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Neutrosophic cubic Heronian mean operators with applications in multiple attribute group decision-making using cosine similarity functions

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Abstract
This article introduces the concept of Heronian mean operators, geometric Heronian mean operators, neutrosophic cubic number–improved generalized weighted Heronian mean operators, neutrosophic cubic number–improved generalized weighted geometric Heronian mean operators. These operators actually generalize the operators of fuzzy sets, cubic sets, and neutrosophic sets. We investigate the average weighted operator on neutrosophic cubic sets and weighted geometric operator on neutrosophic cubic sets to aggregate the neutrosophic cubic information. After this, based on average weighted and geometric weighted and cosine similarity function in neutrosophic cubic sets, we developed a multiple attribute group decision-making method. Finally, we give a mathematical example to illustrate the usefulness and application of the proposed method.

Keywords
Neutrosophic set, neutrosophic cubic set, Heronian mean operator, geometric Heronian mean operator, multiple attribute decision-making problem

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Introduction

The multi-attribute decision-making (MADM) or multi-attribute group decision-making (MAGDM) widely existed in the field of management, military, economy, and engineering techniques[1–3] to get an accurate evaluation information in the premises of decision makers (DMs) to make feasible and rational decision. There is a variety of limitations in real-world problems such as uncertainty and complexity of the decision-making environment, too much abundant data and inconsistent and indeterminate with respect to fuzzy information. To process this kind of information, in 1965 Zadeh[4] first introduced the fuzzy set (FS) theory. After that Atanassov proposed the intuitionistic fuzzy set (IFS).[5,6] In IFS, Atassanav added a

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non-membership function to decrease the shortcomings in which the FS has only the membership function whereas the IFS is composed of the truth-membership function and falsity-membership function and satisfies the conditions $A_{Tru}(u), A_{Fal}(x) \in [0, 1]$ and $0 \leqslant A_{Tru}(u) + A_{Fal}(x) \leqslant 1$. Moreover, in 1998 Smarandache[7] defined the neutrosophic set (NS). In NS, Samarndache added indeterminacy-membership function, that is, NS is characterized by truth-membership $A_{Tru}(u)$, indeterminacy-membership $A_{Ind}(u)$, and falsity-membership $A_{Fal}(u)$. Moreover, the NS is the generalization of FS and IFSs. For applications point of view we refer the readers.[8–10] Further, Jun et al., proposed the concept of neutrosophic cubic set (NCS) by adding truth-membership $A_{Tru}(u)$, indeterminacy-membership $A_{Ind}(u)$, and falsity-membership $A_{Fal}(u)$ in the form of interval NS and truth-membership $A_{Tru}(u)$, indeterminacy-membership $A_{Ind}(u)$ and falsity-membership $A_{Fal}(u)$ in the form of NS.[11] Al-Omeri and Smarandache[12] introduce the idea of neutrosphic sets via neutrosophic topological spaces (NTs), and some other types of NSs such as neutrosophic open sets, neutrosophic continuity, and their application in geographical information system. NCS is the generalization of FS, cubic set, and NS. Many researchers used NCSs in different directions such as,[13–18] to have more applications. So many others discussed different aspects of NCS environment on MADM like, Peng et al.,[19] Zhang et al.,[20] Ye,[21,22] Shi and Ye,[23] Lu and Ye,[24] Pramanik et al.,[25–28] GRA[29] and Dalapati and Pramanik,[30] Liu and Wang[31] proposed the aggregation operator and applied in MAGDM problems. NS theory has various applications in numerous fields such as data record, control theory, problems and decision-making theory. Xu and Yazer[32] and Xu[33] proposed some arithmetic aggregation operators and geometric aggregation operators for intuitionistic fuzzy information and these operators did not consider the correlations of aggregated arguments. After that, in 2007 Beliakov et al.[34] proposed the Heronian mean (HM) operators, which are an important aggregated arguments and possess the characteristic of correlation of aggregation operators. HM operators can deal with the interactions among the attribute values and neutrosophic cubic numbers (NCNs) can easily express the incomplete, indeterminate and inconsistent information. Liu (The research note of HM operators. Shandong University of Finance and Economics, 2012, personal communication) in 2012 extended HM operator to the generalized HM operator.[35] Yu and Wu[36] studied the interval-valued intuitionistic fuzzy information aggregation operators and their applications in decision-making. Further work to aggregate the interval-valued intuitionistic fuzzy information Liu[37] proposed some operators such as generalized interval-valued intuitionistic fuzzy Heronian mean (GIIFHM) operator, generalized interval-valued intuitionistic fuzzy weighted Heronian mean (GIIFWHM) operator, an interval-valued intuitionistic uncertain linguistic weighted geometric average (IVIULWGA) operator, an interval-valued intuitionistic uncertain linguistic ordered weighted geometric (IVIULOWG) operators and also developed the idea of interval-valued intuitionistic uncertain linguistic variables, decision-making problems and their operational laws. Yu[38] proposed the idea of decision-making problems under intuitionistic fuzzy environment and introduced some aggregation operators, such as the intuitionistic fuzzy geometric Heronian mean (IFGHM) operators and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operators and their properties. Liu et al.,[39] proposed the aggregation operator and applied in MAGDM problems. We extend the idea of Li et al.,[40] provided in Liu et al.[39] Therefore, in this article, we will extend neutrosophic numbers (NNs) to NCNs, and propose some HM operators for NCNs, including the improved generalized weighted geometric Heronian mean (IGWGHM) operators which can satisfy some properties, such as reducibility, idempotency, monotonicity and boundedness. At the end, these properties are applied to multi-attribute group decision-making problem (Figure 1).

Preliminaries

In this section, we give some helpful terminologies from the existing literature.

Definition 1 (NS). Let U be a non-empty set.[7] A neutrsophic set in U is a structure of the form $A = \{u; A_{Tru}(u), A_{Ind}(u), A_{Fal}(u) | u \in U\}$, is characterized by a truth-membership Tru, indeterminacy-membership Ind and falsity-membership Fal, where $A_{Tru}, A_{Ind}, A_{Fal} : U \rightarrow [0, 1]$ such that $0 \leqslant A_{Tru}(u) + A_{Ind}(u) + A_{Fal}(u) \leqslant 3$.

Definition 2 (NCS). Let X be a non-empty set.[11] A NCS over U is defined in the form of a pair $\Omega = (\mathring{A}, \Lambda)$ where $\mathring{A} = \{(x; \mathring{A}_{\tilde{T}ru(u)}, \mathring{A}_{\tilde{I}nd(u)}, \mathring{A}_{\tilde{F}al(u)}) | u \in U\}$ is an interval NS in U and $\Lambda = \{(u; \Lambda_{Tru(u)}, \Lambda_{Ind(u)}, \Lambda_{Fal(u)}) | u \in U)\}$ is a NS in U.

Definition 3 (HM operator). A HM operator of dimension n is a mapping $\mathrm{HM} : I^n \rightarrow I$ such that (The research note of HM operators. Shandong University of Finance and Economics, 2012, personal communication)

$$\mathrm{HM}(u_1, u_2, \ldots, u_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \sqrt{u_i u_j} \quad (1)$$

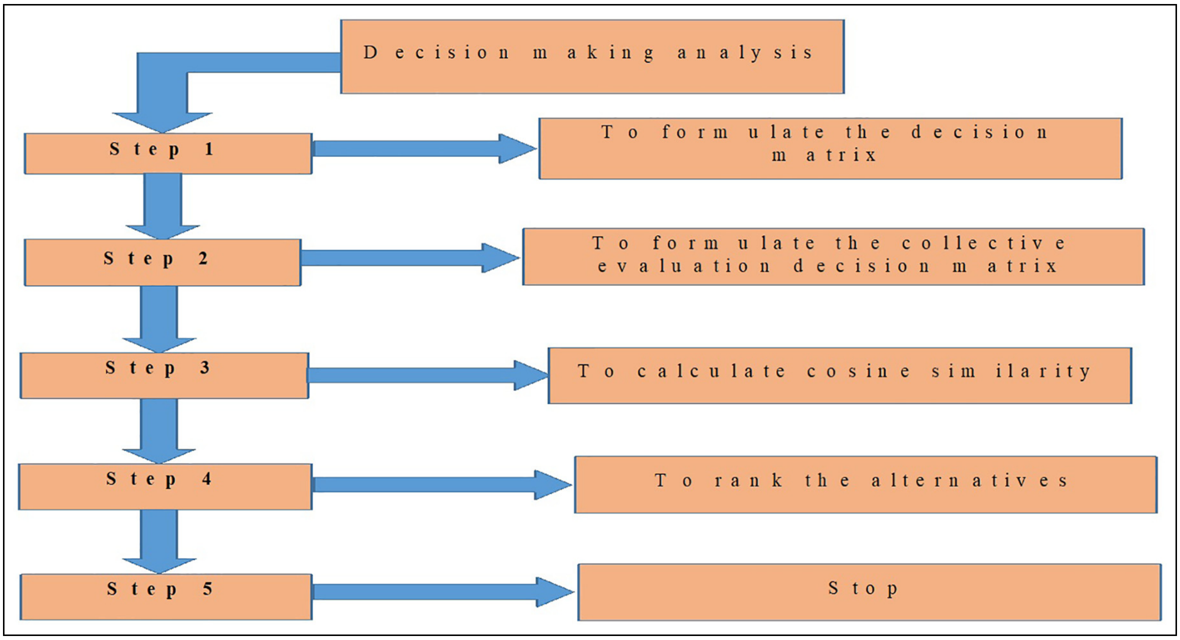

Figure 1. A flowchart of NCNs based on MAGDM problem.

where $I = [0, 1]$ then the function HM is called Heroinan mean (HM) operator.

Definition 4 (geometric Heronian mean operator). A GHM operator of dimension n is a mapping $\text{GHM} : I^n \to I$ such that (The research note of HM operators. Shandong University of Finance and Economics, 2012, personal communication)

$$\text{GHM}(u_1, u_2, \ldots, u_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} u_i^x u_j^y \right)^{\frac{1}{x+y}} \quad (2)$$

where $x, y \geqslant 0$ and $I = [0, 1]$. Then the function $\text{GHM}^{x,y}$ is called generalized Heroinan mean (GHM) operator. It is easy to prove that GHM operator has the following properties:

Theorem 1 (idempotency). Let $u_j = u \ \forall j = 1, 2, \ldots, n$, then

$$\text{GHM}^{x,y}(u_1, u_2, \ldots, u_n) = u$$

Theorem 2 (monotonicity). Suppose $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ be two collections of non-negative numbers, if $u_j \leqslant v_j \ \forall j = 1, 2, \ldots, n$, then

$$\text{GHM}^{x,y}(u_1, u_2, \ldots, u_n) \leqslant \text{GHM}^{x,y}(v_1, v_2, \ldots, v_n)$$

Theorem 3 (boundedness). GHM operator lies between the max and min operators, that is

$$\min(u_1, u_2, \ldots, u_n) \leqslant \text{GHM}^{x,y}(u_1, u_2, \ldots, u_n) \leqslant \max(u_1, u_2, \ldots, u_n)$$

Since the HM and geometric mean (GM) operator only consider the interrelationship of the e input arguments and do not take their own weights into account. In the following, we will introduce another HM operator which is called the weighted generalized Heronian mean (GWHM) operator and shown as follows.

Definition 5. Let $x, y \geqslant 0$ and $u_i (i = 1, 2, \ldots, n)$ be a collection of non-negative numbers.[36] $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $u_i (i = 1, 2, \ldots, n)$ and satisfies $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$, if

$$\text{GWHM}^{p,q}(u_1, u_2, \ldots, u_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (w_i u_i)^x (w_i u_j)^y \right)^{\frac{1}{x+y}} \quad (3)$$

then $\text{GWHM}^{p,q}$ is called a generalized weighted HM (GWHM) operator.

Definition 6.(The GHM operator). Let $x, y \geqslant 0$, and $u_i (i = 1, 2, \ldots, n)$ be a collection of non-negative numbers, if[37]

$$\text{GGHM}^{x,y}(u_1, u_2, \ldots, u_n) = \frac{1}{x+y} \prod_{i=1}^{n} \prod_{j=1}^{n} (x u_i + y u_j)^{\frac{2}{n(n+1)}} \quad (4)$$

The $GGHM^{x,y}$ is called the generalized geometric Heronian (GGHM) operator.

Definition 7. Let $x, y \geqslant 0$, and $u_i (i = 1, 2, \ldots, n)$ be a collection of non-negative numbers.[38] $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $u_i (i = 1, 2, \ldots, n)$ and satisfies $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$, if

$$\begin{aligned} &GGWHM^{p,q}(u_1, u_2, \ldots, u_n) \\ &= \frac{1}{x+y} \prod_{i=1}^{n} \prod_{j=1}^{n} \left((xu_i)^{w_i} + (yu_j)^{w_j}\right)^{\frac{2}{n(n+1)}} \end{aligned} \tag{5}$$

then $GGWHM^{p,q}$ is called the generalized geometric weighted Heronian mean (GGWHM) operator.

Definition 8. Let $x, y \geqslant 0$ and $u_i (i = 1, 2, \ldots, n)$ be a collection of non-negative numbers (The research note of HM operators. Shandong University of Finance and Economics, 2012, personal communication). $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $u_i (i = 1, 2, \ldots, n)$ and satisfies $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$, if

$$\begin{aligned} &IGGWHM^{p,q}(u_1, u_2, \ldots, u_n) \\ &= \frac{1}{x+y} \prod_{i=1}^{n} \prod_{j=1}^{n} (xu_i + yu_j)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}} \end{aligned} \tag{6}$$

then $IGGWHM^{p,q}$ is called the improved generalized geometric weighted Heronian mean (IGGWHM) operator.
The IGGWHM has the properties, such as reducibility, idempotency, monotonicity, and boundedness (The research note of HM operators. Shandong University of Finance and Economics, 2012, personal communication).

Theorem 4 (reducibility). Let $W = (1/n, 1/n, \ldots, 1/n)^T$ then

$$\begin{aligned} &IGGWHM^{p,q}(u_1, u_2, \ldots, u_n) \\ &= GGWHM^{p,q}(u_1, u_2, \ldots, u_n) \end{aligned} \tag{7}$$

Theorem 5 (idempotency). Let $x_j = x$, where $j = 1, 2, \ldots, n$ then

$$IGGWHM^{p,q}(u_1, u_2, \ldots, u_n) = x \tag{8}$$

Theorem 6 (monotonicity). Suppose $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ be two collections of non-negative numbers, if $u_i \geqslant v_i \ \forall i = 1, 2, \ldots, n$, then

$$\begin{aligned} &IGGWHM^{x,y}(u_1, u_2, \ldots, u_n) \\ &\geqslant IGGWHM^{x,y}(v_1, v_2, \ldots, v_n) \end{aligned}$$

Theorem 7 (boundedness). The $IGGWHM^{x,y}$ operator lies between the max and min operators, that is

$$\begin{aligned} &\min(u_1, u_2, \ldots, u_n) \leqslant IGGWHM^{x,y}(u_1, u_2, \ldots, u_n) \\ &\leqslant \max(u_1, u_2, \ldots, u_n) \end{aligned} \tag{9}$$

We analyze some special cases of the IGGWHM operator which are defined as follows:

1. When $y = 0$, then

$$IGGWHM^{x,0}(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n} \prod_{j=1}^{n} (u_i)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}} \tag{10}$$

From here we see that $WGGWHM^{x,0}$ does not have any relationship with x.

2. When $x = 0$, then

$$IGGWHM^{0,y}(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n} \prod_{j=1}^{n} (u_j)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}} \tag{11}$$

Similarly, $IGGWHM^{0,y}$ does not have any relationship with y.

3. When $x = y = 1$, then

$$\begin{aligned} &IGGWHM^{1,1}(u_1, u_2, \ldots, u_n) \\ &= \frac{1}{2} \prod_{i=1}^{n} \prod_{j=i}^{n} (u_i + u_j)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}} \end{aligned} \tag{12}$$

Definition 9 (cubic Hamy mean). Suppose $(\tilde{v}_i, v_i)$ where $i = 1, 2, ..., n$ is a collection of non-negative real numbers and parameter $k = 1, 2, \ldots, n$.[18] Then, the cubic Hamy mean (CHM) is defined as follows

$$CHM^{k}(\tilde{v}_{\hat{i}}, v_{\hat{i}}) = \frac{\sum_{1 \leqslant \hat{i}_1 < \cdots \hat{i}_k \leqslant n} \left(\prod_{j=1}^{k} \tilde{v}_{\hat{i}_j}, \prod_{j=1}^{k} v_{\hat{i}_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} \tag{13}$$

where $(\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_{\overset{\circ}{k}})$ navigate all k-tuple arrangement of $(1, 2, \ldots, n)$ and $\binom{n}{k}$ is the binomial coefficient and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition 10. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite set and two NCSs be $x = \{x_1, x_2, \ldots, x_n\}$ and $y = \{y_1, y_2, \ldots, y_n\}$ where $x_j = ((\tilde{T}_{xj}, \tilde{I}_{xj}, \tilde{F}_{xj}), (T_{xj}, I_{xj}, F_{xj}))$ and $y_j = ((\tilde{T}_{yj}, \tilde{I}_{yj}, \tilde{F}_{yj}), (T_{yj}, I_{yj}, F_{yj}))$ for $j = 1, 2, \ldots, n$ are two collections of NCNs.[24] Then cosine measure of $S(h)$ is proposed based on the distance as follows

$$S(h) = \frac{1}{2n}\sum_{j=1}^{n} w_j \left\{ \begin{array}{c} \cos\left(\frac{|\tilde{T}_{xj}^- - \tilde{T}_{yj}^-| + |\tilde{T}_{xj}^+ - \tilde{T}_{yj}^+| + |\tilde{I}_{xj}^- - \tilde{I}_{yj}^-| + |\tilde{I}_{xj}^+ - \tilde{I}_{yj}^+| + |\tilde{F}_{xj}^- - \tilde{F}_{yj}^-| + |\tilde{F}_{xj}^+ - \tilde{F}_{yj}^+|}{12}\pi\right) \\ + \cos\left(\frac{|T_{xj}^- - T_{yj}^-| + |I_{xj} - I_{yj}^-| + |F_{xj}^- - F_{yj}^-|}{6}\pi\right) \end{array} \right\} \tag{14}$$

Some HM operator based on the NCN

In this section, we define cNCNIGWHM operator and NCNIGWGHM operator, their properties and different operations.

Definition 11 (the NCNIGWHM operator). Let $x, y \geqslant 0$, and $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ where

$$\tilde{R}_{aj} = \{\tilde{A}_{Tru}(u_j), \tilde{A}_{Ind}(u_j), \tilde{A}_{Fal}(u_j)\} \quad and$$
$$S_b = \{A_{Tru}(u_j), A_{Ind}(u_j), A_{Fal}(u_j)\}$$

$(j = 1, 2, \ldots, n)$ be a collection of NCNs with the weight vector $W = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then an NCNIGWH operator of dimension n is a mapping NCNIGWH : $\Psi^n \to \Psi$, and has

$$\mathrm{NCNIGWH}^{x,y}(Q_1, Q_2, \ldots, Q_n) = \left(\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \left(w_i w_j Q_i^x \otimes Q_j^y \right) \right)^{\frac{1}{x+y}} \tag{15}$$

where Ψ is the set of all NCNs.

Theorem 8. Let $x, y \geqslant 0$, and $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ $(j = 1, 2, \ldots, n)$ be a collection of NCNs with the weight vector $W = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then the result aggregated from Definition 11 is still an NCN, and even

$$\mathrm{NCNIGWH}^{x,y}(Q_1, Q_2, \ldots, Q_n) = \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n} \left(\begin{array}{c} 1 - \tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j), \\ 1 - A_{Tru}^x(u_i)A_{Tru}^y(u_j) \end{array} \right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n} \left(\begin{array}{c} 1 - \left(1 - \tilde{A}_{Ind}^x(u_i)\right)^x \left(1 - \tilde{A}_{Ind}^x(u_j)\right)^y, \\ 1 - \left(1 - A_{Ind}^x(u_i)\left(1 - A_{Ind}^y(u_j)\right)\right) \end{array} \right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n} \left(\begin{array}{c} 1 - \left(1 - \tilde{A}_{Fal}^x(u_i)\right)^x \left(1 - \tilde{A}_{Fal}^x(u_j)\right)^y, \\ 1 - \left(1 - A_{Fal}^x(u_i)\left(1 - A_{Fal}^y(u_j)\right)\right) \end{array} \right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \end{array} \right) \tag{16}$$

Proof. Since

$$Q_i^x = \left\{ \begin{array}{c} \left(\tilde{A}_{Tru}^x(u_i), 1 - \left(1 - \tilde{A}_{Ind}(u_i)\right)^x, 1 - \left(1 - \tilde{A}_{Fal}(u_i)\right)^x\right) \\ \left(A_{Tru}^x(u_i), 1 - (1 - A_{Ind}(u_i))^x, 1 - (1 - A_{Fal}(u_i))^x\right) \end{array} \right\}$$

$$Q_j^y = \left\{ \begin{array}{c} \left(\tilde{A}_{Tru}^y(u_j), 1 - \left(1 - \tilde{A}_{Ind}(u_j)\right)^y, 1 - \left(1 - \tilde{A}_{Fal}(u_j)\right)^y\right) \\ \left(A_{Tru}^y(u_j), 1 - (1 - A_{Ind}(u_i))^y, 1 - (1 - A_{Fal}(u_i))^y\right) \end{array} \right\}$$

$$Q_i^x Q_j^y = \left\{ \begin{array}{c} \left(\begin{array}{c} \tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j), 1 - \left(1 - \tilde{A}_{Ind}(u_i)\right)^x \left(1 - \tilde{A}_{Ind}(u_j)\right)^y \\ 1 - \left(1 - \tilde{A}_{Fal}(u_i)\right)^x \left(1 - \tilde{A}_{Fal}(u_j)\right)^y \end{array} \right) \\ \left(\begin{array}{c} A_{Tru}^x(u_i)A_{Tru}^y(u_j), 1 - (1 - A_{Ind}(u_i))^x (1 - A_{Ind}(u_i))^y \\ 1 - (1 - A_{Fal}(u_i))^x (1 - A_{Fal}(u_i))^y \end{array} \right) \end{array} \right\}$$

and

$$w_i w_j Q_i^x \otimes Q_j^y = \left\{ \begin{array}{c} \left(\begin{array}{c} 1 - \left(1 - \tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j)\right)^{w_i w_j} \\ \left(1 - \left(1 - \tilde{A}_{Ind}(u_i)\right)^x \left(1 - \tilde{A}_{Ind}(u_j)\right)^y\right)^{w_i w_j} \\ \left(1 - \left(1 - \tilde{A}_{Fal}(u_i)\right)^x \left(1 - \tilde{A}_{Fal}(u_j)\right)^y\right)^{w_i w_j} \end{array} \right) \\ \left(\begin{array}{c} 1 - \left(1 - A_{Tru}^x(u_i)A_{Tru}^y(u_j)\right)^{w_i w_j} \\ \left(1 - (1 - A_{Ind}(u_i))^x \left(1 - A_{Ind}(u_j)\right)^y\right)^{w_i w_j} \\ \left(1 - (1 - A_{Fal}(u_i))^x \left(1 - A_{Fal}(u_j)\right)^y\right)^{w_i w_j} \end{array} \right) \end{array} \right\}$$

then

$$\underset{i=1}{\overset{n}{\oplus}}\underset{j=i}{\overset{n}{\oplus}}\left(w_iw_jQ_i^x\otimes Q_j^y\right) = \left\{\begin{pmatrix} 1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}_{Tru}^{x}(u_i)\tilde{A}_{Tru}^{y}(u_j)\right)^{w_iw_j} \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^{x}\left(1-\tilde{A}_{Ind}(u_j)\right)^{y}\right)^{w_iw_j} \\ \prod_{i=1}^{n}\prod_{j=i}^{n}1-\left(1-\tilde{A}_{Fal}(u_i)\right)^{x}\left(1-\tilde{A}_{Fal}(u_j)\right)^{y} \end{pmatrix}^{w_iw_j} \\ \begin{pmatrix} 1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{x}(u_i)A_{Tru}^{y}(u_j)\right)^{w_iw_j} \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u_i)\right)^{x}\left(1-A_{Ind}(u_j)\right)^{y}\right)^{w_iw_j} \\ \prod_{i=1}^{n}\prod_{j=i}^{n}1-\left(1-A_{Fal}(u_i)\right)^{x}\left(1-A_{Fal}(u_j)\right)^{y} \end{pmatrix}^{w_iw_j} \right\}$$

Furthermore

$$\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\underset{i=1}{\overset{n}{\oplus}}\underset{j=i}{\overset{n}{\oplus}}\left(w_iw_jQ_i^x\otimes Q_j^y\right)$$

$$= \left\{\begin{pmatrix} 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}_{Tru}^{x}(u_i)\tilde{A}_{Tru}^{y}(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^{x}\left(1-\tilde{A}_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Fal}(u_i)\right)^{x}\left(1-\tilde{A}_{Fal}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \end{pmatrix} \\ \begin{pmatrix} \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{x}(u_i)A_{Tru}^{y}(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u_i)\right)^{x}\left(1-A_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}(u_i)\right)^{x}\left(1-A_{Fal}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \end{pmatrix}^{w_iw_j} \right\}$$

$$\left(\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\underset{i=1}{\overset{n}{\oplus}}\underset{j=i}{\overset{n}{\oplus}}\left(w_iw_jQ_i^x\otimes Q_j^y\right)\right)^{\frac{1}{x+y}}$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \tilde{A}^x_{Tru}(u_i)\tilde{A}^y_{Tru}(u_j)\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \left(1 - \tilde{A}_{Ind}(u_i)\right)^x \left(1 - \tilde{A}_{Ind}(u_j)\right)^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \left(1 - \tilde{A}_{Fal}(u_i)\right)^x \left(1 - \tilde{A}_{Fal}(u_j)\right)^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \end{array} \right) \\ \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - A^x_T(u_i) A^y_T(u_j)\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - (1 - A_{Ind}(u_i))^x (1 - A_{Ind}(u_j))^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - (1 - A_{Fal}(u_i))^x (1 - A_{Fal}(u_j))^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \end{array} \right) \end{array} \right\}$$

which complete the proof of Theorem 8 □

Moreover, the NCNIGWHM operator also has the following properties.

Theorem 9 (idempotency). Let $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ $(j = 1, 2, \ldots, n)$, then

$$\begin{aligned} &\text{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \\ &= (\tilde{R}_{a_j}, S_{b_j}) \\ &= \left\{ \{\tilde{A}_{Tru}(u_j), \tilde{A}_{Ind}(u_j), \tilde{A}_{Fal}(u_j)\}, \{A_{Tru}(u_j), A_{Ind}(u_j), A_{Fal}(u_j)\} \right\} \end{aligned} \quad (17)$$

Proof. Since $Q_j = (\tilde{R}_{a_j}, S_{b_j})\,(j = 1, 2, \ldots, n)$, and then according to equation (16), we have $\text{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n)$

$$= \left\{ \begin{array}{l} \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \tilde{A}^x_{Tru}(u)\tilde{A}^y_{Tru}(u)\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \left(1 - \tilde{A}_{Ind}(u)\right)^x \left(1 - \tilde{A}_{Ind}(u)\right)^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \left(1 - \tilde{A}_{Fal}(u)\right)^x \left(1 - \tilde{A}_{Fal}(u)\right)^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \end{array} \right) \\ \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - A^x_{Tru}(u) A^y_{Tru}(u)\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - (1 - A_{Ind}(u))^x (1 - A_{Ind}(u))^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - (1 - A_{Fal}(u))^x (1 - A_{Fal}(u))^y\right)^{w_i w_j} \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{x+y}} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}_{Tru}^{x+y}(u)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}}, \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u)\right)^{x+y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Fal}(u)\right)^{x+y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix} \begin{pmatrix} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{x+y}(u)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u)\right)^{x+y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}(u)\right)^{x+y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} \left(1-\left(\left(1-\tilde{A}_{T}^{x+y}(u)\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\left(1-\left(1-\tilde{A}_{Ind}(u)\right)^{x+y}\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\left(1-\left(1-\tilde{A}_{Fal}(u)\right)^{x+y}\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix} \begin{pmatrix} \left(1-\left(\left(1-A_{T}^{x+y}(u)\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\left(1-\left(1-A_{Ind}(u)\right)^{x+y}\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\left(1-\left(1-A_{Fal}(u)\right)^{x+y}\right)^{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix} \right\}$$

$$=\left\{\begin{pmatrix}\left(1-\left(1-\tilde{A}_T^{x+y}(u)\right)\right)^{\frac{1}{x+y}}\\ 1-\left(1-\left(1-\left(1-\tilde{A}_{Ind}(u)\right)^{x+y}\right)\right)^{\frac{1}{x+y}}\\ 1-\left(1-\left(1-\left(1-\tilde{A}_{Fal}(u)\right)^{x+y}\right)\right)^{\frac{1}{x+y}}\end{pmatrix}\begin{pmatrix}\left(1-\left(1-A_T^{x+y}(u)\right)\right)^{\frac{1}{x+y}}\\ 1-\left(1-\left(1-\left(1-A_{Ind}(u)\right)^{x+y}\right)\right)^{\frac{1}{x+y}}\\ 1-\left(1-\left(1-\left(1-A_{Fal}(u)\right)^{x+y}\right)\right)^{\frac{1}{x+y}}\end{pmatrix}\right\}$$

$$=\left\{\begin{array}{l}\left(\left(\tilde{A}_T^{x+y}(u)\right)^{\frac{1}{x+y}},1-\left(\left(1-\tilde{A}_{Ind}(u)\right)^{x+y}\right)^{\frac{1}{x+y}},1-\left(\left(1-\tilde{A}_{Fal}(u)\right)^{x+y}\right)^{\frac{1}{x+y}}\right)\\ \left(\left(A_T^{x+y}(u)\right)^{\frac{1}{x+y}},1-\left(\left(1-A_{Ind}(u)\right)^{x+y}\right)^{\frac{1}{x+y}},1-\left(\left(1-A_{Fal}(u)\right)^{x+y}\right)^{\frac{1}{x+y}}\right)\end{array}\right\}$$

$$=\left\{\{\tilde{A}_T(u_j),\tilde{A}_{Ind}(u_j),\tilde{A}_{Fal}(u_j)\},\{A_{Tru}(u_j),A_{Ind}(u_j),A_{Fal}(u_j)\}\right\}=(\tilde{R}_{a_j},S_{b_j})\square$$

Theorem 10 (monotonicity). Let $Q_j=(\tilde{R}_{a_j},S_{b_j})$ and $Q_j'=(\tilde{R}_{a_j}',S_{b_j}')$ $(j=1,2,\ldots,n)$, be two collections of NCNs. If $Q_j\geqslant Q_j'\ \forall j$ (suppose $\tilde{A}_{Tru}(u_j)\geqslant\tilde{A}_{Tru}'(u_j)$, $\tilde{A}_{Ind}(u_j)\geqslant\tilde{A}_{Ind}'(u_j)$, $\tilde{A}_{Fal}(u_j)\geqslant\tilde{A}_{Fal}'(u_j)$, $A_{Tru}(u_j)\geqslant A_{Tru}'(u_j)$, $A_{Ind}(u_j)\geqslant A_{Ind}'(u_j)$, $A_{Fal}(u_j)\geqslant A_{Fal}'(u_j)$), then

$$\begin{aligned}&\text{NCNIGWHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)\\&\geqslant\text{NCNIGWHM}^{x,y}(Q_1',Q_2',\ldots,Q_n')\end{aligned}\tag{18}$$

Proof

1. Since $\tilde{A}_{Tru}(u_j)\geqslant\tilde{A}_{Tru}'(u_j),A_{Tru}(u_j)\geqslant A_{Tru}'(u_j)\ \forall j$, and $x,y>0$, then we have $\tilde{A}_{Tru}^x(u_i)\ \tilde{A}_{Tru}^y(u_j)\geqslant\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j)$, $A_{Tru}^x(u_i)A_{Tru}^y(u_j)\geqslant A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)$, $1-\tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j)\leqslant1-\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j)$, $1-A_{Tru}^x(u_i)A_{Tru}^y(u_j)\leqslant1-A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)$

$$\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j),1-A_{Tru}^x(u_i)A_{Tru}^y(u_j)\right)^{w_iw_j}$$
$$\leqslant\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j),1-A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)\right)^{w_iw_j}$$

and

$$\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j),1-A_{Tru}^x(u_i)A_{Tru}^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}$$
$$\leqslant\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j),1-A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}$$

$$1-\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j),1-A_{Tru}^x(u_i)A_{Tru}^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}$$
$$\geqslant1-\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j),1-A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}$$

so

$$\left(1-\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}^x(u_i)\tilde{A}_{Tru}^y(u_j),1-A_{Tru}^x(u_i)A_{Tru}^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}\right)^{\frac{1}{x+y}}$$
$$\geqslant\left(1-\left(\prod_{i=1}^n\prod_{j=i}^n\left(1-\tilde{A}_{Tru}'^x(u_i)\tilde{A}_{Tru}'^y(u_j),1-A_{Tru}'^x(u_i)A_{Tru}'^y(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^n\sum_{j=1}^n w_iw_j}}\right)^{\frac{1}{x+y}}$$

2. Since $\tilde{A}_{Ind}(u_j)\leqslant\tilde{A}_{Ind}'(u_j),A_{Ind}(u_j)\geqslant A_{Ind}'(u_j)\ \forall j$, and $x,y>0$, then we have $\left(1-\tilde{A}_{Ind}(u_i)\right)^x\geqslant\left(1-\tilde{A}_{Ind}(u_i)\right)^y,\left(1-A_{Ind}(u_i)\right)^x\geqslant\left(1-A_{Ind}(u_i)\right)^y$

and $\left(1-\tilde{A}_{Ind}(u_j)\right)^y \geqslant \left(1-\tilde{A}'_{Ind}(u_j)\right)^y$, $(1-A_{Ind}(u_j))^y \geqslant \left(1-A'_{Ind}(u_j)\right)^y$ then $\begin{pmatrix} \left(1-\tilde{A}_{Ind}(u_i)\right)^x\left(1-\tilde{A}_{Ind}(u_j)\right)^y \geqslant \left(1-\tilde{A}'_{Ind}(u_i)\right)^x\left(1-\tilde{A}'_{Ind}(u_j)\right)^y \\ (1-A_{Ind}(u_i))^x\left(1-A_{Ind}(u_j)\right)^y \geqslant \left(1-A'_{Ind}(u_i)\right)^x\left(1-A'_{Ind}(u_j)\right)^y \end{pmatrix}$

$$\begin{pmatrix} 1-\left(1-\tilde{A}_{Ind}(u_i)\right)^x\left(1-\tilde{A}_{Ind}(u_j)\right)^y \leqslant 1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^x\left(1-\tilde{A}'_{Ind}(u_j)\right)^y \\ 1-(1-A_{Ind}(u_i))^x\left(1-A_{Ind}(u_j)\right)^y \leqslant 1-\left(1-A'_{Ind}(u_i)\right)^x\left(1-A'_{Ind}(u_j)\right)^y \end{pmatrix}$$

$$\begin{pmatrix} \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^x\left(1-\tilde{A}_{Ind}(u_j)\right)^y\right)^{w_iw_j} \leqslant \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^x\left(1-\tilde{A}'_{Ind}(u_j)\right)^y\right)^{w_iw_j} \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-A_{Ind}(u_i))^x\left(1-A_{Ind}(u_j)\right)^y\right)^{w_iw_j} \leqslant \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Ind}(u_i)\right)^x\left(1-A'_{Ind}(u_j)\right)^y\right)^{w_iw_j} \end{pmatrix}$$

$$\begin{pmatrix} \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^x\left(1-\tilde{A}_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \leqslant \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^x\left(1-\tilde{A}'_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-A_{Ind}(u_i))^x\left(1-A_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \leqslant \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Ind}(u_i)\right)^x\left(1-A'_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \end{pmatrix}$$

$$\begin{pmatrix} 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^x\left(1-\tilde{A}_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \geqslant 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^x\left(1-\tilde{A}'_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-A_{Ind}(u_i))^x\left(1-A_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \\ \geqslant 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Ind}(u_i)\right)^x\left(1-A'_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}} \end{pmatrix}$$

$$\left(\begin{array}{c} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{I}(u_i)\right)^{x}\left(1-\tilde{A}_{I}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \geqslant\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^{x}\left(1-\tilde{A}'_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u_i)\right)^{x}\left(1-A_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \geqslant\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Ind}(u_i)\right)^{x}\left(1-A'_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{array}\right)$$

$$\text{so}\left(\begin{array}{c} 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Ind}(u_i)\right)^{x}\left(1-\tilde{A}_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Ind}(u_i)\right)^{x}\left(1-\tilde{A}'_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u_i)\right)^{x}\left(1-A_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Ind}(u_i)\right)^{x}\left(1-A'_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{array}\right)$$

3. Similar to step 2, we can prove

$$\left(\begin{array}{l} 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}_{Fal}(u_i)\right)^x\left(1-\tilde{A}_{Fal}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\tilde{A}'_{Fal}(u_i)\right)^x\left(1-\tilde{A}'_{Fal}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}(u_i)\right)^x\left(1-A_{Fal}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ \leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A'_{Fal}(u_i)\right)^x\left(1-A'_{Fal}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{array}\right)$$

According toTheorems 10–12 and Definition 12, we can get

$$\left(\begin{array}{c} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\widetilde{A}^x Tru(u_i)\widetilde{A}^y Tru\ (u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\widetilde{A}_{Ind}(u_i)\right)^x\left(1-\widetilde{A}_{Ind}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\widetilde{A}_{Fal}(u_i)\right)^x\left(1-\widetilde{A}_{Fal}(u_j)\right)^y\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{array}\right)$$

$$\geqslant \begin{pmatrix} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{\prime x}(u_i)A_{Tru}^{\prime y}(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}^{\prime}(u_i)\right)^{x}\left(1-A_{Ind}^{\prime}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}^{\prime}(u_i)\right)^{x}\left(1-A_{Fal}^{\prime}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix}$$

$$\begin{pmatrix} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{x}(u_i)AA_{Tru}^{y}(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}(u_i)\right)^{x}\left(1-A_{Ind}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}(u_i)\right)^{x}\left(1-A_{Fal}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix}$$

$$\geqslant \begin{pmatrix} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-A_{Tru}^{\prime x}(u_i)A_{Tru}^{\prime y}(u_j)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Ind}^{\prime}(u_i)\right)^{x}\left(1-A_{Ind}^{\prime}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-A_{Fal}^{\prime}(u_i)\right)^{x}\left(1-A_{Fal}^{\prime}(u_j)\right)^{y}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x+y}} \end{pmatrix}$$

that is, $\mathrm{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \geqslant \mathrm{NCNIGWHM}^{x,y}(Q_1', Q_2', \ldots, Q_n')$ which completes the proof. □

Theorem 11 (boundedness). Let $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ $(j = 1, 2, \ldots, n)\ldots$ be a collection of NCNs, and $Q_j^- = (\min\tilde{R}_{a_j}, \min S_{b_j})$, $Q_j^+ = (\max\tilde{R}_{a_j}, \max S_{b_j})$ or

$$Q_j^- = \begin{pmatrix} \min \tilde{A}_{Tru}(u_j), \min \tilde{A}_{Ind}(u_j), \min \tilde{A}_{Fal}(u_j), \\ (\min A_{Tru}(u_j), \min A_{Ind}(u_j), \min A_{Fal}(u_j)) \end{pmatrix}$$

$$Q_j^+ = \begin{pmatrix} \max \tilde{A}_{Tru}(u_j), \max \tilde{A}_{Ind}(u_j), \max \tilde{A}_{Fal}(u_j) \\ (\max A_{Tru}(u_j), \max A_{Ind}(u_j), \max A_{Fal}(u_j)) \end{pmatrix}$$

then

$$Q_j^- \leqslant \mathrm{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \leqslant Q_j^+ \tag{19}$$

Proof. Since $Q_j \geqslant Q_j^-$, then based on Theorems 10 and 11, we have

$$\mathrm{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \geqslant \mathrm{NCNIGWHM}^{x,y}(Q_1^-, Q_2^-, \ldots, Q_n^-) = Q_j^-$$

Like wise, we can get

$$\mathrm{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \leqslant \mathrm{NCNIGWHM}^{x,y}(Q_1^+, Q_2^+, \ldots, Q_n^+) = Q_j^+$$

Then

$$Q_j^- \leqslant \mathrm{NCNIGWHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \leqslant Q_j^+$$

which completes the proof. □

We will discuss some special cases of the NCNIGWHM with respect to parameters x and y, as follows:

1. When $x = 0$, then

$$\mathrm{NCNIGWHM}^{o,y}(Q_1, Q_2, \ldots, Q_n) \tag{20}$$

$$= \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \tilde{A}_{Tru}^{y}(u_j)\right)^{w_i w_j}, \left(1 - A_{Tru}^{y}(u_j)\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \left(1 - \tilde{A}_{Ind}(u_j)\right)^{y}\right)^{w_i w_j}, \left(1 - \left(1 - A_{Ind}(u_j)\right)^{y}\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \left(1 - \tilde{A}_{Fal}(u_j)\right)^{y}\right)^{w_i w_j}, \left(1 - \left(1 - A_{Fal}(u_j)\right)^{y}\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \tilde{A}_{Tru}^{y}(u_j)\right)^{w_i w_j}, \left(1 - A_{Tru}^{y}(u_j)\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \left(1 - \tilde{A}_{Ind}(u_j)\right)^{y}\right)^{w_i w_j}, \left(1 - \left(1 - A_{Ind}(u_j)\right)^{y}\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \\ 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\left(1 - \left(1 - \tilde{A}_{Fal}(u_j)\right)^{y}\right)^{w_i w_j}, \left(1 - \left(1 - A_{Fal}(u_j)\right)^{y}\right)^{w_i w_j} \right) \right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{y}} \end{array} \right)$$

2. When $y = 0$, then we have

$$\mathrm{NCNIGWHM}^{x,0}(Q_1, Q_2, \ldots, Q_n) \tag{21}$$

$$= \left(\begin{array}{c} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\widetilde{A}^{y}_{Tru}(u_i)\right)^{w_iw_j},\left(1-A^{y}_{Tru}(u_i)\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\left(1-\widetilde{A}_{Ind}(u_i)\right)^{y}\right)^{w_iw_j},\left(1-(1-A_{Ind}(u_i))^{y}\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\left(1-\widetilde{A}_{Fal}(u_x)\right)^{y}\right)^{w_iw_j},\left(1-(1-A_{Fal}(u_x))^{y}\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\widetilde{A}^{y}_{Tru}(u_i)\right)^{w_iw_j},\left(1-A^{y}_{Tru}(u_i)\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\left(1-\widetilde{A}_{Ind}(u_i)\right)^{y}\right)^{w_iw_j},\left(1-(1-A_{Ind}(u_i))^{y}\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \\ 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\left(1-\left(1-\widetilde{A}_{Fal}(u_x)\right)^{y}\right)^{w_iw_j},\left(1-(1-A_{Fal}(u_x))^{y}\right)^{w_iw_j}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j}}\right)^{\frac{1}{x}} \end{array}\right)$$

3. when $x = y = 1$, then we have

$$\mathrm{NCNIGWHM}^{1,1}(Q_1, Q_2, \ldots, Q_n) \tag{22}$$

$$
= \left(\begin{array}{c}
\left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \widetilde{A}_{Tru}(u_j)\right)^{w_i w_j}}{\left(1 - A_{Tru}(u_j)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}} \\
1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \left(1 - \widetilde{A}_{Ind}(u_j)\right)\right)^{w_i w_j}}{\left(1 - \left(1 - A_{Ind}(u_j)\right)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}} \\
1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \left(1 - \widetilde{A}_{Fal}(u_j)\right)^{y}\right)^{w_i w_j}}{\left(1 - \left(1 - A_{Fal}(u_j)\right)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \widetilde{A}_{Tru}(u_j)\right)^{w_i w_j}}{\left(1 - A_{Tru}(u_j)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}} \\
1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \left(1 - \widetilde{A}_{Ind}(u_j)\right)\right)^{w_i w_j}}{\left(1 - \left(1 - A_{Ind}(u_j)\right)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}} \\
1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=i}^{n} \left(\frac{\left(1 - \left(1 - \widetilde{A}_{Fal}(u_j)\right)^{y}\right)^{w_i w_j}}{\left(1 - \left(1 - A_{Fal}(u_j)\right)\right)^{w_i w_j}} \right) \right)^{\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j}} \right)^{\frac{1}{2}}
\end{array} \right)
$$

where Ψ is the set of all NCNs.

NCNIGWGHM operator

Definition 12. Let $x, y \geqslant 0$, and $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ where $\tilde{R}_{aj} = \{\tilde{A}_{Tru}(u_j), \tilde{A}_{Ind}(u_j), \breve{A}_{Fal}(u_j)\}$ and $S_b = \{A_{Tru}(u_j), A_{Ind}(u_j), A_{Fal}(u_j)\}\ (j = 1, 2, \ldots, n)$ be a collection of NCNs with the weight vector $W = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then an NCNIGWGHM operator of dimension n is a mapping NCNIGWGHM : $\Psi^n \rightarrow \Psi$, and has

$$
\begin{aligned}
&\mathrm{NCNIGWGHM}^{x,y}(Q_1, Q_2, \ldots, Q_n) \\
&= \left(\frac{1}{x + y} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \left(xQ_i \otimes yQ_j \right) \right)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}}
\end{aligned} \tag{23}
$$

Theorem 12. Let $x, y \geqslant 0$, and $Q_j = (\tilde{R}_{a_j}, S_{b_j})$ where $\tilde{R}_{aj} = \{\tilde{A}_{Tru}(u_j), \tilde{A}_{Ind}(u_j), \breve{A}_{Fal}(u_j)\}$ and $S_b = \{A_{Tru}(u_j), A_{Ind}(u_j), A_{Fal}(u_j)\}\ (j = 1, 2, \ldots, n)$ be a collection of NCNs with the weight vector $W = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then the aggregated value by equation (23) can be expressed as

$$\mathrm{NCNIGWGHM}^{x,y}(Q_1,Q_2,\ldots,Q_n) = \left(\begin{array}{c} 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\begin{array}{c}\left(1-\left(1-\tilde{A}^{x}_{Tru}(u_i)\right),\left(1-\tilde{A}^{y}_{Tru}(u_j)\right)\right),\\ \left(1-\left(1-A^{x}_{Tru}(u_i)\right),\left(1-A^{y}_{Tru}(u_j)\right)\right)\end{array}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{x+y}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^{x}_{Ind}(u_i)\tilde{A}^{y}_{Ind}(u_j),1-A^{x}_{Ind}(u_i)A^{y}_{Ind}(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{x+y}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^{x}_{Fal}(u_i)\tilde{A}^{y}_{Fal}(u_j),1-A^{x}_{Fal}(u_i)A^{y}_{Fal}(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{x+y}} \end{array} \right) \tag{24}$$

Similar, the proofs of Theorem 8 and Theorem 12 are omitted.

Moreover, similar to the proofs of Theorems 9–11, it is easy to prove that the NCNIGWGHM operator also has the following properties.

Theorem 13 (reducibility). Let $W=(1/n,1/n,1/n,\ldots,1/n)^T$ then

$$\begin{aligned}&\mathrm{NCNIGWGHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)\\&=\mathrm{NCNIGWGHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)\end{aligned} \tag{25}$$

Theorem 14 (idempotency). Let $Q_j=(\tilde{R}_{a_j},S_{b_j})\,(j=1,2,\ldots,n)\,(j=1,2,\ldots,n)$ then

$$\mathrm{NCNIGWGHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)=(\tilde{R}_{a_j},S_{b_j}) \tag{26}$$

Theorem 15 (monotonicity). Let $Q_j=(\tilde{R}_{a_j},S_{b_j})$ and $Q'_j=(\tilde{R}'_{a_j},S'_{b_j})$ $(j=1,2,\ldots,n)$ be two collections of NCNs. If $Q_j\geqslant Q'_j\forall j$ (suppose $\tilde{A}_{Tru}(u_j)\geqslant\tilde{A}'_{Tru}(u_j)$, $\tilde{A}_{Ind}(u_j)\geqslant\tilde{A}'_{Ind}(u_j)$, $\tilde{A}_{Fal}(u_j)\geqslant\tilde{A}'_{Fal}(u_j)$, $A_{Tru}(u_j)\geqslant A'_{Tru}(u_j)$, $A_{Ind}(u_j)\geqslant A'_{Ind}(u_j)$, $A_{Fal}(u_j)\geqslant A'_{Fal}(u_j)$), then

$$\begin{aligned}&\mathrm{NCNIGWGHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)\\&\geqslant\mathrm{NCNIGWGHM}^{x,y}(Q'_1,Q'_2,\ldots,Q'_n)\end{aligned} \tag{27}$$

Theorem 16 (boundedness). Let $Q_j=(\tilde{R}_{a_j},S_{b_j})\,(j=1,2,\ldots,n)$ be a collection of NCNs, and

$$Q_j^-=(\min\tilde{R}_{a_j},\min S_{b_j})$$

$$Q_j^+=(\max\tilde{R}_{a_j},\max S_{b_j})$$

or

$$Q_j^-=\left(\begin{array}{c}\min\tilde{A}_{Tru}(u_j),\min\tilde{A}_{Ind}(u_j),\min\tilde{A}_{Fal}(u_j),\\(\min A_{Tru}(u_j),\min A_{Ind}(u_j),\min A_{Fal}(u_j))\end{array}\right)$$

$$Q_j^+=\left(\begin{array}{c}\max\tilde{A}_{Tru}(u_j),\max\tilde{A}_{Ind}(u_j),\max\tilde{A}_{Fal}(u_j),\\(\max A_{Tru}(u_j),\max A_{Ind}(u_j),\max A_{Fal}(u_j))\end{array}\right)$$

then

$$Q_j^-\leqslant\mathrm{NCNIGWHM}^{x,y}(Q_1,Q_2,\ldots,Q_n)\leqslant Q_j^+ \tag{28}$$

Some special cases of the NCNIGWGHM with respect to parameters x and y are discussed as following:

1. When $x=0$, then

$$\mathrm{NCNIGWGHM}^{0,y}(Q_1,Q_2,\ldots,Q_n) = \left(\begin{array}{c} 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\begin{array}{c}1-\left(1-\tilde{A}_{Tru}(u_j)\right)^{y},\\ 1-\left(1-A^{y}_{Tru}(u_j)\right)^{y}\end{array}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{y}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^{y}_{Ind}(u_j),1-A^{y}_{Ind}(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{y}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^{y}_{Fal}(u_j),1-A^{y}_{Fal}(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{1}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{y}} \end{array} \right) \tag{29}$$

2. When $y=0$, then

$$\text{NCNIGWGHM}^{x,0}(Q_1,Q_2,\ldots,Q_n) = \begin{pmatrix} 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\begin{pmatrix}1-\left(1-\tilde{A}_{Tru}(u_i)\right)^x,\\ 1-\left(1-A^y_{Tru}(u_i)\right)^x\end{pmatrix}^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{x}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^x_{Ind}(u_i),1-A^x(u_i)\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{x}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}^x_{Fal}(u_i),1-A^x_{Fal}(u_i)\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{x}} \end{pmatrix} \tag{30}$$

Obviously, $y=0$, $\text{NCNIGWGHM}^{x,y}$ does not have any relationship with w. In addition, the parameters x and y do not have the interchangeability.

3. When $x=y=1$, then

$$\text{NCNIGWGHM}^{1,1}(Q_1,Q_2,\ldots,Q_n) = \begin{pmatrix} 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\begin{pmatrix}1-\left(1-\tilde{A}_{Tru}(u_i)\right)\left(1-\tilde{A}_{Tru}(u_j)\right),\\ 1-(1-A_{Tru}(u_i))(1-A_{Tru}(u_i))\end{pmatrix}^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{2}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}_{Ind}(u_i)\tilde{A}_{Ind}(u_j),1-A(u_i)A(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{2}} \\ \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\tilde{A}_{Fal}(u_i)\tilde{A}_{Fal}(u_j),1-A_{Fal}(u_i)A_{Fal}(u_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{2}} \end{pmatrix} \tag{31}$$

The approach to multiple attribute group decision-making with NCNs

In this section, we shall introduce the approach to multiple attribute group decision-making with the help of the NCNs. We apply NCN-improved generalized weighted Heronian mean operator to deal with the attribute group decision-making problems under the NCNs environment with an illustrated example.

Applications in multiple attribute group decision-making problem

In the problem of multiple attribute group decision-making, the developed procedure can easily be used for the better decision.

Suppose $H=\{H_1,H_2,\ldots,H_m\}$ is a set of alternatives, $G_j=\{G_1,G_2,\ldots,G_n\}$ is a set of attributes or criteria, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ is the weighted vector of the criteria, where, $\omega_i\varepsilon[0,1]$ and $\sum\omega_i=1$. Then, the evaluation value of an attribute $G_j\,(j=1,2,\ldots,n)$ with respect to alternatives $H_i\,(i=1,2,\ldots,m)$ is expressed by an NCN $q_{i_j}=((\tilde{q}_{Truij},\tilde{q}_{Indij},\tilde{q}_{Falij}),\ (q_{Truij},q_{Indij},q_{Falij}))$ $(j=1,2,\ldots,n;\ i=1,2,\ldots,m)$ where $\tilde{q}_{Truij},\tilde{q}_{Indij},\tilde{q}_{Falij}\subseteq[0,1]$ and $q_{Truij},q_{Indij},q_{Falij}\in[0,1]$. So, the decision matrix is obtained as: $D=(q_{ij})_{m\times n}$. The steps of the decision-making based on NCNs are given as follows:

Algorithm

Step 1. The DMs take their analysis of each alternatives based on each criteria. The performance of each alternatives H_i with respect to each crteria G_j.

Step 2. Calculate the NCNIGWHM operator $(h^k_{i1},h^k_{i2},\ldots,h^k_{in})$ to obtain the collective evaluation value of alternatives H_i with respect to each criteria G_j.

Step 3. Calculate the cosine similarity using Definition 10 in article.[24]

Step 4. Rank all the alternatives.

Step 5. End.

Numerical example

This section introduces an illustrative example to show the application of the above MAGDM method based on NCN. An investment company intends to choose one product to invest its money from four alternatives $H_i(i=1,2,3,4)$. Where H_1 = medicine company, H_2 = textile company, H_3 = mobile company, and H_4 = car company. The weights of the indicators are $w=(0.5,0.3,0.1,0.1)$. Three criteria have been evaluated and they are shown as follows: G_1 = Tax Rate, G_2 = Demand/Supply and G_3 = Wages. In order to get a most suitable choice we will use the above-mentioned algorithm as follows:

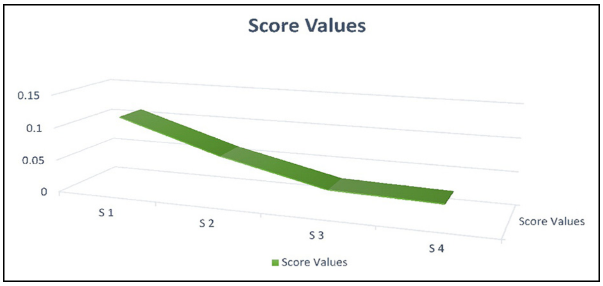

Figure 2. Line chart of alternatives versus score values of alternatives.

Step 1. Let $H = \{H_1, H_2, H_3, H_4\}$ be a set of alternatives and $G = \{G_1, G_2, G_3\}$ be the set of criteria. Let D be set of decision matrix. The decision matrix evaluates each alternative based on given criteria.

$$D = \begin{array}{cccc} & G_1 & G_2 & G_3 \\ H_1 & \left\{\begin{array}{c}([0.45, 0.53], \\ [0.16, 0.25], \\ [0.36, 0.64], \\ [(0.65, 0.37, \\ 0.74))\end{array}\right\} & \left\{\begin{array}{c}([0.34, 0.56], \\ [0.65, 0.73], \\ [0.46, 0.67], \\ (0.66, 0.85, \\ 0.76))\end{array}\right\} & \left\{\begin{array}{c}([0.25, 0.56], \\ [0.46, 0.75], \\ [0.74, 0.85], \\ (0.65, 0.84, \\ 0.95))\end{array}\right\} \\ H_2 & \left\{\begin{array}{c}([0.45, 0.74], \\ [0.76, 0.85], \\ [0.46, 0.84], \\ (0.86, 0.95, \\ 0.96))\end{array}\right\} & \left\{\begin{array}{c}([0.46, 0.74], \\ [0.75, 0.87], \\ [0.16, 0.57], \\ (0.86, 0.95, \\ 0.75))\end{array}\right\} & \left\{\begin{array}{c}([0.17, 0.45], \\ [0.19, 0.76], \\ [0.74, 0.93], \\ (0.54, 0.85, \\ 0.96))\end{array}\right\} \\ H_3 & \left\{\begin{array}{c}([0.25, 0.73], \\ [0.56, 0.77], \\ [0.18, 0.86], \\ (0.86, 0.85, \\ 0.94))\end{array}\right\} & \left\{\begin{array}{c}([0.53, 0.54], \\ [0.45, 0.63], \\ [0.35, 0.84], \\ (0.65, 0.76, \\ 0.94))\end{array}\right\} & \left\{\begin{array}{c}([0.16, 0.54], \\ [0.43, 0.94], \\ [0.24, 0.85], \\ (0.65, 0.96, \\ 0.93))\end{array}\right\} \\ H_4 & \left\{\begin{array}{c}([0.44, 0.93], \\ [0.36, 0.74], \\ [0.46, 0.94], \\ (0.76, 0.85, \\ 0.96))\end{array}\right\} & \left\{\begin{array}{c}([0.14, 0.36], \\ [0.25, 0.73], \\ [0.76, 0.78], \\ (0.45, 0.84, \\ 0.95))\end{array}\right\} & \left\{\begin{array}{c}([0.24, 0.63], \\ [0.26, 0.76], \\ [0.17, 0.85], \\ (0.76, 0.85, \\ 0.94))\end{array}\right\} \end{array}$$

Step 2. Calculate the NCNIGWHM operator by formula (15) to obtain the collective evaluation value $(h^k_{i1}, h^k_{i2}, \ldots, h^k_{in})$ of alternatives H_i with respect to each criterion G_j and $w = (0.5, 0.3, 0.1, 0.1)$, we can get

$$h^1_1([0.01, 0.04], [0.01, 0.04], [0.03, 0.1], (1.9, 1.9, 1.9))$$
$$h^1_2([0.009, 0.06], [0.03, 0.14], [0.014, 0.11], (1.9, 1.9, 1.9))$$
$$h^1_3([0.005, 0.053], [0.03, 0.12], [0.0012, 0.15], (1.9, 1.9, 1.9))$$
$$h^1_4([0.004, 0.02], [0.006, 0.03], [0.02, 0.05], (1.9, 1.9, 1.9))$$

Step 3. To calculate the cosine similarity using Definition 10, we get

$$S(h_1) = 0.11305, S(h_2) = 0.06520$$

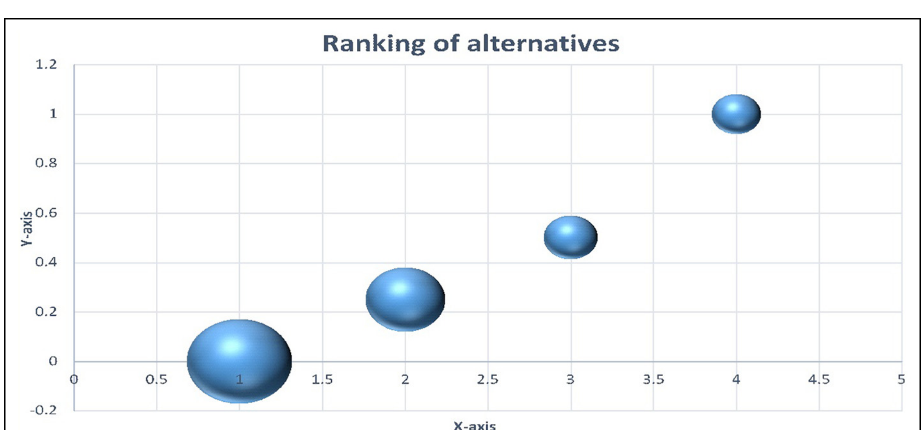

Figure 3. Graphical representation of the ranking values of alternatives.

$$S(h_3) = 0.02935, S(h_4) = 0.02535$$

shown in Figure 2.

Step 4. Rank all the alternatives, we get the sequence of candidates as follows: $h_1 \succ h_2 \succ h_3 \succ h_4$ shown in Figure 3.

Step 5. End.

Conclusion

In this article, we have discussed the idea of NCNs and different operators such as HM, GHM, weighted Heronian mean, generalized Heronian mean, and generalized weighted geometric mean operators. We applied HM to the NCSs. The NCS can be defined as the three elements such as truth, indeterminate, and incomplete information. The Heronian mean can represent the relationship of the aggregated values and MADM method. Finally, a numerical example is given to verify the proposed method.

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