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Intuitionistic topological spaces

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Abstract. First of all, we list some concepts and results introduced by [10, 15]. Second, we give some examples related to intuitionistic topologies and intuitionistic bases, and obtain two properties of an intuitionistic base and an intuitionistic subbase. And we define intuitionistic intervals in $\mathbb{R}$. Finally, we define some types of intuitionistic closures and interiors, and obtain their some properties.

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1. Introduction

In 1983, Atanassove [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets by introduced by Zadeh [21] considering the degree of membership and non-membership (See [2, 3, 4, 5, 6], in order to refer to the details of intuitionistic fuzzy sets). In 1996, Coker [10] introduced the concept of an intuitionistic set (called an intuitionistic crisp set by Salama et al.[18]) as the generalzation of an ordinary set and the specialization of an intuitionistic fuzzy set. After that time, many researchers [7, 8, 11, 12, 13, 17, 19] applied the notion to topology and Selvanayaki and Ilango [20] studied homeomorphisms in intuitionistic topological spaces. In particular, Bayhan and Coker [9] dealt with pairwise separation axioms in intuitionistic topological spaces and some relationships between categories **Dbl-Top** and **Bitop**. Furthermore, Lee and Chu [16] introduced the category **ITop** and investigated some relationships between **ITop** and **Top**. Recently, Kim et al. [15] investigate the category **ISet** composed of intuitionistic sets and morphisms between them in the sense of a topological universe.

In this paper, first of all, we list some concepts and results introduced by [10, 15]. Second, we give some examples (See Examples 3.2, 3.2,3.10,3.13 and 3.15) related to intuitionistic topologies and intuitionistic bases, and obtain two properties of an intuitionistic base and an intuitionistic subbase. And we define intuitionistic intervals in $\mathbb{R}$. Finally, we define some types of intuitionistic closures and interiors, and obtain their some properties.

2. Preliminaries

In this section, we list the concepts of an intuitionistic set, an intuitionistic point, an intuitionistic vanishing point and operations of intuitionistic sets. Also we list some results obtained by [10, 15].

Definition 2.1 ([10])**.** Let X be a non-empty set. Then A is called an intuitionistic set (in short, IS) of X, if it is an object having the form

$$A = (A_T, A_F),$$

such that $A_T \cap A_F = \phi$, where A_T [resp. A_F] is called the set of members [resp. nonmembers] of A.

In fact, A_T [resp. A_F] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by ϕ_I [resp. X_I], is defined by $\phi_I = (\phi, X)$ [resp. $X_I = (X, \phi)$].

In general, $A_T \cup A_F \neq X$.

We will denote the set of all ISs of X as $IS(X)$.

It is obvious that $A = (A, \phi) \in IS(X)$ for each ordinary subset A of X. Then we can consider an IS of X as the generalization of an ordinary subset of X. Furthermore, it is clear that $A = (A_T, A_T, A_F)$ is an neutrosophic crisp set in X, for each $A \in IS(X)$. Thus we can consider a neutrosophic crisp set in X as the generalization of an IS of X.

Remark 2.2. Let X be a set and let $A \in IS(X)$ such that $A_T \cup A_F = X$. We define the mappings $\mu,\ \nu : X \to [0, 1]$ as follows: for each $x \in X$,

$$\mu(x) = \chi_{A_T}(x),\ \nu(x) = \chi_{A_F}(x).$$

Then we can easily see that (μ, ν) is an intuitionistic fuzzy set in X introduced by Atanassov [1]. Thus by identifying A with (μ, ν), we can consider the intuitionistic set A in X as an an intuitionistic fuzzy set in X. However, if $A_T \cup A_F \neq X$, then (μ, ν) is not an intuitionistic fuzzy set in X, since $\mu(x) + \nu(x) = 0$, for each $x \notin A_T \cap A_F$.

Definition 2.3 ([10])**.** Let $A, B \in IS(X)$ and let $(A_j)_{j\in J} \subset IS(X)$.

(i) We say that A is contained in B, denoted by $A \subset B$, if $A_T \subset B_T$ and $A_F \supset B_F$.

(ii) We say that A equals to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of A denoted by A^c, is an IS of X defined as:

$$A^c = (A_F, A_T).$$

(iv) The union of A and B, denoted by $A \cup B$, is an IS of X defined as:

$$A \cup B = (A_T \cup B_T, A_F \cap B_F).$$

(v) The union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} A_j$ (in short, $\bigcup A_j$), is an IS of X defined as:

$$\bigcup_{j\in J} A_j = (\bigcup_{j\in J} A_{j,T}, \bigcap_{j\in J} A_{j,F}).$$

(vi) The intersection of A and B, denoted by $A\cap B$, is an IS of X defined as:

$$A\cap B = (A_T\cap B_T, A_F\cup B_F).$$

(vii) The intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$ (in short, $\bigcap A_j$), is an IS of X defined as:

$$\bigcap_{j\in J} A_j = (\bigcap_{j\in J} A_{j,T}, \bigcup_{j\in J} A_{j,F}).$$

(viii) $A - B = A\cap B^c$.

(ix) $[\ ]A = (A_T, {A_T}^c)$, $<> A = ({A_F}^c, A_F)$.

Example 2.4. Let $X = \{a,b,c,d,e,f\}$ and let $A = (\{b,c,f\},\{b,d\}) \in IS(X)$. Then $A^c = (A_F, A_T)$. Thus

$$\begin{aligned} A\cup A^c &= (A_T\cup A_F, A_F\cap A_T) \\ &= (\{a,c,f\}\cup\{b,d\}, \{b,d\}\cap\{a,c,f\}) \\ &= \{a,b,c,d,f\},\phi) \\ &\neq X_I \end{aligned}$$

and

$$\begin{aligned} A\cap A^c &= (A_T\cap A_F, A_F\cup A_T) \\ &= (\{a,c,f\}\cap\{b,d\}, \{b,d\}\cup\{a,c,f\}) \\ &= (\phi, \{a,b,c,d,f\}) \\ &\neq \phi_I. \end{aligned}$$

Result 2.5 ([15], Proposition 3.6)**.** *Let $A,B,C\in IS(X)$. Then*

(1) (Idempotent laws): $A\cup A = A$, $A\cap A = A$,

(2) (Commutative laws): $A\cup B = B\cup A$, $A\cap B = B\cap A$,

(3) (Associative laws): $A\cup(B\cup C) = (A\cup B)\cup C$, $A\cap(B\cap C) = (A\cap B)\cap C$,

(4) (Distributive laws): $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$,
$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$,

(5) (Absorption laws): $A\cup(A\cap B) = A$, $A\cap(A\cup B) = A$,

(6) (DeMorgan's laws): $(A\cup B)^c = A^c\cap B^c$, $(A\cap B)^c = A^c\cup B^c$,

(7) $(A^c)^c = A$,

(8) (8a) $A\cup\phi_I = A$, $A\cap\phi_I = \phi_I$,
(8b) $A\cup X_I = X_I$, $A\cap X_I = A$,
(8c) ${X_I}^c = \phi_I$, ${\phi_I}^c = X_I$,
(8d) *in general,* $A\cup A^c \neq X_I$, $A\cap A^c \neq \phi_I$.

Result 2.6 ([15], Proposition 3.7)**.** *Let $A\in IS(X)$ and let $(A_j)_{j\in J}\subset IS(X)$. Then*

(1)([10], Corollary 2.7) $(\bigcap A_j)^c = \bigcup A_j^c$, $(\bigcup A_j)^c = \bigcap {A_j}^c$,

(2) $A\cap(\bigcup A_j) = \bigcup(A\cap A_j)$, $A\cup(\bigcap A_j) = \bigcap(A\cup A_j)$.

Definition 2.7 ([10])**.** Let X be a non-empty set, $a\in X$ and let $A\in IS(X)$.

(i) The form $(\{a\},\{a\}^c)$ [resp. $(\phi,\{a\}^c)$]is called an intuitionistic point [resp. vanishing point] of X and denoted by a_I [resp. a_{IV}].

(ii) We say that a_I [resp. a_{IV}] is contained in A, denoted by $a_I \in A$ [resp. $a_{IV} \in A$], if $a \in A_T$ [resp. $a \notin A_F$].

We will denote the set of all intuitionistic points or intuitionistic vanishing points in X as $IP(X)$.

Result 2.8 ([10], Proposition 3.4)**.** *Let $(A_j)_{j\in J} \subset IS(X)$ and let $p \in X$.*
(1) $p_I \in \bigcap A_j$ [resp. $p_{IV} \in \bigcap A_j$] if and only if $p_I \in A_j$ [resp. $p_{IV} \in A_j$], for each $j \in J$.
(2) $p_I \in \bigcup A_j$ [resp. $p_{IV} \in \bigcup A_j$] if and only if there exists $j \in J$ such that $p_I \in A_j$ [resp. $p_{IV} \in A_j$.

Result 2.9 ([10], Proposition 3.5)**.** *Let $A, B \in IS(X)$. Then*
(1) $A \subset B$ if and only if $p_I \in A \Rightarrow p_I \in B$ [resp. $p_{IV} \in A \Rightarrow p_{IV} \in B$], for each $p \in X$.
(2) $A = B$ if and only if $p_I \in A \Leftrightarrow p_I \in B$ [resp. $p_{IV} \in A \Leftrightarrow p_{IV} \in B$], for each $p \in X$.

Result 2.10 ([10], Proposition 3.6)**.** *Let $A \in IS(X)$. Then*

$$A = (\bigcup_{a_I \in A} a_I) \cup (\bigcup_{a_{IV} \in A} a_{IV}).$$

For each $A \in IS(X)$, let $A_I = \bigcup_{a_I \in A} a_I$ and let $A_{IV} = \bigcup_{a_{IV} \in A} a_{IV}$. Then by the above Result, $A = A_I \cup A_{IV}$. In fact, $A_I = (A_T, {A_T}^c)$ and $A_{IV} = (\phi, A_F)$.

Remark 2.11. Let $A \in IS(X)$ such that $A_T \cup A_F = X$, then $A_{IV} \subset A_I$ and thus

$$A = A_I \cup A_{IV} = A_I.$$

We will denote the family of all ISs A in X such that $A_T \cup A_F = X$ as $IS_*(X)$, i.e.,

$$IS_*(X) = \{A \in IS(X) : A_T \cup A_F = X\}.$$

In this case, it is obvious that $A \cap A^c = \phi_I$ and $A \cup A^c = X_I$ and thus

$$(IS_*(X), \subset, \phi_I, X_I)$$

is a Boolean algebra. In fact, there is a one-to-one correspondence between $P(X)$ and $IS_*(X)$, where $P(X)$ denotes the power set of X. Moreover, for any $A, B \in IS_*(X)$, $A = A_I = [\,]A = < > A$ and $A \cup B, A \cap B, A - B \in IS_*(X)$.

Example 2.12. Let $X = \{a, b, c, d, e\}$ and let $A = (\{a, b\}, \{c, d\})$. Then clearly, $a_I, b_I \in A$. Thus

$$a_I \cup b_I = (\{a, b\}, \{c, d, e\}) \neq A.$$

On the other hand,

$A_I = \bigcup_{a_I \in A} a_I = (\{a\} \cup \{b\}, \{b, c, d, e\} \cap \{a, c, d, e\}) = (\{a, b\}, \{c, d, e\})$
$= (A_T, {A_T}^c)$

and

$A_{IV} = \bigcup_{a_{IV} \in A} a_{IV} = (\phi, \{b, c, d, e\} \cap \{a, c, d, e\} \cap \{a, b, c, d\}) = (\phi, \{c, d\})$
$= (\phi, A_F)$.

3. Intuitionistic topological spaces

Coker [11] introduced an intuitionistic topological space, an intuitionistic base, an intuitionistic continuity and an intuitionistic compact space and studied their some properties. In this section, we give additional examples of intuitionistic topologies and obtain two properties related to an intuitionistic base and an intuitionistic subbase. And we define intuitionistic intervals in $\mathbb{R}$.

Definition 3.1 ([11])**.** Let X be a non-empty set and let $\tau \subset IC(X)$. Then τ is called an intuitionistic topology (in short IT) on X, it satisfies the following axioms:
(i) $\phi_I, X_I \in \tau$,
(ii) $A \cap B \in \tau$, for any $A, B \in \tau$,
(iii) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subset \tau$.

In this case, the pair (X, τ) is called an intuitionistic topological space (in short, ITS) and each member O of τ is called an intuitionistic open set (in short, IOS) in X. An IS F of X is called an intuitionistic closed set (in short, ICS) in X, if $F^c \in \tau$.

It is obvious that $\{\phi_I, X_I\}$ is the smallest IT on X and will be called the intuitionistic indiscreet topology and denoted by $\tau_{I,0}$. Also $IS(X)$ is the greatest IT on X and will be called the intuitionistic discreet topology and denoted by $\tau_{I,1}$. The pair $(X, \tau_{I,0})$ [resp. $(X, \tau_{I,1})$] will be called the intuitionistic indiscreet [resp. discreet] space.

We will denote the set of all ITs on X as $IT(X)$. For an ITS X, we will denote the set of all IOSs [resp. ICSs] on X as $IO(X)$ [resp. $IC(X)$].

Example 3.2. (1) ([11], Example 3.2) For any ordinary topological space (X, τ_o), let $\tau = \{(A, A^c) : A \in \tau_o\}$. Then clearly, (X, τ) is an ITS.

(2) Let $X = \{a, b\}$. Then $\tau_{I,1} = \{\phi_I, a_I, b_I, (a, \phi), (a, \phi), a_{IV}, b_{IV}, X_I\}$.

(3) ([11], Example 3.4) Let (X, τ) be an ordinary topological space such that τ is not indiscrete, where $\tau = \{\phi, X\} \cup \{G_j : j \in J\}$. Then there exist two ITs on X as follows: $\tau^1 = \{\phi_I, X_I\} \cup \{(G_j, \phi) : j \in J\}$ and $\tau^2 = \{\phi_I, X_I\} \cup \{(\phi, G_j^c) : j \in J\}$.

(4) Let X be a set and let $A \in IS(X)$. Then A is said to be finite, if A_T is finite. Consider the family $\tau = \{U \in IS(X) : U = \phi_I \text{ or } U^c \text{ is finite}\}$. Then we can easily show that τ is an IT on X.

In this case, τ will be called an intuitionistic cofinite topology on X and denoted by $ICof(X)$.

(5) Let X be a set and let $A \in IS(X)$. Then A is said to be countable, if A_T is countable. Consider the family $\tau = \{U \in IS(X) : U = \phi_I \text{ or } U^c \text{ is countable}\}$. Then we can easily show that τ is an IT on X.

In this case, τ will be called an intuitionistic cocountable topology on X and denoted by $ICoc(X)$.

Result 3.3 ([11], Proposition 3.5)**.** *Let (X, τ) be an ITS. Then the following two ITs on X can be defined by:*

$$\tau_{0,1} = \{[\ ]U : U \in \tau\}, \tau_{0,2} = \{<>U : U \in \tau\}.$$

Furthermore, the following two ordinary topologies on X can be defined by (See [8]):

$$\tau_1 = \{U_T : U \in \tau\},\ \tau_2 = \{U_F^c : U \in \tau\}.$$

Remark 3.4. (1) Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then it is obvious that $\tau = \tau_{0,1} = \tau_{0,2}$.

(2) For an IT τ on a set X, we will denote two ITs $\tau_{0,1}$ and $\tau_{0,2}$ defined in Result 3.3 as $\tau_{0,1} = [\ ]\tau$ and $\tau_{0,2} =< > \tau$, respectively.

(3) For an IT τ on a set X, let τ_1 and τ_2 be ordinary topologies on X defined in Result 3.3. Then (X, τ_1, τ_2) is a bitopological space by Kelly [14] (Also see Proposition 3.1 in [9]).

The following is the immediate result of Definition 3.1.

Proposition 3.5. *Let X be an ITS. Then*
(1) $\phi_I, X_I \in IC(X)$,
(2) $A \cup B \in IC(X)$, *for any* $A, B \in IC(X)$,
(3) $\bigcap_{j \in J} A_j \in IC(X)$, *for each* $(A_j)_{j \in J} \subset IC(X)$.

Definition 3.6 ([11])**.** Let $\tau_1, \tau_2 \in IT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $G \in \tau_2$, for each $G \in \tau_1$.

It is clear that $\tau_{I,0} \subset \tau \subset \tau_{I,1}$.

Result 3.7 ([11], Proposition 3.8)**.** *Let $(\tau_j)_{j \in J} \subset IT(X)$. Then $\bigcap_{j \in J} \tau_j \in IT(X)$. In fact, $\bigcap_{j \in J} \tau_j$ is the coarsest IT on X containing each τ_j.*

Proposition 3.8. *Let $\tau, \gamma \in IT(X)$. We define $\tau \wedge \gamma$ and $\tau \vee \gamma$ as follows:*

$$\tau \wedge \gamma = \{W : W \in \tau \text{ and } W \in \gamma\}$$

and

$$\tau \vee \gamma = \{W : W = U \cup V, U \in \tau \text{ and } V \in \gamma\}.$$

Then
(1) *$\tau \wedge \gamma$ is an IT on X which is the finest IT coarser than both τ and γ,*
(2) *$\tau \vee \gamma$ is an IT on X which is the coarsest IT finer than τ and γ.*

Proof. (1) It is easily to verify that $\tau \wedge \gamma \in IT(X)$. Let η be any IT which is coarser than both τ and γ and let $W \in \eta$. Then $W \in \tau$ and $W \in \gamma$. Thus $W \in \tau \wedge \gamma$. So η is coarser than $\tau \wedge \gamma$.

(2) Similarly, we prove that $\tau \vee \gamma \in IT(X)$ and that it is the coarsest IT finer than τ and γ. □

Definition 3.9 ([11])**.** Let (X, τ) be an ITS.

(i) A subfamily β of τ is called an intutionistic base (in short, IB) for τ, if for each $A \in \tau$, $A = \phi_I$ or there exists $\beta' \subset \beta$ such that $A = \bigcup \beta'$.

(ii) A subfamily σ of τ is called an intutionistic subbase (in short, ISB) for τ, if the family $\beta = \{\bigcap \sigma' : \sigma'$ is a finite subset of $\sigma\}$ is a base for τ.

In this case, the IT τ is said to be generated by σ. In fact, $\tau = \{\phi_I\} \cup \{\bigcup \beta' : \beta' \subset \beta\}$.

Example 3.10. (1) ([11], Example 3.10) Let $\sigma = \{((a, b), (-\infty, a]) : a, b \in \mathbb{R}\}$ be the family of ISs in $\mathbb{R}$. Then σ generates an IT τ on $\mathbb{R}$, which is called the "usual left intuitionistic topology" on $\mathbb{R}$. In fact, the IB β for τ can be written in the form
$\beta = \{\mathbb{R}_I\} \cup \sigma$ and τ consists of the following ISs in $\mathbb{R}$:
$\phi_I, \mathbb{R}_I$;

$(\cup(a_j, b_j), (-\infty, c])$,
where $a_j, b_j, c \in \mathbb{R}$, $\{a_j : j \in J\}$ is bounded from below, $c < inf\{a_j : j \in J\}$;
$(\cup(a_j, b_j), \phi)$,
where $a_j, b_j \in \mathbb{R}$, $\{a_j : j \in J\}$ is not bounded from below.

Similarly, one can define the "usual right intuitionistic topology" on $\mathbb{R}$ using an analogue construction.

(2) ([11], Example 3.11) Consider the family σ of ISs in $\mathbb{R}$

$$\sigma = \{((a, b), (-\infty, a_1] \cup [b_1, \infty)) : a, b, a_1, b_1 \in \mathbb{R}, a_1 \leq a, b_1 \leq b\}.$$

Then σ generates an IT τ on $\mathbb{R}$, which is called the "usual intuitionistic topology" on $\mathbb{R}$. In fact, the IB β for τ can be written in the form $\beta = \{\mathbb{R}_I\} \cup \sigma$ and the elements of τ can be easily written down as in the above example.

(3) Consider the family $\sigma_{[0,1]}$ of ISs in $\mathbb{R}$

$$\sigma_{[0,1]} = \{([a, b], (-\infty, a) \cup (b, \infty)) : a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.$$

Then $\sigma_{[0,1]}$ generates an IT $\tau_{[0,1]}$ on $\mathbb{R}$, which is called the "usual unit closed interval intuitionistic topology" on $\mathbb{R}$. In fact, the IB $\beta_{[0,1]}$ for $\tau_{[0,1]}$ can be written in the form $\beta_{[0,1]} = \{\mathbb{R}\} \cup \sigma_{[0,1]}$ and the elements of τ can be easily written down as in the above example.

In this case, $([0, 1], \tau_{[0,1]})$ is called the "intuitionistic usual unit closed interval" and will be denoted by $[0, 1]_I$, where $[0, 1]_I = ([0, 1], (-\infty, 0) \cup (1, \infty))$.

(4) Let X be a non-empty set and let $\beta = \{p_I : p \in X\} \cup \{p_{IV} : p \in X\}$. Then β is an IB for the intuitionistic discrete topology τ_1 on X.

(5) Let $X = \{a, b, c\}$ and let $\beta = \{(\{a, b\}, \{c\}), (\{b, c\}, \{a\}), X_I\}$. Assume that β is an base for an IT τ on X. Then by the definition of base, $\beta \subset \tau$. Thus $(\{a, b\}, \{c\}), (\{b, c\}, \{a\}) \in \tau$. So $(\{a, b\}, \{c\}) \cap (\{b, c\}, \{a\}) = (\{\{b\}, \{a, c\}) \in \tau$. But for any $\beta' \subset \beta$, $(\{\{b\}, \{a, c\}) \neq \bigcup \beta'$. Hence β is not an IB for an IT on X.

From (1), (2) and (3) in Example 3.10, we can define intutionistic intervals as following.

Definition 3.11. Let $a, b \in \mathbb{R}$ such that $a \leq b$. Then

(i) (the closed interval) $[a, b]_I = ([a, b], (-\infty, a) \cup (b, \infty))$,

(ii) (the open interval) $(a, b)_I = ((a, b), (-\infty, a] \cup [b, \infty))$,

(iii) (the half open interval or the half closed interval)
$(a, b]_I = ((a, b], (-\infty, a] \cup (b, \infty))$, $[a, b)_I = ([a, b), (-\infty, a) \cup [b, \infty))$,

(iv) (the half intuitionistic real line)
$(-\infty, a]_I = ((-\infty, a], (a, \infty))$, $(-\infty, a)_I = ((-\infty, a), [a, \infty))$,
$[a, \infty)_I = ([a, \infty), (-\infty, a))$, $(a, \infty)_I = ((a, \infty), (-\infty, a])$,

(v) (the intuitionistic real line) $(-\infty, \infty)_I = ((-\infty, \infty), \phi) = \mathbb{R}_I$.

Proposition 3.12. *Let X be a non-empty set and let $\beta \subset IS(X)$. Then β is an IB for an IT τ on X if and only if it satisfies the followings:*

(1) $X_I = \bigcup \beta$,

(2) if $B_1, B_2 \in \beta$ and $p_I \in B_1 \cap B_2$ [resp. $p_{IV} \in B_1 \cap B_2$], then there exists $B \in \beta$ such that $p_I \in B \subset B_1 \cap B_2$ [resp. $p_{IV} \in B \subset B_1 \cap B_2$].

Proof. The proof is the same as one in ordinary topological spaces. □

Example 3.13. Let $X=\{a,b,c\}$ and let $\beta=\{(\{a\},\{b,c\}),(\{a,b\},\{c\}),(\{a,c\},\{b\})\}$. Then clearly, β satisfies two conditions of Proposition 3.12. Thus β is an IB for an IT τ on X. Furthermore, $\tau=\{\phi_I,(\{a\},\{b,c\}),(\{a,b\},\{c\}),(\{a,c\},\{b\}),X_I\}$.

Proposition 3.14. *Let X be a non-empty set and let $\sigma\subset IS(X)$ such that $X_I=\bigcup\sigma$. Then there exists a unique IT τ on X such that σ is an ISB for Γ.*

Proof. Let $\beta=\{B\in IS(X): B=\bigcup_{i=1}^n S_i$ and $S_i\in\sigma\}$. Let $\tau=\{U\in IS(X): U=\phi_I$ or there is a subcollection $\beta^{'}$ of β such that $U=\bigcup\beta^{'}\}$. Then we can show that τ is the unique IT on X such that σ is an ISB for τ. □

In Proposition 3.14, τ is called the IT on X generated by σ.

Example 3.15. Let $X=\{a,b,c,d,e\}$ and let $\sigma=\{(\{a\},\{b,c,d,e\}),(\{a,b,c\},\{d,e\}),(\{b,c,e\},\{a,d\}),(\{c,d\},\{a,b,e\})\}$. Then clearly,

$$\bigcup\sigma_T=\{a\}\cup\{a,b,c\}\cup\{b,c,e\}\cup\{c,d\}=X$$

and

$$\bigcap\sigma_F=\{b,c,d,e\}\cap\{d,e\}\cap\{a,d\}\cap\{a,b,e\}=\phi.$$

Thus $\bigcup\sigma=X_I$. Let β be the collection of all finite intersections of members of σ. Then $\beta=\{(\{a\},\{b,c,d,e\}),(\{b,c\},\{a,d,e\}),(\{c\},\{a,b,d,e\}),(\{a,b,c\},\{d,e\}),(\{b,c,e\},\{a,d\}),(\{c,d\},\{a,b,e\})\}$. Thus the generated intutionistic topology τ by σ is

$$\begin{aligned}\tau=\{\phi_I,&(\{a\},\{b,c,d,e\}),(\{c\},\{a,b,d,e\}),(\{a,c\},\{b,d,e\}),(\{b,c\},\{a,d,e\}),\\&(\{c,e\},\{a,b,d\}),(\{a,b,c\},\{d,e\}),(\{a,c,e\},\{b,d\}),(\{b,c,d\},\{a,e\}),\\&(\{b,c,e\},\{a,d\}),(\{a,b,c,d\},\{e\}),(\{a,b,c,e\},\{d\}),(\{b,c,d,e\},\{a\}),X_I\}.\end{aligned}$$

Proposition 3.16. *Let (X,τ) be a ITS such that $\tau\subset IS_*(X)$ and let $A\in IS_*(X)$.*
(1) *If there is $U\in\tau$ such that $a_I\in U\subset A$, for each $a_I\in A$, then $A\in\tau$.*
(2) *If there is $U\in\tau$ such that $a_{IV}\in U\subset A$, for each $a_{IV}\in A$, then $A\in\tau$.*

Proof. (1) By the hypothesis, there is $U_{a_I}\in\tau$ such that $a_I\in U_{a_I}\subset A$. Then $a\in U_{a_I,T}\subset A_T$. Thus $A_T=\bigcup_{a\in A_T}U_{a_I,T}$. Since $\tau\subset IS_*(X)$ and $A\in IS_*(X)$,

$$A_F=A_T^c=\bigcap_{a\notin A_F}U^c_{a_I,T}=\bigcap_{a\notin A_F}U_{a_I,F}.$$

So $A=\bigcup_{a_I\in A}U_{a_I}$. Since $U_{a_I}\in\tau$, $A\in\tau$.
(2) The proof is similar to (1) □

Remark 3.17. If either the condition $\tau\subset IS_*(X)$ or the condition $A\in IS_*(X)$ is drawn, then Proposition 3.16 does not hold, in general.

Example 3.18. (1) Let $X=\{a,b,c\}$ and consider the IT τ on X given by:

$$\tau=\{\phi_I,X_I,A_1,A_2,A_3,A_4,A_5,A_6,$$

where $A_1=(\{a,b\},\{c\})$, $A_2=(\{b,c\},\{a\})$, $A_3=(\{a,c\},\{b\})$,
$A_4=(\{a\},\{b,c\})$, $A_5=(\{b\},\{a,c\})$, $A_5=(\{c\},\{a,b\})$.
Let $A=(\{a\},\{c\})$. Then clearly, $\tau\subset IS_*(X)$ but $A\notin IS_*(X)$. Moreover, $a_I\in A_4\subset A$ and $a_{IV}\in\phi_I\subset A$ but $A\notin\tau$.

(2)) Let $X = \{a, b, c\}$ and consider the IT τ on X given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9,$$

where $A_1 = (\{a\}, \{b\}),\ A_2 = (\{b\}, \{c\}),\ A_3 = (\{c\}, \{a\}),$
$A_4 = (\phi, \{b, c\}),\ A_5 = (\phi, \{a, c\}),\ A_6 = (\phi, \{a, b\}),$
$A_7 = (\{b, c\}, \phi),\ A_8 = (\{a, c\}, \phi),\ A_9 = (\{a, b\}, \phi).$
Let $A = (\{a\}, \{b, c\})$. Then clearly, $\tau \not\subset IS_*(X)$ but $A \in IS_*(X)$. Moreover, $a_I \in A_1 \subset A$, and $a_{IV} \in A_4 \subset A$ and $a_{IV} \in \phi_I \subset A$ but $A \notin \tau$.

4. Intuitionistic neighborhoods

Coker [12] introduced the notions of an intuitionistic neighborhood and intuitionistic vanishing neighborhood, obtained some properties and gave some examples. In this section, we give additional examples and properties. Moreover, we define some types of intuitionistic closures and interiors, and obtain some properties.

Definition 4.1 ([12])**.** Let X be an ITS, $p \in X$ and let $N \in IS(X)$. Then
(i) N is called a neighborhood of p_I, if there exists an IOS G in X such that

$$p_I \in G \subset N, \text{ i.e., } p \in G_T \subset N_T \text{ and } G_F \supset N_F,$$

(ii) N is called a neighborhood of p_{IV}, if there exists an IOS G in X such that

$$p_{IV} \in G \subset N, \text{ i.e., } G_T \subset N_T \text{ and } p \notin G_F \supset N_F.$$

We will denote the set of all neighborhoods of p_I [resp. p_{IV}] by $N(p_I)$ [resp. $N(p_{IV})$].

Result 4.2 ([12], Proposition 3.2)**.** *Let X be an ITS and let $p \in X$.*
[IN1] *If $N \in N(p_I)$, then $p_I \in N$.*
[IN2] *If $N \in N(p_I)$ and $N \subset N$, then $M \in N(p_I)$.*
[IN3] *If $N, M \in N(p_I)$, then $N \cap M \in N(p_I)$.*
[IN4] *If $N \in N(p_I)$, then there exists $M \in N(p_I)$ such that $N \in N(q_I)$, for each $q_I \in M$.*

Result 4.3 ([12], Proposition 3.3)**.** *Let X be an ITS and let $p \in X$.*
[IN1] *If $N \in N(p_{IV})$, then $p_{IV} \in N$.*
[IN2] *If $N \in N(p_{IV})$ and $N \subset N$, then $M \in N(p_{IV})$.*
[IN3] *If $N, M \in N(p_{IV})$, then $N \cap M \in N(p_{IV})$.*
[IN4] *If $N \in N(p_{IV})$, then there exists $M \in N(p_{IV})$ such that $N \in N(q_{IV})$, for each $q_{IV} \in M$.*

Result 4.4 ([12], Proposition 3.4)**.** *Let (X, τ) be an ITS. We define the families*

$$\tau_I = \{G : G \in N(p_I), \text{ for each } p_I \in G\}$$

and

$$\tau_{IV} = \{G : G \in N(p_{IV}), \text{ for each } p_{IV} \in G\}.$$

Then $\tau_I, \tau_{IV} \in IT(X)$.

Remark 4.5. (1) From Result 4.4, we can easily see that for an IT τ on a set X and each $U \in \tau$,

$$\tau_I = \tau \cup \{(U_T, S_U) : S_U \subset U_F\} \cup \{(\phi, S) : S \subset X\}$$

and

$$\tau_{IV} = \tau \cup \{(S_U, U_F) : S_U \supset U_T \text{ and } S_U \cap U_F = \phi\}.$$

(2) For an IT τ on a set X, four ITs can be defined on X:

$$\tau_{I,0,1} = \{[\ ]U : U \in \tau_I\},\ \tau_{IV,0,1} = \{[\ ]U : U \in \tau_{IV}\}$$

and

$$\tau_{I,0,2} = \{<> U : U \in \tau_I\},\ \tau_{IV,0,2} = \{<> U : U \in \tau_{IV}\}.$$

In fact, $\tau_{I,0,1} = \tau_{0,1}$ and $\tau_{IV,0,2} = \tau_{0,2}$.

(3) For an IT τ on a set X, four ordinary topologies can be defined on X:

$$\tau_{I,1} = \{U_T : U \in \tau_I\},\ \tau_{IV,1} = \{U_T : U \in \tau_{IV}\}$$

and

$$\tau_{I,2} = \{U_F^c : U \in \tau_I\},\ \tau_{IV,2} = \{U_F^c : U \in \tau_{IV}\}.$$

In fact, $\tau_{I,1} = \tau_1$ and $\tau_{IV,2} = \tau_2$.

Example 4.6. Let $X = \{a, b, c\}$ and let τ be the IT on X given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where $A_1 = (\{a, b\}, \{c\}),\ A_2 = (\{b\}, \{a\}),\ A_3 = (\{a, b\}, \phi),\ A_4 = (\{b\}, \{a, c\})$.
Then $\tau_I = \tau \cup \{(\{b\}, S_{A_4}) : S_{A_4} \subset \{a, c\}\} \cup \{(\phi, S \subset X\}$
$= \tau \cup \{A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}\}$
and

$$\tau_{IV} = \tau \cup \{(S_{A_2}, \{a\}) : S_{A_2} \supset \{b\},\ S_{A_2} \cap \{a\} = \phi\} = \tau \cup \{A_{14}\},$$

where $A_5 = (\{b\}, \{c\}),\ A_6 = (\{b\}, \phi),\ A_7 = (\phi, \{a\}),\ A_8 = (\phi, \{b\})$,
$A_9 = (\phi, \{c\}),\ A_{10} = (\phi, \{a, b\}),\ A_{11} = (\phi, \{b, c\}),\ A_{12} = (\phi, \{a, c\})$,
$A_{13} = (\phi, \phi),\ A_{14} = (\{b, c\}, \{a\})$.
Thus we have four ITs and ordinary topologies on X as follows:
$\tau_{I,0,1} = \{\phi_I, X_I, A_1, A_4\} = \tau_{0,1}$,
$\tau_{IV,0,1} = \{\phi_I, X_I, A_1, A_4, A_{14}\}$,
$\tau_{I,0,2} = \{\phi_I, X_I, A_1, A_{14}, A_4, <> A_8, <> A_{10}, <> A_{11}\}$,
$\tau_{IV,0,2} = \{\phi_I, X_I, A_1, A_{14}, A_4\} = \tau_{0,2}$
and
$\tau_{I,1} = \{\phi, X, \{a, b\}, \{b\}\} = \tau_1$,
$\tau_{IV,1} = \{\phi, X, \{a, b\}, \{b\}, \{b, c\}\}$,
$\tau_{I,2} = \{\phi, X, \{a, b\}, \{b, c\}, \{b\}, \{a, c\}, \{c\}, \{a\}\}$,
$\tau_{IV,2} = \{\phi, X, \{a, b\}, \{b, c\}, \{b\}\} = \tau_2$.

Result 4.7 ([12], Proposition 3.5)**.** *Let (X, τ) be an ITS. Then $\tau \subset \tau_I$ and $\tau \subset \tau_{IV}$.*

The following is the immediate result of Result 4.7.

Corollary 4.8. *Let (X, τ) be an ITS and let IC_τ [resp. IC_{τ_I} and $IC_{\tau_{IV}}$] be the set of all ICSs w.r.t. τ[resp. τ_I and τ_{IV}]. Then*

$$IC_\tau(X) \subset IC_{\tau_I}(X) \text{ and } IC_\tau(X) \subset IC_{\tau_{IV}}(X).$$

Example 4.9. Let $X = \{a, b, c, d\}$ and consider the family of ISs

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where

$$A_1 = (\{a, b\}, \{d\}),\ A_2 = (\{c\}, \{b, d\}),\ A_3 = (\phi, \{b, d\}),\ A_4 = (\{a, b, c\}, \{d\}).$$

Then from Example 3.6 in [12], (X, τ) is an ITS, and two ITs τ_I and τ_{IV} on X are given, respectively as follows:

$$\tau_I = \tau \bigcup \{A_i : i = 5, 6, \cdots, 23\},$$

where
$A_5 = (\{c\}, \{b\}),\ A_6 = (\{c\}, \{d\}),\ A_7 = (\{a, b\}, \phi),\ A_8 = (\{a, b, c\}, \phi),$
$A_9 = (\{c\}, \phi),\ A_{10} = (\phi, \{a\}),\ A_{11} = (\phi, \{b\}),\ A_{12} = (\phi, \{c\}),$
$A_{13} = (\phi, \{d\}),\ A_{14} = (\phi, \{a, b\}),\ A_{15} = (\phi, \{a, c\}),\ A_{16} = (\phi, \{a, d\}),$
$A_{17} = (\phi, \{b, c\}),\ A_{18} = (\phi, \{c, d\}),\ A_{19} = (\phi, \{a, b, c\}),\ A_{20} = (\phi, \{a, b, d\}),$
$A_{21} = (\phi, \{a, c, d\}),\ A_{22} = (\phi, \{b, c, d\}),\ A_{23} = (\phi, \phi).$
and

$$\tau_{IV} = \tau \cup \{A_{24}, A_{25}\},$$

where
$A_{24} = (\{a, c\}, \{b, d\}),\ A_{25} = (\{a\}, \{b, d\}).$
Thus $IC_\tau(X) = \{\phi_I, X_I, {A_1}^c, {A_2}^c, {A_3}^c, {A_4}^c\}$,
$IC_{\tau_I}(X) = IC_\tau(X) \bigcup \{{A_i}^c : i = 5, 6, \cdots, 23\}$,
$IC_{\tau_{IV}}(X) = IC_\tau(X) \cup \{{A_{24}}^c, {A_{25}}^c\}$,
where
${A_1}^c = (\{d\}, \{a, b\}),\ {A_2}^c = (\{b, d\}, \{c\}),\ {A_3}^c = (\{b, d\}, \phi),\ {A_4}^c = (\{d\}, \{a, b, c\}),$
${A_5}^c = (\{b\}, \{c\}),\ {A_6}^c = (\{d\}, \{c\}),\ {A_7}^c = (\phi, \{a, b\}),\ {A_8}^c = (\phi, \{a, b, c\}),$
${A_9}^c = (\phi, \{c\},\ {A_{10}}^c = (\{a\}, \phi),\ {A_{11}}^c = (\{b\}, \phi),\ {A_{12}}^c = (\{c\}, \phi),$
${A_{13}}^c = (\{d\}, \phi),\ {A_{14}}^c = (\{a, b\}, \phi),\ {A_{15}}^c = (\{a, c\}, \phi),\ {A_{16}}^c = (\{a, d\}, \phi),$
${A_{17}}^c = (\{b, c\}, \phi),\ {A_{18}}^c = (\{c, d\}, \phi),\ {A_{19}}^c = (\{a, b, c\}, \phi),\ {A_{20}}^c = (\{a, b, d\}, \phi),$
${A_{21}}^c = (\{a, c, d\}, \phi),\ {A_{22}}^c = (\{b, c, d\}, \phi),\ {A_{23}}^c = (\phi, \phi),$
${A_{24}}^c = (\{b, d\}, \{a, c\}),\ {A_{25}}^c = (\{b, d\}, \{a\}).$
So $IC_\tau(X) \subset IC_{\tau_I}(X)$ and $IC_\tau(X) \subset IC_{\tau_{IV}}(X)$.

The following is the converse of Result 4.2.

Result 4.10 ([12], Proposition 3.8)**.** *Let X be a non-empty set. Suppose $N^* : X \to P(IS(X))$ is the mapping satisfying the properties* [IN1], [IN2], [IN3] *and* [IN4] *in Result 4.2, where $N^*(p_I \in P(IS(X))$. Then there exists an IT τ_I on X such that $N^*(p_I) = IN(p_I)$, for each $p \in X$, where $IN(p_I)$ denotes the set of INs of p_I in an ITS (X, τ_I).*

The following is the converse of Result 4.3.

Result 4.11 ([12], Proposition 3.7)**.** *Let X be a non-empty set. Suppose $N^* : X \to P(IS(X))$ is the mapping satisfying the properties* [IN1], [IN2], [IN3] *and* [IN4] *in Result 4.3, where $N^*(p_{IV} \in P(IS(X))$. Then there exists an IT τ_{IV} on X such that $N^*(p_{IV}) = IN(p_{IV})$, for each $p \in X$, where $IN(p_{IV})$ denotes the set of INs of p_{IV} in an ITS (X, τ_{IV}).*

Result 4.12 ([12], Proposition 3.9)**.** *Let* (X, τ) *be an ITS. Then* $\tau = \tau_I \cap \tau_{IV}$.

The following is the immediate result of Result 4.12.

Corollary 4.13. *Let* (X, τ) *be an ITS and let* IC_τ*]. Then*

$$IC_\tau(X) = IC_{\tau_I}(X) \cap IC_{\tau_{IV}}(X).$$

Example 4.14. In Example 4.9, we can easily check that

$$IC_\tau(X) = IC_{\tau_I}(X) \cap IC_{\tau_{IV}}(X).$$

Definition 4.15. Let (X, τ) be an ITS and let $A \in IS(X)$.

(i) ([11]) The intuitionistic closure of A w.r.t. τ, denoted by $Icl(A)$, is an IS of X defined as:

$$Icl(A) = \bigcap\{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) ([11]) The intuitionistic interior of A w.r.t. τ, denoted by $Iint(A)$, is an IS of X defined as:

$$Iint(A) = \bigcup\{G : G \in \tau \text{ and } G \subset A\}.$$

(iii) The intuitionistic closure of A w.r.t. τ_I, denoted by $cl_{\tau_I}(A)$, is an IS of X defined as:

$$cl_{\tau_I}(A) = \bigcap\{K : K^c \in \tau_I \text{ and } A \subset K\}.$$

(iv) The intuitionistic interior of A w.r.t. τ_I, denoted by $int_I(A)$, is an IS of X defined as:

$$int_{\tau_I}(A) = \bigcup\{G : G \in \tau_I \text{ and } G \subset A\}.$$

(v) The intuitionistic closure of A w.r.t. τ_{IV}, denoted by $cl_{\tau_{IV}}(A)$, is an IS of X defined as:

$$cl_{\tau_{IV}}(A) = \bigcap\{K : K^c \in \tau_{IV} \text{ and } A \subset K\}.$$

(vi) The intuitionistic interior of A w.r.t. τ_{IV}, denoted by $int_{IV}(A)$, is an IS of X defined as:

$$int_{\tau_{IV}}(A = \bigcup\{G : G \in \tau_{IV} \text{ and } G \subset A\}.$$

From Definition 4.16, we can easily see that

$$Iint(A) \subset int_{\tau_I}(A),\ Iint(A) \subset int_{IV}(A)$$

and

$$cl_{\tau_I}(A) \subset Icl(A),\ cl_{\tau_{IV}}(A) \subset Icl(A).$$

However, the reverse inclusions do not need to hold.

Example 4.16. In Example 4.9, let $A = (\{a, c\}, \{d\}), B = (\{d\}, \{a, c\})$. Then

$$\begin{aligned}
Iint(A) &= \bigcup\{G \in \tau : G \subset A\} = A_2 \cup A_3 = A_2 = (\{c\}, \{b, d\}),\\
int_{\tau_I}(A) &= \bigcup\{G \in \tau_I : G \subset A\}\\
&= A_2 \cup A_3 \cup A_6 \cup A_{13} \cup A_{16} \cup A_{18} \cup A_{20} \cup A_{21} \cup A_{22}\\
&= (\{c\}, \{d\}),\\
int_{\tau_{IV}}(A) &= \bigcup\{G \in \tau_{IV} : G \subset A\} = A_2 \cup A_3 \cup A_{24} = (\{a, b, c\}, \{b, d\})
\end{aligned}$$

and

$$\begin{aligned}
Icl(B) &= \bigcap\{F : F^c \in \tau, B \subset F\} = A_2^c \cap A_3^c \cap X_I = (\{b, d\}, \{c\}),\\
cl_{\tau_I}(B) &= \bigcap\{F : F^c \in \tau_I \text{ and } B \subset F\}\\
&= A_2^c \cap A_3^c \cap A_6^c \cap A_{13}^c \cap A_{16}^c \cap A_{18}^c \cap A_{20}^c \cap A_{21}^c \cap A_{22}^c \cap X_I
\end{aligned}$$

$= (\{d\}, \{c\})$,
$cl_{\tau_{IV}}(B) = \bigcap\{K : K^c \in \tau_{IV}$ and $B \subset K\} = A_{24}^c \cap A_{25}^c \cap X_I = (\{b, d\}, \{a, c\})$.
Thus we can confirm the following inclusions:

$$Iint(A) \subset int_{\tau_I}(A), Iint(A) \subset int_{IV}(A)$$

and

$$cl_{\tau_I}(B) \subset Icl(B), cl_{\tau_{IV}}(B) \subset Icl(B).$$

Result 4.17 ([11], Proposition 3.15)**.** *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*

$$Iint(A^c) = (Icl(A))^c \text{ and } Icl(A^c) = (Iint(A))^c.$$

Result 4.18 ([12], Proposition 3.10)**.** *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*

$$Iint(A) = int_{\tau_I}(A) \cap int_{\tau_{IV}}(A).$$

The following is the immediate result of Definition 4.15 and Results 4.17 and 4.18.

Corollary 4.19. *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*

$$Icl(A) = cl_{\tau_I}(A) \cup cl_{\tau_{IV}}(A).$$

Example 4.20. In Example 4.9, let $A = (\{a, c\}, \{d\}), B = (\{d\}, \{a, c\})$. Then we can see that

$Icl(B) = (\{b, d\}, \{c\}), cl_{\tau_I}(A) = (\{d\}, \{c\}), cl_{\tau_{IV}}(A) = (\{b, d\}, \{a, c\})$

and

$Iint(A) = (\{c\}, \{b, d\}), int_{\tau_I}(A) = (\{c\}, \{d\}), int_{\tau_{IV}}(A) = (\{a, b, c\}, \{b, d\})$.

Thus

$cl_{\tau_I}(B) \cup cl_{\tau_{IV}}(B) = (\{d\}, \{c\}) \cup (\{b, d\}, \{a, c\}) = (\{b, d\}, \{c\}) = Icl(B)$

and

$int_{\tau_I}(A) \cap int_{\tau_{IV}}(A) = (\{c\}, \{d\}) \cap (\{a, b, c\}, \{b, d\}) = (\{c\}, \{b, d\}) = Iint(A)$.

The following is the immediate result of Definition 4.15.

Proposition 4.21. *Let X be an ITS and let $A \in IS(X)$. Then*
(1) *$A \in IC(X)$ if and only if $A = Icl(A)$,*
(2) *$A \in IO(X)$ if and only if $A = Iint(A)$.*

Result 4.22 ([11], Proposition 3.16, Kuratowski Closure Axioms)**.** *Let X be an ITS and let $A, B \in IS(X)$. Then*
[IK0] *if $A \subset B$, then $Icl(A) \subset Icl(B)$,*
[IK1] *$Icl(\phi_I) = \phi_I$,*
[IK2] *$A \subset Icl(A)$,*
[IK3] *$Icl(Icl(A)) = Icl(A)$,*
[IK4] *$Icl(A \cup B) = Icl(A) \cup Icl(A)$.*

Let $Icl^* : IS(X) \to IS(X)$ be the mapping satisfying the properties [IK1], [IK2],[IK3] and [IK4]. Then we will call the mapping Icl^* as the intuitionistic closure operator on X.

Proposition 4.23. *Let Icl^* be the intuitionistic closure operator on X. Then there exists a unique IT τ on X such that $Icl^*(A) = Icl(A)$, for each $A \in IS(X)$, where $Icl(A)$ denotes the intuitionistic closure of A in the ITS (X, τ). In fact,*

$$\tau = \{A^c \in IS(X) : Icl^*(A) = A\}.$$

Proof. The proof is almost similar to the case of ordinary topological spaces. □

Result 4.24 ([11], Proposition 3.16)**.** *Let X be an ITS and let $A, B \in IS(X)$. Then*
[II0] *if $A \subset B$, then $Iint(A) \subset Iint(B)$,*
[II1] *$Iint(X_I) = X_I$,*
[II2] *$Iint(A) \subset A$,*
[II3] *$Iint(Iint(A)) = Iint(A)$,*
[II4] *$Iint(A \cap B) = Iint(A) \cap Iint(A)$.*

Let $Iint^* : IS(X) \to IS(X)$ be the mapping satisfying the properties [II1], [II2],[II3] and [II4]. Then we will call the mapping $Iint^*$ as the intuitionistic interior operator on X.

Proposition 4.25. *Let $Iint^*$ be the intuitionistic interior operator on X. Then there exists a unique IT τ on X such that $Iint^*(A) = Iint(A)$, for each $A \in IS(X)$, where $Iint(A)$ denotes the intuitionistic interior of A in the ITS (X, τ). In fact,*

$$\tau = \{A \in IS(X) : Iint^*(A) = A\}.$$

Proof. The proof is similar to one of Proposition 4.23. □

Definition 4.26 ([12])**.** Let (X, τ) be an ITS, $p \in X$ and let $A \in IS(X)$. Then
(i) $p_I \in A$ is called a τ_I-interior point of A, if $A \in N(p_I)$,
(ii) $p_{IV} \in A$ is called a τ_{IV}-interior point of A, if $A \in N(p_{IV})$,
We will denote the union of all τ_I-interior points [resp. τ_{IV}-interior points] of A as τ_I-$int(A)$ [resp. τ_{IV}-$int(A)$]. It is clear that
τ_I-$int(A) = \bigcup\{p_I : A \in N(p_I)\}$ [resp. τ_{IV}-$int(A) = \bigcup\{p_{IV} : A \in N(p_{IV})\}$].

Result 4.27 ([12], Proposition 4.2)**.** *Let (X, τ) be an ITS and let $A \in IS(X)$.*
(1) *$A \in \tau_I$ if and only if $A_I = \tau_I$-$int(A)$.*
(2) *$A \in \tau_{IV}$ if and only if $A_{IV} = \tau_{IV}$-$int(A)$.*

Result 4.28 ([12], Proposition 4.3)**.** *Let X be a non-empty set, $(G_j)_{j\in J} \subset IS(X)$ and let $G = \bigcup_{j\in J} G_j$. Then*
(1) *$G_I = \bigcup_{j\in J} G_{j,I}$,*
(2) *$G_{IV} = \bigcup_{j\in J} G_{j,IV}$.*

Result 4.29 ([12], Proposition 4.4)**.** *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*
(1) *τ_I-$int(A) = \bigcup_{G\subset A, G\in\tau_I} G_I$,*
(2) *τ_{IV}-$int(A) = \bigcup_{G\subset A, G\in\tau_{IV}} G_{IV}$.*

Remark 4.30 ([12])**.** τ_I-$int(A) \subset int_{\tau_I}(A)$ and τ_{IV}-$int(A) \subset int_{\tau_{IV}}(A)$. But the converse inclusions do not hold, in general.

Example 4.31. Let $X = \{a, b, c, d, e\}$ and let us consider ITS (X, τ) given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where $A_1 = (\{a,b,c\},\{e\})$, $A_2 = (\{c,\},\{d\})$, $A_3 = (\{c\},\{d,e\})$, $A_4 = (\{a,b,c\},\phi)$. Then we can easily find τ_I and τ_{IV}:

$$\tau_I = \tau \cup \{A_5, A_6\},$$

where $A_5 = (\{c\},\phi)$, $A_6 = (\{c\},\{e\})$
and

$$\tau_{IV} = \tau \cup \{A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}\},$$

where $A_7 = (\{a,b,c,d\},\{e\})$, $A_8 = (\{a,b,c\},\{d\})$, $A_9 = (\{b,c,e\},\{d\})$,
$A_{10} = (\{a,b,c,e\},\{d\})$, $A_{11} = (\{a,c\},\{e\})$, $A_{12} = (\{b,c\},\{e\})$,
$A_{13} = (\{c,d\},\{e\})$, $A_{14} = (\{a,c,d\},\{e\})$, $A_{15} = (\{a,b,c,d\},\phi)$,
$A_{16} = (\{a,b,c,e\},\phi)$.

Now let $A = (\{b,c\},\{e\})$. Then
$Iint(A) = \bigcup\{G \in \tau : G \subset A\} = A_3$,
$int_{\tau_I}(A) = \bigcup\{G \in \tau_I : G \subset A\} = A_3 \cup A_6 = A_6$,
$int_{\tau_{IV}}(A) = \bigcup\{G \in \tau_{IV} : G \subset A\} = A_3 \cup A_{12} = A_{12}$,
$\tau_I\text{-}int(A) = \bigcup\{p_I : A \in N(p_I)\} = c_I$,
$\tau_{IV}\text{-}int(A) = \bigcup\{p_{IV} : A \in N(p_{IV})\} = a_{IV}$.
Thus we have the following strict inclusions:
$\tau_I\text{-}int(A) \subset int_{\tau_I}(A)$, $\tau_I\text{-}int(A) \neq int_{\tau_I}(A)$,
$\tau_{IV}\text{-}int(A) \subset int_{\tau_{IV}}(A)$, $\tau_{IV}\text{-}int(A) \neq int_{\tau_{IV}}(A)$.

Result 4.32 ([12], Proposition 4.6)**.** *Let (X,τ) be an ITS and let $A, B \in IS(X)$. Then*
(1) $\tau_I\text{-}int(A) \subset A_I$, $\tau_{IV}\text{-}int(A) \subset A_{IV}$,
(2) *if* $A \subset B$, *then* $\tau_I\text{-}int(A) \subset \tau_I\text{-}int(B)$, $\tau_{IV}\text{-}int(A) \subset \tau_{IV}\text{-}int(B)$,
(3) $\tau_I\text{-}int(A \cap B) = \tau_I\text{-}int(A) \cap \tau_I\text{-}int(B)$,
$\tau_{IV}\text{-}int(A \cap B) = \tau_{IV}\text{-}int(A) \cap \tau_{IV}\text{-}int(B)$,
(4) $\tau_I\text{-}int(X_I) = X_I$, $\tau_{IV}\text{-}int(X_I) = X_I$.

Definition 4.33. Let (X,τ) be an ITS, $p \in X$ and let $A \in IS(X)$. Then
(i) p_I is called a τ_I-closure point of A, if for each $N \in N(p_I)$,

$$A \cap N \neq \phi_I, \text{ i.e., } A_T \cap N_T \neq \phi \text{ or } A_F \cup N_F \neq X,$$

(ii) p_{IV} is called a τ_{IV}-closure point of A, if for each $N \in N(p_{IV})$,

$$A \cap N \neq \phi_I, \text{ i.e., } A_T \cap N_T \neq \phi \text{ or } A_F \cup N_F \neq X.$$

We will denote the union of all τ_I-closure points [resp. τ_{IV}-closure points] of A as $\tau_I\text{-}cl(A)$ [resp. $\tau_{IV}\text{-}cl(A)$]. It is obvious that

$$\tau_I\text{-}cl(A) = \bigcup\{p_I : A \cap N \neq \phi_I, \forall N \in N(p_I)\}$$
$$\text{[resp. } \tau_{IV}\text{-}cl(A) = \bigcup\{p_{IV} : A \cap N \neq \phi_I, \forall N \in N(p_{IV})\}].$$

Remark 4.34. $\tau_I\text{-}cl(A) \subset cl_{\tau_I}(A)$ and $\tau_{IV}\text{-}cl(A) \subset cl_{\tau_{IV}}(A)$.
But the converse inclusions do not hold, in general.

Example 4.35. In Example 4.31, let us consider ITS (X,τ) given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where $A_1 = (\{a, b, c\}, \{e\})$, $A_2 = (\{c\}, \{d\})$, $A_3 = (\{c\}, \{d, e\})$, $A_4 = (\{a, b, c\}, \phi)$ and $X = \{a, b, c, d, e\}$. Then we can easily find $IC_\tau(X)$, $IC_{\tau_I}(X)$ and $IC_{\tau_{IV}}(X)$:

$$IC_\tau(X) = \{\phi, X_I, A_1^c, A_2^c, A_3^c, A_4^c\},$$

where $A_1^c = (\{e\}, \{a, b, c\})$, $A_2^c = (\{d\}, \{c\})$, $A_3^c = (\{d, e\}, \{c\})$, $A_4^c = (\phi, \{a, b, c\})$,

$$IC_{\tau_I}(X) = IC_\tau(X) \cup \{A_5^c, A_6^c\},$$

where $A_5^c = (\phi, \{c\})$, $A_6^c = (\{e\}, \{c\})$
and

$$IC_{\tau_{IV}}(X) = IC_\tau(X) \cup \{A_7^c, A_8^c, A_9^c, A_{10}^c, A_{11}^c, A_{12}^c, A_{13}^c, A_{14}^c, A_{15}^c, A_{16}^c\},$$

where $A_7^c = (\{a, b, c, d\}, \{e\})$, $A_8^c = (\{a, b, c\}, \{d\})$, $A_9^c = (\{b, c, e\}, \{d\})$,
$A_{10}^c = (\{a, b, c, e\}, \{d\})$, $A_{11}^c = (\{e\}, \{a, c\})$, $A_{12}^c = (\{e\}, \{b, c\})$,
$A_{13}^c = (\{e\}, \{c, d\})$, $A_{14}^c = (\{e\}, \{a, c, d\})$, $A_{15}^c = (\phi, \{a, b, c, d\})$,
$A_{16}^c = (\phi, \{a, b, c, e\})$.

Now let $A = (\{e\}, \{b, c\})$. Then

$Icl(A) = \bigcap\{F \in IC_\tau(X) : A \subset F\} = X_I \cap A_3^c = A_3^c$,
$cl_{\tau_I}(A) = \bigcap\{F \in IC_{\tau_I}(X) : A \subset F\} = X_I \cap A_3^c \cap A_6^c = A_6^c$,
$cl_{\tau_{IV}}(A) = \bigcap\{F \in IC_{\tau_{IV}}(X) : A \subset F\} = X_I \cap A_3^c \cap A_{13}^c = A_{13}^c$,
$\tau_I\text{-}cl(A) = \bigcup\{p_I : N_T \cap A_T \neq \phi,\ \forall N \in N(p_I)\} = e_I$,
$\tau_{IV}\text{-}cl(A) = \bigcup\{p_{IV} : N_F \cup A_F \neq X,\ \forall N \in N(p_{IV})\}$
$= a_{IV} \cup b_{IV} \cup d_{IV}.$
$= (\phi, \{c\}).$

Thus we have the following strict inclusions:

$\tau_I\text{-}cl(A) \subset cl_{\tau_I}(A)$, $\tau_I\text{-}cl(A) \neq cl_{\tau_I}(A)$,
$\tau_{IV}\text{-}cl(A) \subset cl_{\tau_{IV}}(A)$, $\tau_{IV}\text{-}cl(A) \neq cl_{\tau_{IV}}(A)$.

Proposition 4.36. *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*
(1) $(\tau_I\text{-}int(A))^c = \tau_I\text{-}cl(A^c)$, $\tau_I\text{-}int(A^c) = (\tau_I\text{-}cl(A))^c$
(2) $(\tau_{IV}\text{-}int(A))^c = \tau_{IV}\text{-}cl(A^c)$, $\tau_{IV}\text{-}int(A^c) = (\tau_{IV}\text{-}cl(A))^c$.

Proof. (1) Let $p_I \in (\tau_I\text{-}int(A))^c$. Then $A \notin N(p_I)$. Thus $G \not\subset A$, i.e., $G_T \not\subset A_T$ or $G_F \not\supset A_F$, for each $G \in \tau$ with $p_I \in G$. So $\phi = G_T \cap G_F \not\supset G_T \cap A_F$, i.e., $G_T \cap A_F \neq \phi$. Hence $p_I \in \tau_I\text{-}cl(A^c)$.

Suppose $p_I \in \tau_I\text{-}cl(A^c)$ and let $N \in N(p_I)$. Then $N_T \cap A_F \neq \phi$, say $q \in N_T \cap A_F$. Assume that $N \subset A$, i.e., $N_T \subset A_T$ and $N_F \supset A_F$. Since $q \in N_T \cap A_F$, $q \in A_T$ and $q \in N_F$. Thus $N_T \cap N_F \neq \phi$ and $A_T \cap A_F \neq \phi$. These are contradictions from $N_T \cap N_F = \phi$ and $A_T \cap A_F = \phi$. So $N_T \not\subset A_T$ or $N_F \not\supset A_F$, i.e., $A \notin N(p_I)$, i.e., $p_I \notin \tau_I\text{-}int(A)$ and thus $p_I \in (\tau_I\text{-}int(A))^c$. Hence $(\tau_I\text{-}int(A))^c = \tau_I\text{-}cl(A^c)$.

The proof of the second part is similar.

(2) The proof is similar to (1). □

Proposition 4.37. *Let (X, τ) be an ITS and let $A \in IS_*(X)$. Then*
(1) $A \in IC_{\tau_I}(X)$ *if and only if* $A_I = \tau_I\text{-}cl(A)$,
(2) $A \in IC_{\tau_{IV}}(X)$ *if and only if* $A_{IV} = \tau_{IV}\text{-}cl(A)$.

Proof. (1) Since $A \in IS_*(X)$, by Remark 2.12, $A = A_I = [\ ]A = <\ > A$. Then clearly, $(A_I)^c = (A^c)_I$. Thus

$A \in IC_{\tau_I}(X)$ if and only if $A^c \in \tau_I$

if and only if $(A^c)_I = \tau_I\text{-}int(A^c)$ [By Result 4.27 (1)]
if and only if $(A_I)^c = (\tau_I\text{-}cl(A))^c$ [By Proposition 4.36 (1)]
if and only if $A_I = \tau_I\text{-}cl(A)$.

(2) The proof is similar to (1). □

Lemma 4.38. *Let X be a set, $(F_j)_{j\in J} \subset IS(X)$ and let $F = \bigcap_{j\in J} F_j$. Then*
(1) $F_I = \bigcap_{j\in J} F_{I,j}$,
(2) $F_{IV} = \bigcap_{j\in J} F_{IV,j}$.

Proof. (1) Let $p_I \in F_I$. Then $p \in F$, i.e., $p \in \bigcap_{j\in J} F_{T,j}$. Thus there exists $j \in J$ such that $p \in F_{T,j}$, i.e., $p_I \in F_{I,j}$. So $p_I \in \bigcap_{j\in J} F_{I,j}$. Hence $F_I \subset \bigcap_{j\in J} F_{I,j}$.

Conversely, suppose $p_I \in \bigcap_{j\in J} F_{I,j}$. Then there exists $j \in J$ such that $p_I \in F_{I,j}$. Thus $p \in F_{T,j}$. So $p \in \bigcap_{j\in J} F_{T,j}$, i.e., $p_I \in F_I$. Hence $\bigcap_{j\in J} F_{I,j} \subset F_I$. Therefore the result holds.

(2) The proof is similar to (1). □

Proposition 4.39. *Let (X, τ) be an ITS and let $A \in IS_*(X)$. Then*
(1) $\tau_I\text{-}cl(A) = \bigcap_{A\subset F,\ F\in IC_{\tau_I}(X)} F_I$,
(2) $\tau_{IV}\text{-}cl(A) = \bigcap_{A\subset F,\ F\in IC_{\tau_{IV}}(X)} F_{IV}$.

Proof. (1)
$$\begin{aligned}\tau_I\text{-}cl(A) &= (\tau_I\text{-}int(A^c))^c \text{ [By Result 4.27 (1)]}\\ &= (\textstyle\bigcup_{G\subset A^c, G\in\tau_I} G_I)^c \text{ [By Result 4.29 (1)]}\\ &= \textstyle\bigcap_{A\subset G^c, G^c\in IC_{\tau_I(X)}} (G^c)_I\\ &= \textstyle\bigcap_{A\subset F, F\in IC_{\tau_I(X)}} F_I.\end{aligned}$$

(2) The proof is similar to (1). □

From Result 4.32 and Proposition 4.36, the followings can be easily proved.

Proposition 4.40. *Let (X, τ) be an ITS and let $A, B \in IS_*(X)$. Then*
(1) $A_I \subset \tau_I\text{-}cl(A)$, $A_{IV}\tau_{IV}\text{-}int(A) \subset \tau_{IV}\text{-}cl(A)$,
(2) *if* $A \subset B$, *then* $\tau_I\text{-}cl(A) \subset \tau_I\text{-}cl(B)$, $\tau_{IV}\text{-}cl(A) \subset \tau_{IV}\text{-}cl(B)$,
(3) $\tau_I\text{-}cl(A \cup B) = \tau_I\text{-}cl(A) \cup \tau_I\text{-}cl(B)$,
$\tau_{IV}\text{-}cl(A \cup B) = \tau_{IV}\text{-}cl(A) \cup \tau_{IV}\text{-}cl(B)$,
(4) $\tau_I\text{-}cl(X_I) = X_I$, $\tau_{IV}\text{-}cl(X_I) = X_I$.

5. Conclusions

From Results 4.4 and 4.5, for any IT τ on a set X, two ITs τ_I and τ_{IV} were defined on X such that $\tau \subset \tau_I$ and $\tau \subset \tau_{IV}$. In the future, by using three ITs τ, τ_I and τ_{IV} in an intuitionistic topological space, we expect that some types continuities, open and closed mappings can be defined.

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