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Hesitant Bipolar-valued Neutrosophic set: Formulation, Theory and Application

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ABSTRACT This paper proposes a hesitant bipolar-valued neutrosophic set (HBVNS) based on the combination of bipolar neutrosophic sets and hesitant fuzzy sets. The proposed set generalizes the notions of fuzzy set, intuitionistic fuzzy set, hesitant fuzzy set, single-valued neutrosophic set, single-valued neutrosophic hesitant fuzzy set, bipolar fuzzy set and bipolar neutrosophic set. Further, we define the basic operational laws, union, intersection and complement for hesitant bipolar-valued neutrosophic elements (HBVNEs) and study its associated properties. Some relevant examples are also given to provide a better understanding of the proposed concept. Two aggregation operators are developed based on HBVNS which are the hesitant bipolar-valued neutrosophic weighted averaging (HBVNWA) and the hesitant bipolar-valued neutrosophic weighted geometric (HBVNWG). A decision making method is developed based on new sets and the proposed HBVNWA and HBVNWG operators. Finally, an illustrative example is given to show the applicability of the proposed decision making method. A comparative analysis with the existing methods is also provided.

INDEX TERMS Decision making, fuzzy set, hesitant bipolar-valued neutrosophic set, neutrosophic set.

I. INTRODUCTION

The neutrosophic set (NS) [1] is a set that is characterized by three independent functions which are the truthfulness, indeterminacy and falsity functions. Its theories and applications have been rapidly explored by scholars and researchers in various research domains. Sumathi and Arockiarani [2] studied the notion of neutrosophic normal subgroup and neutrosophic topological groups. Patro and Smarandache [3] presented neutrosophic statistics versions of binomial and normal distributions. Kundu and Islam [4] developed a goal geometric programming method based on neutrosophic set. Malik et al. [5] proposed the concept of soft rough NSs to graph theory and introduced several new graphs. The neutrosophic set which was first introduced based on the philosophical point of view has been extended to more approachable sets and theories that are applicable in the real multi-criteria decision-making problems [6]–[8]. Some of the well-known particular cases of neutrosophic sets are the single-valued neutrosophic set [9], interval-valued neutrosophic set [10], simplified neutrosophic set [11], neutrosophic soft set [12], multi-valued neutrosophic set [13], complex neutrosophic set [14] and many hybrids neutrosophic sets [15]–[19]. However, truthfulness, indeterminacy and falsity are not exhaustive in characterizing uncertain and vague information. Elements of bipolarity also exist particularly in the real decision-making problems where multiple attributes and alternatives are considered simultaneously. Bipolarity is the tendency of the human brain to make decision based on the good and bad sides which represented by positive and negative values respectively. Zhang [20] introduced the idea of bipolarity into fuzzy set which known as bipolar fuzzy set (BFS). According to Zhang [20], almost all the decision making are based on both, positive and negative sides or so called the bipolar judgmental thinking. For example, when we ask a decision maker about a statement, he or she may state any number in the interval $[0,1]$ represents the satisfaction degree of the statement associated to the fuzzy set and $[-1,0]$ that represents the satisfaction degree of the counter-property related to the fuzzy set. Positive information reflects what is guaranteed possible and preferred while negative information reflects what is surely forbidden which are known based on observation and experience.

This concept has recently investigated among researchers in various fields including the decision making. For instance, Wei et al. [21] proposed the interval-valued bipolar fuzzy set with its operational laws, score and accuracy functions and some aggregation operators based on Hamacher operations. Akram [22] introduced the notion of bipolarity in the fuzzy graphs and investigated the isomorphism of the graphs with its properties. Malik and Shabir [23] employed the bipolar information in its set development and introduced the rough fuzzy bipolar soft set with its application to the decision-making problem. Besides that, Wei et al. [21] solved the multi-attribute decision-making (MADM) problem of emerging technology commercialization under the interval-valued bipolar fuzzy environment. Deli et al. [24] introduced bipolar neutrosophic sets for multi-criteria decision making problems. Surapati and Mondal [16] developed a new set combining the bipolar neutrosophic set theory and the rough neutrosophic set theory called the rough bipolar neutrosophic set and presented the union, intersection, complement and inclusion of the set. Princy and Mohana [25] introduced the neutrosophic bipolar vague set and developed a multi-criteria decision making (MCDM) problem based on the neutrosophic bipolar vague set.

Despite of its established capabilities, bipolar neutrosophic set is unable to handle the hesitancy information that occurs in most of the decision process. Due to uncertainty and the nature of human brain, it may be difficult for decision makers to provide an exact bipolar neutrosophic number or linguistic variable. For instance, a decision maker is asked about a statement and he or she needs to provide a judgment about the statement with five options of linguistic variables namely, 'very good', 'good', 'medium', 'poor' and 'very poor'. The tendency of the decision maker to be confused between two close linguistic variables is high such as between 'very good' and 'good', 'good' and 'medium', 'poor' and 'very poor' and so on. The hesitancy in making a concrete judgment is normal for human being. Therefore, by utilizing the concept of hesitancy, decision makers are not restricted to state only one linguistic variable or a single value. Recently, many researchers have employed the hesitancy information into the development of methods to deal with the MCDM problems [26]–[29].

In fact, bipolarity and hesitancy are two different but complementary concepts that are created to model an easy-going but accurate judgments when dealing with the imprecision and uncertainty in bipolar neutrosophic information. For this purpose, this paper attempts in developing a new set combining the bipolar neutrosophic set (BNS) and hesitant fuzzy set (HFS). The hesitant bipolar-valued neutrosophic set (HBVNS) is implemented to obtain the optimal decision for any decision making problems involving bipolarity, hesitancy and indeterminacy information. The basic operations for HBVNS are presented including the union, intersection, complement, equality and inclusion. The properties related to the introduced definitions are also investigated. The outline of this paper is arranged as follows. Section 2 presents the necessary definitions and concepts for developing the HBVNS set. Section 3 introduces the HBVNS and its properties. Then, a MADM method is developed on the basis of HBVNS in section 4. A numerical example is given in section 6 with the comparison analysis. Finally, section 5 concludes the paper.

II. PRELIMINARIES

This section provides the related definitions that will be used in the development of the proposed set.

A. BIPOLAR NEUTROSOPHIC SET (BNS)

Definition 1. [1] Let V be a universe discourse and v is a generic element in V. A neutrosophic set (NS) R is defined as below.

$$R=\left\{\left\langle v,T_R\left(v\right),I_R\left(v\right),F_R\left(v\right)\right\rangle\middle| v\in V\right\}$$

where $T_R\left(v\right)$, $I_R\left(v\right)$ and $F_R\left(v\right)$ are the degree of truthiness, indeterminacy and falsity respectively. The degrees $T_R\left(v\right)$, $I_R\left(v\right)$ and $F_R\left(v\right)$ are the real standard subsets of $\left]0^-,1^+\right[$. That is, $T_R\left(v\right),I_R\left(v\right),F_R\left(v\right)\to\left]0^-,1^+\right[$ and the sum of $T_R\left(v\right)$, $I_R\left(v\right)$ and $F_R\left(v\right)$ is $0^-\le T_R\left(v\right)+I_R\left(v\right)+F_R\left(v\right)\le 3^+$.

Definition 2. [24] Let V be a universe discourse and a BNS $\mathbb{Q}$ is defined as below.

$$\mathbb{Q}=\left\{\left\langle v,T^+_{\mathbb{Q}}(v),I^+_{\mathbb{Q}}\left(v\right),F^+_{\mathbb{Q}}\left(v\right),T^-_{\mathbb{Q}}\left(v\right),I^-_{\mathbb{Q}}\left(v\right),F^-_{\mathbb{Q}}\left(v\right)\right\rangle\middle| v\in V\right\}$$

where $T^+_{\mathbb{Q}}(v),I^+_{\mathbb{Q}}\left(v\right),F^+_{\mathbb{Q}}\left(v\right)\to\left[0,1\right]$ and $T^-_{\mathbb{Q}}\left(v\right),I^-_{\mathbb{Q}}\left(v\right),F^-_{\mathbb{Q}}\left(v\right)\to\left[-1,0\right]$. The positive membership functions of $T^+_{\mathbb{Q}}(v),I^+_{\mathbb{Q}}\left(v\right)$ and $F^+_{\mathbb{Q}}\left(v\right)$ are the truth-membership, indeterminacy-membership and falsity-membership degrees of an element $v\in V$ corresponding to a BNS $\mathbb{Q}$ and the negative membership functions of $T^-_{\mathbb{Q}}\left(v\right),I^-_{\mathbb{Q}}\left(v\right)$ and $F^-_{\mathbb{Q}}\left(v\right)$ are the degree of truth, indeterminacy and falsity of an element $v\in V$ to some implicit counter-property associated to a BNS $\mathbb{Q}$. A bipolar neutrosophic number (BNN) is denoted by $\mathbb{Q}=\left\langle T^+,I^+,F^+,T^-,I^-,F^-\right\rangle$ for simplicity.

Definition 3. [24] Let $\mathbb{Q}_1=\left\langle T_1^+,I_1^+,F_1^+,T_1^-,I_1^-,F_1^-\right\rangle$ and $\mathbb{Q}_2=\left\langle T_2^+,I_2^+,F_2^+,T_2^-,I_2^-,F_2^-\right\rangle$ be two BNNs. Then

i) Inclusion

$\mathbb{Q}_1\subseteq\mathbb{Q}_2$, if and only if $T_1^+\le T_2^+$, $I_1^+\le I_2^+$, $F_1^+\ge F_2^+$ and $T_1^-\ge T_2^-$, $I_1^-\ge I_2^-$, $F_1^-\le F_2^-$ for all $v\in V$.

ii) Equality

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$\mathbb{Q}_1 = \mathbb{Q}_2$, if and only if $T_1^+ = T_2^+$, $I_1^+ = I_2^+$, $F_1^+ = F_2^+$ and $T_1^- = T_2^-$, $I_1^- = I_2^-$, $F_1^- = F_2^-$ for all $v \in V$.

iii) Complement

The complement of $\mathbb{Q}_1$ is denoted by $\mathbb{Q}_1^c$ and defined as follows:

$$\mathbb{Q}_1^c = \left\langle 1-T_1^+, 1-I_1^+, 1-F_1^+, -1-T_1^-, -1-I_1^-, -1-F_1^- \right\rangle$$

iv) Union

The union of two BNSs is defined by

$$\mathbb{Q}_1 \cup \mathbb{Q}_2 = \left\langle \max\left(T_1^+, T_2^+\right), \frac{I_1^+ + I_2^+}{2}, \min\left(F_1^+, F_2^+\right), \min\left(T_1^-, T_2^-\right), \frac{I_1^- + I_2^-}{2}, \max\left(F_1^-, F_2^-\right) \right.$$

v) Intersection

The intersection of two BNSs is defined by

$$\mathbb{Q}_1 \cap \mathbb{Q}_2 = \left\langle \min\left(T_1^+, T_2^+\right), \frac{I_1^+ + I_2^+}{2}, \max\left(F_1^+, F_2^+\right), \max\left(T_1^-, T_2^-\right), \frac{I_1^- + I_2^-}{2}, \min\left(F_1^-, F_2^-\right) \right\rangle$$

Definition 4. [24] Let $\mathbb{Q}_1 = \left\langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^- \right\rangle$ and $\mathbb{Q}_2 = \left\langle T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^- \right\rangle$ be two BNNs. Then some operations for BNNs are as follows:

i) Addition

$$\mathbb{Q}_1 \oplus \mathbb{Q}_2 = \left\langle T_1^+ + T_2^+ - T_1^+ \cdot T_2^+, I_1^+ \cdot I_2^+, F_1^+ \cdot F_2^+, -T_1^- \cdot T_2^-, -\left(-I_1^- - I_2^- - I_1^- \cdot I_2^-\right), -\left(-F_1^- - F_2^- - F_1^- \cdot F_2^-\right) \right\rangle$$

ii) Product

$$\mathbb{Q}_1 \otimes \mathbb{Q}_2 = \left\langle T_1^+ \cdot T_2^+, I_1^+ + I_2^+ - I_1^+ \cdot I_2^+, F_1^+ + F_2^+ - F_1^+ \cdot F_2^+, -\left(-T_1^- - T_2^- - T_1^- \cdot T_2^-\right), -I_1^- \cdot I_2^-, -F_1^- \cdot F_2^- \right\rangle$$

iii) Power

$$\left(\mathbb{Q}_1\right)^\rho = \left\langle \left(T_1^+\right)^\rho, 1-\left(1-I_1^+\right)^\rho, 1-\left(1-F_1^+\right)^\rho, -\left(1-\left(1-\left(-T_1^-\right)\right)^\rho\right), -\left(-I_1^-\right)^\rho, -\left(-F_1^-\right)^\rho \right\rangle$$

where $\rho > 0$.

iv) Scalar Multiplication

$$\rho\left(\mathbb{Q}_1\right) = \left\langle 1-\left(1-T_1^+\right)^\rho, \left(I_1^+\right)^\rho, \left(F_1^+\right)^\rho, -\left(-T_1^-\right)^\rho, -\left(-I_1^-\right)^\rho, -\left(1-\left(1-\left(-F_1^-\right)\right)^\rho\right) \right\rangle$$

where $\rho > 0$.

Note: The scalar multiplication of BNSs is noticed to have a bit mistake. Thus, we give a revised definition of the scalar multiplication for BNSs as follows:

$$\rho\left(\mathbb{Q}_1\right) = \left\langle 1-\left(1-T_1^+\right)^\rho, \left(I_1^+\right)^\rho, \left(F_1^+\right)^\rho, -\left(-T_1^-\right)^\rho, -\left(1-\left(1-\left(-I_1^-\right)\right)^\rho\right), -\left(1-\left(1-\left(-F_1^-\right)\right)^\rho\right) \right\rangle \quad (1)$$

B. HESITANT FUZZY SET (HFS)

The concept of hesitancy was first proposed by Torra [30] in fuzzy set. The definition of hesitant fuzzy set (HFS) is as below.

Definition 5. [30] Let V be a reference set and a HFS H on V is defined as:

$$H = \left\{ \left\langle v, h_H(v) \right\rangle \middle| v \in V \right\},$$

where $h = h_H(v)$ denoted as the hesitant fuzzy element (HFE) and the values are in the range of $[0,1]$ and reflect the possible membership values of element $v \in V$ of the set H.

Definition 6. [30] Let h, h_1 and h_2 be three HFEs. Then

i) Complement

$$h^c = \bigcup_{\gamma \in h} \{1-\gamma\},$$

ii) Union

$$h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\},$$

iii) Intersection

$$h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}.$$

Definition 7. [30] Let h, h_1 and h_2 be three HFEs and $\rho > 0$. Then, the operations for HFEs are given as below:

i) Addition

$$h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \cdot \gamma_2\}.$$

ii) Product

$$h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \cdot \gamma_2\}.$$

iii) Power

$$h^\rho = \bigcup_{\gamma \in h} \left\{ (\gamma)^\rho \right\}.$$

iv) Scalar Multiplication

$$\rho h = \bigcup_{\gamma \in h} \left\{ 1-(1-\gamma)^\rho \right\}.$$

III. PROPOSED HESITANT BIPOLAR-VALUED NEUTROSOPHIC SET (HBVNS)

In this section, the hesitant bipolar-valued neutrosophic set (HBVNS) is introduced. Then some basic properties and operational laws are investigated for the proposed HBVNS.

Definition 8. Let V be a reference set and with a generic element in V denoted by v. A hesitant bipolar-valued neutrosophic set $\hat{H}$ in X is defined as:

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$$\hat{H}=\left\{v,\left\langle h_T^+(v),h_I^+(v),h_F^+(v),h_T^-(v),h_I^-(v),h_F^-(v)\right\rangle \mid v\in V\right\},$$

where $h_T^+(v),h_I^+(v),h_F^+(v):V\to[0,1]$ and $h_T^-(v),h_I^-(v),h_F^-(v):V\to[-1,0]$. The positive hesitant bipolar-valued neutrosophic elements h_T^+, h_I^+ and h_F^+ denotes the possible satisfactory degree of truth, indeterminacy and falsity of an element $v\in V$ corresponding to a HBVNS $\hat{H}$ respectively while the negative hesitant bipolar-valued neutrosophic elements h_T^-, h_I^- and h_F^- denote the possible satisfactory degree of truth, indeterminacy and falsity of an element $v\in V$ to the implicit counter property to the set $\hat{H}$ respectively. In addition, a HBVNS $\hat{H}$ must satisfy the conditions $0\le\gamma_T^+,\gamma_I^+,\gamma_F^+\le 1$, $-1\le\gamma_T^-,\gamma_I^-,\gamma_F^-\le 0$, $0\le\max\{\gamma_T^+\}+\max\{\gamma_I^+\}+\max\{\gamma_F^+\}\le 3$, and $-3\le\max\{\gamma_T^-\}+\max\{\gamma_I^-\}+\max\{\gamma_F^-\}\le 0$.

In which $\gamma_T^+\in h_T^+(v),\gamma_I^+\in h_I^+(v)$, $\gamma_F^+\in h_F^+(v)$, $\gamma_T^-\in h_T^-(v)$, $\gamma_I^-\in h_I^-(v)$ and $\gamma_F^-\in h_F^-(v)$ for $v\in V$.

For convenience, we use the symbol $\hat{H}$ to represent all the hesitant bipolar-valued neutrosophic sets and $h=\left\langle h_T^+,h_I^+,h_F^+,h_T^-,h_I^-,h_F^-\right\rangle$ for a hesitant bipolar-valued neutrosophic element (HBVNE).

Example 1. Let $V=\{v_1,v_2,v_3\}$. Then

$$\hat{H}=\left\{\begin{array}{c}\langle v_1,\langle\{0.4,0.5\},0.3,0.2,-0.6,-0.4,\{-0.1,-0.2\}\rangle\rangle\\ \langle v_2,\langle 0.3,\{0.2,0.3\},0.7,\{-0.2,-0.4\},-0.1,-0.7\rangle\rangle\\ \langle v_3,\langle 0.8,0.3,0.4,-0.1,-0.4,\{-0.7,-0.8\}\rangle\rangle\end{array}\right\}$$

is a hesitant bipolar-valued neutrosophic elements in *V*.

Proposition 1. A HBVNS is a generalization of BNS

Proof. Suppose that the number of elements in each $h_T^+,h_I^+,h_F^+,h_T^-,h_I^-$ and h_F^- is one, then HBVNS is reduce to BNS.

The definitions of complement, union and intersection for HBVNEs are given as below:

Definition 9. Let $h=\left\langle h_T^+,h_I^+,h_F^+,h_T^-,h_I^-,h_F^-\right\rangle$, $h_1=\left\langle h_{T_1}^+,h_{I_1}^+,h_{F_1}^+,h_{T_1}^-,h_{I_1}^-,h_{F_1}^-\right\rangle$ and $h_2=\left\langle h_{T_2}^+,h_{I_2}^+,h_{F_2}^+,h_{T_2}^-,h_{I_2}^-,h_{F_2}^-\right\rangle$ be three HBVNEs. Then,

i) Inclusion

$h_1\subseteq h_2$, if and only if $\forall\gamma_{T_1}^+\le\forall\gamma_{T_2}^+$, $\forall\gamma_{I_1}^+\le\forall\gamma_{I_2}^+$, $\forall\gamma_{F_1}^+\ge\forall\gamma_{F_2}^+$ and $\forall y_{\Gamma_1}^-\ge\forall y_{\Gamma_2}^-$, $\forall\gamma_{I_1}^-\ge\forall\gamma_{I_2}^-$, $\forall\gamma_{F_1}^-\le\forall\gamma_{F_2}^-$ for all $v\in V$.

ii) Equality

$h_1=h_2$, if and only if $\forall\gamma_{T_1}^+=\forall\gamma_{T_2}^+$, $\forall\gamma_{I_1}^+=\forall\gamma_{I_2}^+$, $\forall\gamma_{F_1}^+=\forall\gamma_{F_2}^+$ and $\forall\gamma_{T_1}^-=\forall\gamma_{T_2}^-$, $\forall\gamma_{I_1}^-=\forall\gamma_{I_2}^-$, $\forall\gamma_{F_1}^-=\forall\gamma_{F_2}^-$ for all $v\in V$.

iii) Complement

The complement of λ is denoted by λ^c and defined as follows:

$$h^c=\left\langle\bigcup_{\gamma_T^+\in h_T^+}\{1-\gamma_T^+\},\bigcup_{\gamma_I^+\in h_I^+}\{1-\gamma_I^+\},\bigcup_{\gamma_F^+\in h_F^+}\{1-\gamma_F^+\},\bigcup_{\gamma_T^-\in h_T^-}\{-1-\gamma_T^-\},\bigcup_{\gamma_I^-\in h_I^-}\{-1-\gamma_I^-\},\bigcup_{\gamma_F^-\in h_F^-}\{-1-\gamma_F^-\}\right\rangle$$

iv) Union

The union of two HBVNEs is defined by

$$h_1\cup h_2=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\max\{\gamma_{T_1}^+,\gamma_{T_2}^+\},\bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\frac{\gamma_{I_1}^++\gamma_{I_2}^+}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\min\{\gamma_{F_1}^+,\gamma_{F_2}^+\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\min\{\gamma_{T_1}^-,\gamma_{T_2}^-\},\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{\frac{\gamma_{I_1}^-+\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\max\{\gamma_{F_1}^-,\gamma_{F_2}^-\}\right\rangle$$

v) Intersection

The intersection of two HBVNEs is defined by

$$h_1\cap h_2=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\min\{\gamma_{T_1}^+,\gamma_{T_2}^+\},\bigcup_{\substack{\gamma_{I_1}^+\in h_{I_1}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\frac{\gamma_{I_1}^++\gamma_{I_2}^+}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\max\{\gamma_{F_1}^+,\gamma_{F_2}^+\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\max\{\gamma_{T_1}^-,\gamma_{T_2}^-\},\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{\frac{\gamma_{I_1}^-+\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\min\{\gamma_{F_1}^-,\gamma_{F_2}^-\}\right\rangle$$

Example 2. Let

$h=\langle 0.5,0.4,\{0.45,0.5\},\{-0.8,-0.7\},-0.1,-0.3\rangle$

$h_1=\langle\{0.35,0.4\},0.6,0.7,\{-0.9,-0.8\},-0.1,\{-0.3,-0.1\}\rangle$

and

$h_2=\langle 0.8,0.2,0.2,\{-0.6,-0.5\},-0.3,-0.5\rangle$ are three HBVNEs, then

i) Complement

$h^c=$ $\langle 0.5,0.6,\{0.55,0.5\},\{-0.2,-0.3\},-0.9,-0.7\rangle$

ii) Union

$h_1\cup h_2=\langle 0.8,0.4,0.2,\{-0.9,-0.8\},-0.2,-0.1\rangle$

iii) Intersection

$h_1\cap h_2=$

$\langle \{0.35, 0.4\}, 0.4, 0.7, \{-0.6, -0.5\}, -0.2, -0.5 \rangle$

Proposition 2. Given three HBVNEs, h, h_1 and h_2, then

1) $h \cup h = h$
2) $h \cap h = h$
3) $h_1 \cup h_2 = h_2 \cup h_1$
4) $h_1 \cap h_2 = h_2 \cap h_1$
5) $h \cup (h_1 \cap h_2) = (h \cup h_1) \cap (h \cup h_2)$
6) $h \cap (h_1 \cup h_2) = (h \cap h_1) \cup (h \cap h_2)$
7) $(h^c)^c = h$
8) $(h_1 \cup h_2)^c = h_1^c \cap h_2^c$
9) $(h_1 \cap h_2)^c = h_1^c \cup h_2^c$
10) $(h^c \cup h_1^c \cup h_2^c) = (h \cap h_1 \cap h_2)^c$
11) $(h^c \cap h_1^c \cap h_2^c) = (h \cup h_1 \cup h_2)^c$

Proof 1):

Let $h = \langle h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^- \rangle$,

$$h \cup h = \left\langle \bigcup_{\gamma_T^+ \in h_T^+} \max\{\gamma_T^+, \gamma_T^+\}, \bigcup_{\gamma_I^+ \in h_I^+} \left\{\frac{\gamma_I^+ + \gamma_I^+}{2}\right\}, \bigcup_{\gamma_F^+ \in h_F^+} \min\{\gamma_F^+, \gamma_F^+\}, \bigcup_{\gamma_T^- \in h_T^-} \min\{\gamma_T^-, \gamma_T^-\}, \bigcup_{\gamma_I^- \in h_I^-} \left\{\frac{\gamma_I^- + \gamma_I^-}{2}\right\}, \bigcup_{\gamma_F^- \in h_F^-} \max\{\gamma_F^-, \gamma_F^-\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma_T^+ \in h_T^+} \gamma_T^+, \bigcup_{\gamma_I^+ \in h_I^+} \gamma_I^+, \bigcup_{\gamma_F^+ \in h_F^+} \gamma_F^+, \bigcup_{\gamma_T^- \in h_T^-} \gamma_T^-, \bigcup_{\gamma_I^- \in h_I^-} \gamma_I^-, \bigcup_{\gamma_F^- \in h_F^-} \gamma_F^- \right\rangle$$

$$\Rightarrow \langle h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^- \rangle = h$$ □

Proof 2): Similar to the proof of 1) and thus omitted.

Proof 3):

Let $h_1 = \langle h_{T_1}^+, h_{I_1}^+, h_{F_1}^+, h_{T_1}^-, h_{I_1}^-, h_{F_1}^- \rangle$ and $h_2 = \langle h_{T_2}^+, h_{I_2}^+, h_{F_2}^+, h_{T_2}^-, h_{I_2}^-, h_{F_2}^- \rangle$ be two HBVNEs. Then,

$$h_1 \cup h_2 = \left\langle \bigcup_{\substack{\gamma_{T_1}^+ \in h_{T_1}^+, \\ \gamma_{T_2}^+ \in h_{T_2}^+}} \max\{\gamma_{T_1}^+, \gamma_{T_2}^+\}, \bigcup_{\substack{\gamma_{I_1}^+ \in h_{I_1}^+, \\ \gamma_{I_2}^+ \in h_{I_2}^+}} \left\{\frac{\gamma_{I_1}^+ + \gamma_{I_2}^+}{2}\right\}, \bigcup_{\substack{\gamma_{F_1}^+ \in h_{F_1}^+, \\ \gamma_{F_2}^+ \in h_{F_2}^+}} \min\{\gamma_{F_1}^+, \gamma_{F_2}^+\}, \bigcup_{\substack{\gamma_{T_1}^- \in h_{T_1}^-, \\ \gamma_{T_2}^- \in h_{T_2}^-}} \min\{\gamma_{T_1}^-, \gamma_{T_2}^-\}, \bigcup_{\substack{\gamma_{I_1}^- \in h_{I_1}^-, \\ \gamma_{I_2}^- \in h_{I_2}^-}} \left\{\frac{\gamma_{I_1}^- + \gamma_{I_2}^-}{2}\right\}, \bigcup_{\substack{\gamma_{F_1}^- \in h_{F_1}^-, \\ \gamma_{F_2}^- \in h_{F_2}^-}} \max\{\gamma_{F_1}^-, \gamma_{F_2}^-\} \right\rangle$$

$$h_2 \cup h_1 = \left\langle \bigcup_{\substack{\gamma_{T_1}^+ \in h_{T_1}^+, \\ \gamma_{T_2}^+ \in h_{T_2}^+}} \max\{\gamma_{T_2}^+, \gamma_{T_1}^+\}, \bigcup_{\substack{\gamma_{I_1}^+ \in h_{I_1}^+, \\ \gamma_{I_2}^+ \in h_{I_2}^+}} \left\{\frac{\gamma_{I_2}^+ + \gamma_{I_1}^+}{2}\right\}, \bigcup_{\substack{\gamma_{F_1}^+ \in h_{F_1}^+, \\ \gamma_{F_2}^+ \in h_{F_2}^+}} \min\{\gamma_{F_2}^+, \gamma_{F_1}^+\}, \bigcup_{\substack{\gamma_{T_1}^- \in h_{T_1}^-, \\ \gamma_{T_2}^- \in h_{T_2}^-}} \min\{\gamma_{T_2}^-, \gamma_{T_1}^-\}, \bigcup_{\substack{\gamma_{I_1}^- \in h_{I_1}^-, \\ \gamma_{I_2}^- \in h_{I_2}^-}} \left\{\frac{\gamma_{I_2}^- + \gamma_{I_1}^-}{2}\right\}, \bigcup_{\substack{\gamma_{F_1}^- \in h_{F_1}^-, \\ \gamma_{F_2}^- \in h_{F_2}^-}} \max\{\gamma_{F_2}^-, \gamma_{F_1}^-\} \right\rangle$$

$$\therefore h_1 \cup h_2 = h_2 \cup h_1$$ □

Proof 4): Similar to the proof of 3) and thus omitted.

Proof 5):

Let $x \in h \cup (h_1 \cap h_2)$.

If $x \in h \cup (h_1 \cap h_2)$, then x is either in h or $(h_1 \cap h_2)$.

This means that $x \in h$ or $x \in (h_1 \cap h_2)$.

If $x \in h$ or $\{x \in h_1 \text{ and } x \in h_2\}$, then $\{x \in h \text{ or } x \in h_1\}$ and $\{x \in h \text{ or } x \in h_2\}$.

So we have,

$x \in h$ or h_1 and $x \in h$ or h_2

$x \in (h \cup h_1)$ and $x \in (h \cup h_2)$

$x \in (h \cup h_1) \cap (h \cup h_2)$

Hence, $h \cup (h_1 \cap h_2) \Rightarrow (h \cup h_1) \cap (h \cup h_2)$

Therefore,

$h \cup (h_1 \cap h_2) \subseteq (h \cup h_1) \cap (h \cup h_2)$

Let $x \in (h \cup h_1) \cap (h \cup h_2)$.

If $x \in (h \cup h_1) \cap (h \cup h_2)$, then x is in $(h \text{ or } h_1)$ and $(h \text{ or } h_2)$.

So we have,

$x \in (h \text{ or } h_1)$ and $x \in (h \text{ or } h_2)$

$\{x \in h \text{ or } x \in h_1\}$ and $\{x \in h \text{ or } x \in h_2\}$

$x \in h$ or $\{x \in h_1 \text{ and } x \in h_2\}$

$x \in h \cup \{x \in (h_1 \text{ and } h_2)\}$

$x \in h \cup \{x \in (h_1 \cap h_2)\}$

$x \in h \cup (h_1 \cap h_2)$

$\therefore h \cup (h_1 \cap h_2) = (h \cup h_1) \cap (h \cup h_2)$ □

Proof 6): Similar to the proof of 5) and thus omitted.

Proof 7):

Let $$h^c = \left\langle \bigcup_{\gamma_T^+ \in h_T^+} \{1 - \gamma_T^+\}, \bigcup_{\gamma_I^+ \in \lambda_I^+} \{1 - \gamma_I^+\}, \bigcup_{\gamma_F^+ \in h_F^+} \{1 - \gamma_F^+\}, \bigcup_{\gamma_T^- \in h_T^-} \{-1 - \gamma_T^-\}, \bigcup_{\gamma_I^- \in h_I^-} \{1 - \gamma_I^-\}, \bigcup_{\gamma_F^- \in h_F^-} \{1 - \gamma_F^-\} \right\rangle.$$

We have,

$$(h^c)^c = \left\langle \bigcup_{\gamma_T^+ \in h_T^+} \{1 - (1 - \gamma_T^+)\}, \bigcup_{\gamma_I^+ \in h_I^+} \{1 - (1 - \gamma_I^+)\}, \bigcup_{\gamma_F^+ \in h_F^+} \{1 - (1 - \gamma_F^+)\}, \bigcup_{\gamma_T^- \in h_T^-} \{-1 - (-1 - \gamma_T^-)\}, \right.$$

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$$\bigcup_{\gamma_I^-\in h_I^-}\left\{-1-\left(-1-\gamma_I^-\right)\right\},\bigcup_{\gamma_F^-\in h_F^-}\left\{-1-\left(-1-\gamma_F^-\right)\right\}\right\rangle$$

$$=\left\langle\bigcup_{\gamma_T^+\in h_T^+}\gamma_T^+,\bigcup_{\gamma_I^+\in h_I^+}\gamma_I^+,\bigcup_{\gamma_F^+\in h_F^+}\gamma_F^+,\bigcup_{\gamma_T^-\in h_T^-}\gamma_T^-,\bigcup_{\gamma_I^-\in h_I^-}\gamma_I^-,\bigcup_{\gamma_F^-\in h_F^-}\gamma_F^-\right\rangle$$

$$\Rightarrow\left\langle h_T^+,h_I^+,h_F^+,h_T^-,h_I^-,h_F^-\right\rangle=h$$

$$\therefore (h^c)^c=h \qquad \square$$

Proof 8):

Let $h_1=\left\langle h_{T_1}^+,h_{I_1}^+,h_{F_1}^+,h_{T_1}^-,h_{I_1}^-,h_{F_1}^-\right\rangle$ and $h_2=\left\langle h_{T_2}^+,h_{I_2}^+,h_{F_2}^+,h_{T_2}^-,h_{I_2}^-,h_{F_2}^-\right\rangle$ be two HBVNEs.

Then we have,

$$h_1\cup h_2=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\max\left\{\gamma_{T_1}^+,\gamma_{T_2}^+\right\},\bigcup_{\substack{\gamma_{I_1}^+\in h_I^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\frac{\gamma_{I_1}^++\gamma_{I_2}^+}{2}\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\min\left\{\gamma_{F_1}^+,\gamma_{F_2}^+\right\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\min\left\{\gamma_{T_1}^-,\gamma_{T_2}^-\right\},$$
$$\left.\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{\frac{\gamma_{I_1}^-+\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\max\left\{\gamma_{F_1}^-,\gamma_{F_2}^-\right\}\right\rangle$$

$$\left(h_1\cup h_2\right)^c=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}1-\max\left\{\gamma_{T_1}^+,\gamma_{T_2}^+\right\},\bigcup_{\substack{\gamma_{I_1}^+\in h_I^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}1-\left\{\frac{\gamma_{I_1}^++\gamma_{I_2}^+}{2}\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}1-\min\left\{\gamma_{F_1}^+,\gamma_{F_2}^+\right\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}-1-\min\left\{\gamma_{T_1}^-,\gamma_{T_2}^-\right\},$$
$$\left.\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}-1-\left\{\frac{\gamma_{I_1}^-+\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}-1-\max\left\{\gamma_{F_1}^-,\gamma_{F_2}^-\right\}\right\rangle$$

Therefore,

$$\left(h_1\cup h_2\right)^c=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\max\left\{1-\gamma_{T_1}^+,1-\gamma_{T_2}^+\right\},\bigcup_{\substack{\gamma_{I_1}^+\in h_I^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\frac{2-\gamma_{I_1}^+-\gamma_{I_2}^+}{2}\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\min\left\{1-\gamma_{F_1}^+,1-\gamma_{F_2}^+\right\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\min\left\{-1-\gamma_{T_1}^-,-1-\gamma_{T_2}^-\right\},$$
$$\left.\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{\frac{-2-\gamma_{I_1}^--\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\max\left\{-1-\gamma_{F_1}^-,-1-\gamma_{F_2}^-\right\}\right\rangle$$

We have,

$$h_1^c=\left\langle\bigcup_{\gamma_{T_1}^+\in h_{T_1}^+}\left\{1-\gamma_{T_1}^+\right\},\bigcup_{\gamma_I^+\in h_I^+}\left\{1-\gamma_{I_1}^+\right\},\bigcup_{\gamma_F^+\in h_F^+}\left\{1-\gamma_{F_1}^+\right\},\right.$$
$$\left.\bigcup_{\gamma_T^-\in h_T^-}\left\{-1-\gamma_{T_1}^-\right\},\bigcup_{\gamma_I^-\in h_I^-}\left\{-1-\gamma_{I_1}^-\right\},\bigcup_{\gamma_F^-\in h_F^-}\left\{-1-\gamma_{F_1}^-\right\}\right\rangle$$

$$h_2^c=\left\langle\bigcup_{\gamma_{T_2}^+\in h_{T_2}^+}\left\{1-\gamma_{T_2}^+\right\},\bigcup_{\gamma_{I_2}^+\in h_{I_2}^+}\left\{1-\gamma_{I_2}^+\right\},\bigcup_{\gamma_{F_2}^+\in h_{F_2}^+}\left\{1-\gamma_{F_2}^+\right\},\right.$$
$$\left.\bigcup_{\gamma_{T_2}^-\in h_{T_2}^-}\left\{-1-\gamma_{T_2}^-\right\},\bigcup_{\gamma_{I_2}^-\in h_{I_2}^-}\left\{-1-\gamma_{I_2}^-\right\},\bigcup_{\gamma_{F_2}^-\in h_{F_2}^-}\left\{-1-_{F_2}^-\right\}\right\rangle$$

Therefore,

$$h_1^c\cap h_2^c=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\min\left\{1-\gamma_{T_1}^+,1-\gamma_{T_2}^+\right\},\bigcup_{\substack{\gamma_{I_1}^+\in h_{I_1}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\frac{2-\gamma_{I_1}^+-\gamma_{I_2}^+}{2}\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\min\left\{1-\gamma_{F_1}^+,1-\gamma_{F_2}^+\right\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\min\left\{-1-\gamma_{T_1}^-,-1-\gamma_{T_2}^-\right\},$$
$$\left.\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{\frac{-2-\gamma_{I_1}^--\gamma_{I_2}^-}{2}\right\},\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\max\left\{1-\gamma_{F_1}^-,1-\gamma_{F_2}^-\right\}\right\rangle$$

$$\therefore\left(h_1\cup h_2\right)^c=h_1^c\cap h_2^c \qquad \square$$

Proof 9): Similar to the proof of 8) and thus omitted.

Proof 10):

Let $x\in\left(h^c\cup h_1^c\cup h_2^c\right)$

$$\Rightarrow x\in h^c\cup x\in h_1^c\cup x\in h_2^c$$
$$\Rightarrow x\notin h\cup x\notin h_1\cup x\notin h_2$$
$$\Rightarrow x\notin\left(h\cap h_1\right)\cap h_2$$
$$\Rightarrow x\notin h\cap h_1\cap h_2$$
$$\Rightarrow x\in\left(h\cap h_1\cap h_2\right)^c$$

Since for all $x\in\left(h^c\cup h_1^c\cup h_2^c\right)$ such that $x\in\left(h\cap h_1\cap h_2\right)^c$.

$$\therefore\left(h^c\cup h_1^c\cup h_2^c\right)=\left(h\cap h_1\cap h_2\right)^c \qquad \square$$

Proof 11): Similar to the proof of 10) and thus omitted.

In the following, some arithmetic operations of HBVNEs are defined.

Definition 10. Let $h=\left\langle h_T^+,h_I^+,h_F^+,h_T^-,h_I^-,h_F^-\right\rangle$, $h_1=\left\langle h_{T_1}^+,h_{I_1}^+,h_{F_1}^+,h_{T_1}^-,h_{I_1}^-,h_{F_1}^-\right\rangle$ and $h_2=\left\langle h_{T_2}^+,h_{I_2}^+,h_{F_2}^+,h_{T_2}^-,h_{I_2}^-,h_{F_2}^-\right\rangle$ be three HBVNEs. Then,

i) Addition

$$h_1\oplus h_2=\left\langle\bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\left\{\gamma_{T_1}^++\gamma_{T_2}^+-\gamma_{T_1}^+\cdot\gamma_{T_2}^+\right\},\bigcup_{\substack{\gamma_{I_1}^+\in h_I^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\gamma_{I_1}^+\cdot\gamma_{I_2}^+\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\left\{\gamma_{F_1}^+\cdot\gamma_{F_2}^+\right\},\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\left\{-\gamma_{T_1}^-\cdot\gamma_{T_2}^-\right\},$$

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$$\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{-\left(-\gamma_{I_1}^- - \gamma_{I_2}^- - \gamma_{I_1}^-\cdot\gamma_{I_2}^-\right)\right\},$$
$$\left.\bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\left\{-\left(-\gamma_{F_1}^- - \gamma_{F_2}^- - \gamma_{F_1}^-\cdot\gamma_{F_2}^-\right)\right\}\right\rangle$$

ii) Product

$$h_1\otimes h_2=\left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}}\left\{\gamma_{T_1}^+\cdot\gamma_{T_2}^+\right\},\ \bigcup_{\substack{\gamma_{I_1}^+\in h_{I_1}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}}\left\{\gamma_{I_1}^+ + \gamma_{I_2}^+ - \gamma_{I_1}^+\cdot\gamma_{I_2}^+\right\},\right.$$
$$\bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}}\left\{\gamma_{F_1}^+ + \gamma_{F_2}^+ - \gamma_{F_1}^+\cdot\gamma_{F_2}^+\right\},$$
$$\bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}}\left\{-\left(-\gamma_{T_1}^- - \gamma_{T_2}^- - \gamma_{T_1}^-\cdot\gamma_{T_2}^-\right)\right\},$$
$$\left.\bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}}\left\{-\gamma_{I_1}^-\cdot\gamma_{I_2}^-\right\},\ \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}}\left\{-\gamma_{F_1}^-\cdot\gamma_{F_2}^-\right\}\right\rangle$$

iii) Power

$$h^\rho=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{\left(\gamma_T^+\right)^\rho\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{1-\left(1-\gamma_I^+\right)^\rho\right\},\right.$$
$$\bigcup_{\gamma_F^+\in h_F^+}\left\{1-\left(1-\gamma_F^+\right)^\rho\right\},\ \bigcup_{\gamma_T^-\in h_T^-}\left\{-\left(1-\left(1-\left(-\gamma_T^-\right)\right)^\rho\right)\right\},$$
$$\left.\bigcup_{\gamma_I^-\in h_I^-}\left\{-\left(-\gamma_I^-\right)^\rho\right\},\ \bigcup_{\gamma_F^-\in h_F^-}\left\{-\left(-\gamma_F^-\right)^\rho\right\}\right\rangle$$

where $\rho>0$.

iv) Scalar Multiplication

$$\rho h=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{1-\left(1-\gamma_T^+\right)^\rho\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{\left(\gamma_I^+\right)^\rho\right\},\right.$$
$$\bigcup_{\gamma_F^+\in h_F^+}\left\{\left(\gamma_F^+\right)^\rho\right\},\ \bigcup_{\gamma_T^-\in h_T^-}\left\{-\left(-\gamma_T^-\right)^\rho\right\},$$
$$\bigcup_{\gamma_I^-\in h_I^-}\left\{-\left(1-\left(1-\left(-\gamma_I^-\right)\right)^\rho\right)\right\},$$
$$\left.\bigcup_{\gamma_F^-\in h_F^-}\left\{-\left(1-\left(1-\left(-\gamma_F^-\right)\right)^\rho\right)\right\}\right\rangle$$

where $\rho>0$.

Example 3. Consider Example 2 and given $\rho=0.5$, then

i) Addition

$$h_1\oplus h_2=\left\langle\{0.87,0.88\},0.12,0.14,\ \{-0.4,-0.45,\ -0.48,-0.54\},\ -0.37,\{-0.55,-0.65\}\right\rangle$$

ii) Multiplication

$$h_1\otimes h_2=\left\langle\{0.28,0.32\},0.68,0.76,\ \{-0.9,-0.92,\ -0.95,-0.96\},-0.03,\{-0.05,-0.15\}\right\rangle$$

iii) Power, $h^{0.5}$

$$h^{0.5}=\left\langle 0.71,0.22,\{0.26,0.29\},\{-0.45,-0.55\},-0.32,-0.55\right\rangle$$

iv) Scalar multiplication, $0.5h$

$$0.5h=\left\langle 0.29,0.63,\{0.67,0.71\},\{-0.84,-0.89\},-0.05,-0.16\right\rangle$$

Proposition 3. Given three HBVNEs, h, h_1 and h_2 with $\rho=0.5$, then we have

1) $\left(h^c\right)^\rho=\left(\rho h\right)^c$

2) $\rho\left(h^c\right)=\left(h^\rho\right)^c$

3) $\left(h_1\oplus h_2\right)^c=h_1{}^c\otimes h_2{}^c$

4) $\left(h_1\otimes h_2\right)^c=h_1{}^c\oplus h_2{}^c$

5) $\rho\left(h_1\oplus h_2\right)=\rho h_1\oplus\rho h_2$

6) $\left(h_1\otimes h_2\right)^\rho=h_1{}^\rho\otimes h_2{}^\rho$

Proof 1)

Let $h=\left\langle h_T^+,h_I^+,h_F^+,h_T^-,h_I^-,h_F^-\right\rangle$, we have

$$h^c=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{1-\gamma_T^+\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{1-\gamma_I^+\right\},\ \bigcup_{\gamma_F^+\in h_F^+}\left\{1-\gamma_F^+\right\},\right.$$
$$\left.\bigcup_{\gamma_T^-\in h_T^-}\left\{-1-\gamma_T^-\right\},\ \bigcup_{\gamma_I^-\in h_I^-}\left\{-1-\gamma_I^-\right\},\ \bigcup_{\gamma_F^-\in h_F^-}\left\{-1-\gamma_F^-\right\}\right\rangle$$
$$\Rightarrow\left(h^c\right)^\rho=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{\left(1-\gamma_T^+\right)^\rho\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{1-\left(\gamma_I^+\right)^\rho\right\},\right.$$
$$\bigcup_{\gamma_F^+\in h_F^+}\left\{1-\left(\gamma_F^+\right)^\rho\right\},\ \bigcup_{\gamma_T^-\in h_T^-}\left\{-1+\left(-\gamma_T^-\right)^\rho\right\},$$
$$\left.\bigcup_{y_I^-\in\hat{\lambda}_I^-}\left\{-\left(1+\gamma_I^-\right)^\rho\right\},\ \bigcup_{y_\Upsilon^-\in\hat{\lambda}_\Upsilon^-}\left\{-\left(1+y_\Upsilon^-\right)^\rho\right\}\right\rangle$$
$$\rho h=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{1-\left(1-\gamma_T^+\right)^\rho\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{\left(\gamma_I^+\right)^\rho\right\},\right.$$
$$\bigcup_{\gamma_F^+\in h_F^+}\left\{\left(\gamma_F^+\right)^\rho\right\},\ \bigcup_{\gamma_T^-\in h_T^-}\left\{-\left(-\gamma_T^-\right)^\rho\right\},$$
$$\bigcup_{\gamma_I^-\in h_I^-}\left\{-\left(1-\left(1-\left(-\gamma_I^-\right)\right)^\rho\right)\right\},$$
$$\left.\bigcup_{\gamma_F^-\in h_F^-}\left\{-\left(1-\left(1-\left(-\gamma_F^-\right)\right)^\rho\right)\right\}\right\rangle$$
$$\Rightarrow\left(\rho h\right)^c=\left\langle \bigcup_{\gamma_T^+\in h_T^+}\left\{\left(1-\gamma_T^+\right)^\rho\right\},\ \bigcup_{\gamma_I^+\in h_I^+}\left\{1-\left(\gamma_I^+\right)^\rho\right\},\right.$$
$$\bigcup_{\gamma_F^+\in h_F^+}\left\{1-\left(\gamma_F^+\right)^\rho\right\},\ \bigcup_{\gamma_T^-\in h_T^-}\left\{-1+\left(-\gamma_T^-\right)^\rho\right\},$$
$$\left.\bigcup_{\gamma_I^-\in h_I^-}\left\{-\left(1+\gamma_I^-\right)^\rho\right\},\ \bigcup_{\gamma_F^-\in h_F^-}\left\{-\left(1-\left(-\gamma_F^-\right)\right)^\rho\right\}\right\rangle$$
$$\therefore\left(h^c\right)^\rho=\left(\rho h\right)^c \qquad \square$$

Proof 2): Similar to the proof of 1) and thus omitted.

Proof 3)

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$$h_1 \oplus h_2 = \left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}} \left\{\gamma_{T_1}^+ + \gamma_{T_2}^+ - \gamma_{T_1}^+\cdot\gamma_{T_2}^+\right\}, \bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}} \left\{\gamma_{I_1}^+\cdot\gamma_{I_2}^+\right\}, \bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}} \left\{\gamma_{F_1}^+\cdot\gamma_{F_2}^+\right\}, \bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}} \left\{-\gamma_{T_1}^-\cdot\gamma_{T_2}^-\right\}, \bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}} \left\{-\left(-\gamma_{I_1}^- - \gamma_{I_2}^- - \gamma_{I_1}^-\cdot\gamma_{I_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}} \left\{-\left(-\gamma_{F_1}^- - \gamma_{F_2}^- - \gamma_{F_1}^-\cdot\gamma_{F_2}^-\right)\right\} \right\rangle$$

$$\left(h_1 \oplus h_2\right)^c = \left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}} \left\{1-\left(\gamma_{T_1}^+ + \gamma_{T_2}^+ - \gamma_{T_1}^+\cdot\gamma_{T_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}} \left\{1-\left(\gamma_{I_1}^+\cdot\gamma_{I_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}} \left\{1-\left(\gamma_{F_1}^+\cdot\gamma_{F_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}} \left\{-1-\left(-\gamma_{T_1}^-\cdot\gamma_{T_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}} \left\{-1-\left(-\left(-\gamma_{I_1}^- - \gamma_{I_2}^- - \gamma_{I_1}^-\cdot\gamma_{I_2}^-\right)\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}} \left\{-1-\left(-\left(-\gamma_{F_1}^- - \gamma_{F_2}^- - \gamma_{F_1}^-\cdot\gamma_{F_2}^-\right)\right)\right\} \right\rangle$$

$$\Rightarrow \left(h_1 \oplus h_2\right)^c = \left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}} \left\{1-\left(\gamma_{T_1}^+ + \gamma_{T_2}^+ - \gamma_{T_1}^+\cdot\gamma_{T_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}} \left\{1-\left(\gamma_{I_1}^+\cdot\gamma_{I_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}} \left\{1-\left(\gamma_{F_1}^+\cdot\gamma_{F_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}} \left\{-1+\left(\gamma_{T_1}^-\cdot\gamma_{T_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}} \left\{-1-\left(\gamma_{I_1}^- + \gamma_{I_2}^- + \gamma_{I_1}^-\cdot\gamma_{I_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}} \left\{-1-\left(\gamma_{F_1}^- + \gamma_{F_2}^- + \gamma_{F_1}^-\cdot\gamma_{F_2}^-\right)\right\} \right\rangle$$

$$h_1^c = \left\langle \begin{array}{l} \bigcup_{\gamma_{T_1}^+\in h_{T_1}^+} \left\{1-\gamma_{T_1}^+\right\}, \bigcup_{\gamma_{I_1}^+\in h_{I_1}^+} \left\{1-\gamma_{I_1}^+\right\}, \bigcup_{\gamma_{F_1}^+\in h_{F_1}^+} \left\{1-\gamma_{F_1}^+\right\}, \\ \bigcup_{\gamma_{T_1}^-\in h_{T_1}^-} \left\{-1-\gamma_{T_1}^-\right\}, \bigcup_{\gamma_{I_1}^-\in h_{I_1}^-} \left\{1-\gamma_{I_1}^-\right\}, \bigcup_{\gamma_{F_1}^-\in h_{F_1}^-} \left\{1-\gamma_{F_1}^-\right\} \end{array} \right\rangle$$

$$h_2^c = \left\langle \begin{array}{l} \bigcup_{\gamma_{T_2}^+\in h_{T_2}^+} \left\{1-\gamma_{T_2}^+\right\}, \bigcup_{\gamma_{I_2}^+\in h_{I_2}^+} \left\{1-\gamma_{I_2}^+\right\}, \bigcup_{\gamma_{F_2}^+\in h_{F_2}^+} \left\{1-\gamma_{F_2}^+\right\}, \\ \bigcup_{\gamma_{T_2}^-\in h_{T_2}^-} \left\{-1-\gamma_{T_2}^-\right\}, \bigcup_{\gamma_{I_2}^-\in h_{I_2}^-} \left\{1-\gamma_{I_2}^-\right\}, \bigcup_{\gamma_{F_2}^-\in h_{F_2}^-} \left\{1-\gamma_{F_2}^-\right\} \end{array} \right\rangle$$

$$h_1^c \otimes h_2^c = \left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}} \left\{\left(1-\gamma_{T_1}^+\right)\cdot\left(1-\gamma_{T_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}} \left\{\left(1-\gamma_{I_1}^+\right)+\left(1-\gamma_{I_2}^+\right)-\left(1-\gamma_{I_1}^+\right)\cdot\left(1-\gamma_{I_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}} \left\{\left(1-\gamma_{F_1}^+\right)+\left(1-\gamma_{F_2}^+\right)-\left(1-\gamma_{F_1}^+\right)\cdot\left(1-\gamma_{F_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}} \left\{-\left(-\left(-1-\gamma_{T_1}^-\right)-\left(-1-\gamma_{T_2}^-\right)-\left(-1-\gamma_{T_1}^-\right)\cdot\left(-1-\gamma_{T_2}^-\right)\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}} \left\{-\left(-1-\gamma_{I_1}^-\right)\cdot\left(-1-\gamma_{I_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}} \left\{-\left(-1-\gamma_{F_1}^-\right)\cdot\left(-1-\gamma_{F_2}^-\right)\right\} \right\rangle$$

$$\Rightarrow h_1^c \otimes h_2^c = \left\langle \bigcup_{\substack{\gamma_{T_1}^+\in h_{T_1}^+,\\ \gamma_{T_2}^+\in h_{T_2}^+}} \left\{1-\left(\gamma_{T_1}^+ + \gamma_{T_2}^+ - \gamma_{T_1}^+\cdot\gamma_{T_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^+\in h_{I}^+,\\ \gamma_{I_2}^+\in h_{I_2}^+}} \left\{1-\left(\gamma_{I_1}^+\cdot\gamma_{I_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^+\in h_{F_1}^+,\\ \gamma_{F_2}^+\in h_{F_2}^+}} \left\{1-\left(\gamma_{F_1}^+\cdot\gamma_{F_2}^+\right)\right\}, \bigcup_{\substack{\gamma_{T_1}^-\in h_{T_1}^-,\\ \gamma_{T_2}^-\in h_{T_2}^-}} \left\{-1+\left(\gamma_{T_1}^-\cdot\gamma_{T_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{I_1}^-\in h_{I_1}^-,\\ \gamma_{I_2}^-\in h_{I_2}^-}} \left\{-1-\left(\gamma_{I_1}^- + \gamma_{I_2}^- + \gamma_{I_1}^-\cdot\gamma_{I_2}^-\right)\right\}, \bigcup_{\substack{\gamma_{F_1}^-\in h_{F_1}^-,\\ \gamma_{F_2}^-\in h_{F_2}^-}} \left\{-1-\left(\gamma_{F_1}^- + \gamma_{F_2}^- + \gamma_{F_1}^-\cdot\gamma_{F_2}^-\right)\right\} \right\rangle.$$

$$\therefore \left(h_1 \oplus h_2\right)^c = h_1^c \otimes h_2^c$$ □

Proof 4), 5) and 6): Similar to the proof of 3) and thus omitted.

Definition 11. Let $h = \left\langle h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^- \right\rangle$ be a HBVNE, then the score function $s(h)$ is defined as follows

$$s(h) = \frac{1}{6}\left(\begin{array}{l} \frac{1}{\ell_{h_T}^+}\sum_{\gamma_T^+\in h_T^+}\gamma_T^+ + 1 - \frac{1}{\ell_{h_I}^+}\sum_{\gamma_I^+\in h_I^+}\gamma_I^+ + 1 - \\ \frac{1}{\ell_{h_F}^+}\sum_{\gamma_F^+\in h_F^+}\gamma_F^+ + 1 + \frac{1}{\ell_{h_T}^-}\sum_{\gamma_T^-\in h_T^-}\gamma_T^- \\ -\frac{1}{\ell_{h_I}^-}\sum_{\gamma_I^-\in h_I^-}\gamma_I^- - \frac{1}{\ell_{h_F}^-}\sum_{\gamma_F^-\in h_F^-}\gamma_F^- \end{array} \right) \tag{2}$$

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If $s(h_1) > s(h_2)$, then h_1 is bigger than h_2, represented by $h_1 > h_2$ and if $s(h_1) = s(h_2)$, then h_1 is equivalent to h_2, represented by $h_1 = h_2$.

Definition 12. Let $h_j = \left\langle h^+_{T_j}, h^+_{I_j}, h^+_{F_j}, h^-_{T_j}, h^-_{I_j}, h^-_{F_j} \right\rangle$ be a family of HBVNEs. A mapping $HBVNWA_\omega : \hat{H}_n \to \hat{H}$ is called hesitant bipolar-valued neutrosophic weighted average (HBVNWA) operator if it satisfies

$$HBVNWA_\omega(h_1, h_2, ..., h_n) = \sum_{j=1}^{n} \omega_j h_j$$

$$= \left\langle \bigcup_{\gamma^+_{T_j} \in h^+_{T_j}} \left\{1 - \prod_{j=1}^{n} \left(1 - \gamma^+_{T_j}\right)^{\omega_j}\right\}, \bigcup_{\gamma^+_{I_j} \in h^+_{I_j}} \left\{\prod_{j=1}^{n} \left(\gamma^+_{I_j}\right)^{\omega_j}\right\}, \bigcup_{\gamma^+_{F_j} \in h^+_{F_j}} \left\{\prod_{j=1}^{n} \left(\gamma^+_{F_j}\right)^{\omega_j}\right\}, \bigcup_{\gamma^-_{T_j} \in h^-_{T_j}} \left\{-\prod_{j=1}^{n} \left(-\gamma^-_{T_j}\right)^{\omega_j}\right\}, \bigcup_{\gamma^-_{I_j} \in h^-_{I_j}} \left\{-\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\gamma^-_{I_j}\right)\right)^{\omega_j}\right)\right\}, \bigcup_{\gamma^-_{F_j} \in h^-_{F_j}} \left\{-\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\gamma^-_{F_j}\right)\right)^{\omega_j}\right)\right\} \right\rangle \quad (3)$$

where ω_j is the weight of $h_j\ (j = 1, 2, ..., n), \omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Then $HBVNWA_\omega(h_1, h_2, ..., h_n)$ is called HBVNWA and the results of the aggregation are still HBVNEs.

Theorem 1. Let $h_j = \left\langle h^+_{T_j}, h^+_{I_j}, h^+_{F_j}, h^-_{T_j}, h^-_{I_j}, h^-_{F_j} \right\rangle$ be a family of HBVNNs. Then,

i) Idempotency

If $h_j = h$ for all $j = 1, 2, ..., n$, then $HBVNWA_\omega(h_1, h_2, ..., h_n) = h$

ii) Monotonicity

If $h_j \le h^*_j$ for all $j = 1, 2, ..., n$, then $HBVNWA_\omega(h_1, h_2, ..., h_n) \le HBVNWA^*_\omega(h^*_1, h^*_2, ..., h^*_n)$

iii) Boundedness

$$\min_{j=1,2,...,n} \{h_j\} \le HBVNWA_\omega(h_1, h_2, ..., h_n) \le \max_{j=1,2,...,n} \{h_j\}$$

Proof i):Idempotency

Since $h_j = h = \left\langle h^+_T, h^+_I, h^+_F, h^-_T, h^-_I, h^-_F \right\rangle$ for all j, we have

$$HBVNWA_\omega(h_j) = \sum_{j=1}^{n} \omega_j h_j$$

$$\Rightarrow \left\langle \bigcup_{\gamma^+_T \in h^+_T} \left\{1 - \prod_{j=1}^{n} \left(1 - \gamma^+_T\right)^{\omega_j}\right\}, \bigcup_{\gamma^+_I \in h^+_I} \left\{\prod_{j=1}^{n} \left(\gamma^+_I\right)^{\omega_j}\right\}, \bigcup_{\gamma^+_F \in h^+_F} \left\{\prod_{j=1}^{n} \left(\gamma^+_F\right)^{\omega_j}\right\}, \bigcup_{\gamma^-_T \in h^-_T} \left\{-\prod_{j=1}^{n} \left(-\gamma^-_T\right)^{\omega_j}\right\}, \bigcup_{\gamma^-_I \in h^-_I} \left\{-\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\gamma^-_I\right)\right)^{\omega_j}\right)\right\}, \bigcup_{\gamma^-_F \in h^-_F} \left\{-\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\gamma^-_F\right)\right)^{\omega_j}\right)\right\} \right\rangle$$

$$\Rightarrow \left\langle \bigcup_{\gamma^+_T \in h^+_T} \left\{1 - \left(1 - \gamma^+_T\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma^+_I \in h^+_I} \left\{\left(\gamma^+_I\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma^+_F \in h^+_F} \left\{\left(\gamma^+_F\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma^-_T \in h^-_T} \left\{-\left(-\gamma^-_T\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma^-_I \in h^-_I} \left\{-\left(1 - \left(1 - \left(-\gamma^-_I\right)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\}, \bigcup_{\gamma^-_F \in h^-_F} \left\{-\left(1 - \left(1 - \left(-\gamma^-_F\right)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\} \right\rangle$$

Since $\sum_{j=1}^{n} \omega_j = 1$, we have

$$\Rightarrow \left\langle \bigcup_{\gamma^+_T \in h^+_T} \left\{1 - \left(1 - \gamma^+_T\right)\right\}, \bigcup_{\gamma^+_I \in h^+_I} \left\{\left(\gamma^+_I\right)\right\}, \bigcup_{\gamma^+_F \in h^+_F} \left\{\left(\gamma^+_F\right)\right\}, \bigcup_{\gamma^-_T \in h^-_T} \left\{-\left(-\gamma^-_T\right)\right\}, \bigcup_{\gamma^-_I \in h^-_I} \left\{-\left(1 - \left(1 - \left(-\gamma^-_I\right)\right)\right)\right\}, \bigcup_{\gamma^-_F \in h^-_F} \left\{-\left(1 - \left(1 - \left(-\gamma^-_F\right)\right)\right)\right\} \right\rangle$$

$$\Rightarrow \left\langle \bigcup_{\gamma^+_T \in h^+_T} \left\{\gamma^+_T\right\}, \bigcup_{\gamma^+_I \in h^+_I} \left\{\gamma^+_I\right\}, \bigcup_{\gamma^+_F \in h^+_F} \left\{\gamma^+_F\right\}, \bigcup_{\gamma^-_T \in h^-_T} \left\{\gamma^-_T\right\}, \bigcup_{\gamma^-_I \in h^-_I} \left\{\gamma^-_I\right\}, \bigcup_{\gamma^-_F \in h^-_F} \left\{\gamma^-_F\right\} \right\rangle$$

$$\Rightarrow \left\langle h^+_T, h^+_I, h^+_F, h^-_T, h^-_I, h^-_F \right\rangle = h$$

which completes the proof of the Theorem 1 i).

Proof ii): Monotonicity

Since $h^+_{T_j} \le h^{+*}_{T_j}$ for all j, then we have

$$\gamma^+_{T_j} \le \gamma^{+*}_{T_j},\ 1 - \gamma^+_{T_j} \ge 1 - \gamma^{+*}_{T_j}$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+}\right)^{\omega_j} \leq 1-\prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{T_j}^{+}\in h_{T_j}^{+}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+}\right)^{\omega_j}\right\} \leq \bigcup_{\gamma_{T_j}^{+}\in h_{T_j}^{+}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{T_j}^{+*}\right)^{\omega_j}\right\}$$

Since $h_{I_j}^{+} \leq h_{I_j}^{+*}$ for all j, then we have

$$\gamma_{I_j}^{+} \leq \gamma_{I_j}^{+*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(\gamma_{I_j}^{+}\right)^{\omega_j} \leq \prod_{j=1}^{n}\left(\gamma_{I_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{I_j}^{+}\in h_{I_j}^{+}}\left\{\prod_{j=1}^{n}\left(\gamma_{I_j}^{+}\right)^{\omega_j}\right\} \leq \bigcup_{\gamma_{I_j}^{+}\in h_{I_j}^{+}}\left\{\prod_{j=1}^{n}\left(\gamma_{I_j}^{+*}\right)^{\omega_j}\right\}$$

Since $h_{F_j}^{+} \geq h_{F_j}^{+*}$ for all j, then we have

$$\gamma_{F_j}^{+} \geq \gamma_{F_j}^{+*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(\gamma_{F_j}^{+}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(\gamma_{F_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{F_j}^{+}\in h_{F_j}^{+}}\left\{\prod_{j=1}^{n}\left(\gamma_{F_j}^{+}\right)^{\omega_j}\right\} \geq \bigcup_{\gamma_{F_j}^{+}\in h_{F_j}^{+}}\left\{\prod_{j=1}^{n}\left(\gamma_{F_j}^{+*}\right)^{\omega_j}\right\}$$

Since $h_{T_j}^{-} \geq h_{T_j}^{-*}$ for all j, we have

$$\gamma_{T_j}^{-} \geq \gamma_{T_j}^{-*}, -\gamma_{T_j}^{-} \leq -\gamma_{T_j}^{-*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(-\gamma_{T_j}^{-}\right)^{\omega_j} \leq \prod_{j=1}^{n}\left(-\gamma_{T_j}^{-*}\right)^{\omega_j}$$

$$\Rightarrow -\prod_{j=1}^{n}\left(-\gamma_{T_j}^{-}\right)^{\omega_j} \geq -\prod_{j=1}^{n}\left(-\gamma_{T_j}^{-*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{T_j}^{-}\in h_{T_j}^{-}}\left\{-\prod_{j=1}^{n}\left(-\gamma_{T_j}^{-}\right)^{\omega_j}\right\} \geq \bigcup_{\gamma_{T_j}^{-}\in h_{T_j}^{-}}\left\{-\prod_{j=1}^{n}\left(-\gamma_{T_j}^{-*}\right)^{\omega_j}\right\}$$

Since $h_{I_j}^{-} \geq h_{I_j}^{-*}$ for all j, then we have

$$\gamma_{I_j}^{-} \geq \gamma_{I_j}^{-*},\ -\gamma_{I_j}^{-} \leq -\gamma_{I_j}^{-*}$$

$$\Rightarrow 1-\left(-\gamma_{I_j}^{-}\right) \geq 1-\left(-\gamma_{I_j}^{-*}\right)$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-}\right)\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-}\right)\right)^{\omega_j} \leq 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-}\right)\right)^{\omega_j}\right) \geq -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-*}\right)\right)^{\omega_j}\right)$$

$$\Rightarrow \bigcup_{\gamma_{I_j}^{-}\in h_{I_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-}\right)\right)^{\omega_j}\right)\right\} \geq \bigcup_{\gamma_{I_j}^{-}\in h_{I_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{I_j}^{-*}\right)\right)^{\omega_j}\right)\right\}$$

Since $h_{F_j}^{-} \leq h_{F_j}^{-*}$ for all j, then we have

$$\gamma_{F_j}^{-} \leq \gamma_{F_j}^{-*}, -\gamma_{F_j}^{-} \geq -\gamma_{F_j}^{-*}$$

$$\Rightarrow 1-\left(-\gamma_{F_j}^{-}\right) \leq 1-\left(-\gamma_{F_j}^{-*}\right)$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-}\right)\right)^{\omega_j} \leq \prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-}\right)\right)^{\omega_j} \geq 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-}\right)\right)^{\omega_j}\right) \leq -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-*}\right)\right)^{\omega_j}\right)$$

$$\Rightarrow \bigcup_{\gamma_{F_j}^{-}\in h_{F_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-}\right)\right)^{\omega_j}\right)\right\} \leq \bigcup_{\gamma_{F_j}^{-}\in h_{F_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{F_j}^{-*}\right)\right)^{\omega_j}\right)\right\}$$

Based on the above analysis, we have

$$HBVNWA_{\omega}\left(h_1,h_2,...,h_n\right) \leq HBVNWA_{\omega}^{*}\left(h_1^{*},h_2^{*},...,h_n^{*}\right)$$

which completes the proof of the Theorem 1 ii).

Proof iii): Boundedness

Similar to the proof of Theorem 1 ii), thus omitted.

Definition 13. Let $h_j = \left\langle h_{T_j}^{+}, h_{I_j}^{+}, h_{F_j}^{+}, h_{T_j}^{-}, h_{I_j}^{-}, h_{F_j}^{-}\right\rangle$ be a family of HBVNNs. A mapping $HBVNWG_{\omega}: \hat{H}_n \to \hat{H}$ is called hesitant bipolar-valued neutrosophic weighted geometric (HBVNWG) operator if it satisfies

$$HBVNWG_{\omega}\left(h_1,h_2,...,h_n\right) = \prod_{j=1}^{n}\left(h_j\right)^{\omega_j}$$

$$= \left\langle \bigcup_{\gamma_{T_j}^{+}\in h_{T_j}^{+}}\left\{\prod_{j=1}^{n}\left(\gamma_{T_j}^{+}\right)^{\omega_j}\right\}, \bigcup_{\gamma_{I_j}^{+}\in h_{I_j}^{+}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{I_j}^{+}\right)^{\omega_j}\right\},\right.$$

$$\bigcup_{\gamma_{F_j}^{+}\in h_{F_j}^{+}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+}\right)^{\omega_j}\right\},$$

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$$\bigcup_{\gamma_{T_j}^- \in h_{T_j}^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^-\right)\right)^{\omega_j}\right)\right\},$$

$$\bigcup_{\gamma_{I_j}^- \in h_{I_j}^-}\left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{I_j}^-\right)^{\omega_j}\right)\right\}, \bigcup_{\gamma_{F_j}^- \in h_{F_j}^-}\left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{F_j}^-\right)^{\omega_j}\right)\right\}\Bigg\rangle \quad (4)$$

where ω_j is the weight of $h_j\left(j=1,2,...,n\right), \omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$. Then $HBVNWG_\omega\left(h_1,h_2,...,h_n\right)$ is called HBVNWG and the results of the aggregation are still HBVNNs.

Theorem 2. Let $h_j = \left\langle h_{T_j}^+, h_{I_j}^+, h_{F_j}^+, h_{T_j}^-, h_{I_j}^-, h_{F_j}^- \right\rangle$ be a family of HBVNNs. Then,

i) Idempotency

If $h_j = h$ for all $j=1,2,...,n$, then $HBVNWG_\omega\left(h_1,h_2,...,h_n\right) = h$

ii) Monotonicity

If $h_j \le h_j^*$ for all $j=1,2,...,n$, then $HBVNWG_\omega\left(h_1,h_2,...,h_n\right) \le HBVNWG_\omega^*\left(h_1^*,h_2^*,...,h_n^*\right)$

iii) Boundedness

$$\min_{j=1,2,...,n}\left\{h_j\right\} \le HBVNWG_\omega\left(h_1,h_2,...,h_n\right) \le \max_{j=1,2,...,n}\left\{h_j\right\}$$

Proof i) Idempotency:

Since $h_j = h = \left\langle h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^- \right\rangle$ for all j, we have

$$HBVNWG_\omega\left(h_j\right) = \prod_{j=1}^{n}\left(h_j\right)^{\omega_j}$$

$$\Rightarrow\Bigg\langle \bigcup_{\gamma_T^+ \in h_T^+}\left\{\prod_{j=1}^{n}\left(\gamma_T^+\right)^{\omega_j}\right\}, \bigcup_{\gamma_I^+ \in h_I^+}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_I^+\right)^{\omega_j}\right\},$$

$$\bigcup_{\gamma_F^+ \in h_F^+}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_F^+\right)^{\omega_j}\right\},$$

$$\bigcup_{\gamma_T^- \in h_T^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_T^-\right)\right)^{\omega_j}\right)\right\}, \bigcup_{\gamma_I^- \in h_I^-}\left\{-\left(\prod_{j=1}^{n}\left(-\gamma_I^-\right)^{\omega_j}\right)\right\},$$

$$\bigcup_{\gamma_F^- \in h_F^-}\left\{-\left(\prod_{j=1}^{n}\left(-\gamma_F^-\right)^{\omega_j}\right)\right\}\Bigg\rangle$$

$$\Rightarrow\Bigg\langle \bigcup_{\gamma_T^+ \in h_T^+}\left\{\left(\gamma_T^+\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma_I^+ \in h_I^+}\left\{1-\left(1-\gamma_I^+\right)^{\sum_{j=1}^{n}\omega_j}\right\},$$

$$\bigcup_{\gamma_F^+ \in h_F^+}\left\{1-\left(1-\gamma_F^+\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma_T^- \in h_T^-}\left\{-\left(1-\left(1-\left(-\gamma_T^-\right)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\},$$

$$\bigcup_{\gamma_I^- \in h_I^-}\left\{-\left(-\gamma_I^-\right)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma_F^- \in h_F^-}\left\{-\left(\left(-\gamma_F^-\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\}\Bigg\rangle$$

Since $\sum_{j=1}^{n}\omega_j = 1$, we have

$$\Rightarrow\Bigg\langle \bigcup_{\gamma_T^+ \in h_T^+}\left\{\gamma_T^+\right\}, \bigcup_{\gamma_I^+ \in h_I^+}\left\{1-\left(1-\gamma_I^+\right)\right\}, \bigcup_{\gamma_F^+ \in h_F^+}\left\{1-\left(1-\gamma_F^+\right)\right\},$$

$$\bigcup_{\gamma_T^- \in h_T^-}\left\{-\left(1-\left(1-\left(-\gamma_T^-\right)\right)\right)\right\}, \bigcup_{\gamma_I^- \in h_I^-}\left\{-\left(-\gamma_I^-\right)\right\},$$

$$\bigcup_{\gamma_F^- \in h_F^-}\left\{-\left(-\gamma_F^-\right)\right\}\Bigg\rangle$$

$$\Rightarrow\Bigg\langle \bigcup_{\gamma_T^+ \in h_T^+}\left\{\gamma_T^+\right\}, \bigcup_{\gamma_I^+ \in h_I^+}\left\{\gamma_I^+\right\}, \bigcup_{\gamma_F^+ \in h_F^+}\left\{\gamma_F^+\right\}, \bigcup_{\gamma_T^- \in h_T^-}\left\{\gamma_T^-\right\},$$

$$\bigcup_{\gamma_I^- \in h_I^-}\left\{\gamma_I^-\right\}, \bigcup_{\gamma_F^- \in h_F^-}\left\{\gamma_F^-\right\}\Bigg\rangle$$

$$\Rightarrow\left\langle h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^- \right\rangle = h$$

which completes the proof of the Theorem 2 i).

Proof ii) Monotonicity:

Since $h_{T_j}^+ \le h_{T_j}^{+*}$ for all j, we have

$$\gamma_{T_j}^+ \le \gamma_{T_j}^{+*}$$

$$\Rightarrow\prod_{j=1}^{n}\left(\gamma_{T_j}^+\right)^{\omega_j} \le \prod_{j=1}^{n}\left(\gamma_{T_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{T_j}^+ \in h_{T_j}^+}\left\{\prod_{j=1}^{n}\left(\gamma_{T_j}^+\right)^{\omega_j}\right\} \le \bigcup_{\gamma_{T_j}^+ \in h_{T_j}^+}\left\{\prod_{j=1}^{n}\left(\gamma_{T_j}^{+*}\right)^{\omega_j}\right\}$$

Since $h_{I_j}^+ \le h_{I_j}^{+*}$ for all j, we have

$$\gamma_{I_j}^+ \le \gamma_{I_j}^{+*}$$

$$\Rightarrow\left(1-\gamma_{I_j}^+\right) \ge \left(1-\gamma_{I_j}^{+*}\right)$$

$$\Rightarrow\prod_{j=1}^{n}\left(1-\gamma_{I_j}^+\right)^{\omega_j} \ge \prod_{j=1}^{n}\left(1-\gamma_{I_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\gamma_{I_j}^+\right)^{\omega_j} \le 1-\prod_{j=1}^{n}\left(1-\gamma_{I_j}^{+*}\right)^{\omega_j}$$

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$$\Rightarrow \bigcup_{\gamma_{I_j}^{+} \in h_{I_j}^{+}} \left\{1-\prod_{j=1}^{n}\left(1-\gamma_{I_j}^{+}\right)^{\omega_j}\right\} \le \bigcup_{\gamma_{I_j}^{+} \in h_{I_j}^{+}} \left\{1-\prod_{j=1}^{n}\left(1-\gamma_{I_j}^{+*}\right)^{\omega_j}\right\},$$

Since $h_{F_j}^{+} \ge h_{F_j}^{+*}$ for all j, then we have

$$\gamma_{F_j}^{+} \ge \gamma_{F_j}^{+*}$$

$$\Rightarrow 1-\gamma_{F_j}^{+} \le 1-\gamma_{F_j}^{+*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+}\right)^{\omega_j} \le \prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+}\right)^{\omega_j} \ge 1-\prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+*}\right)^{\omega_j}$$

$$\Rightarrow \bigcup_{\gamma_{F_j}^{+} \in h_{F_j}^{+}} \left\{1-\prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+}\right)^{\omega_j}\right\} \ge \bigcup_{\gamma_{F_j}^{+} \in h_{F_j}^{+}} \left\{1-\prod_{j=1}^{n}\left(1-\gamma_{F_j}^{+*}\right)^{\omega_j}\right\}$$

Since $h_{T_j}^{-} \ge h_{T_j}^{-*}$ for all j, we have

$$\gamma_{T_j}^{-} \ge \gamma_{T_j}^{-*}, -\gamma_{T_j}^{-} \le -\gamma_{T_j}^{-*}$$

$$\Rightarrow 1-\left(-\gamma_{T_j}^{-}\right) \ge 1-\left(-\gamma_{T_j}^{-*}\right)$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-}\right)\right)^{\omega_j} \ge \prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-}\right)\right)^{\omega_j} \le 1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-*}\right)\right)^{\omega_j}$$

$$\Rightarrow -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-}\right)\right)^{\omega_j}\right) \ge -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-*}\right)\right)^{\omega_j}\right)$$

$$\Rightarrow \bigcup_{\gamma_{T_j}^{-} \in h_{T_j}^{-}} \left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-}\right)\right)^{\omega_j}\right)\right\} \ge$$

$$\bigcup_{\gamma_{T_j}^{-} \in h_{T_j}^{-}} \left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{T_j}^{-*}\right)\right)^{\omega_j}\right)\right\}$$

Since $h_{I_j}^{-} \ge h_{I_j}^{-*}$ for all j, we have

$$\gamma_{I_j}^{-} \ge \gamma_{I_j}^{-*}, -\gamma_{I_j}^{-} \le -\gamma_{I_j}^{-*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(-\gamma_{I_j}^{-}\right)^{\omega_j} \le \prod_{j=1}^{n}\left(-\gamma_{I_j}^{-*}\right)^{\omega_j}$$

$$\Rightarrow -\left(\prod_{j=1}^{n}\left(-\gamma_{I_j}^{-}\right)^{\omega_j}\right) \ge -\left(\prod_{j=1}^{n}\left(-\gamma_{I_j}^{-*}\right)^{\omega_j}\right)$$

$$\Rightarrow \bigcup_{\gamma_{I_j}^{-} \in h_{I_j}^{-}} \left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{I_j}^{-}\right)^{\omega_j}\right)\right\} \ge \bigcup_{\gamma_{I_j}^{-} \in h_{I_j}^{-}} \left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{I_j}^{-*}\right)^{\omega_j}\right)\right\}$$

Since $h_{F_j}^{-} \le h_{F_j}^{-*}$ for all j, then we have

$$\gamma_{F_j}^{-} \le \gamma_{F_j}^{-*}, -\gamma_{F_j}^{-} \ge -\gamma_{F_j}^{-*}$$

$$\Rightarrow \prod_{j=1}^{n}\left(-\gamma_{F_j}^{-}\right)^{\omega_j} \ge \prod_{j=1}^{n}\left(-\gamma_{F_j}^{-*}\right)^{\omega_j}$$

$$\Rightarrow -\left(\prod_{j=1}^{n}\left(-\gamma_{F_j}^{-}\right)^{\omega_j}\right) \le -\left(\prod_{j=1}^{n}\left(-\gamma_{F_j}^{-*}\right)^{\omega_j}\right)$$

$$\Rightarrow \bigcup_{\gamma_{F_j}^{-} \in h_{F_j}^{-}} \left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{F_j}^{-}\right)^{\omega_j}\right)\right\} \le \bigcup_{\gamma_{F_j}^{-} \in h_{F_j}^{-}} \left\{-\left(\prod_{j=1}^{n}\left(-\gamma_{F_j}^{-*}\right)^{\omega_j}\right)\right\}$$

Based on the above analysis, we have

$$HBVNWG_{\omega}\left(h_1, h_2, ..., h_n\right) \le HBVNWG_{\omega}^{*}\left(h_1^{*}, h_2^{*}, ..., h_n^{*}\right)$$

which completes the proof of the Theorem 2 ii).

Proof iii): Boundedness
Similar to the proof of Theorem 2 ii), thus omitted.

IV. PROPOSED METHOD

In this section, an approach is developed to solve MCDM problem under the hesitant bipolar-valued neutrosophic environment. Let $A_i = \{A_1, A_2, ..., A_m\}$ and $c_j = \{c_1, c_2, ..., c_n\}$ be a set of alternatives and attributes respectively. An evaluation is collected with hesitant bipolar-valued neutrosophic elements, h_{ij} which represents the decision makers' judgment on i-th alternative with respect to j-th attribute. The general decision matrix form is as below:

$$D = \left\langle h_{ij} \right\rangle_{m\times n} = \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{m1} \\ h_{12} & h_{22} & \cdots & h_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ h_{1n} & h_{2n} & \cdots & h_{mn} \end{bmatrix}$$

The evaluation can be analysed using the HBVNWA or HBVNWG operators. The algorithms of the proposed decision-making approach are given as below:

Step 1: Compute the weighted average values, $h_i = HBVNWA_{\omega}\left(h_{i1}, h_{i2}, ..., h_{in}\right)$ by using the HBVNWA operator in (3) or the weighted geometric values, $h_i = HBVNWG_{\omega}\left(h_{i1}, h_{i2}, ..., h_{in}\right)$ by using the HBVNWG operator in (4).

Step 2: Calculate the score function $s\left(h_i\right)\left(i = 1, 2, 3, 4\right)$ for each $A_i\left(i = 1, 2, ..., m\right)$ by using (2).

Step 3: Rank the alternatives in descending order based on the obtained score function.

The general proposed methodology is illustrated in Figure 1.

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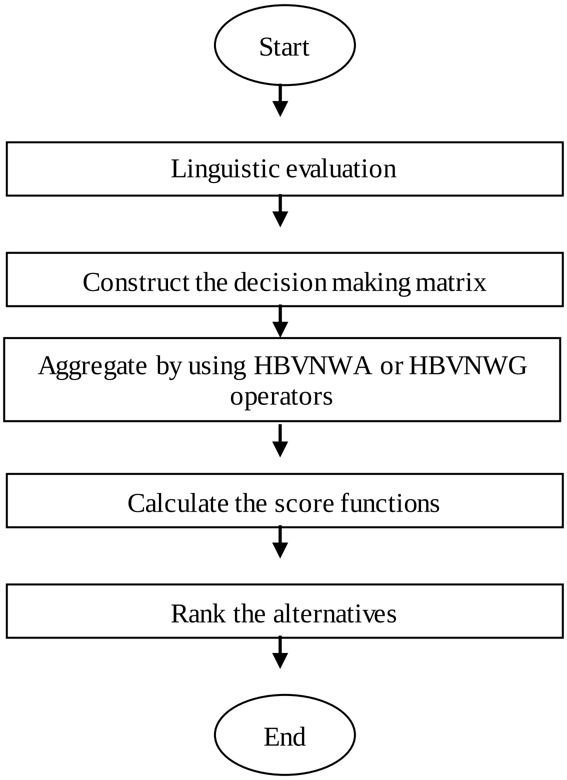

FIGURE 1. Framework of the proposed method.

V. NUMERICAL EXAMPLE

In this section, we present an example adopted from Ye [31] where MADM problem under hesitant bipolar-valued neutrosophic environment is involved. This application is essential to show the applicability of the developed decision-making approach.

There is an investment company that wants to invest a sum of money to the best option by considering four alternatives which are a car company (A_1), a food company (A_2), a computer company (A_3) and an arms company (A_4). There is a panel who is invited to made decision according to a set of attributes: the risk (c_1); the growth (c_2) and the environmental impact (c_3). The weights of attribute are $\omega=(0.35,0.25,0.4)^T$.

The decision-making evaluation of the alternatives based on the set of attributes $c_j\,(j=1,2,3)$ is obtained via questionnaire of a panel. The evaluation information is represented in the form of HBVNEs. Then, according to the proposed MADM approach, we have

TABLE 1. The HBNVS decision making matrix, D

	A_1	A_2
c_1	$\langle\{0.5,0.6\},\{0.2,0.3\},\{0.3,0.4\},-0.4,-0.2,\{-0.4,-0.5\}\rangle$	$\langle\{0.6,0.7\},\{0.1,0.2\},\{0.2,0.3\},\{-0.2,-0.3\},-0.2,-0.7\rangle$
c_2	$\langle 0.5,\{0.1,0.3\},0.2,\{-0.2,-0.3\},-0.7,\{-0.4,-0.6\}\rangle$	$\langle\{0.6,0.7\},\{0.1,0.2\},\{0.2,0.3\},\{-0.2,-0.3\},-0.2,-0.7\rangle$
c_3	$\langle 0.2,\{0.2,0.3\},\{0.5,0.6\},-0.5,-0.8,\{-0.1,-0.3\}\rangle$	$\langle\{0.3,0.6\},\{0.3,0.5\},0.1,-0.2,-0.3,-0.7\rangle$

	A_3	A_4
c_1	$\langle\{0.3,0.6\},0.2,0.4,\{-0.3,-0.4\},\{-0.2,-0.3\},-0.5\rangle$	$\langle 0.7,\{0,0.1\},\{0.1,0.2\},-0.2,-0.1,\{-0.7,-0.8\}\rangle$
c_2	$\langle\{0.5,0.6\},\{0.2,0.3\},0.4,-0.3,-0.7,-0.7\rangle$	$\langle 0.6,0.1,\{0.1,0.3\},-0.2,-0.8,\{-0.6,-0.7\}\rangle$
c_3	$\langle\{0.5,0.6\},0.3,0.2,\{-0.1,-0.3\},-0.6,-0.4\rangle$	$\langle 0.4,0.3,\{0.1,0.2\},\{-0.1,-0.2\},-0.7,-0.3\rangle$

A. THE HBVNS DECISION MAKING PROCEDURE

Step 1: Compute $h_i = HBVNWA_\omega(h_{i1},h_{i2},h_{i3},h_{i4})$ for each $A_i\,(i=1,2,3,4)$. Based on the aggregation operator in (3), we have

$$h_1=\left\langle\{0.36,0.4\},\left\{\begin{matrix}0.17,0.19,0.2,0.22,\\0.23,0.26,0.26,0.3\end{matrix}\right\},\{0.33,0.36,0.37,0.4\},\{-0.37,-0.41\},-0.64,\left\{\begin{matrix}-0.29,-0.34,-0.36,-0.36,\\-0.4,-0.4,-0.42,-0.46\end{matrix}\right\}\right\rangle$$

$$h_2=\left\langle\left\{\begin{matrix}0.5,0.53,0.55,0.56,\\0.6,0.64,0.63,0.66\end{matrix}\right\},\left\{\begin{matrix}0.16,0.18,0.19,0.2,\\0.23,0.24,0.24,0.29\end{matrix}\right\},\left\{\begin{matrix}0.15,0.17,\\0.17,0.19\end{matrix}\right\},\left\{\begin{matrix}-0.2,-0.22,\\-0.23,-0.26\end{matrix}\right\},-0.5,-0.7\right\rangle$$

$$h_3=\left\langle\left\{\begin{matrix}0.44,0.47,0.49,0.51,\\0.54,0.56,0.58,0.6\end{matrix}\right\},\{0.24,0.26\},0.3,\left\{\begin{matrix}-0.19,-0.21,\\-0.3,0.33\end{matrix}\right\},\{-0.53,-0.55\},-0.53\right\rangle$$

$$h_4=\left\langle 0.57,\{0,0.16\},\left\{\begin{matrix}0.1,0.13,0.13,0.13,\\0.17,0.17,0.17,0.22\end{matrix}\right\},\{-0.15,-0.2\},-0.6,\left\{\begin{matrix}-0.55,-0.58,\\-0.61,-0.63\end{matrix}\right\}\right\rangle$$

Step 2: Calculate the score function $s(h_i)\,(i=1,2,3,4)$ for each $A_i\,(i=1,2,3,4)$. By applying the score function in (2), we have

$s(h_1)=0.570$, $s(h_2)=0.695$, $s(h_3)=0.629$ and $s(h_4)=0.727$.

Step 3: According to the obtained score functions, we can rank the alternatives in descending order as $A_4 > A_2 > A_3 > A_1$.

With similar computation, the final ranking result when utilizing the HBVNWG operator is $A_2 > A_4 > A_3 > A_1$. The best alternative when utilizing the HBVNWA operator is A_4 while HBVNWG operator is A_2. The worst alternative is always A_1 when using either HBVNWA or HBVNWG operators.

B. COMPARATIVE ANALYSIS

A comparative analysis is presented to verify the feasibility of the proposed decision-making method. Based on the same example from Ye [31], the proposed HBVNS decision-making method is compared to the existing methods under different information; intuitionistic fuzzy set (IFS), interval-valued IFS (IVIFS), interval-valued Pythagorean fuzzy set (IVPFS), simplified neutrosophic set (SNS), interval-valued neutrosophic set (INS) and single-valued neutrosophic hesitant fuzzy set (SVNHFS). Table 2 shows the characteristics and ranking order of the existing group decision making methods compared to the proposed method.

TABLE 2. The comparison with the existing methods

Method	Characteristics	Set	Ranking Order
Correlation coefficient [32]	✓ Dual membership degrees	IFS	$A_2 > A_4 > A_3 > A_1$
IVIFWA [33]	✓ Dual interval membership degrees	IVIFS	$A_2 > A_4 > A_3 > A_1$
IVPFWA [34]	✓ Larger space to dual membership degrees	IVPFS	$A_2 > A_4 > A_3 > A_1$
GSNNWA GSNNWG [35]	✓ Indeterminate membership degree	SNS	$A_4 > A_2 > A_3 > A_1$ $A_4 > A_2 > A_3 > A_1$
Similarity measures [36]	✓ Interval indeterminate membership degree	INS	$A_4 > A_2 > A_3 > A_1$ $A_2 > A_4 > A_3 > A_1$
SVNHFWA SVNHFWG [31]	✓ Hesitant indeterminate membership degree	SVNHFS	$A_4 > A_2 > A_3 > A_1$ $A_2 > A_4 > A_3 > A_1$
HBVNWA HBVNWG	✓ Indeterminate membership degree ✓ Hesitant membership degrees ✓ Bipolar membership degrees	HBVNS	$A_4 > A_2 > A_3 > A_1$ $A_2 > A_4 > A_3 > A_1$

Ye [32] developed a group decision making method using the correlation coefficient method under dual membership degrees of IFS. Ye [33] extended the IFS by introducing interval-valued IFS where the dual membership degrees are applied in interval form. Interval-valued Pythagorean fuzzy weighted average (IVPFWA) was developed by Garg [34] to aggregate the interval-valued Pythagorean fuzzy numbers which are also based on IFS. From the characteristic column in Table 2, it can be seen that the existing group decision making methods were improvised based on the basic fuzzy set with its well-known characteristic of single membership degrees. However, the proposed decision making method generalizes the above existing methods as it combines three important features which are the indeterminate membership degree, hesitant membership degree and bipolar membership degree. The obtained ranking order of the proposed method is $A_4 > A_2 > A_3 > A_1$ when using HBVNWA operator and $A_2 > A_4 > A_3 > A_1$ when using HBVNWG operator. In general, the obtained results of the proposed method are quite close to the ranking results of the existing methods where the first and second ranking order are alternate between A_4 and A_2 while the third and fourth ranks are consistent. On the other hand, the obtained results of the proposed decision making methods are consistent with Ye [31]. This is because both methods shared similar characteristics of the indeterminate and hesitant membership degrees. In addition, both methods used the weighted averaging and weighted geometric operators in their method development. Nevertheless, the proposed method with HBVNS is more practical compared to the other existing group decision making methods as it can consider the bipolar judgmental thinking of positive and negative sides of the problem. Apparently, the proposed method under HBVNS environment is easy and practical to be used in solving the multi-criteria decision making problems as it emphasizes the bipolarity, hesitancy, indeterminacy and fuzziness information. Moreover, the proposed HBVNS is a generalized form of IFS, IVIFS, SNS, INS and SVNHFS.

VI. CONCLUSION

This paper has successfully established a concept of hesitant bipolar-valued neutrosophic set (HBVNS) by combining the hesitant fuzzy set and the bipolar neutrosophic set. There are several significant features of the proposed set. First, it is practical in solving bipolar, hesitant and indeterminate multi-criteria decision making problems. By adopting the characteristics of the hesitant fuzzy sets, the proposed set provides a lenient way of judgmental process instead of apprehensive typical concrete judgments. Apart from that, the importance of bipolar judgmental thinking during the judgment process can be considered by using the proposed HBVNS. Secondly, some basic properties of HBVNS such as the operational laws, union, intersection and complement are studied. The proposition related to the proposed properties is investigated and proved. Thirdly, this paper proposed two aggregation operators which are the HBVNWA and HBVNWG operators to aggregate the hesitant bipolar-valued neutrosophic elements (HBVNEs). A few important desirable properties of aggregation operators have been proved. A MADM method is developed under the hesitant bipolar-valued neutrosophic environment, in which the evaluation values are in the form of HBVNEs. The proposed HBVNWA and HBVNWG operators can be applied to obtain the best alternative. A numerical example is given and a comparative analysis is conducted to show the applicability and feasibility of the proposed approach. The proposed MADM method under the hesitant bipolar-valued neutrosophic environment can be utilized in dealing with the bipolar, indeterminate and hesitant decision making especially in solving the real scientific and engineering problems. For future work, more MADM methods can be developed based on HBVNS to solve real applications in the areas such as the support system, expert system and information fusion system.

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