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Neutrosophic Hesitant Fuzzy Subalgebras and Filters in Pseudo-BCI Algebras

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Abstract: The notions of the neutrosophic hesitant fuzzy subalgebra and neutrosophic hesitant fuzzy filter in pseudo-BCI algebras are introduced, and some properties and equivalent conditions are investigated. The relationships between neutrosophic hesitant fuzzy subalgebras (filters) and hesitant fuzzy subalgebras (filters) is discussed. Five kinds of special sets are constructed by a neutrosophic hesitant fuzzy set, and the conditions for the two kinds of sets to be filters are given. Moreover, the conditions for two kinds of special neutrosophic hesitant fuzzy sets to be neutrosophic hesitant fuzzy filters are proved.

Keywords: pseudo-BCI algebra; hesitant fuzzy set; neutrosophic set; filter

1. Introduction

G. Georgescu and A. Iogulescu presented pseudo-BCKalgebras, which was an extension of the famous BCK algebra theory. In [1], the notion of the pseudo-BCI algebra was introduced by W.A. Dudek and Y.B. Jun. They investigated some properties of pseudo-BCI algebras. In [2], Y.B. Jun et al. presented the concept of the pseudo-BCI ideal in pseudo-BCI algebras and researched its characterizations. Then, some classes of pseudo-BCI algebras and pseudo-ideals (filters) were studied; see [3–14].

In 1965, Zadeh introduced fuzzy set theory [15]. In the study of modern fuzzy logic theory, algebraic systems played an important role, such as [16–22]. In 2010, Torra introduced hesitant fuzzy set theory [23]. The hesitant fuzzy set was a useful tool to express peoples' hesitancy in real life, and uncertainty problems were resolved. Furthermore, hesitant fuzzy sets have been applied to decision making and algebraic systems [24–31]. As a generalization of fuzzy set theory, Smarandache introduced neutrosophic set theory [32]; the neutrosophic set theory is a useful tool to deal with indeterminate and inconsistent decision information [33,34]. The neutrosophic set includes the truth membership, indeterminacy membership and falsity membership. Then, Wang et al. [35,36] introduced the interval neutrosophic set and single-valued neutrosophic set. Ye [37] introduced the single-valued neutrosophic hesitant fuzzy set as an extension of the single-valued neutrosophic set and hesitant fuzzy set. Recently, the neutrosophic triplet structures were introduced and researched [38–40].

In this paper, some preliminary concepts in pseudo-BCI algebras, hesitant fuzzy set theory and neutrosophic set theory are briefly reviewed in Section 2. In Section 3, the notion of neutrosophic hesitant fuzzy subalgebras in pseudo-BCI algebras is introduced. The relationships between neutrosophic hesitant fuzzy subalgebras and hesitant fuzzy subalgebras are investigated. Five kinds of special sets are constructed. Some properties are studied. Third, the two kinds of sets to be filters are given. In Section 4, the concept of neutrosophic hesitant fuzzy filters in pseudo-BCI algebras is proposed. The equivalent conditions of the neutrosophic hesitant fuzzy filters in the construction of hesitant fuzzy filters are given. The conditions for two kinds of special neutrosophic hesitant fuzzy sets to be neutrosophic hesitant fuzzy filters are given.

2. Preliminaries

Let us review some fundamental notions of pseudo-BCI algebra and interval-valued hesitant fuzzy filter in this section.

Definition 1. *([13]) A pseudo-BCI algebra is a structure $(X; \rightarrow, \hookrightarrow, 1)$, where "$\rightarrow$" and "$\hookrightarrow$" are binary operations on X and "1" is an element of X, verifying the axioms: $\forall x, y, z \in X$,*

(1) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \hookrightarrow (y \rightarrow x)) = 1$, $(y \hookrightarrow z) \hookrightarrow ((z \hookrightarrow x) \rightarrow (y \hookrightarrow x)) = 1$;
(2) $x \rightarrow ((x \rightarrow y) \hookrightarrow y) = 1$, $x \hookrightarrow ((x \hookrightarrow y) \rightarrow y) = 1$;
(3) $x \rightarrow x = 1$;
(4) $x \rightarrow y = y \rightarrow x = 1 \Longrightarrow x = y$;
(5) $x \rightarrow y = 1 \Longleftrightarrow x \hookrightarrow y = 1$.

If $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra satisfying $\forall x, y \in X$, $x \rightarrow y = x \hookrightarrow y$, then $(X; \rightarrow, 1)$ is a BCI algebra. If $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra satisfying $\forall x \in X$, $x \rightarrow 1 = 1$, then $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCK algebra.

Remark 1. *([1]) In any pseudo-BCI algebra $(X; \rightarrow, \hookrightarrow)$, we can define a binary relation '$\leq$' by putting:*

$$x \leq y \text{ if and only if } x \rightarrow y \text{ (or } x \hookrightarrow y).$$

Proposition 1. *([13]) Let $(X; \rightarrow, \hookrightarrow)$ be a pseudo-BCI algebra, then X satisfies the following properties, $\forall x, y, z \in X$,*

(1) $1 \leq x \Rightarrow x = 1$;
(2) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z, y \hookrightarrow z \leq x \hookrightarrow z$;
(3) $x \leq y, y \leq z \Rightarrow x \leq z$;
(4) $x \hookrightarrow (y \rightarrow z) = y \rightarrow (x \hookrightarrow z)$;
(5) $x \leq y \rightarrow z \Rightarrow y \leq x \hookrightarrow z$;
(6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), x \hookrightarrow y \leq (z \hookrightarrow x) \hookrightarrow (z \hookrightarrow y)$;
(7) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, z \hookrightarrow x \leq z \hookrightarrow y$;
(8) $1 \rightarrow x = x, 1 \hookrightarrow x = x$;
(9) $((y \rightarrow x) \hookrightarrow x) \rightarrow x = y \rightarrow x, ((y \hookrightarrow x) \rightarrow x) \hookrightarrow x = y \hookrightarrow x$;
(10) $x \rightarrow y \leq (y \rightarrow x) \hookrightarrow 1, x \hookrightarrow y \leq (y \hookrightarrow x) \rightarrow 1$;
(11) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \hookrightarrow (y \hookrightarrow 1), (x \hookrightarrow y) \hookrightarrow 1 = (x \hookrightarrow 1) \rightarrow (y \rightarrow 1)$;
(12) $x \rightarrow 1 = x \hookrightarrow 1$.

Definition 2. *([13]) A subset F of a pseudo-BCI algebra X is called a filter of X if it satisfies:*
(F1) $1 \in F$;
(F2) $x \in F, x \rightarrow y \in F \Rightarrow y \in F$;
(F3) $x \in F, x \hookrightarrow y \in F \Rightarrow y \in F$.

Definition 3. *([1]) By a pseudo-BCI subalgebra of a pseudo-BCI algebra X, we mean a subset S of X that satisfies $\forall x, y \in S$, $x \rightarrow y \in S, x \hookrightarrow y \in S$.*

Definition 4. *([12]) A pseudo-BCK algebra is called a type-2 positive implicative if it satisfies:*

$$x \rightarrow (y \hookrightarrow z) = (x \rightarrow y) \hookrightarrow (x \rightarrow z),$$
$$x \hookrightarrow (y \rightarrow z) = (x \hookrightarrow y) \rightarrow (x \hookrightarrow z).$$

If X is a type-2 positive implicative pseudo-BCK algebra, then $x \rightarrow y = x \hookrightarrow y$ for all $x \in X$.

Definition 5. *([23]) Let X be a reference set. A hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that returns a subset of $[0,1]$ when it is applied to X, i.e.,*

$$A = \{(x, h_A(x)) | x \in X\}.$$

where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$. $h_A(x)$ is called a hesitant fuzzy element, a basis unit of the hesitant fuzzy set.

Example 1. *Let $X = \{a, b, c\}$ be a reference set, $h_A(a) = [0.1, 0.2]$, $h_A(b) = [0.3, 0.6]$, $h_A(c) = [0.7, 0.8]$. Then, A is considered as a hesitant fuzzy set,*

$$A = \{(a, [0.1, 0.2]), (b, [0.3, 0.6]), (c, [0.7, 0.8])\}.$$

Definition 6. *([13]) A fuzzy set $\mu : X \rightarrow [0,1]$ is called a fuzzy pseudo-filter (fuzzy filter) of a pseudo-BCI algebra X if it satisfies:*
(FF1) $\mu(1) \geq \mu(x), \forall x \in X$;
(FF2) $\mu(y) \geq \mu(x \rightarrow y) \wedge \mu(x), \forall x, y \in X$;
(FF3) $\mu(y) \geq \mu(x \hookrightarrow y) \wedge \mu(x), \forall x, y \in X$.

Definition 7. *([32]) Let X be a non-empty fixed set, a neutrosophic set A on X is defined as:*

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0,1]$, denoting the truth, indeterminacy and falsity membership degree of the element $x \in X$, respecting, and satisfying the limit: $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 8. *([34]) Let X be a fixed set; a neutrosophic hesitant fuzzy set N on X is defined as*

$$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\},$$

in which $\tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x) \in P([0,1])$, denoting the possible truth membership hesitant degrees, indeterminacy membership hesitant degrees and falsity membership hesitant degrees of $x \in X$ to the set N, respectively, with the conditions $0 \leq \delta, \gamma, \eta \leq 1$ and $0 \leq \delta^+ + \gamma^+ + \eta^+ \leq 3$, where $\gamma \in \tilde{t}_N(x)$, $\delta \in \tilde{i}_N(x)$, $\eta \in \tilde{f}_N(x)$, $\gamma^+ \in \bigcup_{\gamma \in \tilde{t}_N(x)} max\{\gamma\}$, $\delta^+ \in \bigcup_{\delta \in \tilde{i}_N(x)} max\{\delta\}$, $\eta^+ \in \bigcup_{\eta \in \tilde{f}_N(x)} max\{\eta\}$ for $x \in X$.

Example 2. *Let $X = \{a, b, c\}$ be a reference set, $h_A(a) = ([0.4, 0.5], [0.1, 0.2], [0.2, 0.4])$, $h_A(b) = ([0.5, 0.6], \{0.2, 0.3\}, [0.3, 0.4])$, $h_A(c) = ([0.5, 0.8], [0.2, 0.4], \{0.3, 0.5\})$. Then, A is considered as a neutrosophic hesitant fuzzy set,*

$$A = \{(a, [0.4, 0.5], [0.1, 0.2], [0.2, 0.4]), (b, [0.5, 0.6], \{0.2, 0.3\}, [0.3, 0.4]), (c, [0.5, 0.8], [0.2, 0.4], \{0.3, 0.5\})\}.$$

Conveniently, $N(x) = \{\tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)\}$ is called a neutrosophic hesitant fuzzy element, which is denoted by the simplified symbol $N(x) = \{\tilde{t}_N, \tilde{i}_N, \tilde{f}_N\}$.

Definition 9. *([34]) Let $N_1 = \{\tilde{t}_{N_1}, \tilde{i}_{N_1}, \tilde{f}_{N_1}\}$ and $N_2 = \{\tilde{t}_{N_2}, \tilde{i}_{N_2}, \tilde{f}_{N_2}\}$ be two neutrosophic hesitant fuzzy sets, then:*

$$N_1 \cup N_2 = \{\tilde{t}_{N_1} \cup \tilde{t}_{N_2}, \tilde{i}_{N_1} \cap \tilde{i}_{N_2}, \tilde{f}_{N_1} \cap f_{N_2}\};$$
$$N_1 \cap N_2 = \{\tilde{t}_{N_1} \cap \tilde{t}_{N_2}, \tilde{i}_{N_1} \cup \tilde{i}_{N_2}, \tilde{f}_{N_1} \cup f_{N_2}\}.$$

3. Neutrosophic Hesitant Fuzzy Subalgebras of Pseudo-BCI Algebras

In the following, let X be a pseudo-BCI algebra, unless otherwise specified.

Definition 10. *A hesitant fuzzy set $A = \{(x, h_A(x)) | x \in X\}$ is called a hesitant fuzzy pseudo-subalgebra (hesitant fuzzy subalgebra) of X if it satisfies:*
(HFS2) $h_A(x) \cap h_A(y) \subseteq h_A(x \rightarrow y)$, $\forall x, y \in X$;
(HFS3) $h_A(x) \cap h_A(y) \subseteq h_A(x \hookrightarrow y)$, $\forall x, y \in X$.

Definition 11. *A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is called a neutrosophic hesitant fuzzy pseudo-subalgebra (neutrosophic hesitant fuzzy subalgebra) of X if it satisfies:*
(1) $\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \rightarrow y)$, $\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow y)$, $\forall x, y \in X$;
(2) $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \rightarrow y)$, $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow y)$, $\forall x, y \in X$;
(3) $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \rightarrow y)$, $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \hookrightarrow y)$, $\forall x, y \in X$.

Example 3. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2.*

Table 1. $\rightarrow$.

$\rightarrow$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	d	c	1	1	1
d	c	c	c	1	1
1	a	b	c	d	1

Table 2. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	c	b	c	1	1
1	a	b	c	d	1

Then, $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra. Let:

$$N = \{(1, [0,1], \{0, \tfrac{1}{16}\}, [0, \tfrac{1}{6}]), (a, [\tfrac{1}{3}, \tfrac{1}{4}], [0, \tfrac{1}{2}], [0, \tfrac{5}{6}]), (b, [0, \tfrac{1}{2}], [0, \tfrac{2}{3}], [0, \tfrac{2}{3}]), \\ (c, [\tfrac{1}{3}, \tfrac{2}{3}], [0, \tfrac{1}{6}], [0, \tfrac{1}{5}]), (d, [\tfrac{1}{3}, 1], [0, \tfrac{1}{3}], [0, \tfrac{1}{5}])\}.$$

then, N is a neutrosophic hesitant fuzzy subalgebra of X.

Considering three hesitant fuzzy sets $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ by:

$$H_{\tilde{t}_N} = \{(x, \tilde{t}_N(x)) | x \in X\}, H_{\tilde{i}_N} = \{(x, 1 - \tilde{i}_N(x)) | x \in X\}, H_{\tilde{f}_N} = \{(x, 1 - \tilde{f}_N(x)) | x \in X\}.$$

Therefore, $H_{\tilde{t}_N}$ is called a generated hesitant fuzzy set by function $\tilde{t}_N(x)$; $H_{\tilde{i}_N}$ is called a generated hesitant fuzzy set by function $\tilde{i}_N(x)$; $H_{\tilde{f}_N}$ is called a generated hesitant fuzzy set by function $\tilde{f}_N(x)$.

Theorem 1. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(y), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, N is a neutrosophic hesitant fuzzy subalgebra of X if and only if it satisfies the conditions: $\forall x \in X$, $H_{\tilde{t}_N}$ and $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ are hesitant fuzzy subalgebras of X.*

Proof. **Necessity:** (i) By Definition 10 and Definition 11, we can obtain that $H_{\tilde{t}_N}$ is a hesitant fuzzy subalgebra of X.

(ii) $\forall x, y \in X$, $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \to y), (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \hookrightarrow y)$.

Similarly, $(1 - \tilde{f}_N(x)) \cap (1 - \tilde{f}_N(y)) \subseteq 1 - \tilde{f}_N(x \to y), (1 - \tilde{f}_N(x)) \cap (1 - \tilde{f}_N(y)) \subseteq 1 - \tilde{f}_N(x \to y)$. Therefore, $\forall x \in X$, $H_{\tilde{i}_N} = \{(x, 1 - \tilde{i}(x)) | x \in X\}$ and $H_{\tilde{f}_N} = \{(x, 1 - \tilde{f}_N(x)) | x \in X\}$ are hesitant fuzzy subalgebras of X.

Sufficiency: (i) Let $x, y \in H_{\tilde{t}_N}$. Obviously, $\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to y), \tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow y)$.

(ii) Let $x, y \in H_{\tilde{i}_N}$. By Definition 10, we have $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \to y), (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \to y)$, thus $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \to y), \tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow y)$.

Similarly, Let $x, y \in H_{\tilde{f}_N}$; we have $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \to y), \tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}(x \hookrightarrow y)$.

That is, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Theorem 2. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ be a neutrosophic hesitant fuzzy set on X. Then, the following conditions are equivalent:*

(1) $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy subalgebra of X;

(2) $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$, the nonempty hesitant fuzzy level sets $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are subalgebras of X, where $P([0,1])$ is the power set of $[0,1]$,

$$H_{\tilde{t}_N}(\lambda_1) = \{x \in X | \lambda_1 \subseteq \tilde{t}_N(x)\},$$
$$H_{\tilde{i}_N}(\lambda_2) = \{x \in X | \lambda_2 \subseteq 1 - \tilde{i}_N(x)\},$$
$$H_{\tilde{f}_N}(\lambda_3) = \{x \in X | \lambda_3 \subseteq 1 - \tilde{f}_N(x)\}.$$

Proof. (1)⇒(2) Suppose $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are nonempty sets. If $x, y \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x), \lambda_1 \subseteq \tilde{t}_N(y)$. Since N is a neutrosophic hesitant fuzzy subalgebra of X, by Definition 11, we can obtain:

$$\lambda_1 \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to y), \lambda_1 \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow y);$$

then $x \to y, x \hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$, $H_{\tilde{t}_N}(\lambda_1)$ is a subalgebra of X.

If $x, y \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1 - \tilde{i}_N(x), \lambda_2 \subseteq 1 - \tilde{i}_N(y)$. Since N is a neutrosophic hesitant fuzzy subalgebra of X, by Definition 11, we can obtain:

$$\lambda_2 \subseteq (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \to y),$$
$$\lambda_2 \subseteq (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) \subseteq 1 - \tilde{i}_N(x \hookrightarrow y);$$

Thus, $x \to y, x \hookrightarrow y \in H_{\tilde{i}_N}(\lambda_2)$, $H_{\tilde{i}_N}(\lambda_2)$ is a subalgebra of X.

Similarly, we can obtain then that $H_{\tilde{f}_N}(\lambda_3)$ is a subalgebra of X.

(2)⇒(1) Suppose that $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are nonempty subalgebras of X, $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$. Let $x, y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(y) = \mu_2$. Let $\mu_1 \cap \mu_2 = \lambda_1$. Therefore, we have $x, y \in H_X^{(1)}(\lambda_1)$. Since $H_X^{(1)}(\lambda_1)$ is a subalgebra, we can obtain $x \to y, x \hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$. Hence, we can obtain:

$$\tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to y), \tilde{t}_N(x) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow y);$$

Let $x, y \in X$ with $\tilde{i}(x) = \mu_3, \tilde{i}(y) = \mu_4$. Let $(1 - \mu_3) \cap (1 - \mu_4) = \lambda_2$. Then, we have $x, y \in H_{\tilde{i}_N}(\lambda_2)$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a subalgebra, we can obtain $x \to y, x \hookrightarrow y \in H_{\tilde{f}_N}(\lambda_2)$. Hence, we can obtain $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) = \lambda_2 \subseteq 1 - \tilde{i}_N(x \to y)$, $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(y)) = \lambda_2 \subseteq 1 - \tilde{i}_N(x \hookrightarrow y)$. Then, we have $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \to y)$, $\tilde{i}_N(x) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow y)$.

Similarly, let $x, y \in X$ with $\tilde{f}_N(x) = \mu_5, \tilde{f}_N(y) = \mu_6$; we can obtain $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \to y)$, $\tilde{f}_N(x) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \hookrightarrow y)$.

Thus, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Definition 12. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. $X_N^{(1)}(a^k, b)$, $X_N^{(2)}(a^k, b)$, $X_N^{(3)}(a^k, b)$, $X_N^{(4)}(a^k, b)$, $X_N^{(5)}(a)$ *are called generated subsets by* N: $\forall a, b \in X, k \in \mathbb{N}$,

$$
\begin{aligned}
X_N^{(1)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k * (b * x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k * (b * x)) = \tilde{i}_N(1), \tilde{f}_N(a^k * (b * x)) = \tilde{f}_N(1)\}; \\
X_N^{(2)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \to (b \hookrightarrow x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \to (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{f}_N(a^k \to (b \hookrightarrow x)) = \tilde{f}_N(1)\}; \\
X_N^{(3)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \hookrightarrow (b \to x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \hookrightarrow (b \to x)) = \tilde{t}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \to x)) = \tilde{f}_N(1)\}; \\
X_N^{(4)}(a^k, b) = \{x \in X | &\tilde{t}_N(a^k \to (b \to x)) = \tilde{t}_N(1), \\
&\tilde{i}_N(a^k \to (b \to x)) = \tilde{i}_N(1), \tilde{f}_N(a^k \to (b \to x)) = \tilde{f}_N(1), \\
&\quad \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{i}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{f}_N(1)\}; \\
X_N^{(5)}(a) = \{x \in X | &\tilde{t}_N(a) \subseteq \tilde{t}_N(x), \\
&\tilde{i}_N(a) \supseteq \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x)\}.
\end{aligned}
$$

where "a" appears "k" times, "$*$" represents any binary operation "$\to$" or "$\hookrightarrow$" on X,

$$
\begin{aligned}
a^k * (b * x) &= a * (a * (\cdots (a * (b * x)) \cdots)); \\
a^k \to (b \hookrightarrow x)) &= a \to (a \to (\cdots (a \to (b \hookrightarrow x)) \cdots)); \\
a^k \hookrightarrow (b \to x)) &= a \hookrightarrow (a \hookrightarrow (\cdots (a \hookrightarrow (b \to x)) \cdots)); \\
a^k \to (b \to x)) &= a \to (a \to (\cdots (a \to (b \to x)) \cdots)); \\
a^k \hookrightarrow (b \hookrightarrow x) &= a \hookrightarrow (a \hookrightarrow (\cdots (a \hookrightarrow (b \hookrightarrow x)) \cdots)).
\end{aligned}
$$

Theorem 3. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. *If* N *satisfies the following conditions:*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \hookrightarrow y) = \tilde{t}_N(x) \cup \tilde{t}_N(y), \forall x, y \in X$;
(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \hookrightarrow y) = \tilde{i}_N(x) \cap \tilde{i}_N(y), \forall x, y \in X$;
(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \hookrightarrow y) = \tilde{f}_N(x) \cap \tilde{f}_N(y), \forall x, y \in X$;

then $X_N^{(1)}(a^k, b) = X, k \in \mathbb{N}$.

Proof. By Proposition 1, we can obtain $\forall x \in X$,

$$
\begin{aligned}
&\tilde{t}_N(a^k * (b * x) = \tilde{t}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{t}_N(1) \cup \tilde{t}_N(a^k * (b * x))) = \tilde{t}_N(1). \\
&\tilde{i}_N(a^k * (b * x)) = \tilde{i}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{i}_N(1) \cap \tilde{t}_N(a^k * (b * x))) = \tilde{i}_N(1). \\
&\tilde{f}_N(a^k * (b * x)) = \tilde{f}_N(1 \hookrightarrow (a^k * (b * x))) \\
=&\tilde{f}_N(1) \cap \tilde{t}_N(a^k * (b * x))) = \tilde{f}_N(1).
\end{aligned}
$$

Thus, $x \in X_N^{(1)}(a^k, b)$, $X \subseteq X_N^{(1)}(a^k, b)$.

Conversely, it is easy to check that $X_N^{(1)}(a^k, b) \subseteq X$.

Finally, we can obtain $X = X_N^{(1)}(a^k, b)$. □

Corollary 1. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. *If* N *satisfies the following conditions:*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \to y) = \tilde{t}_N(x) \cup \tilde{t}_N(y), \forall x, y \in X$;

(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \to y) = \tilde{i}_N(x) \cap \tilde{i}_N(y), \forall x, y \in X$;
(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \to y) = \tilde{f}_N(x) \cap \tilde{f}_N(y), \forall x, y \in X$;
then $X_N^{(1)}(a^k, b) = X, k \in \mathbb{N}$.

Theorem 4. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. N *satisfies the following conditions:*
(1) $\tilde{t}_N(1) \supseteq \tilde{t}_N(x), \tilde{i}_N(1) \subseteq \tilde{i}_N(x), \tilde{f}_N(1) \subseteq \tilde{f}_N(x), \forall x \in X$;
(2) $x \hookrightarrow y = 1 \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y), \forall x, y \in X$.
If $\forall a, b, c \in X, k \in \mathbb{N}, b \leq c$, *then* $X_N^{(2)}(a^k, c) \subseteq X_N^{(2)}(a^k, b)$.

Proof: Let $x \in X_N^{(2)}(a^k, c)$. If $b \leq c$, by Proposition 1, we can obtain:

$$\begin{aligned}\tilde{t}_N(1) =& \tilde{t}_N(a^k \to (c \hookrightarrow x)) \\ =& \tilde{t}_N(c \hookrightarrow (a^k \to x)) \\ \subseteq& \tilde{t}_N(b \hookrightarrow (a^k \to x)) \\ =& \tilde{t}_N(a^k \to (b \hookrightarrow x)).\end{aligned}$$

Similarly, we can obtain:

$$\tilde{i}_N(a^k \to (b \hookrightarrow x)) \subseteq \tilde{i}_N(a^k \to (c \hookrightarrow x)) \subseteq \tilde{i}_N(1);$$
$$\tilde{f}_N(a^k \to (b \hookrightarrow x)) \subseteq \tilde{f}_N(a^k \to (c \hookrightarrow x)) \subseteq \tilde{f}_N(1).$$

That is, $x \in X_N^{(2)}(a^k, b)$, $X_N^{(2)}(a^k, c) \subseteq X_N^{(2)}(a^k, b)$.

Corollary 2. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on* X. N *satisfies the following conditions:*
(1) $\tilde{t}_N(1) \supseteq \tilde{t}_N(x), \tilde{i}_N(1) \subseteq \tilde{i}_N(x), \tilde{f}_N(1) \subseteq \tilde{f}_N(x), \forall x \in X$;
(2) $x \to y = 1 \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y), \forall x, y \in X$.
If $\forall a, b, c \in X, k \in \mathbb{N}, b \leq c$, *then* $X_N^{(3)}(a^k, c) \subseteq X_N^{(3)}(a^k, b)$.

The following example shows that $X_N^{(4)}(a^k, b)$ may not be a filter of X.

Example 4. *Let* $X = \{a, b, c, d, 1\}$ *with two binary operations in Tables 3 and 4.*

Table 3. $\to$.

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	d	1	1	1	1
c	d	c	1	1	1
d	c	c	c	1	1
1	a	b	c	d	1

Table 4. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	c	b	c	1	1
1	a	b	c	d	1

Then, $(X; \to, \hookrightarrow, 1)$ *is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0,1], [\tfrac{1}{6}, \tfrac{1}{5}], [0, \tfrac{1}{5}]), (a, [\tfrac{1}{3}, \tfrac{1}{4}], [0, \tfrac{5}{6}], [0, \tfrac{3}{4}]), (b, [0, \tfrac{1}{2}], [\tfrac{1}{6}, \tfrac{3}{4}], [0, \tfrac{1}{3}]), \\ (c, [\tfrac{1}{3}, \tfrac{2}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [\tfrac{1}{3}, 1], [\tfrac{1}{6}, \tfrac{1}{3}], [0, \tfrac{5}{6}])\}.$$

then $X_N^{(4)}(c,d) = \{a, c, d, 1\}$ *is not a filter of X. Since* $c \to b = c \in X_N^{(4)}(c,d)$*, but* $b \notin X_N^{(4)}(c,d)$*.*

Theorem 5. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on X. Let X be a type-2 positive implicative pseudo-BCK algebra. If functions* $\tilde{t}_N(x), \tilde{i}_N(x)$ *and* $\tilde{f}_N(x)$ *are injective, then* $X_N^{(4)}(a^k, b)$ *is a filter of X for all* $a, b \in X, k \in \mathbb{N}$*.*

Proof. (1) If X is a pseudo-BCK algebra, then by Definition 1 and Proposition 1, we can obtain $1 \in X_N^{(4)}(a^k, b)$.

(2) Let $x, y \in X$ with $x, x \to y \in X_N^{(4)}(a^k, b)$. Thus, $a^k \hookrightarrow (b \hookrightarrow x) = 1, a^k \hookrightarrow (b \hookrightarrow (x \to y)) = 1$. Since functions $\tilde{t}_N, \tilde{i}_N$ and $\tilde{f}_N$ are injective, by Definition 5, we have:

$$\begin{aligned}
\tilde{t}_N(1) &= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow x) \to (b \hookrightarrow y))) \\
&= \tilde{t}_N((a^k \hookrightarrow (b \hookrightarrow x)) \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(1 \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow y)).
\end{aligned}$$

Similarly, we can obtain $\tilde{i}_N(a^k \hookrightarrow ((b \hookrightarrow y)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow ((b \hookrightarrow y)) = \tilde{f}_N(1)$. Thus, we have $y \in X_N^{(4)}(a^k, b)$.

(3) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(4)}(a^k, b)$; we have $y \in X_N^{(4)}(a^k, b)$. □

This means that $X_N^{(4)}(a^k, b)$ is a filter of X for all $a, b \in X, k \in \mathbb{N}$.

Theorem 6. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X)\}$ *be a neutrosophic hesitant fuzzy set on X. Let X be a type-2 positive implicative pseudo-BCK algebra. If functions* $\tilde{t}_N(x), \tilde{i}_N(x)$ *and* $\tilde{f}_N(x)$ *satisfy the following identifies:* $\forall x, y \in X$*,*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{i}_N(x) \supseteq i_N(1), \tilde{f}_N(x) \supseteq f_N(1)$*;*

(2) $\tilde{t}_N(x \to y) = \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \to y) = \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \to y) = \tilde{f}_N(x) \cup \tilde{f}_N(y)$*;*

(3) $\tilde{t}_N(x \hookrightarrow y) = \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) = \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) = \tilde{f}_N(x) \cup \tilde{f}_N(y)$*;*

then $X_N^{(4)}(a^k, b)$ *is a filter of X for all* $a, b \in X, k \in \mathbb{N}$*.*

Proof. (1) If X is a pseudo-BCK algebra, by Definition 1 and Proposition 1, $1 \in X_N^{(4)}(a^k, b)$.

(2) Let $x, y \in X$ with $x, x \to y \in X_N^{(4)}(a^k, b)$. We have $\tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) = \tilde{t}_N(1), \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) = \tilde{t}_N(1)$. By Definition 5, we have:

$$\begin{aligned}
\tilde{t}_N(1) &= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow (x \to y))) \\
&= \tilde{t}_N(a^k \hookrightarrow ((b \hookrightarrow x) \to (b \hookrightarrow y))) \\
&= \tilde{t}_N((a^k \hookrightarrow (b \hookrightarrow x)) \to (a^k \hookrightarrow (b \hookrightarrow y))) \\
&= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow x)) \cap \tilde{t}(a^k \hookrightarrow (b \hookrightarrow y)) \\
&= \tilde{t}_N(1) \cap \tilde{t}(a^k \hookrightarrow (b \hookrightarrow y)) \\
&= \tilde{t}_N(a^k \hookrightarrow (b \hookrightarrow y)).
\end{aligned}$$

Similarly, we can obtain $\tilde{i}_N(a^k \hookrightarrow (b \hookrightarrow y)) = \tilde{i}_N(1), \tilde{f}_N(a^k \hookrightarrow (b \hookrightarrow y)) = \tilde{f}_N(1)$. Thus, we have $y \in X_N^{(4)}(a^k, b)$.

(3) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(4)}(a^k, b)$; we have $y \in X_N^{(4)}(a^k, b)$.

This means that $X_N^{(4)}(a^k,b)$ is a filter of X for all $a,b\in X, k\in\mathbb{N}$. □

Theorem 7. *Let $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x))|x\in X)\}$ be a neutrosophic hesitant fuzzy set on X and F be a filter of X. If functions $\tilde{t}_N(x),\tilde{i}_N(x)$ and $\tilde{f}_N(x)$ are injective, then $\bigcup X_N^{(4)}(a^k,b)=F$ for all $a,b\in F,k\in\mathbb{N}$.*

Proof. (1) Let $x\in\bigcup X_N^{(4)}(a^k,b)$. By Definition 12, we have $\tilde{t}_N(a\to(a^{k-1}\to(b\to x)))=\tilde{t}_N(1),\tilde{i}_N(a\to(a^{k-1}\to(b\to x)))=\tilde{i}_N(1),\tilde{f}_N(a\to(a^{k-1}\to(b\to x)))=\tilde{f}_N(1)$. Since F is a filter of X and $\tilde{t}_N,\tilde{i}_N,\tilde{f}_N$ are injective, thus we can obtain $a\to(a^{k-1}\to(b\to x))=1$ and $a^{k-1}\to(b\to x)\in F$. Continuing, we can obtain $b\to x\in F$. Since $b\in F$, thus $x\in F$, $\bigcup X_N^{(4)}(a^k,b)\subseteq F$.

(2) Let $x\in F$. When $a=1,b=x$, we can obtain $\tilde{t}_N(1^k\to(x\to x))=\tilde{t}_N(1^k\hookrightarrow(x\hookrightarrow x))=\tilde{t}_N(1)$. Similarly, we have $\tilde{i}_N(1^k\to(x\to x))=\tilde{i}_N(1^k\hookrightarrow(x\hookrightarrow x))=\tilde{i}_N(1),\tilde{f}_N(1^k\to(x\to x))=\tilde{f}_N(1^k\hookrightarrow(x\hookrightarrow x))=\tilde{f}_N(1)$. Thus, we have $F\subseteq\bigcup X_N^{(4)}(a^k,b)$.

This means that $\bigcup X_N^{(4)}(a^k,b)=F$ for all $a,b\in F,k\in\mathbb{N}$. □

Theorem 8. *Let $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x))|x\in X)\}$ be a neutrosophic hesitant fuzzy set on X.*

(1) If $X_N^{(5)}(a)$ is a filter of X, then N satisfies: $\forall x,y\in X$,

(i) $\tilde{t}_N(a)\subseteq\tilde{t}_N(x\to y)\cap\tilde{t}_N(x),\tilde{i}_N(a)\supseteq\tilde{i}_N(x\to y)\cup\tilde{i}_N(x),\tilde{f}_N(a)\supseteq\tilde{f}_N(x\to y)\cup\tilde{f}_N(x)\Rightarrow\tilde{t}_N(a)\subseteq\tilde{t}_N(y),\tilde{i}_N(a)\supseteq\tilde{i}_N(y),\tilde{f}_N(a)\supseteq\tilde{f}_N(y)$;

(ii) $\tilde{t}_N(a)\subseteq\tilde{t}_N(x\hookrightarrow y)\cap\tilde{t}_N(x),\tilde{i}_N(a)\supseteq\tilde{i}_N(x\hookrightarrow y)\cup\tilde{i}_N(x),\tilde{f}_N(a)\supseteq\tilde{f}_N(x\hookrightarrow y)\cup\tilde{f}_N(x)\Rightarrow\tilde{t}_N(a)\subseteq\tilde{t}_N(y),\tilde{i}_N(a)\supseteq\tilde{i}_N(y),\tilde{f}_N(a)\supseteq\tilde{f}_N(y)$.

(2) If N satisfies Conditions (i), (ii) and $\tilde{t}_N(x)\subseteq\tilde{t}_N(1),\tilde{i}_N(x)\supseteq\tilde{i}_N(1),\tilde{f}_N(x)\supseteq\tilde{f}_N(1)$ for all $x,y\in X$, then $X_N^{(5)}(a)$ is a filter of X.

Proof. (1) (i) Let $x,y\in X$ with $\tilde{t}_N(a)\subseteq\tilde{t}_N(x\to y)\cap\tilde{t}_N(x),\tilde{i}_N(a)\supseteq\tilde{i}_N(x\to y)\cup\tilde{i}_N(x),\tilde{f}_N(a)\supseteq\tilde{f}_N(x\to y)\cup\tilde{f}_N(x)$; we have $x\in X_N^{(5)}(a),x\to y\in X_N^{(5)}(a)$. Since $X_N^{(5)}(a)$ is a filter, thus we can have $y\in X_N^{(5)}(a)$, $\tilde{t}_N(a)\subseteq\tilde{t}_N(y),\tilde{i}_N(a)\supseteq\tilde{i}_N(y),\tilde{f}_N(a)\supseteq\tilde{f}_N(y)$.

(ii) Similarly, we know that (ii) is correct.

(2) Since $\tilde{t}_N(x)\subseteq\tilde{t}_N(1),\tilde{i}_N(x)\supseteq\tilde{i}_N(1),\tilde{f}_N(x)\supseteq\tilde{f}_N(1)$ for all $x\in X$, thus $1\in X_N^{(5)}(a)$. Let $x,y\in X$ with $x,x\to y\in X_N^{(5)}(a)$; we can obtain $\tilde{t}_N(a)\subseteq\tilde{t}_N(x),\tilde{t}_N(a)\subseteq\tilde{t}_N(x\to y),\tilde{i}_N(a)\supseteq\tilde{i}_N(x),\tilde{i}_N(a)\supseteq\tilde{i}_N(x\to y),\tilde{f}_N(a)\supseteq\tilde{f}_N(x),\tilde{f}_N(a)\supseteq\tilde{f}_N(x\to y)$. By Condition (i), we have $\tilde{t}_N(a)\subseteq\tilde{t}_N(y),\tilde{i}_N(a)\supseteq\tilde{i}_N(y),\tilde{f}_N(a)\supseteq\tilde{f}_N(y)$. Thus, we can obtain $y\in X_N^{(5)}(a)$. Similarly, let $x,y\in X$ with $x,x\hookrightarrow y\in X_N^{(5)}(a)$, by Condition (1)(ii); we can obtain $y\in X_N^{(5)}(a)$.

This means that $X_N^{(5)}(a)$ is a filter of X. □

4. Neutrosophic Hesitant Fuzzy Filters of Pseudo-BCI Algebras

In the following, let X be a pseudo-BCI algebra, unless otherwise specified.

Definition 13. *([22]) A hesitant fuzzy set $A=\{(x,h_A(x))|x\in X\}$ is called a hesitant fuzzy pseudo-filter (briefly, hesitant fuzzy filter) of X if it satisfies:*

(HFF1) $h_A(x)\subseteq h_A(1),\forall x\in X$;

(HFF2) $h_A(x)\cap h_A(x\to y)\subseteq h_A(y),\forall x,y\in X$;

(HFF3) $h_A(x)\cap h_A(x\hookrightarrow y)\subseteq h_A(y),\forall x,y\in X$.

Definition 14. *A neutrosophic hesitant fuzzy set $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x))|x\in X\}$ is called a neutrosophic hesitant fuzzy pseudo-filter (neutrosophic hesitant fuzzy filter) of X if it satisfies:*

(NHFF1) $\tilde{t}_N(x)\subseteq\tilde{t}_N(1),\tilde{i}_N(x)\supseteq\tilde{i}_N(1),\tilde{f}_N(x)\supseteq\tilde{f}_N(1),\forall x\in X$;

(NHFF2) $\tilde{t}_N(x\to y)\cap\tilde{t}_N(x)\subseteq\tilde{t}_N(y),\tilde{i}_N(x\to y)\cup\tilde{i}_N(x)\supseteq\tilde{i}_N(y),\tilde{f}_N(x\to y)\cup\tilde{f}_N(x)\supseteq\tilde{f}_N(y)$, $\forall x,y\in X$;

(NHFF3) $\tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) \cup \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) \cup \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$, $\forall x, y \in X$.

A neutrosophic hesitant fuzzy set $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X)\}$ is called a neutrosophic hesitant fuzzy closed filter of X if it is a neutrosophic hesitant fuzzy filter such that:

$$\tilde{t}_N(x \to 1) \supseteq \tilde{t}_N(x), \tilde{i}_N(x \to 1) \subseteq \tilde{i}_N(x), \tilde{f}_N(x \to 1) \subseteq \tilde{f}_N(x).$$

Example 5. *Let* $X = \{a, b, c, d, 1\}$ *with two binary operations in Tables 5 and 6. Then,* $(X; \to, \hookrightarrow, 1)$ *is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0,1], [0, \tfrac{3}{7}], [0, \tfrac{1}{10}]), (a, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (b, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (c, [0, \tfrac{1}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [0, \tfrac{3}{4}]), [0, \tfrac{3}{6}], [0, \tfrac{1}{5}])\}.$$

Then, N is a neutrosophic hesitant fuzzy filter of X.

Table 5. $\to$.

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	c	1	1	1	1
c	a	b	1	d	1
d	b	b	c	1	1
1	a	b	c	d	1

Table 6. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	1	1	1	1
b	d	1	1	1	1
c	b	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Theorem 9. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(y), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on X. Then, N is a neutrosophic hesitant fuzzy filter of X if and only if it satisfies the following conditions:* $\forall x \in X$, $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ *are hesitant fuzzy filters of X.*

Proof. Necessity: If N is a neutrosophic hesitant fuzzy filter:

(1) Obviously, $H_{\tilde{t}_N}$ is a hesitant fuzzy filter of X.

(2) By Definition 14, we have $(1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(1)), 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(x \to y)) = (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \to y)) \subseteq (1 - \tilde{i}_N(y))$; similarly, by Definition 14, we have $(1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \hookrightarrow y)) \subseteq (1 - \tilde{i}_N(y))$. Thus, $H_{\tilde{i}_N}$ is hesitant fuzzy filter of X.

(3) Similarly, we have that $H_{\tilde{f}_N}$ is a hesitant fuzzy filter of X.

Sufficiency: If $H_{\tilde{t}_N}$, $H_{\tilde{i}_N}$, $H_{\tilde{f}_N}$ are hesitant fuzzy filters of X. It is easy to prove that $\tilde{t}_N(x)$, $\tilde{i}_N(x)$, $\tilde{f}_N(x)$ satisfies Definition 14. Therefore, $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy filter of X. $\square$

Theorem 10. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy set on X. Then, the following are equivalent:*

(1) $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ *is a neutrosophic hesitant fuzzy filter of X;*

(2) $\forall \lambda_1, \lambda_2, \lambda_3 \in P([0,1])$, the nonempty hesitant fuzzy level sets $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are filters of X, where $P([0,1])$ is the power set of $[0,1]$,

$$\begin{aligned} H_{\tilde{t}_N}(\lambda_1) &= \{x \in X | \lambda_1 \subseteq \tilde{t}_N(x)\}; \\ H_{\tilde{i}_N}(\lambda_2) &= \{x \in X | \lambda_2 \subseteq 1 - \tilde{i}_N(x)\}; \\ H_{\tilde{f}_N}(\lambda_3) &= \{x \in X | \lambda_3 \subseteq 1 - \tilde{f}_N(x)\}. \end{aligned}$$

Proof. (1)$\Rightarrow$(2) (i) Suppose $H_{\tilde{t}_N}(\lambda_1) \neq \emptyset$. Let $x \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x)$. Since N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\lambda_1 \subseteq \tilde{t}_N(x) \subseteq \tilde{t}_N(1)$. Thus, $1 \in H_{\tilde{t}_N}(\lambda_1)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{t}_N}(\lambda_1)$, then $\lambda_1 \subseteq \tilde{t}_N(x), \lambda_1 \subseteq \tilde{t}_N(x \to y)$. Since N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\lambda_1 \subseteq \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$. Thus $y \in H_{\tilde{t}_N}(\lambda_1)$. Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in H_{\tilde{t}_N}(\lambda_1)$. We have $y \in H_{\tilde{t}_N}(\lambda_1)$.

Thus, we can obtain that $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X.

(ii) Suppose $H_{\tilde{i}_N}(\lambda_2) \neq \emptyset$. Let $x \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1 - \tilde{i}_N(x)$. Since N is a neutrosophic hesitant fuzzy filter of X, we have $\tilde{i}_N(1) \subseteq \tilde{i}_N(x)$. Thus, $\lambda_2 \subseteq 1 - \tilde{i}_N(x) \subseteq 1 - \tilde{i}_N(1), 1 \in H_{\tilde{i}_N}(\lambda_2)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{i}_N}(\lambda_2)$, then $\lambda_2 \subseteq 1 - \tilde{i}_N(x), \lambda_2 \subseteq 1 - \tilde{i}_N(x \to y)$. Since N is a neutrosophic hesitant fuzzy filter of X, we have $\tilde{i}_N(x \to y) \cup \tilde{i}_N(x) \supseteq \tilde{i}_N(y)$. Thus, $1 - (\tilde{i}_N(x \to y) \cup \tilde{i}_N(x)) = (1 - \tilde{i}_N(x \to y)) \cap (1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(y)), \lambda_2 \subseteq (1 - \tilde{i}_N(y)), y \in H_{\tilde{i}_N}(\lambda_2)$. Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in H_{\tilde{i}_N}(\lambda_2)$. We have $y \in H_{\tilde{i}_N}(\lambda_2)$.

Thus, we can obtain that $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X.

(iii) We have that $H_{\tilde{f}_N}(\lambda_3)$ is a filter of X. The progress of proof is similar to (ii).

(2)$\Rightarrow$(1) Suppose $H_{\tilde{t}_N}(\lambda_1) \neq \emptyset, H_{\tilde{i}_N}(\lambda_2) \neq \emptyset, H_{\tilde{f}_N}(\lambda_3) \neq \emptyset$ for all $\lambda_1, \lambda_2, \lambda_3 \in P([0,1])$.

(i′) Let $x \in X$ with $\tilde{t}_N(x) = \mu_1$. Let $\lambda_1 = \mu_1$. Since $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X, we have $1 \in H_{\tilde{t}_N}(\lambda_1)$. Thus, $\lambda_1 = \mu_1 = \tilde{t}_N(x) \subseteq \tilde{t}_N(1)$.

Let $x, y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(x \to y) = \mu_4$. Let $\mu_1 \cap \mu_4 = \lambda_1$. Since $H_{\tilde{t}_N}(\lambda_1)$ is a filter of X for all $\lambda_1 \in P([0,1])$, we have $y \in H_{\tilde{t}_N}(\lambda_1)$. Thus, $\lambda_1 = \tilde{t}_N(x) \cap \tilde{t}_N(x \to y) \subseteq \tilde{t}_N(y)$.

Similarly, let $x, y \in X$ with $\tilde{t}_N(x) = \mu_1, \tilde{t}_N(x \hookrightarrow y) = \mu_4'$. We can obtain $\tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$.

(ii′) Let $x \in X$ with $\tilde{i}_N(x) = \mu_2$. Let $\lambda_2 = 1 - \mu_2$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X for all $\lambda_2 \in P([0,1])$, we have $1 \in H_{\tilde{i}_N}(\lambda_2), \lambda_2 \subseteq 1 - \tilde{i}_N(1)$. Thus, $1 - \lambda_2 = \mu_2 = \tilde{i}_N(x) \supseteq \tilde{i}_N(1)$.

Let $x, y \in X$ with $\tilde{i}_N(x) = \mu_2, \tilde{i}_N(x \to y) = \mu_5$. Let $(1 - \mu_2) \cap (1 - \mu_5) = \lambda_2$. Since $H_{\tilde{i}_N}(\lambda_2)$ is a filter of X for all $\lambda_2 \in P([0,1])$, we have $y \in H_{\tilde{i}_N}(\lambda_2), \lambda_2 \subseteq 1 - \tilde{i}_N(y)$. Thus, $\lambda_2 = (1 - \mu_2) \cap (1 - \mu_5) = (1 - \tilde{i}_N(x)) \cap (1 - \tilde{i}_N(x \to y)) = 1 - (\tilde{i}_N(x) \cup \tilde{i}_N(x \to y)) \subseteq (1 - \tilde{i}_N(y)), \tilde{i}_N(x) \cup \tilde{i}_N(x \to y) \supseteq \tilde{i}_N(y)$.

Similarly, let $x, y \in X$ with $\tilde{i}_N(x) = \mu_2, \tilde{i}_N(x \hookrightarrow y) = \mu_5'$; we have $\tilde{i}_N(x) \cup \tilde{i}_N(x \hookrightarrow y) \supseteq \tilde{i}_N(y)$.

(iii′) Similarly, we can obtain $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x) \cup \tilde{f}_N(x \to y) \supseteq \tilde{f}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(x \hookrightarrow y) \supseteq \tilde{f}_N(y)$.

Therefore, $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy filter of X. □

Definition 15. *$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy set on X. Define a neutrosophic hesitant fuzzy set $N^* = \{(x, \tilde{t}^*_N(x), \tilde{i}^*_N(x), \tilde{f}^*_N(x)) | x \in X\}$ by:*

$$\tilde{t}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{t}_N(x), & x \in H_{\tilde{t}_N}(\lambda_1) \\ \varphi_1, & x \notin H_{\tilde{t}_N}(\lambda_1) \end{cases}$$

$$\tilde{i}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{i}_N(x), & x \in H_{\tilde{i}_N}(\lambda_2) \\ 1 - \varphi_2, & x \notin H_{\tilde{i}_N}(\lambda_2) \end{cases}$$

$$\tilde{f}^*_N : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \tilde{f}_N(x), & x \in H_{\tilde{f}_N}(\lambda_3) \\ 1 - \varphi_3, & x \notin H_{\tilde{f}_N}(\lambda_3) \end{cases}$$

where $\lambda_1, \lambda_2, \lambda_3, \varphi_1, \varphi_2, \varphi_3 \in P([0,1])$, $\varphi_1 \subseteq \lambda_1, \varphi_2 \subseteq \lambda_2, \varphi_3 \subseteq \lambda_3$. Then, N^* is called a generated neutrosophic hesitant fuzzy set by hesitant fuzzy level sets $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2)$ and $H_{\tilde{f}_N}(\lambda_3)$.

Theorem 11. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ be a neutrosophic hesitant fuzzy filter of X. Then, N^* is a neutrosophic hesitant fuzzy filter of X.*

Proof. (1) If N is a neutrosophic hesitant fuzzy filter of X, by Theorem 10, we know that $H_{\tilde{t}_N}(\lambda_1), H_{\tilde{i}_N}(\lambda_2), H_{\tilde{f}_N}(\lambda_3)$ are filters of X. Thus, $1 \in H_{\tilde{t}_N}(\lambda_1), 1 \in H_{\tilde{i}_N}(\lambda_2), 1 \in H_{\tilde{f}_N}(\lambda_3)$, $\tilde{t}^*_N(1) = \tilde{t}_N(1) \supseteq \tilde{t}^*_N(x)$, $\tilde{i}^*_N(1) = \tilde{i}_N(1) \subseteq \tilde{i}^*_N(x)$, $\tilde{f}^*_N(1) = \tilde{f}_N(1) \subseteq \tilde{f}^*_N(x), \forall x \in X$

(2) (i) Let $x, y \in X$ with $x, x \to y \in H_{\tilde{t}_N}(\lambda_1)$. By Theorem 9, Theorem 10 and Definition 15, we know $\lambda_1 \subseteq \tilde{t}^*_N(x \to y) \cap \tilde{t}^*_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y) = \tilde{t}^*_N(y)$.

Let $x, y \in X$ with $x, x \to y \in H_{\tilde{i}_N}(\lambda_2)$. By Theorem 9, Theorem 10 and Definition 15, we know $\lambda_2 \subseteq (1 - \tilde{i}^*_N(x \to y)) \cap (1 - \tilde{t}^*_N(x)) = (1 - \tilde{i}_N(x \to y)) \cap (1 - \tilde{t}_N(x)) = 1 - (\tilde{i}_N(x \to y) \cup \tilde{i}_N(x)) \subseteq 1 - \tilde{i}_N(y) = 1 - \tilde{t}^*_N(y)$. Thus, we have $1 - \lambda_2 \supseteq \tilde{i}^*_N(x \to y) \cup \tilde{i}^*_N(x) = \tilde{i}_N(x \to y) \cup \tilde{i}_N(x) \supseteq i_N(y) = \tilde{i}^*_N(y)$.

Similarly, let $x, y \in X$ with $x, x \to y \in H_{\tilde{f}_N}(\lambda_3)$; we have $1 - \lambda_3 \supseteq \tilde{f}^*_N(x \to y) \cup \tilde{f}^*_N(x) = \tilde{f}_N(x \to y) \cup \tilde{f}_N(x) \supseteq f_N(y) = \tilde{f}^*_N(y)$.

(ii) Let $x, y \in X$ with $x \notin H_{\tilde{t}_N}(\lambda_1)$ or $x \to y \notin H_{\tilde{t}_N}(\lambda_1)$. By Definition 15, we have $\tilde{t}^*_N(x) = \varphi_1$ or $\tilde{t}^*_N(x \to y) = \varphi_1$. Thus, we can obtain $\tilde{t}^*_N(x) \cap \tilde{t}^*_N(x \to y) = \varphi_1 \subseteq \tilde{t}^*_N(y)$.

Let $x, y \in X$ with $x \notin H_{\tilde{i}_N}(\lambda_2)$ or $x \to y \notin H_{\tilde{i}_N}(\lambda_2)$. By Definition 15, we have $\tilde{i}^*_N(x) = 1 - \varphi_2$ or $\tilde{i}^*_N(x \to y) = 1 - \varphi_2$. Since $1 - \lambda_2 \subseteq 1 - \varphi_2$; thus, we can obtain $\tilde{i}^*_N(x) \cup \tilde{i}^*_N(x \to y) = 1 - \varphi_2 \supseteq \tilde{t}^*_N(y)$.

Similarly, let $x, y \in X$ with $x \notin H_{\tilde{f}_N}(\lambda_3)$ or $x \to y \notin H_{\tilde{f}_N}(\lambda_3)$; we have $\tilde{f}^*(x) \cup \tilde{f}^*(x \to y) = 1 - \varphi_3 \supseteq \tilde{f}^*(y)$.

(3) We can obtain $\tilde{t}^*(x) \cap \tilde{t}^*(x \hookrightarrow y) \subseteq \tilde{t}^*(y)$, $\tilde{i}^*(x) \cup \tilde{i}^*(x \hookrightarrow y) \supseteq \tilde{i}^*(y)$, $\tilde{f}^*(x) \cup \tilde{f}^*(x \hookrightarrow y) \supseteq \tilde{f}^*(y)$. The process of proof is similar to (2).

Thus N^* is a neutrosophic hesitant fuzzy filter of X. □

Theorem 12. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X\}$ be a neutrosophic hesitant fuzzy filter of X. Then, N satisfies the following properties, $\forall x, y, z \in X$,*

(1) $x \leq y \Rightarrow \tilde{t}_N(x) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$;

(2) $\tilde{t}_N(x \to z) \supseteq \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y), \tilde{t}_N(x \hookrightarrow z) \supseteq \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y)$;
$\tilde{i}_N(x \to z) \subseteq \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y), \tilde{i}_N(x \hookrightarrow z) \subseteq \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y)$;
$\tilde{f}_N(x \to z) \subseteq \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y), \tilde{f}_N(x \hookrightarrow z) \subseteq \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y)$;

(3) $\tilde{t}_N((x \to y) \hookrightarrow y) \supseteq \tilde{t}_N(x), \tilde{t}_N((x \hookrightarrow y) \to y) \supseteq \tilde{t}_N(x)$;
$\tilde{i}_N((x \to y) \hookrightarrow y) \subseteq \tilde{i}_N(x), \tilde{i}_N((x \hookrightarrow y) \to y) \subseteq \tilde{i}_N(x)$;
$\tilde{f}_N((x \to y) \hookrightarrow y) \subseteq \tilde{f}_N(x), \tilde{f}_N((x \hookrightarrow y) \to y) \subseteq \tilde{f}_N(x)$;

(4) $z \leq x \to y \Rightarrow \tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$;
$z \leq x \hookrightarrow y \Rightarrow \tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y), \tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$.

Proof. (1) Let $x, y \in X$ with $x \leq y$. By Proposition 1, we know $x \to y = 1$ (or $x \hookrightarrow y = 1$). If N is a neutrosophic hesitant fuzzy filter of X, by Definition 14, we have $\tilde{t}_N(x) = \tilde{t}_N(1) \cap \tilde{t}_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$ ($\tilde{t}_N(x) = \tilde{t}_N(1) \cap \tilde{t}_N(x) = \tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x) \subseteq \tilde{t}_N(y)$). Thus, $\tilde{t}_N(x) \subseteq \tilde{t}_N(y)$.

Similarly, we have $\tilde{i}_N(x) \supseteq \tilde{i}_N(y), \tilde{f}_N(x) \supseteq \tilde{f}_N(y)$.

(2) By Proposition 1, Definition 14, we know, $\forall x, y, z \in X$,

$$\tilde{t}_N(x \to z) \supseteq \tilde{t}_N(y \hookrightarrow (x \to z)) \cap \tilde{t}_N(y) = \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y),$$
$$\tilde{t}_N(x \hookrightarrow z) \supseteq \tilde{t}_N(y \to (x \hookrightarrow z)) \cap \tilde{t}_N(y) = \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y).$$

Similarly, we have, $\forall x, y, z \in X$:

$$\tilde{i}_N(x \to z) \subseteq \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y), \tilde{i}_N(x \hookrightarrow z) \subseteq \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y);$$
$$\tilde{f}_N(x \to z) \subseteq \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y), \tilde{f}_N(x \hookrightarrow y) \subseteq \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y).$$

(3) By Definition 1 and Definition 14, with regard to the function $\tilde{t}_N(x)$, we can obtain, $\forall x, y \in X$,

$$\begin{aligned}\tilde{t}_N((x \to y) \hookrightarrow y) &\supseteq \tilde{t}_N(x \to ((x \to y) \hookrightarrow y)) \cap \tilde{t}_N(x)\\ &= \tilde{t}_N((x \to y) \hookrightarrow (x \to y)) \cap \tilde{t}_N(x)\\ &= \tilde{t}_N(1) \cap \tilde{t}_N(x)\\ &= \tilde{t}_N(x).\end{aligned}$$

Similarly, we have $\tilde{t}_N((x \hookrightarrow y) \to y) \supseteq \tilde{t}_N(x)$.

With regard to the function $\tilde{i}_N(x)$, we can obtain, $\forall x, y \in X$,

$$\begin{aligned}\tilde{i}_N((x \to y) \hookrightarrow y) &\subseteq \tilde{i}_N(x \to ((x \to y) \hookrightarrow y)) \cup \tilde{i}_N(x)\\ &= \tilde{i}_N((x \to y) \hookrightarrow (x \to y)) \cup \tilde{i}_N(x)\\ &= \tilde{i}_N(1) \cup \tilde{i}_N(x)\\ &= \tilde{i}_N(x).\end{aligned}$$

Similarly, we have $\tilde{i}_N((x \hookrightarrow y) \to y) \subseteq \tilde{i}_N(x)$.

Similarly, with regard to the function $\tilde{f}_N(x)$, we can obtain $\tilde{f}_N((x \to y) \hookrightarrow y) \subseteq \tilde{f}_N(x)$, $\tilde{f}_N((x \hookrightarrow y) \to y) \subseteq \tilde{f}_N(x)$.

(4) Let $x, y, z \in X$ with $z \le x \to y$. By Remark 1 and Definition 14, we can obtain:

$$\begin{aligned}\tilde{t}_N(x) \cap \tilde{t}_N(z) &= \tilde{t}_N(x) \cap (\tilde{t}_N(1) \cap \tilde{t}_N(z))\\ &= \tilde{t}_N(x) \cap (\tilde{t}_N(z \hookrightarrow (x \to y)) \cap \tilde{t}_N(z))\\ &\subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \to y),\\ &\subseteq \tilde{t}_N(y).\\ \tilde{i}_N(x) \cup \tilde{i}_N(z) &= \tilde{i}_N(x) \cup (\tilde{i}_N(1) \cup \tilde{i}_N(z))\\ &= \tilde{i}_N(x) \cup (\tilde{i}_N(z \to (x \to y)) \cup \tilde{i}_N(z))\\ &\supseteq \tilde{i}_N(x) \cup \tilde{i}_N(x \to y),\\ &\supseteq \tilde{i}_N(y).\end{aligned}$$

Similarly, we can obtain $\tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$.

Let $x, y, z \in X$ with $z \le x \hookrightarrow y$. We can obtain $\tilde{t}_N(x) \cap \tilde{t}_N(z) \subseteq \tilde{t}_N(y)$, $\tilde{i}_N(x) \cup \tilde{i}_N(z) \supseteq \tilde{i}_N(y)$, $\tilde{f}_N(x) \cup \tilde{f}_N(z) \supseteq \tilde{f}_N(y)$. The process of the proof is similar to the above. □

Theorem 13. *A neutrosophic hesitant fuzzy set* $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X)\}$ *is a neutrosophic hesitant fuzzy filter of X if and only if hesitant fuzzy sets* $H_{\tilde{t}_N}, H_{\tilde{i}_N}, H_{\tilde{f}_N}$ *satisfy the following conditions, respectively.*

(1) $\tilde{t}_N(x) \subseteq \tilde{t}_N(1), \tilde{t}_N(x \to (y \hookrightarrow z)) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \to z), \tilde{t}_N(x \hookrightarrow (y \to z)) \cap \tilde{t}_N(y) \subseteq \tilde{t}_N(x \hookrightarrow z), \forall x, y, z \in X$;

(2) $\tilde{i}_N(x) \supseteq \tilde{i}_N(1), \tilde{i}_N(x \to (y \hookrightarrow z)) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \to z), \tilde{i}_N(x \hookrightarrow (y \to z)) \cup \tilde{i}_N(y) \supseteq \tilde{i}_N(x \hookrightarrow z), \forall x, y, z \in X$;

(3) $\tilde{f}_N(x) \supseteq \tilde{f}_N(1), \tilde{f}_N(x \to (y \hookrightarrow z)) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \to z), \tilde{f}_N(x \hookrightarrow (y \to z)) \cup \tilde{f}_N(y) \supseteq \tilde{f}_N(x \hookrightarrow z), \forall x, y, z \in X$.

Proof. Necessity: By Theorem 9, Theorem 12 and Definition 14, (1)∼(3) holds.

Sufficiency: (1) $\forall x, y, z \in X$, by Proposition 1, we can obtain $\tilde{t}_N(y) = \tilde{t}_N(1 \to y) \supseteq \tilde{t}_N(1 \to (x \hookrightarrow y)) \cap \tilde{t}_N(x) = \tilde{t}_N(x \hookrightarrow y) \cap \tilde{t}_N(x)$ and $\tilde{t}_N(y) = \tilde{t}_N(1 \hookrightarrow y) \supseteq \tilde{t}_N(1 \hookrightarrow (x \to y)) \cap \tilde{t}_N(x) = \tilde{t}_N(x \to y) \cap \tilde{t}_N(x)$. We have $\tilde{i}_N(x) \supseteq \tilde{i}_N(1)$ for all $x \in X$. Thus, $H_{\tilde{t}_N}$ is a hesitant fuzzy filter of X.

(2) $\forall x, y, z \in X$, by Proposition 1, we can obtain $\tilde{i}_N(y) = \tilde{i}_N(1 \to y) \subseteq \tilde{i}_N(1 \to (x \hookrightarrow y)) \cup \tilde{i}_N(x) = \tilde{i}_N(x \hookrightarrow y) \cup \tilde{i}_N(x)$; thus, we have $(1 - \tilde{i}_N(x \hookrightarrow y)) \cap (1 - \tilde{i}_N(x)) \subseteq (1 - \tilde{i}_N(y))$.

Similarly, we can have $(1-\tilde{i}_N(x\to y))\cap(1-\tilde{i}_N(x))\subseteq(1-\tilde{i}_N(y))$.

It is easy to obtain $(1-\tilde{i}_N(x))\subseteq(1-\tilde{t}_N(1))$ for all $x\in X$. Thus, $H_{\tilde{i}_N}$ is a hesitant fuzzy filter of X.

(3) We have that $H_{\tilde{f}_N}$ is a hesitant fuzzy filter of X. The process of the proof is similar (2).

Therefore, $H_{\tilde{t}_N}, H_{\tilde{i}_N}, H_{\tilde{f}_N}$ are hesitant fuzzy filters of X. By Theorem 9, we know that N is a neutrosophic hesitant fuzzy filter of X. □

Theorem 14. *Let $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x))|x\in X)\}$ be a neutrosophic hesitant fuzzy filter of X. Then:*

$$\prod_{k=1}^{n} x_k\to y=1\Rightarrow \tilde{t}_N(y)\supseteq\bigcap_{k=1}^{n}\tilde{t}_N(x_k),\tilde{i}_N(y)\subseteq\bigcup_{i=k}^{n}\tilde{i}_N(x_k),\tilde{f}_N(y)\subseteq\bigcup_{k=1}^{n}\tilde{f}_N(x_k).$$

where $n\in\mathbb{N}$,

$$\prod_{k=1}^{n} x_k\to y=x_n\to(x_{n-1}\to(\cdots(x_1\to y)\cdots)).$$

Proof. If N is a neutrosophic hesitant fuzzy filter of X:

(i) By Theorem 12, we know that $\tilde{t}_N(x_1)\subseteq\tilde{t}_N(y),\tilde{i}_N(x_1)\supseteq\tilde{i}_N(y),\tilde{f}_N(x_1)\supseteq\tilde{f}_N(y)$ for $n=1$.

(ii) By Theorem 12, we know that $\tilde{t}_N(x_2)\subseteq\tilde{t}_N(x_1\to y),\tilde{i}_N(x_2)\supseteq\tilde{i}_N(x_1\to y),\tilde{f}_N(x_2)\supseteq\tilde{f}_N(x_1\to y)$ for $n=2$. By Definition 14, we have $\tilde{t}_N(x_1)\cap\tilde{t}_N(x_1\to y)\subseteq\tilde{t}_N(y),\tilde{i}_N(x_1)\cup\tilde{i}_N(x_1\to y)\supseteq\tilde{i}_N(y),\tilde{f}_N(x_1)\cup\tilde{f}_N(x_1\to y)\supseteq\tilde{f}_N(y)$. Thus, $\tilde{t}_N(x_1)\cap\tilde{t}_N(x_2)\subseteq\tilde{t}_N(y),\tilde{i}_N(x_1)\cup\tilde{i}_N(x_2)\supseteq\tilde{i}_N(y),\tilde{f}_N(x_1)\cup\tilde{f}_N(x_2)\supseteq\tilde{f}_N(y)$.

(iii) Suppose that the above formula is true for $n=j$; thus, $\prod_{k=1}^{j} x_k\to y=1,\forall x_j,\cdots,x_1,y\in X$, and we can obtain $\bigcap_{k=1}^{j}\tilde{t}_N(x_k)\subseteq\tilde{t}_N(y)$, $\bigcup_{k=1}^{j}\tilde{i}_N(x_k)\supseteq\tilde{i}_N(y)$, $\bigcup_{k=1}^{j}\tilde{f}_N(x_k)\supseteq\tilde{f}_N(y)$. Therefore, suppose that $\prod_{k=1}^{j+1} x_k\to y=1,\forall x_{j+1},\cdots,x_1,y\in X$, then we have $\bigcap_{k=2}^{j+1}\tilde{t}_N(x_k)\subseteq\tilde{t}_N(x_1\to y)$, $\bigcup_{k=2}^{j+1}\tilde{i}_N(x_k)\supseteq\tilde{i}_N(x_1\to y)$, $\bigcup_{k=2}^{j+1}\tilde{f}_N(x_k)\supseteq\tilde{f}_N(x_1\to y)$. By Definition 14, we can obtain:

$$\begin{aligned}\tilde{t}_N(y)\supseteq\tilde{t}_N(x_1)\cap\tilde{t}_N(x_1\to y)\supseteq\tilde{t}_N(x_1)\cap(\bigcap_{k=2}^{j+1}\tilde{t}_N(x_k))=\bigcap_{k=1}^{j+1}\tilde{t}_N(x_k),\\ \tilde{i}_N(y)\subseteq\tilde{i}_N(x_1)\cup\tilde{i}_N(x_1\to y)\subseteq\tilde{i}_N(x_1)\cup(\bigcup_{k=2}^{j+1}\tilde{i}_N(x_k))=\bigcup_{k=1}^{j+1}\tilde{i}_N(x_k),\\ \tilde{f}_N(y)\subseteq\tilde{f}_N(x_1)\cup\tilde{f}_N(x_1\to y)\subseteq\tilde{f}_N(x_1)\cup(\bigcup_{k=2}^{j+1}\tilde{f}_N(x_k))=\bigcup_{k=1}^{j+1}\tilde{f}_N(x_k),\end{aligned}$$

which complete the proof. □

Corollary 3. *Let $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x)))|x\in X)\}$ be a neutrosophic hesitant fuzzy filter of X. Then:*

$$\prod_{k=1}^{n} x_k * y=1\Rightarrow \tilde{t}_N(y)\supseteq\bigcap_{k=1}^{n}\tilde{t}_N(x_k),\tilde{i}_N(y)\subseteq\bigcup_{k=1}^{n}\tilde{i}_N(x_k),\tilde{f}_N(y)\subseteq\bigcup_{k=1}^{n}\tilde{f}_N(x_k).$$

where "$*$" represents any binary operation "$\to$" or "$\hookrightarrow$" on X, $n\in\mathbb{N}$,

$$\prod_{k=1}^{n} x_k * y=x_n*(x_{n-1}*(\cdots(x_1*y)\cdots)).$$

Theorem 15. *Let $N=\{(x,\tilde{t}_N(x),\tilde{i}_N(x),\tilde{f}_N(x))|x\in X)\}$ be a neutrosophic hesitant fuzzy filter of X and X be a pseudo-BCK algebra, then N is a neutrosophic hesitant fuzzy subalgebra of X.*

Proof. If $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy filter of X, then we can obtain $\forall x, y \in X$,

$$\begin{aligned}
\tilde{t}_N(x \to y) &\supseteq \tilde{t}_N(y \hookrightarrow (x \to y)) \cap \tilde{t}_N(y) \\
&= \tilde{t}_N(x \to (y \hookrightarrow y)) \cap \tilde{t}_N(y) \\
&= \tilde{t}_N(x \to 1) \cap \tilde{t}_N(y) \\
&\supseteq \tilde{t}_N(x) \cap \tilde{t}_N(y). \\
\tilde{i}_N(x \to y) &\subseteq \tilde{i}_N(y \hookrightarrow (x \to y)) \cup \tilde{i}_N(y) \\
&= \tilde{i}_N(x \to (y \hookrightarrow y)) \cup \tilde{i}_N(y) \\
&= \tilde{i}_N(x \to 1) \cup \tilde{i}_N(y) \\
&\subseteq \tilde{i}_N(x) \cup \tilde{i}_N(y).
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_N(x \to y) &\subseteq \tilde{f}_N(y \hookrightarrow (x \to y)) \cup \tilde{f}(y) \\
&= \tilde{f}_N(x \to (y \hookrightarrow y)) \cup \tilde{f}_N(y) \\
&= \tilde{f}_N(x \to 1) \cup \tilde{f}_N(y) \\
&\subseteq \tilde{f}_N(x) \cup \tilde{f}_N(y).
\end{aligned}$$

Similarly, we can obtain $\tilde{t}_N(x \hookrightarrow y) \supseteq \tilde{t}_N(x) \cap \tilde{t}_N(y), \tilde{i}_N(x \hookrightarrow y) \subseteq \tilde{i}_N(x) \cup \tilde{i}_N(y), \tilde{f}_N(x \hookrightarrow y) \subseteq \tilde{f}_N(x) \cup \tilde{f}_N(y)$. Thus, N is a neutrosophic hesitant fuzzy subalgebra of X. □

Theorem 16. *Let $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ be a neutrosophic hesitant fuzzy closed filter of X. Then, N is a neutrosophic hesitant fuzzy subalgebra of X.*

Proof. The process of proof is similar to Theorem 15. □

If $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy subalgebra of X, then N may not be a neutrosophic hesitant fuzzy filter of X.

Example 6. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2. Then, $(X; \to, \hookrightarrow, 1)$ is a pseudo-BCI algebra. N is a neutrosophic hesitant fuzzy subalgebra of X. However, N is not a neutrosophic hesitant fuzzy filter of X. Since $\tilde{t}(b \to a) \cap \tilde{t}(b) = [\frac{1}{3}, \frac{1}{2}]$, $\tilde{t}(a) = [\frac{1}{3}, \frac{1}{4}]$, we cannot obtain $\tilde{t}(b \to a) \cap \tilde{t}(b) \subseteq \tilde{t}(a)$.*

Definition 16. *$N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x))|x \in X)\}$ is a neutrosophic hesitant fuzzy set on X. Define a neutrosophic hesitant fuzzy set $N^{(a,b)} = \{(x, \tilde{t}_N^{(a,b)}(x), \tilde{i}_N^{(a,b)}(x), \tilde{f}_N^{(a,b)}(x))|x \in X\}$ by $\forall a, b \in X$,*

$$\tilde{t}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_1, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_2, & otherwise: \end{cases}$$

$$\tilde{i}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_3, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_4, & otherwise: \end{cases}$$

$$\tilde{f}_N^{(a,b)} : X \Longrightarrow P([0,1]), x \mapsto \begin{cases} \psi_5, & a \to (b \to x) = 1, a \hookrightarrow (b \hookrightarrow x) = 1; \\ \psi_6, & otherwise: \end{cases}$$

where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6 \in P([0,1])$, $\psi_1 \supseteq \psi_2, \psi_3 \subseteq \psi_4, \psi_5 \subseteq \psi_6$. Then, $N^{(a,b)}$ is called a generated neutrosophic hesitant fuzzy set.

A generated neutrosophic hesitant fuzzy set $N^{(a,b)}$ may not be a neutrosophic hesitant fuzzy filter of X.

Example 7. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 1 and 2. Then, $(X; \rightarrow, \hookrightarrow, 1)$ is a pseudo-BCI algebra. N is a neutrosophic hesitant fuzzy set of X. However, $N^{(a,b)}$ is not a neutrosophic hesitant fuzzy filter of X. Since $\tilde{t}^{(1,a)}(a \rightarrow b) \cap \tilde{t}^{(1,a)}(a) = [0, 1]$, $\tilde{t}^{(1,a)}(b) = [\frac{1}{3}, \frac{2}{3}]$, we cannot obtain $\tilde{t}^{(1,a)}(a \rightarrow b) \cap \tilde{t}^{(1,a)}(a) \subseteq \tilde{t}^{(1,a)}(b)$.*

Theorem 17. *Let X be a pseudo-BCK algebra. If X is a type-2 positive implicative pseudo-BCK algebra, then $N^{(a,b)}$ is a neutrosophic hesitant fuzzy filter of X for all $a, b \in X$.*

Proof. If X is a pseudo-BCK algebra, (1) by Definition 1 and Proposition 1, we can obtain $a \rightarrow (b \rightarrow 1) = 1$ $(a \hookrightarrow (b \hookrightarrow 1) = 1)$. $\tilde{t}_N^{(a,b)}(1) = \psi_1 \supseteq \tilde{t}_N^{(a,b)}(x), \tilde{i}_N^{(a,b)}(1) = \psi_3 \subseteq \tilde{i}_N^{(a,b)}(x), \tilde{f}_N^{(a,b)}(1) = \psi_5 \subseteq \tilde{f}_N^{(a,b)}(x)$ for all $x \in X$.

(2) (i) Let $x, y \in X$ with $a \rightarrow (b \rightarrow x) \neq 1$ or $a \hookrightarrow (b \hookrightarrow x) \neq 1$ or $a \rightarrow (b \rightarrow (x \rightarrow y)) \neq 1$ or $a \hookrightarrow (b \hookrightarrow (x \rightarrow y)) \neq 1$. Thus, we can obtain:

$$\tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \rightarrow y) = \psi_2 \subseteq \tilde{t}_N^{(a,b)}(y), \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \hookrightarrow y) = \psi_2 \subseteq \tilde{t}_N^{(a,b)}(y);$$
$$\tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \rightarrow y) = \psi_4 \supseteq \tilde{i}_N^{(a,b)}(y), \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \hookrightarrow y) = \psi_4 \supseteq \tilde{i}_N^{(a,b)}(y);$$
$$\tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \rightarrow y) = \psi_6 \supseteq \tilde{f}_N^{(a,b)}(y), \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \hookrightarrow y) = \psi_6 \supseteq \tilde{f}_N^{(a,b)}(y).$$

(ii) Let $x, y \in X$ with $a \rightarrow (b \rightarrow x) = 1$, $a \hookrightarrow (b \hookrightarrow x) = 1$ and $a \rightarrow (b \rightarrow (x \rightarrow y)) = 1$, $a \hookrightarrow (b \hookrightarrow (x \hookrightarrow y)) = 1$. Then, by Proposition 1 and Definition 4, we can obtain:

$$\begin{aligned}
&\tilde{t}_N^{(a,b)}(a \hookrightarrow (b \hookrightarrow y)) \\
=&\tilde{t}_N^{(a,b)}(1 \rightarrow (a \hookrightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}((a \hookrightarrow (b \hookrightarrow x)) \rightarrow (a \hookrightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \hookrightarrow ((b \hookrightarrow x) \rightarrow (b \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \hookrightarrow (b \hookrightarrow (x \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(1).
\end{aligned}$$

$$\begin{aligned}
&\tilde{t}_N^{(a,b)}(a \rightarrow (b \rightarrow y)) \\
=&\tilde{t}_N^{(a,b)}(1 \hookrightarrow (a \rightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(((a \rightarrow (b \rightarrow x)) \hookrightarrow (a \rightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \rightarrow ((b \rightarrow x) \hookrightarrow (b \rightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(a \rightarrow (b \rightarrow (x \hookrightarrow y))) \\
=&\tilde{t}_N^{(a,b)}(1).
\end{aligned}$$

Therefore, we can obtain,

$$\tilde{t}_N^{(a,b)}(y) = \psi_1 = \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \rightarrow y), \tilde{t}_N^{(a,b)}(y) = \psi_1 = \tilde{t}_N^{(a,b)}(x) \cap \tilde{t}_N^{(a,b)}(x \hookrightarrow y).$$

Similarly, we can obtain,

$$\tilde{i}_N^{(a,b)}(y) = \psi_3 = \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \rightarrow y), \tilde{i}_N^{(a,b)}(y) = \psi_3 = \tilde{i}_N^{(a,b)}(x) \cup \tilde{i}_N^{(a,b)}(x \hookrightarrow y);$$
$$\tilde{f}_N^{(a,b)}(y) = \psi_5 = \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \rightarrow y), \tilde{f}_N^{(a,b)}(y) = \psi_5 = \tilde{f}_N^{(a,b)}(x) \cup \tilde{f}_N^{(a,b)}(x \hookrightarrow y).$$

This means that $N^{(a,b)}$ is a neutrosophic hesitant fuzzy filter of X. □

Example 8. *Let $X = \{a, b, c, d, 1\}$ with two binary operations in Tables 7 and 8. Then, $(X; \rightarrow, \hookrightarrow, 1)$ is a type-2 positive implicative pseudo-BCI algebra. Let N be a neutrosophic hesitant fuzzy set. We take b, c as* *an example; thus, we have* $\{b, c, d, 1\}$ *satisfy* $d \to (c \to x) = 1, d \hookrightarrow (c \hookrightarrow x) = 1$. *Let* $\psi_1 = [0.1, 0.4]$, $\psi_2 = [0.2, 0.3]$, $\psi_3 = [0.4, 0.5]$, $\psi_4 = [0.3, 0.6]$, $\psi_5 = [0.2, 0.8]$, $\psi_6 = [0.1, 0.9]$,

$$N^{(d,c)} = \{(1, \psi_1, \psi_3, \psi_5), (a, \psi_2, \psi_4, \psi_6), (b, \psi_1, \psi_3, \psi_5), (c, \psi_1, \psi_3, \psi_5), (e, \psi_1, \psi_3, \psi_5)\} =$$
$$\{(1, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]), (a, [0.2, 0.3], [0.3, 0.6], [0.1, 0.9]), (b, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]),$$
$$(c, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8]), (d, [0.1, 0.4], [0.4, 0.5], [0.2, 0.8])\}.$$

Then, we can obtain that $N^{(d,c)}$ *is a neutrosophic hesitant fuzzy filter of* X.

Table 7. $\to$.

$\to$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Table 8. $\hookrightarrow$.

$\hookrightarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Theorem 18. *Let* $N = \{(x, \tilde{t}_N(x), \tilde{i}(x), \tilde{f}(x)) | x \in X\}$ *be a neutrosophic hesitant fuzzy filter of* X. *Then,* $X_N^{(5)}(a) = \{x | \tilde{t}_N(a) \subseteq \tilde{t}_N(x), \tilde{i}_N(a) \supseteq \tilde{i}_N(x), \tilde{f}_N(a) \supseteq \tilde{f}_N(x)\}$ *is a filter of* X *for all* $a \in X$.

Proof. (1) Let $x, y \in X$ with $x, x \to y \in X_N^5(a)$. Then, we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x), \tilde{t}_N(a) \subseteq \tilde{t}_N(x \to y)$. Since $N = \{(x, \tilde{t}_N(x), \tilde{i}_N(x), \tilde{f}_N(x)) | x \in X\}$ is a neutrosophic hesitant fuzzy filter, thus we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \to y) \subseteq \tilde{t}_N(y) \subseteq \tilde{t}_N(1)$. Similarly, we can get $\tilde{i}_N(a) \supseteq \tilde{i}_N(x) \cup \tilde{i}(x \to y) \supseteq \tilde{i}_N(y) \supseteq \tilde{i}_N(1), \tilde{f}_N(a) \supseteq \tilde{f}_N(x) \cup \tilde{f}_N(x \to y) \supseteq \tilde{f}_N(y) \supseteq \tilde{f}_N(1)$.

(2) Similarly, let $x, y \in X$ with $x, x \hookrightarrow y \in X_N^{(5)}(a)$; we have $\tilde{t}_N(a) \subseteq \tilde{t}_N(x) \cap \tilde{t}_N(x \hookrightarrow y) \subseteq \tilde{t}_N(y) \subseteq \tilde{t}_N(1), \tilde{i}_N(a) \supseteq \tilde{i}_N(x) \cup \tilde{i}_N(x \hookrightarrow y) \supseteq \tilde{i}_N(y) \supseteq \tilde{i}_N(1), \tilde{f}_N(a) \supseteq \tilde{f}_N(x) \cup \tilde{f}_N(x \hookrightarrow y) \supseteq \tilde{f}_N(y) \supseteq \tilde{f}_N(1)$.

This means that $X_N^{(5)}(a)$ satisfies the conditions of Definition 2 (F1), (F2) and (F3); $X_N^{(5)}(a)$ is a filter of X. □

Example 9. *Let* $X = \{a, b, c, d, 1\}$ *with two binary operations in Tables 5 and 6. Then,* $(X; \to, \hookrightarrow, 1)$ *is a pseudo-BCI algebra. Let:*

$$N = \{(1, [0, 1], [0, \tfrac{3}{7}], [0, \tfrac{1}{10}]), (a, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]), (b, [0, \tfrac{1}{4}], [0, \tfrac{3}{4}], [0, \tfrac{1}{2}]),$$
$$(c, [0, \tfrac{1}{3}], [0, \tfrac{3}{5}], [0, \tfrac{1}{4}]), (d, [0, \tfrac{3}{4}]), [0, \tfrac{3}{6}], [0, \tfrac{1}{5}])\}.$$

Then, N *is a neutrosophic hesitant fuzzy filter of* X. *Let* $X_N^{(5)}(c) = \{c, d, 1\}$. *It is easy to get that* $X_N^{(5)}(a)$ *is a filter.*

5. Conclusions

In this paper, the neutrosophic hesitant fuzzy set theory was applied to pseudo-BCI algebra, and the neutrosophic hesitant fuzzy subalgebras (filters) in pseudo-BCI algebras were developed. The relationships between neutrosophic hesitant fuzzy subalgebras (filters) and hesitant fuzzy subalgebras (filters) was discussed, and some properties were demonstrated. In future work, different types of neutrosophic hesitant fuzzy filters will be defined and discussed.

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