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Frank Choquet Bonferroni Mean Operators of Bipolar Neutrosophic Sets and Their Application to Multi-criteria Decision-Making Problems

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Abstract In this study, comprehensive multi-criteria decision-making (MCDM) methods are investigated under bipolar neutrosophic environment. First, the operations of BNNs are redefined based on the Frank operations considering the extant operations of bipolar neutrosophic numbers (BNNs) lack flexibility and robustness. Subsequently, the Frank bipolar neutrosophic Choquet weighted Bonferroni mean operator and the Frank bipolar neutrosophic Choquet geometric Bonferroni mean operator are proposed based on the Frank operations of BNNs. The proposed operators simultaneously consider the interactions and interrelationships among the criteria by combining the Choquet integral operator and Bonferroni mean operators. Furthermore, MCDM methods are developed based on the proposed aggregation operators. A numerical example of plant location selection is conducted to explain the application of the proposed methods, and the influences of parameters are also discussed. Finally, the proposed methods are compared with several extant methods to verify their feasibility.

Keywords Multi-criteria decision-making · Bipolar neutrosophic set · Frank operations · Choquet integral operator · Bonferroni mean operator

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1 Introduction

With the increasing complexity and uncertainty in decision environment, the representation of decision information has varied and is no longer limited to real numbers. Compared with real numbers, fuzzy numbers have been widely utilized to describe fuzzy information since the introduction of fuzzy sets (FSs) [1]. Nevertheless, FSs cannot tackle complex problems because they only have one membership. For a highly flexible description of uncertain information, Atanassov [2] proposed intuitionistic fuzzy sets (IFSs), which include both membership and non-membership degrees. Thereafter, Atanassov and Gargov [3] presented interval-valued intuitionistic fuzzy sets (IVIFSs) by extending IFSs. However, IFSs and IVIFSs cannot represent inconsistent information in practical decision-making problems. Smarandache [4] then introduced neutrosophic sets (NSs), which utilize the functions of truth, indeterminacy, and falsity to depict uncertain, incomplete and inconsistent information. Furthermore, Wang et al. [5, 6] defined the single-valued neutrosophic sets (SVNSs) and the interval neutrosophic sets (INSs) to apply NSs in practical problems.

Positive and negative effects can be generated in human minds when we make a decision. Positive information expresses what is possible, satisfactory, permitted, desired, or acceptable. By contrast, negative information states what is impossible, rejected, or forbidden [7]. Acceptable or satisfactory conception refers to positive preferences, whereas unacceptable conception refers to negative preferences. Negative preferences correspond to constraints, while positive preferences correspond to wishes [8]. For example, when a decision maker (DM) evaluates an object, he may express what he considers (more or less) acceptable or satisfactory; however, he may express what he considers unacceptable because of several constraints. To describe the aforementioned information, Zhang [9] introduced the concept of bipolar fuzzy sets (BFSs) consisting of the following two parts: positive membership degree and negative membership degree. Given the advantages of BFSs and NSs, Deli et al. [10] proposed bipolar neutrosophic sets (BNSs) that can describe bipolar, fuzzy, uncertain, and inconsistent information. Recently, Dey et al. [11] proposed the bipolar neutrosophic TOPSIS (BN-TOPSIS) method to solve multi-criteria decision-making (MCDM) problems. Pramanik et al. [12] proposed bipolar neutrosophic projection, bidirectional projection, and hybrid projection measures for handling MCDM problems. Ulucay et al. [13] defined the similarity measures of BNSs and applied them to practical problems.

The aggregation operators have been widely used in MCDM methods under various fuzzy environments [14–16]. However, most aggregation operators assume that the criteria are independent. In the decision-making process, the interactions (e.g., redundancy or complementarity) among criteria are universal. For example, a company wants to assess four plant locations in terms of four criteria: cost, expansion possibility, transportation, and labor. Expansion possibility and labor can be regarded as redundant, since the weight of a combination of expansion possibility and labor is less than the sum of their weighs. To deal with such situations, the interactions among criteria must be considered, and the Choquet integral operator represents an effective solution. Choquet integral operators, proposed by Grabisch et al. [17], can fully consider the importance of criteria and the interactions among them [18]. Numerous researchers extended the Choquet integral operators into all kinds of fuzzy environments [19–21]. Sun et al. [22] introduced the interval neutrosophic number Choquet integral operators. Meng et al. [23] introduced the generalized Banzhaf interval-valued intuitionistic fuzzy geometric Choquet operators. Furthermore, many researchers proposed several methods by combining the Choquet integral operator with other theories. Yuan and Li [24] integrated the Choquet integral operators with prospect theory and proposed the intuitionistic trapezoidal prospect Choquet integral operators. Cheng and Tang [25] introduced the generalized Shapley function into Choquet integral operators and defined the generalized Shapley interval-valued intuitionistic fuzzy geometric Choquet operators.

Aside from the interactions among the criteria, interrelationships among them also exist, such as in the case of plant location selection. In this situation, transportation and labor may influence cost, and expansion possibility may depend on the effect of transportation and labor. Dealing with such kinds of problems requires the introduction of the Bonferroni mean (BM) operator [26], which is capable of capturing the interrelationships of input arguments. However, the BM operator overlooks the importance of input arguments. To overcome this shortcoming, Xu and Yager [27] defined the weighted Bonferroni mean (WBM) operator on the basis of the BM operator. Nonetheless, the WBM operator does not meet the property of reducibility. Consequently, Zhou and He [28] introduced the normalized weighted Bonferroni mean (NWBM) operator. Afterward, Sun [29] introduced the normalized geometric Bonferroni mean (NGBM) operator. Since then, the NWBM and NGBM operators have been widely utilized in various fuzzy environments [30–32]. Liu and Wang [33] applied the NWBM operator to solve MCDM problems with SVNSs. Tian et al. [34] extended the NWBM operator into the simplified neutrosophic linguistic environment. Furthermore, Liu et al. [35] proposed the multi-valued neutrosophic NWBM operator and multi-valued neutrosophic NGBM operator.

Frank triangular norms share the characteristics of general triangular norms, such as algebraic triangular norms, Einstein triangular norms, and Hamacher triangular norms. Frank triangular norms are the only type of triangular norms that satisfy compatibility [36]. They are the generalization of algebraic triangular norms and Lukasiewicz triangular norms [37]. Frank triangular norms are also more flexible and robust than other triangular norms because they have a parameter that can be selected by DMs [38]. Frank operations are examples of Frank triangular norms, and several aggregation operators based on Frank operations are utilized for decision-making problems under fuzzy environments [36, 39]. Peng et al. [40] defined the Frank Heronian mean operator with linguistic intuitionistic fuzzy information. Qin et al. [38] developed the Frank power aggregation operator of hesitant fuzzy sets. Ji et al. [41] proposed the Frank prioritized BM operator with SVNSs.

BNSs can describe bipolar, fuzzy, uncertain, and inconsistent information involved in practical decision-making problems. However, studies on the extension of BNSs are relatively fewer than other fuzzy sets. The Choquet integral operator can consider the interactions among criteria, and the BM operators can take into account the interrelationships among input arguments. The interactions and interrelationships among criteria exist extensively in practical decision-making problems. However, no study on the bipolar neutrosophic aggregation operator has been conducted to consider the interactions and interrelationships among criteria simultaneously. In addition, compared with other operations, Frank operations are more flexible and robust. Nevertheless, Frank operations for BNSs have not been investigated.

On the basis of the aforementioned analysis, this study aims to overcome existing limitations and developing comprehensive methods. The purposes of this study are as follows. First, BNSs are utilized to depict evaluation information in the decision-making process to avoid information loss. Second, the Frank operations of BNNs are defined. Third, Frank bipolar neutrosophic Choquet BM operators are proposed by combining the Choquet integral operators and BM operators on the basis of the Frank operations of BNNs. Then, MCDM methods are established based on the proposed operators. Finally, the effectiveness and flexibility of the proposed methods are further verified through a numerical example of plant location selection, parametric analysis, and comparative analysis.

The remainder of this paper is organized as follows. Section 2 reviews several basic concepts which will be used in this study. Section 3 defines the Frank operations of bipolar neutrosophic numbers (BNNs). Section 4 discusses the proposed new Frank Choquet BM operators and their desirable properties. In addition, this section proposes comprehensive MCDM methods based on the proposed operators. Section 5 provides a numerical example of plant location selection and discusses the influences of parameters. It likewise demonstrates a comparative analysis with other methods. Finally, Sect. 6 concludes this paper.

2 Preliminaries

This section reviews the basic concepts of BNSs, BNNs, Choquet integral operators, BM operators, and Frank operations. These concepts will be used in subsequent sections.

2.1 Bipolar Neutrosophic Sets and Bipolar Neutrosophic Numbers

Definition 1 [10] Let X be a fixed set. Then, the BNS can be defined as follows:

$$A(x) = \{\langle x,\, T_A^+(x),\, I_A^+(x), F_A^+(x), T_A^-(x),\, I_A^-(x), F_A^-(x)\rangle | x \in X\},$$

where $T_A^+(x)$, $I_A^+(x), F_A^+(x) : X \to [0,1]$ and $T_A^-(x)$, $I_A^-(x)$, $F_A^-(x) : X \to [-1,0]$. The positive membership degrees $T_A^+(x)$, $I_A^+(x)$, and $F_A^+(x)$ are the truth membership, indeterminacy membership, and falsity membership degrees of a point $x \in X$ corresponding to a BNS A, and the negative membership degrees $T_A^-(x), I_A^-(x), F_A^-(x)$ denote the truth membership, indeterminate membership, and false membership degrees of a point $x \in X$ to some implicit counter-property corresponding to a BNS A. In particular, if X has only one element, then $A(x) = \langle T_A^+(x),\, I_A^+(x), F_A^+(x),$ $T_A^-(x),\, I_A^-(x), F_A^-(x)\rangle$ is called a BNN. For convenience, a BNN is denoted as $A = \langle T_A^+, I_A^+, F_A^+, T_A^-, I_A^-, F_A^-\rangle$.

Deli et al. [10] defined the algebraic operations of BNNs as follows.

Definition 2 [10] Let $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^-\rangle$ and $b = \langle T_b^+, I_b^+, F_b^+, T_b^-, I_b^-, F_b^-\rangle$ be two BNNs; then, the operations of BNNs are defined as follows:

(1) $a \oplus b = \langle T_a^+ + T_b^+ - T_a^+ T_b^+, I_a^+ I_b^+, F_a^+ F_b^+, -T_a^- T_b^-, -(-I_a^- - I_b^- - I_a^- I_b^-), -(-F_a^- - F_b^- - F_a^- F_b^-)\rangle$;

(2) $\lambda a = \left\langle 1-(1-T_a^+)^\lambda, (I_a^+)^\lambda, (F_a^+)^\lambda, -(-T_a^-)^\lambda, -(1-(1-(-I_a^-))^\lambda), -(1-(1-(-F_a^-))^\lambda)\right\rangle (\lambda > 0)$;

(3) $a \otimes b = \langle T_a^+ T_b^+, I_a^+ + I_b^+ - I_a^+ I_b^+, F_a^+ + F_b^+ - F_a^+ F_b^+, -(-T_a^- - T_b^- - T_a^- T_b^-), -I_a^- I_b^-, -F_a^- F_b^-\rangle$;

(4) $a^\lambda = \left\langle (T_a^+)^\lambda, 1-(1-I_a^+)^\lambda, 1-(1-F_a^+)^\lambda, -(1-(1-(-T_a^-))^\lambda), -(-I_a^-)^\lambda, -(-F_a^-)^\lambda\right\rangle (\lambda > 0)$.

Deli et al. [10] proposed a comparison method that includes score, accuracy, and certainty functions to compare two BNNs.

Definition 3 [10] Let $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^-\rangle$ be a BNN; then, the score function $S(a)$, accuracy function $A(a)$, and certainty function $C(a)$ are defined as follows:

$$S(a) = \frac{T_a^+ + 1 - I_a^+ + 1 - F_a^+ + 1 + T_a^- - I_a^- - F_a^-}{6}, \tag{1}$$

$$A(a) = T_a^+ - F_a^+ + T_a^- - F_a^-, \tag{2}$$

$$C(a) = T_a^+ - F_a^-. \tag{3}$$

Based on Eqs. (1)–(3), the ranking methods of BNNs can be derived as follows.

Definition 4 [10] Let $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^-\rangle$ and $b = \langle T_b^+, I_b^+, F_b^+, T_b^-, I_b^-, F_b^-\rangle$ be two BNNs. Therefore,

(1) If $S(a) > S(b)$, then $a > b$;
(2) If $S(a)$=$S(b)$ and $A(a) > A(b)$, then $a > b$;
(3) If $S(a) = S(b)$, $A(a) = A(b)$, and $C(a) > C(b)$, then $a > b$;
(4) If $S(a) = S(b)$, $A(a) = A(b)$, and $C(a) = C(b)$, then $a = b$.

2.2 Choquet Integral Operators and BM Operators

This subsection reviews the Choquet integral operators and BM operators.

The Choquet integral operator is the generalization of the weighted average (WA), ordered WA (OWA), and max–min operators. The fuzzy measures defined by Sugeno [42] play an important role in the Choquet integral operator. Fuzzy measures can capture the importance of each criterion and the importance of a combination of criteria. Therefore, fuzzy measures can be employed to measure the interactive characteristics among criteria. The concepts of fuzzy measures and the Choquet integral operator are defined as follows.

Definition 5 [42] Let C be a set of criteria. Then, a set function h: $P(C) \to [0,1]$ is called a fuzzy measure if this set function satisfies the following axioms.

(1) $h(\emptyset) = 0$: the empty set has no importance.
(2) $h(C) = 1$: the full set has maximal importance.
(3) If $A \subseteq C, B \subseteq C$ and $A \subseteq B$, then $h(A) \leq h(B)$: a new added criterion cannot diminish the importance of a criterion set.

Sugeno [42] proposed a special kind of fuzzy measure based on $P(C)$ which satisfies the finite λ-fuzzy measure and fulfills the following additional property:

$$h(A \cup B) = h(A) + h(B) + \theta h(A)h(B), \tag{4}$$

where $A \cap B = \emptyset$ for all $A, B \in P(C)$, and $\theta > -1$.

(1) If $\theta = 0$, then $h(A \cup B) = h(A) + h(B)$. This condition is called an additive measure, which indicates no interaction between A and B.
(2) If $\theta > 0$, then $h(A \cup B) > h(A) + h(B)$. A positive synergetic interaction exists between A and B, which indicates that the set $\{A,\ B\}$ has a multiplicative effect.
(3) If $\theta < 0$, then $h(A \cup B) < h(A) + h(B)$. A negative synergetic interaction exists between A and B, which indicates that set $\{A,\ B\}$ has a substitutive effect.

Therefore, θ can be utilized to represent the interactions among criteria and it is used in the Choquet integral operator.

Let C be a set of criteria, and $\cup_{j=1}^{n} c_j = C$. The θ-fuzzy measure h can be defined as follows:

$$h(C) = \begin{cases} \frac{1}{\theta}\left(\prod_{j=1}^{n}\left(1 + \theta h(c_j)\right) - 1\right), & \text{if } \theta \neq 0 \\ \sum_{j=1}^{n} h(c_j), & \text{if } \theta = 0 \end{cases}, \tag{5}$$

where $c_k \cap c_j = \emptyset$ and $k \neq j$. If a subset has one element c_j, then $h(c_j)$ can be designated as a fuzzy density. Particularly, for every subset $A \subseteq C$, the following expression can be derived:

$$h(A) = \begin{cases} \frac{1}{\theta}\left(\prod_{c_j \in A}\left(1 + \theta h(c_j)\right) - 1\right), & \text{if } \theta \neq 0 \\ \sum_{c_j \in A} h(c_j), & \text{if } \theta = 0 \end{cases}. \tag{6}$$

On the basis of Eq. (5), θ can be determined from $h(C) = 1$, which can be simplified as follows:

$$\theta + 1 = \prod_{j=1}^{n}\left(1 + \theta h(c_j)\right). \tag{7}$$

Definition 6 [17] Let g be a positive real-valued function on C and h be a fuzzy measure on C. Thereafter, the discrete Choquet integral of g with respect to h is defined as follows:

$$C_\eta(g) = \sum_{j=1}^{n} g\left(c_{(j)}\right)\left[h\left(A_{(j)}\right) - h\left(A_{(j+1)}\right)\right], \tag{8}$$

where $g\left(c_{(1)}\right) \leq g\left(c_{(2)}\right) \leq \cdots \leq g\left(c_{(n)}\right)$, $A_{(j)} = \left\{c_{(j)}, \ldots, c_{(n)}\right\}$ and $A_{(n+1)} = \emptyset$.

The BM operator introduced by Bonferroni [26] emphasizes the interrelationships among aggregated arguments. Considering the importance of input arguments, NWBM [28] and NGBM [29] operators were introduced.

Definition 7 [28] Let $s, t \geq 0$, and $\vartheta_i (i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers. Then, the NWBM operator is expressed as follows:

$$\mathrm{NWBM}^{s,t}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1 - w_i} \vartheta_i^s \vartheta_j^t\right)^{\frac{1}{s+t}}. \tag{9}$$

where $w = \{w_1, w_2, \ldots, w_n\}$ is the weight vector of ϑ_i and $\sum_{i=1}^{n} w_i = 1$.

Definition 8 [29] Let $s, t \geq 0$, and $\vartheta_i (i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers. Then, the NGBM operator is expressed as follows:

$$\mathrm{NGBM}^{s,t}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \frac{1}{s+t} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(s\vartheta_i + t\vartheta_j\right)^{\frac{w_i w_j}{1 - w_i}}, \tag{10}$$

where $w = \{w_1, w_2, \ldots, w_n\}$ is the weight vector of ϑ_i and $\sum_{i=1}^{n} w_i = 1$.

2.3 Frank Operations

Frank triangular norms have been investigated by numerous researchers [43, 44] because of their flexibility and robustness. Frank operations are examples of Frank triangular norms. Similar to algebraic operations, Einstein operations, and Hamacher operations, Frank operations also contain product and sum (accordingly designated as Frank product and Frank sum).

Definition 9 [45] Let α and β be two real numbers. The Frank product $\otimes_F$ and the Frank sum $\oplus_F$ between α and β are then defined as

$$\alpha \oplus_F \beta = 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\alpha} - 1)(\lambda^{1-\beta} - 1)}{\lambda - 1}\right),$$

$$\alpha \otimes_F \beta = \log_\lambda\left(1 + \frac{(\lambda^{\alpha} - 1)(\lambda^{\beta} - 1)}{\lambda - 1}\right),$$

where $\alpha, \beta \in [0, 1]$ and $\lambda \in (1, +\infty)$.

Some special cases can be easily proved as well [37].

(1) If $\lambda \to 1$, then $\alpha \oplus_F \beta \to \alpha + \beta - \alpha\beta$ and $\alpha \otimes_F \beta \to \alpha\beta$. The Frank product and Frank sum are reduced to the algebraic product and algebraic sum, respectively.
(2) If $\lambda \to \infty$, then $\alpha \oplus_F \beta \to \min(\alpha + \beta, 1)$ and $\alpha \otimes_F \beta \to \max(0, \alpha + \beta - 1)$. The Frank product and Frank sum are reduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

3 Frank Operations of BNNs

This section defines the Frank operations of BNNs and discusses several of their properties.

Given the efficiency and comprehensiveness of Frank operations, defining Frank operations under bipolar neutrosophic environment is beneficial. On the basis of existing Frank operations, the Frank operations of BNNs are defined as follows.

Definition 10 Let $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^- \rangle$ and $b = \langle T_b^+, I_b^+, F_b^+, T_b^-, I_b^-, F_b^- \rangle$ be two BNNs and $\lambda > 1$. Then, the Frank operations of BNNs are defined as follows.

$$(1)\, a \oplus_F b = \left\langle 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{1-T_a^+} - 1\right)\left(\lambda^{1-T_b^+} - 1\right)}{\lambda - 1}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{I_a^+} - 1\right)\left(\lambda^{I_b^+} - 1\right)}{\lambda - 1}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{F_a^+} - 1\right)\left(\lambda^{F_b^+} - 1\right)}{\lambda - 1}\right),\right.$$
$$\left. -\log_\lambda\left(1 + \frac{(\lambda^{-T_a^-} - 1)(\lambda^{-T_b^-} - 1)}{\lambda - 1}\right), \log_\lambda\left(1 + \frac{(\lambda^{1+I_a^-} - 1)(\lambda^{1+I_b^-} - 1)}{\lambda - 1}\right) - 1, \log_\lambda\left(1 + \frac{(\lambda^{1+F_a^-} - 1)(\lambda^{1+F_b^-} - 1)}{\lambda - 1}\right) - 1\right\rangle;$$

$$(2)\, a \otimes_F b = \left\langle \log_\lambda\left(1 + \frac{\left(\lambda^{T_a^+} - 1\right)\left(\lambda^{T_b^+} - 1\right)}{\lambda - 1}\right), 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{1-I_a^+} - 1\right)\left(\lambda^{1-I_b^+} - 1\right)}{\lambda - 1}\right), 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{1-F_a^+} - 1\right)\left(\lambda^{1-F_b^+} - 1\right)}{\lambda - 1}\right),\right.$$
$$\left. \log_\lambda\left(1 + \frac{(\lambda^{1+T_a^-} - 1)(\lambda^{1+T_b^-} - 1)}{\lambda - 1}\right) - 1, -\log_\lambda\left(1 + \frac{(\lambda^{-I_a^-} - 1)(\lambda^{-I_b^-} - 1)}{\lambda - 1}\right), -\log_\lambda\left(1 + \frac{(\lambda^{-F_a^-} - 1)(\lambda^{-F_b^-} - 1)}{\lambda - 1}\right)\right\rangle;$$

$$(3)\, k \cdot_F a = \left\langle 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{T_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{I_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{F_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right),\right.$$
$$\left. -\log_\lambda\left(1 + \frac{(\lambda^{-T_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{-I_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right) - 1, \log_\lambda\left(1 + \frac{(\lambda^{-F_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right) - 1\right\rangle k > 0;$$

$$(4)\, a^{\wedge_F k} = \left\langle \log_\lambda\left(1 + \frac{\left(\lambda^{T_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right), 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{I_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right), 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{F_a^+} - 1\right)^k}{(\lambda - 1)^{k-1}}\right),\right.$$
$$\left. \log_\lambda\left(1 + \frac{(\lambda^{-T_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right) - 1, -\log_\lambda\left(1 + \frac{(\lambda^{-I_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right), -\log_\lambda\left(1 + \frac{(\lambda^{-F_a^-} - 1)^k}{(\lambda - 1)^{k-1}}\right)\right\rangle k > 0.$$

Theorem 1 *Let* $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^- \rangle$ *and* $b = \langle T_b^+, I_b^+, F_b^+, T_b^-, I_b^-, F_b^- \rangle$ *be two BNNs, and let* $c = a \oplus_F b$, $d = a \otimes_F b$, $e = k \cdot_F a (k > 0)$, *and* $f = a^{\wedge_F k} (k > 0)$. *Thereafter*, c, d, e, *and* f *are also BNNs*.

As Theorem 1 can be easily verified, the proof is omitted here.

The properties of Frank operations of BNNs will be discussed as follows.

Theorem 2 *Let* $a = \langle T_a^+, I_a^+, F_a^+, T_a^-, I_a^-, F_a^- \rangle$ *and* $b = \langle T_b^+, I_b^+, F_b^+, T_b^-, I_b^-, F_b^- \rangle$ *be two BNNs, and* $k, k_1, k_2 > 0$, and *the following properties can be proven easily*.

(1) $a \oplus_F b = b \oplus_F a$,
(2) $a \otimes_F b = b \otimes_F a$,
(3) $k \cdot_F (a \oplus_F b) = k \cdot_F a \oplus_F k \cdot_F b$,
(4) $(a \otimes_F b)^{\wedge_F k} = a^{\wedge_F k} \otimes_F b^{\wedge_F k}$,
(5) $(k_1 + k_2) \cdot_F a = k_1 \cdot_F a \oplus_F k_2 \cdot_F a$,
(6) $a^{\wedge_F (k_1 + k_2)} = a^{\wedge_F k_1} \otimes_F a^{\wedge_F k_2}$.

4 Frank Bipolar Neutrosophic Choquet Bonferroni Mean Operators

This section proposes the Frank bipolar neutrosophic Choquet weighted Bonferroni mean (FBNCWBM) and Frank bipolar neutrosophic Choquet geometric Bonferroni mean (FBNCGBM) operators and discusses several properties of these two aggregation operators.

The FBNCWBM and FBNCGBM operators are proposed by combining the Choquet integral and BM operators on the basis of Frank operations. Such combination considers the interactions among criteria and the interrelationships of input arguments comprehensively, as well as takes advantage of the Frank operations under bipolar neutrosophic environment.

4.1 Frank Bipolar Neutrosophic Choquet Weighted Bonferroni Mean Operator

Definition 11 Let s, $t \geq 0$, and $\sigma_i = \langle T_i^+, I_i^+, F_i^+, T_i^-, I_i^-, F_i^- \rangle$ $(i = 1, 2, \ldots n)$ be a collection of BNNs. Then, the FBNCWBM operator is defined as follows:

$$\mathrm{FBNCWBM}^{s,t}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n}{}_F \frac{w_{(i)} w_{(j)}}{1 - w_{(i)}} \cdot_F \left(\left(\sigma_{(i)}\right)^{\wedge_F^s} \otimes_F \left(\sigma_{(j)}\right)^{\wedge_F^t} \right) \right)^{\wedge_F \frac{1}{s+t}}, \tag{11}$$

where $w_{(i)} = h(A_{(i)}) - h(A_{(i+1)})$ and $h(A_{(i)})$ is obtained using Eqs. (6) and (7).

Theorem 3 *Let* $\sigma_i = \langle T_i^+, I_i^+, F_i^+, T_i^-, I_i^-, F_i^- \rangle$ *be a collection of BNNs. Then, the value aggregated by* Eq. (11) *is also a BNN, and*

$$\begin{aligned}&\mathrm{FBNCWBM}^{s,t}(\sigma_1,\sigma_2,\ldots,\sigma_n)\\&=\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus_F}}\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\right)^{\wedge_F \frac{1}{s+t}}\\&=\Big\langle f\left(T_{(i)}^+,T_{(j)}^+\right),g\left(I_{(i)}^+,I_{(j)}^+\right),g\left(F_{(i)}^+,F_{(j)}^+\right),\\&\quad -g\left(-T_{(i)}^-,-T_{(j)}^-\right),-f\left(-I_{(i)}^-,-I_{(j)}^-\right),-f\left(-F_{(i)}^-,-F_{(j)}^-\right)\Big\rangle,\end{aligned} \tag{12}$$

where $w_{(i)} = h\left(A_{(i)}\right) - h\left(A_{(i+1)}\right)$,

$$f\left(x_{(i)},x_{(j)}\right) = \log_\lambda\left(1+(\lambda-1)\left(\frac{1-\prod\limits_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{x(i)}-1\right)^s\left(\lambda^{x(j)}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{x(i)}-1\right)^s\left(\lambda^{x(j)}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}{1+(\lambda-1)\prod\limits_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{x(i)}-1\right)^s\left(\lambda^{x(j)}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{x(i)}-1\right)^s\left(\lambda^{x(j)}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}\right)^{\frac{1}{s+t}}\right),$$

and

$$g\left(y_{(i)},y_{(j)}\right) = 1-\log_\lambda\left(1+(\lambda-1)\left(\frac{1-\prod\limits_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-y(i)}-1\right)^s\left(\lambda^{1-y(j)}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-y(i)}-1\right)^s\left(\lambda^{1-y(j)}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}{1+(\lambda-1)\prod\limits_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-y(i)}-1\right)^s\left(\lambda^{1-y(j)}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-y(i)}-1\right)^s\left(\lambda^{1-y(j)}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}\right)^{\frac{1}{s+t}}\right).$$

The proof of Theorem 3 is shown in Appendix.

Similar to [35] and [41], the following theorems of the FBNCWBM operator can be obtained.

Theorem 4 (Monotonicity). *Let* $\alpha_i = \left\langle T_{\alpha_i}^+, I_{\alpha_i}^+, F_{\alpha_i}^+, T_{\alpha_i}^-, I_{\alpha_i}^-, F_{\alpha_i}^- \right\rangle$ *and* $\beta_i = \left\langle T_{\beta_i}^+, I_{\beta_i}^+, F_{\beta_i}^+, T_{\beta_i}^-, I_{\beta_i}^-, F_{\beta_i}^- \right\rangle$ $(i = 1, 2, \ldots, n)$ *be two collections of BNNs. If* $\alpha_i \le \beta_i$ *for all i, that* $T_{\alpha_i}^+ \le T_{\beta_i}^+$, $I_{\alpha_i}^+ \ge I_{\beta_i}^+$, $F_{\alpha_i}^+ \ge F_{\beta_i}^+$, $T_{\alpha_i}^- \ge T_{\beta_i}^-$, $I_{\alpha_i}^- \le I_{\beta_i}^-$ *and* $F_{\alpha_i}^- \le F_{\beta_i}^-$ *for all i, then*

$$\mathrm{FBNCWBM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n) \le \mathrm{FBNCWBM}^{s,t}(\beta_1,\beta_2,\ldots,\beta_n). \tag{13}$$

Theorem 5 (Boundedness). *Let* $\alpha_i = \langle T_i^+, I_i^+, F_i^+, T_i^-, I_i^-, F_i^- \rangle (i = 1, 2, \ldots, n)$ *be a collection of BNNs. Thereafter, we have*

$$\begin{aligned}&\mathrm{FBNCWBM}^{s,t}(\alpha^-,\alpha^-,\ldots,\alpha^-) \le \mathrm{FBNCWBM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&\quad\le \mathrm{FBNCWBM}^{s,t}(\alpha^+,\alpha^+,\ldots,\alpha^+),\end{aligned}$$

where

$$\begin{aligned}\alpha^+ &= \langle T_{\alpha^+}^+, I_{\alpha^+}^+, F_{\alpha^+}^+, T_{\alpha^+}^-, I_{\alpha^+}^-, F_{\alpha^+}^- \rangle\\&= \Big\langle \max\left(T_{\alpha_1}^+, T_{\alpha_2}^+, \ldots, T_{\alpha_n}^+\right), \min\left(I_{\alpha_1}^+, I_{\alpha_2}^+, \ldots, I_{\alpha_n}^+\right),\\&\quad \min\left(F_{\alpha_1}^+, F_{\alpha_2}^+, \ldots, F_{\alpha_n}^+\right), \min\left(T_{\alpha_1}^-, T_{\alpha_2}^-, \ldots, T_{\alpha_n}^-,\right)\\&\quad \max\left(I_{\alpha_1}^-, I_{\alpha_2}^-, \ldots, I_{\alpha_n}^-\right), \max\left(F_{\alpha_1}^-, F_{\alpha_2}^-, \ldots, F_{\alpha_n}^-\right)\Big\rangle;\\ \alpha^- &= \langle T_{\alpha^-}^+, I_{\alpha^-}^+, F_{\alpha^-}^+, T_{\alpha^-}^-, I_{\alpha^-}^-, F_{\alpha^-}^- \rangle\\&= \Big\langle \min\left(T_{\alpha_1}^+, T_{\alpha_2}^+, \ldots, T_{\alpha_n}^+\right), \max\left(I_{\alpha_1}^+, I_{\alpha_2}^+, \ldots, I_{\alpha_n}^+\right),\\&\quad \max\left(F_{\alpha_1}^+, F_{\alpha_2}^+, \ldots, F_{\alpha_n}^+\right), \max\left(T_{\alpha_1}^-, T_{\alpha_2}^-, \ldots, T_{\alpha_n}^-\right),\\&\quad \min\left(I_{\alpha_1}^-, I_{\alpha_2}^-, \ldots, I_{\alpha_n}^-\right), \min\left(F_{\alpha_1}^-, F_{\alpha_2}^-, \ldots, F_{\alpha_n}^-\right)\Big\rangle.\end{aligned}$$

Theorem 6 (Reducibility). *Let* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. *Then we have*

$$\begin{aligned}&\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus_F}}\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\right)^{\wedge_F \frac{1}{s+t}}\\&=\frac{1}{n(n-1)}\cdot_F\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus_F}}\left(\left(\sigma_i\right)^{\wedge_F s}\otimes\left(\sigma_j\right)^{\wedge_F t}\right)\right)^{\wedge_F \frac{1}{s+t}}.\end{aligned} \tag{14}$$

Theorem 7 (Idempotency). *Let* $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ *be a collection of BNNs. If* $\sigma_i = \sigma$ $(i = 1, 2, \ldots, n)$, *then* $\mathrm{FBNCWBM}^{s,t}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma$.

Several special cases of the FBNCWBM operator will be discussed as follows.

(1) If $t = 0$, then Eq. (11) is reduced to the Frank Bipolar neutrosophic generalized Choquet integral (FBNGC) operator as follows:

$$\begin{aligned}&\mathrm{FBNGC}(\sigma_1, \sigma_2, \ldots, \sigma_n)\\&\quad=\left(\overset{n}{\underset{i=1}{\oplus_F}}\left(h\left(A_{(i)}\right)-h\left(A_{(i+1)}\right)\right)\cdot_F\left(\sigma_{(i)}\right)^{\wedge_F s}\right)^{\wedge_F \frac{1}{s}}.\end{aligned} \tag{15}$$

(2) If $s = 1$ and $t = 0$, then Eq. (11) is reduced to the Frank Bipolar neutrosophic Choquet (FBNC) integral operator as follows:

$$\mathrm{FBNC}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \overset{n}{\underset{i=1}{\oplus_F}}\left(h\left(A_{(i)}\right)-h\left(A_{(i+1)}\right)\right)\cdot_F \sigma_{(i)}. \tag{16}$$

(3) If $\lambda \to 1$, the Frank operations are reduced to the algebraic operations. Then, Eq. (11) is reduced to the Bipolar neutrosophic Choquet weighted BM (BNCWBM) operator as follows.

$$\lim_{\lambda\to 1}\mathrm{FBNCWBM}^{s,t}(\sigma_1,\sigma_2,\ldots,\sigma_n)$$
$$=\mathrm{BNCWBM}^{s,t}(\sigma_1,\sigma_2,\ldots,\sigma_n)=\left(\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n}\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\left(\sigma_{(i)}^{s}\otimes\sigma_{(j)}^{t}\right)\right)^{\frac{1}{s+t}}$$
$$=\left\langle\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-T_{(i)}^{s}T_{(j)}^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}},\right.$$
$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_{(i)}\right)^{s}\left(1-I_{(j)}\right)^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}},$$
$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-F_{(i)}\right)^{s}\left(1-F_{(j)}\right)^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}},$$
$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(-T_{(i)}^{-}\right)\right)^{s}\left(1-\left(-T_{(j)}^{-}\right)\right)^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}}\right),$$
$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(-I_{(i)}^{-}\right)^{s}\left(-I_{(j)}^{-}\right)^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}},$$
$$\left.-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(-F_{(i)}^{-}\right)^{s}\left(-F_{(j)}^{-}\right)^{t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)^{\frac{1}{s+t}}\right\rangle,$$

where $w_{(i)}=h\left(A_{(i)}\right)-h\left(A_{(i+1)}\right)$.

4.2 Frank Bipolar Neutrosophic Choquet Geometric Bonferroni Mean Operator

Definition 12 Let $s,\ t\geq 0$, and $\sigma_i=\left\langle T_i^+,I_i^+,F_i^+,T_i^-,I_i^-,F_i^-\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of BNNs. Then, the FBNCGBM operator is defined as follows:

$$\mathrm{FBNCGBM}^{s,t}(\sigma_1,\sigma_2,\ldots,\sigma_n)=\frac{1}{s+t}\otimes_{F\,\substack{i,j=1\\ i\neq j}}^{n}\left(\left(s\cdot_F\sigma_{(i)}\right)\oplus_F\left(t\cdot_F\sigma_{(j)}\right)\right)^{\wedge_F\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}},\tag{17}$$

where $w_{(i)}=h\left(A_{(i)}\right)-h\left(A_{(i+1)}\right)$.

Several theorems of the FBNCGBM operator will be proposed in the following parts, and their proofs can be easily completed in the same manner as the theorems of the FBNCWBM operator.

Theorem 8 *Let $\sigma_i=\left\langle T_i^+,I_i^+,F_i^+,T_i^-,I_i^-,F_i^-\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of BNNs. Then, the value aggregated by* Eq. (17) *is also a BNN, and*

$$\mathrm{FBNCGBM}^{s,t}(\sigma_1,\sigma_2,\ldots,\sigma_n)=\frac{1}{s+t}\cdot_F\otimes_{F\,\substack{i,j=1\\ i\neq j}}^{n}\left(\left(s\cdot_F\sigma_{(i)}\right)\oplus_F\left(t\cdot_F\sigma_{(j)}\right)\right)^{\wedge_F\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}$$
$$=\left\langle h\left(T_{(i)}^+,T_{(j)}^+\right),R\left(I_{(i)}^+,I_{(j)}^+\right),R\left(F_{(i)}^+,F_{(j)}^+\right),-R\left(-T_{(i)}^-,-T_{(j)}^-\right),-h\left(-I_{(i)}^-,-I_{(j)}^-\right),-h\left(-F_{(i)}^-,-F_{(j)}^-\right)\right\rangle,\tag{18}$$

where $w_{(i)}=h\left(A_{(i)}\right)-h\left(A_{(i+1)}\right)$,,

$$h\left(x_{(i)},x_{(j)}\right)=1-\log_\lambda\left(1+(\lambda-1)\left(\frac{1-\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-x_{(i)}}-1\right)^{s}\left(\lambda^{1-x_{(j)}}-1\right)^{t}}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-x_{(i)}}-1\right)^{s}\left(\lambda^{1-x_{(j)}}-1\right)^{t}}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}{1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-x_{(i)}}-1\right)^{s}\left(\lambda^{1-x_{(j)}}-1\right)^{t}}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-x_{(i)}}-1\right)^{s}\left(\lambda^{1-x_{(j)}}-1\right)^{t}}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}\right)^{\frac{1}{s+t}}\right),$$

and

$$R\left(y_{(i)},y_{(j)}\right)=\log_\lambda\left(1+(\lambda-1)\left(\frac{1-\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{y_{(i)}}-1\right)^{s}\left(\lambda^{y_{(j)}}-1\right)^{t}}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{y_{(i)}}-1\right)^{s}\left(\lambda^{y_{(j)}}-1\right)^{t}}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}{1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{y_{(i)}}-1\right)^{s}\left(\lambda^{y_{(j)}}-1\right)^{t}}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{y_{(i)}}-1\right)^{s}\left(\lambda^{y_{(j)}}-1\right)^{t}}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}}\right)^{\frac{1}{s+t}}\right).$$

Theorem 9 (Monotonicity). *Let $\alpha_i=\left\langle T_{\alpha_i}^+,I_{\alpha_i}^+,F_{\alpha_i}^+,T_{\alpha_i}^-,I_{\alpha_i}^-,F_{\alpha_i}^-\right\rangle$ and $\beta_i=\left\langle T_{\beta_i}^+,I_{\beta_i}^+,F_{\beta_i}^+,T_{\beta_i}^-,I_{\beta_i}^-,F_{\beta_i}^-\right\rangle$ $(i=1,2,\ldots,n)$ be two collections of BNNs. If $\alpha_{(i)}\leq\beta_{(i)}$ for all i, that $T_{\alpha_{(i)}}^+\leq T_{\beta_{(i)}}^+$, $I_{\alpha_{(i)}}^+\geq I_{\beta_{(i)}}^+$, $F_{\alpha_{(i)}}^+\geq F_{\beta_{(i)}}^+$, $T_{\alpha_{(i)}}^-\geq T_{\beta_{(i)}}^-$, $I_{\alpha_{(i)}}^-\leq I_{\beta_{(i)}}^-$, and $F_{\alpha_{(i)}}^-\leq F_{\beta_{(i)}}^-$ for all i, then*

$$\mathrm{FBNCGBM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\mathrm{FBNCGBM}^{s,t}(\beta_1,\beta_2,\ldots,\beta_n).\tag{19}$$

Theorem 10 (Boundedness). *Let $\alpha_i=\langle T_i^+,I_i^+,F_i^+,T_i^-,I_i^-,F_i^-\rangle$ $(i=1,2,\ldots,n)$ be a collection of BNNs. Then, we have*

$$\mathrm{FBNCGBM}^{s,t}(\alpha^-,\alpha^-,\ldots,\alpha^-)\leq\mathrm{FBNCGBM}^{s,t}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\mathrm{FBNCGBM}^{s,t}(\alpha^+,\alpha^+,\ldots,\alpha^+),$$

where

$$\alpha^{+} = \langle T^{+}_{\alpha^{+}}, I^{+}_{\alpha^{+}}, F^{+}_{\alpha^{+}}, T^{-}_{\alpha^{+}}, I^{-}_{\alpha^{+}}, F^{-}_{\alpha^{+}} \rangle$$
$$= \Big\langle \max\Big(T^{+}_{\alpha_1}, T^{+}_{\alpha_2}, \ldots, T^{+}_{\alpha_n}\Big), \min\Big(I^{+}_{\alpha_1}, I^{+}_{\alpha_2}, \ldots, I^{+}_{\alpha_n}\Big), \min\Big(F^{+}_{\alpha_1}, F^{+}_{\alpha_2}, \ldots, F^{+}_{\alpha_n}\Big),$$
$$\min\Big(T^{-}_{\alpha_1}, T^{-}_{\alpha_2}, \ldots, T^{-}_{\alpha_n}\Big), \max\Big(I^{-}_{\alpha_1}, I^{-}_{\alpha_2}, \ldots, I^{-}_{\alpha_n}\Big), \max\Big(F^{-}_{\alpha_1}, F^{-}_{\alpha_2}, \ldots, F^{-}_{\alpha_n}\Big)\Big\rangle.$$
$$\alpha^{-} = \langle T^{+}_{\alpha^{-}}, I^{+}_{\alpha^{-}}, F^{+}_{\alpha^{-}}, T^{-}_{\alpha^{-}}, I^{-}_{\alpha^{-}}, F^{-}_{\alpha^{-}} \rangle$$
$$= \Big\langle \min\Big(T^{+}_{\alpha_1}, T^{+}_{\alpha_2}, \ldots, T^{+}_{\alpha_n}\Big), \max\Big(I^{+}_{\alpha_1}, I^{+}_{\alpha_2}, \ldots, I^{+}_{\alpha_n}\Big), \max\Big(F^{+}_{\alpha_1}, F^{+}_{\alpha_2}, \ldots, F^{+}_{\alpha_n}\Big),$$
$$\max\Big(T^{-}_{\alpha_1}, T^{-}_{\alpha_2}, \ldots, T^{-}_{\alpha_n}\Big), \min\Big(I^{-}_{\alpha_1}, I^{-}_{\alpha_2}, \ldots, I^{-}_{\alpha_n}\Big), \min\Big(F^{-}_{\alpha_1}, F^{-}_{\alpha_2}, \ldots, F^{-}_{\alpha_n}\Big)\Big\rangle.$$

Theorem 11 (Reducibility). *Let* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. *Thus,*

$$\frac{1}{s+t} \cdot_F \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes_F}} \left(\left(s \cdot_F \sigma_{(i)}\right) \oplus_F \left(t \cdot_F \sigma_{(j)}\right)\right)^{\wedge_F \frac{w_{(i)} w_{(j)}}{1 - w_{(i)}}}$$
$$= \frac{1}{s+t} \cdot_F \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes_F}} \left(\left(s \cdot_F \sigma_{(i)}\right) \oplus_F \left(t \cdot_F \sigma_{(j)}\right)\right)^{\wedge_F \frac{1}{n(n-1)}}.$$

Theorem 12 (Idempotency). *Let* $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ *be a collection of BNNs. If* $\sigma_i = \sigma$ $(i = 1, 2, \ldots, n)$, *then* $\mathrm{FBNCGBM}^{s,t}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma$.

4.3 MCDM Methods Based on the FBNCWBM and FBNCGBM Operators

In this subsection, two comprehensive MCDM methods are put forward based on the proposed FBNCWBM and FBNCGBM operators. The following steps depict the main procedures of the proposed methods.

For MCDM problems with bipolar neutrosophic information, let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_m\}$ be a set of criteria. BNNs $\sigma_{ij} = \left\langle T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}, T^{-}_{ij}, I^{-}_{ij}, F^{-}_{ij} \right\rangle$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ are used to represent the evaluation values of the ith alternative under the jth criterion.

Step 1 Collect bipolar neutrosophic evaluation information.

Bipolarity, fuzziness, uncertainty, and inconsistency exist in the practical decision-making process. Therefore, the evaluation can be considered bipolar neutrosophic information. Positive information expresses what is satisfactory, desired, or considered acceptable. Negative information indicates what is impossible or rejected. In other words, when an evaluation object is compared with the worst object, this evaluation will produce positive preferences. By contrast, when an evaluation object is compared with the best object, this evaluation will produce negative preferences. T^{+}_{ij}, I^{+}_{ij}, and F^{+}_{ij} denote the truth membership, indeterminate membership, and false membership degrees of alternative A_i under criterion C_j with respect to positive preferences. T^{-}_{ij}, I^{-}_{ij}, and F^{-}_{ij} signify the truth membership, indeterminate membership, and false membership degrees of alternative A_i under criterion C_j with respect to negative preferences. Therefore, the bipolar neutrosophic evaluation information can be collected according to the evaluation of experts.

Step 2 Obtain the score and accuracy values of the collected evaluation.

The score values $S(\sigma_{ij})$ and accuracy values $A(\sigma_{ij})$ of alternative A_i can be obtained based on Eqs. (1) and (2).

Step 3 Reorder the evaluation under each criterion.

The score values are utilized to rank σ_{ij}, such that $S(\sigma_{i(j)}) < S(\sigma_{i(j+1)})$. If $S(\sigma_{ij}) = S(\sigma_{ik})$, then the accuracy values are utilized to rank σ_{ij}, such that $A(\sigma_{i(j)}) < A(\sigma_{i(j+1)})$.

Step 4 Calculate the fuzzy measures of criteria.

The fuzzy density of each criterion can be obtained according to expert opinions. Thereafter, all fuzzy measures can be calculated based on Eqs. (6) and (7).

Step 5 Calculate the comprehensive performance value of each alternative.

Method 1 Utilize the FBNCWBM operator to compute the comprehensive performance values, then

$$\sigma_i = \mathrm{FBNCWBM}^{s,t}(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{im}) = \left(\underset{\substack{k,j=1 \\ k \neq j}}{\overset{m}{\oplus_F}} \frac{w_{i(k)} w_{i(j)}}{1 - w_{i(k)}} \cdot_F \left(\left(\sigma_{i(k)}\right)^{\wedge_F s} \otimes_F \left(\sigma_{i(j)}\right)^{\wedge_F t} \right) \right)^{\wedge_F \frac{1}{s+t}}, \tag{20}$$

where $w_{i(j)} = h(A_{i(j)}) - h(A_{i(j+1)})$.

Method 2 Utilize the FBNCGBM operator to calculate the comprehensive performance values, then

$$\sigma_i = \mathrm{FBNCGBM}^{s,t}(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{im}) = \frac{1}{s+t} \cdot_F \underset{\substack{k,j=1 \\ k \neq j}}{\overset{m}{\otimes_F}} \left(\left(s \cdot_F \sigma_{i(k)}\right) \oplus \left(t \cdot_F \sigma_{i(j)}\right)\right)^{\wedge_F \frac{w_{i(k)} w_{i(j)}}{1 - w_{i(k)}}}, \tag{21}$$

where $w_{i(j)} = h(A_{i(j)}) - h(A_{i(j+1)})$.

Step 6 Calculate the score values of each alternative.

Equation (1) is used to compute the score values $S(\sigma_i)$ of each alternative under these two methods.

Step 7 Rank all alternatives, and select the best one.

On the basis of the obtained score values and the comparison method in Definition 4, alternatives can be ranked in descending order, and the best alternative can be selected using these two methods.

5 Numerical Example

A numerical example of plant location selection adopted from Deli et al. [10] is presented here. Moreover, the flexibility and effectiveness of the proposed methods are confirmed through parametric analysis and comparative analysis.

A manufacturing company considers selecting a location for building a new plant. Through preliminary screening, four locations A_1, A_2, A_3, and A_4 are selected for further evaluation. The company invited four groups of experts to evaluate the information. The experts include investment experts, manufacturing experts, transportation experts, and human resource professionals. The four plant locations are evaluated by the experts according to four criteria: cost (C_1), expansion possibility (C_2), transportation (C_3), and labor (C_4). These criteria are interactive and interrelated.

5.1 Steps of the Proposed Methods

Step 1 Collect the bipolar neutrosophic evaluation information.

The evaluation information obtained through expert discussion is shown in Table 1.

Step 2 Obtain the score and accuracy values of the collected evaluation.

The score values $S(\sigma_{ij})$ and accuracy values $A(\sigma_{ij})$ of alternative A_i under criterion C_j can be obtained based on Eqs. (1) and (2). $S(\sigma_{ij})$ and $A(\sigma_{ij})$ are shown in Tables 2 and 3, respectively.

Step 3 Reorder the evaluation under each criterion.

The score values are utilized to rank σ_{ij}, such that $S(\sigma_{i(j)}) < S(\sigma_{i(j+1)})$. If $S(\sigma_{ij}) = S(\sigma_{ik})$, the accuracy values are used to rank σ_{ij}, such that $A(\sigma_{i(j)}) < A(\sigma_{i(j+1)})$. Subsequently, σ_{ij} is reordered based on $S(\sigma_{ij})$ and $A(\sigma_{ij})$ (Table 4).

Step 4 Calculate the fuzzy measures of criteria.

According to expert opinions, the fuzzy density of each criterion can be obtained as follows:

$h(c_1) = 0.4$, $h(c_2) = 0.25$, $h(c_3) = 0.37$, and $h(c_4) = 0.2$.

Thereafter, the parameter $\theta = -0.44$ can be calculated based on Eq. (7). Using Eq. (6), the following values can be obtained:

$h(c_1, c_2) = 0.6$, $h(c_1, c_3) = 0.7$, $h(c_1, c_4) = 0.56$, $h(c_2, c_3) = 0.58$, $h(c_2, c_4) = 0.43$, $h(c_3, c_4) = 0.54$,

$h(c_1, c_2, c_3) = 0.88$, $h(c_1, c_2, c_4) = 0.75$, $h(c_1, c_3, c_4) = 0.84$, $h(c_2, c_3, c_4) = 0.73$, $h(c_1, c_2, c_3, c_4) = 1$.

Step 5 Calculate the comprehensive performance value of each alternative.

Method 1 Using FBNCWBM operator in Eq. (20) and supporting s, $t = 1$, $\lambda = 2$ to obtain the comprehensive performance value σ_i of each alternative, then

$$\sigma_1 = \langle 0.4476, 0.5977, 0.4478, 0.6844, 0.4856, 0.6059 \rangle;$$
$$\sigma_2 = \langle 0.7977, 0.5142, 0.6395, 0.521, 0.5587, 0.3247 \rangle;$$
$$\sigma_3 = \langle 0.5599, 0.4157, 0.3142, 0.5222, 0.4133, 0.3653 \rangle;$$
$$\sigma_4 = \langle 0.5525, 0.4888, 0.3621, 0.5027, 0.5843, 0.2261 \rangle.$$

Table 1 Bipolar neutrosophic evaluation information

	C_1	C_2	C_3	C_4
A_1	<0.5, 0.7, 0.2, −0.7, −0.3, −0.6>	<0.4, 0.4, 0.5, −0.7, −0.8, −0.4>	<0.7, 0.7, 0.5, −0.8, −0.7, −0.6>	<0.1, 0.5, 0.7, −0.5, −0.2, −0.8>
A_2	<0.9, 0.7, 0.5, −0.7, −0.7, −0.1>	<0.7, 0.6, 0.8, −0.7, −0.5, −0.1>	<0.9, 0.4, 0.6, −0.1, −0.7, −0.5>	<0.5, 0.2, 0.7, −0.5, −0.1, −0.9>
A_3	<0.3, 0.4, 0.2, −0.6, −0.3, −0.7>	<0.2, 0.2, 0.2, −0.4, −0.7, −0.4>	<0.9, 0.5, 0.5, −0.6, −0.5, −0.2>	<0.7, 0.5, 0.3, −0.4, −0.2, −0.2>
A_4	<0.9, 0.7, 0.2, −0.8, −0.6, −0.1>	<0.3, 0.5, 0.2, −0.5, −0.5, −0.2>	<0.5, 0.4, 0.5, −0.1, −0.7, −0.2>	<0.4, 0.2, 0.8, −0.5, −0.5, −0.6>

Table 2 Score values $S(\sigma_{ij})$

	C_1	C_2	C_3	C_4
A_1	0.47	0.50	0.50	0.40
A_2	0.47	0.37	0.67	0.52
A_3	0.52	0.58	0.50	0.48
A_4	0.48	0.47	0.57	0.50

Table 3 Accuracy values $A(\sigma_{ij})$

	C_1	C_2	C_3	C_4
A_1	0.4	0.2	0.4	−0.9
A_2	1	0.5	−0.1	−0.6
A_3	0	0	0.8	0.6
A_4	1.4	0.4	−0.1	−0.5

Method 2 Using FBNCGBM operator in Eq. (21) and supporting s, $t = 1$, $\lambda = 2$ to obtain the comprehensive performance value of each alternative, then

$$\sigma_1 = \langle 0.4401, 0.5987, 0.4388, 0.6865, 0.4761, 0.6011 \rangle;$$
$$\sigma_2 = \langle 0.791, 0.5164, 0.6364, 0.5167, 0.547, 0.2957 \rangle;$$
$$\sigma_3 = \langle 0.5653, 0.4186, 0.3079, 0.5229, 0.406, 0.3761 \rangle;$$
$$\sigma_4 = \langle 0.5718, 0.4875, 0.3732, 0.4866, 0.5837, 0.2188 \rangle.$$

Step 6 Calculate the score values of each alternative.

Equation (1) is used to compute the score value $S(\sigma_i)$.

Method 1 The score values obtained using the FBNCWBM operator are as follows.

$S(\sigma_1) = 0.4682$; $S(\sigma_2) = 0.5011$; $S(\sigma_3) = 0.5144$; $S(\sigma_4) = 0.5016$.

Method 2 The score values obtained using the FBNCGBM operator are as follows.

$S(\sigma_1) = 0.4655$; $S(\sigma_2) = 0.4940$; $S(\sigma_3) = 0.5163$; $S(\sigma_4) = 0.5045$.

Step 7 Rank all alternatives in descending order and select the best one.

On the basis of the score values and the comparison method in Definition 4, the alternatives can be ranked in descending order and the best alternative can be selected.

Method 1 According to the score values obtained using the FBNCWBM operator, the ranking order is $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, A_3 is optimal.

Method 2 According to the score values obtained using the FBNCGBM operator, the ranking order is $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, A_3 is optimal.

5.2 Influences of the Parameters

This subsection discusses the influences of parameters s, t, and λ in detail.

First, the influences of parameters s and t on the proposed operators is discussed. Table 5 presents the corresponding rankings with respect to the FBNCWBM and FBNCGBM

Table 4 Reordered evaluations σ_{ij}

	$C_{(1)}$	$C_{(2)}$	$C_{(3)}$	$C_{(4)}$
A_1	<0.7, 0.7, 0.5, −0.8, −0.7, −0.6>	<0.4, 0.4, 0.5, −0.7, −0.8, −0.4>	<0.5, 0.7, 0.2, −0.7, −0.3, −0.6>	<0.1, 0.5, 0.7, −0.5, −0.2, −0.8>
A_2	<0.9, 0.4, 0.6, −0.1, −0.7, −0.5>	<0.5, 0.2, 0.7, −0.5, −0.1, −0.9>	<0.9, 0.7, 0.5, −0.7, −0.7, −0.1>	<0.7, 0.6, 0.8, −0.7, −0.5, −0.1>
A_3	<0.2, 0.2, 0.2, −0.4, −0.7, −0.4>	<0.3, 0.4, 0.2, −0.6, −0.3, −0.7>	<0.9, 0.5, 0.5, −0.6, −0.5, −0.2>	<0.7, 0.5, 0.3, −0.4, −0.2, −0.2>
A_4	<0.5, 0.4, 0.5, −0.1, −0.7, −0.2>	<0.4, 0.2, 0.8, −0.5, −0.5, −0.6>	<0.9, 0.7, 0.2, −0.8, −0.6, −0.1>	<0.3, 0.5, 0.2, −0.5, −0.5, −0.2>

Table 5 Rankings with different parameters of the FBNCWBM and FBNCGBM operators

s, t	FBNCWBM	FBNCGBM
$s = 0,\ t = 0.1$	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 0,\ t = 1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 0,\ t = 2$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 0,\ t = 5$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = t = 0.1$	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = t = 1$	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = t = 2$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = t = 5$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 1,\ t = 0.1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 2,\ t = 0.1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 2,\ t = 1$	$A_3 \succ A_2 \succ A_4 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 5,\ t = 1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$s = 5,\ t = 2$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_4 \succ A_2 \succ A_1$

operators as the value of s changes from 0 to 5 and the value of t changes from 0.1 to 5. With dissimilar parameters s and t, different final rankings can be obtained. Thus, parameters s and t of the FBNCWBM and FBNCGBM operators can affect the decision results. Specifically, when different values of parameters s and t are used in the proposed methods, dissimilar interrelationships among criteria can be captured, such that the ranking results are different. Clearly, the worst location is always A_1. The best alternative obtained using the FBNCWBM operator is either location A_2 or A_3, whereas the optimal location obtained using the FBNCGBM operator is A_3.

Further analysis of the selection of appropriate values of parameters s and t according to the preferences of the DMs is conducted in this part. With alternative A_1 as an example, the scores of alternative A_1 obtained using the FBNCWBM and FBNCGBM operators with different values of s and t are shown in Figs. 1 and 2, respectively. Figure 1 indicates that the greater the values of parameters s and t, the greater the scores of the alternative obtained using the FBNCWBM operator. By contrast, Fig. 2 exhibits that the greater the values of parameters s and t, the smaller the scores of the alternative obtained using the FBNCGBM operator. The comparison of Figs. 1 and 2 shows that the scores in Fig. 1 are greater than those in Fig. 2, indicating that the FBNCWBM operator can achieve positive results, whereas the FBNCGBM operator can obtain passive results. Therefore, when the DMs want to attain positive results, the FBNCWBM operator with greater values of parameters s and t can be utilized. Conversely, when the DMs want to acquire negative results, the FBNCGBM operator with greater values of parameters s and t can be applied.

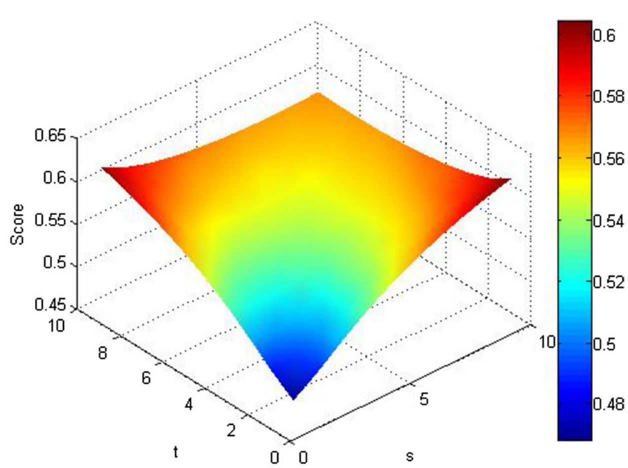

Fig. 1 Scores of A_1 obtained using the FBNCWBM operator

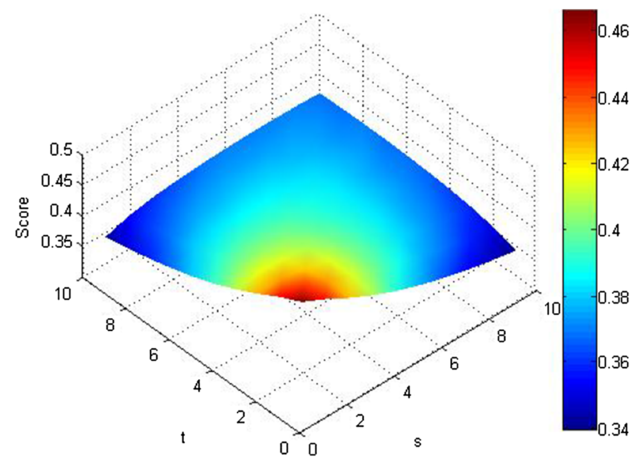

Fig. 2 Scores of A_1 obtained using the FBNCGBM operator

Second, the influence of parameter λ of the proposed operators on the plant location selection problem is analyzed. Figures 3 and 4, respectively, illustrate the scores of alternatives obtained using the FBNCWBM and FBNCGBM operators as the value of λ changes from 1 to 50.

Figures 3 and 4 illustrate that the rankings remain constant with different values of parameter λ. Therefore, λ does not influence the ranking results. However, the alternative scores change different with FBNCWBM and FBNCGBM operators. Figure 3 exhibits that the scores obtained using the FBNCWBM operator increase as the value of λ increases, while Fig. 4 shows that the scores obtained using the FBNCGBM operator decrease as the value of λ increases. In practice, when DMs have the same ranking, the scores may be dissimilar with different values of λ. Therefore, when DMs seek to obtain positive results, the FBNCWBM operator with high values of λ or the FBNCGBM operator with low values of λ can be adopted.

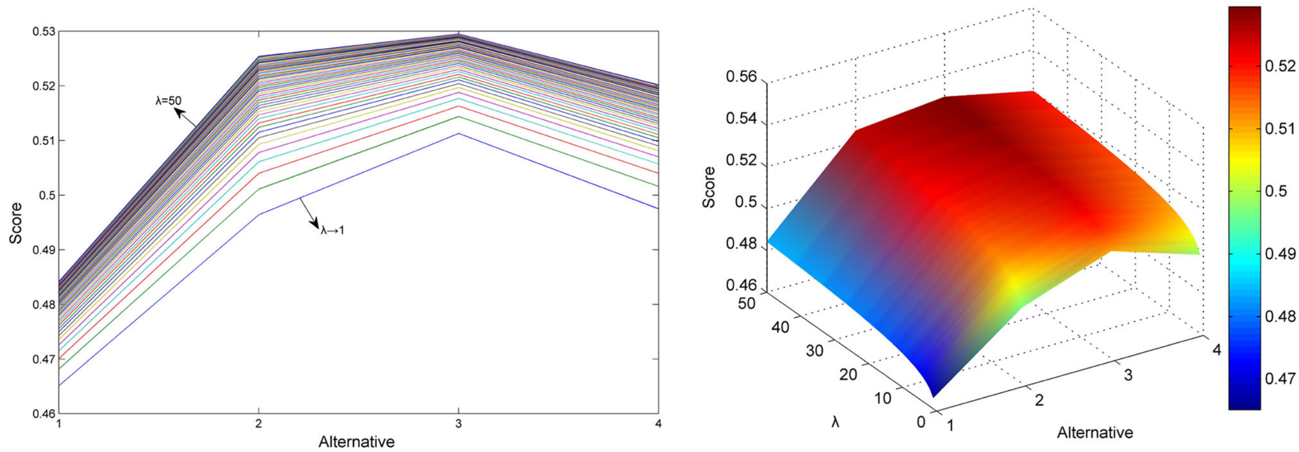

Fig. 3 Scores of alternatives obtained using the FBNCWBM operator with λ

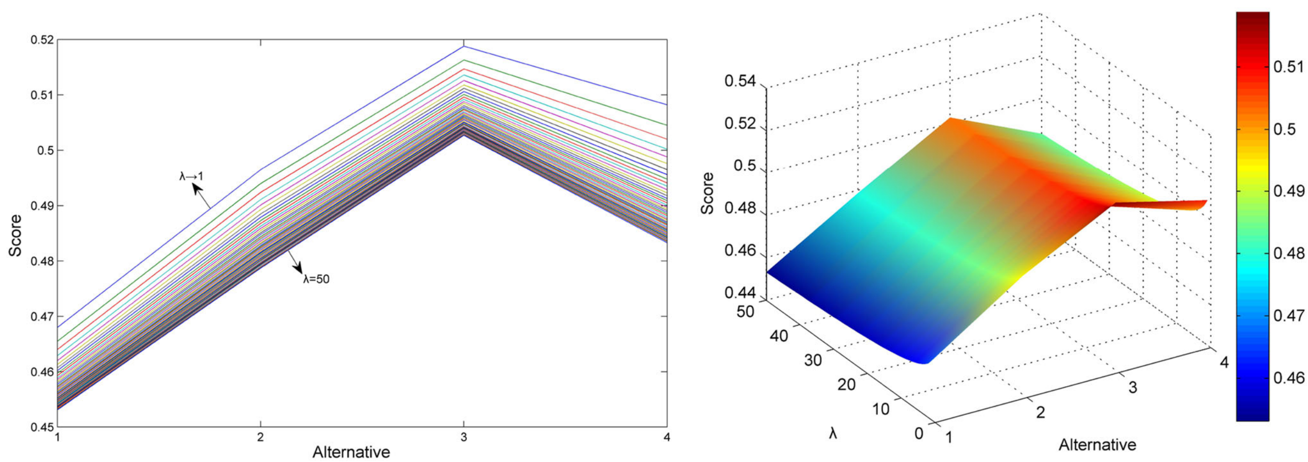

Fig. 4 Scores of alternatives obtained using the FBNCGBM operator with λ

5.3 Comparative Analysis

This subsection presents a comparative analysis with extant methods under bipolar neutrosophic environment to validate the effectiveness and advantages of the proposed methods using the same evaluation information shown in Table 1.

Given that the proposed MCDM methods are based on the proposed operators, a comparison of the existing MCDM methods based on the bipolar neutrosophic operators is first conducted. Deli et al. [10] proposed BNSs and two aggregation operators based on the algebraic operations, namely bipolar neutrosophic WA (A_w) operator and bipolar neutrosophic weighted geometric (G_w) operator. Deli et al. [10] assumed that the weight vector of the criteria is $W_1 = (0.5, 0.25, 0.125, 0.125)$. Thereafter, the proposed methods are compared with the BN-TOPSIS method proposed by Dey et al. [11]. Dey et al. [11] used the maximizing deviation method to obtain criteria weight vector $W_2 = (0.2585,\ 0.2552,\ 0.2278,\ 0.2585)$. The proposed methods are then compared with the BN-TOPSIS method using the criteria weight vector $W_1 = (0.5, 0.25, 0.125, 0.125)$ and $W_2 = (0.2585,\ 0.2552,\ 0.2278,\ 0.2585)$. Table 6 presents the rankings of these methods.

Table 6 Rankings obtained by the different methods

Method	Final ranking
A_w operator [10]	$A_3 \succ A_4 \succ A_2 \succ A_1$
G_w operator [10]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BN-TOPSIS method (W_1)	$A_4 \succ A_2 \succ A_3 \succ A_1$
BN-TOPSIS method (W_2) [11]	$A_3 \succ A_2 \succ A_4 \succ A_1$
FBNCWBM operator (s, $t = 1$)	$A_3 \succ A_4 \succ A_2 \succ A_1$
FBNCGBM operator (s, $t = 1$)	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 6 indicates that the best alternative is A_4 for the BN-TOPSIS method (W_1) and A_3 for the other methods. The worst alternative is A_1 for all methods. Meanwhile, the proposed methods generate the same rankings as the A_{w} and G_{w} operators. However, the proposed methods produce dissimilar rankings to the BN-TOPSIS methods.

The rankings obtained by the proposed methods and the A_{w} and G_{w} operators are the same, which demonstrates the effectiveness of the proposed operators. In contrast to the A_{w} and G_{w} operators, the proposed operators consider the interactions and interrelationships among criteria and can select the appropriate parameters according to the preferences of the DMs. In other words, the proposed operators fully take advantages of the BM operator, Choquet integral operator, and Frank operations. They are more flexible and comprehensive than the A_{w} and G_{w} operators. The comparison of the BN-TOPSIS (W_1) and BN-TOPSIS (W_2) methods shows that they differ in rankings from the different weights. Therefore, the importance of weights has a significant role in the final ranking result. In contrast to the proposed methods, the TOPSIS method ignores the influence of the relationships among criteria. However, in practical MCDM problems, only a few cases with independent criteria occur. The BM operator considers the interrelationships among criteria, and the Choquet integral operator can flexibly describe the relative importance and interactions of the decision criteria. Therefore, the proposed methods are more applicable in practice.

6 Conclusion

BNSs can describe bipolar, uncertain, and inconsistent information. The FBNCWBM and FBNCGBM operators were defined by combining the BM and the Choquet integral operators based on the Frank operations to ensure that the bipolar neutrosophic aggregation operator is flexible and reliable. Subsequently, a numerical example was provided to prove the proposed methods and discuss the influence of different parameters (s, t, and λ). Finally, the accuracy and reliability of the proposed methods were further demonstrated through a comparative analysis with other methods.

The contributions and novelties of this study are as follows. First, BNSs were employed to depict the decision-making evaluation information. Second, Frank operations were extended to bipolar neutrosophic environment. Third, two novel bipolar neutrosophic aggregation operators were proposed by combining the BM and Choquet integral operators based on the Frank operations that simultaneously considered the interrelationships and interactions among criteria. Fourth, MCDM methods were developed based on the proposed operators. Finally, the practicality, flexibility, and efficiency of the proposed methods were confirmed through a numerical example and a comparative analysis.

In the future, the FBNCWBM and FBNCGBM operators can be extended to other fuzzy environments, such as bipolar interval neutrosophic sets and bipolar neutrosophic soft expert sets. In addition, future research can explore other possible approaches to deal with bipolar neutrosophic information.

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Appendix

Proof of Theorem 3.

Proof Theorem 3 will be proven by utilizing the mathematical induction of n as follows:

$x_i = T_i^+, I_i^-, F_i^-$ and $y_i = I_i^+, F_i^+, T_i^-$ are utilized to simplify the process. First, Eq. (22) must be proven.

$$\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus_F}} \frac{w_{(i)}w_{(j)}}{1-w_{(i)}} \cdot_F \left(\left(\sigma_{(i)}\right)^{\wedge_F s} \otimes_F \left(\sigma_{(j)}\right)^{\wedge_F t} \right)$$
$$= \left(1 - \log_\lambda \left(1 + (\lambda - 1) \prod_{\substack{i,j=1\\ j\neq i}}^{n} \left(\frac{(\lambda-1)^{s+t} - (\lambda^{x_{(i)}} - 1)^s (\lambda^{x_{(j)}} - 1)^t}{(\lambda-1)^{s+t} + (\lambda-1)(\lambda^{x_{(i)}} - 1)^s (\lambda^{x_{(j)}} - 1)^t} \right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}} \right), \right.$$
$$\left. \log_\lambda \left(1 + (\lambda - 1) \prod_{\substack{i,j=1\\ j\neq i}}^{n} \left(\frac{(\lambda-1)^{s+t} - \left(\lambda^{1-y_{(i)}} - 1\right)^s \left(\lambda^{1-y_{(j)}} - 1\right)^t}{(\lambda-1)^{s+t} + (\lambda-1)\left(\lambda^{1-y_{(i)}} - 1\right)^s \left(\lambda^{1-y_{(j)}} - 1\right)^t} \right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}} \right) \right). \tag{22}$$

(a) For $n = 2$, the following equations can be easily calculated:

$$\frac{w_{(1)}w_{(2)}}{1-w_{(1)}} \cdot_F \left(\left(\sigma_{(1)}\right)^{\wedge_F s} \otimes_F \left(\sigma_{(2)}\right)^{\wedge_F t} \right)$$
$$= \left(1 - \log_\lambda \left(1 + (\lambda - 1) \left(\frac{(\lambda-1)^{s+t} - (\lambda^{x_{(1)}} - 1)^s (\lambda^{x_{(2)}} - 1)^t}{(\lambda-1)^{s+t} + (\lambda-1)(\lambda^{x_{(1)}} - 1)^s (\lambda^{x_{(2)}} - 1)^t} \right)^{\frac{w_{(1)}w_{(2)}}{1-w_{(1)}}} \right), \right.$$
$$\left. \log_\lambda \left(1 + (\lambda - 1) \left(\frac{(\lambda-1)^{s+t} - \left(\lambda^{1-y_{(1)}} - 1\right)^s \left(\lambda^{1-y_{(2)}} - 1\right)^t}{(\lambda-1)^{s+t} + (\lambda-1)\left(\lambda^{1-y_{(1)}} - 1\right)^s \left(\lambda^{1-y_{(2)}} - 1\right)^t} \right)^{\frac{w_{(1)}w_{(2)}}{1-w_{(1)}}} \right) \right) \tag{23}$$

and

$$\frac{w_{(2)}w_{(1)}}{1-w_{(2)}} \cdot_F \left(\left(\sigma_{(2)}\right)^{\wedge_F s} \otimes_F \left(\sigma_{(1)}\right)^{\wedge_F t} \right)$$
$$= \left(1 - \log_\lambda \left(1 + (\lambda - 1) \left(\frac{(\lambda-1)^{s+t} - (\lambda^{x_{(2)}} - 1)^s (\lambda^{x_{(1)}} - 1)^t}{(\lambda-1)^{s+t} + (\lambda-1)(\lambda^{x_{(2)}} - 1)^s (\lambda^{x_{(1)}} - 1)^t} \right)^{\frac{w_{(2)}w_{(1)}}{1-w_{(2)}}} \right), \right.$$
$$\left. \log_\lambda \left(1 + (\lambda - 1) \left(\frac{(\lambda-1)^{s+t} - \left(\lambda^{1-y_{(2)}} - 1\right)^s \left(\lambda^{1-y_{(1)}} - 1\right)^t}{(\lambda-1)^{s+t} + (\lambda-1)\left(\lambda^{1-y_{(2)}} - 1\right)^s \left(\lambda^{1-y_{(1)}} - 1\right)^t} \right)^{\frac{w_{(2)}w_{(1)}}{1-w_{(2)}}} \right) \right). \tag{24}$$

Then,

$$\overset{2}{\underset{\substack{i,j=1\\ i\neq j}}{\bigoplus_F}} \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)$$
$$=\frac{w_{(1)}w_{(2)}}{1-w_{(1)}}\cdot_F\left(\left(\sigma_{(1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(2)}\right)^{\wedge_F t}\right)\oplus_F\frac{w_{(2)}w_{(1)}}{1-w_{(2)}}\cdot_F\left(\left(\sigma_{(2)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(1)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{2}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{2}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)\right).$$

That is, when $n=2$, Eq. (22) is correct.

(b) Equation (22) is assumed to be correct when $n=k$. That is,

$$\overset{k}{\underset{\substack{i,j=1\\ i\neq j}}{\bigoplus_F}} \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{k}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{k}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)\right). \tag{25}$$

Thereafter, when $n=k+1$, the following equation can be obtained:

$$\overset{k+1}{\underset{\substack{i,j=1\\ i\neq j}}{\bigoplus_F}} \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)$$
$$=\overset{k}{\underset{\substack{i,j=1\\ i\neq j}}{\bigoplus_F}} \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\overset{k}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(k+1)}}{1-w_{(i)}}\cdot_F$$
$$\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(k+1)}\right)^{\wedge_F t}\right)$$
$$\overset{k}{\underset{j=1}{\bigoplus_F}}\frac{w_{(k+1)}w_{(j)}}{1-w_{(k+1)}}\cdot_F\left(\left(\sigma_{(k+1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right). \tag{26}$$

Equations (27) and (28) must be proven to prove Eq. (26).

$$\overset{k}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(k+1)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(k+1)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{k}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(k+1)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(k+1)}}-1)^t}\right)^{\frac{w_{(i)}w_{(k+1)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{k}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(k+1)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(k+1)}}-1)^t}\right)^{\frac{w_{(i)}w_{(k+1)}}{1-w_{(i)}}}\right)\right). \tag{27}$$

$$\overset{k}{\underset{j=1}{\bigoplus_F}}\frac{w_{(k+1)}w_{(j)}}{1-w_{(k+1)}}\cdot_F\left(\left(\sigma_{(k+1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{j=1}^{k}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(k+1)}}-1)^s(\lambda^{x_{(j)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(k+1)}}-1)^s(\lambda^{x_{(j)}}-1)^t}\right)^{\frac{w_{(k+1)}w_{(j)}}{1-w_{(k+1)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{j=1}^{k}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(k+1)}}-1)^s(\lambda^{1-y_{(j)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(k+1)}}-1)^s(\lambda^{1-y_{(j)}}-1)^t}\right)^{\frac{w_{(k+1)}w_{(j)}}{1-w_{(k+1)}}}\right)\right). \tag{28}$$

(1) Equation (27) can be proven by utilizing the mathematical induction of k as follows:

① For $k=2$, the following equation can be obtained easily:

$$\overset{2}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(3)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(3)}\right)^{\wedge_F t}\right)$$
$$=\frac{w_{(1)}w_{(3)}}{1-w_{(1)}}\cdot_F\left(\left(\sigma_{(1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(3)}\right)^{\wedge_F t}\right)\oplus_F\frac{w_{(2)}w_{(3)}}{1-w_{(2)}}\cdot_F\left(\left(\sigma_{(2)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(3)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{2}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(3)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(3)}}-1)^t}\right)^{\frac{w_{(i)}w_{(3)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{2}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(3)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(3)}}-1)^t}\right)^{\frac{w_{(i)}w_{(3)}}{1-w_{(i)}}}\right)\right).$$

② Equation (27) is assumed to be correct when $k=l$. That is,

$$\overset{l}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(l+1)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(l+1)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+1)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+1)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+1)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+1)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+1)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+1)}}{1-w_{(i)}}}\right)\right).$$

Subsequently, when $k=l+1$, the following equation can be obtained:

$$\overset{l+1}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(l+2)}\right)^{\wedge_F t}\right)$$
$$=\overset{l}{\underset{i=1}{\bigoplus_F}}\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(l+2)}\right)^{\wedge_F t}\right)\oplus_F\frac{w_{(l+1)}w_{(l+2)}}{1-w_{(l+1)}}\cdot_F\left(\left(\sigma_{(l+1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(l+2)}\right)^{\wedge_F t}\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}}\right)\right)$$
$$\oplus_F\left(1-\log_\lambda\left(1+(\lambda-1)\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(l+1)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(l+1)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}\right)^{\frac{w_{(l+1)}w_{(l+2)}}{1-w_{(l+1)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(l+1)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(l+1)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}\right)^{\frac{w_{(l+1)}w_{(l+2)}}{1-w_{(l+1)}}}\right)\right)$$
$$=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l+1}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{x_{(i)}}-1)^s(\lambda^{x_{(l+2)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}}\right),\right.$$
$$\left.\log_\lambda\left(1+(\lambda-1)\prod_{i=1}^{l+1}\left(\frac{(\lambda-1)^{s+t}-(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}{(\lambda-1)^{s+t}+(\lambda-1)(\lambda^{1-y_{(i)}}-1)^s(\lambda^{1-y_{(l+2)}}-1)^t}\right)^{\frac{w_{(i)}w_{(l+2)}}{1-w_{(i)}}}\right)\right).$$

Thereafter, when $k=l+1$, Eq. (27) is correct. Therefore, Eq. (27) is correct for all k.

(2) Similarly, Eq. (28) can be proven.

Subsequently, when Eqs. (25), (27), and (28) are used, Eq. (26) can be converted into the following form:

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}{}_F \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\\
&=\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}{}_F \frac{w_{(i)}w_{(j)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\bigoplus_{i=1}^{k}{}_F\frac{w_{(i)}w_{(k+1)}}{1-w_{(i)}}\cdot_F\left(\left(\sigma_{(i)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(k+1)}\right)^{\wedge_F t}\right)\\
&\quad\bigoplus_{j=1}^{k}{}_F\frac{w_{(k+1)}w_{(j)}}{1-w_{(k+1)}}\cdot_F\left(\left(\sigma_{(k+1)}\right)^{\wedge_F s}\otimes_F\left(\sigma_{(j)}\right)^{\wedge_F t}\right)\\
&=\left(1-\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{k+1}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{x_{(i)}}-1\right)^s\left(\lambda^{x_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right),\right.\\
&\left.\log_\lambda\left(1+(\lambda-1)\prod_{\substack{i,j=1\\ j\neq i}}^{k+1}\left(\frac{(\lambda-1)^{s+t}-\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}{(\lambda-1)^{s+t}+(\lambda-1)\left(\lambda^{1-y_{(i)}}-1\right)^s\left(\lambda^{1-y_{(j)}}-1\right)^t}\right)^{\frac{w_{(i)}w_{(j)}}{1-w_{(i)}}}\right)\right).
\end{aligned}$$

Thus, when $n = k + 1$, Eq. (22) is correct. Then, Eq. (22) is correct for all n.

Therefore, Eq. (18) can be easily proven.

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