_1, 0.55, 0.2, 0.25, \rangle, \langle x_2, 0.7, 0.1, 0.15 \rangle\},$$
$$A_4 = \{\langle x_1, 0.44, 0.2, 0.35, \rangle, \langle x_2, 0.6, 0.2, 0.2 \rangle\},$$
$$A_5 = \{\langle x_1, 0.5, 0.15, 0.35, \rangle, \langle x_2, 0.75, 0.1, 0.2 \rangle\},$$
$$A_6 = \{\langle x_1, 0.55, 0.2, 0.25, \rangle, \langle x_2, 0.57, 0.2, 0.15 \rangle\}.$$

Then, the results of the SVN-MST clustering algorithm utilised to group these operational plans $A_j(j=1, 2, \ldots, 6)$ are given in Table 3.

7.2 Results of the IF-MST Clustering Algorithm

To perform a comparison of the DVN-MST clustering algorithm with the IF-MST clustering algorithm that was proposed in Ref. [33], assume that the indeterminacy is not considered independently but is represented as an intuitionistic fuzzy data (adopted in Ref. [33]).

Example 12: The military committee evaluates the performance of the six operational plans $A_j(j=1, 2, \ldots, 6)$ with respect to the attributes $x_i(i=1, 2)$ and gives the IFSs as follows:

Table 3: Clustering Results of the Six Operational Plans Using SVN-MST.

Threshold r	Corresponding clustering result
$r = d_{23} = 0.1127$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
$r = d_{16} = 0.1051$	$\{A_2\}, \{A_1, A_3, A_4, A_5, A_6\}$
$r = d_{36} = 0.0842$	$\{A_1\}, \{A_2\}, \{A_3, A_4, A_5, A_6\}$
$r = d_{46} = 0.0784$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4, A_6\}$
$r = d_{35} = 0.0764$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4\}, \{A_6\}$
$r = 0$	$\{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_2\}, \{A_1\}$

Table 4: Clustering Results of the Operational Plans Using IF-MST.

Threshold r	Corresponding clustering result
$r = d_{16} = 0.1115$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
$r = d_{25} = d_{35} = 0.1$	$\{A_1\}, \{A_2, A_3, A_4, A_5, A_6\}$
$r = d_{46} = 0.088$	$\{A_1\}, \{A_2\}, \{A_5\}, \{A_3, A_4, A_6\}$
$r = d_{46} = 0.0715$	$\{A_1\}, \{A_2\}, \{A_4\}, \{A_5\}, \{A_3, A_6\}$
$r = 0$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}$

$$A_1 = \{\langle x_1, 0.7, 0.15\rangle, \langle x_2, 0.6, 0.2\rangle\}, \quad A_2 = \{\langle x_1, 0.4, 0.35,\rangle, \langle x_2, 0.8, 0.1\rangle\},$$
$$A_3 = \{\langle x_1, 0.55, 0.25,\rangle, \langle x_2, 0.7, 0.15\rangle\}, \quad A_4 = \{\langle x_1, 0.44, 0.35,\rangle, \langle x_2, 0.6, 0.2\rangle\},$$
$$A_5 = \{\langle x_1, 0.5, 0.35,\rangle, \langle x_2, 0.75, 0.2\rangle\}, \quad A_6 = \{\langle x_1, 0.55, 0.25,\rangle, \langle x_2, 0.57, 0.15\rangle\}.$$

The results of the IF-MST clustering algorithm utilised to group these operational plans $A_j(j = 1, 2, \ldots, 6)$ are given in Table 4.

7.3 Results of the Fuzzy MST Clustering Algorithm

To perform comparison of the DVN-MST clustering algorithm with the fuzzy MST clustering algorithm, assume that indeterminacy is not considered; it is represented as fuzzy data (adopted in Ref. [33]).

Example 13: The military committee evaluates the performance of the six operational plans $A_j(j = 1, 2, \ldots, 6)$ with respect to the attributes $x_i(i = 1, 2)$ and gives the fuzzy data as follows:

$$A_1 = \{\langle x_1, 0.7\rangle, \langle x_2, 0.6\rangle\}, \quad A_2 = \{\langle x_1, 0.4\rangle, \langle x_2, 0.8\rangle\},$$
$$A_3 = \{\langle x_1, 0.55\rangle, \langle x_2, 0.7\rangle\}, \quad A_4 = \{\langle x_1, 0.44\rangle, \langle x_2, 0.6\rangle\},$$
$$A_5 = \{\langle x_1, 0.5\rangle, \langle x_2, 0.75\rangle\}, \quad A_6 = \{\langle x_1, 0.55\rangle, \langle x_2, 0.57\rangle\}$$

Then, the results of the fuzzy MST clustering algorithm utilised to group these operational plans $A_j(j = 1, 2, \ldots, 6)$ are given in Table 5.

7.4 Comparison with DVN-MST

The comparison of the results of the DVN-MST clustering algorithm with the results of the SVN-MST clustering algorithm, IF-MST clustering algorithm, and fuzzy MST clustering algorithm is given in Table 6 for comparative purposes.

From Table 6, it is seen that the clustering results of the four clustering algorithms are rather different. The important reason can be obtained by the following comparative analysis of the clustering algorithms and their capacity to deal with indeterminate, inconsistent, and incomplete information.

Table 5: Clustering Results of the Six Operational Plans Using Fuzzy MST.

Threshold r	Corresponding clustering result
$r = d_{16} = 0.1031$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
$r = d_{36} = 0.0964$	$\{A_1\}, \{A_2, A_3, A_4, A_5, A_6\}$
$r = d_{46} = 0.0774$	$\{A_1\}, \{A_2, A_3, A_5\}, \{A_4, A_6\}$
$r = d_{25} = 0.0766$	$\{A_1\}, \{A_2, A_3, A_5\}, \{A_4\}, \{A_6\}$
$r = d_{35} = 0.0500$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4\}, \{A_6\}$
$r = 0$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}$

Table 6: Clustering Results of Different Clustering Algorithms.

Class	DVN-MST clustering algorithm	SVN-MST clustering algorithm
1	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
2	$\{A_1\}, \{A_2, A_3, A_4, A_5, A_6\}$	$\{A_2\}, \{A_1, A_3, A_4, A_5, A_6\}$
3	$\{A_1\}, \{A_2\}, \{A_3, A_4, A_5, A_6\}$	$\{A_1\}, \{A_2\}, \{A_3, A_4, A_5, A_6\}$
4	$\{A_1\}, \{A_2\}, \{A_3, A_5, A_4\}, \{A_6\}$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4, A_6\}$
5	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_5\}, \{A_6\}$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4\}, \{A_6\}$
6	$\{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_2\}, \{A_1\}$	$\{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_2\}, \{A_1\},$
	IF-MST clustering algorithm	**Fuzzy MST clustering algorithm**
1	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
2	$\{A_1\}, \{A_2, A_3, A_4, A_5, A_6\}$	$\{A_1\}, \{A_2, A_3, A_4, A_5, A_6\}$
3		$\{A_1\}, \{A_2, A_3, A_5\}, \{A_4, A_6\}$
4	$\{A_1\}, \{A_2\}, \{A_5\}, \{A_3, A_4, A_6\}$	$\{A_1\}, \{A_2, A_3, A_5\}, \{A_4\}, \{A_6\}$
5	$\{A_1\}, \{A_2\}, \{A_4\}, \{A_5\}, \{A_3, A_6\}$	$\{A_1\}, \{A_2\}, \{A_3, A_5\}, \{A_4\}, \{A_6\}$
6	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_6\}, \{A_5\}, \{A_6\}$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}$

Double-valued neutrosophic information is a generalisation of neutrosophic information. It is observed that neutrosophic information/single-valued neutrosophic information is a generalisation of intuitionistic fuzzy information, and intuitionistic fuzzy information is itself a generalisation of fuzzy information.

DVNS is an instance of a neutrosophic set, which provides more accuracy and precision to represent the existing uncertain, imprecise, incomplete, and inconsistent information. It has the additional feature of being able to describe with more sensitivity the indeterminate and inconsistent information. While the SVNS can handle indeterminate information and inconsistent information, it cannot describe with accuracy the existing indeterminacy.

It is known that the connector in the fuzzy set is defined with respect to T (membership only), so the information of indeterminacy and non-membership is lost. The connectors in IFS are defined with respect to truth membership and false membership only; here, the indeterminacy is taken as what is left after the truth and false membership.

The IFS cannot deal with the indeterminate and inconsistent information; however, it has provisions to describe and deal with incomplete information. In SVNS, truth, indeterminacy, and falsity membership are represented independently, and they can also be defined with respect to any of them (no restriction). This makes SVNS equipped to deal with information better than IFS, whereas in DVNS, more scope is given to describe and deal with the existing indeterminate and inconsistent information because the indeterminacy concept is classified as two distinct values. This provides more accuracy and precision to indeterminacy in DVNS than in SVNS.

It is clearly noted that in the case of the SVN-MST clustering algorithm that was proposed in Ref. [28], the indeterminacy concept/value is not classified into two but is represented as a single-valued neutrosophic data leading to a loss of accuracy of the indeterminacy. SVNSs are incapable of giving this amount of accuracy or

precision about the indeterminacy concept. Similarly, when the IF-MST clustering algorithm was considered, it was not possible to deal with the indeterminacy membership function independently as it is dealt in SVN-MST or DVN-MST clustering algorithm, leading to a loss of information about the existing indeterminacy. In the fuzzy MST clustering algorithm, only the membership degree is considered; details of non-membership and indeterminacy are completely lost. It is clearly observed that the DVNS representation and the DVN-MST clustering algorithm are better equipped to deal with indeterminate, inconsistent, and incomplete information.

8 Conclusions

In this paper, a modified form of a neutrosophic set, called DVNS, with two distinct indeterminate values, was introduced. More sensitivity and precision is provided to indeterminacy because the indeterminate concept/value is classified into two based on membership: one as indeterminacy leaning towards truth membership and the other as indeterminacy leaning towards false membership. This kind of classification of indeterminacy is not feasible with SVNS. DVNS is better equipped at dealing with indeterminate and inconsistent information, with more accuracy than SVNS, which fuzzy sets and IFSs are incapable of.

A generalised distance measure between DVNSs and a distance matrix is defined, based on which a clustering algorithm was developed. A DVN-MST clustering algorithm, to cluster data represented by double-valued neutrosophic information, was constructed. Clearly, the DVN-MST method proposed in this paper overcomes the disadvantage in the existing clustering algorithm using MST by giving appropriate importance to uncertain, imprecise, and incomplete information present in the data. This is the main advantage in using the proposed DVN-MST method.

Through the illustrative computational samples of the DVN-MST clustering algorithm and other clustering algorithms, the clustering results have shown that the DVN-MST clustering algorithm is more general and more reasonable than the SVN-MST, IF-MST, and fuzzy MST clustering algorithms. Furthermore, in situations that are represented by indeterminate information and inconsistent information, the DVN-MST clustering algorithm exhibits its great superiority in clustering those double-valued neutrosophic data because the DVNSs are a powerful tool to deal with uncertain, imprecise, incomplete, and inconsistent information with accuracy.

In the future, DVNS sets and the DVN-MST clustering algorithm can be applied to many areas such as online social networks, information retrieval, investment decision making, and data mining where the fuzzy theory has been used and where uncertainty and indeterminacy involved were not studied.

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