Soft Computing
https://doi.org/10.1007/s00500-019-04025-z

METHODOLOGIES AND APPLICATION

Group decision making based on power Heronian aggregation operators under neutrosophic cubic environment

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Abstract

Neutrosophic cubic sets can deal with the complex information by combining the neutrosophic sets and cubic sets, the power average (PA) can weaken some effects of awkward data from biased decision makers, and Heronian mean (HM) can deal with the interrelationship between the aggregated attributes or arguments. In this article, in order to consider the advantages of the PA and HM, we combined and extended them to process neutrosophic cubic information. Firstly, we defined a distance measure for neutrosophic cubic numbers, then we presented the neutrosophic cubic power Heronian aggregation operator and neutrosophic cubic power weighted Heronian aggregation operator, and some characters and special cases of these new aggregation operators were investigated. Furthermore, we gave a new approach for multi-attribute group decision making based on new proposed operators. Finally, two examples were given to explain the validity and advantages of the developed approach by comparing with the existing method.

Communicated by V. Loia.

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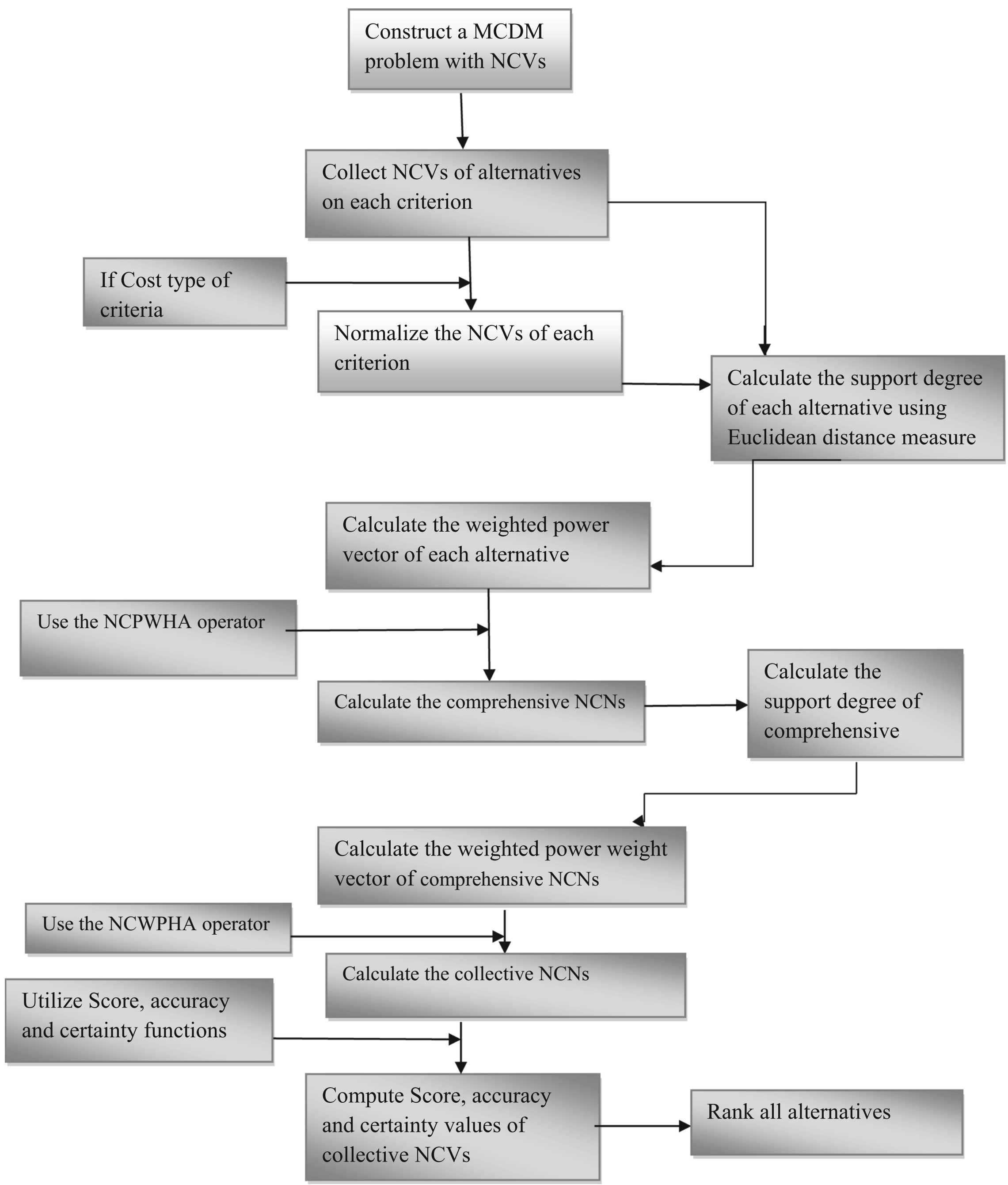
Published online: 08 May 2019

Graphical abstract



Keywords MAGDM · Neutrosophic cubic sets · Power average · Heronian mean

1 Introduction

The entire world is designated with complex circumstances. In order to process the complexity and uncertainty, Zadeh (1965) was the first one who introduced the concept of fuzzy set (FS), which is described by membership function (MF) in the closed interval [0, 1]. After then, Fs theory was successfully applied into various fields, such as pattern recognition, medical diagnosis, algebra, and decision making. But in FS, only MF was taken into account and it cannot handle the non-membership function (NMF) degree. To handle this kind of information, Attanassov (1986) proposed the concept of intuitionistic fuzzy sets (IFSs) which consisted of both MF and NMF degrees. In IFS, the hesitation degree can be obtained automatically by subtracting the sum of MF and NMF degrees from one. Now, IFS gained more and more attention from the researchers. Several similarity measures, correlation coefficient, and entropy measures for IFSs were defined by many researchers and also were applied to various fields (Szmidt and Kacprzyk 2000, 2001; Xu 2007; Liu et al. 2017). Atanassov and Gargov (1989) further generalized the concept of IFSs into interval-valued IFSs (IVIFSs), which extended the MF and NMF of IFSs to an interval number. Ye (2011) defined the fuzzy cross-entropy of IVIFSs and applied it to MADM. In IFSs and IVIFSs, the MF and NMF degrees are defined independently, while indeterminacy membership function (IMF) is dependent on MF and NMF degrees. To overcome this limitation of IFSs, Smarandache (1998) proposed the concept of neutrosophic sets (NSs) by generalized IFSs. In NSs, MF, NMF, and IMF are defined independently. Similar to FSs, IFSs, IVIFSs, and NSs have also some limitations. NSs are difficult to use in practical and engineering problems due to the containment of nonstandard united interval. To overcome this limitation, several subclasses of NSs were proposed by changing the nonstandard unit interval into standard interval, such as single-valued NSs (SVNSs) (Wang et al. 2010), interval NSs (INSs) (Wang et al. 2005), simplified NSs (Ye 2014), and multi-valued NSs (Peng et al. 2015).

In the recent years, Jun et al. (2012) proposed the concept cubic set (CS), and it is characterized by IVFS (Turksen 1986) and FS, which is more effective tool to deal with uncertain and vague information. The hybrid structure of CS has the advantage that it can contain more information than FS and IVFS. Taking the advantages of this hybrid structure of CS, several scholars extended the concept of CSs and proposed several generalizations of CSs, such as cubic hesitant fuzzy sets (CHFSs) (Mehmood et al. 2016), cubic soft sets (CFSs) (Muhiuddin and Al-roqi 2014), and neutrosophic cubic sets (NCSs) (Jun et al. 2017). In the above generalizations of CSs, NCSs gained much attention from the researchers, because they are more effective and informative by the available information in the form of INSs and SVNSs.

The information aggregation operator is one of the most important tools, which receives a great attention from the researchers in the recent years (Liu and Li 2017; Liu and Shi 2017; Liu et al. 2016a; Liu and Tang 2016; Liu and Wang 2017). Zhou and He (2015) and Chen (2014) presented an ordered precise weighted aggregation operator and a prioritized aggregation operator for IVIFSs, respectively. He et al. (2016) presented the concept of power Bonferroni mean (PBM) for IVIFSs and extended PBM for interval-valued hesitant fuzzy sets. Liu and Zhang (2018) proposed MAGDM method-based Bonferroni mean operator under intuitionistic uncertain linguistic environment. Liu and Chen (2017) proposed extended PBM for IVIFSs. Yu and Wu (2012) proposed the concept of HM operator for IVIFSs. Liu (2017) proposed the concept of power HM operator (PHM) for IVIFSs by combining the PA defined by Yager (2001) and the HM operator defined by Sykora (2009). Ju et al. (2018) proposed a MAGDM method based on power generalized Heronian mean operators under hesitant fuzzy linguistic environment. Recently, Liu et al. (2018) combined PA operator with HM operators to develop PHM to deal with linguistic neutrosophic information. For NSs and their subclasses, several other studies were made by many researchers, such as aggregation operators (Zhang et al. 2016), similarity measures (Ye and Fu 2016; Bolturk and Kahraman 2018), correlation coefficients (Ye 2013), and entropy measures (Tian et al. 2016; Wu et al. 2018), that were explored. Abdel-Basset et al. (2018) developed a novel GDM method for triangular neutrosophic numbers. Recently, some studies on NCSs were made. Lu and Ye (2017) proposed cosine similarity measure for NCSs and applied it to MADM, and Ali et al. (2016) proposed some similarity measures for NCSs and applied them to pattern recognition. Banerjee et al. (2017) proposed GRA for NC environment. Iqbal et al. (2016) defined NC sub-algebras, and NC closed ideals in B-algebras. Zhan et al. (2017) and Ye (2018) defined some operational laws, score, accuracy, and certainty functions for NCNs and proposed some aggregation operators which considered the relationship between the input arguments and applied them to MADM under neutrosophic cubic environment. From the existing studies, there are no such aggregation operators to deal with NC information which have the capacity for considering interrelationship among input arguments and removing the effect of awkward data at the same time.

Furthermore, in real word, the complexity in MAGDM problems is increasing day by day. In order to solve a decision-making problem, it is necessary for us to consider the following needs for selecting the best alternative. (a) About how to describe the complex uncertain information, as mentioned above, NCS contained more information than SVNS and INS. Therefore, NCS is a good tool to solve this problem; (b) sometimes, due to the partiality of the DMs, they give some awkward attribute values of alternatives, which may be unnecessarily low or unnecessarily high. In order to reduce these effects, we can choose PA to attain this function by earmarking the corresponding weights according to the support degrees; (c) In some situations, the relationship between the attributes (or input arguments) can also be taken under consideration and the HM or BM can achieve this goal. However, because HM has more advantages than the BM, we can select HM to deal with interactions. So the motivation and goal of this paper are (a) to define some distance and similarity measures for NCNs; (b) to combine the PA and HM operators, extend them to neutrosophic cubic environments, further propose some neutrosophic cubic PHM aggregation operators, such as NCPHA operator and NCPWHA operator, and investigate some properties and some special cases of them; (c) to develop new approach for MAGDM problems by these operators; (d) to illustrate the validity and advantages of the developed method.

To finish these goals, the rest of this article is organized as follows. In Sect. 2, we give some basic definitions and results of the NCSs, NCNs, PA, and HM. In Sect. 3, some novel PHM aggregation operators for NCNs are proposed. In Sect. 4, a novel MAGDM approach based on the proposed operators is developed. In Sect. 5, we explain the validity of the proposed method by two examples. Section 6 gives the conclusions.

2 Preliminaries

In this section, we give some basic definitions and results of NCSs, PA operator, and HM operator.

2.1 NCSs and their operations

Definition 1 (Jun et al. 2017). Let $\aleph$ be the universe of discourse set. A NCS in $\aleph$ is a pair $N = (\Theta, \Xi)$, in which Θ is an INS and Ξ is SVNS, and then, $n = \{\langle[\Theta_T, \Theta_{\bar{T}}], [\Theta_I \Theta_{\bar{I}}], [\Theta_F \Theta_{\bar{F}}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle\}$ is called a NCN.

Definition 2 (Zhan et al. 2017). Let $n_1 = \left\{\left\langle\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right], \left[\Theta_{I_1^L}, \Theta_{I_1^U}\right], \left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right\rangle, \langle \Xi_{T_1}, \Xi_{I_1}, \Xi_{F_1}\rangle\right\}$ and $n_2 = \left\{\left\langle\left[\Theta_{T_2^L}, \Theta_{T_2^U}\right], \left[\Theta_{I_2^L}, \Theta_{I_2^U}\right], \left[\Theta_{F_2^L}, \Theta_{F_2^U}\right]\right\rangle, \langle \Xi_{T_2}, \Xi_{I_2}, \Xi_{F_2}\rangle\right\}$ be any two NCNs and $\xi > 0$. Then, the operational laws for CNNs are shown as follows:

(1)

$$
\begin{aligned}
n_1 + n_2 = \Big\{\Big\langle & \left[\Theta_{T_1^L} + \Theta_{T_2^L} - \Theta_{T_1^L}\Theta_{T_2^L}, \Theta_{T_1^U} + \Theta_{T_2^U} - \Theta_{T_1^U}\Theta_{T_2^U}\right], \\
& \left[\Theta_{I_1^L} + \Theta_{I_2^L} - \Theta_{I_1^L}\Theta_{I_2^L}, \Theta_{I_1^U} + \Theta_{I_2^U} - \Theta_{I_1^U}\Theta_{I_2^U}\right], \\
& \left[\Theta_{F_1^L} + \Theta_{F_2^L} - \Theta_{F_1^L}\Theta_{F_2^L}, \Theta_{F_1^U} + \Theta_{F_2^U} - \Theta_{F_1^U}\Theta_{F_2^U}\right]\Big\rangle, \\
& \langle \Xi_{T_1}\Xi_{T_2}, \Xi_{I_1}\Xi_{I_2}, \Xi_{F_2}\Xi_{F_2}\rangle\Big\}; \qquad (1)
\end{aligned}
$$

(2)

$$
\begin{aligned}
n_1 \times n_2 = \Big\{\Big\langle & \left[\Theta_{T_1^L}\Theta_{T_2^L}, \Theta_{T_1^U}\Theta_{T_2^U}\right], \left[\Theta_{I_1^L}\Theta_{I_2^L}, \Theta_{I_1^U}\Theta_{I_2^U}\right], \\
& \left[\Theta_{F_1^L}\Theta_{F_2^L}, \Theta_{F_1^U}\Theta_{F_2^U}\right]\Big\rangle, \\
& \langle \Xi_{T_1} + \Xi_{T_2} - \Xi_{T_1}\Xi_{T_2}, \Xi_{I_1} + \Xi_{I_2} - \Xi_{I_1}\Xi_{I_2}, \Xi_{F_2} \\
& + \Xi_{F_2} - \Xi_{F_2}\Xi_{F_2}\rangle\Big\}; \qquad (2)
\end{aligned}
$$

(3)

$$
\begin{aligned}
\xi n_1 = \Big\{\Big\langle & \left[1 - \left(1 - \left(\Theta_{T_1^L}\right)\right)^{\xi}, 1 - \left(1 - \left(\Theta_{T_1^U}\right)\right)^{\xi}\right], \\
& \left[1 - \left(1 - \left(\Theta_{I_1^L}\right)\right)^{\xi}, 1 - \left(1 - \left(\Theta_{I_1^U}\right)\right)^{\xi}\right], \\
& \left[1 - \left(1 - \left(\Theta_{F_1^L}\right)\right)^{\xi}, 1 - \left(1 - \left(\Theta_{F_1^U}\right)\right)^{\xi}\right]\Big\rangle, \\
& \left\langle (\Xi_{T_1})^{\xi}, (\Xi_{I_1})^{\xi}, (\Xi_{F_1})^{\xi}\right\rangle\Big\}; \qquad (3)
\end{aligned}
$$

(4)

$$
\begin{aligned}
n_1^{\xi} = \Big\{\Big\langle & \left(\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right]\right)^{\xi}, \left(\left[\Theta_{I_1^L}, \Theta_{I_1^U}\right]\right)^{\xi}, \left(\left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right)^{\xi}\Big\rangle, \\
& \left\langle 1 - (1 - (\Xi_{T_1}))^{\xi}, 1 - (1 - (\Xi_{I_1}))^{\xi}, 1 - (1 - (\Xi_{F_1}))^{\xi}\right\rangle\Big\}. \qquad (4)
\end{aligned}
$$

Theorem 1 (Zhan et al. 2017). *Let* $n_1 = \left\{\left\langle\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right], \left[\Theta_{I_1^L}, \Theta_{I_1^U}\right], \left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right\rangle, \langle \Xi_{T_1}, \Xi_{I_1}, \Xi_{F_1}\rangle\right\}$ *and* $n_2 = \left\{\left\langle\left[\Theta_{T_2^L}, \Theta_{T_2^U}\right], \left[\Theta_{I_2^L}, \Theta_{I_2^U}\right], \left[\Theta_{F_2^L}, \Theta_{F_2^U}\right]\right\rangle, \langle \Xi_{T_2}, \Xi_{I_2}, \Xi_{F_2}\rangle\right\}$ *be any two NCNs. Then, they have the following properties.*

(1) $n_1 + n_2 = n_2 + n_1;$ (5)

(2) $n_1 \times n_2 = n_2 \times n_1;$ (6)

(3) $\psi(n_1 + n_2) = \psi.n_1 + \psi.n_2, \psi > 0;$ (7)

(4) $\psi_1 n_1 + \psi_2 n_1 = (\psi_1 + \psi_2)n_1, \psi_1, \psi_2 > 0;$ (8)

(5) $n_1^{\psi_1} \times n_1^{\psi_2} = (n_1)^{\psi_1 + \psi_2}, \psi_1, \psi_2 > 0;$ (9)

(6) $n_1^{\psi} \times n_2^{\psi} = (n_1 \times n_2)^{\psi}.$ (10)

Definition 3 (Zhan et al. 2017). Let $n_1 = \left\{\left\langle\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right], \left[\Theta_{I_1^L}, \Theta_{I_1^U}\right], \left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right\rangle, \langle \Xi_{T_1}, \Xi_{I_1}, \Xi_{F_1}\rangle\right\}$ be a NCN. Then, the score, accuracy, and certainty functions of NCN are defined as follows:

(1)

$$\widehat{S}(n_1) = \frac{\Theta_{T_1^L} + 2 - \Theta_{I_1^L} - \Theta_{F_1^L} + \Theta_{T_1^U} + 2 - \Theta_{I_1^U} - \Theta_{F_1^U} + \Xi_{T_1} + 2 - \Xi_{I_1} - \Xi_{F_1}}{9}; \tag{11}$$

(2)

$$\widehat{A}(n_1) = \Theta_{T_1^L} - \Theta_{F_1^L} + \Theta_{T_1^U} - \Theta_{F_1^U} + \Xi_{T_1} - \Xi_{F_1}; \tag{12}$$

(3)

$$\widehat{C}(n_1) = \Theta_{T_1^L} + \Theta_{T_1^U} + \Xi_{T_1}. \tag{13}$$

Theorem 2 (Zhan et al. 2017). *Let* $n_1 = \left\{\left\langle\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right], \left[\Theta_{I_1^L}, \Theta_{I_1^U}\right], \left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right\rangle, \langle \Xi_{T_1}, \Xi_{I_1}, \Xi_{F_1}\rangle\right\}$ *and* $n_2 = \left\{\left\langle\left[\Theta_{T_2^L}, \Theta_{T_2^U}\right], \left[\Theta_{I_2^L}, \Theta_{I_2^U}\right], \left[\Theta_{F_2^L}, \Theta_{F_2^U}\right]\right\rangle, \langle \Xi_{T_2}, \Xi_{I_2}, \Xi_{F_2}\rangle\right\}$ *be any two NCNs. Then, the comparison rules for NCNs can be defined as follows:*

(1) If $\widehat{S}(n_1) > \widehat{S}(n_2)$, then n_1 is greater than n_2 and is denoted by $n_1 > n_2$;

(2) If $\widehat{S}(n_1) = \widehat{S}(n_2)$ and $\widehat{A}(n_1) > \widehat{A}(n_2)$, then n_1 is greater than n_2 and is denoted by $n_1 > n_2$;

(3) If $\widehat{S}(n_1) = \widehat{S}(n_2)$,$\widehat{A}(n_1) = \widehat{A}(n_2)$, and $\widehat{C}(n_1) > \widehat{C}(n_2)$, then n_1 is greater than n_2 and is denoted by $n_1 > n_2$;

(4) If $\widehat{S}(n_1) = \widehat{S}(n_2)$,$\widehat{A}(n_1) = \widehat{A}(n_2)$, and $\widehat{C}(n_1) = \widehat{C}(n_2)$, then n_1 is equal to n_2 and is denoted by $n_1 = n_2$.

Example 1 If $n_1 = (\langle [0.3, 0.4], [0.2, 0.3], [0.1, 0.3]\rangle, \langle 0.2, 0.2, 0.3\rangle)$ and $n_2 = (\langle [0.5, 0.6], [0.1, 0.2], [0.1, 0.2]\rangle, \langle 0.5, 0.1, 0.2\rangle)$ are two NCNs, then based on Definition 3, we have the following results:

$$\widehat{S}(n_1) = \frac{0.3 + 2 - 0.2 - 0.1 + 0.4 + 2 - 0.3 - 0.3 + 0.2 + 2 - 0.2 - 0.3}{9} = 0.6111$$

and $\widehat{S}(n_2) = 0.7444$ implies that $\widehat{S}(n_1) < \widehat{S}(n_2)$.

So, by Theorem 2, we have $n_1 < n_2$.

Example 2 If $n_1 = (\langle [0.6, 0.7], [0.3, 0.4], [0.1, 0.2]\rangle, \langle 0.6, 0.3, 0.2\rangle)$ and $n_2 = (\langle [0.4, 0.6], [0.2, 0.3], [0.1, 0.2]\rangle, \langle 0.5, 0.2, 0.1\rangle)$ are two NCNs, then

$$\widehat{S}(n_1) = \widehat{S}(n_2) = 0.7111. \text{ So, we need to calculate } \widehat{A}(n_1) = 1.4, \widehat{A}(n_2) = 1.1.$$

Hence, by Theorem 2, we have $n_1 > n_2$.

Example 3 If $n_1 = (\langle [0.5, 0.7], [0.1, 0.2], [0.3, 0.4]\rangle, \langle 0.6, 0.1, 0.3\rangle)$ and $n_2 = (\langle [0.4, 0.6], [0.1, 0.2], [0.2, 0.3]\rangle, \langle 0.5, 0.1, 0.2\rangle)$ are two NCNs, then

$$\widehat{S}(n_1) = \widehat{S}(n_2) = 0.7111 \text{ and } \widehat{A}(n_1) = \widehat{A}(n_2) = 0.8;$$

so, we need to calculate $C(n_1) = 1.8,\ C(n_2) = 1.5$.

Hence, by Theorem 2, we have $n_1 > n_2$.

Definition 4 Let $n_1 = \left\{\left\langle\left[\Theta_{T_1^L}, \Theta_{T_1^U}\right], \left[\Theta_{I_1^L}, \Theta_{I_1^U}\right], \left[\Theta_{F_1^L}, \Theta_{F_1^U}\right]\right\rangle, \langle \Xi_{T_1}, \Xi_{I_1}, \Xi_{F_1}\rangle\right\}$ and $n_2 = \left\{\left\langle\left[\Theta_{T_2^L}, \Theta_{T_2^U}\right], \left[\Theta_{I_2^L}, \Theta_{I_2^U}\right], \left[\Theta_{F_2^L}, \Theta_{F_2^U}\right]\right\rangle, \langle \Xi_{T_2}, \Xi_{I_2}, \Xi_{F_2}\rangle\right\}$ be any two NCNs. Then, the Euclidean distance measure between two NCNs is defined and given as follows:

$$D(n_1, n_2) = \sqrt{\frac{1}{9}\left(\left(\Theta_{T_1^L} - \Theta_{T_2^L}\right)^2 + \left(\Theta_{T_1^U} - \Theta_{T_2^U}\right)^2 + \left(\Theta_{I_1^L} - \Theta_{I_2^L}\right)^2 + \left(\Theta_{I_1^U} - \Theta_{I_2^U}\right)^2 + \left(\Theta_{F_1^L} - \Theta_{F_2^L}\right)^2 + \left(\Theta_{F_1^U} - \Theta_{F_2^U}\right)^2 + (\Xi_{T_1} - \Xi_{T_2})^2 + (\Xi_{I_1} - \Xi_{I_2})^2 + (\Xi_{F_1} - \Xi_{F_2})^2\right)}. \tag{14}$$

Example 4 Let $n_1 = (\langle [0.3, 0.4], [0.2, 0.3], [0.1, 0.3]\rangle, \langle 0.2, 0.2, 0.3\rangle)$ and $n_2 = (\langle [0.5, 0.6], [0.1, 0.2], [0.1, 0.2]\rangle, \langle 0.5, 0.1, 0.2\rangle)$ be two NCNs. Then, based on Definition 4, we have the following distance measure:

$$D(n_1, n_2) = \sqrt{\frac{1}{9}\left((0.3-0.5)^2+(0.4-0.6)^2+(0.2-0.1)^2+(0.3-0.2)^2+(0.1-0.1)^2+(0.3-0.2)^2+(0.2-0.5)^2+(0.2-0.1)^2+(0.3-0.2)^2\right)} = 0.1563.$$

2.2 The PA operator

Yagar (2001) was the first one who presented the concept of the PA which is one of the important aggregation operators. The PA operator diminishes some negative effects of unnecessarily high or unnecessarily low arguments given by experts. The conventional PA operator can only deal with crisp numbers and is defined as follows.

Definition 5 (Yager 2001). Let $b_i (i = 1, 2, \ldots, m)$ be a group of nonnegative crisp numbers; the PA is a function defined by

$$PA(b_1, b_2, \ldots, b_m) = \frac{\sum_{i=1}^{m} (1 + T(b_i))b_i}{\sum_{i=1}^{m} (1 + T(b_i))}, \tag{15}$$

where

$$T(b_i) = \sum_{\substack{j=1 \\ j \neq i}}^{m} \text{Sup}(b_i, b_j) \tag{16}$$

and $\text{Sup}(b, c)$ is the support degree for b from c, which satisfies some axioms. (1) $\text{Sup}(b, c) \in [0, 1]$; (2) $\text{Sup}(b, c) = \text{Sup}(c, b)$; (3) $\text{Sup}(b, c) \geq \text{Sup}(d, e)$, if $|b - c| < |d - e|$.

2.3 HM operator

HM (Sykora 2009) is also an important tool, which can represent the interrelationships of the input values, and it is defined as follows:

Definition 5 (Sykora 2009). Let $I = [0, 1], x, y \geq 0, H^{x,y}: I^m \to I$. If $H^{x,y}$ satisfies

$$H^{x,y}(b_1, b_2, \ldots, b_m) = \left(\frac{2}{m^2 + m}\sum_{i=1}^{m}\sum_{j=i}^{m} b_i^x b_j^y\right)^{\frac{1}{x+y}}, \tag{17}$$

then the mapping $H^{x,y}$ is said to be HM operator with parameters. The HM satisfies the properties of idem potency, boundedness, and monotonicity.

3 The neutrosophic cubic PHM operators

In this part, we introduce neutrosophic cubic PHM operator and the neutrosophic cubic power weighted HM based on the operational rules for NCNs, shown as follows.

Definition 7 Let $n_i = \left\{\left\langle\left[\Theta_{T_i^L}, \Theta_{T_i^U}\right], \left[\Theta_{I_i^L}, \Theta_{I_i^U}\right], \left[\Theta_{F_i^L}, \Theta_{F_i^U}\right]\right\rangle, \langle \Xi_{T_i}, \Xi_{I_i}, \Xi_{F_i}\rangle\right\}$ $(i = 1, 2, \ldots, m)$ be a group of NCNs, $x, y \geq 0$, and $\text{NCPHA}: \aleph^m \to \aleph$, if

$$\text{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) = \left(\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(m\frac{(T(n_i)+1)}{\sum_{z=1}^{m}(T(n_z)+1)}n_i\right)^x \otimes_h \left(m\frac{(T(n_j)+1)}{\sum_{z=1}^{m}(T(n_z)+1)}n_j\right)^y\right)^{\frac{1}{x+y}}, \tag{18}$$

where $\aleph$ is the set of all NCNs and $T(n_z) = \sum_{\substack{i=1 \\ i \neq z}}^{m} \text{Sup}(n_z, n_i)$, in which $\text{Sup}(n_z, n_i)$ is the support degree for n_z from n_i, which satisfies the following axioms.

(1) $\sup(n_z, n_i) \in [0, 1]$; (2) $\sup(n_z, n_i) = \sup(n_i, n_z)$; (3) $\sup(s, t) \geq \sup(p, q)$; if $D(s, t) < D(p, q)$, in which $D(s, t)$ is the distance measure defined in Definition 4, then NCPHA is called the neutrosophic cubic PHM aggregation operator.

In order to simplify Eq. (18), we can define

$$\varphi_g = \frac{(T(n_g) + 1)}{\sum_{g=1}^{m} (T(n_g) + 1)} \tag{19}$$

and call $(\varphi_1, \varphi_2, \ldots, \varphi_m)$ as the power importance degree, which satisfies the condition that $\varphi_g \geq 0$, $\sum_{g=1}^{m} \varphi_g = 1$. Then, Eq. (18) can be expressed as follows:

$$\text{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) = \left(\frac{2}{m^2 + m}\sum_{i=1}^{m}\sum_{j=1}^{m} (m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y\right)^{\frac{1}{x+y}}. \tag{20}$$

Based on the operational rules of the NCNs described in Definition 1, we have the following theorem.

Theorem 3 *Let* $n_i = \left\{\left\langle\left[\Theta_{T_i^L}, \Theta_{T_i^U}\right], \left[\Theta_{I_i^L}, \Theta_{I_i^U}\right], \left[\Theta_{F_i^L}, \Theta_{F_i^U}\right]\right\rangle, \langle \Xi_{T_i}, \Xi_{I_i}, \Xi_{F_i}\rangle\right\}$ $(i = 1, 2, \ldots, m)$ *be a group of NCNs and* $x, y \geq 0$, *then the aggregated value from Eq. (20) is still NCN and even*

$$\begin{aligned}
&\mathrm{NCPHA}^{x,y}(n_1,n_2,\ldots,n_m)\\
&=\Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{T_i^L}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{T_j^L}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\ \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{T_i^U}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{T_j^U}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{I_i^L}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{I_j^L}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\ \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{I_i^U}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{I_j^U}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{F_i^L}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{F_j^L}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\ \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\big(1-\Theta_{F_i^U}\big)^{m\varphi_i}\Big)^{x}\times\Big(1-\big(1-\Theta_{F_j^U}\big)^{m\varphi_j}\Big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\big(1-(1-(\Xi_{T_i})^{n\varphi_i})^{x}\times\big(1-(\Xi_{T_j})^{n\varphi_j}\big)^{y}\big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\ 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\big(1-(1-(\Xi_{I_i})^{n\varphi_i})^{x}\times\big(1-(\Xi_{I_j})^{n\varphi_j}\big)^{y}\big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\\
&1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\big(1-(1-(\Xi_{F_i})^{n\varphi_i})^{x}\times\big(1-(\Xi_{F_j})^{n\varphi_j}\big)^{y}\big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg\rangle\Bigg).
\end{aligned}\tag{21}$$

Proof Since

$$\begin{aligned}
m\varphi_i n_i=\Big(\Big\langle&\Big[1-\big(1-\Theta_{T_i^L}\big)^{m\varphi_i},1-\big(1-\Theta_{T_i^U}\big)^{m\varphi_i}\Big],\\
&\Big[1-\big(1-\Theta_{I_i^L}\big)^{m\varphi_i},1-\big(1-\Theta_{I_i^U}\big)^{m\varphi_i}\Big],\\
&\Big[1-\big(1-\Theta_{F_i^L}\big)^{m\varphi_i},1-\big(1-\Theta_{F_i^U}\big)^{m\varphi_i}\Big]\Big\rangle,\\
&\langle(\Xi_{T_i})^{m\varphi_i},(\Xi_{I_i})^{m\varphi_i},(\Xi_{F_i})^{m\varphi_i}\rangle\Big),
\end{aligned}$$

So,

$$(m\varphi_i n_i)^{x}=\left(\begin{array}{l}\Big\langle\Big[\Big(1-\big(1-\Theta_{T_i^L}\big)^{m\varphi_i}\Big)^{x},\Big(1-\big(1-\Theta_{T_i^U}\big)^{m\varphi_i}\Big)^{x}\Big],\Big[\Big(1-\big(1-\Theta_{I_i^L}\big)^{m\varphi_i}\Big)^{x},\Big(1-\big(1-\Theta_{I_i^U}\big)^{m\varphi_i}\Big)^{x}\Big]^{x},\Big[\Big(1-\big(1-\Theta_{F_i^L}\big)^{m\varphi_i}\Big)^{x},\Big(1-\big(1-\Theta_{F_i^U}\big)^{m\varphi_i}\Big)^{x}\Big]\Big\rangle,\\ \big\langle 1-(1-(\Xi_{T_i})^{m\varphi_i})^{x},1-(1-(\Xi_{I_i})^{m\varphi_i})^{x},1-(1-(\Xi_{F_i})^{m\varphi_i})^{x}\big\rangle\end{array}\right).$$

Similarly, we have

$$(m\varphi_j n_j)^{y}=\left(\begin{array}{l}\Big\langle\Big[\Big(1-\big(1-\Theta_{T_j^L}\big)^{m\varphi_j}\Big)^{y},\Big(1-\big(1-\Theta_{T_j^U}\big)^{m\varphi_j}\Big)^{y}\Big],\Big[\Big(1-\big(1-\Theta_{I_j^L}\big)^{m\varphi_j}\Big)^{y},\Big(1-\big(1-\Theta_{I_j^U}\big)^{m\varphi_j}\Big)^{y}\Big],\Big[\Big(1-\big(1-\Theta_{F_j^L}\big)^{m\varphi_j}\Big)^{y},\Big(1-\big(1-\Theta_{F_j^U}\big)^{m\varphi_j}\Big)^{y}\Big]\Big\rangle,\\ \big\langle 1-\big(1-\big(\Xi_{T_j}\big)^{m\varphi_j}\big)^{y},1-\big(1-\big(\Xi_{I_j}\big)^{m\varphi_j}\big)^{y},1-\big(1-\big(\Xi_{F_j}\big)^{m\varphi_j}\big)^{y}\big\rangle\end{array}\right)$$

and then

$$
\begin{aligned}
&(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y = \Big(\Big\langle\Big[\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y, \Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big],\\
&\quad\Big[\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y, \Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big],\\
&\quad\Big[\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y, \Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big]\Big\rangle,\\
&\quad\langle 1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y, 1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y, 1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y\rangle)\\
&\quad\sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y = \Bigg(\Bigg\langle\Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x\\
&\quad\times\Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg],\\
&\quad\Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg]\\
&\quad\Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg]\Bigg\rangle,\\
&\quad\Bigg\langle\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y)\Bigg\rangle\Bigg).
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y\\
&\quad=\Bigg(\Bigg\langle\Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg],\\
&\quad\Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg],\\
&\quad\Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg]\Bigg\rangle,\\
&\quad\Bigg\langle\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg\rangle\Bigg).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\left(\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y\right)^{\frac{1}{x+y}}\\
&\quad=\Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},. \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg],\\
&\quad\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg],\\
&\quad\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\quad\Bigg\langle 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\\
&\quad 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg\rangle\Bigg).
\end{aligned}
$$

This completes the proof.

In order to obtain the weight vector φ, it is necessary to calculate the support degree between NCNs. In general, the support degree between NCNs can be regarded as similarity degree between NCNs. That is,

$$\mathrm{Sup}(n_i, n_l) = 1 - D(n_i, n_l)\ (i, l = 1, 2, \ldots, m). \tag{22}$$

Example 5 Let $n_1 = (\langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle, \langle 0.5, 0.2, 0.3\rangle)$, $n_2 = (\langle[0.5, 0.6], [0.1, 0.2], [0.2, 0.3]\rangle, \langle 0.6, 0.1, 0.2\rangle)$, and $n_3 = (\langle[0.7, 0.8], [0.1, 0.2], [0.1, 0.2]\rangle, \langle 0.8, 0.1, 0.2\rangle)$ be three NCNs, $x = 1, y = 2$. Then, by Theorem 3 in (21), we can aggregate these three NCNs and generate the comprehensive value $n = (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle)$ which is calculated as follows.

Step 1. Calculate the supports $\mathrm{Sup}(n_i, n_j), i, j = 1, 2, 3$ by using Eq. (22), and then, we get $\mathrm{Sup}(n_1, n_2) = \mathrm{Sup}(n_2, n_1) = 0.9001, \mathrm{Sup}(n_1, n_3) = \mathrm{Sup}(n_3, n_1) = 0.7919, \mathrm{Sup}(n_2, n_3) = \mathrm{Sup}(n_3, n_2) = 0.8753$.

Step 2. Calculate the power weight vector by using Eq. (16), and we have

$$\begin{aligned} T(n_1) &= \mathrm{Sup}(n_1, n_2) + \mathrm{Sup}(n_1, n_3) = 1.6920, T(n_2) \\ &= \mathrm{Sup}(n_2, n_1) + \mathrm{Sup}(n_2, n_3) = 1.7754, T(n_3) \\ &= \mathrm{Sup}(n_3, n_1) + \mathrm{Sup}(n_3, n_2) = 1.6673 \end{aligned}$$

and

$$\varphi_1 = \frac{(T(n_1) + 1)}{(T(n_1) + 1) + (T(n_2) + 1) + (T(n_3) + 1)} = 0.3309,$$

$$\varphi_2 = \frac{(T(n_2) + 1)}{(T(n_1) + 1) + (T(n_2) + 1) + (T(n_3) + 1)} = 0.3412,$$

$$\varphi_3 = \frac{(T(n_3) + 1)}{(T(n_1) + 1) + (T(n_2) + 1) + (T(n_3) + 1)} = 0.3279.$$

Step 3. Calculate the comprehensive value $n = (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle)$ by Eq. (21), and we have

$$\begin{aligned}
&[\Theta_{T^L}, \Theta_{T^U}] \\
&= \left[\left(1 - \left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1 - \left(1 - \left(1 - \Theta_{T_i^L}\right)^{3\varphi_i}\right)^{1} \times \left(1 - \left(1 - \Theta_{T_j^L}\right)^{3\varphi_j}\right)^{2}\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{3}}, \left(1 - \left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1 - \left(1 - \left(1 - \Theta_{T_i^U}\right)^{3\varphi_i}\right)^{1} \times \left(1 - \left(1 - \Theta_{T_j^U}\right)^{3\varphi_j}\right)^{2}\right)\right)^{\frac{2}{3^2+3}}\right)\right]^{\frac{1}{3}} \\
&= \left[\left(1 - \left(\begin{array}{l} \left(1 - \left(1 - (1 - 0.4)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.4)^{3\varphi_1}\right)^{2}\right) \times \left(1 - \left(1 - (1 - 0.4)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.5)^{3\varphi_2}\right)^{2}\right) \\ \quad \times \left(1 - \left(1 - (1 - 0.4)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.7)^{3\varphi_3}\right)^{2}\right) \\ \times \left(1 - \left(1 - (1 - 0.5)^{3\varphi_2}\right)^{1} \times \left(1 - (1 - 0.5)^{3\varphi_2}\right)^{2}\right) \times \left(1 - \left(1 - (1 - 0.5)^{3\varphi_2}\right)^{1} \times \left(1 - (1 - 0.7)^{3\varphi_3}\right)^{2}\right) \\ \times \left(1 - \left(1 - (1 - 0.7)^{3\varphi_3}\right)^{1} \times \left(1 - (1 - 0.7)^{3\varphi_3}\right)^{2}\right) \end{array}\right)^{\frac{2}{12}}\right)^{\frac{1}{3}}, \right. \\
&\quad \left. \left(1 - \left(\begin{array}{l} \left(1 - \left(1 - (1 - 0.5)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.5)^{3\varphi_1}\right)^{2}\right) \times \left(1 - \left(1 - (1 - 0.5)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.6)^{3\varphi_2}\right)^{2}\right) \\ \times \left(1 - \left(1 - (1 - 0.5)^{3\varphi_1}\right)^{1} \times \left(1 - (1 - 0.8)^{3\varphi_3}\right)^{2}\right) \\ \times \left(1 - \left(1 - (1 - 0.6)^{3\varphi_2}\right)^{1} \times \left(1 - (1 - 0.6)^{3\varphi_2}\right)^{2}\right) \times \left(1 - \left(1 - (1 - 0.6)^{3\varphi_2}\right)^{1} \times \left(1 - (1 - 0.8)^{3\varphi_3}\right)^{2}\right) \\ \times \left(1 - \left(1 - (1 - 0.8)^{3\varphi_3}\right)^{1} \times \left(1 - (1 - 0.8)^{3\varphi_3}\right)^{2}\right) \end{array}\right)^{\frac{2}{12}}\right)^{\frac{1}{3}}\right] \\
&= [0.5680, 0.6703];
\end{aligned}$$

$$[\Theta_{I^L},\Theta_{I^U}]=\left[\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-\left(1-\Theta_{I_i^L}\right)^{3\varphi_i}\right)^1\times\left(1-\left(1-\Theta_{I_j^L}\right)^{3\varphi_j}\right)^2\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{3}},\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-\left(1-\Theta_{I_i^U}\right)^{3\varphi_i}\right)^1\times\left(1-\left(1-\Theta_{I_j^U}\right)^{3\varphi_j}\right)^2\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{3}}\right]$$

$$=\left[\left(1-\begin{pmatrix}\left(1-\left(1-(1-0.2)^{3\varphi_1}\right)^1\times\left(1-(1-0.2)^{3\varphi_1}\right)^2\right)\times\left(1-\left(1-(1-0.2)^{3\varphi_1}\right)^1\times\left(1-(1-0.1)^{3\varphi_2}\right)^2\right)\left(1-\left(1-(1-0.2)^{3\varphi_1}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\times\\\left(1-\left(1-(1-0.1)^{3\varphi_2}\right)^1\times\left(1-(1-0.1)^{3\varphi_2}\right)^2\right)\times\left(1-\left(1-(1-0.1)^{3\varphi_2}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\left(1-\left(1-(1-0.1)^{3\varphi_3}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}},\right.$$

$$\left.\left(1-\begin{pmatrix}\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.3)^{3\varphi_1}\right)^2\right)\times\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.2)^{3\varphi_2}\right)^2\right)\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\times\\\left(1-\left(1-(1-0.2)^{3\varphi_2}\right)^1\times\left(1-(1-0.2)^{3\varphi_2}\right)^2\right)\times\left(1-\left(1-(1-0.2)^{3\varphi_2}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\left(1-\left(1-(1-0.2)^{3\varphi_3}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)\right]$$

$$=[0.1354,0.2319];$$

$$[\Theta_{F^L},\Theta_{F^U}]=\left[\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{3\varphi_i}\right)^1\times\left(1-\left(1-\Theta_{F_j^L}\right)^{3\varphi_j}\right)^2\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{3}},\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-\left(1-\Theta_{F_i^U}\right)^{3\varphi_i}\right)^1\times\left(1-\left(1-\Theta_{F_j^U}\right)^{3\varphi_j}\right)^2\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{3}}\right]$$

$$=\left[\left(1-\begin{pmatrix}\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.3)^{3\varphi_1}\right)^2\right)\times\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.2)^{3\varphi_2}\right)^2\right)\left(1-\left(1-(1-0.3)^{3\varphi_1}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\times\\\left(1-\left(1-(1-0.2)^{3\varphi_2}\right)^1\times\left(1-(1-0.2)^{3\varphi_2}\right)^2\right)\times\left(1-\left(1-(1-0.2)^{3\varphi_2}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\left(1-\left(1-(1-0.1)^{3\varphi_3}\right)^1\times\left(1-(1-0.1)^{3\varphi_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}},\right.$$

$$\left.\left(1-\begin{pmatrix}\left(1-\left(1-(1-0.4)^{3\varphi_1}\right)^1\times\left(1-(1-0.4)^{3\varphi_1}\right)^2\right)\times\left(1-\left(1-(1-0.4)^{3\varphi_1}\right)^1\times\left(1-(1-0.3)^{3\varphi_2}\right)^2\right)\left(1-\left(1-(1-0.4)^{3\varphi_1}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\times\\\left(1-\left(1-(1-0.3)^{3\varphi_2}\right)^1\times\left(1-(1-0.3)^{3\varphi_2}\right)^2\right)\times\left(1-\left(1-(1-0.3)^{3\varphi_2}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\left(1-\left(1-(1-0.2)^{3\varphi_3}\right)^1\times\left(1-(1-0.2)^{3\varphi_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}}\right]$$

$$=[0.2074,0.3018];$$

$$\Xi_T=1-\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-(\Xi_{T_i})^{3\varphi_i}\right)^x\times\left(1-(\Xi_{T_j})^{3\varphi_j}\right)^y\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{1+2}}$$

$$=1-\left(1-\begin{pmatrix}\left(1-\left(1-0.5^{3\omega_1}\right)^1\left(1-0.5^{3\omega_1}\right)^2\right)\times\left(1-\left(1-0.5^{3\omega_1}\right)^1\left(1-0.6^{3\omega_2}\right)^2\right)\\\times\left(1-\left(1-0.5^{3\omega_1}\right)^1\left(1-0.8^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.6^{3\omega_2}\right)^1\left(1-0.6^{3\omega_2}\right)^2\right)\\\left(1-\left(1-0.6^{3\omega_2}\right)^1\left(1-0.8^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.8^{3\omega_3}\right)^1\left(1-0.8^{3\omega_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}}=0.6229;$$

$$\Xi_I=1-\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-(\Xi_{I_i})^{3\varphi_i}\right)^x\times\left(1-(\Xi_{I_j})^{3\varphi_j}\right)^y\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{1+2}}$$

$$=1-\left(1-\begin{pmatrix}\left(1-\left(1-0.2^{3\omega_1}\right)^1\left(1-0.2^{3\omega_1}\right)^2\right)\times\left(1-\left(1-0.2^{3\omega_1}\right)^1\left(1-0.1^{3\omega_2}\right)^2\right)\\\times\left(1-\left(1-0.2^{3\omega_1}\right)^1\left(1-0.1^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.1^{3\omega_2}\right)^1\left(1-0.1^{3\omega_2}\right)^2\right)\\\left(1-\left(1-0.1^{3\omega_2}\right)^1\left(1-0.1^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.1^{3\omega_3}\right)^1\left(1-0.1^{3\omega_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}}=0.1234;$$

$$\Xi_F=1-\left(1-\left(\prod_{i=1}^{3}\prod_{j=i}^{3}\left(1-\left(1-(\Xi_{F_i})^{3\varphi_i}\right)^x\times\left(1-(\Xi_{F_j})^{3\varphi_j}\right)^y\right)\right)^{\frac{2}{3^2+3}}\right)^{\frac{1}{1+2}}$$

$$=1-\left(1-\begin{pmatrix}\left(1-\left(1-0.3^{3\omega_1}\right)^1\left(1-0.3^{3\omega_1}\right)^2\right)\times\left(1-\left(1-0.3^{3\omega_1}\right)^1\left(1-0.2^{3\omega_2}\right)^2\right)\\\times\left(1-\left(1-0.3^{3\omega_1}\right)^1\left(1-0.2^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.2^{3\omega_2}\right)^1\left(1-0.2^{3\omega_2}\right)^2\right)\\\left(1-\left(1-0.2^{3\omega_2}\right)^1\left(1-0.2^{3\omega_3}\right)^2\right)\times\left(1-\left(1-0.2^{3\omega_3}\right)^1\left(1-0.2^{3\omega_3}\right)^2\right)\end{pmatrix}^{\frac{2}{12}}\right)^{\frac{1}{3}}=0.2249.$$

So, the comprehensive value $n = (\langle[0.5680, 0.6703], [0.1354, 0.2319], [0.2074, 0.3018]\rangle, \langle 0.6229, 0.1234, 0.2249\rangle)$.

So $\varphi_z = \frac{1}{m}$ for all $z = 1, 2, \ldots, m$, and then,

$$
\begin{aligned}
&\mathrm{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) = \mathrm{NCPHA}^{x,y}(n, n, \ldots, n)\\
&= \Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{T^L})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{T^L})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{T^U})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{T^U})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\\
&\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{I^L})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{I^L})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{I^U})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{I^U})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\\
&\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{F^L})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{F^L})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(1-\Theta_{F^U})^{m\frac{1}{m}}\big)^{x}\times\big(1-(1-\Theta_{F^U})^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(\Xi_T)^{m\frac{1}{m}}\big)^{x}\times\big(1-(\Xi_T)^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(\Xi_I)^{m\frac{1}{m}}\big)^{x}\times\big(1-(\Xi_I)^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\\
&1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\big(1-(\Xi_{F_i})^{m\frac{1}{m}}\big)^{x}\times\big(1-(\Xi_F)^{m\frac{1}{m}}\big)^{y}\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg\rangle\Bigg)\\
&= \Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{T^L}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{T^U}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg], \Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{I^L}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{I^U}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{F^L}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, \Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-\Theta_{F^U}^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle, \Bigg\langle 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-\Xi_T)^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}},\\
&1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-\Xi_I)^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}, 1-\Bigg(1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-\Xi_F)^{x+y})\Bigg)^{\frac{2}{m^2+m}}\Bigg)^{\frac{1}{x+y}}\Bigg\rangle\Bigg)\\
&= \Big(\Big\langle\Big[(1-(1-\Theta_{T^L}^{x+y}))^{\frac{1}{x+y}}, (1-(1-\Theta_{T^U}^{x+y}))^{\frac{1}{x+y}}\Big], \Big[(1-(1-\Theta_{I^L}^{x+y}))^{\frac{1}{x+y}}, (1-(1-\Theta_{I^U}^{x+y}))^{\frac{1}{x+y}}\Big], \Big[(1-(1-\Theta_{F^L}^{x+y}))^{\frac{1}{x+y}}, (1-(1-\Theta_{F^U}^{x+y}))^{\frac{1}{x+y}}\Big]\Big\rangle,\\
&\Big\langle 1-((1-\Xi_T)^{x+y})^{\frac{1}{x+y}}, 1-((1-\Xi_I)^{x+y})^{\frac{1}{x+y}}, 1-((1-\Xi_F)^{x+y})^{\frac{1}{x+y}}\Big\rangle\Big)\\
&= (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle) = n.
\end{aligned}
$$

Theorem 4 (Idempotency) *Let* $n_i = \left\{\left\langle\left[\Theta_{T_i^L}, \Theta_{T_i^U}\right], \left[\Theta_{I_i^L}, \Theta_{I_i^U}\right], \left[\Theta_{F_i^L}, \Theta_{F_i^U}\right]\right\rangle, \langle \Xi_{T_i}, \Xi_{I_i}, \Xi_{F_i}\rangle\right\}$ $(i = 1, 2, \ldots, m)$ *be a group of NCNs; if* $n_i = n = (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle)$ *for all* $i = 1, 2, \ldots, m$, *then*

$\mathrm{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) = n$.

Proof Since $n_i = n = (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle)$ for all $i = 1, 2, \ldots, m$, we have

$\mathrm{Sup}(n_z, n_j) = 1 \quad \text{for all } z, j = 1, 2, \ldots, m.$

This completes the proof of Theorem 4.

Theorem 5 (Boundedness) *Let* $n_i = \left\{\left\langle\left[\Theta_{T_i^L}, \Theta_{T_i^U}\right], \left[\Theta_{I_i^L}, \Theta_{I_i^U}\right], \left[\Theta_{F_i^L}, \Theta_{F_i^U}\right]\right\rangle, \langle \Xi_{T_i}, \Xi_{I_i}, \Xi_{F_i}\rangle\right\}$ $(i = 1, 2, \ldots, m)$ *be a group of NCNs, and* $p = \min(n_1, n_2, \ldots, n_m) = (\langle[\Theta_{T^L}, \Theta_{T^U}], [\Theta_{I^L}, \Theta_{I^U}], [\Theta_{F^L}, \Theta_{F^U}]\rangle, \langle \Xi_T, \Xi_I, \Xi_F\rangle)$, $q = \max(n_1, n_2, \ldots, n_m) = \left(\left\langle\left[\overline{\Theta_{T^L}}, \overline{\Theta_{T^U}}\right], \left[\overline{\Theta_{I^L}}, \overline{\Theta_{I^U}}\right], \left[\overline{\Theta_{F^L}}, \overline{\Theta_{F^U}}\right]\right\rangle, \langle \overline{\Xi_T}, \overline{\Xi_I}, \overline{\Xi_F}\rangle\right)$, *then the NCPHA lies:* $p \le \mathrm{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) \le q$, *where*

$$p = \left(\left\langle \left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{T^L})^{m\varphi_i})^x \times (1 - (1 - \Theta_{T^L})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{T^U})^{m\varphi_i})^x \times (1 - (1 - \Theta_{T^U})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right] \right. \right.$$
$$\left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{I^L})^{m\varphi_i})^x \times (1 - (1 - \Theta_{I^L})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{I^U})^{m\varphi_i})^x \times (1 - (1 - \Theta_{I^U})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right]$$
$$\left. \left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{F^L})^{m\varphi_i})^x \times (1 - (1 - \Theta_{F^L})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (1 - \Theta_{F^U})^{m\varphi_i})^x \times (1 - (1 - \Theta_{F^U})^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right] \right\rangle,$$
$$\left\langle 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (\Xi_T)^{m\varphi_i})^x \times (1 - (\Xi_T)^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (\Xi_I)^{m\varphi_i})^x \times (1 - (\Xi_I)^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \right.$$
$$\left. \left. 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} (1 - (1 - (\Xi_F)^{m\varphi_i})^x \times (1 - (\Xi_F)^{m\varphi_j})^y) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right\rangle \right).$$

$$q = \left(\left\langle \left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{T^L}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{T^L}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{T^U}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{T^U}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right] \right. \right.$$
$$\left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{I^L}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{I^L}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{I^U}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{I^U}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right]$$
$$\left. \left[\left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{F^L}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{F^L}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - \left(1 - (1 - \overline{\Theta_{F^U}})^{m\varphi_i} \right)^x \times \left(1 - (1 - \overline{\Theta_{F^U}})^{m\varphi_j} \right)^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right] \right\rangle,$$
$$\left\langle 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - (1 - (\overline{\Xi_T})^{m\varphi_i})^x \times (1 - (\overline{\Xi_T})^{m\varphi_j})^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - (1 - (\overline{\Xi_I})^{m\varphi_i})^x \times (1 - (\overline{\Xi_I})^{m\varphi_j})^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}}, \right.$$
$$\left. \left. 1 - \left(1 - \left(\prod_{i=1}^{m} \prod_{j=i}^{m} \left(1 - (1 - (\overline{\Xi_F})^{m\varphi_i})^x \times (1 - (\overline{\Xi_F})^{m\varphi_j})^y \right) \right)^{\frac{2}{m^2+m}} \right)^{\frac{1}{x+y}} \right\rangle \right).$$

Proof Since

$$m\varphi_i n_i = \left(\left\langle \left[1 - \left(1 - \Theta_{T_i^L} \right)^{m\varphi_i}, 1 - \left(1 - \Theta_{T_i^U} \right)^{m\varphi_i} \right], \left[1 - \left(1 - \Theta_{I_i^L} \right)^{m\varphi_i}, 1 - \left(1 - \Theta_{I_i^U} \right)^{m\varphi_i} \right], \right. \right.$$
$$\left. \left[1 - \left(1 - \Theta_{F_i^L} \right)^{m\varphi_i}, 1 - \left(1 - \Theta_{F_i^U} \right)^{m\varphi_i} \right] \right\rangle, \langle (\Xi_{T_i})^{m\varphi_i}, (\Xi_{I_i})^{m\varphi_i}, (\Xi_{F_i})^{m\varphi_i} \rangle \Big) \geq (\langle [1 - (1 - \Theta_{T^L})^{m\varphi_i}, 1 - (1 - \Theta_{T^U})^{m\varphi_i}],$$
$$[1 - (1 - \Theta_{I^L})^{m\varphi_i}, 1 - (1 - \Theta_{I^U})^{m\varphi_i}], [1 - (1 - \Theta_{F^L})^{m\varphi_i}, 1 - (1 - \Theta_{F^U})^{m\varphi_i}] \rangle, \langle (\Xi_T)^{m\varphi_i}, (\Xi_I)^{m\varphi_i}, (\Xi_F)^{m\varphi_i} \rangle),$$

$$(m\varphi_i n_i)^x = \begin{pmatrix} \left\langle \left[\left(1 - \left(1 - \Theta_{T_i^L} \right)^{m\varphi_i} \right)^x, \left(1 - \left(1 - \Theta_{T_i^U} \right)^{m\varphi_i} \right)^x \right], \left[\left(1 - \left(1 - \Theta_{I_i^L} \right)^{m\varphi_i} \right)^x, \left(1 - \left(1 - \Theta_{I_i^U} \right)^{m\varphi_i} \right)^x \right], \left[\left(1 - \left(1 - \Theta_{F_i^L} \right)^{m\varphi_i} \right)^x, \left(1 - \left(1 - \Theta_{F_i^U} \right)^{m\varphi_i} \right)^x \right] \right\rangle, \\ \langle 1 - (1 - (\Xi_{T_i})^{m\varphi_i})^x, 1 - (1 - (\Xi_{I_i})^{m\varphi_i})^x, 1 - (1 - (\Xi_{F_i})^{m\varphi_i})^x \rangle \end{pmatrix}$$
$$\geq \begin{pmatrix} \langle [(1 - (1 - \Theta_{T^L})^{m\varphi_i})^x, (1 - (1 - \Theta_{T^U})^{m\varphi_i})^x], [(1 - (1 - \Theta_{I^L})^{m\varphi_i})^x, (1 - (1 - \Theta_{I^U})^{m\varphi_i})^x], [(1 - (1 - \Theta_{F^L})^{m\varphi_i})^x, (1 - (1 - \Theta_{F^U})^{m\varphi_i})^x] \rangle, \\ \langle 1 - (1 - (\Xi_T)^{m\varphi_i})^x, 1 - (1 - (\Xi_I)^{m\varphi_i})^x, 1 - (1 - (\Xi_F)^{m\varphi_i})^x \rangle. \end{pmatrix}$$

Similarly, we have

Then,

$$(m\varphi_j n_j)^y = \left(\left\langle \left[\left(1 - \left(1 - \Theta_{T_j^L} \right)^{m\varphi_j} \right)^y, \left(1 - \left(1 - \Theta_{T_j^U} \right)^{m\varphi_j} \right)^y \right], \left[\left(1 - \left(1 - \Theta_{I_j^L} \right)^{m\varphi_j} \right)^y, \left(1 - \left(1 - \Theta_{I_j^U} \right)^{m\varphi_j} \right)^y \right], \left[\left(1 - \left(1 - \Theta_{F_j^L} \right)^{m\varphi_j} \right)^y, \left(1 - \left(1 - \Theta_{F_j^u} \right)^{m\varphi_j} \right)^y \right] \right\rangle, \right.$$
$$\left. \langle 1 - (1 - (\Xi_{T_j})^{m\varphi_j})^y, 1 - (1 - (\Xi_{I_j})^{m\varphi_j})^y, 1 - (1 - (\Xi_{F_j})^{m\varphi_j})^y \rangle \right)$$
$$\geq (\langle [(1 - (1 - \Theta_{T^L})^{m\varphi_j})^y, (1 - (1 - \Theta_{T^U})^{m\varphi_j})^y], [(1 - (1 - \Theta_{I^L})^{m\varphi_j})^y, (1 - (1 - \Theta_{I^U})^{m\varphi_j})^y], [(1 - (1 - \Theta_{F^L})^{m\varphi_j})^y, (1 - (1 - \Theta_{F^U})^{m\varphi_j})^y] \rangle,$$
$$\langle 1 - (1 - (\Xi_T)^{m\varphi_j})^y, 1 - (1 - (\Xi_I)^{m\varphi_j})^y, 1 - (1 - (\Xi_F)^{m\varphi_j})^y \rangle).$$

$$
\begin{aligned}
(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y = & \Big(\Big\langle\Big[\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y, \Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big], \Big[\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y, \\
& \Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big], \Big[\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y, \Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big]\Big\rangle, \\
& \langle 1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y, 1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y, 1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y\rangle) \\
\geq & (\langle[(1-(1-\Theta_{T^L})^{m\varphi_i})^x \times (1-(1-\Theta_{T^L})^{m\varphi_j})^y, (1-(1-\Theta_{T^U})^{m\varphi_i})^x \times (1-(1-\Theta_{T^U})^{m\varphi_j})^y], [(1-(1-\Theta_{I^L})^{m\varphi_i})^x \times (1-(1-\Theta_{I^L})^{m\varphi_j})^y, \\
& (1-(1-\Theta_{I^U})^{m\varphi_i})^x \times (1-(1-\Theta_{I^U})^{m\varphi_j})^y], [(1-(1-\Theta_{F^L})^{m\varphi_i})^x \times (1-(1-\Theta_{F^L})^{m\varphi_j})^y, (1-(1-\Theta_{F^U})^{m\varphi_i})^x \times (1-(1-\Theta_{F^U})^{m\varphi_j})^y]\rangle \\
& \langle 1-(1-(\Xi_T)^{m\varphi_i})^x \times (1-(\Xi_T)^{m\varphi_j})^y, 1-(1-(\Xi_I)^{m\varphi_i})^x \times (1-(\Xi_I)^{m\varphi_j})^y, 1-(1-(\Xi_F)^{m\varphi_i})^x \times (1-(\Xi_F)^{m\varphi_j})^y\rangle)
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y \\
& = \Bigg(\Bigg\langle\Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg] \\
& \quad \Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg] \\
& \quad \Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y\Big), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg]\Bigg\rangle, \\
& \quad \Bigg\langle\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y)\Bigg\rangle\Bigg) \\
& \geq \Bigg(\Bigg\langle\Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^L})^{m\varphi_i})^x \times (1-(1-\Theta_{T^L})^{m\varphi_j})^y), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^U})^{m\varphi_i})^x \times (1-(1-\Theta_{T^U})^{m\varphi_j})^y)\Bigg] \\
& \quad \Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^L})^{m\varphi_i})^x \times (1-(1-\Theta_{I^L})^{m\varphi_j})^y), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^U})^{m\varphi_i})^x \times (1-(1-\Theta_{I^U})^{m\varphi_j})^y)\Bigg] \\
& \quad \Bigg[1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^L})^{m\varphi_i})^x \times (1-(1-\Theta_{F^L})^{m\varphi_j})^y), 1-\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^U})^{m\varphi_i})^x \times (1-(1-\Theta_{F^U})^{m\varphi_j})^y)\Bigg]\Bigg\rangle, \\
& \quad \Bigg\langle\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_T)^{m\varphi_i})^x \times (1-(\Xi_T)^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_I)^{m\varphi_i})^x \times (1-(\Xi_I)^{m\varphi_j})^y), \prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_F)^{m\varphi_i})^x \times (1-(\Xi_F)^{m\varphi_j})^y)\Bigg\rangle\Bigg).
\end{aligned}
$$

Further,

$$
\begin{aligned}
& \frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y \\
& = \Bigg(\Bigg\langle\Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{T_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{T_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg], \\
& \quad \Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{I_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{I_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg], \\
& \quad \Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^L}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^L}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}\Big(1-\Big(1-\Big(1-\Theta_{F_i^U}\Big)^{m\varphi_i}\Big)^x \times \Big(1-\Big(1-\Theta_{F_j^U}\Big)^{m\varphi_j}\Big)^y\Big)\Bigg)^{\frac{2}{m^2+m}}\Bigg]\Bigg\rangle, \\
& \quad \Bigg\langle\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x \times (1-(\Xi_{T_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x \times (1-(\Xi_{I_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x \times (1-(\Xi_{F_j})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg\rangle\Bigg) \\
& \geq \Bigg(\Bigg\langle\Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^L})^{m\varphi_i})^x \times (1-(1-\Theta_{T^L})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^U})^{m\varphi_i})^x \times (1-(1-\Theta_{T^U})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg] \\
& \quad \Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^L})^{m\varphi_i})^x \times (1-(1-\Theta_{I^L})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^U})^{m\varphi_i})^x \times (1-(1-\Theta_{I^U})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg], \\
& \quad \Bigg[1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^L})^{m\varphi_i})^x \times (1-(1-\Theta_{F^L})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, 1-\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^U})^{m\varphi_i})^x \times (1-(1-\Theta_{F^U})^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg]\rangle, \\
& \quad \Bigg\langle\Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_T)^{m\varphi_i})^x \times (1-(\Xi_T)^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_I)^{m\varphi_i})^x \times (1-(\Xi_I)^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}, \Bigg(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_F)^{m\varphi_i})^x \times (1-(\Xi_F)^{m\varphi_j})^y)\Bigg)^{\frac{2}{m^2+m}}\Bigg\rangle\Bigg).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\left(\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=i}^{m}(m\varphi_i n_i)^x \otimes_h (m\varphi_j n_j)^y\right)^{\frac{1}{x+y}}\\
&=\Bigg(\Bigg\langle\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^L}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{T_j^L}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^U}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{T_j^U}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg]\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^L}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{I_j^L}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^U}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{I_j^U}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{F_j^L}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{m\varphi_i}\right)^x\times\left(1-\left(1-\Theta_{F_j^L}\right)^{m\varphi_j}\right)^y\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T_i})^{m\varphi_i})^x\times(1-(\Xi_{T_j})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I_i})^{m\varphi_i})^x\times(1-(\Xi_{I_j})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\\
&1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F_i})^{m\varphi_i})^x\times(1-(\Xi_{F_j})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg\rangle\Bigg).\\
&\geq\Bigg(\Bigg\langle\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^L})^{m\varphi_i})^x\times(1-(1-\Theta_{T^L})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{T^U})^{m\varphi_i})^x\times(1-(1-\Theta_{T^U})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^L})^{m\varphi_i})^x\times(1-(1-\Theta_{I^L})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{I^U})^{m\varphi_i})^x\times(1-(1-\Theta_{I^U})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^L})^{m\varphi_i})^x\times(1-(1-\Theta_{F^L})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(1-\Theta_{F^U})^{m\varphi_i})^x\times(1-(1-\Theta_{F^U})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{T})^{m\varphi_i})^x\times(1-(\Xi_{T})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{I})^{m\varphi_i})^x\times(1-(\Xi_{I})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\\
&1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}(1-(1-(\Xi_{F})^{m\varphi_i})^x\times(1-(\Xi_{F})^{m\varphi_j})^y)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg\rangle\Bigg)\quad = p.
\end{aligned}
$$

In a similar way, we can show that $\text{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) \leq q$.

So, we have $p \leq \text{NCPHA}^{x,y}(n_1, n_2, \ldots, n_m) \leq q$.

However, the NCPHA does not have the property of monotonicity.

The main reason is that the weight vector of the two collections comes from the different support degrees and they have no constant inequality relationship.

Then, we can explore some special examples of the $\text{NCPHA}^{x,y}$ operator based on the parameters x and y.

(1) When $y \to 0$, Eq. (21) degenerates to neutrosophic cubic power generalized linear descending weighted operator, and so we have

$$
\begin{aligned}
&\mathrm{NCPHA}^{x,0}(n_1,n_2,\ldots,n_m)\\
&\left(\left\langle\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^L}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^U}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}}\right],\right.\right.\\
&\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^L}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^U}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}}\right],\\
&\left.\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^U}\right)^{m\varphi_i}\right)^{x}\right)^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x}}\right]\right\rangle,\\
&\left\langle 1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{T_i})^{n\varphi_i})^{x})^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{I_i})^{n\varphi_i})^{x})^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\right.\\
&\left.\left.1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{F_i})^{n\varphi_i})^{x})^{m+1-i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\right\rangle\right).
\end{aligned}
\tag{23}
$$

(2) When $x \to 0$, Eq. (21) degenerates to neutrosophic cubic power generalized linear ascending weighted operator, and so we have

$$
\begin{aligned}
&\mathrm{NCPHA}^{0,y}(n_1,n_2,\ldots,n_m)\\
&=\left(\left\langle\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{T_j^L}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{T_j^U}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}}\right],\right.\right.\\
&\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{I_j^L}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{I_j^U}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}}\right],\\
&\left.\left[\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{F_j^L}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}},\left(1-\left(\prod_{i=1}^{m}\left(1-\left(1-\left(1-\Theta_{F_j^U}\right)^{m\varphi_j}\right)^{y}\right)^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}}\right]\right\rangle,\\
&\left\langle 1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{T_j})^{n\varphi_j})^{y})^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}},1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{I_j})^{n\varphi_j})^{y})^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}},\right.\\
&\left.1-\left(1-\left(\prod_{i=1}^{m}(1-(1-(\Xi_{F_j})^{n\varphi_j})^{y})^{i}\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{y}}\right\rangle.
\end{aligned}
\tag{24}
$$

From Eqs. (23) and (24), we know $\mathrm{NCPHA}^{x,0}(n_1,n_2,\ldots,n_m)$ and $\mathrm{NCPHA}^{0,y}(n_1,n_2,\ldots,n_m)$, respectively, can weigh the information $((m\varphi_1 n_1)^x,(m\varphi_2 n_2)^x,\ldots,(m\varphi_m n_m)^x)$, $((m\varphi_1 n_1)^y,(m\varphi_2 n_2)^y,\ldots,(m\varphi_m n_m)^y)$ with heavy weight vectors $(m,(m-1),\ldots,1)$ and $(1,2,\ldots,m)$. Hence, whenever $x=0\, or\, y=0$, $\mathrm{NCPHA}^{x,y}(n_1,n_2,\ldots,n_m)$ has the linear weighted function. We also know from Eqs. (13) and (14) the parameters x and y are not interchangeable.

(3) When $x=y=\frac{1}{2}$, Eq. (21) degenerates to neutrosophic cubic power basic Heronian operator; so, we have

$$
\begin{aligned}
&\mathrm{NCPHA}^{\frac{1}{2},\frac{1}{2}}(n_1,n_2,\ldots,n_m)\\
&=\Bigg(\Bigg\langle\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{T_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{T_j^L}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right),\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{T_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{T_j^U}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right)\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{I_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{I_j^L}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right),\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{I_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{I_j^U}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right)\Bigg]\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{F_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{F_j^L}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right),\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{\left(1-\left(1-\Theta_{F_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{F_j^U}\right)^{m\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right)\Bigg]\Bigg\rangle,\\
&\Bigg\langle\left(\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{(1-(\Xi_{T_i})^{n\varphi_i})\times\left(1-\left(\Xi_{T_j}\right)^{n\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right),\left(\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{(1-(\Xi_{I_i})^{n\varphi_i})^{x}\times\left(1-\left(\Xi_{I_j}\right)^{n\varphi_j}\right)^{y}}\right)\right)^{\frac{2}{m^2+m}}\right),\\
&\left(\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\sqrt{(1-(\Xi_{F_i})^{n\varphi_i})\times\left(1-\left(\Xi_{F_j}\right)^{n\varphi_j}\right)}\right)\right)^{\frac{2}{m^2+m}}\right)\Bigg\rangle\Bigg).
\end{aligned}
\tag{25}
$$

(4) When $x=y=1$, Eq. (21) degenerates to neutrosophic cubic power line Heronian operator; so, we have

$$
\begin{aligned}
&\mathrm{NCPHA}^{1,1}(n_1,n_2,\ldots,n_m)\\
&=\Bigg(\Bigg\langle\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{T_j^L}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{T_j^U}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{I_j^L}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{I_j^U}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{F_j^L}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^U}\right)^{m\varphi_i}\right)\times\left(1-\left(1-\Theta_{F_j^U}\right)^{m\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-(1-(\Xi_{T_i})^{n\varphi_i})\times\left(1-\left(\Xi_{T_j}\right)^{n\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-(1-(\Xi_{I_i})^{n\varphi_i})\times\left(1-\left(\Xi_{I_j}\right)^{n\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}},\\
&1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-(1-(\Xi_{F_i})^{n\varphi_i})\times\left(1-\left(\Xi_{F_j}\right)^{n\varphi_j}\right)\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{2}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{26}
$$

In NCPHA operator, we only take the weight vector from PA and do not take the weight of each NCN into consideration. However, in practical decision-making problem, the weight vector of every NCN plays an important role. Therefore, to overcome this limitation of the above aggregation operator, we further propose neutrosophic cubic power weighted Heronian aggregation operator.

Definition 8 Let $n_i=\left\{\left\langle\left[\Theta_{T_i^L},\Theta_{T_i^U}\right],\left[\Theta_{I_i^L},\Theta_{I_i^U}\right],\left[\Theta_{F_i^L},\Theta_{F_i^U}\right]\right\rangle,\langle\Xi_{T_i},\Xi_{I_i},\Xi_{F_i}\rangle\right\}$ $(i=1,2,\ldots,m)$ be a group of NCNs, $x,y\geq 0$, and $NCPWHA:\aleph^m\to\aleph$, if

$$
NCPWHA^{x,y}(n_1,n_2,\ldots,n_m)=\left(\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}n_i\right)^{x}\otimes_h\left(\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}n_j\right)^{y}\right)^{\frac{1}{x+y}},
\tag{27}
$$

where $\aleph$ is the set of all NCNs, $\varphi_g=\frac{\left(T\left(n_g\right)+1\right)}{\sum_{g=1}^{m}\left(T\left(n_g\right)+1\right)}$ and $\sum_{g=1}^{m}\varphi_g=1$.

$T(n_z)=\sum_{\substack{i=1\\ i\neq z}}^{m}\sup(n_z,n_i)$, and $\sup(n_z,n_i)$ is the support degree for n_z from n_i, which satisfies the following axioms.

(1) $\sup(n_z, n_i) \in [0,1]$; (2) $\sup(n_z, n_i) = \sup(n_i, n_z)$; (3) $\sup(s,t) \geq \sup(p,q)$, if $D(s,t) < D(p,q)$, in which $D(s,t)$ is the distance measure from Definition 4. Further, $\delta = (\delta_1, \delta_2, \ldots, \delta_m)^{\mathrm{T}}$ is the importance degree of the NCNs $(n_1, n_2, \ldots, n_m)$, $\delta_g \in [0,1]$ and $\sum_{g=1}^{m} \delta_g = 1$. Then, NCPWHA is called the neutrosophic cubic power weighted Heronian aggregation (NCPWHA) operator.

Then, we have the following theorem by the operational laws of the NCNs described in Definition 8.

Theorem 6 *Let* $n_i = \left\{\left\langle\left[\Theta_{T_i^L}, \Theta_{T_i^U}\right], \left[\Theta_{I_i^L}, \Theta_{I_i^U}\right], \left[\Theta_{F_i^L}, \Theta_{F_i^U}\right]\right\rangle, \langle \Xi_{T_i}, \Xi_{I_i}, \Xi_{F_i}\rangle\right\}$ $(i = 1,2,\ldots,m)$ *be a group of NCNs and* $x, y \geq 0$, *then the aggregated value from Eq. (27) is still NCN and even*

$$
\begin{aligned}
&NCPWHA^{x,y}(n_1, n_2, \ldots, n_m)\\
&= \Bigg(\Bigg\langle\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^L}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{T_j^L}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}, \left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{T_i^U}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{T_j^U}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^L}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{I_j^L}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}, \left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{I_i^U}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{I_j^U}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^L}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{F_j^L}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}, \left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-\left(1-\Theta_{F_i^U}\right)^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-\left(1-\Theta_{F_j^U}\right)^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-(\Xi_{T_i})^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-(\Xi_{T_j})^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}, 1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-(\Xi_{I_i})^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-(\Xi_{I_j})^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}},\\
&1-\left(1-\left(\prod_{i=1}^{m}\prod_{j=i}^{m}\left(1-\left(1-(\Xi_{F_i})^{\frac{m\varphi_i\delta_i}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{x}\times\left(1-(\Xi_{F_j})^{\frac{m\varphi_j\delta_j}{\sum_{z=1}^{m}\varphi_z\delta_z}}\right)^{y}\right)\right)^{\frac{2}{m^2+m}}\right)^{\frac{1}{x+y}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{28}
$$

4 A group decision-making approach based on the NCPWHA operator

In this part, we present the application of the NCPWHA operator in the MAGDM.

Assume a MAGDM problem with NCNs, in which the criteria and experts importance degrees are known. There are m alternatives $(M_1, M_2, \ldots, M_m)$ and n criteria $(B_1, B_2, \ldots, B_n)$ and the importance degree of the criteria $B_j\,(j = 1,2,\ldots,n)$ is represented by $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^{\mathrm{T}}$ with $\varphi_j \geq 0$, $j = 1,2,\ldots,n$, $\sum_{j=1}^{n} \varphi_j = 1$. Assume that there are z experts $(V_1, V_2, \ldots, V_z)$ and the importance degree of the expert is denoted by $\delta = (\delta_1, \delta_2, \ldots, \delta_z)^{\mathrm{T}}$ with $\delta_l \geq 0 (l = 1,2,\ldots,z)$, $\sum_{l=1}^{z} \delta_l = 1$. Assume that $D^l = \left[n_{ij}^l\right]_{m\times n}$ is the decision matrix, where n_{ij}^l takes the form NCNs, given by the expert V_l for the alternative M_i with respect to criteria B_j. Then, the goal of this MAGDM is to rank the alternatives.

In the following, we give the decision steps based on the NCPWHA operator as follows.

Step 1. Normalize the criterion values.

Generally, there are two types of criterion values. The one is of cost type and the other is of benefit type. We can change them into the same type. In general, cost-type criteria will be changed into benefit one which is done by

$$n_{ij}^{l}=\left(\left\langle\left[\Theta_{l_{T_{ij}^{L}}},\Theta_{l_{T_{ij}^{U}}}\right],\left[\Theta_{l_{I_{ij}^{L}}},\Theta_{l_{I_{ij}^{U}}}\right],\left[\Theta_{l_{F_{ij}^{L}}},\Theta_{l_{F_{ij}^{U}}}\right]\right\rangle,\left\langle\Xi_{l_{T_{ij}}},\Xi_{l_{I_{ij}}},\Xi_{l_{F_{ij}}}\right\rangle\right)$$
$$=\begin{cases}\left(\left\langle\left[\Theta_{l_{T_{ij}^{L}}},\Theta_{l_{T_{ij}^{U}}}\right],\left[\Theta_{l_{I_{ij}^{L}}},\Theta_{l_{I_{ij}^{U}}}\right],\left[\Theta_{l_{F_{ij}^{L}}},\Theta_{l_{F_{ij}^{U}}}\right]\right\rangle,\left\langle\Xi_{l_{T_{ij}}},\Xi_{l_{I_{ij}}},\Xi_{l_{F_{ij}}}\right\rangle\right) & \text{for benefit type}\\ \left(\left\langle\left[1-\Theta_{l_{T_{ij}^{U}}},1-\Theta_{l_{T_{ij}^{L}}}\right],\left[1-\Theta_{l_{I_{ij}^{U}}},1-\Theta_{l_{I_{ij}^{L}}}\right],\left[1-\Theta_{l_{F_{ij}^{U}}},1-\Theta_{l_{F_{ij}^{L}}}\right]\right\rangle,\left\langle1-\Xi_{l_{T_{ij}}},1-\Xi_{l_{I_{ij}}},1-\Xi_{l_{F_{ij}}}\right\rangle\right) & \text{for cost type}\end{cases}.$$

So, the decision matrix $D^{l}=\left[n_{ij}^{l}\right]_{m\times n}$ can be changed into $R^{l}=\left[n_{ij}^{l}\right]_{m\times n}$.

Step 2: Calculate the support $\text{Sup}\left(n_{ij}^{l},n_{ik}^{l}\right)(i=1,2,\ldots,m;l=1,2,\ldots,z;j,k=1,2,\ldots,n)$ by

$$\text{Sup}\left(n_{ij}^{l},n_{ik}^{l}\right)=1-D\left(n_{ij}^{l},n_{ik}^{l}\right)\;, \tag{29}$$

where $D\left(n_{ij}^{l},n_{ik}^{l}\right)$ is the Euclidean distance from Definition 4.

Step 3: Calculate $T\left(n_{ij}^{l}\right)$ by

$$T\left(n_{ij}^{l}\right)=\sum_{\substack{k=1\\k\neq j}}^{n}\text{Sup}\left(n_{ij}^{l},n_{ik}^{l}\right)(i=1,2,\ldots,m;l=1,2,\ldots,z;j=1,2,\ldots,n). \tag{30}$$

Step 4: Calculate

$$\kappa_{ij}^{l}=\frac{n\varphi_{j}\left(1+T\left(n_{ij}^{l}\right)\right)}{\sum_{z=1}^{n}\varphi_{z}(1+T(n_{iz}^{l}))}(i=1,2,\ldots,m;l=1,2,\ldots,k;j=1,2,\ldots,n). \tag{31}$$

Step 5: Use the NCPWHA operator

$$n_{i}^{l}=\left(\left\langle\left[\Theta_{l_{T_{i}^{L}}},\Theta_{l_{T_{i}^{U}}}\right],\left[\Theta_{l_{I_{i}^{L}}},\Theta_{l_{I_{i}^{U}}}\right],\left[\Theta_{l_{F_{i}^{L}}},\Theta_{l_{F_{i}^{U}}}\right]\right\rangle,\left\langle\Xi_{l_{T_{i}}},\Xi_{l_{I_{i}}},\Xi_{l_{F_{i}}}\right\rangle\right)=NCPWHA\left(n_{i1}^{l},n_{i2}^{l},\ldots,n_{in}^{l}\right) \tag{32}$$

to calculate the comprehensive NCNs $n_{i}^{l}(i=1,2,\ldots,m;k=1,2,\ldots,z)$.

Step 6: Calculate the supports $\text{Sup}\left(n_{i}^{l},n_{i}^{h}\right)(i=1,2,\ldots,m;l,h=1,2,\ldots,z)$ by

$$\text{Sup}\left(n_{i}^{l},n_{i}^{h}\right)=1-D\left(n_{i}^{l},n_{i}^{h}\right), \tag{33}$$

where $D\left(n_{i}^{l},n_{i}^{h}\right)$ is the Euclidean distance from Definition 4.

Step 7: Calculate $T\left(n_{i}^{l}\right)$ by

$$T\left(n_{i}^{l}\right)=\sum_{\substack{h=1\\h\neq i}}^{n}Sup\left(n_{i}^{l},n_{i}^{h}\right)(i=1,2,\ldots,m;h=1,2,\ldots,z)\,. \tag{34}$$

Step 8: Calculate

$$\kappa_{i}^{l}=\frac{n\delta_{l}\left(1+T\left(n_{i}^{l}\right)\right)}{\sum_{l=1}^{n}\delta_{l}(1+T(n_{i}^{l}))}(i=1,2,\ldots,m;l=1,2,\ldots,z). \tag{35}$$

Step 9: Use the NCWPHA operator to get calculate the collective NCNs $n_{i}(i=1,2,\ldots,m)$.

$$n_{i}=\left(\left\langle\left[\Theta_{T_{i}^{L}},\Theta_{T_{i}^{U}}\right],\left[\Theta_{I_{i}^{L}},\Theta_{I_{i}^{U}}\right],\left[\Theta_{F_{i}^{L}},\Theta_{F_{i}^{U}}\right]\right\rangle,\left\langle\Xi_{T_{i}},\Xi_{I_{i}},\Xi_{F_{i}}\right\rangle\right)=NCWPHA\left(n_{i}^{1},n_{i}^{2},\ldots,n_{i}^{z}\right). \tag{36}$$

Step 10: Calculate the score values of each NCN by Definition 3.

Step 11. Rank all the alternatives n_{i} according to their score values based on the comparison rules from Theorem 2 and select the best alternative.

5 An application example

In this subsection, a practical example is adapted from Smarandache (1998) to show the effectiveness of the proposed aggregation operator in MAGDM.

Example 6 Assume that there are four alternatives $(M_{1},M_{2},\ldots,M_{m})$ expressing the air quality of the Guangzhou of the years 2006, 2007, 2008, and 2009, respectively, which are assessed. The three criteria $SO2(B_{1})$, $NO2(B_{2})$, and $PM10(B_{3})$ are taken under account. The importance degree of the criteria is given by $\varphi=(0.5,0.3,0.2)^{\mathrm{T}}$. The four possible alternatives $M_{i}(i=1,2,3,4)$ are evaluated by three air quality stations regarded as DMs (V_{1},V_{2},V_{3}). The importance degree of the DMs is given by $\delta = (0.4, 0.3, 0.3)^{\mathrm{T}}$. The evaluation values by the form of NCNs are given in Tables 1, 2 and 3.

The following steps are involved to rank the alternatives by the proposed method.

Step 1. Change the decision matrix $D^l = \left[n_{ij}^l\right]_{m\times n}$ into the normalize decision matrix $R^l = \left[n_{ij}^l\right]_{m\times n}$.

Since all the criteria are of benefit type. Hence, there is no need to change it.

Step 2. Calculate the supports $\mathrm{Sup}\left(n_{ij}^l, n_{ik}^l\right)$ $(i = 1,2,3,4; l = 1,2,3; j,k = 1,2,3)$ by formula (29); for computational simplicity, we shall represent $\mathrm{Sup}\left(n_{ij}^l, n_{ik}^l\right)$ with $S_{ij,ik}^l (j,k = 1,2,3; i = 1,2,3,4; l = 1,2,3)$. We have

$$S_{11,12}^1 = S_{12,11}^1 = 0.7946, S_{12,13}^1 = S_{13,12}^1 = 0.9058,$$
$$S_{11,13}^1 = S_{13,11}^1 = 0.8115,$$
$$S_{21,22}^1 = S_{22,21}^1 = 0.8269, S_{22,23}^1 = S_{23,22}^1 = 0.9001,$$
$$^1_{21,23} = S_{23,21}^1 = 0.7841,$$
$$S_{31,32}^1 = S_{32,31}^1 = 0.8473, S_{32,33}^1 = S_{33,32}^1 = 0.9334,$$
$$^1_{31,33} = S_{33,31}^1 = 0.8710,$$
$$S_{41,42}^1 = S_{42,41}^1 = 0.8586, S_{42,43}^1 = S_{43,42}^1 = 0.9001,$$
$$^1_{41,43} = S_{43,41}^1 = 0.8145,$$

$$S_{11,12}^2 = S_{12,11}^2 = 0.7841, S_{12,13}^2 = S_{13,12}^2 = 0.7765,$$
$$S_{11,13}^2 = S_{13,11}^2 = 0.9119,$$
$$S_{21,22}^2 = S_{22,21}^2 = 0.8626, S_{22,23}^2 = S_{23,22}^2 = 0.8473,$$
$$S_{21,23}^2 = S_{23,21}^2 = 0.8799,$$
$$S_{31,32}^2 = S_{32,31}^2 = 0.8799, S_{32,33}^2 = S_{33,32}^2 = 0.9184,$$
$$S_{31,33}^2 = S_{33,31}^2 = 0.8799,$$
$$S_{41,42}^2 = S_{42,41}^2 = 0.7598, S_{42,43}^2 = S_{43,42}^2 = 0.8586,$$
$$S_{41,43}^2 = S_{43,41}^2 = 0.8586,$$

$$S_{11,12}^3 = S_{12,11}^3 = 0.8115, S_{12,13}^3 = S_{13,12}^3 = 0.8029,$$
$$S_{11,13}^3 = S_{13,11}^3 = 0.7893,$$
$$S_{21,22}^3 = S_{22,21}^3 = 0.9001, S_{22,23}^3 = S_{23,22}^3 = 0.8029,$$
$$S_{21,23}^3 = S_{23,21}^3 = 0.8402,$$
$$S_{31,32}^3 = S_{32,31}^3 = 0.9334, S_{32,33}^3 = S_{33,32}^3 = 0.8086,$$
$$S_{31,33}^3 = S_{33,31}^3 = 0.8206,$$
$$S_{41,42}^3 = S_{42,41}^3 = 0.8710, S_{42,43}^3 = S_{43,42}^3 = 0.9119,$$
$$S_{41,43}^3 = S_{43,41}^3 = 0.8237.$$

Step 3. Calculate $T\left(n_{ij}^l\right)$ $(j = 1,2,3; i = 1,2,\ 3,4; l = 1,2,3)$ by Eq. (30). For simplicity, $T\left(n_{ij}^l\right)$ can be represented as T_{ij}^l. We have

$$T_{11}^1 = 1.6062, T_{12}^1 = 1.7173, T_{13}^1 = 1.7004,$$
$$T_{21}^1 = 1.6110, T_{22}^1 = 1.6841, T_{23}^1 = 1.7269,$$
$$T_{31}^1 = 1.7183, T_{32}^1 = 1.8043, T_{33}^1 = 1.7807,$$
$$T_{41}^1 = 1.6732, T_{42}^1 = 1.7146, T_{43}^1 = 1.7587,$$
$$T_{11}^2 = 1.6959, T_{12}^2 = 1.6884, T_{13}^2 = 1.5606,$$
$$T_{21}^2 = 1.7425, T_{22}^2 = 1.7272, T_{23}^2 = 1.7100,$$
$$T_{31}^2 = 1.7598, T_{32}^2 = 1.7983, T_{33}^2 = 1.7983,$$
$$T_{41}^2 = 1.6184, T_{42}^2 = 1.6184, T_{43}^2 = 1.7173,$$
$$T_{11}^3 = 1.6008, T_{12}^3 = 1.5922, T_{13}^3 = 1.6144,$$
$$T_{21}^3 = 1.7403, T_{22}^3 = 1.7156, T_{23}^3 = 1.7754,$$
$$T_{31}^3 = 1.7540, T_{32}^3 = 1.6292, T_{33}^3 = 1.7420,$$
$$T_{41}^3 = 1.6947, T_{42}^3 = 1.7356, T_{43}^3 = 1.7828.$$

Step 4. Calculate κ_{ij}^l $(j = 1,2,3; i = 1,2,3,4; l = 1,2,3)$. We have

$$\kappa_{11}^1 = 1.4715, \kappa_{12}^1 = 0.6137, \kappa_{13}^1 = 0.9148,$$
$$\kappa_{21}^1 = 1.4721, \kappa_{22}^1 = 0.6054, \kappa_{23}^1 = 0.9225,$$
$$\kappa_{31}^1 = 1.4804, \kappa_{32}^1 = 0.6109, \kappa_{33}^1 = 0.9087,$$
$$\kappa_{41}^1 = 1.4812, \kappa_{42}^1 = 0.6017, \kappa_{43}^1 = 0.9172,$$
$$\kappa_{11}^2 = 1.5238, \kappa_{12}^2 = 0.6078, \kappa_{13}^2 = 0.8684,$$
$$\kappa_{21}^2 = 1.5070, \kappa_{22}^2 = 0.5995, \kappa_{23}^2 = 0.8935,$$
$$\kappa_{31}^2 = 1.4896, \kappa_{32}^2 = 0.6042, \kappa_{33}^2 = 0.9062,$$
$$\kappa_{41}^2 = 1.4888, \kappa_{42}^2 = 0.5955, \kappa_{43}^2 = 0.8933,$$
$$\kappa_{11}^3 = 1.4986, \kappa_{12}^3 = 0.5975, \kappa_{13}^3 = 0.9039,$$
$$\kappa_{21}^3 = 1.4969, \kappa_{22}^3 = 0.5934, \kappa_{23}^3 = 0.9097,$$
$$\kappa_{31}^3 = 1.5157, \kappa_{32}^3 = 0.5788, \kappa_{33}^3 = 0.9055,$$
$$\kappa_{41}^3 = 1.4810, \kappa_{42}^3 = 0.6014, \kappa_{43}^3 = 0.9177.$$

Step 5. Use NCWPHA operator to determine the comprehensive NCNs n_i^l, which is given in Table 4 (assume $x = 1, y = 2$)

Step 6. Calculate the supports $\mathrm{Sup}\left(n_i^l, n_i^k\right)$ $(i = 1,2,3,4; l,k = 1,2,3)$ by formula (33); for computational simplicity, we shall represent $\mathrm{Sup}\left(n_i^l, n_i^k\right)$ $(i = 1,2,3,4; k,l = 1,2,3)$ with $S_{ij,ik}^l (j,k = 1,2,3; i = 1,2,3,\ 4; l = 1,2,3)$. We have

$$S_{1,2}^1 = S_{2,1}^1 = 0.8994, S_{2,3}^1 = S_{3,2}^1 = 0.9548, S_{1,3}^1 = S_{3,1}^1 = 0.9181,$$
$$S_{1,2}^2 = S_{2,1}^2 = 0.8136, S_{2,3}^2 = S_{3,2}^2 = 0.9196, S_{1,3}^2 = S_{3,1}^2 = 0.8633,$$
$$S_{1,2}^3 = S_{2,1}^3 = 0.9370, S_{2,3}^3 = S_{3,2}^3 = 0.9463, S_{1,3}^3 = S_{3,1}^3 = 0.9611,$$
$$S_{1,2}^4 = S_{2,1}^4 = 0.9301, S_{2,3}^4 = S_{3,2}^4 = 0.9559, S_{1,3}^4 = S_{3,1}^4 = 0.9414.$$

Step 7. Calculate $T(n_i^l)$ by using formula (34); we can get

$$T_1^1 = 1.8175, T_2^1 = 1.8730, T_3^1 = 1.8542, T_1^2 = 1.6769, T_2^2 = 1.7829, T_3^2 = 1.7331,$$
$$T_1^3 = 1.8981, T_2^3 = 1.9074, T_3^3 = 1.8833, T_1^4 = 1.8715, T_2^4 = 1.8973, T_3^4 = 1.8859.$$

Step 8. Calculate κ_i^l by using Eq. (35), we have

$$\kappa_1^1 = 1.1883, \kappa_2^1 = 0.9088, \kappa_3^1 = 0.9029, \kappa_1^2 = 1.1786, \kappa_2^2 = 0.9189, \kappa_3^2 = 0.9025,$$
$$\kappa_1^3 = 1.2007, \kappa_2^3 = 0.9189, \kappa_3^3 = 0.8959, \kappa_1^4 = 1.1950, \kappa_2^4 = 0.9043, \kappa_3^4 = 0.9007.$$

Step 9. Use the NCWPHA operator to get the collective NCNs $n_i(i = 1, 2, 3, 4)$ by Eq. (36), and we can obtain

$$n_1 = (\langle[0.3080, 0.4044], [0.2017, 0.2961], [0.3589, 0.4482]\rangle, \langle 0.7772, 0.2724, 0.3915\rangle),$$
$$n_2 = (\langle[0.3817, 0.4780], [0.1719, 0.2648], [0.2634, 0.3559]\rangle, \langle 0.3833, 0.2488, 0.3197\rangle),$$
$$n_3 = (\langle[0.4505, 0.5635], [0.1272, 0.2387], [0.1536, 0.2682]\rangle, \langle 0.5431, 0.1572, 0.2045\rangle),$$
$$n_4 = (\langle[0.6044, 0.7124], [0.1468, 0.2304], [0.1542, 0.2475]\rangle, \langle 0.6504, 0.1724, 0.1942\rangle).$$

Step 10. Calculate the score values of each NCN by using Definition 2, and we have

$$\widehat{S}(M_1) = 0.6134, \quad \widehat{S}(M_2) = 0.6243,$$
$$\widehat{S}(M_3) = 0.7120, \quad \widehat{S}(M_4) = 0.7580.$$

Step 11. Rank all the alternatives $n_i(1, 2, 3, 4)$ according to their score values. We have $M_4 > M_3 > M_2 > M_1$. So M_4 is the best alternative and the worst one is M_1.

5.1 The effect of the parameters x, y on the this decision-making problem

To perceive the effect of the parameters x, y on the decision making, we set different values for the parameters x, y in step 5 and step 9 and then rank the alternatives $M_i(1, 2, 3, 4)$. The ranking order for different parameters values is given in Table 5.

From Table 5, we can see that the ranking orders obtained for different values of the parameters x, y are slightly different, and the best alternative is M_4 and the worst one is M_1 or M_2. The difference in the ranking order occurs when we take the neutrosophic cubic weighted power generalized linear descending operator. That is, when $x = 1, y = 0$, the ranking order slightly changed to $M_4 > M_3 > M_1 > M_2$. Otherwise, the ranking orders remain the same. So, the proposed aggregation operators in this article are more practical and effective. For computational simplicity, we set the parameters $x = y = 1$.

5.2 Comparison with existing methods

5.2.1 Validity of the proposed approach

To further illustrate the validity of the proposed approach, we solve an example adopted from Zhan et al. (2017) by comparing our method with that from Zhan et al. (2017).

Example 7 A passenger wants to travel to Karachi in the available four vans $M_i(i = 1, 2, 3, 4)$. The attributes under consideration are (i) B_1 is the facility, (ii) B_2 is the rent saving, (iii) B_3 is the comfort, and (iv) B_4 is the safety. The importance degree of the attributes is given as $\omega = (0.5, 0.25, 0.125, 0.125)^{\mathrm{T}}$. The four possible alternatives $M_i(i = 1, 2, 3, 4)$ are evaluated, and the assessment values provided by NCNs for the attributes $B_j(j = 1, 2, 3, 4)$ are given in Table 6.

(1) *Ranking four vans by the method based on the NCWA operator* (Zhan et al. 2017)

The steps are shown as follows.

Step 1. We use the method based on the NCWA operator (Zhan et al. 2017) to obtain the comprehensive NCNs, which are shown as

Table 1 Air quality data from station V_1

	B_1	B_2	B_3
M_1	$(\langle[0.2,0.3],[0.2,0.3],[0.4,0.5]\rangle, \langle 0.3,0.3,0.5\rangle)$	$(\langle[0.3,0.4],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.4,0.2,0.1\rangle)$	$(\langle[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.5,0.2,0.2\rangle)$
M_2	$(\langle[0.3,0.4],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.3,0.2,0.3\rangle)$	$(\langle[0.2,0.3],[0.3,0.4],[0.4,0.5]\rangle, \langle 0.2,0.4,0.5\rangle)$	$(\langle[0.1,0.2],[0.2,0.3],[0.5,0.6]\rangle, \langle 0.1,0.3,0.6\rangle)$
M_3	$(\langle[0.3,0.4],[0.1,0.2],[0.1,0.2]\rangle, \langle 0.4,0.1,0.1\rangle)$	$(\langle[0.5,0.6],[0.2,0.3],[0.2,0.3]\rangle, \langle 0.6,0.2,0.3\rangle)$	$(\langle[0.5,0.6],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.6,0.1,0.2\rangle)$
M_4	$(\langle[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.5,0.1,0.2\rangle)$	$(\langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.7,0.2,0.1\rangle)$	$(\langle[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle, \langle 0.8,0.2,0.3\rangle)$

Table 2 Air quality data from station V_2

	B_1	B_2	B_3
M_1	$(\langle[0.3,0.4],[0.2,0.3],[0.5,0.6]\rangle, \langle 0.3,0.3,0.6\rangle)$	$(\langle[0.5,0.6],[0.3,0.4],[0.2,0.3]\rangle, \langle 0.5,0.4,0.3\rangle)$	$(\langle[0.2,0.3],[0.2,0.3],[0.4,0.5]\rangle, \langle 0.2,0.2,0.5\rangle)$
M_2	$(\langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.5,0.3,0.1\rangle)$	$(\langle[0.4,0.5],[0.1,0.2],[0.3,0.4]\rangle, \langle 0.5,0.2,0.3\rangle)$	$(\langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.6,0.3,0.2\rangle)$
M_3	$(\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle, \langle 0.5,0.2,0.3\rangle)$	$(\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.7,0.1,0.2\rangle)$	$(\langle[0.5,0.6],[0.1,0.2],[0.1,0.2]\rangle, \langle 0.6,0.1,0.1\rangle)$
M_4	$(\langle[0.8,0.9],[0.1,0.1],[0.1,0.2]\rangle, \langle 0.9,0.1,0.1\rangle)$	$(\langle[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.5,0.1,0.2\rangle)$	$(\langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.7,0.2,0.1\rangle)$

Table 3 Air quality data from station V_3

	B_1	B_2	B_3
M_1	$(\langle[0.2,0.3],[0.1,0.2],[0.3,0.4]\rangle, \langle 0.2,0.1,0.4\rangle)$	$(\langle[0.3,0.4],[0.3,0.4],[0.5,0.6]\rangle, \langle 0.3,0.4,0.6\rangle)$	$(\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle, \langle 0.6,0.3,0.4\rangle)$
M_2	$(\langle[0.3,0.4],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.4,0.1,0.3\rangle)$	$(\langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle, \langle 0.5,0.2,0.4\rangle)$	$(\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.6,0.2,0.3\rangle)$
M_3	$(\langle[0.4,0.5],[0.1,0.2],[0.1,0.2]\rangle, \langle 0.5,0.1,0.2\rangle)$	$(\langle[0.4,0.5],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.5,0.2,0.1\rangle)$	$(\langle[0.7,0.8],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.8,0.1,0.2\rangle)$
M_4	$(\langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle, \langle 0.6,0.2,0.3\rangle)$	$(\langle[0.7,0.8],[0.1,0.2],[0.2,0.3]\rangle, \langle 0.7,0.1,0.2\rangle)$	$(\langle[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle, \langle 0.8,0.2,0.1\rangle)$

$$n_1 = (\langle[0.1881, 0.5226], [0.3393, 0.6760], [0.2455, 0.4709]\rangle, \langle 0.7610, 0.5835, 0.3298\rangle),$$
$$n_2 = (\langle[0.2882, 0.8391], [0.3740, 0.7551], [0.3529, 0.5838]\rangle, \langle 0.5853, 0.5370, 0.600\rangle),$$
$$n_3 = (\langle[0.3380, 0.5602], [0.3085, 0.6452], [0.2410, 0.5804]\rangle, \langle 0.1996, 0.3557, 0.2359\rangle),$$
$$n_4 = (\langle[0.4328, 0.7960], [0.3326, 0.7461], [0.2570, 0.5281]\rangle, \langle 0.1914, 0.5131, 0.2667\rangle).$$

Step 2. By the score function from Definition 3, we can get the score values of the alternatives as follows.

$$\widehat{S}(n_1) = 0.5363, \widehat{S}(n_2) = 0.5011, \widehat{S}(n_3) = 0.5257, \widehat{S}(n_4) = 0.5307.$$

Step 3. The ranking order of their alternatives according to the score values is

$M_1 > M_4 > M_3 > M_2$.

So, the best alternative is M_1, while the worst one is M_2.

Table 4 Comprehensive NCNs produced by the NCWPHA operator

	V_1	V_2	V_3
M_1	$\langle[0.2753, 0.3736],[0.1882, 0.2832],[0.3076, 0.3915]\rangle, \langle 0.3861, 0.2562, 0.2602\rangle$	$\langle[0.3570, 0.4517],[0.2316, 0.3283],[0.4106, 0.4990]\rangle, \langle 0.3381, 0.3181, 0.4753\rangle$	$\langle[0.3043, 0.4030],[0.1940, 0.2887],[0.3708, 0.4671]\rangle, \langle 0.3098, 0.2357, 0.4668\rangle$
M_2	$\langle[0.2494, 0.3395],[0.1961, 0.2908],[0.3382, 0.4374]\rangle, \langle 0.2166, 0.3004, 0.4231\rangle$	$\langle[0.4854, 0.5821],[0.1789, 0.2732],[0.1824, 0.2745]\rangle, \langle 0.5276, 0.2827, 0.1994\rangle$	$\langle[0.4026, 0.5038],[0.1378, 0.2339],[0.2339, 0.3306]\rangle, \langle 0.4777, 0.1764, 0.3492\rangle$
M_3	$\langle[0.4082, 0.5076],[0.1375, 0.2335],[0.1529, 0.2503]\rangle, \langle 0.5025, 0.1494, 0.1998\rangle$	$\langle[0.4839, 0.6196],[0.1009, 0.2582],[0.1889, 0.3405]\rangle, \langle 0.5784, 0.1517, 0.2182\rangle$	$\langle[0.4746, 0.5788],[0.1377, 0.2340],[0.1204, 0.2182]\rangle, \langle 0.5437, 0.1516, 0.1805\rangle$
M_4	$\langle[0.5381, 0.6430],[0.1530, 0.2504],[0.1774, 0.2718]\rangle, \langle 0.6000, 0.1764, 0.1952\rangle$	$\langle[0.6425, 0.7442],[0.1164, 0.1688],[0.1423, 0.2376]\rangle, \langle 0.6979, 0.1440, 0.1518\rangle$	$\langle[0.6430, 0.7553],[0.1657, 0.2577],[0.1381, 0.2342]\rangle, \langle 0.6598, 0.1788, 0.2180\rangle$

(2) *Ranking four vans by the proposed method in this paper*

Now we use the aggregation operators defined in this article to solve this example.

Step 1. Calculate the support $\text{Sup}(n_{ij}, n_{ik})(i = 1,2,3,4; j,k = 1,2,3,4)$ by $\text{Sup}(n_{ij}, n_{ik}) = 1 - D(n_{ij}, n_{ik})$ where $D(n_{ij}, n_{ik})$ is the Euclidean distance defined in Definition 4. We can get

$$S^1_{11,12} = S^1_{12,11} = 0.7946, S^1_{11,13} = S^1_{13,11} = 0.7356, S^1_{11,14} = S^1_{14,11} = 0.6538, S^1_{12,13} = S^1_{13,12} = 0.7716, S^1_{12,14} = S^1_{14,12} = 0.8057, S^1_{13,14} = S^1_{14,13} = 0.7867,$$

$$S^2_{11,12} = S^2_{12,11} = 0.7973, S^2_{11,13} = S^2_{13,11} = 0.7668, S^2_{11,14} = S^2_{14,11} = 0.7271, S^2_{12,13} = S^2_{13,12} = 0.7507, S^2_{12,14} = S^2_{14,12} = 0.7507, S^2_{13,14} = S^2_{14,13} = 0.7692,$$

$$S^3_{11,12} = S^3_{12,11} = 0.8001, S^3_{11,13} = S^3_{13,11} = 0.5920, S^3_{11,14} = S^3_{14,11} = 0.7356, S^3_{12,13} = S^3_{13,12} = 0.6635, S^3_{12,14} = S^3_{14,12} = 0.7115, S^3_{13,14} = S^3_{14,13} = 0.7867,$$

$$S^4_{11,12} = S^4_{12,11} = 0.7377, S^4_{11,13} = S^4_{13,11} = 0.7377, S^4_{11,14} = S^4_{14,11} = 0.6187, S^4_{12,13} = S^4_{13,12} = 0.7644, S^4_{12,14} = S^4_{14,12} = 0.6586, S^4_{13,14} = S^4_{14,13} = 0.7153.$$

Step 2. Calculate $T(n_{ij})$ by $T(n_{ij}) = \sum_{k=1, k\neq j}^{n} \text{Sup}(n_{ij}, n_{ik})(i = 1,2,3,\ 4; j,k = 1,2,3,4)$, and we have

$$T^1_{11} = 2.1839, T^1_{12} = 2.3719, T^1_{13} = 2.2938, T^1_{14} = 2.2462, T^2_{11} = 2.2914, T^2_{12} = 2.2987, T^2_{13} = 2.2866, T^2_{14} = 2.2471,$$

$$T^3_{11} = 2.1276, T^3_{12} = 2.1751, T^3_{13} = 2.0421, T^3_{14} = 2.2337, T^4_{11} = 2.0940, T^4_{12} = 2.1607, T^4_{13} = 2.2174, T^4_{14} = 1.9926.$$

Step 3. Calculate $\kappa_{ij} = \frac{n\omega_j(1+T(n_{ij}))}{\sum_{z=1}^{3}\omega_z(1+T(n_{iz}))}(i = 1,2,3,4; j = 1,2,3,4)$, and we obtain

$$\kappa^1_{11} = 1.9579, \kappa^1_{12} = 1.0367, \kappa^1_{13} = 0.5064, \kappa^1_{14} = 0.4990, \kappa^2_{11} = 2.0026, \kappa^2_{12} = 1.0035, \kappa^2_{13} = 0.4999, \kappa^2_{14} = 0.4937;$$

$$\kappa^3_{11} = 1.9908, \kappa^3_{12} = 1.0105, \kappa^3_{13} = 0.4841, \kappa^3_{14} = 0.5146, \kappa^4_{11} = 1.9875, \kappa^4_{12} = 1.0152, \kappa^4_{13} = 0.5167, \kappa^4_{14} = 0.4806.$$

Step 4. The comprehensive NCNs by the NCWPHA operator defined in this paper are obtained as follows (assume $x = 2, y = 2$):

Table 5 Ordering of the alternatives based on NCWPHA by using different values of x, y

x, y	Score values	Ranking order
$x=0, y=1$	$\widehat{S}(M_1)=0.6409, \widehat{S}(M_2)=0.6440, \widehat{S}(M_3)=0.7281, \widehat{S}(M_4)=0.7697;$	$M_4>M_3>M_2>M_1$
$x=1, y=0$	$\widehat{S}(M_1)=0.6401, \widehat{S}(M_2)=0.6222, \widehat{S}(M_3)=0.7092, \widehat{S}(M_4)=0.7664;$	$M_4>M_3>M_1>M_2$
$x=2, y=1$	$\widehat{S}(M_1)=0.6150, \widehat{S}(M_2)=0.6185, \widehat{S}(M_3)=0.7082, \widehat{S}(M_4)=0.7551;$	$M_4>M_3>M_2>M_1$
$x=3, y=4$	$\widehat{S}(M_1)=0.5685, \widehat{S}(M_2)=0.6144, \widehat{S}(M_3)=0.6936, \widehat{S}(M_4)=0.7384;$	$M_4>M_3>M_2>M_1$
$x=7, y=5$	$\widehat{S}(M_1)=0.5493, \widehat{S}(M_2)=0.6121, \widehat{S}(M_3)=0.6832, \widehat{S}(M_4)=0.7276;$	$M_4>M_3>M_2>M_1$
$x=10, y=1$	$\widehat{S}(M_1)=0.5483, \widehat{S}(M_2)=0.6138, \widehat{S}(M_3)=0.6833, \widehat{S}(M_4)=0.7301$	$M_4>M_3>M_2>M_1$

$$n_1=(\langle[0.2247,0.5326],[0.3637,0.6790],[0.2521,0.4774]\rangle,\langle 0.7504,0.5549,0.3251\rangle),$$
$$n_2=(\langle[0.3282,0.8057],[0.4012,0.7504],[0.3685,0.5873]\rangle,\langle 0.5284,0.5340,0.5401\rangle),$$
$$n_3=(\langle[0.3593,0.5630],[0.3817,0.6639],[0.2529,0.5924]\rangle,\langle 0.2141,0.3413,0.2466\rangle),$$
$$n_4=(\langle[0.4754,0.7613],[0.3650,0.7337],[0.2785,0.5552]\rangle,\langle 0.2231,0.5120,0.2629\rangle).$$

Step 5. The score values of the alternatives are obtained by Definition 3 as follows.

$$\widehat{S}(M_1)=0.5395, \widehat{S}(M_2)=0.4979, \widehat{S}(M_3)=0.5175, \widehat{S}(M_4)=0.5281.$$

Step 6. So the ranking order of their alternative according to the score values is

$M_1>M_4>M_3>M_2$.

There are the same ranking results from the method proposed in this paper and the method in Zhan et al. (2017), so this can show that our proposed method is valid.

5.2.2 The advantage of the proposed approach

Example 8 A customer wants to buy Huawei P Series mobile model in the available four models. The available four models are as follows: (1) M_1 is the P7 model, (2) M_2 is the P10 model, (3) M_3 is the P9 model, and (4) M_4 is the P8 model. The customer takes the following three attributes under consideration: (i) B_1 is the RAM, (ii) B_2 is the price, and (iii) B_3 is the camera. The importance degree of the attributes is given as $\omega=(0.4,0.2,0.4)^{\mathrm{T}}$. The four possible alternatives $M_i (i=1,2,3,4)$ are evaluated, and the assessment values provided by the neutrosophic cubic information with respect to the attributes $B_j (j=1,2,3)$ are given in the following decision matrix $D_{4\times 3}$ given in Table 7.

(1) *Ranking four Huawei P Series mobile model by the proposed method in this paper*

Now we use the aggregation operators defined in this article to solve this example.

Step 1. Calculate the support $\mathrm{Sup}(n_{ij}, n_{ik}) (i=1,2,3,4; j,k=1,2,3)$ by $\mathrm{Sup}(n_{ij}, n_{ik})=1-D(n_{ij}, n_{ik})$

where $D(n_{ij}, n_{ik})$ is the Euclidean distance from Definition 4. We can get

$$S^1_{11,12}=S^1_{12,11}=0.9001, S^1_{12,13}=S^1_{13,12}=0.8586, S^1_{11,13}=S^1_{13,11}=0.8895,$$
$$S^2_{11,12}=S^2_{12,11}=0.8368, S^2_{12,13}=S^2_{13,12}=0.8269, S^2_{11,13}=S^2_{13,11}=0.9423,$$
$$S^3_{11,12}=S^3_{12,11}=0.8510, S^3_{12,13}=S^3_{13,12}=0.8753, S^3_{11,13}=S^3_{13,11}=0.8301,$$
$$S^4_{11,12}=S^4_{12,11}=0.8799, S^4_{12,13}=S^4_{13,12}=0.8946, S^4_{11,13}=S^4_{13,11}=0.9423.$$

Step 2. Calculate $T(n_{ij})$ by $T(n_{ij})=\sum_{k=1, k\neq j}^{n}\mathrm{Sup}(n_{ij}, n_{ik}) (i=1,2,3,4; j,k=1,2,3)$, and we have

$$T^1_{11}=1.7896, T^1_{12}=1.7587, T^1_{13}=1.7482, T^2_{11}=1.7791, T^2_{12}=1.6637, T^2_{13}=1.7692,$$
$$T^3_{11}=1.6811, T^3_{12}=1.7263, T^3_{13}=1.7055, T^4_{11}=1.8222, T^4_{12}=1.7745, T^4_{13}=1.8369.$$

Step 3. Calculate $\kappa_{ij}=\frac{n\omega_j(1+T(n_{ij}))}{\sum_{z=1}^{3}\omega_z(1+T(n_{iz}))} (i=1,2,3,4; j=1,2,3)$, and we obtain

$$\kappa^1_{11}=1.2099, \kappa^1_{12}=0.5982, \kappa^1_{13}=1.1919, \kappa^2_{11}=1.2118, \kappa^2_{12}=0.5807, \kappa^2_{13}=1.2075;$$
$$\kappa^3_{11}=1.1917, \kappa^3_{12}=0.6059, \kappa^3_{13}=1.2025, \kappa^4_{11}=1.2015, \kappa^4_{12}=0.5906, \kappa^4_{13}=1.2078.$$

Table 6 The NC information for Example 7

	B_1	B_2	B_3	B_4
M_1	$(\langle[0.2,0.5],[0.3,0.7],[0.1,0.2]\rangle,\langle0.9,0.7,0.2\rangle)$	$(\langle[0.2,0.4],[0.4,0.5],[0.2,0.5]\rangle,\langle0.7,0.4,0.5\rangle)$	$(\langle[0.2,0.7],[0.4,0.9],[0.5,0.7]\rangle,\langle0.7,0.7,0.5\rangle)$	$(\langle[0.1,0.6],[0.3,0.4],[0.5,0.8]\rangle,\langle0.5,0.5,0.7\rangle)$
M_2	$(\langle[0.3,0.9],[0.2,0.7],[0.3,0.5]\rangle,\langle0.5,0.7,0.5\rangle)$	$(\langle[0.3,0.7],[0.6,0.8],[0.2,0.4]\rangle,\langle0.7,0.6,0.8\rangle)$	$(\langle[0.3,0.9],[0.4,0.6],[0.6,0.8]\rangle,\langle0.9,0.4,0.6\rangle)$	$(\langle[0.2,0.5],[0.4,0.9],[0.5,0.8]\rangle,\langle0.5,0.2,0.7\rangle)$
M_3	$(\langle[0.3,0.4],[0.4,0.8],[0.2,0.6]\rangle,\langle0.1,0.4,0.2\rangle)$	$(\langle[0.2,0.4],[0.2,0.3],[0.2,0.5]\rangle,\langle0.2,0.2,0.2\rangle)$	$(\langle[0.4,0.9],[0.1,0.2],[0.4,0.5]\rangle,\langle0.9,0.5,0.5\rangle)$	$(\langle[0.6,0.7],[0.3,0.6],[0.3,0.7]\rangle,\langle0.7,0.5,0.3\rangle)$
M_4	$(\langle[0.5,0.9],[0.1,0.8],[0.2,0.6]\rangle,\langle0.1,0.7,0.2\rangle)$	$(\langle[0.3,0.5],[0.1,0.2],[0.1,0.2]\rangle,\langle0.3,0.5,0.2\rangle)$	$(\langle[0.5,0.6],[0.2,0.4],[0.3,0.5]\rangle,\langle0.5,0.4,0.5\rangle)$	$(\langle[0.3,0.7],[0.7,0.8],[0.6,0.7]\rangle,\langle0.4,0.2,0.8\rangle)$

Table 7 The NC information from Example 8

	B_1	B_2	B_3
M_1	$(\langle[0.5,0.7],[0.2,0.3],[0.3,0.4]\rangle,\langle0.6,0.3,0.4\rangle)$	$(\langle[0.6,0.8][0.3,0.4],[0.20.5]\rangle,\langle0.7,0.2,0.5\rangle)$	$(\langle[0.4,0.7],[0.4,0.5],[0.3,0.4]\rangle,\langle0.5,0.3,0.3\rangle)$
M_2	$(\langle[0.7,0.8],[0.1,0.2],[0.2,0.3]\rangle,\langle0.8,0.1,0.2\rangle)$	$(\langle[0.6,0.9],[0.4,0.5],[0.3,0.4]\rangle,\langle0.9,0.2,0.2\rangle)$	$(\langle[0.7,0.8],[0.1,0.2],[0.2,0.4]\rangle,\langle0.7,0.1,0.3\rangle)$
M_3	$(\langle[0.6,0.9],[0.2,0.4],[0.3,0.4]\rangle,\langle0.6,0.4,0.4\rangle)$	$(\langle[0.7,0.8],[0.3,0.5],[0.2,0.3]\rangle,\langle0.7,0.1,0.2\rangle)$	$(\langle[0.8,0.9],[0.1,0.3],[0.1,0.2]\rangle,\langle0.8,0.2,0.2\rangle)$
M_4	$(\langle[0.6,0.8],[0.2,0.4],[0.2,0.3]\rangle,\langle0.8,0.1,0.2\rangle)$	$(\langle[0.8,0.9],[0.3,0.4],[0.1,0.3]\rangle,\langle0.7,0.2,0.4\rangle)$	$(\langle[0.7,0.8],[0.2,0.4],[0.1,0.2]\rangle,\langle0.8,0.1,0.2\rangle)$

Step 4. The comprehensive NCNs by the NCWPHA operator defined in this paper are obtained as follows (assume $x = 1, y = 1$):

$$n_1 = (\langle[0.4846, 0.7207], [0.3038, 0.4024], [0.2809, 0.4208]\rangle, \langle 0.5801, 0.2793, 0.3843\rangle),$$
$$n_2 = (\langle[0.6724, 0.8224], [0.1690, 0.2695], [0.2205, 0.3603]\rangle, \langle 0.7743, 0.1292, 0.2428\rangle),$$
$$n_3 = (\langle[0.7012, 0.8710], [0.1841, 0.3843], [0.2060, 0.3006]\rangle, \langle 0.6960, 0.2341, 0.2721\rangle),$$
$$n_4 = (\langle[0.6545, 0.8231], [0.2208, 0.3984], [0.1436, 0.2612]\rangle, \langle 0.7792, 0.1291, 0.2462\rangle).$$

Step 5. The score values of the alternatives are obtained by Definition 3 as follows:

$$\widehat{S}(n_1) = 0.6349, \widehat{S}(n_2) = 0.7642, \widehat{S}(n_3) = 0.7430, \widehat{S}(n_4) = 0.7619.$$

Step 6. So the ranking order of their alternative according to the score values is

$M_2 > M_4 > M_3 > M_1$.

So, the best alternative is M_2, while the worst one is M_1.

(2) *Ranking four Huawei P Series mobile model by the method based on the NCWA operator* (Zhan et al. 2017)

The steps are shown as follows.

Step 1. We use the method based on the NCWA operator (Zhan et al. 2017) to obtain the comprehensive NCNs, which are shown as

$$n_1 = (\langle[0.4856, 0.7234], [0.3057, 0.4067], [0.28110.4215]\rangle, \langle 0.5753, 0.2766, 0.3728\rangle),$$
$$n_2 = (\langle[0.6822, 0.8259], [0.1701, 0.2718], [0.2211, 0.3618]\rangle, \langle 0.7765, 0.1149, 0.2352\rangle),$$
$$n_3 = (\langle[0.7138, 0.8851], [0.1835, 0.3847], [0.2050, 0.3057]\rangle, \langle 0.6943, 0.2297, 0.2639\rangle),$$
$$n_4 = (\langle[0.6896, 0.8259], [0.2211, 0.4000], [0.1414, 0.2616]\rangle, \langle 0.7789, 0.1149, 0.2297\rangle).$$

Step 2. By the score function defined in Definition 3, we can get the score values of the alternatives as follows.

$$\widehat{S}(n_1) = 0.6355, \widehat{S}(n_2) = 0.7677, \widehat{S}(n_3) = 0.7467, \widehat{S}(n_4) = 0.7695.$$

Step 3. The ranking order of their alternatives according to the score values is

$M_4 > M_2 > M_3 > M_1$.

So, the best alternative is M_4, while the worst one is M_1.

Obviously, there are the different ranking results produced by these two methods. In order to explain that our ranking result is more reasonable than the method in Zhan et al. (2017), we give the following discussions.

(i) In the above example, we take the weights of PA operator that are equal to the proposed aggregation operators and because $x = 1$ and $y = 0$, we can get $\widehat{S}(n_1) = 0.5804, \widehat{S}(n_2) = 0.6681, \widehat{S}(n_3) = 0.6251, \widehat{S}(n_4) = 0.6694$ and then $M_4 > M_2 > M_3 > M_1$. Obviously, there are the same ranking orders obtained by the proposed aggregation operators in this article and in Wu et al. (2018). We can explain the results as follows.

In this special case, because the weights of PA operator are equal, the proposed method does not use the PA operator. In addition, because $x = 1$ and $y = 0$, the proposed method does not consider the relationship between the attributes. So in this case, the proposed method is similar to that proposed by Zhan et al. (2017), and they can produce

(ii) If we can only set $x = 0$ or $y = 0$ in the proposed aggregation operator, then the ranking results obtained for these two cases by the proposed aggregation operators in this paper and in Zhan et al. (2017) are the same. Because in these two cases $x = 0$ or $y = 0$, the proposed method does not consider the relationship between the attributes, we can get the same ranking results by the proposed aggregation operator in this paper and in Zhan et al. (2017), and this can show the PA operator cannot influence the ranking results.

(iii) For the initial results of this example, we get the different ranking results. From above (i) and (ii), we can know the reason produced the different ranking results is the relationship between the attributes. The proposed method in this paper considers the relationship between the attributes, while the method in Wu et al. (2018) does not take into account. Because this example is with three attributes, i.e., (i) B_1 is the RAM, (ii) B_2 is the price, and (iii) B_3 is the camera. Obviously, there exists the relationship between the attributes. So, the ranking result obtained by the proposed method in this paper is more reasonable than that produced by Zhan et al. (2017), i.e., the best alternative is M_2, not M_4.

Based on the above analysis, the developed aggregation operators are more practical and effective than the existing aggregation operators. The main difference between the developed aggregation operators and the existing aggregation operators is that the proposed aggregation operators have two characteristics: (1) It can consider the interrelationship among the input argument and (2) it can remove the influence of awkward data at the same time. Some developed aggregation operators can only deal with NC information for one property. That is, they can consider interrelationship among input argument or remove the effect of awkward data. Therefore, the developed method which is based on the proposed aggregation operator is more practical and effective for solving MADM and MAGDM problems.

6 Conclusion

In this paper, firstly we presented the novel comparison method and distance measure for NCNs and then we proposed the NCPHA operator and NCPWHA operator by extending the PHM operator to NCNs; further, we explored some characteristics and special examples for the proposed operators and developed a new approach for MAGDM problems with NCNs. Finally, we proved the validity and showed the advantages of the developed method by some examples, i.e., the proposed approach has three main advantages: (1) It adopts novel comparison method and distance measure for NCNs which can overcome some existing weaknesses; (2) it can process the relationships between any two arguments of real decision-making problems, which makes the decision result more reasonable; (3) it can eliminate the negative effects because of the unreasonable attribute values by the power weighting. In a word, the proposed method is more practical and effective in solving the MADM and MAGDM problems. In the future research, we will extend PHM aggregation operator to some new uncertain environment, such as two-dimensional uncertain linguistic information (Liu and Teng 2016; Liu et al. 2016b) and so on.

Acknowledgements This paper is supported by the National Natural Science Foundation of China (Nos. 71771140 and 71471172) and the Special Funds of Taishan Scholars Project of Shandong Province (No. ts201511045).

Compliance with ethical standards

Conflict of interest We declare that we do have no commercial or associative interests that represent a conflict of interests in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work.

Research involving human participants and/or animals This article does not contain any studies with human participants or animals performed by any of the authors.

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