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Multi-Valued Logics: A Review

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Abstract

The present work, starting from the fuzzy sets theory introduced by Zadeh in 1965, reviews its generalizations and other relevant theories developed during the last 50 years. Those theories gave genesis to multi-valued logics, which help to treat better the uncertain or vague situations that frequently appear in problems of every day life, science and technology. Therefore they need not be seen as competitive, but as complementary to the traditional bi-valued logic of Aristotle. The article offers the basic framework to those wanting to study further the multi-valued logics that have found recently many and important applications to almost all sectors of the human activity.

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1. Fuzzy Sets and Logic

Forms of multi-valued logic have been studied since the 1920s, notably by the Polish philosopher Jan Lukasiewicz (1878-1956) and by the Polish-American logician and mathematician Alfred Tarski (1901-1983). However the term ***Fuzzy Logic (FL)*** was introduced with the 1965 proposal of ***Fuzzy Set (FS)*** theory **[16]** by the electrical engineer Lofti Zadeh, professor at the University of Berkley, California, as an infinite-valued logic in which the truth values of variables may be any real number between 0 and 1. It is employed to handle the concept of partial truth, where the truth value may range between completely true and completely false. By contrast, in the traditional logic of Aristotle, the truth values of variables may only be the integer values 0 (false) or 1 (true).

FL is based on the observation that people make frequently decisions in terms of imprecise and non-numerical information. It is recalled that a FS A on the crisp set of the discourse U is defined with the help of its *membership function m:* U $\rightarrow$ [0, 1] as the set of the ordered pairs A = {(x, m(x)): $x \in U$}. For reasons of simplicity many authors identify a FS with its membership function. A FS is usually represented symbolically by a sum, or by a series or by an integral when U is a finite or numerable set or it has the power of the continuous respectively. FSs are mathematical models representing vagueness and imprecise information that have the capability of recognizing, representing, manipulating, interpreting, and utilizing data and information which are vague and lack certainty. However, the creditability of a fuzzy set in representing the corresponding real situation depends on the successful definition of its membership function. The process of turning a FS to a crisp number, which is necessary for applying a fuzzy solution to the corresponding real world situation (e.g. in fuzzy control) is called ***defuzzification.*** For general facts on FSs and the connected to them uncertainty we refer to the book **[7].**

Zadeh introduced also the ***Fuzzy Numbers (FNs)*** **[17]** as a special form of FSs on the set of the real numbers. He defined the basic arithmetic operations on them in terms of his ***extension principle***, which provides the means for any function mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y. FNs play an important role in fuzzy mathematics analogous to the role of the ordinary numbers in the traditional mathematics. For general facts on GNs we refer to the book **[6]**.

As it was expected, the far-reaching theory of FSs aroused some objections to the scientific community. While there have been generic complaints about the fuzziness of assigning values to linguistic terms, the most cogent criticisms come from Haack **[5]** in 1979, who argued that there are only two areas – the nature of Truth and Falsity and the fuzzy systems' utility – in which FL could be possibly needed, and then maintained that in both cases it can be shown that FL is unnecessary. Fox **[4]** responded against to her objections, his most powerful argument being that traditional and FL need not be seen as competitive, but as complementary and that FL, despite the objections of classical logicians, has found its way into practical applications to almost all fields of human activity, from control theory to artificial intelligence and has proved very successful there.

2. Generalizations and Relative Theories

In 1975 Zadeh generalized the ordinary FS, otherwise called ***type-1 FS***, to the ***type-2 FS*** **[17]** so that more uncertainty can be handled connected to the membership function. The membership function of a type-2 FS is three - dimensional, its third dimension being the value of the membership function at each point of its two – dimensional domain, which is called ***Footprint of Uncertainty (FOU)***. The FOU is completely determined by its two bounding functions, a ***lower*** membership function and an ***upper*** membership function, both of which are type-1 FSs. When no uncertainty exists about the membership function, then a type-2 FS reduces to a type-1 FS, in a way analogous to probability reducing to determinism when unpredictability vanishes. In order to distinguish between a type-1 and a type-2 FS, a tilde symbol is put over the FS, so that A denotes the type-1 FS and $\tilde{A}$ denotes the comparable type-2 FS. Nevertheless, Zadeh didn't stop there, but in the same paper **[17]** generalized the type-2 FS to the ***type-n FS***, n = 1, 2, 3, …

However, when Zadeh proposed type-2 FS in 1975, the time was not right for researchers to drop what they were doing with type-1 FS and focus on type-2 FS. This changed in the late 1990s as a result of Prof. Jerry Mendel and his student's works on type-2 FS **[9]**. Since then, more and more researchers around the world are writing articles about type-2 FS and systems.

Another application for FS that has also been inspired by Prof. Zadeh is the ***Computing With Words (CWW)***, a methodology in which the objects of computation are words and propositions drawn from a natural language **[18]**. The idea was that computers would be activated by words, which would be converted into a mathematical representation using FSs and that these FSs would be mapped by a CWW engine into some other FS, after which the latter would be converted back into a word. Much research is under way about CWW. As Mendel has argued **[10]** a type-2 fuzzy set should be used as a model for a word.

In 1982 Julong Deng, professor of the Huazhong University of Science and Technology, Wuhan, China, introduced the theory of ***Grey System (GS)*** **[2]** for handling the approximate data that are frequently appear in the study of large and complex systems, like the socio-economic, the biological ones, etc. The systems which lack information, such as structure message, operation mechanism and behaviour document, are referred to as GSs. Usually, on the grounds of existing grey relations and elements one can identify where "grey" means poor, incomplete, uncertain, etc. The GS theory was mainly developed in China and it has found many applications in agriculture, economy, management, industry, ecology and in many other fields of the human activity **[3]**.

An effective tool of the GS theory is the use of ***Grey Numbers (GNs)*** that are indeterminate numbers defined in terms of the closed real intervals. More explicitly, a GN, say A, is of the form $A \in [a, b]$, where a and b are real numbers with $a \leq b$. In other words, the range in which A lies is known, but not its exact value. A GN may enrich its uncertainty representation with respect to the interval $[a, b]$ by a function $g: [a, b] \to [0, 1]$, which defines a ***degree of greyness*** $g(x)$ for each x in $[a, b]$.

The well known arithmetic of the real intervals introduced by Moore et al. **[12]** has been used to define the basic arithmetic operations among the GNs. The real number with the greatest probability to be the representative real value of the GN $A \in [a, b]$ is denoted by $W(A)$. The technique of determining the value of $w(A)$ is called ***whitening*** of A. When the distribution of A is unknown (i.e. no function g has been defined for it) one usually takes $W(A) = \frac{a+b}{2}$. For general facts on GNs we refer to the book **[8]**.

Kassimir Atanassov, professor of mathematics at the Bulgarian Academy of Sciences, introduced in 1986 as a complement of Zadeh's membership degree m(x), $x \in U$, the ***degree of non-membership*** n(x) and proposed the notion of ***intuitionistic FS (IFS)*** for a more accurate quantification of the uncertainty **[1]**.

An IFS A is formally defined as the set of the ordered triples

$A = \{(x, m(x), n(x)): x \in U, 0 \leq m(x) + n(x) \leq 1\}$.

One can write m(x) + n(x) + h(x) = 1, where h(x) is called the ***hesitation*** or ***uncertainty degree*** of x. If h(x) = 0 for all x in U, then m(x) = 1 - n(x) and A becomes an ordinary FS.

A ***rough set***, first described by the Polish computer scientist Zdzislaw Pawlak in 1991 **[13]** is a formal approximation of a crisp set in terms of a pair of sets which give the ***lower*** and the ***upper*** approximation of the original set. In the standard version of rough set theory the lower and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be FSs. The theory of rough sets has found important applications in many scientific fields and in particular in Informatics.

The Romanian – American writer and mathematician Florentin Smarandache, professor at the branch of Gallup of the New Mexico University, introduced in 1995 the ***degree of indeterminancy / neutrality*** (i) and defined the ***neutrosophic set*** in three components (M, N, I), where $M = \{m(x): x \in U\}$, $N = \{n(x): x \in U\}$ and $I = \{i(x): x \in U\}$ are subsets of the interval [0, 1] **[14]**. In other words, if A is a neutrosophic set on U, then each element x of U is expressed with respect to A in the form (m(x), n(x), i(x)). A neutrosophic set generalizes the notions of FS and of IFS. When the components m, n and i are independent, they are leaving room for incomplete information when their sum is <1, for paraconsistent information when their sum is >1 and for complete information when their sum is equal to 1.

In 1999 Dmtri Molodstov, professor of the Computing Center of the Russian Academy of Sciences in Moscow, in order to overcome the existing in each particular case difficulty of defining the proper membership function of a FS, proposed the ***Soft Sets*** as a new mathematical tool for dealing with the uncertaintie**s [11].**

Let E be a set of parameters, then a pair (F, E) is called a soft set on the universal set U, if, and only if, F is a mapping of E into the set of all subsets of U. In other words, the soft set is a paramametrized family of subsets of U. Every set F(ε) of this family, $\varepsilon \in E$, may be considered as the set of the ε-elements of the soft set (F, E).

As an example, let U be the set of the girls of a high school and let E be the set of the characterizations {pretty, ugly, tall, short, clever} assigned to each of them. It becomes evident that for an ε in E the corresponding set F(ε) could be arbitrary depending on the observer's personal criteria, or empty, whereas some of them could have non empty intersection.

A FS on U with membership function y = m(x) is a soft set on U of the form (F, [0. 1]), where $F(\alpha) = \{x \in U: m(x) \geq \alpha\}$ is the corresponding α – cut of the FS, , for each α in [0. 1].

The above topics, presented in chronological order, constitute the main generalizations and relative to FSs theories. Further, in some cases the corresponding notions have been combined to form new hybrid theories. For example, if in the definition of the soft set the set of all subsets of U is replaced by the set of all fuzzy subsets of U, one gets the notion of the ***fuzzy soft set***. Also the notion of neutrosophic set has been combined with soft sets to form a new hybrid set called ***interval valued neutrosophic set***, etc.

References

[1] **Atanassov, K.T.**, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20(1), 87-96, 1986

[2] **Deng, J.**, Control Problems of Grey Systems, *Systems and Control Letters*, 288-294, 1982.

[3] **Deng, J.**, Introduction to Grey System Theory, *The Journal of Grey System*, 1, 1-24, 1989.

[4] **Fox, J.**, Towards a reconciliation of fuzzy logic and standard logic, *Int. J. of Man - Machine Studies*, 15, 213-220, 1981.

[5] **Haack, S.**, Do we need fuzzy logic? *Int. J. of Man-Machine Studies*,11, 437-445, 1979.

[6] **Kaufmann, A. & Gupta, M.**, *Introduction to Fuzzy Arithmetic*, Van Nostrand-Reinhold Company, New York, 1991.

[7] **Klir, G. J. & Folger, T. A.,** *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988.

[8] **Liu, S. F. & Lin, Y.** (Eds.), *Advances in Grey System Research*, Berlin – Heidelberg: Springer, 2010.

[9] **Mendel, J. M.,** *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper-Saddle River, NJ, 2001.

[10] **Mendel, J. M.,** Fuzzy Sets for Words: a New Beginning, *Proc. IEEE FUZZ Conference*, St. Louis, MO, May 26–28, pp. 37–42, 2003.

[11] **Molodtsov, D.,** Soft Set Theory – First Results, *Computers and Mathematics with Applications*, 37(4-5), 19-31, 1999.

[12] **Moore, R.A., Kearfort, R.B. & Clood, M.J.**, *Introduction to Interval Analysis*, 2nd Printing, Philadelphia, SIAM, 1995.

[13] **Pawlak, Z.,** *Rough Sets: Aspects of Reasoning about Data*, Kluer Academic Publishers, Dordrecht, 1991

[14] **Smarandache, F.,** *Neutrosophy / Neutrosophic probability, set, and logic*, Proquest, Michigan, USA, 1998

[15] Zadeh , L.A., Fuzzy Sets, *Information and Control*, 8, 338-353, 1965.

[16] Zadeh, L.A., The Concept of a Linguistic Variable and its Application to Approximate Reasoning, Parts 1-3, *Information Science*, 8, 199-249, 301-357 and 9, 43-80, 1975.

[17] Zadeh, L. A., Fuzzy logic = computing with words, *IEEE Trans. on Fuzzy Systems*, 4, 103–111, 1996.

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