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Neutrosophic Commutative $\mathcal{N}$-ideals in KU-algebras

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Abstract. In this paper, the new concept of neutrosophic commutative $\mathcal{N}$-ideal in KU-algebras is introduced, and investigated some related properties. Also, a relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic commutative $\mathcal{N}$-ideal are considered.

Keywords and phrases: $\mathcal{N}_n$-structure, $\mathcal{N}_n$-ideal, neutrosophic commutative $\mathcal{N}$-ideal.

1. Introduction

A (crisp) set A in a universe P can be defined in the form of its characteristic function $\mu_A : P \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far, lost of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [2] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [11] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1999) [9] and defined the neutrosophic set on three components:(t, i, f) = (truth, indeterminacy, falsehood).

For more details, refer to the following site: http:// fs.gallup.unm.edu/FlorentinSmarandache.htm

Jun et al. [2] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [3] introduced the notion of $\mathcal{N}_n$-structure and applied it to a semigroup. Vasu and Ramesh Kumar [6] applied the notion of $\mathcal{N}_n$-structure to KU-algebras. They introduced the notions of a $\mathcal{N}_n$-subalgebra and a (closed) $\mathcal{N}_n$-ideal in a KU-algebra, and investigated related properties. They also considered characterizations of a $\mathcal{N}_n$-subalgebra and a $\mathcal{N}_n$-ideal, and discussed relations between a $\mathcal{N}_n$-subalgebra and a $\mathcal{N}_n$-ideal. They provided conditions for a $\mathcal{N}_n$-ideal to be a closed $\mathcal{N}_n$-ideal. KU-algebras entered into mathematics in 2009 through the work of Prabpayak and Leerawat [7, 8], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets ($= MV$-algebras).

The background of this study is displayed in the second section. In the third section, we introduce the notion of a neutrosophic commutative $\mathcal{N}$-ideal in KU-algebras, and investigate several properties. We consider relations between a $\mathcal{N}_n$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We discuss characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2,0)$. A KU-algebra [7, 8] refers to a system $P := (P, *, 0) \in K(\tau)$ satisfies

(KU1) $(l_{11} * l_{22}) * ((l_{22} * l_{33}) * (l_{11} * l_{33})) = 0$,

(KU2) $l_{11} * 0 = 0$,

(KU3) $0 * l_{11} = l_{11}$,

(KU4) $l_{11} * l_{22} = 0$ and $l_{22} * l_{11} = 0$ implies $l_{11} = l_{22}$,

(KU5) $l_{11} * l_{11} = 0$, for all $l_{11}, l_{22}, l_{33} \in P$.

On a KU-algebra $(P, *, 0)$ we can define a binary relation $\leq$ by putting $l_{11} \leq l_{22} \Leftrightarrow l_{22} * l_{11} = 0$, $\forall\, l_{11}, l_{22} \in P$.

In a KU-algebra P, the following hold:

(KU1') $(l_{22} * l_{33}) * (l_{11} * l_{33}) \leq (l_{11} * l_{22})$,

(KU2') $0 \leq l_{11}$,

(KU3') $l_{11} \leq l_{22}$, $l_{22} \leq l_{11}$ implies $l_{11} = l_{22}$,

(KU4') $l_{22} * l_{11} \leq l_{11}$.

Theorem 2.1 [4] In a KU-algebra P, the following axioms are satisfied: For all $l_{11}, l_{22}, l_{33} \in P$,

(i) $l_{11} \leq l_{22}$ imply $l_{22} * l_{33} \leq l_{11} * l_{33}$,

(ii) $l_{11} * (l_{22} * l_{33}) = l_{22} * (l_{11} * l_{33})$, for all $l_{11}, l_{22}, l_{33} \in P$,

(iii) $((l_{22} * l_{11}) * l_{11}) \leq l_{22}$,

(iv) $(((l_{22} * l_{11}) * l_{11}) * l_{11}) = (l_{22} * l_{11})$.

A subset I of a KU-algebra P is called an ideal [7, 8] of P if it satisfies the following:

(I1) $0 \in I$,

(I2) $(\forall\, l_{11}, l_{22} \in P)$ $(l_{22} * l_{11} \in I, l_{22} \in I \Rightarrow l_{11} \in I)$.

A KU-algebra P is said to be commutative [5] if it satisfies the following equality:

$$(\forall\, l_{11}, l_{22} \in P)((l_{22} * l_{11}) * l_{11} = (l_{11} * l_{22}) * l_{22}). \tag{1}$$

A subset I of a KU-algebra P is called a commutative ideal [5] of P if it satisfies (I1) and

$$(\forall\, l_{11}, l_{22}, l_{33} \in P)(l_{22} * (l_{33} * l_{11}) \in I, l_{33} \in I \Rightarrow ((l_{11} * l_{22}) * l_{22}) * l_{11} \in I). \tag{2}$$

Lemma 2.1 An ideal I is commutative iff the following assertion is valid.

$$(\forall\, l_{11}, l_{22} \in P)(l_{22} * l_{11} \in I \Rightarrow ((l_{11} * l_{22}) * l_{22}) * l_{11} \in I). \tag{3}$$

For any family $\{\lambda_j \mid j \in \Delta\}$ of real numbers, we define

$$\bigvee\{\lambda_j \mid j \in \Delta\} := \begin{cases} \max\{\lambda_j \mid j \in \Delta\} & \text{if } \Delta \text{ is finite} \\ \sup\{\lambda_j \mid j \in \Delta\} & \text{otherwise} \end{cases}$$
$$\bigwedge\{\lambda_j \mid j \in \Delta\} := \begin{cases} \min\{\lambda_j \mid j \in \Delta\} & \text{if } \Delta \text{ is finite} \\ \inf\{\lambda_j \mid j \in \Delta\} & \text{otherwise} \end{cases}$$

We mean by $\mathcal{F}(P,[-1,0])$ the collection of functions from a set P to $[-1,0]$. We say that an element of $\mathcal{F}(P,[-1,0])$ is a negative-valued function from P to $[-1,0]$ (briefly, $\mathcal{N}$-function on P). An $\mathcal{N}$-structure refers to an ordered pair (P,f) of P and an $\mathcal{N}$-function f on P ([2]). In what follows, we let P denote the nonempty universe of discourse unless otherwise specified.

A neutrosophic $\mathcal{N}$ (briefly, $\mathcal{N}_n$)-structure over P ([3]) is defined to be the structure:

$$P_N := \frac{P}{(\mathbb{T}_N,\mathbb{I}_N,\mathbb{F}_N)} = \left\{\frac{l_{11}}{(\mathbb{T}_N(l_{11}),\mathbb{I}_N(l_{11}),\mathbb{F}_N(l_{11}))} \mid l_{11} \in P\right\} \tag{4}$$

where $\mathbb{T}_N, \mathbb{I}_N$ and $\mathbb{F}_N$ are $\mathcal{N}$-functions called the negative truth (resp. indeterminacy and falsity) membership function on P.

We note that every $\mathcal{N}_n$-structure P_N over P satisfies the condition:

$$(\forall\, l_{11} \in P)\,(-3 \le \mathbb{T}_N(l_{11}) + \mathbb{I}_N(l_{11}) + \mathbb{F}_N(l_{11}) \le 0)\,.$$

3. Neutrosophic Commutative $\mathcal{N}$-ideals

In what follows, let P denote a KU-algebra unless otherwise specified.

Definition 3.1 A $\mathcal{N}_n$-structure P_N over P is called a $\mathcal{N}_n$-ideal [6] of P if the following assertion is valid.

$$(\forall\, l_{11}, l_{22} \in P)\left(\begin{array}{l} \mathbb{T}_N(0) \le \mathbb{T}_N(l_{11}) \le \bigvee\{\mathbb{T}_N(l_{22} * l_{11}),\ \mathbb{T}_N(l_{22})\} \\ \mathbb{I}_N(0) \ge \mathbb{I}_N(l_{11}) \ge \bigwedge\{\mathbb{I}_N(l_{22} * l_{11}),\ \mathbb{I}_N(l_{22})\} \\ \mathbb{F}_N(0) \le \mathbb{F}_N(l_{11}) \le \bigvee\{\mathbb{F}_N(l_{22} * l_{11}),\ \mathbb{F}_N(l_{22})\} \end{array}\right). \tag{5}$$

Definition 3.2 A $\mathcal{N}_n$-structure P_N over P is called a neutrosophic commutative $\mathcal{N}$ (briefly, $\mathcal{N}_{nc}$)-ideal of P if the following assertions are valid.

$$(\forall l_{11} \in P)\,(\mathbb{T}_N(0) \le \mathbb{T}_N(l_{11}), \mathbb{I}_N(0) \ge \mathbb{I}_N(l_{11}), \mathbb{F}_N(0) \le \mathbb{F}_N(l_{11})) \tag{6}$$

$$(\forall\, l_{11}, l_{22}, l_{33} \in P)\left(\begin{array}{l} \mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \le \bigvee\{\mathbb{T}_N(l_{22} * (l_{33} * l_{11})), \mathbb{T}_N(l_{33})\} \\ \mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \ge \bigwedge\{\mathbb{I}_N(l_{22} * (l_{33} * l_{11})), \mathbb{I}_N(l_{33})\} \\ \mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \le \bigvee\{\mathbb{F}_N(l_{22} * (l_{33} * l_{11})), \mathbb{F}_N(l_{33})\} \end{array}\right) \tag{7}$$

Example 3.1 Consider a KU-algebra $P = \{0, a_5, b_5, c_5\}$ with the following Cayley table.

$*$	0	a_5	b_5	c_5
0	0	a_5	b_5	c_5
a_5	0	0	a_5	c_5
b_5	0	0	0	c_5
c_5	0	a_5	b_5	0

The $\mathcal{N}_n$-structure $P_N = \left\{\frac{0}{(-0.7,-0.2,-0.6)}, \frac{a_5}{(-0.5,-0.3,-0.4)}, \frac{b_5}{(-0.5,-0.3,-0.4)}, \frac{c_5}{(-0.3,-0.8,-0.5)}\right\}$ be a $\mathcal{N}_n$-structure over P. Then P_N is a $\mathcal{N}_{nc}$-ideal of P.

Theorem 3.1 Every $\mathcal{N}_{nc}$-ideal is a $\mathcal{N}_n$-ideal. But not conversely.

Proof. Let P_N be a $\mathcal{N}_{nc}$-ideal of P. For every $l_{11}, l_{33} \in P$, we have

$$\begin{array}{l} \mathbb{T}_N(l_{11}) = \mathbb{T}_N(((l_{11} * 0) * 0) * l_{11}) \leq \bigvee \{\mathbb{T}_N(0 * (l_{33} * l_{11})), \mathbb{T}_N(l_{33})\} = \bigvee \{\mathbb{T}_N(l_{33} * l_{11}), \mathbb{T}_N(l_{33})\}, \\ \mathbb{I}_N(l_{11}) = \mathbb{I}_N(((l_{11} * 0) * 0) * l_{11}) \geq \bigwedge \{\mathbb{I}_N(0 * (l_{33} * l_{11})), \mathbb{I}_N(l_{33})\} = \bigwedge \{\mathbb{I}_N(l_{33} * l_{11}), \mathbb{I}_N(l_{33})\} \\ \mathbb{F}_N(l_{11}) = \mathbb{F}_N(((l_{11} * 0) * 0) * l_{11}) \leq \bigvee \{f_N(0 * (l_{33} * l_{11})), \mathbb{F}_N(l_{33})\} = \bigvee \{\mathbb{F}_N(l_{33} * l_{11}), \mathbb{F}_N(l_{33})\} \end{array}$$

by putting $l_{22} = 0$ in (7) and using (KU3). Therefore, P_N is a $\mathcal{N}_{nc}$-ideal of P.

Example 3.2 Consider a KU-algebra $P = \{0, a_5, b_5, c_5, d_5\}$ with the following Cayley table.

$*$	0	a_5	b_5	c_5	d_5
0	0	a_5	b_5	c_5	d_5
a_5	0	0	a_5	a_5	b_5
b_5	0	0	0	a_5	a_5
c_5	0	0	a_5	0	b_5
d_5	0	0	0	0	0

The $\mathcal{N}_n$-structure
$P_N = \left\{ \frac{0}{(-0.7,-0.2,-0.6)}, \frac{a_5}{(-0.5,-0.3,-0.4)}, \frac{b_5}{(-0.5,-0.3,-0.4)}, \frac{c_5}{(-0.3,-0.8,-0.5)}, \frac{d_5}{(-0.3,-0.8,-0.5)} \right\}$. Then P_N is a $\mathcal{N}_n$-ideal of P but not a $\mathcal{N}_{nc}$-ideal of P, since $\mathbb{F}_N(((a_5 * 0) * 0) * a_5) = -0.4 \not\leq -0.5 \bigvee\{\mathbb{F}_N(0 * (c_5 * a_5)), \mathbb{F}_N(c_5)\}$.

Theorem 3.2 Let P_N be a $\mathcal{N}_n$-ideal of P. Then, P_N is a $\mathcal{N}_{nc}$-ideal of P iff the following assertion is valid.

$$(\forall\, l_{11}, l_{22} \in P) \left(\begin{array}{l} \mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \leq \mathbb{T}_N(l_{22} * l_{11}) \\ \mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \geq \mathbb{I}_N(l_{22} * l_{11}) \\ \mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \leq \mathbb{F}_N(l_{22} * l_{11}) \end{array} \right) \tag{8}$$

Proof. Assume that P_N is a $\mathcal{N}_{nc}$-ideal of P. The assertion (8) is by taking $n = 0$ in (7) and using (KU3) and (6).

Conversely, suppose that a $\mathcal{N}_n$-ideal P_N of P satisfies the condition (8). Then,

$$(\forall\, l_{11}, l_{22} \in P) \left(\begin{array}{l} \mathbb{T}_N(l_{22} * l_{11}) \leq \bigvee \{\mathbb{T}_N(l_{33} * (l_{22} * l_{11})), \mathbb{T}_N(l_{33})\} \\ \mathbb{I}_N(l_{22} * l_{11}) \geq \bigwedge \{\mathbb{I}_N(l_{33} * (l_{22} * l_{11})), \mathbb{I}_N(l_{33})\} \\ \mathbb{F}_N(l_{22} * l_{11}) \leq \bigvee \{\mathbb{F}_N(l_{33} * (l_{22} * l_{11})), \mathbb{F}_N(l_{33})\} \end{array} \right) \tag{9}$$

It follows that the condition (7) is induced by (8) and (9). Therefore, P_N is a $\mathcal{N}_{nc}$-ideal of P.

Lemma 3.1 [6] For any $\mathcal{N}_n$-ideal P_N of P, we have

$$(\forall\, l_{11}, l_{22}, l_{33} \in P) \left(l_{22} * l_{11} \preceq l_{33} \Rightarrow \left\{ \begin{array}{l} \mathbb{T}_N(l_{11}) \leq \bigvee \{\mathbb{T}_N(l_{22}), \mathbb{T}_N(l_{33})\} \\ \mathbb{I}_N(l_{11}) \geq \bigwedge \{\mathbb{I}_N(l_{22}), \mathbb{I}_N(l_{33})\} \\ \mathbb{F}_N(l_{11}) \leq \bigvee \{\mathbb{F}_N(l_{22}), \mathbb{F}_N(l_{33})\} \end{array} \right. \right) \tag{10}$$

Theorem 3.3 In a commutative KU-algebra, every $\mathcal{N}_n$-ideal is a $\mathcal{N}_{nc}$-ideal.

Proof. Let P_N be a $\mathcal{N}_n$-ideal of a commutative KU-algebra P. For any $l_{11}, l_{22}, l_{33} \in P$ We have

$$\begin{array}{l} (((l_{11} * l_{22}) * l_{22}) * l_{11}) * ((l_{22} * (l_{33} * l_{11})) * l_{33}) \\ = (l_{22} * (l_{33} * l_{11})) * ((((l_{11} * l_{22}) * l_{22}) * l_{11}) * l_{33}) \\ = (l_{22} * (l_{33} * l_{11})) * ((((l_{22} * l_{11}) * l_{11}) * l_{11}) * l_{33}) \\ = (l_{33} * (l_{22} * l_{11})) * ((((l_{22} * l_{11}) * l_{11}) * p) * l_{33}) \\ \leq l_{33} * ((l_{11} * l_{11}) * l_{33}) = 0 \end{array}$$

that is, $(((l_{11} * l_{22}) * l_{22}) * l_{11}) * (l_{22} * (l_{33} * l_{11})) \preceq l_{33}$. It follows from Lemma 3.1 that

$$\begin{aligned}
\mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\leq \bigvee \{\mathbb{T}_N(l_{22} * (l_{33} * l_{11})), \mathbb{T}_N(l_{33})\} \\
\mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\geq \bigwedge \{\mathbb{I}_N(l_{22} * (l_{33} * l_{11})), \mathbb{I}_N(l_{33})\} \\
\mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\leq \bigvee \{\mathbb{F}_N(l_{22} * (l_{33} * l_{11})), \mathbb{F}_N(l_{33})\}
\end{aligned}$$

Therefore, P_N is a $\mathcal{N}_{nc}$-ideal of P.

Let P_N be a $\mathcal{N}_n$-structure over P and let $\lambda, \mu, \delta \in [-1, 0]$ be such that $-3 \leq \lambda + \mu + \delta \leq 0$. Consider the following sets.

$$\begin{aligned}
\mathbb{T}_N^{\lambda} &:= \{l_{11} \in P \mid \mathbb{T}_N(l_{11}) \leq \lambda\}, \\
\mathbb{I}_N^{\mu} &:= \{l_{11} \in P \mid \mathbb{I}_N(l_{11}) \geq \mu\}, \\
\mathbb{F}_N^{\delta} &:= \{l_{11} \in P \mid \mathbb{F}_N(l_{11}) \leq \delta\}.
\end{aligned}$$

The set

$$P_N(\lambda, \mu, \delta) := \{l_{11} \in P \mid \mathbb{T}_N(l_{11}) \leq \lambda, \mathbb{I}_N(l_{11}) \geq \mu, \mathbb{F}_N(l_{11}) \leq \delta\}$$

is called the (λ, μ, δ)-level set of P_N. It is clear that

$$P_N(\lambda, \mu, \delta) = \mathbb{T}_N^{\lambda} \cap \mathbb{I}_N^{\mu} \cap \mathbb{F}_N^{\delta}.$$

Theorem 3.4 If P_N is a $\mathcal{N}_n$-ideal of P, then $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are commutative ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$ whenever they are nonempty.

We call $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ level commutative ideals of P_N.

Proof. Assume that $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are nonempty for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$. Then, $l_{11} \in \mathbb{T}_N^{\lambda}$, $l_{22} \in \mathbb{I}_N^{\mu}$ and $l_{33} \in \mathbb{F}_N^{\delta}$ for some $l_{11}, l_{22}, l_{33} \in P$. Thus, $\mathbb{T}_N(0) \leq \mathbb{T}_N(l_{11}) \leq \lambda$, $\mathbb{I}_N(0) \geq \mathbb{I}_N(l_{22}) \geq \mu$ and $\mathbb{F}_N(0) \leq \mathbb{F}_N(l_{33}) \leq \delta$, that is, $0 \in \mathbb{T}_N^{\lambda} \cap \mathbb{I}_N^{\mu} \cap \mathbb{F}_N^{\delta}$. Let $l_{22} * (l_{33} * l_{11}) \in \mathbb{T}_N^{\lambda}$ and $l_{33} \in \mathbb{T}_N^{\lambda}$. Then, $\mathbb{T}_N(l_{22} * (l_{33} * l_{11})) \leq \lambda$ and $\mathbb{T}_N(l_{33}) \leq \lambda$, which imply that

$$\mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \leq \bigvee \{\mathbb{T}_N(l_{22} * (l_{33} * l_{11})), \mathbb{T}_N(l_{33})\} \leq \lambda,$$

that is, $((l_{11} * l_{22}) * l_{22}) * l_{11} \in \mathbb{T}_N^{\lambda}$. If $b_5 * (c_5 * a_5) \in \mathbb{I}_N^{\mu}$ and $c_5 \in \mathbb{I}_N^{\mu}$, then $\mathbb{I}_N(b_5 * (c_5 * a_5)) \geq \mu$ and $\mathbb{I}_N(c_5) \geq \mu$. Thus

$$\mathbb{I}_N(((a_5 * b_5) * b_5) * a_5) \geq \bigwedge \{\mathbb{I}_N(b_5 * (c_5 * a_5)), \mathbb{I}_N(c_5)\} \geq \mu,$$

and so $((a_5 * b_5) * b_5) * a_5 \in \mathbb{I}_N^{\mu}$. Finally, suppose that $v * (w * u) \in \mathbb{F}_N^{\delta}$ and $w \in \mathbb{F}_N^{\delta}$. Then, $\mathbb{F}_N(v * (w * u)) \leq \delta$ and $\mathbb{F}_N(w) \leq \delta$. Thus,

$$\mathbb{F}_N(((u * v) * v) * u) \leq \bigvee \{\mathbb{F}_N(v * (w * u)), \mathbb{F}_N(w)\} \leq \delta,$$

that is, $((u * v) * v) * u \in \mathbb{F}_N^{\delta}$. Therefore, $\mathbb{T}_N^{\lambda}, \mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are commutative ideals of P.

Corollary 3.1 Let P_N be a $\mathcal{N}_n$-structure over P and let $\lambda, \mu, \delta \in [-1, 0]$ be $\in -3 \leq \lambda + \mu + \delta \leq 0$. If P_N is a $\mathcal{N}_{nc}$-ideal of P, then the nonempty (λ, μ, δ)-level set of P_N is a commutative ideal of P.

Lemma 3.2 [6] Let P_N be a $\mathcal{N}_n$-structure over P and assume that $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are ideals of P $\forall$ $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$. Then P_N is a $\mathcal{N}_n$-ideal of P.

Theorem 3.5 Let P_N be a $\mathcal{N}_n$-structure over P and assume that $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are commutative ideals of P $\forall$ $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$. Then, P_N is a $\mathcal{N}_{nc}$-ideal of P.

Proof. If $\mathbb{T}_N^\lambda$, $\mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are commutative ideals of P, then they are ideals of P. Hence, P_N is a $\mathcal{N}_n$-ideal of P by Lemma 3.2. Let $l_{11}, l_{22} \in P$ and $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$ such that $\mathbb{T}_N(l_{22} * l_{11}) = \lambda, \mathbb{I}_N(l_{22} * l_{11}) = \mu$ and $\mathbb{F}_N(l_{22} * l_{11}) = \delta$. Then, $l_{22} * l_{11} \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$. Since $\mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$ is a commutative ideal of P, it follows from Lemma 2.1 that $((l_{11} * l_{22}) * l_{22}) * l_{11} \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$ Hence

$$\begin{aligned}
&\mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \leq \lambda = \mathbb{T}_N(l_{22} * l_{11}), \\
&\mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \geq \mu = \mathbb{I}_N(l_{22} * l_{11}), \\
&\mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) \leq \delta = \mathbb{F}_N(l_{22} * l_{11}).
\end{aligned}$$

Therefore, P_N is a $\mathcal{N}_{nc}$-ideal of P by Theorem 3.2.

Theorem 3.6 Let $f : P \to P$ be an injective mapping. Given a $\mathcal{N}_n$-structure P_N over P the following are equivalent.

(i) P_N is a $\mathcal{N}_{nc}$-ideal of P, satisfying the following condition.

$$(\forall l_{11} \in P) \begin{pmatrix} \mathbb{T}_N(f(l_{11})) = \mathbb{T}_N(l_{11}) \\ \mathbb{I}_N(f(l_{11})) = \mathbb{I}_N(l_{11}) \\ \mathbb{F}_N(f(l_{11})) = \mathbb{F}_N(l_{11}) \end{pmatrix}. \tag{11}$$

(ii) $\mathbb{T}_N^\lambda$, $\mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are commutative ideals of P_N, satisfying the following condition.

$$f\left(\mathbb{T}_N^\lambda\right) = \mathbb{T}_N^\lambda, f\left(\mathbb{I}_N^\mu\right) = \mathbb{I}_N^\mu, f\left(\mathbb{F}_N^\delta\right) = \mathbb{F}_N^\delta. \tag{12}$$

Proof. Let P_N be a $\mathcal{N}_{nc}$-ideal of P, satisfying the condition (11). Then, $\mathbb{T}_N^\lambda$ $\mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are commutative ideals of P_N by Theorem 3.4. Let $\lambda \in Im\left(\mathbb{T}_N\right),\ \mu \in Im\left(\mathbb{I}_N\right),\ \delta \in Im\left(\mathbb{F}_N\right)$ and $l_{11} \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$. Then $\mathbb{T}_N(f(l_{11})) = \mathbb{T}_N(l_{11}) \leq \lambda$, $\mathbb{I}_N(f(l_{11})) = \mathbb{I}_N(l_{11}) \geq \mu$ and $\mathbb{F}_N(f(l_{11})) = \mathbb{F}_N(l_{11}) \leq \delta$. Thus, $f(l_{11}) \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$, which shows that $f\left(\mathbb{T}_N^\lambda\right) \subseteq \mathbb{T}_N^\lambda$, $f\left(\mathbb{I}_N^\mu\right) \subseteq \mathbb{I}_N^\mu$ and $f\left(\mathbb{F}_N^\delta\right) \subseteq \mathbb{F}_N^\delta$. Let $l_{22} \in P$ be such that $f(l_{22}) = x$. Then, $\mathbb{T}_N(l_{22}) = \mathbb{T}_N(f(l_{22})) = \mathbb{T}_N(l_{11}) \leq \lambda$, $\mathbb{I}_N(l_{22}) = \mathbb{I}_N(f(l_{22})) = \mathbb{I}_N(l_{11}) \geq \mu$ and $\mathbb{F}_N(l_{22}) = \mathbb{F}_N(f(l_{22})) = \mathbb{F}_N(l_{11}) \leq \delta$, which imply that $l_{22} \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$. Thus, $l_{11} = f(l_{22}) \in f\left(\mathbb{T}_N^\lambda\right) \cap f\left(\mathbb{I}_N^\mu\right) \cap f\left(\mathbb{F}_N^\delta\right)$, and so $\mathbb{T}_N^\lambda \subseteq f\left(\mathbb{T}_N^\lambda\right)$, $\mathbb{I}_N^\mu \subseteq f\left(\mathbb{I}_N^\mu\right)$ and $\mathbb{F}_N^\delta \subseteq f\left(\mathbb{F}_N^\delta\right)$. Therefore (12) is valid.

Conversely, assume that $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are commutative ideals of P_N, satisfying the condition (12). Then, P_N is a $\mathcal{N}_{nc}$-ideal of P by Theorem 3.5. Let $l_{11}, l_{22}, l_{33} \in P$ be such that $\mathbb{T}_N(l_{11}) = \lambda$, $\mathbb{I}_N(l_{22}) = \mu$ and $\mathbb{F}_N(l_{33}) = \delta$. Note that

$$\begin{aligned}
&\mathbb{T}_N(l_{11}) = \lambda \Longleftrightarrow l_{11} \in \mathbb{T}_N^\lambda \text{ and } l_{11} \notin \mathbb{T}_N^{\tilde{\lambda}} \text{ for all } \lambda > \tilde{\lambda}, \\
&\mathbb{I}_N(l_{22}) = \mu \Longleftrightarrow l_{22} \in \mathbb{I}_N^\mu \text{ and } l_{22} \notin \mathbb{I}_N^{\tilde{\mu}} \text{ for all } \mu < \tilde{\mu}, \\
&\mathbb{F}_N(l_{33}) = \delta \Longleftrightarrow l_{33} \in \mathbb{F}_N^\delta \text{ and } l_{33} \notin \mathbb{F}_N^{\tilde{\delta}} \text{ for all } \delta > \tilde{\delta}.
\end{aligned}$$

It follows from (12) that $f(l_{11}) \in \mathbb{T}_N^\lambda, f(l_{22}) \in \mathbb{I}_N^\mu$ and $f(l_{33}) \in \mathbb{F}_N^\delta$. Hence, $\mathbb{T}_N(f(l_{11})) \leq \lambda, \mathbb{I}_N(f(l_{22})) \geq \mu$ and $\mathbb{F}_N(f(l_{33})) \leq \delta$. Let $\tilde{\lambda} = \mathbb{T}_N(f(l_{11})), \tilde{\mu} = \mathbb{I}_N(f(l_{22}))$ and $\tilde{\delta} = \mathbb{F}_N(f(l_{33}))$. If $\lambda > \tilde{\lambda}$, then $f(l_{11}) \in \mathbb{T}_N^l = f\left(\mathbb{T}_N^\lambda\right)$, and thus $l_{11} \in \mathbb{T}_N^{\tilde{\lambda}}$ since f is one to one. This is a contradiction. Hence, $\mathbb{T}_N(f(l_{11})) = \lambda = \mathbb{T}_N(l_{11})$. If $\mu < \tilde{\mu}$, then $f(l_{22}) \in \mathbb{I}_N^{\tilde{\mu}} = f\left(\mathbb{I}_N^{\tilde{\mu}}\right)$ which implies from the injectivity of f that $l_{22} \in \mathbb{I}_N^{\tilde{\mu}}$, a contradiction. Hence, $\mathbb{I}_N(f(l_{11})) = \mu = \mathbb{I}_N(l_{11})$.

If $\delta > \tau$, then $f(l_{33}) \in \mathbb{F}_N^{\tilde{\delta}} = f\left(\mathbb{F}_N^{\tilde{\delta}}\right)$. Since f is one to one, we have $l_{33} \in \mathbb{F}_N^{\tilde{\delta}}$ which is a contradiction. Thus, $\mathbb{F}_N(f(l_{11})) = \delta = \mathbb{F}_N(l_{11})$. This completes the proof.

For any elements $\zeta_t, \zeta_i, \zeta_f \in P$, we consider sets:

$$\begin{aligned} P_N^{\zeta_t} &:= \{l_{11} \in P \mid \mathbb{T}_N(l_{11}) \le \mathbb{T}_N(\zeta_t)\}, \\ P_N^{\zeta_i} &:= \{l_{11} \in P \mid \mathbb{I}_N(l_{11}) \ge \mathbb{I}_N(\zeta_i)\}, \\ P_N^{\zeta_f} &:= \{l_{11} \in P \mid \mathbb{F}_N(l_{11}) \le \mathbb{F}_N(\zeta_f)\}. \end{aligned}$$

Obviously, $\zeta_t \in P_N^{\zeta_t}, \zeta_i \in P_N^{\zeta_i}$ and $\zeta_f \in P_N^{\zeta_f}$.

Lemma 3.3 [6] Let ζ_t, ζ_i and ζ_f be any elements of P. If P_N is a $\mathcal{N}_n$-ideal of P, then $P_N^{\zeta_t}$ $P_N^{\zeta_i}$ and $P_N^{\zeta_f}$ are ideals of P.

Theorem 3.7 Let ζ_t, ζ_i and ζ_f be any elements of P. If P_N is a $\mathcal{N}_{nc}$-ideal of P then $P_N^{\zeta_t}, P_N^{\zeta_i}$ and $P_N^{\zeta_f}$ are commutative ideals of P.

Proof. If P_N is a $\mathcal{N}_{nc}$-ideal of P, then it is a $\mathcal{N}_n$-ideal of P and so $P_N^{\zeta_t}, P_N^{\zeta_i}$ and $P_N^{\zeta_f}$ are ideals of P by Lemma 3.3. Let $l_{22} * l_{11} \in P_N^{\zeta_t} \cap P_N^{\zeta_i} \cap P_N^{\zeta_f}$ for any $l_{11}, l_{22} \in P$. Then, $\mathbb{T}_N(l_{22} * l_{11}) \le \mathbb{T}_N(\zeta_t), \mathbb{I}_N(l_{22} * l_{11}) \ge \mathbb{I}_N(\zeta_i)$ and $\mathbb{F}_N(l_{22} * l_{11}) \le \mathbb{F}_N(\zeta_f)$. It follows from Theorem 3.2 that

$$\begin{aligned} \mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\le \mathbb{T}_N(l_{22} * l_{11}) \le \mathbb{T}_N(\zeta_t), \\ \mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\ge \mathbb{I}_N(l_{22} * l_{11}) \ge \mathbb{I}_N(\zeta_i), \\ \mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\le \mathbb{F}_N(l_{22} * l_{11}) \le \mathbb{F}_N(\zeta_f). \end{aligned}$$

Hence, $((l_{11} * l_{22}) * l_{22}) * l_{11} \in P_N^{\zeta_t} \cap P_N^{\zeta_i} \cap P_N^{\zeta_f}$, and therefore $P_N^{\zeta_t}, P_N^{\zeta_i}$ and $P_N^{\zeta_f}$ are commutative ideals of P by Lemma 2.1.

Theorem 3.8 Any commutative ideal of P can be realized as level commutative ideals of some $\mathcal{N}_{nc}$-ideal of P.

Proof. Let A be a commutative ideal of P and let P_N be a $\mathcal{N}_n$-structure over P in which

$$\begin{aligned} \mathbb{T}_N &: P \to [-1,0],\ p \mapsto \begin{cases} \lambda & \text{if } l_{11} \in A, \\ 0 & \text{otherwise}, \end{cases} \\ \mathbb{I}_N &: P \to [-1,0],\ p \mapsto \begin{cases} \mu & \text{if } l_{11} \in A, \\ -1 & \text{otherwise}, \end{cases} \\ \mathbb{F}_N &: P \to [-1,0],\ p \mapsto \begin{cases} \delta & \text{if } l_{11} \in A, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\lambda, \delta \in [-1, 0)$ and $\mu \in (-1, 0]$. Division into the following cases will verify that P_N is a $\mathcal{N}_{nc}$-ideal of P. If $l_{22} * (l_{33} * l_{11}) \in A$ and $l_{33} \in A$, then $((l_{11} * l_{22}) * l_{22}) * l_{11} \in A$. Thus,

$$\begin{aligned} \mathbb{T}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{T}_N(l_{33}) = \mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) = \lambda, \\ \mathbb{I}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{I}_N(l_{33}) = \mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) = \mu, \\ \mathbb{F}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{F}_N(l_{33}) = \mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) = \delta, \end{aligned}$$

and so (7) is clearly verified. If $l_{22} * (l_{33} * l_{11}) \notin A$ and $l_{33} \notin A$, then $\mathbb{T}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{T}_N(l_{33}) = 0, \mathbb{I}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{I}_N(l_{33}) = -1$ and $\mathbb{F}_N(l_{22} * (l_{33} * l_{11})) = \mathbb{F}_N(l_{33}) = 0$. Hence

$$\begin{aligned} \mathbb{T}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\le \bigvee \{\mathbb{T}_N(l_{22} * (l_{33} * l_{11})), \mathbb{T}_N(l_{33})\}, \\ \mathbb{I}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\ge \bigwedge \{\mathbb{I}_N(l_{22} * (l_{33} * l_{11})), \mathbb{I}_N(l_{33})\}, \\ \mathbb{F}_N(((l_{11} * l_{22}) * l_{22}) * l_{11}) &\le \bigvee \{\mathbb{F}_N(l_{22} * (l_{33} * l_{11})), \mathbb{F}_N(l_{33})\}. \end{aligned}$$

If $l_{22}*(l_{33}*l_{11}) \in A$ and $l_{33} \notin A$, then $\mathbb{T}_N(l_{22}*(l_{33}*l_{11})) = \lambda, \mathbb{T}_N(l_{33}) = 0, \mathbb{I}_N(l_{22}*(l_{33}*l_{11})) = \mu, \mathbb{I}_N(l_{33}) = -1$ $\mathbb{F}_N(l_{22}*(l_{33}*l_{11})) = \delta$ and $\mathbb{F}_N(l_{33}) = 0$. Therefore,

$$\begin{aligned}
&\mathbb{T}_N(((l_{11}*l_{22})*l_{22})*l_{11}) \leq \bigvee \{\mathbb{T}_N(l_{22}*(l_{33}*l_{11})), \mathbb{T}_N(l_{33})\}, \\
&\mathbb{I}_N(((l_{11}*l_{22})*l_{22})*l_{11}) \geq \bigwedge \{\mathbb{I}_N(l_{22}*(l_{33}*l_{11})), \mathbb{I}_N(l_{33})\}, \\
&\mathbb{F}_N(((l_{11}*l_{22})*l_{22})*l_{11}) \leq \bigvee \{\mathbb{F}_N(l_{22}*(l_{33}*l_{11})), \mathbb{F}_N(l_{33})\}.2
\end{aligned}$$

Similarly, if $l_{22}*(l_{33}*l_{11}) \notin A$ and $l_{33} \in A$, then (7) is verified. Therefore, P_N is a $\mathcal{N}_{nc}$-ideal of P. Obviously, $\mathbb{T}_N^{\lambda} = A, \mathbb{I}_N^{\mu} = A$ and $\mathbb{F}_N^{\delta} = A$.

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