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# Group Decision Making Based on Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operators

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**Abstract:** In this paper, a new concept of the triangular neutrosophic cubic fuzzy numbers (TNCFNs), their score and accuracy functions are introduced. Based on TNCFNs, some new Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy Einstein weighted averaging (TNCFEWA), triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging (TNCFEOWA) and triangular neutrosophic cubic fuzzy Einstein hybrid weighted averaging (TNCFEHWA) operators are developed. Furthermore, their application to multiple-attribute decision-making with triangular neutrosophic cubic fuzzy (TNCF) information is discussed. Finally, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

**Keywords:** triangular neutrosophic cubic fuzzy number; Einstein t-norm; arithmetic averaging operator; Multi-attribute decision making; numerical application

## 1. Introduction

Atanassov [1] introduced the IFS, which is a generalization of FS. Atanassov [2] introduced operations and relations over IFSs taking as a point of departure respective definitions of relations and operations over fuzzy sets. Bustince et al. [3] introduced the characterization of certain structures of intuitionistic relations according to the structures of two concrete fuzzy relations. Deschrijver et al. [4] established the relationships between intuitionistic fuzzy sets (Atanassov, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian)), L-fuzzy sets. Deschrijver et al. [5] defined the mathematical relationship between intuitionistic fuzzy sets and other models of imprecision. Jun et al. [6] introduced the cubic set. Mohiuddin et al. [7] showed that the union of two internal cubic soft sets might not be internal. Turksen [8] showed that the proposed representation (1) exists for certain families of the conjugate pairs of t-norms and t-norms, and (2) resolves some of the difficulties associated with particular interpretations of conjunction, disjunction, and implication in fuzzy set theories.

Xu [9] developed some aggregation operators, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid aggregation operator, to aggregate intuitionistic fuzzy values. Xu et al. [10] developed some new geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator and the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator. Xu et al. [11] provided a survey of the aggregation techniques of intuitionistic fuzzy information and their applications in various fields, such as decision making, cluster analysis, medical diagnosis, forecasting, and manufacturing grid. Liu et al. [12] introduced and discussed the concept of intuitionistic fuzzy point operators. Zeng et al. [13] defined the situation with intuitionistic fuzzy information and developed an intuitionistic fuzzy ordered weighted distance (IFOWD) operator. The fuzzy set was introduced by Zadeh [14]. Zadeh [15] introduced the interval-valued fuzzy set Li et al. [16] proposed group decision-making methods of the interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm and Choquet integral. Zhao et al. [17] developed some hesitant triangular fuzzy aggregation operators based on the Einstein operation: the hesitant triangular fuzzy Einstein weighted averaging (HTFEWA) operator. Xu et al. [18] introduced two new aggregation operators: dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator.

The Neutrosophic Set (NS) was projected by Smarandache [19,20]. Neutrosophic sets are characterized by fact participation, an indeterminacy-enrollment work and misrepresentation participation, which are inside the ordinary or nonstandard unit interim $]^{-}0, 1^{+}[$ in order to apply NS to genuine applications. In order to apply NS to real-world applications, Aliya et al. [21] introduced the concept of the triangular cubic fuzzy number. Aliya et al. [22] introduced the triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator.

Beg et al. [23] introduced a computational means to manage situations in which experts assess alternatives in possible membership and non-membership values. Przemyslaw et al. [24] introduced a simple test that sometimes might be helpful in detecting non-separability at a glance.

The differences between Reference 21, 22 and the current paper are as Table 1:

**Table 1.** Difference between references 21, 22 and current paper.

| Reference 21 | Reference 22 | Current Paper |
|---|---|---|
| Defines a new extension of the triangular cubic fuzzy number by using a cubic set. | Defines a new extension of the triangular cubic hesitant fuzzy number by using a cubic set. | Defines a new extension of the triangular neutrosophic cubic fuzzy number by using a neutrosophic set. |
| Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions. | Introduced the triangular cubic hesitant fuzzy number, operational laws, and their score, accuracy functions, membership uncertainty index and hesitation index. | Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions, membership uncertainty index and hesitation index. |
| Introduced the triangular cubic fuzzy hybrid aggregation operator. | Introduced three Einstein aggregation operators, such as the triangular cubic fuzzy hybrid aggregation operator, and the TCHFEWA, TCHFEOWA and TCHFEHWA operators | Introduced three Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy hybrid aggregation operator, and the TNCFEWA, TNCFEOWA and TNCFEHWA operators |

Based on the above analysis, in this paper we develop TNCFNs, which is the generalization of the triangular neutrosophic intuitionistic fuzzy number and triangular neutrosophic interval fuzzy number. We perform some operations based on Einstein T-norm and Einstein T-conorm for TNCFNs. We also develop score and accuracy functions to compare two TNCFNs. Due to the developed operation, we propose the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, to aggregate a collection of TNCFNs.

This paper is organized as follows. In Section 2, we define some concepts of FS, CS, and TNCFNs. In Section 3, we discuss some Einstein operations on TNCFNs and their properties. In Section 4, we first develop some novel arithmetic averaging operators, such as the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, for aggregating a group of TNCFNs. In Section 5, we apply the

TNCFEHWA operator to MADM with TNCFNs material. In Section 6, we offer a numerical example consistent with our approach. In Section 7, we discuss comparison analysis. In Section 8, we present a conclusion.

## 2. Preliminaries

**Definition 1.** [15]. *Let $H$ be a fixed set, a FS $F$ in $H$ is defined as:$F = \{(h, \Gamma_F(h)|h \in H\}$ where $\Gamma_F(h)$ is a mapping from $h$ to the closed interval $[0, 1]$ and for each $h \in H$, $\Gamma_F(h)$ is called the degree of membership of $h$ in $H$.*

**Definition 2.** *Let $H$ is a fixed set and an interval-valued fuzzy set $I$ in $H$ is defined as $I = \{h, \mathrm{R}_I^-(h), \mathrm{R}_I^+(h)|h \in H\}$, where $\mathrm{R}_I^- : H \to [0, 1]$ and $\mathrm{R}_I^+ : H \to [0, 1]$. The $\mathrm{R}_I^-(h)$ is lower membership and $\mathrm{R}_I^+(h)$ is upper membership such that $0 \leq \mathrm{R}_I^-(h) \leq \mathrm{R}_I^+(h) \leq 1$.*

**Definition 3.** *[1]. An IFS Đ in $H$* is given by $Đ = \{(h, \mathrm{R}_{Đ}(h), \Omega_{Đ}(h)|h \in H\}$, *where $\mathrm{R}_{Đ} : H \to [0, 1]$ and $\Omega_{Đ} : H \to [0, 1]$, with the condition $0 \leq \mathrm{R}_{Đ}(h) + \Omega_{Đ}(h) \leq 1$.*

*The numbers $\mathrm{R}_{Đ}(h)$ and $\Omega_{Đ}(h)$ represent, respectively, the membership degree and non-membership degree of the element $h$ to the set Đ.*

*Triangular Neutrosophic Cubic Fuzzy Number*

**Definition 4.** *Let* $A_1 = \left\{\begin{array}{c} [p_1(h), q_1(h), \\ r_1(h)], \\ \langle[\mathrm{Y}_1^-(h), \\ \mathrm{R}_1^-(h), \delta_1^-(h)], \\ [\mathrm{Y}_1^+(h), \\ \mathrm{R}_1^+(h), \delta_1^+(h)], \\ [\mathrm{Y}_1(h), \\ \mathrm{R}_1(h), \delta_1(h)]\rangle \\ |h \in \mathrm{H} \end{array}\right\}$ and $A_2 = \left\{\begin{array}{c} [p_2(h), q_2(h), \\ r_2(h)], \\ \langle[\mathrm{Y}_2^-(h), \\ \mathrm{R}_2^-(h), \delta_2^-(h)], \\ [\mathrm{Y}_2^+(h), \\ \mathrm{R}_2^+(h), \delta_2^+(h)], \\ [\mathrm{Y}_2(h), \\ \mathrm{R}_2(h), \delta_2(h)]\rangle \\ |h \in \mathrm{H} \end{array}\right\}$ *are two TNCFNs, some operations on TNCFNs are defined as follows:*

*(a)* $A_1 \subseteq A_2$ *iff* $\forall h \in H$, $p_1(h)) \geq p_2(h)$ $q_1(h) \geq q_2(h)$, $r_1(h) \geq r_2(h)$, $\mathrm{Y}_1^-(h) \geq \mathrm{Y}_2^-(h)$, $\mathrm{R}_1^-(h) \geq \mathrm{R}_2^-(h)$, $\delta_1^-(h) \geq \delta_1^-(h)$, $\mathrm{Y}_1^+(h) \geq \mathrm{Y}_2^+(h)$ $\mathrm{R}_1^+(h) \geq \mathrm{R}_2^+(h)$,, $\delta_1^+(h) \geq \delta_2^+(h)$ and $\mathrm{Y}_1(h) \geq \mathrm{Y}_2(h)$ $\delta_1(h) \leq \delta_2(h)$.

*(b)* $A_1 \cap_{T,S} A_2 = T[p_1(h), p_2(h)]$, $T[q_1(h), q_2(h)]$, $T[r_1(h), r_2(h)]$, $\langle T[\mathrm{Y}_1^-(h), \mathrm{Y}_2^-(h)]$, $T[\mathrm{R}_1^-(h), \mathrm{R}_2^-(h)]$, $T[\delta_1^-(h), \delta_2^-(h)]$, $T[\mathrm{Y}_1^+(h), \mathrm{Y}_2^+(h)]$, $T[\mathrm{R}_1^+(h), \mathrm{R}_2^+(h)]$, $T[\delta_1^+(h), \delta_2^+(h)]$, $S[\mathrm{Y}_1(h), \mathrm{Y}_2(h)]$, $S[\mathrm{R}_1(h), \mathrm{R}_2(h)]$, $S[\delta_1(h), \delta_2(h)]\rangle$

**Example 1.** *Let* $\ddot{A}_1 = \left\{\begin{array}{c} \langle[0.1, 0.2, 0.3], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7]\rangle \end{array}\right\}$ *and* $\ddot{A}_2 = \left\{\begin{array}{c} \langle[0.103, 0.104, 0.105], \\ [0.100, 0.102, 0.104], \\ [0.102, 0.104, 0.106], \\ [0.101, 0.103, 0.105]\rangle \end{array}\right\}$ *be two TNCFSs*

*(a)* $\ddot{A}_1 \subseteq \ddot{A}_2$, *if* $\forall \ddot{z} \in Z$, $0.1 \geq 0.103$, $0.2 \geq 0.104$, $0.3 \geq 0.105$, $0.2 \geq 0.100$, $0.4 \geq 0.102$, $0.6 \geq 0.104$, $0.4 \geq 0.102$, $0.6 \geq 0.104$, $0.8 \geq 0.106$ *and* $0.3 \leq 0.101$, $0.5 \leq 0.103$, $0.7 \leq 0.105$.

*(b)* $\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = T[0.1, 0.103]$, $T[0.2, 0.104]$, $T[0.3, 0.105]$, $[T[0.2, 0.10]$, $T[0.4, 0.12]$, $T[0.6, 0.14]$, $T[0.4, 0.12]$, $T[0.6, 0.14]$, $T[0.8, 0.16]$ *and* $S[0.3, 0.11]$, $S[0.7, 0.15]$.

**Definition 5.** *Let* $C = \left\{ \begin{array}{c} [p_C(h),\, q_C(h),\, r_C(h)] \\ \left\langle [A_C^-(h), \mathrm{R}_C^-(h),\, \widetilde{U}_C^-(h)], \right. \\ [A_C^+(h), \mathrm{R}_C^+(h),\, \widetilde{U}_C^+(h)], \\ \left. [A_C(h), \mathrm{R}_C(h),\, \widetilde{U}_C(h)] \right\rangle \Big| h \in \mathrm{H} \end{array} \right\}$ *be a TNCFN and then the score function $S(C)$, accuracy function $N(C)$, membership uncertainty index $T(C)$ and hesitation uncertainty index $G(C)$ of a TNCFN $C$ are defined by*

$$S(C) = \frac{\begin{array}{c} \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \\ +[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]] \\ -[A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \Big\rangle \end{array}}{27},$$

$$N(C) = \frac{\begin{array}{c} \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \\ +[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]] + [A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \Big\rangle \end{array}}{27}$$

$$\begin{aligned} T(C) = {} & \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)] + \\ & [A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] - [A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \Big\rangle, \\ G(C) = {} & \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)] \\ & +[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] - [A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \Big\rangle. \end{aligned}$$

**Example 2.** *Let* $C = \left\{ \begin{array}{c} \langle [0.101,\, 0.102, \\ 0.103], [0.5,\, 0.7,\, 0.9], \\ [0.7,\, 0.9,\, 0.11], [0.6, \\ 0.8,\, 0.10] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function $S(C)$, accuracy function $H(C)$, membership uncertainty index $T(C)$ and hesitation uncertainty index $G(C)$ of a TNCFN $C$ are defined by*

$$S(C) = \frac{\begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9]] \\ +[0.7 + 0.9 + 0.11] - [0.6 + 0.8 + 0.10] \rangle \end{array}}{27} = \frac{0.306(3.81 - 1.5)}{27} = \frac{0.7068}{27} = 0.0261,$$

$$H(C) = \frac{\begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9]] \\ +[0.7 + 0.9 + 0.11] + [0.6 + 0.8 + 0.10] \rangle \end{array}}{27} = \frac{0.306(3.81 + 1.5)}{27} = \frac{1.6248}{27} = 0.0601,$$

$$T(C) = \left\{ \begin{array}{c} [0.101 + 0.102 + 0.103] \langle [0.7 + 0.9 + 0.11] \\ +[0.6 + 0.8 + 0.10] - [0.5 + 0.7 + 0.9] \rangle \end{array} \right\} = 0.6(1.71 + 1.5 - 2.1) = 0.306(3.21 - 2.1) = 0.3396,$$

$$G(C) = \left\{ \begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.7 + 0.9 + 0.11] \\ +[0.5 + 0.7 + 0.9] - [0.6 + 0.8 + 0.10] \rangle \end{array} \right\} = 0.306(1.71 + 2.1 - 1.5) = 0.7068.$$

*See Figure 1.*

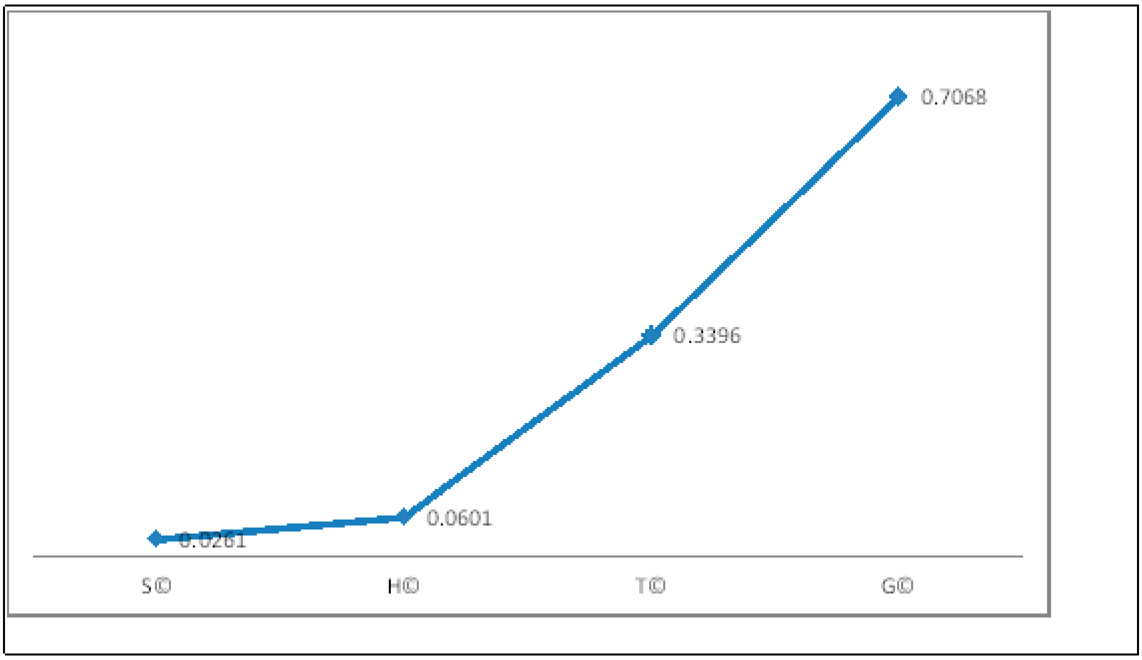


**Figure 1.** The score function, accuracy function, membership uncertainty index and hesitation uncertainty index are ranking of TNCFN.

## 3. Some Einstein Operations on TNCFNs

**Definition 6.** *Let* $C = \left\{ \begin{array}{c} [a(h), e(h), \\ G(h)], \langle [Y =^{-} (h), \\ k^{-}(h), \Gamma^{-}(h)], \\ [Y =^{+} (h), k^{+}(h), \\ \Gamma^{+}(h)], [Y = (h), \\ k(h), \Gamma(h)] \rangle \\ | h \in H \end{array} \right\}$, $C_1 = \left\{ \begin{array}{c} [a_1(h), e_1(h), \\ G_1(h)] \langle [Y =_1^{-} (h), \\ k_1^{-}(h), \Gamma_1^{-}(h)], \\ [Y =_1^{+} (h), k_1^{+}(h), \\ \Gamma_1^{+}(h)], [Y =_1 (h), \\ k_1(h), \Gamma_1(h)] \rangle \\ | h \in H \end{array} \right\}$ *and* $C_2 = \left\{ \begin{array}{c} [a_2(h), e_2(h), \\ G_2(h)] \langle [Y =_2^{-} (h), \\ k_2^{-}(h), \Gamma_2^{-}(h)], \\ [Y =_2^{+} (h), k_2^{+}(h), \\ \Gamma_2^{+}(h)], [Y =_2 (h), \\ k_2(h), \Gamma_2(h)] \rangle \\ | h \in H \end{array} \right\}$ *be any three TNCFNs. Then some Einstein operations of* $C_1$ *and* $C_2$ *can be defined as:* $C_1 + C_2 = \left\langle \begin{bmatrix} \frac{a_1(h)+a_2(h)}{1+a_1(h)(1-a_2(h))}, \\ \frac{e_1(h)+e_2(h)}{1+e_1(h)(1-e_2(h))}, \\ \frac{G_1(h)+G_2(h)}{1+G_1(h)(1-G_2(h))} \end{bmatrix}, \begin{bmatrix} \frac{Y=_1^{-}(h)+Y=_2^{-}(h)}{1+Y=_1^{-}(h)(1-Y=_2^{-}(h))}, \\ \frac{Y=_1^{+}(h)+Y=_2^{+}(h)}{1+Y=_1^{+}(h)(1-Y=_2^{+}(h))} \quad , \\ \frac{k_1^{-}(h)+k_2^{-}(h)}{1+k_1^{-}(h)(1-k_2^{-}(h))}, \\ \frac{k_1^{+}(h)+k_2^{+}(h)}{1+k_1^{+}(h)(1-k_2^{+}(h))} \quad , \\ \frac{\Gamma_1^{-}(h)+\Gamma_2^{-}(h)}{1+\Gamma_1^{-}(h)(1-\Gamma_2^{-}(h))}, \\ \frac{\Gamma_1^{+}(h)+\Gamma_2^{+}(h)}{1+\Gamma_1^{+}(h)(1-\Gamma_2^{+}(h))} \end{bmatrix}, \right.$

$$\left.\begin{bmatrix} \frac{Y\!=_1(h)Y\!=_2(h)}{(1+(1-Y\!=_1(h))\,(1-Y\!=_2(h)))}, \\ \frac{\text{ķ}_1(h)\text{ķ}_2(h)}{(1+(1-\text{ķ}_1(h))\,(1-\text{ķ}_2(h)))}, \\ \frac{\Gamma_1(h)\Gamma_2(h)}{(1+((1-\Gamma_1(h))\,(1-\Gamma_2(h)))} \end{bmatrix}\right\rangle, \; \lambda C = \left\langle \begin{bmatrix} \frac{[1+\text{ą}_C(h)]^{\lambda}-[1-\text{ą}_C(h)]^{\lambda}}{[1+\text{ą}_C(h)]^{\lambda}+[1-\text{ą}(h)]^{\lambda}}, \\ \frac{[1+\text{ę}_C(h)]^{\lambda}-[1-\text{ę}_C(h)]^{\lambda}}{[1+\text{ę}_C(h)]^{\lambda}+[1-\text{ę}(h)]^{\lambda}}, \\ \frac{[1+\text{Ģ}_C(h)]^{\lambda}-[1-\text{Ģ}_C(h)]^{\lambda}}{[1+\text{Ģ}_C(h)]^{\lambda}+[1-\text{Ģ}(h)]^{\lambda}} \end{bmatrix}, \begin{bmatrix} \frac{[1+Y\!=^-_C(h)]^{\lambda}-[1-Y\!=^-_C(h)]^{\lambda}}{[1+Y\!=^-_C(h)]^{\lambda}+[1-Y\!=^-_C(h)]^{\lambda}}, \\ \frac{[1+Y\!=^+_C(h)]^{\lambda}-[1-Y\!=^+_C(h)]^{\lambda}}{[1+Y\!=^+_C(h)]^{\lambda}+[1-Y\!=^+_C(h)]^{\lambda}}, \\ \frac{[1+\text{ķ}^-_C(h)]^{\lambda}-[1-\text{ķ}^-_C(h)]^{\lambda}}{[1+\text{ķ}^-_C(h)]^{\lambda}+[1-\text{ķ}^-_C(h)]^{\lambda}}, \\ \frac{[1+\text{ķ}^+_C(h)]^{\lambda}-[1-\text{ķ}^+_C(h)]^{\lambda}}{[1+\text{ķ}^+_C(h)]^{\lambda}+[1-\text{ķ}^+_C(h)]^{\lambda}}, \\ \frac{[1+\Gamma^-_C(h)]^{\lambda}-[1-\Gamma^-_C(h)]^{\lambda}}{[1+\Gamma^-_C(h)]^{\lambda}+[1-\Gamma^-_C(h)]^{\lambda}}, \\ \frac{[1+\Gamma^+_C(h)]^{\lambda}-[1-\Gamma^+_C(h)]^{\lambda}}{[1+\Gamma^+_C(h)]^{\lambda}+[1-\Gamma^+_C(h)]^{\lambda}} \end{bmatrix},$$

$$\left.\begin{bmatrix} \frac{2[Y\!=_C(h)]^{\lambda}}{[(2-Y\!=_C(h)]^{\lambda}+[Y\!=_C(h)]^{\lambda}}, \\ \frac{2[\text{ķ}_C(h)]^{\lambda}}{[(2-\text{ķ}_C(h)]^{\lambda}+[\text{ķ}_C(h)]^{\lambda}}, \\ \frac{2[\Gamma_C(h)]^{\lambda}}{[(2-\Gamma_C(h)]^{\lambda}+[\Gamma_C(h)]^{\lambda}} \end{bmatrix}\right\rangle.$$

**Proposition 1.** *Let* $\ddot{A}$, $\ddot{A}_1$ *and* $\ddot{A}_2$ *be three TNCFNs,* $\lambda$, $\lambda_1$, $\lambda_2 > 0$, *then we have:*

(1) $\ddot{A}_1 + \ddot{A}_2 = \ddot{A}_2 + \ddot{A}_1$,
(2) $\lambda(\ddot{A}_1 + \ddot{A}_2) = \lambda\ddot{A}_2 + \lambda\ddot{A}_1$,
(3) $\lambda_1\ddot{A} + \lambda_2\ddot{A} = (\lambda_1 + \lambda_2)\ddot{A}$.

**Proof.** The proof of these propositions is provided in Appendix A. □

**Remark 1.** *If* $\alpha_1 \leq_{L_{\text{TNCFN}}} \alpha_2$, *then* $\alpha_1 \leq \alpha_2$, *the total order is the partial order on* $L_{\text{TNCFN}}$, *see Figure 2.*

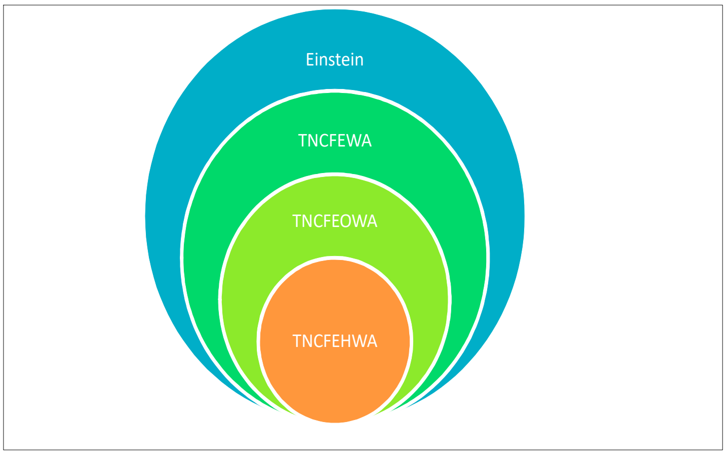


**Figure 2.** New extend aggregation operators, such as TNCFEWA, TNCFEOWA and TNCFEHWA operators.

## 4. Triangular Neutrosophic Cubic Fuzzy Averaging Operators Based on Einstein Operations

In this section, we define the aggregation operators.

*4.1. Triangular Neutrosophic Cubic Fuzzy Einstein Weighted Averaging Operator*

**Definition 7.** Let $\ddot{A} = \left\{ \begin{array}{c} [\alpha(h), \beta(h), \\ \Delta(h)]\langle[\xi_1^-(h), \\ \xi_2^-(h), \xi_3^-(h)], \\ [\xi_1^+(h), \xi_2^+(h), \\ \xi_3^+(h)], [\xi_1(h), \\ \xi_2(h), \xi_3(h)]\rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in $L_{\text{TNCFN}}$ and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ be the weight vector, with $\ddot{\omega}_j \in [0,1]$, $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. Hence TNCFEWA operator of dimension $n$ is a mapping TNCFEWA : $L^n_{\text{TNCFN}} \to L_{\text{TNCFN}}$ and defined by $\text{TNCFEWA}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \ddot{\omega}_1\ddot{A}_1 + \ddot{\omega}_2\ddot{A}_2, \ldots, \ddot{\omega}_n\ddot{A}_n$.

*If* $\ddot{\omega} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. *Hence the TNCFEWA operator is reduced to TNCFEA operator of dimension n. It can be defined as follows:* $\text{TNCFEA}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \frac{1}{n}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$.

**Theorem 1.** *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle[p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)]\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* $L_{\text{TNCFN}}$. *The amassed an incentive by utilizing the TNCFEWA operator is additionally a TNCFN and TNCFEWA.*

$$(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \left\langle \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}, \end{array} \right], \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+p_1^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^-(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+q_2^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_2^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_2^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_2^-(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r_3^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_3^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_3^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_3^-(h)]^{\ddot{\omega}}} \end{array} \right], \right.$$

$$\left. \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^+(h)]^{\ddot{\omega}}}, \frac{\prod_{j=1}^{n}[1+q_2^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_2^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_2^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_2^+(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r_3^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_3^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_3^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_3^+(h)]^{\ddot{\omega}}} \end{array} \right], \left[ \begin{array}{c} \frac{2\prod_{j=1}^{n}[p_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_1(h)]^{\ddot{\omega}}}, \frac{2\prod_{j=1}^{n}[q_2(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_2(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_2(h)]^{\ddot{\omega}}}, \\ , \frac{2\prod_{j=1}^{n}[r_3(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_3(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_3(h)]^{\ddot{\omega}}} \end{array} \right] \right\rangle$$

*where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *be the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0,1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\alpha_1(h) = \alpha_1(h)$, $\beta_1(h) = \beta_1(h)$, $\Delta_1(h) = \Delta_1(h)$, $p_1^-(h) = p_1^-(h)$, $q_2^-(h) = q_2^-(h)$, $r_3^-(h) = r_3^-(h)$, $p_1^+(h) = p_1^+(h)$, $q_2^+(h) = q_2^+(h)$, $r_3^+(h) = r_3^+(h)$ *and* $p_1(h) = p_1(h)$, $q_2(h) = q_2(h)$,

$r_3(h) = r_3(h)$. *Then the TNCFN* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle [p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *are reduced to the triangular neutrosophic cubic fuzzy numbers* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle [p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *and the TNCFEWA operator is reduced to the TNCFEWA operator.*

**Proof.** The proof of this theorem is provided in Appendix B. □

**Example 3.** *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.02, 0.03, 0.04], \\ [0.02, 0.04, 0.06], \\ [0.04, 0.06, 0.08], \\ [0.03, 0.05, 0.07] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.205, 0.207, 0.209], \\ [0.211, 0.213, 0.215], \\ [0.213, 0.215, 0.217], \\ [0.212, 0.214, 0.216] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.004, 0.005, 0.006], \\ [0.102, 0.104, 0.106], \\ [0.104, 0.106, 0.108], \\ [0.103, 0.105, 0.107] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \frac{\left\{ \begin{array}{c} \langle [0.02 + 0.03 + 0.04], \\ [0.02 + 0.04 + 0.06] + \\ [0.04 + 0.06 + 0.08] - \\ [0.03 + 0.05 + 0.07] \rangle \end{array} \right\}}{27} = 0.0005$, $S(C_2) = \frac{\left\{ \begin{array}{c} \langle [0.205 + 0.207 + 0.209], \\ [0.211 + 0.213 + 0.215] + \\ [0.213 + 0.215 + 0.217] - \\ [0.212 + 0.214 + 0.216] \rangle \end{array} \right\}}{27} = 0.0147$, $S(C_3) = \frac{\left\{ \begin{array}{c} \langle [0.004, 0.005, 0.006], \\ [0.102, 0.104, 0.106], \\ [0.104, 0.106, 0.108], \\ [0.103, 0.105, 0.107] \rangle \end{array} \right\}}{27} = \frac{0.015(0.63 - 0.315)}{27} = 0.0001.$

*See Figure 3.*

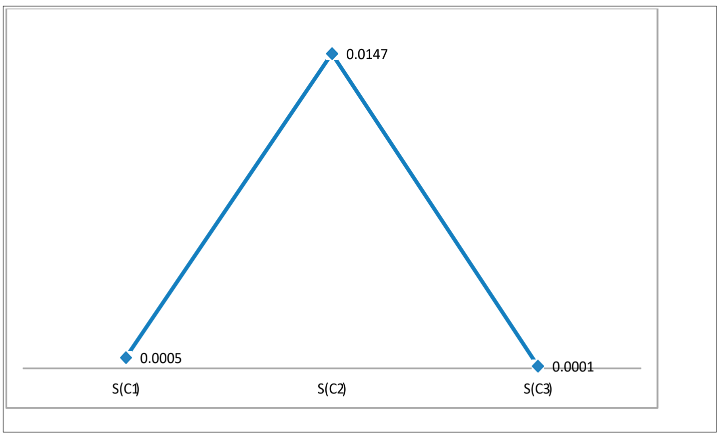


**Figure 3.** S(C2) is the first score value, S(C1) is the second score value and S(C3) is the third score value.

**Example 4.** *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.04, 0.05, 0.06], \\ [0.01, 0.03, 0.05], \\ [0.03, 0.05, 0.07], \\ [0.02, 0.04, 0.06] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.05, 0.07, 0.09], \\ [0.11, 0.13, 0.15], \\ [0.13, 0.15, 0.17], \\ [0.12, 0.14, 0.16] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.2004, 0.2005, 0.2006], \\ [0.2102, 0.2104, 0.2106], \\ [0.2104, 0.2106, 0.2108], \\ [0.2103, 0.2105, 0.2107] \rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \frac{\left\{ \begin{array}{c} \langle [0.04 + 0.05 + 0.06], \\ [0.01 + 0.03 + 0.05]+ \\ [0.03 + 0.05 + 0.07]+ \\ [0.02 + 0.04 + 0.06] \rangle \end{array} \right\}}{27} = 0.002$, $H(C_2) = \frac{\left\{ \begin{array}{c} \langle [0 + 05 + 0.07 + 0.09], \\ [0.11 + 0.13 + 0.15]+ \\ [0.13 + 0.15 + 0.17]+ \\ [0.12 + 0.14 + 0.16] \rangle \end{array} \right\}}{27} = 0.0098$, $H(C_3) = \frac{\left\{ \begin{array}{c} \langle [0.2004 + 0.2005 + 0.2006], \\ [0.2102 + 0.2104 + 0.2106]+ \\ [0.2104 + 0.2106 + 0.2108]+ \\ [0.2103 + 0.2105 + 0.2107] \rangle \end{array} \right\}}{27} = 0.0422$.

*See Figure 4.*

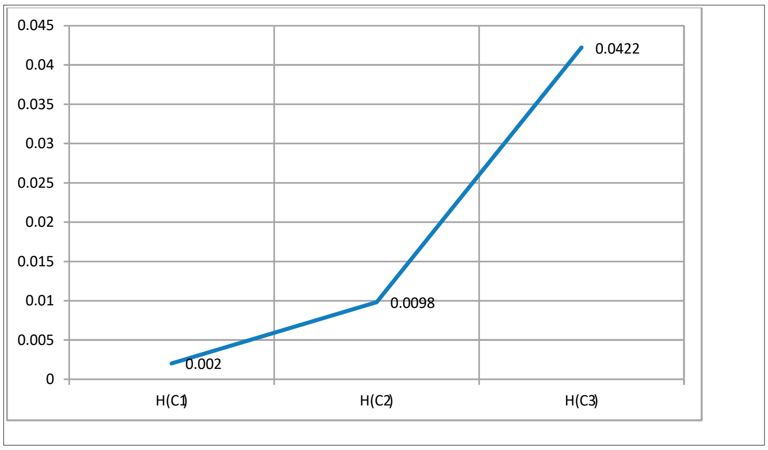


**Figure 4.** H(C2) is the first score value, H(C1) is the second score value and H(C3) is the third score value.

**Proposition 2.** *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \Delta_1(h)] \\ \langle [p_1^-(h), q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), r_1^+(h)], \\ [p_1(h), q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* $L_{\text{TNCFN}}$ *and where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *with* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$.

*Then (1) (Idempotency): If all* $A_j$, $j = 1, 2, \ldots, n$ *are equal, i.e.,* $A_j = A$, *for all* $j = 1, 2, \ldots, n$, *then* TNCFEWA $(A_1, A_2, \ldots, A_n) = A$.

*(2) (Boundary): If* $\alpha_{\min} = \min_{1 \le j \le n} \alpha_j$, $\beta_{\min} = \min_{1 \le j \le n} \beta_j$, $\Delta_{\min} = \min_{1 \le j \le n} \Delta_j$, $p_{\min}^- = \min_{1 \le j \le n} p_j^-$, $q_{\min}^- = \min_{1 \le j \le n} q_j^-$, $r_{\min}^- = \min_{1 \le j \le n} r_j^-$, $p_{\min}^+ = \min_{1 \le j \le n} p_j^+$, $q_{\min}^+ = \min_{1 \le j \le n} q_j^+$, $r_{\min}^+ = \min_{1 \le j \le n} r_j^+$, $p_{\max} = \max_{1 \le j \le n} p_j$, $q_{\max} = \max_{1 \le j \le n} q_j$, $r_{\max} = \max_{1 \le j \le n} r_j$, $\mu_{\max} = \max_{1 \le j \le n} \mu_j$, $p_{\max}^- = \max_{1 \le j \le n} p_j^-$, $q_{\max}^- = \max_{1 \le j \le n} q_j^-$, $r_{\max}^- = \max_{1 \le j \le n} r_j^-$, $\mu_{\max}^- = \max_{1 \le j \le n} \mu_j^-$, $p_{\max}^+ = \max_{1 \le j \le n} p_j^+$, $q_{\max}^+ = \max_{1 \le j \le n} q_j^+$, $r_{\max}^+ = \max_{1 \le j \le n} r_j^+$, $p_{\min} = \min_{1 \le j \le n} p_j$,

$q_{\min} = \min_{1\leq j\leq n} q_j$, $r_{\min} = \min_{1\leq j\leq n} r_j$, $\mu_{\min} = \min_{1\leq j\leq n} \mu_j$ *for all* $j = 1, 2, .., n$, *we can determine that*

$$\left\{\begin{array}{c} [\alpha_{\min}(h), \\ \beta_{\min}(h), \Delta_{\min}(h)] \\ \langle [p^-_{\min}(h), \\ q^-_{\min}(h), r^-_{\min}(h)] \\ [p^+_{\min}(h), \\ q^+_{\min}(h), r^+_{\min}(h)] \\ [p_{\max}(h), q_{\max}(h), \\ r_{\max}(h), s_{\max}(h)]\rangle | \\ h \in H \end{array}\right\} \leq TNCFEWA(A_1, A_2, \ldots, A_n) \leq \left\{\begin{array}{c} [\alpha_{\max}(h), \beta_{\max}(h), \\ \Delta_{\max}(h)], \langle [p^-_{\max}(h), \\ q^-_{\max}(h), r^-_{\max}(h)], \\ [p^+_{\max}(h), q^+_{\max}(h), \\ r^+_{\max}(h)], [p_{\min}(h), \\ q_{\min}(h), r_{\min}(h)]\rangle | \\ h \in H \end{array}\right\}$$

*(3) (Monotonicity):* $A = \left\{\begin{array}{c} [\alpha_A(h), \\ \beta_A(h), \Delta_A(h)], \\ \langle [\{ p^-_A(h), \\ q^-_A(h), r^-_A(h)] \\ , [p^+_A(h), \\ q^+_A(h), r^+_A(h)] \\ , [p_A(h), \\ q_A(h), r_A(h)]\rangle | \\ h \in H \end{array}\right\}$ *and* $B = \left\{\begin{array}{c} [\alpha_B(h), \\ \beta_B(h), \Delta_B(h)], \\ \langle [p^-_B(h), \\ q^-_B(h), r^-_B(h)] \\ [p^+_B(h), \\ q^+_B(h), r^+_B(h)] \\ [p_B(h), \\ q_B(h), r_B(h)\rangle \\ |h \in H \end{array}\right\}$ *be two collection of TNCFNs in* $L_{\text{TNCFN}}$ *and* $A_j \leq L_{\text{TNCFN}} B_j$ *i.e.,* $\alpha_A(h) \leq \alpha_B(h)$, $\beta_A(h) \leq \beta_B(h)$, $\Delta_A(h) \leq \Delta_B(h)$, $p^-_A(h) \leq p^-_B(h)$, $q^-_A(h) \leq q^-_B(h)$, $r^-_A(h) \leq r^-_B(h)$, $p^+_A(h) \leq p^+_B(h)$, $q^+_A(h) \leq q^+_B(h)$, $r^+_A(h) \leq r^+_B(h)$ *and* $p_A(h) \leq p_B(h)$, $q_A(h) \leq q_B(h)$, $r_A(h) \leq r_B(h)$ *then TNCFEWA* $(A_1, A_2, \ldots, A_n) \leq$ *TNCFEWA* $(B_1, B_2, \ldots, B_n)$.

### *4.2. Triangular Neutrosophic Cubic Fuzzy Einstein Ordered Weighted Averaging Operator*

**Definition 8.** *Let* $\ddot{A} = \left\{\begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \left\langle [p^-_{\ddot{A}}(h), \right. \\ q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \\ [p^+_{\ddot{A}}(h), q^-_{\ddot{A}}(h), \\ r^-_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ \left. q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array}\right\}$ *be a collection of TNCFNs in* $L_{\text{TNCFN}}$*, a TNCFEOWA operator of dimension n is a mapping TNCFEOWA:* $L^n_{\text{TNCFN}} \to L_{\text{TNCFN}}$*, that has an associated vector* $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *TNCFEOWA* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \eth_1 \ddot{A}_{(\sigma)1} + \eth_2 \ddot{A}_{(\sigma)2}, \ldots, +\eth_n \ddot{A}_{(\sigma)n}$*, where* $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\ddot{A}_{\sigma(1)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the j the largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$*. If* $w = (w_1, w_2, \ldots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$*. Then the TNCFEOWA operator is reduced to the TCFA operator (2) of dimension n.*

**Theorem 2.** *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h),\, \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)],\, \Big\langle [p^{-}_{\ddot{A}}(h), \\ q^{-}_{\ddot{A}}(h),\, r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h),\, q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)],\, p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h),\, r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* $L_{\mathrm{TNCFN}}$. *Then their aggregated value by using the TNCFEOWA operator is also a TNCFN and TNCFEOWA*

$$(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \left\langle \begin{array}{c} \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}, \end{array} \right], \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+q^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^{-}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^{-}_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right], \\ \left[ \begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}}, \frac{\prod_{j=1}^{n}[1+q^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^{+}_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^{+}_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right], \\ \left[ \begin{array}{c} \frac{2\prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\ddot{\omega}}}, \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right] \end{array} \right\rangle$$

*where* $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *with* $\ddot{A}_{\sigma(1)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$, $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$, *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. *Then the TNCFEOWA operator is reduced to the TNCFA operator of dimension n. Where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\alpha_1(h) = \alpha_1(h)$, $\beta_1(h) = \beta_1(h)$, $\Delta_1(h) = \Delta_1(h)$, $p^{-}_{\ddot{A}}(h) = p^{-}_{\ddot{A}}(h)$, $q^{-}_{\ddot{A}}(h) = q^{-}_{\ddot{A}}(h)$, $r^{-}_{\ddot{A}}(h) = r^{-}_{\ddot{A}}(h)$, $p^{+}_{\ddot{A}}(h) = p^{+}_{\ddot{A}}(h)$, $q^{+}_{\ddot{A}}(h) = q^{+}_{\ddot{A}}(h)$, $r^{+}_{\ddot{A}}(h) = r^{+}_{\ddot{A}}(h)$ *and* $p_{\ddot{A}}(h) = p_{\ddot{A}}(h)$,

$q_{\ddot{A}}(h) = q_{\ddot{A}}(h)$, $r_{\ddot{A}}(h) = r_{\ddot{A}}(h)$. *The TNCFN* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)] \Big\langle [p^{-}_{\ddot{A}}(h), \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$ *are reduced to the triangular neutrosophic cubic fuzzy numbers* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \Big\langle [p^{-}_{\ddot{A}}(h), \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$. *Then the TNCFEWA operator is reduced to the triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging operator.*

**Proof.** The process of this proof is the same as Theorem 1. □

**Example 5.** *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.01, 0.02, 0.03], \\ [0.103, 0.105, 0.107], \\ [0.105, 0.107, 0.109], \\ [0.104, 0.106, 0.108] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.306, 0.308, 0.310], \\ [0.310, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.44, 0.55, 0.66], \\ [0.122, 0.124, 0.126], \\ [0.124, 0.126, 0.128], \\ [0.123, 0.125, 0.127] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \dfrac{\left\{ \begin{array}{c} \langle [0.01 + 0.02 + 0.03], \\ [0.103 + 0.105 + 0.107] + \\ [0.105 + 0.107 + 0.109] - \\ [0.104 + 0.106 + 0.108] \rangle \end{array} \right\}}{27} = 0.00004$, $S(C_2) = \dfrac{\left\{ \begin{array}{c} \langle [0.306 + 0.308 + 0.310], \\ [0.310 + 0.313 + 0.315] + \\ [0.313 + 0.315 + 0.317] - \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0322$, $S(C_3) = \dfrac{\left\{ \begin{array}{c} \langle [0.44 + 0.55 + 0.66], \\ [0.122 + 0.124 + 0.126] + \\ [0.124 + 0.126 + 0.128] - \\ [0.123 + 0.125 + 0.127] \rangle \end{array} \right\}}{27} = 0.0229$.

*See Figure 5.*

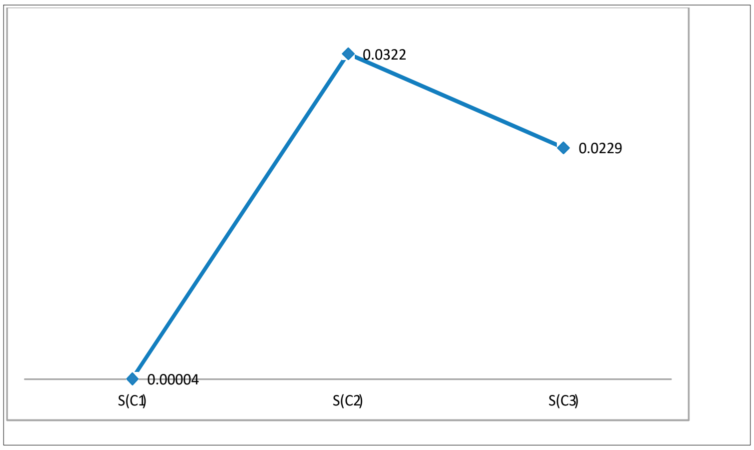


**Figure 5.** Different score ranking of TNCFEOWA operator.

**Example 6.** *Let* $C_1 = \left\{ \begin{array}{c} \langle[0.02, 0.03, 0.04], \\ [0.06, 0.08, 0.010], \\ [0.08, 0.010, 0.012], \\ [0.07, 0.09, 0.011]\rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle[0.1105, 0.1107, 0.1109], \\ [0.1111, 0.1113, 0.1115], \\ [0.1113, 0.1115, 0.1117], \\ [0.1112, 0.1114, 0.1116]\rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle[0.214, 0.215, 0.216], \\ [0.2202, 0.2204, 0.2206], \\ [0.2204, 0.2206, 0.2208], \\ [0.2203, 0.2205, 0.2207]\rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \frac{\left\{ \begin{array}{c} \langle[0.02, 0.03, 0.04], \\ [0.06, 0.08, 0.010], \\ [0.08, 0.010, 0.012], \\ [0.07, 0.09, 0.011]\rangle \end{array} \right\}}{27} = -0.0021$, $H(C_2) = \frac{\left\{ \begin{array}{c} \langle[0.1105, 0.1107, 0.1109], \\ [0.1111, 0.1113, 0.1115], \\ [0.1113, 0.1115, 0.1117], \\ [0.1112, 0.1114, 0.1116]\rangle \end{array} \right\}}{27} = 0.0041$, $H(C_3) = \frac{\left\{ \begin{array}{c} \langle[0.214, 0.215, 0.216], \\ [0.2202, 0.2204, 0.2206], \\ [0.2204, 0.2206, 0.2208], \\ [0.2203, 0.2205, 0.2207]\rangle \end{array} \right\}}{27} = 0.0158$.

*See Figure 6.*

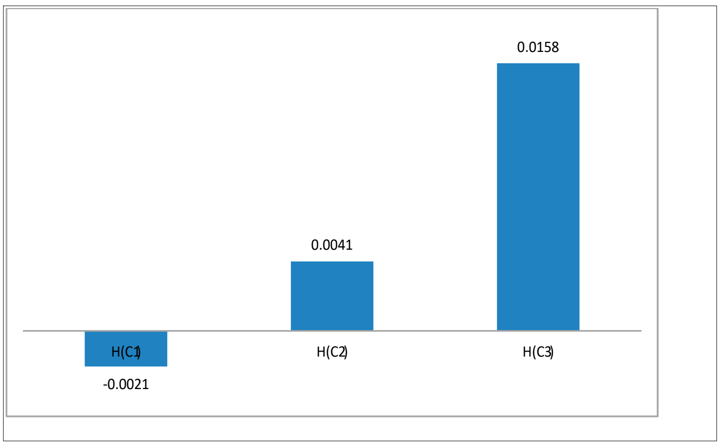


**Figure 6.** Different accuracy ranking of TNCFEOWA operator.

### 4.3. Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operator

**Definition 9.** *Let* $\ddot{A} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ _{\ddot{A}}(h)], \left\langle [p^{-}_{\ddot{A}}(h), \right. \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ \left. q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* $L_{\text{TNCFN}}$ *and* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then TNCFEHWA operator of dimension n is a mapping* $TNCFEHWA : L^{n}_{\text{TNCFN}} \to L_{\text{TNCFN}}$, *that is an associated vector* $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *TNCFEHWA* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = p_1 \ddot{A}_{\sigma(1)} + p_2 \ddot{A}_{\sigma(1)}, \ldots, p_n \ddot{A}_{\sigma(1)}$. *If* $p = \theta \ddot{\omega}_{\sigma(j)} + (1 - \theta) w_{\sigma(j)}$ *with a balancing coefficient* $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\ddot{A}_{\sigma(j)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the j th largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$*).*

**Example 7.** *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.11, 0.13, 0.14], \\ [0.62, 0.64, 0.66], \\ [0.64, 0.66, 0.68], \\ [0.63, 0.65, 0.67] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.51, 0.52, 0.53], \\ [0.311, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \dfrac{\left\{ \begin{array}{c} \langle [0.11 + 0.13 + 0.14], \\ [0.62 + 0.64 + 0.66]+ \\ [0.64 + 0.66 + 0.68]- \\ [0.63 + 0.65 + 0.67] \rangle \end{array} \right\}}{27} = 0.0281$, $S(C_2) = \dfrac{\left\{ \begin{array}{c} \langle [0.51 + 0.52 + 0.53], \\ [0.311 + 0.313 + 0.315]+ \\ [0.313 + 0.315 + 0.317]- \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0543$, $S(C_3) = \dfrac{\left\{ \begin{array}{c} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}}{27} = 0.0104.$

*See Figure 7.*

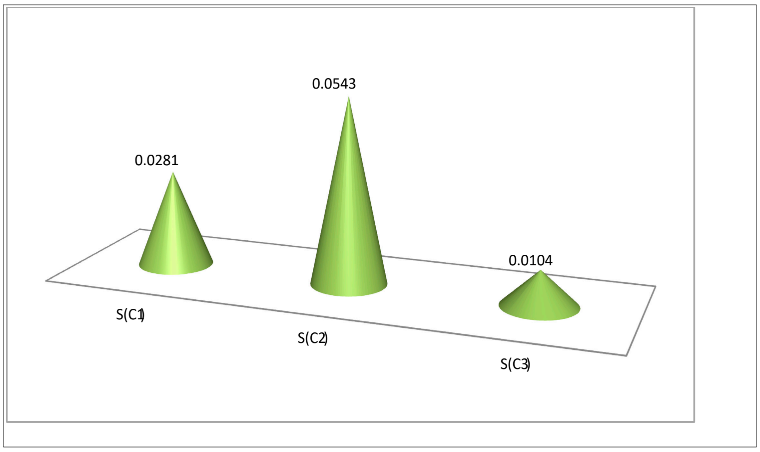


**Figure 7.** Different score ranking of TNCFEHWA operator.

**Example 8.** *Let* $C_1 = \left\{ \begin{array}{c} \langle[0.112, 0.113, 0.114], \\ [0.21, 0.24, 0.28], \\ [0.24, 0.28, 0.32], \\ [0.23, 0.26, 0.30]\rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle[0.0019, 0.0021, 0.0034], \\ [0.1231, 0.1233, 0.1235], \\ [0.1233, 0.1235, 0.1237], \\ [0.1232, 0.1234, 0.1236]\rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle[0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667]\rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \dfrac{\left\{ \begin{array}{c} \langle[0.112 + 0.113 + 0.114], \\ [0.21 + 0.24 + 0.28]+ \\ [0.24 + 0.28 + 0.32]- \\ [0.23 + 0.26 + 0.30]\rangle \end{array} \right\}}{27} = 0.0111$, $H(C_2) = \dfrac{\left\{ \begin{array}{c} \langle[0.0019 + 0.0021 + 0.0034] \\ [0.1231 + 0.1233 + 0.1235]+ \\ [0.1233 + 0.1235 + 0.1237]- \\ [0.1232 + 0.1234 + 0.1236]\rangle \end{array} \right\}}{27} = 0.1124$, $H(C_3) = \dfrac{\left\{ \begin{array}{c} \langle[0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667]\rangle \end{array} \right\}}{27} = 0.0226.$

*See Figure 8.*

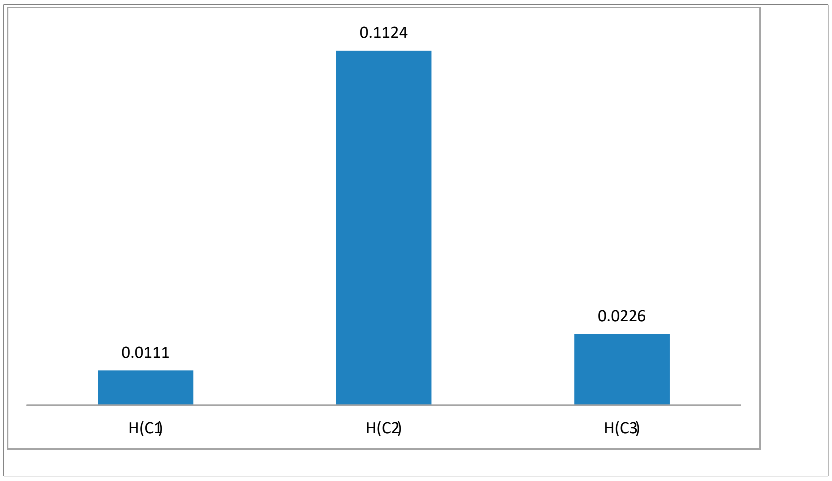


**Figure 8.** Different accuracy ranking of TNCFEHWA operator.

## 5. An Approach to MADM with TNCF Data

Let us suppose the discrete set is $h = \{h_1, h_2, \ldots, h_n\}$ and $G = \{g_1, g_2, \ldots, g_n\}$ are the attributes. Consider that the value of alternatives $h_i$ $(i = 1, 2, \ldots, n)$ on attributes $g_j$ $(j = 1, 2, \ldots, m)$ given by decision maker are TNCFNs in $L_{TNCFN}$: $\ddot{A} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ {}_{\ddot{A}}(h)], \left\langle [p^-_{\ddot{A}}(h), \right. \\ q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \\ [p^+_{\ddot{A}}(h), q^+_{\ddot{A}}(h), \\ r^+_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ \left. q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$, a MADM problem is expressed in the TNCF-decision matrix $\ddot{D} = (\ddot{A}_{ij})_{m \times n} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), {}_{\ddot{A}}(h)], \\ \left\langle [p^-_{\ddot{A}}(h), q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \right. \\ [p^+_{\ddot{A}}(h), q^+_{\ddot{A}}(h), r^+_{\ddot{A}}(h)], \\ \left. p_{\ddot{A}}(h), q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$.

Step 1: Calculate the TNCF decision matrix.

Step 2: Utilize the TNCFEWA operator to mix all values $\ddot{\beta}_{ij}$ $(j = 1, 2, \ldots, m)$ and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ is the weight vector.

Step 3: Calculate the score function.

Step 4: Find the ranking.

See Figure 9.

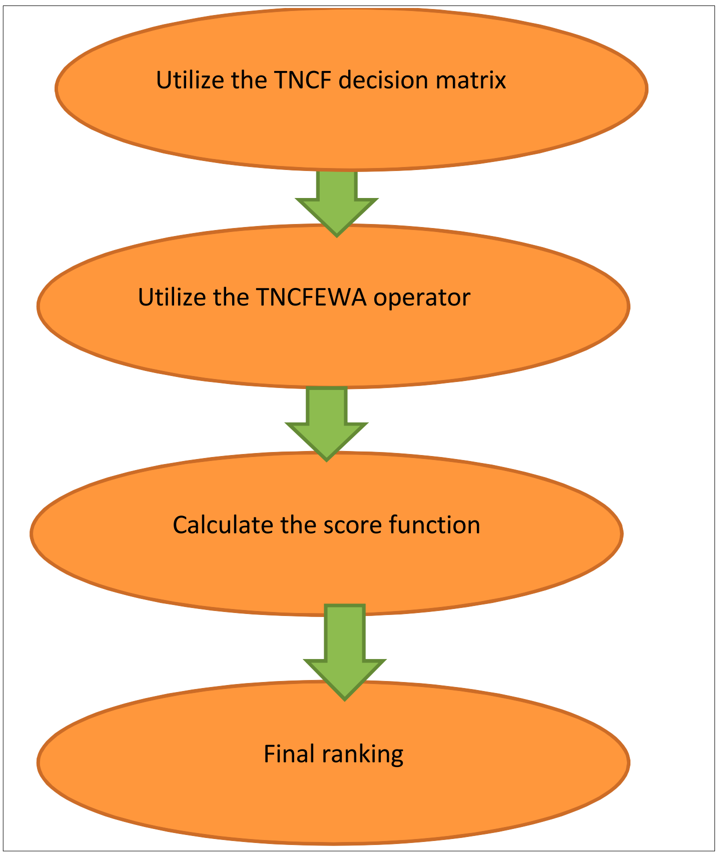


**Figure 9.** Proposed method.

## 6. Numerical Application

The inspiration structure is designed to be dependent upon an assessment that has been devised for the purpose of a stimulus/influencing technique of a twofold entire traveler dispersion to work over the Lahore in Faisalabad by lessening the adventure stage in extraordinarily brimful waterway movement. Inspiration structure choices are sure the settled of options $A = \{A_1, A_2, A_3, A_4\}$

$A_1$ : Old-style propeller and high trundle
$A_2$ : Get-up-and-go,
$A_3$ : Cyclonical propeller,
$A_4$ : Outmoded
See Figure 10.

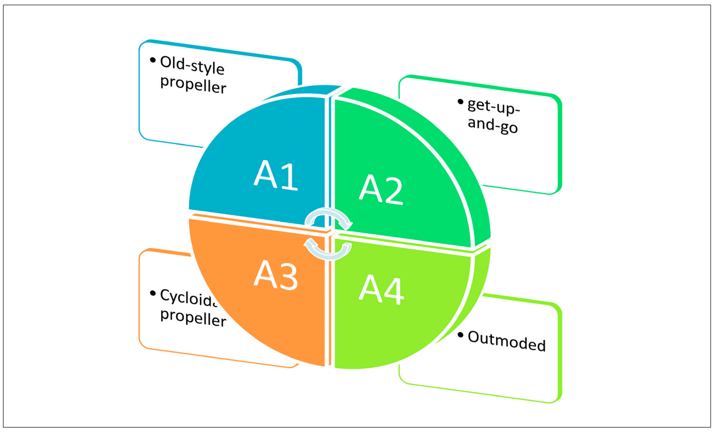


**Figure 10.** Four alternatives.

The ideal is prepared on the possibility of lone zone and four issue characteristics, which are as follows:

$c_1$ : Theory rate
$c_2$ : Reparation and support uses
$c_3$ : Agility
$c_4$ : Tremor and unrest.
See Figure 11.

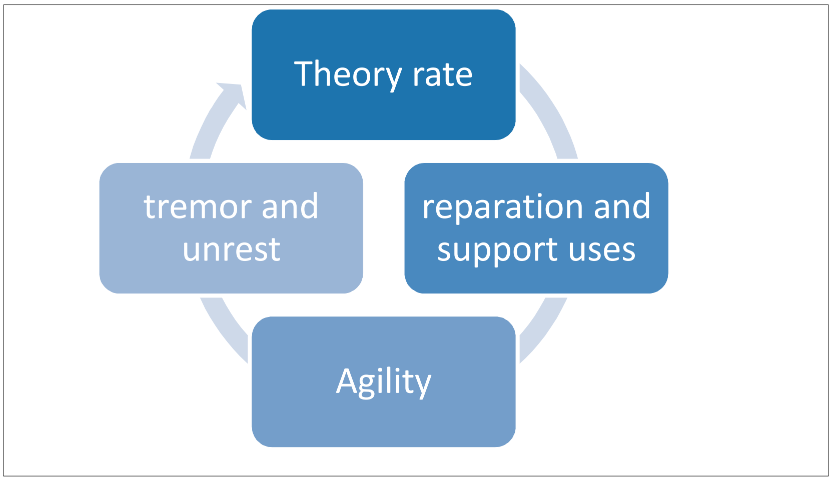


**Figure 11.** Different criteria.

The weight vector is $\ddot{\omega} = (0.25, 0.50, 0.25)$. So, the triangular neutrosophic cubic fuzzy MADM issue is intended to choose the appropriate energy structure from between 3 choices.

Step 1: Calculate the TNCF decision matrix.
The TNCF decision matrix is as Table 2

**Table 2.** Triangular Neutrosophic Cubic Fuzzy Decision Matrix.

| | $c_1$ | $c_2$ |
|---|---|---|
| $A_1$ | $\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ |
| $A_2$ | $\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ |
| $A_3$ | $\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$ |
| $A_4$ | $\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$ |
| | $c_3$ | $c_4$ |
| $A_1$ | $\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ |
| $A_2$ | $\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$ |
| $A_3$ | $\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$ |
| $A_4$ | $\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ | $\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ |

Step 2: Calculate the TNCFEWA operator to total all the rating values and $w = (0.1,\ 0.2,\ 0.4,\ 0.3)^T$. The TNCFEWA operator are defined in Table 3.

**Table 3.** TNCFEWA Operator.

| | |
|---|---|
| $A_1$ | $\left\{ \left\langle \begin{array}{c} [0.2539, 0.2751, 0.2965], [0.1628, 0.2513, 0.3973], \\ [0.2513, 0.3973, 0.0503], [0.7335, 0.8054, 0.6003] \end{array} \right\rangle \right\}$ |
| $A_2$ | $\left\{ \left\langle \begin{array}{c} [0.1536, 0.1995, 0.2477], [0.2944, 0.3597, 0.6447], \\ [0.3597, 0.6447, 0.0988], [0.4838, 0.6067, 0.4831] \end{array} \right\rangle \right\}$ |
| $A_3$ | $\left\{ \left\langle \begin{array}{c} [0.3481, 0.4499, 0.5594], [0.3626, 0.5867, 0.7852], \\ [0.5867, 0.7852, 0.7582], [0.1049, 0.2122, 0.3571] \end{array} \right\rangle \right\}$ |
| $A_4$ | $\left\{ \left\langle \begin{array}{c} [0.2282, 0.2945, 0.3622], [0.3704, 0.5631, 0.7729], \\ [0.5631, 0.7729, 0.4197], [0.2593, 0.3985, 0.2735] \end{array} \right\rangle \right\}$ |

Step 3: The score value are calculated as $s_1 = -0.0192$, $s_2 = 0.0184$, $s_3 = 0.1603$, $s_4 = 0.0829$.

Step 4: Ranking $\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$.
See Figure 12.

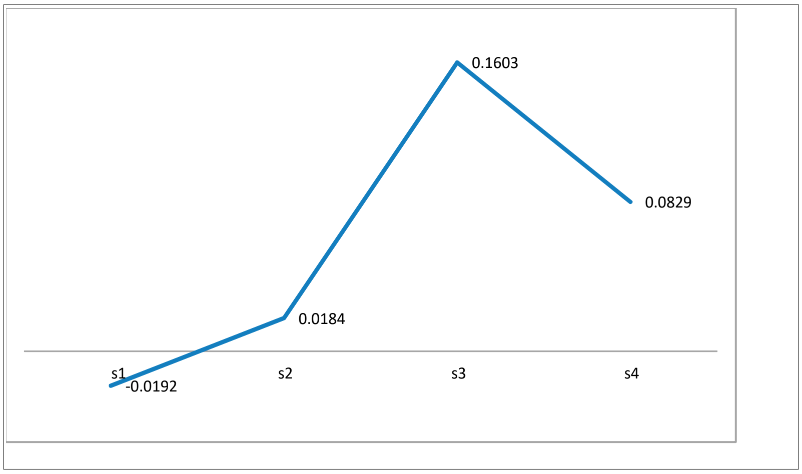


**Figure 12.** Rating value different range of values.

## 7. Comparsion Analysis

So as to check the legitimacy and viability of the proposed methodology, a near report is led utilizing the techniques triangular cubic fuzzy number [21], which are unique instances of TNCFNs, to the equivalent illustrative model.

### *A Comparison Analysis with the Existing MCDM Method Triangular Cubic Fuzzy Number*

Aliya et al [21] after transformation, the triangular cubic fuzzy information is given in Table 4.

**Table 4.** Triangular cubic fuzzy decision matrix.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $A_1$ | $\left\{\begin{array}{c}[0.1, 0.2, 0.3],\\ \langle[0.4, 0.8], 0.7\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.21, 0.22, 0.23],\\ \langle[0.5, 0.9], 0.8\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.15, 0.16, 0.17],\\ \langle[0.12, 0.16], 0.15\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.21, 0.22, 0.23],\\ \langle[0.5, 0.9], 0.8\rangle\end{array}\right\}$ |
| $A_2$ | $\left\{\begin{array}{c}[0.21, 0.22, 0.23],\\ \langle[0.5, 0.9], 0.8\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.1, 0.2, 0.3],\\ \langle[0.4, 0.8], 0.7\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.3, 0.4, 0.5],\\ \langle[0.2, 0.6], 0.4\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.15, 0.16, 0.17],\\ \langle[0.12, 0.16], 0.15\rangle\end{array}\right\}$ |
| $A_3$ | $\left\{\begin{array}{c}[0.15, 0.16, 0.17],\\ \langle[0.12, 0.16], 0.15\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.3, 0.4, 0.5],\\ \langle[0.2, 0.6], 0.4\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.1, 0.2, 0.3],\\ \langle[0.4, 0.8], 0.7\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.3, 0.4, 0.5],\\ \langle[0.2, 0.6], 0.5\rangle\end{array}\right\}$ |
| $A_4$ | $\left\{\begin{array}{c}[0.3, 0.4, 0.5],\\ \langle[0.2, 0.6], 0.4\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.15, 0.16, 0.17],\\ \langle[0.12, 0.16], 0.15\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.21, 0.22, 0.23],\\ \langle[0.5, 0.9], 0.8\rangle\end{array}\right\}$ | $\left\{\begin{array}{c}[0.1, 0.2, 0.3],\\ \langle[0.4, 0.8], 0.7\rangle\end{array}\right\}$ |

Calculate the TCFA operator and $w = (0.1,\ 0.2,\ 0.4,\ 0.3)^T$ .
The TCFA operator is presented in Table 5.

**Table 5.** TCFA operator.

| | |
|---|---|
| $A_1$ | $\langle[0.067, 0.081, 0.093], [0.1833, 0.4721], 0.1384\rangle$ |
| $A_2$ | $\langle[0.152, 0.196, 0.24], [0.2672, 0.6329], 0.5073\rangle$ |
| $A_3$ | $\langle[0.42, 0.464, 0.588], [0.4631, 0.7646], 0.2132\rangle$ |
| $A_4$ | $\langle[0.228, 0.294, 0.36], [0.3727, 0.7771], 0.3613\rangle$ |

Calculate the score function $s_1 = 0.0138$, $s_2 = 0.0256$, $s_3 = 0.1659$, $s_4 = 0.0772$.
See Figure 13.

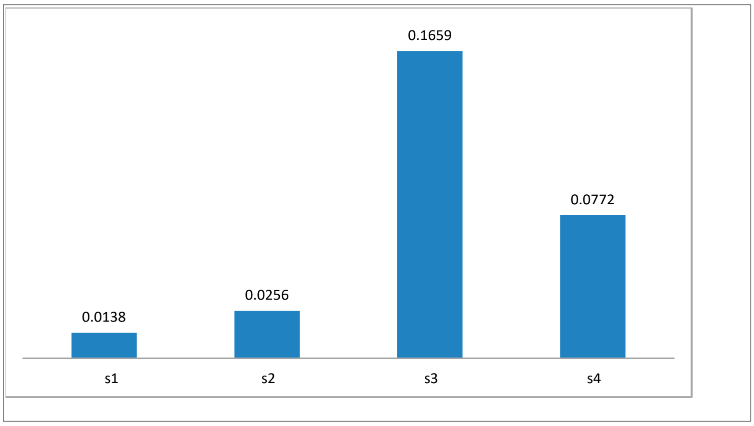


**Figure 13.** $s_3$ is the first ranking, $s_4$ is the 2$^{nd}$ ranking, $s_2$ is the third ranking and $s_1$ is the 4$^{th}$ ranking in the TCFN.

The existing Table 6 is as

**Table 6.** Comparison method with existing methods.

| Method | Ranking |
|---|---|
| TNCFNs | $\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$ |
| TCFN [21] | $\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$ |

See Figure 14.

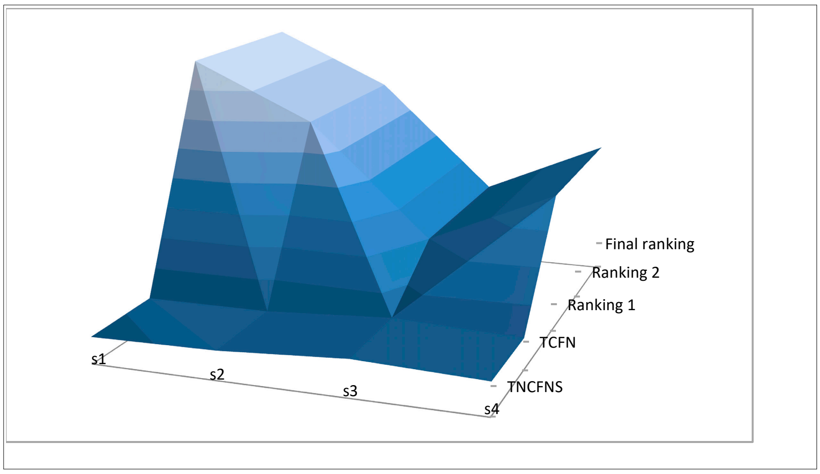


**Figure 14.** Comparison analysis with existing method.

The comparison method of score function is presented in Table 7.

**Table 7.** Comparison method with score function.

| Score function | Ranking |
|---|---|
| TNCFEWA operator | $S(C_3) > S(C_2) > S(C_1)$ |
| TNCFEOWA operator | $S(C_1) > S(C_2) > S(C_3)$ |
| TNCFEHWA operator | $S(C_2) > S(C_1) > S(C_3)$ |

See Figure 15.

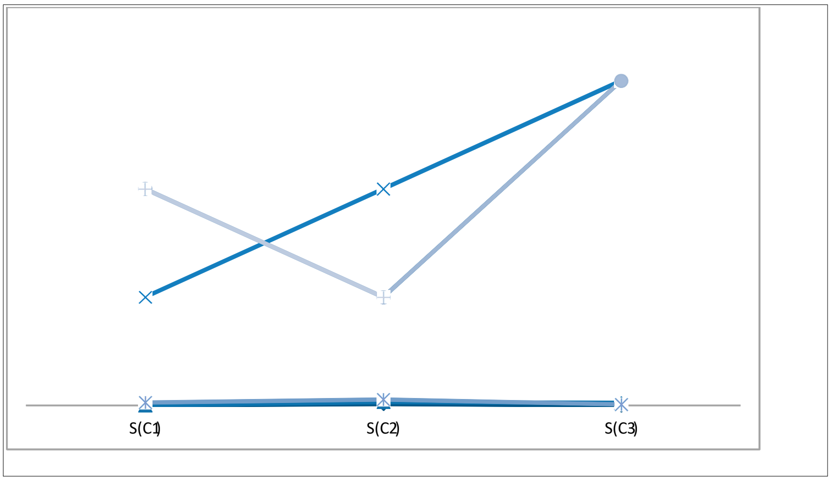


**Figure 15.** Different score value.

## 8. Conclusions

In this paper, we introduce a new concept of TNCFNs and operational laws. We introduce three aggregation operators, namely, the TNCFEWA operator, TNCFEOWA operator and TNCFEWA operator. We introduce group decision making under TNCFNs. Finally, a numerical example is provided to demonstrate the utility of the established approach. In cluster decision-making issues, consultants sometimes return from completely different specialty fields and have different backgrounds and levels of data; as such, they sometimes have branching opinions. These operators may be applied to several different fields, like data fusion, data processing, and pattern recognition, triangular neutrosophic cube like linguistic fuzzy Vikor methodology and quadrangle neutrosophic cube linguistic fuzzy Vikor methodology, which may be a suitable topic for longer term analysis, see Figure 16.

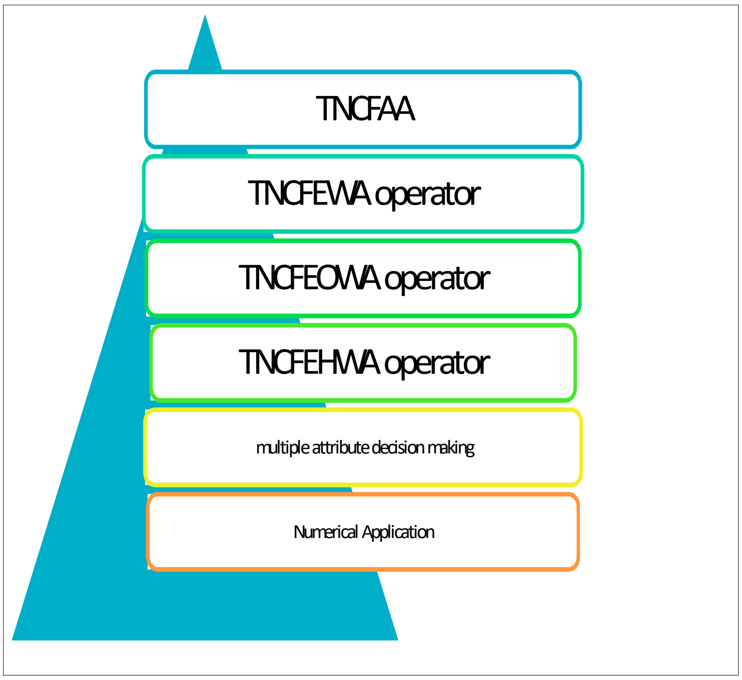


**Figure 16.** Flowcharts of whole papers.

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## Appendix A. Proof of Proposition 1

(1) $A_1 + A_2 = A_2 + A_1$;

$$A_1 + A_2 = \left\{ \begin{array}{c} \left\langle \left[ \frac{\alpha_1(h)+\alpha_2(h)}{(1+\alpha_1(h))(1-\alpha_2(h))}, \frac{\beta_1(h)+\beta_2(h)}{(1+\beta_1(h))(1-\beta_2(h))}, \frac{\Delta_1(h)+\Delta_2(h)}{(1+\Delta_1(h))(1-\Delta_2(h))} \right], \right. \\ \left[ \begin{array}{c} \left[ \frac{p_1^-(h)+p_2^-(h)}{(1+p_1^-(h))(1-p_2^-(h))}, \frac{q_1^-(h)+q_2^-(h)}{(1+q_1^-(h))(1-q_2^-(h))}, \frac{r_1^-(h)+r_2^-(h)}{(1+r_1^-(h))(1-r_2^-(h))} \right], \\ \left[ \frac{p_1^+(h)+p_2^+(h)}{(1+p_1^+(h))(1-p_2^+(h))}, \frac{q_1^+(h)+q_2^+(h)}{(1+q_1^+(h))(1-q_2^+(h))}, \frac{r_1^+(h)+r_2^+(h)}{(1+r_1^+(h))(1-r_2^+(h))} \right] \end{array} \right], \\ \left. \left[ \left[ \frac{p_1(h)p_2(h)}{1+((1-p_1(h))\,(1-p_2(h)))}, \frac{q_1(h)q_2(h)}{1+((1-q_1(h))\,(1-q_2(h)))}, \frac{r_1(h)r_2(h)}{1+((1-r_1(h))\,(1-r_2(h)))} \right] \right] \right\rangle \end{array} \right\} =$$

$$\left\{ \begin{array}{c} \left\langle \left[ \frac{\alpha_2(h)+\alpha_1(h)}{(1+\alpha_2(h))(1-\alpha_1(h))}, \frac{\beta_2(h)+\beta_1(h)}{(1+\beta_2(h))(1-\beta_1(h))}, \frac{\Delta_2(h)+\Delta_1(h)}{(1+\Delta_2(h))(1-\Delta_1(h))} \right], \right. \\ \left[ \begin{array}{c} \left[ \frac{p_2^-(h)+p_1^-(h)}{(1+p_2^-(h))(1-p_1^-(h))}, \frac{q_2^-(h)+q_1^-(h)}{(1+q_2^-(h))(1-q_1^-(h))}, \frac{r_2^-(h)+r_1^-(h)}{(1+r_2^-(h))(1-r_1^-(h))} \right], \\ \left[ \frac{p_2^+(h)+p_1^+(h)}{(1+p_2^+(h))(1-p_1^+(h))}, \frac{q_2^+(h)+q_1^+(h)}{(1+q_2^+(h))(1-q_1^+(h))}, \frac{r_2^+(h)+r_1^+(h)}{(1+r_2^+(h))(1-r_1^+(h))} \right], \end{array} \right], \\ \left. \left[ \left[ \frac{p_2(h)p_1(h)}{1+((1-p_2(h))\,(1-p_1(h)))}, \frac{q_2(h)q_1(h)}{1+((1-q_2(h))\,(1-q_1(h)))}, \frac{r_2(h).r_1(h)}{1+((1-r_2(h))\,(1-r_1(h)))} \right] \right] \right\rangle \end{array} \right\} = A_2 + A_1$$

Hence $A_1 + A_2 = A_2 + A_1$.
(2) $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1 + A_2) = \left\langle \left[ \begin{array}{c} \frac{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda[((1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda[((1+\beta_2(h))(1-\beta_2(h))]^\lambda}{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda[(1+\beta_2(h))(1-\beta_2(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda[((1+\Delta_2(h))(1-\Delta_2(h))]^\lambda}{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda} \end{array} \right], \left[ \begin{array}{c} \left[ \frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \right. \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}, \\ \left. \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda} \right] \\ \left[ \frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+p_2^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+p_2^+(h))(1-p_2^+(h))]^\lambda}, \right. \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \\ \left. \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda} \right] \end{array} \right], \right.$$

$$\left. \left[ \begin{array}{c} \left[ \frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^\lambda+[p_1(h)p_2(h)]^\lambda}, \right. \\ \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^\lambda+[q_1(h)q_2(h)]^\lambda}, \\ \left. \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^\lambda+[r_1(h)r_2(h)]^\lambda} \right] \end{array} \right] \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \left[ \begin{array}{c} \frac{[(1+\alpha_1(h))]^\lambda-[(1-\alpha_1(h))]^\lambda}{[(1+\alpha_1(h))]^\lambda+[(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))]^\lambda-[(1-\beta_1(h))]^\lambda}{[(1+\beta_1(h))]^\lambda+[(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))]^\lambda-[(1-\Delta_1(h))]^\lambda}{[(1+\Delta_1(h))]^\lambda+[(1-\Delta_1(h))]^\lambda} \end{array} \right], \left[ \begin{array}{c} \left[ \frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \right. \\ \left. \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]} \right] \\ \left[ \frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \right. \\ \left. \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]} \right] \end{array} \right], \right.$$

$$\left. \left[ \begin{array}{c} \left[ \frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \right. \\ \left. \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda} \right] \end{array} \right] \right\rangle$$

$$\lambda A_2 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))]^{\lambda}-[(1-\alpha_2(h))]^{\lambda}}{[(1+\alpha_2(h))]^{\lambda}+[(1-\alpha_2(h))]^{\lambda}}, \\ \frac{[(1+\beta_2(h))]^{\lambda}-[(1-\beta_2(h))]^{\lambda}}{[(1+\beta_2(h))]^{\lambda}+[(1-\beta_2(h))]^{\lambda}}, \\ \frac{[(1+\Delta_2(h))]^{\lambda}-[(1-\Delta_2(h))]^{\lambda}}{[(1+\Delta_2(h))]^{\lambda}+[(1-\Delta_2(h))]^{\lambda}} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))^{\lambda}-(1-p_2^-(h))^{\lambda}]}{[(1+p_2^-(h))^{\lambda}+(1-p_2^-(h))^{\lambda}]}, \\ \frac{[(1+q_2^-(h))^{\lambda}-(1-q_2^-(h))^{\lambda}]}{[(1+q_2^-(h))^{\lambda}+(1-q_2^-(h))^{\lambda}]}, \frac{[(1+r_2^-(h))^{\lambda}-(1-r_2^-(h))^{\lambda}]}{[(1+r_2^-(h))^{\lambda}+(1-r_2^-(h))^{\lambda}]}], \\ [\frac{[(1+p_2^+(h))^{\lambda}-(1-p_2^+(h))^{\lambda}]}{[(1+p_2^+(h))^{\lambda}+(1-p_2^+(h))^{\lambda}]}, \\ \frac{[(1+q_2^+(h))^{\lambda}-(1-q_2^+(h))^{\lambda}]}{[(1+q_2^+(h))^{\lambda}+(1-q_2^+(h))^{\lambda}]}, \frac{[(1+r_2^+(h))^{\lambda}-(1-r_2^+(h))^{\lambda}]}{[(1+r_2^+(h))^{\lambda}+(1-r_2^+(h))^{\lambda}]}] \end{bmatrix}, \begin{bmatrix} [\frac{2p_2^{\lambda}(h)}{[(2-p_2(h)]^{\lambda}+[p_2(h)]^{\lambda}}, \\ \frac{2q_2^{\lambda}(h)}{[(2-q_2(h)]^{\lambda}+[q_2(h)]^{\lambda}}, \frac{2r_2^{\lambda}(h)}{[(2-r_2(h)]^{\lambda}+[r_2(h)]^{\lambda}}] \end{bmatrix} \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}[((1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}}{[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}}, \\ \frac{[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}[((1+\beta_1(h))(1-\beta_1(h))]^{\lambda}}{[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}}, \\ \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}[((1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}}{[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}}{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}] \\ [\frac{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}}{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}] \end{bmatrix}, \begin{bmatrix} [\frac{2[p_2(h)p_1(h)]^{\lambda}}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^{\lambda}+[p_2(h)p_1(h)]^{\lambda}}, \\ \frac{2[q_2(h)q_1(h)]^{\lambda}}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^{\lambda}+[q_2(h)q_1(h)]^{\lambda}}, \\ \frac{2[r_2(h)r_1(h)]^{\lambda}}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^{\lambda}+[r_2(h)r_1(h)]^{\lambda}}] \end{bmatrix} \right\rangle$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

(3) $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$

$$\lambda_1 A = \left\langle \begin{bmatrix} \frac{[(1+\alpha_A(h))]^{\lambda_1}-[(1-\alpha_A(h))]^{\lambda_1}}{[(1+\alpha_A(h))]^{\lambda_1}+[(1-\alpha_A(h))]^{\lambda_1}}, \\ \frac{[(1+\beta_A(h))]^{\lambda_1}-[(1-\beta_A(h))]^{\lambda_1}}{[(1+\beta_A(h))]^{\lambda_1}+[(1-\beta_A(h))]^{\lambda_1}}, \\ \frac{[(1+\Delta_A(h))]^{\lambda_1}-[(1-\Delta_A(h))]^{\lambda_1}}{[(1+\Delta_A(h))]^{\lambda_1}+[(1-\Delta_A(h))]^{\lambda_1}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}}, \\ \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}] \\ [\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}] \end{bmatrix}, \begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}}, \\ \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}] \end{bmatrix} \right\rangle$$

$$\text{and } \lambda_2 A = \left\langle \begin{bmatrix} \frac{[(1+\alpha_A(h))]^{\lambda_2}-[(1-\alpha_A(h))]^{\lambda_2}}{[(1+\alpha_A(h))]^{\lambda_2}+[(1-\alpha_A(h))]^{\lambda_2}}, \\ \frac{[(1+\beta_A(h))]^{\lambda_2}-[(1-\beta_A(h))]^{\lambda_2}}{[(1+\beta_A(h))]^{\lambda_2}+[(1-\beta_A(h))]^{\lambda_2}}, \\ \frac{[(1+\Delta_A(h))]^{\lambda_2}-[(1-\Delta_A(h))]^{\lambda_2}}{[(1+\Delta_A(h))]^{\lambda_2}+[(1-\Delta_A(h))]^{\lambda_2}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}] \end{bmatrix}, \begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}}, \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}}, \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}] \end{bmatrix} \right\rangle =$$

$$\left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}-[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}+[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}, \frac{[1+\beta_A(h)]^{\lambda_1+\lambda_2}-[1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2}+[1-\beta_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}-[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}+[1-\Delta_A(h)]^{\lambda_1+\lambda_2}} \end{bmatrix}, \right.$$
$$\begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix},$$
$$\left. \begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}}, \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix} \right\rangle = (\lambda_1+\lambda_2)A.$$

## Appendix B. Proof of Theorem 1

Assume that $n = 1$, TCFEWA $(A_1, A_2, \ldots, A_n) = \overset{k}{\underset{j=1}{\oplus}} w_1 A_1$ $\langle(\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1+A_2) = \begin{bmatrix} \frac{[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}[((1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}}{[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}}, \\ \frac{[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}[((1+\beta_2(h))(1-\beta_2(h))]^{\lambda}}{[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}}, \\ \frac{[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}[((1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}}{[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}} \end{bmatrix}, \left\langle \begin{bmatrix} [\frac{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}, \\ \frac{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}] \\ ,[\frac{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}, \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}] \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} [\frac{2[p_1(h)p_2(h)]^{\lambda}}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^{\lambda}+[p_1(h)p_2(h)]^{\lambda}}, \\ \frac{2[q_1(h)q_2(h)]^{\lambda}}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^{\lambda}+[q_1(h)q_2(h)]^{\lambda}}, \\ \frac{2[r_1(h)r_2(h)]^{\lambda}}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^{\lambda}+[r_1(h)r_2(h)]^{\lambda}}] \end{bmatrix} \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_1(h))]^\lambda-[(1-\alpha_1(h))]^\lambda}{[(1+\alpha_1(h))]^\lambda+[(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))]^\lambda-[(1-\beta_1(h))]^\lambda}{[(1+\beta_1(h))]^\lambda+[(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))]^\lambda-[(1-\Delta_1(h))]^\lambda}{[(1+\Delta_1(h))]^\lambda+[(1-\Delta_1(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \\ \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]}] \\ [\frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \\ \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]}] \end{bmatrix}, \begin{bmatrix} [\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \\ \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \\ \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda}] \end{bmatrix} \right\rangle$$

$$\lambda A_2 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))]^\lambda-[(1-\alpha_2(h))]^\lambda}{[(1+\alpha_2(h))]^\lambda+[(1-\alpha_2(h))]^\lambda}, \\ \frac{[(1+\beta_2(h))]^\lambda-[(1-\beta_2(h))]^\lambda}{[(1+\beta_2(h))]^\lambda+[(1-\beta_2(h))]^\lambda}, \\ \frac{[(1+\Delta_2(h))]^\lambda-[(1-\Delta_2(h))]^\lambda}{[(1+\Delta_2(h))]^\lambda+[(1-\Delta_2(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))^\lambda-(1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda+(1-p_2^-(h))^\lambda]}, \\ \frac{[(1+q_2^-(h))^\lambda-(1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda+(1-q_2^-(h))^\lambda]}, \frac{[(1+r_2^-(h))^\lambda-(1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda+(1-r_2^-(h))^\lambda]}] \\ [\frac{[(1+p_2^+(h))^\lambda-(1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda+(1-p_2^+(h))^\lambda]}, \\ \frac{[(1+q_2^+(h))^\lambda-(1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda+(1-q_2^+(h))^\lambda]}, \frac{[(1+r_2^+(h))^\lambda-(1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda+(1-r_2^+(h))^\lambda]}] \end{bmatrix}, \begin{bmatrix} [\frac{2p_2^\lambda(h)}{[(2-p_2(h)]^\lambda+[p_2(h)]^\lambda}, \frac{2q_2^\lambda(h)}{[(2-q_2(h)]^\lambda+[q_2(h)]^\lambda}, \\ \frac{2r_2^\lambda(h)}{[(2-r_2(h)]^\lambda+[r_2(h)]^\lambda}] \end{bmatrix} \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda[((1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda[((1+\beta_1(h))(1-\beta_1(h))]^\lambda}{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda[(1+\beta_1(h))(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda[((1+\Delta_1(h))(1-\Delta_1(h))]^\lambda}{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}] \\ [\frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}] \end{bmatrix}, \begin{bmatrix} [\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^\lambda+[p_2(h)p_1(h)]^\lambda}, \\ \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^\lambda+[q_2(h)q_1(h)]^\lambda}, \\ \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^\lambda+[s_2(h)s_1(h)]^\lambda}] \end{bmatrix} \right\rangle$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$

$$\lambda_1 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1}-[1-\alpha_A(h)]^{\lambda_1}}{[1+\alpha_A(h)]^{\lambda_1}+[1-\alpha_A(h)]^{\lambda_1}}, \frac{[1+\beta_A(h)]^{\lambda_1}-[1-\beta_A(h)]^{\lambda_1}}{[1+\beta_A(h)]^{\lambda_1}+[1-\beta_A(h)]^{\lambda_1}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1}-[1-\Delta_A(h)]^{\lambda_1}}{[1+\Delta_A(h)]^{\lambda_1}+[1-\Delta_A(h)]^{\lambda_1}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}}, \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}] \\ [\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}}, \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}] \end{bmatrix}, \begin{bmatrix} \frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}}, \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}} \end{bmatrix} \right\rangle$$

and $$\lambda_2 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_2}-[1-\alpha_A(h)]^{\lambda_2}}{[1+\alpha_A(h)]^{\lambda_2}+[1-\alpha_A(h)]^{\lambda_2}}, \\ \frac{[1+\beta_A(h)]^{\lambda_2}-[1-\beta_A(h)]^{\lambda_2}}{[1+\beta_A(h)]^{\lambda_2}+[1-\beta_A(h)]^{\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_2}-[1-\Delta_A(h)]^{\lambda_2}}{[1+\Delta_A(h)]^{\lambda_2}+[1-\Delta_A(h)]^{\lambda_2}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}] \end{bmatrix}, $$

$$\left[\begin{array}{c}[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}},\\ \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}},\frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}]\end{array}\right]\Big\rangle=$$

$$\Big\langle\left[\begin{array}{c}\frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}-[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}+[1-\alpha_A(h)]^{\lambda_1+\lambda_2}},\frac{[\beta_A(h)]^{\lambda_1+\lambda_2}-[1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2}+[1-\beta_A(h)]^{\lambda_1+\lambda_2}},\\ \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}-[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}+[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}\end{array}\right],$$

$$\left[\begin{array}{c}[\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}},\frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}},\\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}],[\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}},\\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}},\frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}]\end{array}\right],$$

$$\left[\begin{array}{c}[\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}},\\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}},\frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}]\end{array}\right]\Big\rangle=$$

$$\left[\frac{[[1+\alpha_1(h)]^{\lambda_1}-[1-\alpha_1(h)]^{\lambda_1}}{[1+\alpha_1(h)]^{\lambda_1}+[1-\alpha_1(h)]^{\lambda_1}},\frac{[[1+\beta_1(h)]^{\lambda_1}-[1-\beta_1(h)]^{\lambda_1}}{[1+\beta_1(h)]^{\lambda_1}+[1-\beta_1(h)]^{\lambda_1}},\frac{[[1+\Delta_1(h)]^{\lambda_1}-[1-\Delta_1(h)]^{\lambda_1}}{[1+\Delta_1(h)]^{\lambda_1}+[1-\Delta_1(h)]^{\lambda_1}}\right]$$

$$\left[\begin{array}{c}\frac{[[1+p_1^-(h)]^{\lambda_1}-[1-p_1^-(h)]^{\lambda_1}}{[1+p_1^-(h)]^{\lambda_1}+[1-p_1^-(h)]^{\lambda_1}},\frac{[1+q_1^-(h)]^{\lambda_1}-[1-q_1^-(h)]^{\lambda_1}}{[1+q_1^-(h)]^{\lambda_1}+[1-q_1^-(h)]^{\lambda_1}},\\ \frac{[1+r_1^-(h)]^{\lambda_1}-[1-r_1^-(h)]^{\lambda_1}}{[1+r_1^-(h)]^{\lambda_1}+[1-r_1^-(h)]^{\lambda_1}}\end{array}\right];\left[\begin{array}{c}\frac{[1+p_1^+(h)]^{\lambda_1}-[1-p_1^+(h)]^{\lambda_1}}{[1+p_1^+(h)]^{\lambda_1}+[1-p_1^+(h)]^{\lambda_1}},\frac{[1+q_1^+(h)]^{\lambda_1}-[1-q_1^+(h)]^{\lambda_1}}{[1+q_1^+(h)]^{\lambda_1}+[1-q_1^+(h)]^{\lambda_1}},\\ \frac{[1+r_1^+(h)]^{\lambda_1}-[1-r_1^+(h)]^{\lambda_1}}{[1+r_1^+(h)]^{\lambda_1}+[1-r_1^+(h)]^{\lambda_1}}\end{array}\right];$$

$$\left[\begin{array}{c}\frac{2[p_1(h)]^{\lambda_1}}{[(2-p_1(h)]^{\lambda_1}+[p_1(h)]^{\lambda_1}},\frac{2[q_1(h)]^{\lambda_1}}{[(2-q_1(h)]^{\lambda_1}+[q_1(h)]^{\lambda_1}},\\ \frac{2[r_1(h)]^{\lambda_1}}{[(2-r_1(h)]^{\lambda_1}+[r_1(h)]^{\lambda_1}}\end{array}\right].$$

Assume that $n=k$, TCFEWA $(A_1, A_2, \ldots, A_n)=\overset{k}{\underset{j=1}{\oplus}} w_j A_j$

$$\Big\langle\left[\frac{[\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}},\frac{[\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}},\frac{[\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}}\right]$$

$$\left[\frac{[\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}\right],$$

$$\left[\frac{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}\right];$$

$$\left[\frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}},\frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}},\frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}}\right].$$

Then when $n=k+1$, we have TCFEWA $(A_1, A_2, \ldots, A_{k+1})=$ TCFEWA $(A_1, A_2, \ldots, A_k)\oplus A_{k+1})$

$$
\begin{aligned}
&\left\langle \left[ \frac{[\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}} \right]\right. \\
&\left[ \frac{[\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \right]; \\
&\left[ \frac{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \right]; \\
&\left[ \frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}} \right] \oplus_{k+1} \\
&\left\langle \left[ \frac{[\prod_{j=1}^{k+1}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\alpha_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k+1}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\beta_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k+1}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\Delta_1(h)]^{\varpi}} \right]\right. \\
&\left[ \frac{[\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}} \right]; \\
&\left[ \frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}} \right]; \\
&\left[ \frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \right] \\
&= \left[ \frac{[\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \right], \\
&\left[ \frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \right], \\
&\left[ \frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \right].
\end{aligned}
$$

Especially, if $w = \left(\frac{1}{n}, \frac{1}{n}, ...., \frac{1}{n}\right)^T$, then the the TNCFEWA operator is reduced to the triangular neutrosophic cubic fuzzy einstein averaging operator, which is shown as follows:

$$\left\langle \left[ \frac{[\prod_{j=1}^{n}[1+\alpha_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\frac{1}{n}}}, \frac{[\prod_{j=1}^{n}[1+\beta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\frac{1}{n}}}, \frac{[\prod_{j=1}^{n}[1+\Delta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\frac{1}{n}}} \right] \right.$$

$$\left[ \frac{[\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}} \right];$$

$$\left[ \frac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}} \right];$$

$$\left[ \frac{2\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}, \frac{2\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-q_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}, \frac{2\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-r_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}} \right].$$

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