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Heronian mean operators of linguistic neutrosophic multisets and their multiple attribute decision-making methods

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Abstract
A valid aggregation operator can reflect the decision result more clearly and make the decision effect more correctly. In this article, a linguistic neutrosophic multiset is first proposed to handle the multiplicity information, which is an expanding of neutrosophic multiset. Two Heronian mean operators are proposed to aggregate the linguistic neutrosophic multiset, one is a linguistic neutrosophic multiplicity number generalized-weighted Heronian mean operator, the other is a linguistic neutrosophic multiplicity number improved-generalized-weighted Heronian mean operator, and then their properties are discussed. Furthermore, two decision-making methods are introduced based on linguistic neutrosophic multiplicity number generalized-weighted Heronian mean or linguistic neutrosophic multiplicity number improved-generalized-weighted Heronian mean operators under linguistic neutrosophic multiplicity number environment. Finally, an illustrative example is used to indicate the practicality and validity of these two methods.

Keywords
Neutrosophic multiset, linguistic neutrosophic multiset, linguistic neutrosophic multiplicity number, linguistic neutrosophic multiplicity number generalized-weighted Heronian mean operator, linguistic neutrosophic multiplicity number improved-generalized-weighted Heronian mean operator, multi attributes decision-making

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Introduction

As an important branch of modern decision theory, theories and methods of multi-attribute decision-making are widely used in economy, management, military, engineering, and so on. One of the core issues of the multi-attribute decision-making method is how to aggregate the attribute values of individual attributes (the use of aggregation operators). A valid aggregation operator can reflect the decision result more clearly and make the decision effect more correctly. For the importance of aggregation operator in multiple attribute decision-making, many scholars have done many studies on them, including a variety of information aggregation operators and their properties, etc. According to the decision-making information environment, aggregation operators can be probably divided into the following categories: real aggregation operator; interval number aggregation operator; fuzzy aggregation operator; intuitionistic fuzzy aggregation operator; neutrosophic aggregation operator; language aggregation

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operator, and so on. The real number aggregation operator is applicable to the decision environment with the real attribute values, and which is the most basic information aggregation operator. Other types of aggregation operators are generally derived by extending the real number aggregation operators. Common kinds of real aggregation operators: Yager[1] proposed an ordered weighted average (OWA) operator. Chiclana et al.[2] and Xu and Da[3] proposed the ordered-weighted-geometric (OWG) operator. Chen et al.[4] proposed the ordered-weighted harmonic averaging (OWHA) operator. Yager[5] merged the generalized averaging and the OWA operator together and defined the generalized-ordered-weighted averaging (GOWA) operator. Bonferroni[6] proposed the Bonferroni mean (BM) operator; then, Zhou and He[7] proposed the normalized weighted Bonferroni mean (NWBM) operator. The generalized-weighted-Heronian-mean (GWHM) operator was proposed by Yu and Wu[8] Common kinds of interval number aggregation operators: uncertain ordered weighted average (UOWA) operator and uncertain ordered-weighted-geometric (UOWG) operator,[9,10] uncertain Bonferroni mean (UBM) operator and interval Heronian mean (IHM) operator;[11,12] Common kinds of fuzzy aggregation operators: fuzzy ordered weighted average (FOWA) operator,[13] fuzzy ordered-weighted-geometric (FOWG) operator,[14] trapezoidal fuzzy ordered weighted average (TFOWA) operator,[15] trapezoidal fuzzy Bonferroni mean (TFBM) operator;[16] common kinds of intuitionistic fuzzy aggregation operators: intuitionistic fuzzy ordered weighted average (IFOWA) operator,[17] intuitionistic ordered-weighted-geometric (IFOWG) operator,[18] intuitionistic fuzzy Bonferroni mean (IFBM) operator;[19] common kinds of neutrosophic aggregation operators: interval neutrosophic prioritized ordered weighted average (INPOWA) operator,[20] neutrosophic Bonferroni mean (NBM) operator,[21] and neutrosophic Heronian mean (NHM) operator;[22] common kinds of language aggregation operators: linguistic ordered weighted average (LOWA) operator,[23] extended ordered weighted average (EOWA) operator, and extended ordered-weighted-geometric (EOWG) operator,[24] induced uncertain linguistic ordered weighted average (IULOWA) operator.[25] Then, these language aggregation operators were merged with different information, such as intuitionistic uncertain linguistic Heronian mean (IULHM) operator,[26] neutrosophic linguistic number weighted arithmetic average (NLNWAA) and neutrosophic linguistic number weighted geometric average (NLNWGA) operators,[27] linguistic neutrosophic number weighted arithmetic averaging (LNNWAA) operator and an linguistic neutrosophic weighted geometric averaging (LNNWGA) operator,[28] interval neutrosophic linguistic numbers Heronian mean (INLNHM) operator,[29] linguistic neutrosophic number Bonferroni mean (LNNBM) operator, and so on;[30] the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean (NULNIGWHM) operator and the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean (NULNIGGWHM) operator.[31]

In this article, we first introduce a linguistic neutrosophic multiset (LNM), which is an expanding of neutrosophic multiset (NM) proposed by Smarandache.[32] An LNM can use pure linguistic to express and deal with the multiplicity information while other neutrosophic sets cannot do it. Compared to linguistic fuzzy multiset and linguistic intutionistic fuzzy multiset, LNM can use three values to express truth, indeterminacy, and falsity, respectively, while the other two multisets cannot do this. Next, we define linguistic neutrosophic multiplicity number generalized-weighted Heronian mean (LNMNGWHM) operator and linguistic neutrosophic multiplicity number improved-generalized-weighted Heronian mean (LNMNIGWHM) operator and discuss the properties of them. In the literature, Fan et al.[33] defined a single value neutrosophic multiset (SVNM) and utilized the weighted cosine measure in SVNM environment. In the literature, Ye[34] defined a dynamic single valued neutrosophic multiset (DSVNM) and utilized the weighted correlation coefficient of DSVNMs. Compared with the literature,[33,34] on one hand, the LNMNGWHM and LNMNIGWHM operators can express the pure linguistic values; then it is very suitable for handling the linguistic information; on the other hand, Heronian mean operator can embody the interaction between attributes and not repeatedly consider the relationship between each attribute and the others. Compared with the literatures,[22,26–31] the LNMNGWHM and LNMNIGWHM operators can handle the multiplicity problems while other neutrosophic sets in the literatures,[22,26–31] cannot deal with them.

The other sections are as follows. The concepts of LNN, GWHM, IGWHM, and NM are described in section "Some basic concepts." An LNM is proposed in section "LNM." The LNMNGWHM and LNMNIGWHM operators are proposed in section "GWHM and IGWHM operators of LNMNs." Multi attributes decision-making (MADM) methods using the LNMNGWHM and LNMNIGWHM operators are proposed in section "Using LNMNGWHM or LNMNIGWHM operator for MADM methods." Two proposed methods are used in section "Illustrative examples" to deal with an illustrative example. Conclusions and future expectations are given in section "Conclusion."

Some basic concepts

Linguistic neutrosophic sets

Definition 1. Set a universe $X = \{x_1, x_2, \ldots, x_n\}\}$ and $\Theta = \{\theta_j | j \in [0, \varrho]\}$ as a language term set with cardinality $\varrho + 1$, and then linguistic neutrosophic sets (LNS) can be expressed as follows[28]

$$M = \{\langle x, \theta_\mu(x), \theta_\nu(x), \theta_\xi(x) | x \in X \rangle\}$$

in which $\theta_\mu(x), \theta_\nu(x), \theta_\xi(x) \in \Theta$ and $\mu, \nu, \xi \in [0, \varrho]$, $\theta_\mu(x), \theta_\nu(x)$ and $\theta_\xi(x)$ represent the truth-membership function, the indeterminacy-membership function, and the falsity-membership function, respectively, $x \in X$ in M.

For convenient expression, a basic element $\langle x, \theta_\mu(x), \theta_\nu(x), \theta_\xi(x) | x \in X \rangle$ in M is simplified into $m = \langle \theta_\mu, \theta_\nu, \theta_\xi \rangle$, which is called a linguistic neutrosophic number (LNN).

Definition 2. There are three LNNs $m = \langle \theta_\mu, \theta_\nu, \theta_\xi \rangle$, $m_1 = \langle \theta_{\mu 1}, \theta_{\nu 1}, \theta_{\xi 1} \rangle$, and $m_2 = \langle \theta_{\mu 2}, \theta_{\nu 2}, \theta_{\xi 2} \rangle$ in Θ, $\eta \geqslant 0$, and then some operational rules of LNNs are defined as follows[28]

$$m_1 \oplus m_2 = \langle \theta_{\mu 1}, \theta_{\nu 1}, \theta_{\xi 1} \rangle \oplus \langle \theta_{\mu 2}, \theta_{\nu 2}, \theta_{\xi 2} \rangle = \langle \theta_{\mu 1 + \mu 2 - \frac{\mu 1 \mu 2}{\varrho}}, \theta_{\frac{\nu 1 \nu 2}{\varrho}}, \theta_{\frac{\xi 1 \xi 2}{\varrho}} \rangle \tag{1}$$

$$m_1 \otimes m_2 = \langle \theta_{\mu 1}, \theta_{\nu 1}, \theta_{\xi 1} \rangle \otimes \langle \theta_{\mu 2}, \theta_{\nu 2}, \theta_{\xi 2} \rangle = \left\langle \theta_{\frac{\mu 1 \mu 2}{\varrho}}, \theta_{\nu 1 + \nu 2 - \frac{\nu 1 \nu 2}{\varrho}}, \theta_{\xi 1 + \xi 2 - \frac{\xi 1 \xi 2}{\varrho}} \right\rangle \tag{2}$$

$$\eta m = \eta \langle \theta_\mu, \theta_\nu, \theta_\xi \rangle = \left\langle \theta_{\varrho - \varrho\left(1 - \frac{\mu}{\varrho}\right)^\eta}, \theta_{\varrho\left(\frac{\nu}{\varrho}\right)^\eta}, \theta_{\varrho\left(\frac{\xi}{\varrho}\right)^\eta} \right\rangle \tag{3}$$

$$m^\eta = \langle \theta_\mu, \theta_\nu, \theta_\xi \rangle^\eta = \left\langle \theta_{\varrho\left(\frac{\mu}{\varrho}\right)^\eta}, \theta_{\varrho - \varrho\left(1 - \frac{\nu}{\varrho}\right)^\eta}, \theta_{\varrho - \varrho\left(1 - \frac{\xi}{\varrho}\right)^\eta} \right\rangle \tag{4}$$

Definition 3. There is an LNN $m = \langle \theta_\mu, \theta_\nu, \theta_\xi \rangle$ in Θ, we define the score $O(m)$ and the accuracy $C(m)$ functions as follows[28]

$$O(m) = \frac{1}{3\varrho}(2\varrho + \mu - \nu - \xi) \tag{5}$$

$$C(m) = \frac{1}{\varrho}(\mu - \nu) \tag{6}$$

Definition 4. Set $m_i = \langle \theta_{\mu i}, \theta_{\nu i}, \theta_{\xi i} \rangle$ and $m_j = \langle \theta_{\mu j}, \theta_{\nu j}, \theta_{\xi j} \rangle$ as two LNNs in Θ, then[28]

If $O(m_i) > O(m_j)$, then $m_i > m_j$;
If $O(m_i) = O(m_j)$ and $C(m_i) > C(m_j)$ then $m_i > m_j$;
If $O(m_i) = O(m_j)$ and $C(m_i) = C(m_j)$ then $m_i = m_j$;
If $O(m_i) = O(m_j)$ and $C(m_i) < C(m_j)$ then $m_i < m_j$.

GWHM operator and IGWHM operator

Definition 5. Let $d, f \geqslant 0$, and d, f do not equal 0 at the same time, $m_i (i = 1, 2, \ldots, n)$ be a set of non-negative numbers, the weight vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be of m_i (i = 1, 2,..., n), $\sum_{i=1}^n \phi_i = 1$ and $\phi_i \in [0, 1]$. If[8]

$$GWHM^{d,f}(m_1, m_2, \ldots m_n) = \left(\frac{1}{n(n+2)} \sum_{i=1}^n \sum_{j=1}^n (\phi_i m_i)^d (\phi_j m_j)^f \right)^{\frac{1}{d+f}} \tag{7}$$

then we call GWHM as a generalized weighted Heronian mean (GWHM) operator.

For GWHM has no idempotence property, in order to compensate for this deficiency, the IGWHM operator is proposed.

Definition 6. Set $d, f \geqslant 0$, and d, f do not equal 0 at the same time, $m_i (i = 1, 2, \ldots, n)$ be a set of non-negative numbers, the weight vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be of m_i (i = 1,2,..., n), $\sum_{i=1}^n \phi_i = 1$ and $\phi_i \in [0, 1]$. If[35]

$$IGWHM^{d,f}(m_1, m_2, \ldots m_n) = \left(\frac{1}{\lambda} \mathop{\oplus}_{i=1}^{n} \mathop{\oplus}_{j=i}^{n} \left(\phi_i \phi_j m_i^d \otimes m_j^f \right) \right)^{\frac{1}{d+f}} \tag{8}$$

where $1/\lambda = 1/\sum_{i=1}^n \sum_{j=i}^n \phi_i \phi_j$, then we call IGWHM as an improved generalized weighted HM(IGWHM) operator.

NM

Definition 7. Set a universe $X = \{x_1, x_2, \ldots, x_n\}$ and a NM is a neutrosophic set, there is at least one element is repeated with different or the same neutrosophic component and we write a NM M as follows[32]

$$\{(m, nm(m), \; for \; m \in M)\} \; or \; (M, nm(m)) \tag{9}$$

LNM

Definition 8. Set a universe $X = \{x_1, x_2, \ldots, x_n\}$ and $\Theta = \{\theta_j | j \in [0, \varrho]\}$ be a language term set with cardinality $\varrho + 1$, and Z = {1, 2, 3,..., ∞}, then we define an LNM Q as follows

$$Q = \Big\{ x, \Big(\Big(h_{Q1}, \Big\langle \theta_{\mu_{Q1}}(x), \theta_{\nu_{Q1}}(x), \theta_{\xi_{Q1}}(x) \Big\rangle \Big), \Big(h_{Q2}, \Big\langle \theta_{\mu_{Q2}}(x), \theta_{\nu_{Q2}}(x), \theta_{\xi_{Q1}}(x) \Big\rangle \Big), \ldots, \Big(h_{Qy}, \Big\langle \theta_{\mu_{Qy}}(x), \theta_{\nu_{Qy}}(x), \theta_{\xi_{Qy}}(x) \Big\rangle \Big) \Big) | x \in X \Big\}$$

where $\theta_{\mu_{Qt}}(x), \theta_{\nu_{Qt}}(x), \theta_{\xi_{Qt}}(x) \in \Theta$, $\mu_{Qt}, \nu_{Qt}, \xi_{Qt} \in [0, \varrho]$, $\theta_{\mu_{Qt}}(x)$ expresses the truth-membership function, $\theta_{\xi_{Qt}}(x)$ expresses the falsity-membership function, and $\theta_{\nu_{Qt}}(x)$

expresses the indeterminacy-membership function. $\theta_{\mu_{Q1}}(x), \theta_{\mu_{Q2}}(x), \ldots, \theta_{\mu_{Qt}}(x)$, $\theta_{\nu_{Q1}}(x), \theta_{\nu_{Q2}}(x), \ldots, \theta_{\nu_{Qt}}(x)$ and $\theta_{\xi_{Q1}}(x), \theta_{\xi_{Q2}}(x), \ldots, \theta_{\xi_{Qt}}(x) \in [0, 1]$ and $0 \leqslant \theta_{\mu_{Qt}}(x) + \theta_{\nu_{Qt}}(x) + \theta_{\xi_{Qt}}(x) \leqslant 3$ $(t = 1, 2, \ldots, y)$, $y \in Z$, $h_{Q1}, h_{Q2}, \ldots, h_{Qy} \in Z$ and $h_{Q1} + h_{Q2} + \cdots + h_{Qy} \geqslant 2$.

For convenience, we simplify an LNM Q as follows

$$Q = \left\{x, \left(h_{Qt}, \left\langle \theta_{\mu_{Qt}}(x), \theta_{\nu_{Qt}}(x), \theta_{\xi_{Qt}}(x)\right\rangle\right) | x \in X\right\}, \quad \text{for } t = 1, 2, \ldots, y$$

For example, an LNM Q is given as

$$Q = \{(x_1, (3, \langle \theta_6, \theta_2, \theta_2\rangle), (1, \langle \theta_8, \theta_4, \theta_3\rangle)), (x_2, (1, \langle \theta_6, \theta_3, \theta_1\rangle), (2, \langle \theta_5, \theta_3, \theta_4\rangle))\}$$

Then

$$nm_Q(x_1) = \{(3, \langle \theta_6, \theta_2, \theta_2\rangle), (1, \langle \theta_8, \theta_4, \theta_3\rangle)\}$$

$$nm_Q(x_2) = \{(1, \langle \theta_6, \theta_3, \theta_1\rangle), (2, \langle \theta_5, \theta_3, \theta_4\rangle)\}$$

Definition 9. Set a universe $X = \{x_1, x_2, \ldots, x_n\}$ and $\Theta = \{\theta_j | j \in [0, \varrho]\}$ be a language term set with cardinality $\varrho + 1$, and $N = \{1, 2, 3, \ldots, \infty\}$, there are two LNMs Q_1 and Q_2

$$Q_1 = \left\{x, \left(h_{Q_1 t}, \left\langle \theta_{\mu_{Q_1 t}}(x), \theta_{\nu_{Q_1 t}}(x), \theta_{\xi_{Q_1 t}}(x)\right\rangle\right) | x \in X\right\}, \quad \text{for } t = 1, 2, \ldots, y$$

$$Q_2 = \left\{x, \left(h_{Q_2 t}, \left\langle \theta_{\mu_{Q_2 t}}(x), \theta_{\nu_{Q_2 t}}(x), \theta_{\xi_{Q_2 t}}(x)\right\rangle\right) | x \in X\right\}, \quad \text{for } t = 1, 2, \ldots, y$$

Then they have the following relations:

1. $Q_1 = Q_2$, if and only if $h_{Q_1 t} = h_{Q_2 t}, \theta_{\mu_{Q_1 t}}(x) = \theta_{\mu_{Q_2 t}}(x)$, $\theta_{\nu_{Q_1 t}}(x) = \theta_{\nu_{Q_2 t}}(x), \theta_{\xi_{Q_1 t}}(x) = \theta_{\xi_{Q_2 t}}(x)$, for $t = 1, 2, \ldots, y$.
2. $Q_1 \cup Q_2 = \{x, ((h_{Q_1 t} \vee h_{Q_2 t}), \langle \theta_{\mu_{Q_1 t}}(x) \vee \theta_{\mu_{Q_2 t}}(x), \theta_{\nu_{Q_1 t}}(x) \wedge \theta_{\nu_{Q_2 t}}(x), \theta_{\xi_{Q_1 t}}(x) \wedge \theta_{\xi_{Q_2 t}}(x)\rangle) | x \in x\}$, *for* $t = 1, 2, \ldots, y$.
3. $Q_1 \cap Q_2 = \{x, ((h_{Q_1 t} \wedge h_{Q_2 t}), \langle \theta_{\mu_{Q_1 t}}(x) \wedge \theta_{\mu_{Q_2 t}}(x), \theta_{\nu_{Q_1 t}}(x) \vee \theta_{\nu_{Q_2 t}}(x), \theta_{\xi_{Q_1 t}}(x) \vee \theta_{\xi_{Q_2 t}}(x)\rangle) | x \in X\}$, *for* $t = 1, 2, \ldots, y$.

For convenience, a basic element can be expressed using a linguistic neutrosophic multiplicity number (LNMN) $q = \left(\left(h_{q1}\left\langle \theta_{\mu_{q1}}, \theta_{\nu_{q1}}, \theta_{\xi_{q1}}\right\rangle\right), \left(h_{q2}\left\langle \theta_{\mu_{q2}}, \theta_{\nu_{q2}}, \theta_{\xi_{q2}}\right\rangle\right), \ldots, \left(h_{qy}\left\langle \theta_{\mu_{qy}}, \theta_{\nu_{qy}}, \theta_{\xi_{qy}}\right\rangle\right)\right)$ in *LNM Q*.

Definition 10. Set $q = ((h_{q1}\langle \theta_{\mu_{q1}}, \theta_{\nu_{q1}}, \theta_{\xi_{q1}}\rangle), (h_{q2}\langle \theta_{\mu_{q2}}, \theta_{\nu_{q2}}, \theta_{\xi_{q2}}\rangle), \ldots, (h_{qy}\langle \theta_{\mu_{qy}}, \theta_{\nu_{qy}}, \theta_{\xi_{qy}}\rangle))$ as an LNMN, we can use Definition 2 to change a LNMN q into an LNN $\tilde{q}$

$$\tilde{q} = \left(\theta_{\varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}}\right) \tag{10}$$

Proof. Set elements $q(1), q(2), \ldots, q(y)$ be in q, then $q(t) = \left(h_{qt}, \left\langle \theta_{\mu_{qt}}, \theta_{\nu_{qt}}, \theta_{\xi_{qt}}\right\rangle\right)$, for $t = 1$ to y.

According with the operational rules of LNNs, while $t = 1, 2$, we can get

$$+ q(1) = \theta_{\varrho\left(1-\left(1-\frac{\mu_{q1}}{\varrho}\right)^{h_{q1}}\right)}, \theta_{\varrho\left(\frac{\nu_{q1}}{\varrho}\right)^{h_{q1}}}, \theta_{\varrho\left(\frac{\xi_{q1}}{\varrho}\right)^{h_{q1}}}$$

$$+ q(2) = \theta_{\varrho\left(1-\left(1-\frac{\mu_{q2}}{\varrho}\right)^{h_{q2}}\right)}, \theta_{\varrho\left(\frac{\nu_{q2}}{\varrho}\right)^{h_{q2}}}, \theta_{\varrho\left(\frac{\xi_{q2}}{\varrho}\right)^{h_{q2}}}$$

Then

$$\begin{aligned} q(1) + q(2) &= \theta_{\varrho\left(1-\left(1-\frac{\mu_{q1}}{\varrho}\right)^{h_{q1}}\right) + \varrho\left(1-\left(1-\frac{\mu_{q2}}{\varrho}\right)^{h_{q2}}\right) - \varrho\left(1-\left(1-\frac{\mu_{q1}}{\varrho}\right)^{h_{q1}}\right)\left(1-\left(1-\frac{\mu_{q2}}{\varrho}\right)^{h_{q2}}\right)}, \\ &\quad \theta_{\varrho\left(\frac{\nu_{q1}}{\varrho}\right)^{h_{q1}}\left(\frac{\nu_{q2}}{\varrho}\right)^{h_{q2}}}, \theta_{\varrho\left(\frac{\xi_{q1}}{\varrho}\right)^{h_{q1}}\left(\frac{\xi_{q2}}{\varrho}\right)^{h_{q2}}} \\ &= \theta_{\varrho\left(1-\prod_{t=1}^{2}\left(\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{2}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}}, \theta_{\varrho\prod_{t=1}^{2}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}} \end{aligned}$$

When $t = i$, we suppose equation (10) is established, then

$$q(1) + q(2) + \cdots + q(i) = \theta_{\varrho\left(1-\prod_{t=1}^{i}\left(\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{i}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}}, \theta_{\varrho\prod_{t=1}^{i}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}}$$

can be gotten.

Next

$$\begin{aligned} &q(1) + q(2) + \cdots + q(i) + q(i+1) = \\ &\theta_{\varrho\left(1-\prod_{t=1}^{i}\left(\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)\right) + \varrho\left(1-\left(1-\frac{\mu_{q(i+1)}}{\varrho}\right)^{h_{q(i+1)}}\right) - \varrho\left(1-\prod_{t=1}^{i}\left(\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)\right)\left(1-\left(1-\frac{\mu_{q(i+1)}}{\varrho}\right)^{h_{q(i+1)}}\right)} \\ &\theta_{\varrho\left(\prod_{t=1}^{i}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}\right)\left(\frac{\nu_{q(i+1)}}{\varrho}\right)^{h_{q(i+1)}}}, \theta_{\varrho\left(\prod_{t=1}^{n}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}\right)\left(\frac{\xi_{q(i+1)}}{\varrho}\right)^{h_{q(i+1)}}} \\ &= \left(\theta_{\varrho\left(1-\prod_{t=1}^{i+1}\left(\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{i+1}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}}, \theta_{\varrho\prod_{t=1}^{i+1}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}}\right) \end{aligned}$$

The result according to equation (10) while $t = i + 1$, and then we can obtain that the result of aggregation is also true according to the mathematical induction.

Definition 11. Set $q_1 = \left(h_{q_1 t}, \left\langle \theta_{\mu_{q_1 t}}, \theta_{\nu_{q_1 t}}, \theta_{\xi_{q_1 t}}\right\rangle\right)$ and $q_2 = \left(h_{q_2 t}, \left\langle \theta_{\mu_{q_2 t}}, \theta_{\nu_{q_2 t}}, \theta_{\xi_{q_2 t}}\right\rangle\right)$ as two LNMNs in Θ, for $t = 1, 2, \ldots, y$ and a real number $\eta \geqslant 0$, then we define LNMNs' operational laws as follows

$$q_1 \oplus q_2 = \left\langle \theta_{\varrho\left(1-\prod_{t=1}^{y}\left(\left(\left(1-\frac{\mu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\left(\left(1-\frac{\mu_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}\right)\right)\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\left(\frac{\nu_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\left(\frac{\xi_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}\right)} \right\rangle$$

$$q_1 \otimes q_2 = \left\langle \theta_{\varrho\left(1-\prod_{t=1}^{y}\left(\left(1-\frac{\mu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)\left(1-\prod_{t=1}^{y}\left(\left(1-\frac{\mu_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}\right)\right)\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}} + \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_2 t}}{\varrho}\right)^{h_{q_2 t}} - \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\prod_{t=1}^{y}\left(\frac{\nu_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}} + \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_2 t}}{\varrho}\right)^{h_{q_2 t}} - \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\prod_{t=1}^{y}\left(\frac{\xi_{q_2 t}}{\varrho}\right)^{h_{q_2 t}}} \right\rangle$$

$$\eta q_1 = \left\langle \theta_{\varrho\left(1-\left(\prod_{t=1}^{y}\left(\left(1-\frac{\mu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}\right)\right)}, \theta_{\varrho\left(\prod_{t=1}^{y}\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}}, \theta_{\varrho\left(\prod_{t=1}^{y}\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}} \right\rangle$$

$$q_1^{\eta} = \left\langle \theta_{\varrho\left(1-\prod_{k=1}^{y}\left(\left(1-\frac{\mu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}\right)}, \theta_{\varrho\left(1-\left(1-\prod_{k=1}^{y}\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}\right)}, \theta_{\varrho\left(1-\left(1-\prod_{k=1}^{y}\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{\eta}\right)} \right\rangle$$

GWHM and IGWHM operators of LNMNs

LNMNGWHM

Definition 12. Let $d, f \geqslant 0$, and d, f do not equal 0 at the same time, a collection $q_i = (h_{q_i t}, \langle \theta_{\mu_{q_i t}}, \theta_{\nu_{q_i t}}, \theta_{\xi_{q_i t}} \rangle)$, $(i = 1, 2, \ldots, n; t = 1, 2, \ldots, y)$ of LNMNs in Θ, then we can define the LNMNGWHM operator as follows

$$LNMNGWHM^{d,f}(q_1, q_2, \ldots, q_n) = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(\phi_i q_i)^d(\phi_j q_j)^f\right)^{\frac{1}{d+f}} \tag{11}$$

where $\sum_{i=1}^{n}\phi_i = 1$ and $\phi_i \in [0, 1]$.

Using Definitions 2, 10, and 12, the following theorem can be gotten.

Theorem 1. Set a collection $q_i = (h_{q_i t}, \langle \theta_{\mu_{q_i t}}, \theta_{\nu_{q_i t}}, \theta_{\xi_{q_i t}} \rangle)(i = 1, 2, \ldots, n; t = 1, 2, \ldots, y)$ of LNMNs in Θ, with LNMNGWHM operator, we can get the aggregation result of q_i, which is given by the following form

$$LNMNGWHM^{d,f}(q_1, q_2, \ldots, q_n) = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i q_i)^d(\phi_j q_j)^f\right)^{\frac{1}{d+f}} =$$

$$\left(\left\langle \theta_{\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}} \right\rangle\right) \tag{12}$$

where

$$\mu i = \varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{q_i t}}{\varrho}\right)^{h_{q_i t}}\right)$$

$$\mu j = \varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{q_j t}}{\varrho}\right)^{h_{q_j t}}\right)$$

$$\nu i = \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_i t}}{\varrho}\right)^{h_{q_i t}}$$

$$\nu j = \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_j t}}{\varrho}\right)^{h_{q_j t}}$$

$$\xi i = \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_i t}}{\varrho}\right)^{h_{q_i t}}$$

and

$$\xi j = \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_j t}}{\varrho}\right)^{h_{q_j t}}$$

$$\frac{1}{\lambda} = \frac{2}{n(n+1)}$$

$$\sum_{i=1}^{n}\phi_i = 1$$

and

$$\phi_i \in [0, 1]$$

Proof.

1. Using equation (10), a LNMN q_i can be changed into an LNN $\widetilde{q}_i$

$$\tilde{q}_i = (\theta_{\mu i}, \theta_{\nu i}, \theta_{\xi i}) = \left(\theta_{\varrho\left(1-\prod_{t=1}^{y}\left((1-\frac{\mu_{q_it}}{\varrho})^{h_{q_it}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_it}}{\varrho}\right)^{h_{q_it}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_it}}{\varrho}\right)^{h_{q_it}}}\right)$$

2. $\phi_i q_i = \phi_i\tilde{q}_i = \langle\theta_{\varrho-\varrho\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}}, \theta_{\varrho\left(\frac{\nu i}{\varrho}\right)^{\phi_i}}, \theta_{\varrho\left(\frac{\xi i}{\varrho}\right)^{\phi_i}}\rangle$
3. $\phi_j q_j = \phi_j\tilde{q}_j = \langle\theta_{\varrho-\varrho\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}}, \theta_{\varrho\left(\frac{\nu j}{\varrho}\right)^{\phi_j}}, \theta_{\varrho\left(\frac{\xi j}{\varrho}\right)^{\phi_j}}\rangle$
4. $(\phi_i q_i)^d = (\phi_i\tilde{q}_i)^d = \left\langle\theta_{\varrho\left(\frac{\varrho-\varrho\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}}{\varrho}\right)^d}, \theta_{\varrho-\varrho\left(1-\frac{\varrho\left(\frac{\nu i}{\varrho}\right)^{\phi_i}}{\varrho}\right)^d}, \theta_{\varrho-\varrho\left(1-\frac{\varrho\left(\frac{\xi i}{\varrho}\right)^{\phi_i}}{\varrho}\right)^d}\right\rangle\left\langle\theta_{\varrho\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d}, \theta_{\varrho-\varrho\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d}, \theta_{\varrho-\varrho\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d}\right\rangle$
5. $(\phi_j q_j)^f = \left(\phi_j\tilde{q}_j\right)^f = \theta_{\varrho\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f}, \theta_{\varrho-\varrho\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f}, \theta_{\varrho-\varrho\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f}$
6. $(\phi_i q_i)^d(\phi_j q_j)^f = (\phi_i\tilde{q}_i)^d(\phi_j\tilde{q}_j)^f =$

$$\left\langle\theta_{\frac{\varrho\left(1-\left(1-\frac{\mu i}{g}\right)^{\phi_i}\right)^d\varrho\left(1-\left(1-\frac{\mu j}{g}\right)^{\phi_j}\right)^f}{\varrho}}, \theta_{\varrho-\varrho\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d+\varrho-\varrho\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f-\frac{\left(\varrho-\varrho\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\right)\left(\varrho-\varrho\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f\right)}{\varrho}}, \theta_{\varrho-\varrho\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d+\varrho-\varrho\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f-\frac{\left(\varrho-\varrho\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\right)\left(\varrho-\varrho\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f\right)}{\varrho}}\right\rangle$$

$$= \left\langle\theta_{\varrho\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f}, \theta_{\varrho-\varrho\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f}, \theta_{\varrho-\varrho\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f}\right\rangle$$

7. $\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i q_i)^d(\phi_j q_j)^f = \sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i\tilde{q}_i)^d(\phi_j\tilde{q}_j)^f = \left\langle\theta_{\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f\right)\right)},\right.$

$$\left.\theta_{\varrho\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f\right)}, \theta_{\varrho\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f\right)}\right\rangle$$

8. $\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i q_i)^d(\phi_j q_j)^f = \frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i q_i)^d(\phi_j q_j)^f = \frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i\tilde{q}_i)^d(\phi_j\tilde{q}_j)^f = \left\langle\theta_{\varrho-\varrho\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}},\right.$

$$\left.\theta_{\varrho\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}}, \theta_{\varrho\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}}\right\rangle$$

9. $\left(\frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i q_i)^d(\phi_j q_j)^f\right)^{\frac{1}{d+f}} = \left(\frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=i}^{n}(\phi_i\tilde{q}_i)^d(\phi_j\tilde{q}_j)^f\right)^{\frac{1}{d+f}} =$

$$\left\langle\theta_{\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(1-\frac{\mu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(1-\frac{\mu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\nu i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\nu j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\left(\frac{\xi i}{\varrho}\right)^{\phi_i}\right)^d\left(1-\left(\frac{\xi j}{\varrho}\right)^{\phi_j}\right)^f\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}\right\rangle$$

This proves Theorem 1.

Improved generalized weighted HM operators of LNMNs

Definition 13. Let $d, f \geqslant 0$, and d, f do not equal 0 at the same time, a collection $q_i = (h_{q_it}, \langle\theta_{\mu_{q_it}}, \theta_{\nu_{q_it}}, \theta_{\xi_{q_it}}\rangle)$ $(i = 1, 2, \ldots, n; t = 1, 2, \ldots, y)$ of LNMNs in Θ, then we can define the LNMNGWHM operator as follows

$$LNMNIGWHM^{d,f}(q_1, q_2, \ldots q_n) = \left(\frac{1}{\lambda}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=i}^{n}\left(\phi_i\phi_j {q_i}^d \otimes {q_j}^f\right)\right)^{\frac{1}{d+f}} \tag{13}$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n}\phi_i\phi_j$, $\sum_{i=1}^{n}\phi_i = 1$ and $\phi_i \in [0, 1]$.

Using Definitions 2, 10, and 13, Theorem 2 can be gotten as follows.

Theorem 2. Set a collection $q_i = (h_{q_it}, \langle\theta_{\mu_{q_it}}, \theta_{\nu_{q_it}}, \theta_{\xi_{q_it}}\rangle)(i = 1, 2, \ldots, n; t = 1, 2, \ldots, y)$ of LNMNs in Θ, with LNMNIGWHM operator, we can get the aggregation result of q_i, which is given by the following form

$$LNMNIGWHM^{d,f}(q_1, q_2, \ldots q_n) = \left(\frac{1}{\lambda}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=i}^{n}\left(\phi_i\phi_j {q_i}^d \otimes {q_j}^f\right)\right)^{\frac{1}{d+f}} =$$

$$\left\langle\theta_{\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\frac{\mu i}{\varrho}\right)^d\left(\frac{\mu j}{\varrho}\right)^f\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\nu i}{\varrho}\right)^d\left(1-\frac{\nu j}{\varrho}\right)^f\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}},\right.$$

$$\left.\theta_{\varrho-\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\xi i}{\varrho}\right)^d\left(1-\frac{\xi j}{\varrho}\right)^f\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}\right\rangle$$

where

$$\mu i = \varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{q_i t}}{\varrho}\right)^{h_{q_i t}}\right)$$

$$\mu j = \varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{q_j t}}{\varrho}\right)^{h_{q_j t}}\right)$$

$$vi = \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_i t}}{\varrho}\right)^{h_{q_i t}}$$

$$vj = \varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_j t}}{\varrho}\right)^{h_{q_j t}}$$

$$\xi i = \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_i t}}{\varrho}\right)^{h_{q_i t}}$$

and

$$\xi j = \varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_j t}}{\varrho}\right)^{h_{q_j t}}$$

$$\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n}\phi_i\phi_j,\ \sum_{i=1}^{n}\phi_i = 1$$

and

$$\phi_i \in [0,1]$$

We can use the proof of Theorem 1 to prove Theorem 2, so we omit it.

Theorem 3 (idempotency). Set a collection $q_i = \left(h_{q_i t}, \left\langle \theta_{\mu_{q_i t}}, \theta_{\nu_{q_i t}}, \theta_{\xi_{q_i t}} \right\rangle\right)(i = 1,2,\ldots,n; t = 1,2,\ldots,y)$ of LNMNs in Θ, if $q_i = q$, then

$$LNMNIGWHM^{d,f}(q_1,q_2,\ldots,q_n) = LNMNIGWHM^{d,f}(q,q,\ldots,q) = q$$

Proof. Since $q_i = q$ ($i = 1, 2,\ldots, n$), we can get

$$LNMNIGWHM^{d,f}(q_1,q_2,\ldots,q_n) = LNMNIGWHM^{d,f}(q,q,\ldots,q) =$$

$$\left\langle \theta_{\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d}\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{f}\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\prod_{t=1}^{y}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d}\left(1-\prod_{t=1}^{y}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{f}\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\prod_{t=1}^{y}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d}\left(1-\prod_{t=1}^{y}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}\right)^{f}\right)^{\phi_i\phi_j}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{d+f}}} \right\rangle =$$

$$\left\langle \theta_{\varrho\left(\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d+f}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(\left(1-\prod_{t=1}^{y}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d+f}\right)^{\frac{1}{d+f}}}, \theta_{\varrho-\varrho\left(\left(1-\prod_{t=1}^{y}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}\right)^{d+f}\right)^{\frac{1}{d+f}}} \right\rangle =$$

$$\left\langle \theta_{\varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{qt}}{\varrho}\right)^{h_{qt}}\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{qt}}{\varrho}\right)^{h_{qt}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{qt}}{\varrho}\right)^{h_{qt}}} \right\rangle = \tilde{q} = q$$

This proves Theorem 3.

Theorem 4 (monotonicity). Set $q_i = \left(h_{q_i t}, \left\langle \theta_{\mu_{q_i t}}, \theta_{\nu_{q_i t}}, \theta_{\xi_{q_i t}} \right\rangle\right)(i = 1,2,\ldots,n; t = 1,2,\ldots,y)$ and $k_i = \left(h_{k_i t}, \left\langle \theta_{\mu_{k_i t}}, \theta_{\nu_{k_i t}}, \theta_{\xi_{k_i t}} \right\rangle\right)(i = 1,2,\ldots,n; t = 1,2,\ldots,y)$ as two collection of LNMNs in Θ, if $h_{q_i t} \leqslant h_{k_i t}, \mu_{q_i t} \leqslant \mu_{k_i t}, \nu_{q_i t} \geqslant \nu_{k_i t}$ and $\xi_{q_i t} \geqslant \xi_{k_i t}$, then

$$LNMNIGWHM^{d,f}(q_1,q_2,\ldots,q_n) \leqslant LNMNIGWHM^{d,f}(k_1,k_2,\ldots,k_n)$$

Proof. Using equation (10), LNMN q_i and k_i can be changed into LNN $\widetilde{q}_i$ and $\widetilde{k}_i$

$$\tilde{q}_i = (\theta_{\mu i}, \theta_{\nu i}, \theta_{\xi i}) = \left(\theta_{\varrho\left(1-\prod_{t=1}^{y}\left(\left(1-\frac{\mu_{q_i t}}{\varrho}\right)^{h_{q_i t}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_i t}}{\varrho}\right)^{h_{q_i t}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_i t}}{\varrho}\right)^{h_{q_i t}}}\right)$$

$$\tilde{k}_i = \left(\theta_{\mu i'}, \theta_{\nu i'}, \theta_{\xi i'}\right) = \left(\theta_{\varrho\left(1-\prod_{t=1}^{y}\left(\left(1-\frac{\mu_{k_i t}}{\varrho}\right)^{h_{k_i t}}\right)\right)}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\nu_{k_i t}}{\varrho}\right)^{h_{k_i t}}}, \theta_{\varrho\prod_{t=1}^{y}\left(\frac{\xi_{k_i t}}{\varrho}\right)^{h_{k_i t}}}\right)$$

Because $h_{q_it} \leqslant h_{k_it}, \mu_{q_it} \leqslant \mu_{k_it}, \nu_{q_it} \geqslant \nu_{k_it}$ and $\xi_{q_it} \geqslant \xi_{k_it}$, we can easy obtain

$$\left(1-\frac{\mu_{q_it}}{\varrho}\right)^{h_{q_it}} \geqslant \left(1-\frac{\mu_{k_it}}{\varrho}\right)^{h_{k_it}}$$

$$\varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{q_it}}{\varrho}\right)^{h_{q_it}}\right) \leqslant \varrho\left(1-\prod_{t=1}^{y}\left(1-\frac{\mu_{k_it}}{\varrho}\right)^{h_{k_it}}\right)$$

Similarly

$$\left(\frac{\nu_{q_it}}{\varrho}\right)^{h_{q_it}} \geqslant \left(\frac{\nu_{k_it}}{\varrho}\right)^{h_{k_it}}$$

$$\varrho\prod_{t=1}^{y}\left(\frac{\nu_{q_it}}{\varrho}\right)^{h_{q_it}} \geqslant \varrho\prod_{t=1}^{y}\left(\frac{\nu_{k_it}}{\varrho}\right)^{h_{k_it}}$$

And

$$\left(\frac{\xi_{q_it}}{\varrho}\right)^{h_{q_it}} \geqslant \left(\frac{\xi_{k_it}}{\varrho}\right)^{h_{k_it}}$$

$$\varrho\prod_{t=1}^{y}\left(\frac{\xi_{q_it}}{\varrho}\right)^{h_{q_it}} \geqslant \varrho\prod_{t=1}^{y}\left(\frac{\xi_{k_it}}{\varrho}\right)^{h_{k_it}}$$

So, $LNMNIGWHM^{d,f}(q_1, q_2, \ldots, q_n) \leqslant LNMNIGWHM^{d,f}(k_1, k_2, \ldots, k_n)$, the proof of Theorem 4 is completed.

Theorem 5 (boundedness). Set a collection $q_i = (h_{q_it}, \langle\theta_{\mu_{q_it}}, \theta_{\nu_{q_it}}, \theta_{\xi_{q_it}}\rangle)(i = 1, 2, \ldots, n; t = 1, 2, \ldots, y)$ of LNMNs in Θ, let

$$q^- = \left(min\left(h_{q_it}\right), \left\langle min\left(\theta_{\mu_{q_it}}\right), max\left(\theta_{\nu_{q_it}}\right), max\left(\theta_{\xi_{q_it}}\right)\right\rangle\right)$$

and $q^+ = \left(max\left(h_{q_it}\right), \left\langle max\left(\theta_{\mu_{q_it}}\right), min\left(\theta_{\nu_{q_it}}\right), min\left(\theta_{\xi_{q_it}}\right)\right\rangle\right)$

then

$$q^- \leqslant LNMNIGWHM^{d,f}(q_1, q_2, \ldots, q_n) \leqslant q^+$$

Proof. Based on Theorems 3 and 4, we can obtain

$$q^- = LNMNIGWHM^{d,f}(q^-, q^-, \ldots, q^-) \text{ and}$$

$$q^+ = LNMNIGWHM^{d,f}(q^+, q^+, \ldots, q^+)$$

$$LNMNIGWHM^{d,f}(q^-, q^-, \ldots, q^-) \leqslant LNMNIGWHM^{d,f}(q_1, q_2, \ldots, q_n) \leqslant LNMNIGWHM^{d,f}(q^+, q^+, \ldots, q^+)$$

Then $q^- \leqslant LNMNIGWHM^{d,f}(q_1, q_2, \ldots, q_n) \leqslant q^+$. The proof of Theorem 5 is completed.

Using LNMNGWHM or LNMNIGWHM operator for MADM methods

Here, the MADM method is studied based on LNMNGWHM or LNNIGWHM operator. Set a set of alternatives $T = \{t_1, t_2, \ldots, t_m\}$ and a set of attributes $U = \{u_1, u_2, \ldots, u_n\}$, a set of the weights vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ of U, $\sum_{j=1}^{n}\phi_j = 1$ and $\phi_j \geqslant 0$. Then decision-makers give the assessed values for every alternative with attributes. However, sometimes one or more attributes may be evaluated for many times, and then the evaluation value can be represented using the form of LNM.

Step 1: Using a LNM $t_r = \{x_i, ((h_{t_rk}, \langle\theta_{\mu_{t_rk}}(x_i), \theta_{\nu_{t_rk}}(x_i), \theta_{\xi_{t_rk}}(x_i)\rangle))|x_i \in X\}$ $(r = 1, 2, \ldots, m; i = 1, 2, \ldots, n; k = 1, 2, \ldots, j)$ as the assessed values for t_r with attribute u_i to express the evaluation values, and then the decision matrix can be gotten.
Step 2: With LNMNGWHM or LNMNIGWHM operator, we can calculate $t_r = LNMNGWHM^{d,f}(t_{r1}, t_{r2}, \ldots, t_{ri})$ or $t_r = LNMNIGWHM^{d,f}(t_{r1}, t_{r2}, \ldots, t_{ri})$.
Step 3: Calculate $O(t_r)$ ($C(t_r)$ if necessary) according to formula (5) (formula (6) if necessary).
Step 4: The alternatives can be ranked with the value $O(t_r)$ (value $C(t_r)$ if necessary).
Step 5: End.

Illustrative examples

Now there is an illustrative example adapted from the literature.[33] A man wants to buy a car from four kinds c_1, c_2, c_3 and c_4, and the evaluation indicators are fuel economy (u_1), price (u_2), comfort (u_3), and safety (u_4). These evaluation indicators' weight vector is $\phi = (0.5, 0.25, 0.125, 0.125)^T$. The man tests these kinds of cars on the road with fewer obstacles and more obstacles, respectively. And then he can get two sets of data on some attributes after testing. Then the decision-maker evaluates the four kinds of cars using LNMNs based on Θ. Set $\Theta = \{\theta_0$ = extremely poor, θ_1 = very poor, θ_2 = poor, θ_3 = slightly poor, θ_4 = medium, θ_5 = slightly excellence, θ_6 = excellence, θ_7 = very excellence, θ_8 = extremely excellence$\}$. Then the decision matrix of LNMN can be built in Table 1.

Next, the LNMNGWHM operator or LNMNIGWHM operator is used to aggregate the LNMNs.

Table 1. LNMN decision matrix.

	u_1	u_2	u_3	u_4
c_1	$(1, \langle\theta_5, \theta_7, \theta_2\rangle), (1, \langle\theta_7, \theta_3, \theta_6\rangle)$	$1, \langle\theta_4, \theta_4, \theta_5\rangle$	$(1, \langle\theta_7, \theta_7, \theta_5\rangle), (1, \langle\theta_8, \theta_7, \theta_6\rangle)$	$(1, \langle\theta_1, \theta_5, \theta_7\rangle), (1, \langle\theta_5, \theta_2, \theta_8\rangle)$
c_2	$(1, \langle\theta_8, \theta_7, \theta_5\rangle), (1, \langle\theta_7, \theta_7, \theta_1\rangle)$	$1, \langle\theta_7, \theta_6, \theta_8\rangle$	$2, \langle\theta_8, \theta_4, \theta_6\rangle$	$(1, \langle\theta_5, \theta_2, \theta_7\rangle, (1, \langle\theta_5, \theta_1, \theta_8\rangle)$
c_3	$(1, \langle\theta_3, \theta_4, \theta_2\rangle, (1, \langle\theta_6, \theta_3, \theta_7\rangle)$	$1, \langle\theta_2, \theta_2, \theta_2\rangle$	$(1, \langle\theta_8, \theta_5, \theta_5\rangle, (1, \langle\theta_6, \theta_5, \theta_2\rangle)$	$(1, \langle\theta_7, \theta_5, \theta_3\rangle, (1, \langle\theta_4, \theta_2, \theta_2\rangle)$
c_4	$(1, \langle\theta_8, \theta_7, \theta_2\rangle, (1, \langle\theta_8, \theta_6, \theta_1\rangle)$	$1, \langle\theta_3, \theta_5, \theta_2\rangle$	$(1, \langle\theta_5, \theta_4, \theta_5\rangle, (1, \langle\theta_1, \theta_7, \theta_2\rangle)$	$2, \langle\theta_4, \theta_2, \theta_8\rangle$

LNMN: linguistic neutrosophic multiplicity number.

Table 2. The results with different values *d* and *f* based on LNMNGWHM.

d, f	LNMNGWHM	Ranking
$d = 1, f = 0.5$	$O(c_1) = 0.4833$, $O(c_2) = 0.4696$, $O(c_3) = 0.5558$, $O(c_4) = 0.5209$	$c_3 > c_4 > c_1 > c_2$
$d = 1, f = 2$	$O(c_1) = 0.5045$, $O(c_2) = 0.5247$, $O(c_3) = 0.5507$, $O(c_4) = 0.5491$	$c_3 > c_4 > c_2 > c_1$
$d = 0, f = 1$	$O(c_1) = 0.4361$, $O(c_2) = 0.4560$, $O(c_3) = 0.5095$, $O(c_4) = 0.4870$	$c_3 > c_4 > c_2 > c_1$
$d = 0.5, f = 1$	$O(c_1) = 0.4608$, $O(c_2) = 0.4684$, $O(c_3) = 0.5311$, $O(c_4) = 0.5086$	$c_3 > c_4 > c_2 > c_1$
$d = 2, f = 1$	$O(c_1) = 0.5198$, $O(c_2) = 0.5203$, $O(c_3) = 0.5736$, $O(c_4) = 0.5569$	$c_3 > c_4 > c_2 > c_1$
$d = 2, f = 2$	$O(c_1) = 0.5313$, $O(c_2) = 0.5460$, $O(c_3) = 0.5726$, $O(c_4) = 0.5697$	$c_3 > c_4 > c_2 > c_1$

LNMNGWHM: linguistic neutrosophic multiplicity number generalized-weighted Heronian mean.

Table 3. The result with different values *d* and *f* based on LNMNIGWHM.

d, f	LNMNIGWHM	Ranking
$d = 1, f = 0.5$	$O(c_1) = 0.7634$, $O(c_2) = 0.7027$, $O(c_3) = 0.8597$, $O(c_4) = 0.7771$	$c_3 > c_4 > c_1 > c_2$
$d = 1, f = 2$	$O(c_1) = 0.7661$, $O(c_2) = 0.7649$, $O(c_3) = 0.8623$, $O(c_4) = 0.8074$	$c_3 > c_4 > c_1 > c_2$
$d = 0, f = 1$	$O(c_1) = 0.7418$, $O(c_2) = 0.7687$, $O(c_3) = 0.8650$, $O(c_4) = 0.8235$	$c_3 > c_4 > c_2 > c_1$
$d = 0.5, f = 1$	$O(c_1) = 0.7504$, $O(c_2) = 0.7195$, $O(c_3) = 0.8614$, $O(c_4) = 0.7904$	$c_3 > c_4 > c_1 > c_2$
$d = 2, f = 1$	$O(c_1) = 0.7735$, $O(c_2) = 0.7484$, $O(c_3) = 0.8603$, $O(c_4) = 0.7923$	$c_3 > c_4 > c_1 > c_2$
$d = 2, f = 2$	$O(c_1) = 0.7754$, $O(c_2) = 0.7775$, $O(c_3) = 0.8615$, $O(c_4) = 0.8087$	$c_3 > c_4 > c_2 > c_1$

LNMNIGWHM: linguistic neutrosophic multiplicity number improved-generalized-weighted Heronian mean.

LNMNGWHM or LNMNIGWHM operator for decision-making process

Step 1: Using LNMNGWHM operator to aggregate the LNMNs (suppose $d = f = 1$), we can obtain the comprehensive evaluation values $c_r(r = 1, 2, 3, 4)$, which are as follows

$$c_1 = \langle\theta_{8.0000}, \theta_{6.2424}, \theta_{6.1509}\rangle, c_2 = \langle\theta_{8.0000}, \theta_{6.5262}, \theta_{5.7796}\rangle$$

$$c_3 = \langle\theta_{8.0000}, \theta_{5.5042}, \theta_{5.3498}\rangle, c_4 = \langle\theta_{8.0000}, \theta_{6.5826}, \theta_{4.7603}\rangle$$

Step 2: Calculate the score value $O(c_r)$ according to equation (5) for $c_r(r = 1, 2, 3, 4)$: $O(c_1) = 0.4836$, $O(c_2) = 0.4872$, $O(c_3) = 0.5478$, and $O(c_4) = 0.5274$.
Step 3: According to the score value $O(c_r)$ $(r = 1, 2, 3, 4)$, we rank $c_3 > c_4 > c_2 > c_1$, so c_3 is the best selection among all the alternatives.
Step 4: End.

Now, the LNMNIGWHM operator (suppose $d = f = 1$) is used to handle it.

Step 1: Using LNMNIGWHM operator to aggregate the LNMNs (suppose $d = f = 1$), we can obtain the comprehensive evaluation values $c_r(r = 1, 2, 3, 4)$, which are as follows

$$c_1 = \langle\theta_{8.0000}, \theta_{3.0336}, \theta_{2.7335}\rangle, c_2 = \langle\theta_{8.0000}, \theta_{4.4668}, \theta_{2.1337}\rangle$$

$$c_3 = \langle\theta_{8.0000}, \theta_{1.7231}, \theta_{1.6239}\rangle, c_4 = \langle\theta_{8.0000}, \theta_{4.2422}, \theta_{0.8627}\rangle$$

Step 2: Calculate the score value $O(c_r)$ according to equation (5) for $c_r(r = 1, 2, 3, 4)$: $O(c_1) = 0.7597$, $O(c_2) = 0.7250$, $O(c_3) = 0.8605$, and $O(c_4) = 0.7873$.
Step 3: According to the score value $O(c_r)$ $(r = 1, 2, 3, 4)$, we rank $c_3 > c_4 > c_1 > c_2$, so c_3 is the best selection among all the alternatives.
Step 4: End.

Set d and f with different values

Different parameters *d* and *f* may have distinct effects on the decision results. In this section, the paper will set *d* and *f* with different values and the ranking results can be seen in Tables 2 and 3.

The data in Tables 2 and 3 indicate that the ranking results are almost the same when d and f are taken different values. If either d or f equals zero, these two proposed operators cannot gain the interrelationships of individual arguments; therefore, the values of parameters d and f are usually set as $d = f = 1$.

Comparative analysis

Now, we use the method proposed in the literature[28] for LNM

$$LNMNWAA(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} w_i q_i =$$
$$\left\langle \theta_{\varrho\left(1-\prod_{i=1}^{n}\left(\prod_{t=1}^{y}\left(1-\frac{\mu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{w_i}\right)}, \theta_{\varrho\left(\prod_{i=1}^{n}\left(\prod_{t=1}^{y}\left(\frac{\nu_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{w_i}\right)}, \theta_{\varrho\left(\prod_{i=1}^{n}\left(\prod_{t=1}^{y}\left(\frac{\xi_{q_1 t}}{\varrho}\right)^{h_{q_1 t}}\right)^{w_i}\right)} \right\rangle \quad (15)$$

Its proof is similar to the literature,[28] so we omit it.

Step 1: Using LNMNWAA operator to aggregate the LNMNs, we can obtain the comprehensive evaluation values $c_r (r = 1, 2, 3, 4)$, which are as follows

$$c_1 = \langle \theta_{8.0000}, \theta_{2.8581}, \theta_{2.4499} \rangle, c_2 = \langle \theta_{8.0000}, \theta_{2.5116}, \theta_{1.7723} \rangle$$

$$c_3 = \langle \theta_{8.0000}, \theta_{1.5355}, \theta_{1.3875} \rangle, c_4 = \langle \theta_{8.0000}, \theta_{3.0899}, \theta_{0.7050} \rangle$$

Step 2: Calculate the score value $O(c_r)$ according to equation (5) for $c_r (r = 1, 2, 3, 4)$: $O(c_1) = 0.7788$, $O(c_2) = 0.8215$, $O(c_3) = 0.8782$, and $O(c_4) = 0.8419$

Step 3: According to the score value $O(c_r)$ $(r = 1, 2, 3, 4)$, we rank $c_3 > c_4 > c_2 > c_1$, so c_3 is the best selection among all the alternatives.

Step 4: End.

Compared with the operators of literature,[28] LNMNGWHM and LNMNIGWHM operators can express and handle the multiplicity problems. Compared with the operators of the literature,[33] LNMNGWHM and LNMNIGWHM operators can not only express the pure linguistic values, but also can get more complicated results through considering the interrelationships and prioritizations of the attributes. However, they make the result more practical.

Conclusion

This article proposed the LNMNGWHM and LNMNIGWHM operators for LNMNs. First, we introduced an LNM using pure linguistic value to express the multiplicity information, proposed LNMNGWHM operator and LNMNIGWHM operator, and discussed their related properties. Second, we proposed two methods of MADM based on LNMNGWHM operator and LNMNIGWHM operator in LNMN environment, and the main advantage of these two methods is using pure linguistic values to express and handle the multiplicity problems. Finally, we used these two methods to solve the practical problem and took the different values of d and f to analyze the ranking results. Results demonstrated that whatever values d and f were taken, ranking results were the same, so d and f made little influence on ranking results. Furthermore, we compared these two proposed methods with literature[28,33] and found that their results were consistent. Therefore, the proposed methods can make the decision-making results more reliable, and they are practical in real-life situations. In the future, we will make greater efforts to study linguistic NM and apply it to many other fields, such as strategic planning, big data analysis, e-government website evaluation, analytic hierarchy process, smart city.[36–42]

Author contributions

C.F. and K.H. originally proposed the LNMNGWHM and LNNIGWHM operators and investigated their properties; E.F., S.F., and J.Y. provided the calculation and analyzed the data; C.F. wrote the paper.

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References

1. Yager RR. On ordered weighted averaging aggregation operators in multi criteria decision making. *IEEE T Syst Man Cyb* 1988; 18(1): 183–190.

2. Chiclana F, Herrera F and Herrera-Viedma E. The ordered weighted geometric operator: properties and applications. In: *Proceeding 8th international conference on information processing and management of uncertainty in knowledge-based systems*, Madrid, 3–7 July 2000, pp.985–991.
3. Xu ZS and Da QL. The ordered weighted geometric averaging operator. *Int J Intell Syst* 2002; 17: 709–716.
4. Chen HY, Liu CL and Sheng HH. The application of IOWHA operator in combination forecast. *Manage Sci China* 2004; 12(5): 35–40.
5. Yager RR. Generalized OWA aggregation operators. *Fuzzy Optim Decis Ma* 2004; 3: 93–107.
6. Bonferroni C. Sulle medie multiple di potenze. *Bollettino Matematica Italiana* 1950; 5(3): 267–270.
7. Zhou W and He JM. Intuitionistic fuzzy normalized weighted Bonferroni mean and its application in multicriteria decision making. *J Appl Math* 2012; 2012: 136254.
8. Yu DJ and Wu YY. Interval-valued intuitionistic fuzzy Heronian mean operators and their application in multicriteria decision making. *African J Business Manage* 2012; 6(11): 4158–4168.
9. Xu ZS and Da QL. The uncertain OWA operator. *Int J Intell Syst* 2002; 17(6): 569–575.
10. Xu ZS and Da QL. An uncertain ordered weighted geometric (UOWG) operator and its application. *Information* 2004; 7(2): 175–182.
11. Xu ZS. Uncertain Bonferroni mean operators. *Int J Comput Intell Syst* 2010; 3(6): 761–769.
12. Wang DX, Li ZQ and Liu WF. Interval Heronian mean average operator. *Pract Knowl Math* 2015; 45(21): 274–281.
13. Xu ZS. The triangular fuzzy number complementary judgment matrix sorting method of FOWA operator. *Syst Eng Theory Pract* 2003; 23(10): 86–89.
14. Xu ZS. One application of FOWG operator for fuzzy AHP. *Syst Eng Electr Technol* 2002; 24(7): 31–33.
15. Xu YJ and Da QL. The application of TFOWA operator for decision-making. *Journal of Dong Nan Univ* 2006; 36(6): 1034–1038.
16. Liu JP, Lin S and Chen HY. The application of fuzzy Bonferroni Mean operator for multi attributes decision-making. *Syst Eng Electr Technol* 2012; 34(1): 115–119.
17. Xu ZS. Intuitionistic fuzzy aggregation operator. *IEEE Trans Fuzzy Syst* 2007; 15(6): 1179–1187.
18. Xu ZS and Yager RR. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int J Gen Syst* 2006; 35(4): 417–433.
19. Xu ZS and Yager RR. Intuitionistic fuzzy Bonferroni means. *IEEE T Syst Man Cy B* 2011; 41(2): 568–578.
20. Liu PD and Teng F. Multiple attribute decision making method based on normal neutrosophic generalized weighted power averaging operator. *Int J Mach Learn Cyber* 2018; 9: 281–293.
21. Liu PD and Li HG. Multiple attribute decision-making method based on some normal neutrosophic Bonferroni mean operators. *Neur Comput Appl* 2017; 28(1): 179–194.
22. Li YH, Liu PD and Chen YB. Some single valued neutrosophic number Heronian mean operators and their application in multiple attribute group decision making. *Informatica* 2016; 27(1): 85–110.
23. Bordogna G, Fedrizzi M and Passi G. A linguistic modeling of consensus in group decision making on OWA operator. *IEEE T Syst Man Cy A* 1997; 27(1): 126–132.
24. Xu ZS. EOWA and EOWG operators for aggregation linguistic labels based on linguistic preference relations. *Int J Uncertain Fuzz* 2004; 12(6): 791–810.
25. Xu ZS. Induced uncertain linguistic OWA operators applied to group decision making. *Inform Fusion* 2006; 7(2): 231–238.
26. Liu PD, Liu ZM and Zhang X. Some intuitionistic uncertain linguistic Heronian mean operators and their application to group decision making. *Appl Math Comput* 2014; 230: 570–586.
27. Ye J. Aggregation operators of neutrosophic linguistic numbers for multiple attribute group decision making. *SpringerPlus* 2016; 5: 1691.
28. Fang ZB and Ye J. Multiple attribute group decision-making method based on linguistic neutrosophic numbers. *Symmetry* 2017; 9: 111.
29. Ma YX, Wang JQ, Wang J, et al. An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options. *Neural Comput Appl* 2017; 28: 2745–2765.
30. Fan CX, Ye J, Hu KL, et al. Bonferroni mean operators of linguistic neutrosophic numbers and their multiple attribute group decision-making methods. *Information* 2017; 8(3): 107.
31. Liu P and Shi L. Some neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. *Neural Comput Appl* 2017; 28(5): 1079–1093.
32. Smarandache F. *Neutrosophic perspectives: triplets, duplets, multisets, hybrid operators, modal logic, hedge algebras and applications*. Brussels: PonsEditions, 2017, pp.115–123.
33. Fan CX, Fan E and Ye J. The cosine measure of single-valued neutrosophic multisets for multiple attribute decision-making. *Symmetry* 2018; 10: 154.
34. Ye J. Correlation coefficient between dynamic single valued neutrosophic multisets and its multiple attribute decision-making method. *Information* 2017; 8(2): 41.
35. Liu PD. The evaluation method and application research of enterprise information level based on fuzzy multi-attribute decision-making. Beijing, China: Beijing Jiaotong University, 2010, pp.113–129.
36. Abdel-Basset M, Mai M and Smarandache F. An extension of neutrosophic AHP–SWOT analysis for strategic planning and decision-making. *Symmetry* 2018; 10(4): 18.
37. Abdel-Basset M, Mai M, Smarandache F, et al. Neutrosophic association rule mining algorithm for big data analysis. *Symmetry* 2018; 10(4): 106.
38. Abdel-Basset M, Zhou Y, Mai M, et al. A group decision making framework based on neutrosophic VIKOR approach for e-government website evaluation. *J Intell Fuzz Syst* 2018; 34(6): 4213–4224.
39. Abdel-Basset M, Mai M and Zhou YQ. Multi-criteria group decision making based on neutrosophic analytic hierarchy process. *J Intell Fuzz Syst* 2017; 33(6): 4055–4066.
40. Abdel-Basset M and Mai M. The role of single valued neutrosophic sets and rough sets in smart city: imperfect

and incomplete information systems. *Measurement* 2018; 124: 55–47.

41. Abdel-Basset M, Manogaran G, Gamal A, et al. A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. *Design Autom Embed Syst* 2018; 4: 1–22.
42. Abdel-Basset M, Gunasekaran M, Mai M, et al. Three-way decisions based on neutrosophic sets and AHP-QFD framework for supplier selection problem. *Future Gener Comp Syst* 2018; 89: 19–30.