



Introduction to AntiGroups

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◇ In commemoration of the 60th birthday of the author

Abstract

The notion of AntiGroups is formally presented in this paper. A particular class of AntiGroups of type-AG[4] is studied with several examples and basic properties presented. In AntiGroups of type-AG[4], the existence of an inverse is taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity are taking to be either partially true, partially indeterminate or partially false for some elements. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that distinct left(right) cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4].

Keywords: NeutroGroup, AntiGroup, AntiSubgroup, AntiQuotientGroup, AntiGroupHomomorphism.

1 Introduction

Neutrosophic logic (NL) introduced by Smarandache in 1995 is an alternative to the existing classical logics and the generalization of fuzzy logic (FL) of Zadeh [12] and intuitionistic fuzzy logic (IFL) of Atanassov [5]. Neutrosophic logic is a non-classical logic that can be used as a mathematical tool to model situations characterized by uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. In neutrosophic logic [11], each proposition is estimated to have percentage of truth in a subset T , the percentage of indeterminacy in a subset I , and the percentage of falsity in a subset F where (T, I, F) are standard or non-standard subsets of the non-standard interval $]^-0, 1^+]$, where $n_{\text{inf}} = \inf T + \inf I + \inf F \geq^- 0$, and $n_{\text{sup}} = \sup T + \sup I + \sup F \leq 3^+$. Statically, (T, I, F) are subsets but dynamically, they are functions/operators depending on many known or unknown parameters. In NL, if $\langle A \rangle$ is an idea, or proposition, theory, law, axiom, event, concept, entity etc., there correspond $\langle \text{Non-}A \rangle = \langle \text{Anti-}A \rangle$ which is the opposite of $\langle A \rangle$ and $\langle \text{Neut-}A \rangle$ which stands for what is neither $\langle A \rangle$ nor $\langle \text{Anti-}A \rangle$, that is neutrality in between the two extremes. Consequently in NL, it is possible to have the triad $(\langle A \rangle, \langle \text{Neut-}A \rangle, \langle \text{Anti-}A \rangle)$. The non-restriction in NL allows for paraconsistent, dialetheist, and incomplete information to be characterized. This special and unique feature of NL has made it applicable in solving problems involving uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. arising from science, social science, engineering, technology, computer science, artificial intelligence, ICT, robotics etc.

The indeterminacy factor I in NL is fundamental in the formulation and establishment of any neutrosophic algebraic structure. Given any classical algebraic structure $(X, *)$, a structure $X(I) = \langle X \cup I \rangle$ generated by X and I under the binary operation $*$ of X is called a neutrosophic algebraic structure with its name derived from the name of X . For instance, if X is a group, then $X(I)$ is called a neutrosophic group. Since I^{-1} , the inverse of I does not exist, finding x^{-1} , the inverse of any neutrosophic element $x \in X(I)$ becomes difficult and impossible. Consequently, algebraic manipulations of the elements of $X(I)$ become restrictive. The recent introduction of the concepts of NeutroStructures and AntiStructures have lessened the restrictiveness of the algebraic manipulations of the elements of neutrosophic algebraic structures imposed by the neutrosophic element I .

In any classical algebraic structure $(X, *)$, the of composition of the elements with respect to the binary operation $*$ is well defined for all the elements of X that is, $x * y \in X \quad \forall x, y \in X$. All the axioms like associativity, commutativity, distributivity, monotonicity etc. defined on X with respect to $*$ are totally true for all the elements of X . The compositions of elements of X this way are restrictive and do not reflect the reality. They do not give rooms for compositions that are either partially defined, partially undefined (indeterminate), and partially outerdefined or totally outerdefined with respect to $*$. However in the domain of knowledge, science and reality, the law of composition and axioms defined on X may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to the binary operation $*$. In 2019, Smarandache [10] addressed the problem of allowing the law of composition on X to be either partially defined and partially undefined or totally undefined by introducing the notions of NetroDefined and AntiDefined laws, as well as the notions of NeutroAxiom and AntiAxiom inspired by NL he introduced in 1995. The work of Smarandache in [10] has given birth to the new fields of research called NeutroStructures and AntiStructures. For any classical algebraic law or axiom defined on X , there correspond neutrosophic triads ($\langle \text{Law} \rangle$, $\langle \text{NeutroLaw} \rangle$, $\langle \text{AntiLaw} \rangle$) and ($\langle \text{Axiom} \rangle$, $\langle \text{NeutroAxiom} \rangle$, $\langle \text{AntiAxiom} \rangle$) respectively. In [9], Smarandache studied NeutroAlgebras and AntiAlgebras and in [8], he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. In [7], Rezaei and Smarandache studied Neutro-BE-algebras and Anti-BE-algebras and they showed that any classical algebra S with n operations (laws and axioms) where $n \geq 1$ will have $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras. In [2], Agboola et al. studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} and in [3], Agboola studied NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he studied NeutroSubgroups, NeutroCyclic-Groups, NeutroQuotientGroups and NeutroGroupHomomorphisms. He showed that generally, Lagrange's theorem and fundamental homomorphism theorem of the classical groups do not hold in the class of NeutroGroups studied. In [4], Agboola introduced and studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). He presented Several results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms. He showed that that the fundamental homomorphism of the classical rings holds in the class of NeutroRings considered. Motivated and inspired by the work of Rezaei and Smarandache in [7], Agboola in [1] revisited the NeutroGroups by studying a particular class of NeutroGroups and presented their basic and elementary properties. In the present paper however, the concept of AntiGroups is formally presented. A particular class of AntiGroups is studied with presentation of several examples and basic properties. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups studied. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups studied in this paper.

2 Preliminaries

In this section, we will give some definitions and results that will be used later in the paper.

Definition 2.1. [8]

- (i) A classical operation is an operation well defined for all the set's elements.
- (ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.
- (iii) An AntiOperation is an operation that is outer defined for all set's elements.
- (iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).
- (v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

- (vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.
- (vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no Anti-Operation or AntiAxiom.
- (viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 2.2. [7] Let \mathbb{U} be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of \mathbb{U} . If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.

Definition 2.3. [Classical group][6]

Let G be a nonempty set and let $*$: $G \times G \rightarrow G$ be a binary operation on G . The couple $(G, *)$ is called a classical group if the following conditions hold:

- (G1) $x * y \in G \forall x, y \in G$ [closure law].
- (G2) $x * (y * z) = (x * y) * z \forall x, y, z \in G$ [axiom of associativity].
- (G3) There exists $e \in G$ such that $x * e = e * x = x \forall x \in G$ [axiom of existence of neutral element].
- (G4) There exists $y \in G$ such that $x * y = y * x = e \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G .

If in addition $\forall x, y \in G$, we have

- (G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.4. [AntiSophication of the law and axioms of the classical group][1]

- (AG1) For all the duplets $(x, y) \in G$, $x * y \notin G$ [AntiClosureLaw].
- (AG2) For all the triplets $(x, y, z) \in G$, $x*(y*z) \neq (x*y)*z$ [AntiAxiom of associativity (AntiAssociativity)].
- (AG3) There does not exist an element $e \in G$ such that $x * e = e * x = x \forall x \in G$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].
- (AG4) There does not exist $u \in G$ such that $x * u = u * x = e \forall x \in G$ [AntiAxiom of existence of inverse element (AntiInverseElement)] where e is an AntiNeutralElement in G .
- (AG5) For all the duplets $(x, y) \in G$, $x * y \neq y * x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.5. [AntiGroup][1]

An AntiGroup AG is an alternative to the classical group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$.

Definition 2.6. [AntiAbelianGroup][1]

An AntiAbelianGroup AG is an alternative to the classical abelian group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$ and $AG5$.

Theorem 2.7. [1] Let $(G, *)$ be a finite or infinite classical group. Then there are 65 types of AntiGroups.

Theorem 2.8. [1] Let $(G, *)$ be a finite or infinite classical abelian group. Then there are 211 types of AntiAbelianGroups.

Definition 2.9. Let $(AG, *)$ be an AntiGroup. AG is said to be finite of order n if the cardinality of AG is n that is $o(AG) = n$. Otherwise, AG is called an infinite AntiGroup and we write $o(AG) = \infty$.

Since there are many types of AntiGroups, in what follows, AntiGroups will be classified and named type-AG[,] according to which of $AG1 - AG5$ is(are) satisfied. If only $AG1$ is satisfied, the AntiGroup will be called of type-AG[1], type-AG[3,4] if only $AG3$ and $AG4$ are satisfied and so on. AntiGroups of type-AG[1,2,3,4] or of type-AG[1,2,3,4,5] will be called trivial AntiGroups or trivial AntiAbelianGroups respectively.

Example 2.10. Let $AG = \mathbb{Q}_+^*$ be the set of all irrational positive numbers and consider algebraic structure (AG, \cdot) where " \cdot " is the ordinary multiplication of real numbers. Then (AG, \cdot) is an infinite trivial AntiGroup.

Example 2.11. Let $AG = \mathbb{N}$.

(i) Let $*$ be a binary operation on AG defined $\forall x, y \in AG$ by

$$x * y = x + y + xy.$$

Then $(AG, *)$ is a finite AntiGroup of type-AG[3,4].

(ii) Let $*$ be a binary operation on AG defined $\forall x, y \in AG$ by

$$x * y = x + y + 1.$$

Then $(AG, *)$ is a finite AntiGroup of type-AG[3,4].

3 A Study of Finite AntiGroups of Type-AG[4]

In this section, we are going to study a particular class of AntiGroups $(AG, *)$ where $G4$ is totally false for all the elements of AG while $G1, G2, G3$ and $G5$ are either partially true, partially indeterminate or partially false for some elements of AG .

Definition 3.1. Let $(AG, *)$ and (AH, \circ) be AntiGroups of type-AG[4]. The direct product of AG and AH denoted by $AG \times AH$ is defined by

$$AG \times AH = \{(g, h) : g \in AG, h \in AH\}.$$

Proposition 3.2. Let $(AG, *)$ and (AH, \circ) be AntiGroups of type-AG[4] and let \otimes be a binary operation on $AG \times AH$ defined by

$$(g, h) \otimes (x, y) = (g * x, h \circ y) \quad \forall (g, h), (x, y) \in AG \times AH.$$

Then $(AG \times AH, \otimes)$ is an AntiGroup of type-AG[4].

Proof. The proof follows from the definition of AntiGroups of type-AG[4] and the definition of direct product of AntiGroups of type-AG[4]. \square

Proposition 3.3. Let $(AG, *)$ be an AntiGroup of type-AG[4] and let $g, x, y \in AG$. Then

$$(i) \quad g * x = g * y \not\Rightarrow x = y.$$

$$(ii) \quad x * g = y * g \not\Rightarrow x = y.$$

Proof. Since g^{-1} does not exist and $*$ is NeutroAssociative, the required results follow. \square

Proposition 3.4. Let $(AG, *)$ be an AntiGroup of type-AG[4], $x, y \in AG$ and let $m, n \in \mathbb{N}$. Then

$$(i) \quad x^{m+1} \neq x^m * x.$$

$$(ii) \quad x^{-m} \neq (x^{-1})^m.$$

(iii) $x^m * x^{-m} \neq N_e$ where N_e is a NeutroNeutralElement in AG .

$$(iv) \quad x^m * x^n \neq x^{m+n}.$$

$$(v) \quad (x^m)^n \neq x^{mn}.$$

$$(vi) \quad (x * y)^m \neq x^m * y^m.$$

Proof. Since x^{-1} does not exist and $*$ is NeutroAssociative, the required results follow. \square

Corollary 3.5. An AntiGroup $(AG, *)$ of type-AG[4] cannot be generated by an element $x \in AG$ and hence cannot be cyclic.

Definition 3.6. Let $(AG, *)$ be an AntiGroup of type-AG[4]. A nonempty subset AH of AG is called an AntiSubgroup of AG if $(AH, *)$ is also an AntiGroup of the same type as AG . Otherwise, if $(AH, *)$ is an AntiGroup of a type different from the type of AG , then AH is called a QuasiAntiSubgroup of AG .

Definition 3.7. Let $(AG, *)$ be an AntiGroup of type-AG[4] and let AH and AK be AntiSubgroups of AG . The set $A * B$ is defined by

$$A * B = \{x \in AG : x = h * k \text{ for some } h \in AH, k \in AK\}.$$

Proposition 3.8. Let AH, AK and AL be AntiSubgroups of an AntiGroup $(AG, *)$ of type-AG[4]. Then

- (i) $AH * AH \neq AH$.
- (ii) $AH * AK \neq AK * AH$.
- (iii) $AH * (AK * AL) \neq (AH * AK) * AL$.

Proof. Obvious. □

Definition 3.9. Let $(AG, *)$ be an AntiGroup of type-AG[4] and let $a \in AG$ be a fixed element.

- (i) An AntiCenter of AG denoted by $AZ(AG)$ is a set defined by

$$AZ(AG) = \{x \in AG : x * g \neq g * x \ \forall g \in AG\}.$$

- (ii) An AntiCentralizer of $a \in AG$ denoted by AC_a is a set defined by

$$AC_a = \{g \in AG : g * a \neq a * g\}.$$

Example 3.10. Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $AG = \{a, b, c, e\}$ be a subset of \mathbb{U} .

- (i) Let $*$ be a binary operation defined on AG as shown in the Cayley table below.

*	a	b	c	e
a	d	a	c	e
b	a	d	e	c
c	b	a	f	e
e	a	b	c	f

It is evident from the table that $G1, G2, G3, G5$ are either partially true or partially false with respect to $*$ but $G4$ is totally false for all the elements of AG . Hence $(AG, *)$ is a finite AntiGroup of type-AG[4].

- (ii) Let $*$ be a binary operation defined on AG as shown in the Cayley table below.

*	e	a	b	c
e	d	a	b	c
a	a	f	c	b
b	b	a	?	c
c	c	b	a	?

It is evident from the table that $G1, G2, G3, G5$ are either partially true, partially indeterminate or partially false with respect to $*$ but $G4$ is totally false for all the elements of AG . Hence $(AG, *)$ is a finite AntiGroup of type-AG[4].

Example 3.11. Let $(AG, *)$ be the AntiGroup of Example 3.10(i) and let $AH_1 = \{a, b, e\}$ and $AH_2 = \{b, c, e\}$ be two subsets of AG . Let $*$ be defined on AH_1 and AH_2 as shown in the Cayley tables below:

$AH_1 :$	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>*</td><td>a</td><td>b</td><td>e</td></tr> <tr><td>a</td><td>d</td><td>a</td><td>e</td></tr> <tr><td>b</td><td>a</td><td>d</td><td>c</td></tr> <tr><td>e</td><td>a</td><td>b</td><td>f</td></tr> </table>	*	a	b	e	a	d	a	e	b	a	d	c	e	a	b	f	$,$	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>*</td><td>b</td><td>c</td><td>e</td></tr> <tr><td>b</td><td>d</td><td>e</td><td>c</td></tr> <tr><td>c</td><td>a</td><td>f</td><td>e</td></tr> <tr><td>e</td><td>b</td><td>c</td><td>f</td></tr> </table>	*	b	c	e	b	d	e	c	c	a	f	e	e	b	c	f
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It can easily be seen from the tables that AH_1 is an AntiSubgroup of AG while AH_2 is a QuasiAntiSubgroup of AG . It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{a, b, c, e\} = AG, \\ NH_1 \cap NH_2 &= \{b, e\}, \end{aligned}$$

from which it is deduced that $NH_1 \cup NH_2$ is an AntiSubgroup of AG but $AH_1 \cap AH_2$ is a QuasiAntiSubgroup of AG as it is evident in the Cayley table below:

$$NH_1 \cap NH_2 : \begin{array}{|c|c|c|} \hline * & b & e \\ \hline b & d & c \\ \hline e & b & f \\ \hline \end{array}.$$

$AZ(AG) = \{a, b, c, e\} = AG$ is an AntiSubgroup of AG . Also, $AC_a = AC_b = AC_c = AC_e = \{a, b, c, e\} = AG$ are AntiSubgroups of AG .

Example 3.12. Let $(AG, *)$ be the AntiGroup of Example 3.10(ii) and let $AH_1 = \{e, a, b\}$ and $AH_2 = \{e, b, c\}$ be two subsets of AG . Let $*$ be defined on AH_1 and AH_2 as shown in the Cayley tables below:

$$AH_1 : \begin{array}{|c|c|c|c|} \hline * & e & a & b \\ \hline e & d & a & b \\ \hline a & a & f & c \\ \hline b & b & a & ? \\ \hline \end{array}, \quad AH_2 : \begin{array}{|c|c|c|c|} \hline * & e & b & c \\ \hline e & d & b & c \\ \hline b & b & ? & c \\ \hline c & c & a & ? \\ \hline \end{array}.$$

It can easily be seen from the tables that AH_1 and AH_2 are AntiSubgroups of AG . It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{e, a, b, c\} = AG, \\ NH_1 \cap NH_2 &= \{e, b\}, \end{aligned}$$

from which it is deduced that $NH_1 \cup NH_2$ is an AntiSubgroup of AG and $AH_1 \cap AH_2$ is an AntiSubgroup of AG as it is evident in the Cayley table below:

$$NH_1 \cap NH_2 : \begin{array}{|c|c|c|} \hline * & e & b \\ \hline e & d & f \\ \hline b & b & ? \\ \hline \end{array}.$$

$AZ(AG) = \{a, b, c\}$ is a QuasiAntiSubgroup of AG . Also, $AC_a = AC_b = \{a, b, c\}$, $AC_c = \{b, c\}$ are QuasiAntiSubgroups of AG and $AC_e = \{\} = \emptyset$ is neither an AntiSubgroup nor a QuasiAntiSubgroup of AG .

Example 3.13. (i) Let $AG = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and let \oplus be a binary operation on AG as defined in the Cayley table.

$$\begin{array}{|c|c|c|c|c|} \hline \oplus & 0 & 1 & 2 & 3 \\ \hline 0 & 0 \text{ or } 1 & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 & 0 \text{ or } 2 \\ \hline 2 & 2 & 3 & 0 \text{ or } 3 & 1 \\ \hline 3 & 3 & ? & 1 & 2 \\ \hline \end{array}.$$

Then (AG, \oplus) is a finite AntiGroup of type-AG[4] and $AH = \{0, 1, 2\}$ is an AntiSubgroup of AG .

(ii) Let $AG = \{1, 2, 3, 4\} \subseteq \mathbb{Z}_5$ and let \otimes be a binary operation on AG as defined in the Cayley table.

$$\begin{array}{|c|c|c|c|c|} \hline \otimes & 1 & 2 & 3 & 4 \\ \hline 1 & ? & 2 & 3 & 4 \\ \hline 2 & 2 & 4 & 0 & 3 \\ \hline 3 & 3 & ? & 4 & 2 \\ \hline 4 & 4 & 3 & 2 & 0 \\ \hline \end{array}.$$

Then (AG, \otimes) is a finite AntiGroup of type-AG[4] and $AH = \{1, 2, 3\}$ is an AntiSubgroup of AG .

Definition 3.14. Let AH be an AntiSubgroup of the AntiGroup $(AG, *)$ of type-AG[4] and let $x \in AG$.

(i) $x * AH$ the left coset of AH in AG is defined by

$$x * AH = \{x * h : h \in AH\}.$$

(ii) $AH * x$ the right coset of AH in AG is defined by

$$AH * x = \{h * x : h \in AH\}.$$

(iii) AH is called a normal AntiSubgroup if $x * AH = AH * x$ for at least one $x \in AG$.

(iv) The number of distinct left or right cosets of AH in AG is called the index of AH in AG and it is denoted by $[AG : AH]$.

(vi) The set of all distinct left cosets of AH in AG denoted by $(AG/AH)_L$ is defined by

$$(AG/AH)_L = \{x * AH : x \in AG\}.$$

(vii) The set of all distinct right cosets of AH in AG denoted by $(AG/AH)_R$ is defined by

$$(AG/AH)_R = \{AH * x : x \in AG\}.$$

Suppose that AG/AH is the set of all distinct left cosets of AH in AG and suppose that \odot is a binary operation on AG/AH defined by

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x * AH, y * AH \in AG/AH.$$

If the couple $(AG/AH, \odot)$ is an AntiGroup of type-AG[4], then AG/AH is called an AntiQuotientGroup of AG factored by AH .

Lemma 3.15. Let AH be an AntiSubgroup of the AntiGroup $(AG, *)$ of type-AG[4] and let $e \in AG$ be a NeutroNeutralElement. Then globally,

$$e * AH \neq AH.$$

Example 3.16. Let (AG, \oplus) be an AntiGroup of Example 3.13 (i) and let $AH = \{0, 1, 2\}$ be its AntiSubgroup. The left and right cosets of AH in AG are:

$$\begin{aligned} 0 \oplus AH &= \{0, 1, 2\} \text{ or } \{1, 2\} = AH \oplus 0, \\ 1 \oplus AH &= \{1, 2, 3\}, = AH \oplus 1 \\ 2 \oplus AH &= \{0, 2, 3\} \text{ or } \{2, 3\} = AH \oplus 2, \\ 3 \oplus AH &= \{1, 3, ?\} = AH \oplus 3, \\ \therefore AG/AH &= \{0 \oplus AH, 1 \oplus AH, 2 \oplus AH, 3 \oplus AH\}. \end{aligned}$$

It is noted that AH is a normal AntiSubgroup of AG and distinct left and right cosets of AH do not partition AG .

Now consider the Cayley table below.

\oplus	$0 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$0 \oplus AH$	$0 \oplus AH$ or $1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $2 \oplus AH$
$2 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $3 \oplus AH$	$1 \oplus AH$
$3 \oplus AH$	$3 \oplus AH$?	$1 \oplus AH$	$2 \oplus AH$

It is evident from the table that $(AG/AH, \oplus)$ is an AntiGroup of type-AG[4].

Example 3.17. Let (AG, \otimes) be an AntiGroup of Example 3.13 (ii) and let $AH = \{1, 2, 3\}$ be its AntiSubgroup. The left cosets of AH in AG are:

$$\begin{aligned} 1 \otimes AH &= \{?, 2, 3\} = AH \otimes 1, \\ 2 \otimes AH &= \{0, 2, 4\} = AH \otimes 2, \\ 3 \otimes AH &= \{3, 4, ?\} = AH \otimes 3, \\ 4 \otimes AH &= \{2, 3, 4\} = AH \otimes 4, \\ \therefore AG/AH &= \{1 \otimes AH, 2 \otimes AH, 3 \otimes AH, 4 \otimes AH\}. \end{aligned}$$

It is noted that AH is a normal AntiSubgroup of AG and distinct left and right cosets of AH do not partition AG .

Now consider the Cayley table below.

\otimes	$1 \otimes AH$	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$1 \otimes AH$?	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$2 \otimes AH$	$2 \otimes AH$	$4 \otimes AH$	$0 \otimes AH$	$3 \otimes AH$
$3 \otimes AH$	$3 \otimes AH$?	$4 \otimes AH$	$2 \otimes AH$
$4 \otimes AH$	$4 \otimes AH$	$3 \otimes AH$	$2 \otimes AH$	$0 \otimes AH$

It is evident from the table that $(AG/AH, \otimes)$ is an AntiGroup of type-AG[4].

Proposition 3.18. Let AH be a normal AntiSubgroup of an AntiGroup $(AG, *)$ of type-AG[4] and let AG/AH be the set of distinct left cosets of AH in AG . For $x * AH, y * AH \in AG/AH$ with $x, y \in AG$, let \odot be a binary operation defined on AG/AH by

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x, y \in AG.$$

Then, $(AG/AH, \odot)$ is an AntiGroup of type-AG[4].

Proof. Suppose that AH is a normal AntiSubgroup of an AntiGroup $(AG, *)$ of type-AG[4] and suppose that the composition of elements in AG/AH is given by $(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x, y \in AG$. Then there exist some duplets $(x, y), (u, v), (p, q) \in AG$ such that $x * y \in AG$ (inner-defined) and $[u * v = \text{indeterminate or } p * q \notin AG \text{ (outer-defined/falsehood)}]$. Hence, \odot satisfies the NeutroClosureLaw. Next, there exist some triplets $(x, y, z), (p, q, r), (u, v, w) \in AG$ such that $x * (y * z) = (x * y) * z$ (inner-defined) and $[[p * (q * r)] \text{ or } (p * q) * r = \text{indeterminate or } u * (v * w) \neq (u * v) * w \text{ (outer-defined/falsehood)}]$. This again shows that \odot satisfies the NeutroAssociativityAxiom. Also, there exists an element $e \in AG$ such that $x * e = e * x = x$ (inner-defined) and $[[x * e] \text{ or } [e * x] = \text{indeterminate or } x * e \neq x \neq e * x \text{ (outer-defined/falsehood)}]$ for at least one $x \in AG$. This shows the existence of NeutroNeutralElement in AG and hence there exists a NeutroNeutralElement $e * AH \in AG/AH$. Again for all $x \in AG$, there does not exist $u \in AG$ such that $x * u = u * x = e$. This is an AntiAxiom of existence of inverse element in AG and consequently, no element $x * AH \in AG/AH$ has an inverse. Lastly, there exist some duplets $(x, y), (u, v), (p, q) \in AG$ such that $x * y = y * x$ (inner-defined) and $[[u * v] \text{ or } [v * u] = \text{indeterminate or } p * q \neq q * p \text{ (outer-defined/falsehood)}]$. This shows that \odot satisfies the NeutroCommutativityAxiom. Hence, $(AG/AH, \odot)$ is an AntiGroup of type-AG[4]. \square

Definition 3.19. Let $(AG, *)$ and (AH, \circ) be any two AntiGroups of type-AG[4]. The mapping $\phi : AG \rightarrow AH$ is called an AntiGroupHomomorphism if ϕ does not preserve the binary operations $*$ and \circ that is for all the duplet $(x, y) \in AG$, we have

$$\phi(x * y) \neq \phi(x) \circ \phi(y).$$

The kernel of ϕ denoted by $Ker\phi$ is defined by

$$Ker\phi = \{x : \phi(x) = e_{AH} \text{ for at least one } e_{AH} \in AH\}$$

where e_{AH} is a NeutroNeutralElement in AH .

The image of ϕ denoted by $Im\phi$ is defined by

$$Im\phi = \{y \in AH : y = \phi(x) \text{ for some } x \in AG\}.$$

If in addition ϕ is an AntiBijection, then ϕ is called an AntiGroupIsomorphism. AntiGroupEpimorphism, AntiGroupMonomorphism, AntiGroupEndomorphism, and AntiGroupAutomorphism are defined similarly.

Example 3.20. (i) Let (AG, \oplus) be the AntiGroup of Example 3.13 (i) and let $\phi : AG \rightarrow AG$ be a mapping defined by

$$\phi(x) = 2 \oplus x \quad \forall x \in AG.$$

Then

$$\begin{aligned} \phi(0) &= 2, \\ \phi(1) &= 3, \\ \phi(2) &= 0 \text{ or } 3, \\ \phi(3) &= 1, \end{aligned}$$

from which we obtain that $\phi(x \oplus y) \neq \phi(x) \oplus y$ for all $x, y \in AG$. Accordingly, ϕ is an AntiGroupHomomorphism. $Im\phi = \{1, 2, 3\}$ which is an AntiSubgroup of AG . $Ker\phi = \{\} = \emptyset$.

- (ii) Let (AG, \otimes) be the AntiGroup of Example 3.13 (ii) and let $\psi : AG \rightarrow AG$ be a mapping defined by

$$\psi(x) = x \otimes 4 \quad \forall x \in AG.$$

Then

$$\begin{aligned} \psi(1) &= 4, \\ \psi(2) &= 3, \\ \psi(3) &= 2, \\ \psi(4) &= 0, \end{aligned}$$

from which we obtain that $\psi(x \otimes y) \neq \psi(x) \otimes y$ for all $x, y \in AG$. Accordingly, ψ is an AntiGroupHomomorphism. $Im\psi = \{0, 2, 3, 4\}$ which is not an AntiSubgroup of AG . $Ker\psi = \{\} = \emptyset$.

Example 3.21. (i) Let $(AG, *)$ be the AntiGroup of Example 3.10 (i) and let $\phi : AG \times AG \rightarrow AG$ be a projection defined by

$$\phi((x, y)) = x \quad \forall x, y \in AG.$$

Then ϕ is not an AntiGroupHomomorphism because $\phi((a, b) \otimes (b, c)) = \phi((a, b)) * \phi((b, c)) = a$. However, $Im\phi = \{a, b, c, e\} = AG$.

- (ii) Let $(AG, *)$ be the AntiGroup of Example 3.10 (ii) and let $\psi : AG \times AG \rightarrow AG$ be a projection defined by

$$\psi((x, y)) = y \quad \forall x, y \in AG.$$

Then ψ is not an AntiGroupHomomorphism because $\psi((a, b) \otimes (b, c)) = \psi((a, b)) * \psi((b, c)) = c$. However, $Im\psi = \{a, b, c, e\} = AG$.

Example 3.22. (i) Let $(AG/AH, \oplus)$ be the AntiQuotientGroup of Example 3.16 and let $\phi : AG \rightarrow AG/AH$ be a mapping defined by

$$\phi(x) = x \oplus AH \quad \forall x \in AG.$$

Then

$$\begin{aligned} \phi(0) &= 0 \oplus AH, \\ \phi(1) &= 1 \oplus AH, \\ \phi(2) &= 2 \oplus AH, \\ \phi(3) &= 3 \oplus AH, \end{aligned}$$

from which we obtain

$$\phi(1 \oplus 2) = \phi(1) \oplus \phi(2) = 3 \oplus AH.$$

This shows that ϕ is not an AntiGroupHomomorphism.

- (ii) Let $(AG/AH, \otimes)$ be the AntiQuotientGroup of Example 3.17 and let $\psi : AG \rightarrow AG/AH$ be a mapping defined by

$$\psi(x) = x \otimes AH \quad \forall x \in AG.$$

Then

$$\begin{aligned} \psi(1) &= 1 \otimes AH, \\ \psi(2) &= 2 \otimes AH, \\ \psi(3) &= 3 \otimes AH, \\ \psi(4) &= 4 \otimes AH, \end{aligned}$$

from which we obtain

$$\psi(2 \otimes 4) = \psi(2) \otimes \psi(4) = 3 \otimes AH.$$

This shows that ψ is not an AntiGroupHomomorphism.

Remark 3.23. The fundamental theorem of homomorphisms of the classical groups cannot hold in the class of AntiGroups of type-AG[4] as demonstrated in Examples 3.22 (i) and (ii).

4 Conclusion

The notion of AntiGroups was formally presented in this paper. A particular class of AntiGroups of type-AG[4] was studied. In AntiGroups of type-AG[4], the existence of an inverse element was taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity were taking to be either partially true, partially indeterminate or partially false for some elements. It was shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it was shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it was shown that distinct left(right)cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4]. More classes of AntiGroups will be studied in our future papers.

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