On Neutrosophic Semi-Open sets in Neutrosophic Topological Spaces

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Abstract - *The purpose of this paper is to define the product related neutrosophic topological space and proved some theorems based on this. We introduce the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets in neutrosophic topological spaces and derive some of their characterization. Finally, we analyze neutrosophic semi-interior and neutrosophic semi-closure operators also.*

Mathematics Subject Classification : 03E72

Keywords: *Neutrosophic semi-open set, Neutrosophic semi-closed set, Neutrosophic semi-interior operator and Neutrosophic semi-closure operator.*

INTRODUCTION

Theory of Fuzzy sets [17], Theory of Intuitionistic fuzzy sets [2], Theory of Neutrosophic sets [9] and the theory of Interval Neutrosophic sets [11] can be considered as tools for dealing with uncertainities. However, all of these theories have their own difficulties which are pointed out in [9]. In 1965, Zadeh [17] introduced fuzzy set theory as a mathematical tool for dealing with uncertainities where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov [2] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. The neutrosophic set was introduced by Smarandache [9] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set. In 2012, Salama, Alblowi [15], introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of Intuitionistic fuzzy topological space and a Neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element.

This paper consists of six sections. The section I consists of the basic definitions and some properties which are used in the later sections. The section II, we define product related neutrosophic topological space and proved some theorem related to this definition. The section III deals with the definition of neutrosophic semi-open set in neutrosophic topological spaces and its various properties. The section IV deals with the definition of neutrosophic semi-closed set in neutrosophic topological spaces and its various properties. The section V and VI are dealt with the concepts of neutrosophic semi-interior and neutrosophic semi-closure operators.

I. PRELIMINARIES

In this section, we give the basic definitions for neutrosophic sets and its operations.

Definition 1.1 [15] Let X be a non-empty fixed set. A neutrosophic set [*NS* for short] *A* is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\mu_A(x)$, $\sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function, the degree indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set A.

Remark 1.2 [15] A neutrosophic set $A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A \rangle$ in $]^-0,1^+[$ on X.

Remark 1.3 [15] For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$ for the neutrosophic set $A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$.

Example 1.4 [15] Every IFS A is a non-empty set in X is obviously on *NS* having the form
$A = \{ \langle x, \mu_A(x), 1-(\mu_A(x) + \gamma_A(x)), \gamma_A(x) \rangle : x \in X \}$. Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the *NS* 0_N and 1_N in X as follows:

0_N may be defined as :
(0_1) $0_N = \{ \langle x, 0, 0, 1 \rangle : x \in X \}$
(0_2) $0_N = \{ \langle x, 0, 1, 1 \rangle : x \in X \}$
(0_3) $0_N = \{ \langle x, 0, 1, 0 \rangle : x \in X \}$
(0_4) $0_N = \{ \langle x, 0, 0, 0 \rangle : x \in X \}$

1_N may be defined as :
(1_1) $1_N = \{ \langle x, 1, 0, 0 \rangle : x \in X \}$
(1_2) $1_N = \{ \langle x, 1, 0, 1 \rangle : x \in X \}$
(1_3) $1_N = \{ \langle x, 1, 1, 0 \rangle : x \in X \}$
(1_4) $1_N = \{ \langle x, 1, 1, 1 \rangle : x \in X \}$

Definition 1.5 [15] Let $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ be a *NS* on X, then the complement of the set A [C (A) for short] may be defined as three kinds of complements :
(C_1) $C(A) = \{ \langle x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) \rangle : x \in X \}$
(C_2) $C(A) = \{ \langle x, \gamma_A(x), \sigma_A(x), \mu_A(x) \rangle : x \in X \}$
(C_3) $C(A) = \{ \langle x, \gamma_A(x), 1 - \sigma_A(x), \mu_A(x) \rangle : x \in X \}$

One can define several relations and operations between *NSs* follows :

Definition 1.6 [15] Let x be a non-empty set, and neutrosophic sets A and B in the form $A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$ and $B = \{ \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle : x \in X \}$. Then we may consider two possible definitions for subsets ($A \subseteq B$).
$A \subseteq B$ may be defined as :
(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \le \sigma_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)\ \forall\, x \in X$
(2) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)\ \forall\, x \in X$

Proposition 1.7 [15] For any neutrosophic set A, then the following conditions are holds :
(1) $0_N \subseteq A$, $0_N \subseteq 0_N$
(2) $A \subseteq 1_N$, $1_N \subseteq 1_N$

Definition 1.8 [15] Let X be a non-empty set, and $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle$ are *NSs*. Then
(1) $A \cap B$ may be defined as :
(I_1) $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\gamma_A(x) \vee \gamma_B(x) \rangle$
(I_2) $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\gamma_A(x) \vee \gamma_B(x) \rangle$

(2) $A \cup B$ may be defined as :
(U_1) $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\gamma_A(x) \wedge \gamma_B(x) \rangle$
(U_2) $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\gamma_A(x) \wedge \gamma_B(x) \rangle$

We can easily generalize the operations of intersection and union in Definition 1.8 to arbitrary family of *NSs* as follows :

Definition 1.9 [15] Let $\{ A_j : j \in J \}$ be a arbitrary family of *NSs* in X, then
(1) $\cap A_j$ may be defined as :
(i) $\cap A_j = \langle x, \wedge_{j \in J} \mu_{A_j}(x), \wedge_{j \in J} \sigma_{A_j}(x), \vee_{j \in J} \gamma_{A_j}(x) \rangle$
(ii) $\cap A_j = \langle x, \wedge_{j \in J} \mu_{A_j}(x), \vee_{j \in J} \sigma_{A_j}(x), \vee_{j \in J} \gamma_{A_j}(x) \rangle$

(2) $\cup A_j$ may be defined as :
(i) $\cup A_j = \langle x, \vee, \vee, \wedge \rangle$
(ii) $\cup A_j = \langle x, \vee, \wedge, \wedge \rangle$

Proposition 1.10 [15] For all A and B are two neutrosophic sets then the following conditions are true :
(1) $C(A \cap B) = C(A) \cup C(B)$
(2) $C(A \cup B) = C(A) \cap C(B)$.

Here we extend the concepts of fuzzy topological space [5] and Intuitionistic fuzzy topological space [6,7] to the case of neutrosophic sets.

Definition 1.11 [15] A neutrosophic topology [*NT* for short] is a non-empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms :
(NT_1) $0_N, 1_N \in \tau$,
(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(NT_3) $\cup G_i \in \tau$ for every $\{ G_i : i \in J \} \subseteq \tau$

In this case the pair (X, τ) is called a neutrosophic topological space [*NTS* for short]. The elements of τ are called neutrosophic open sets [*NOS* for short]. A neutrosophic set F is closed if and only if C (F) is neutrosophic open.

Example 1.12 [15] Any fuzzy topological space (X, τ_0) in the sense of Chang is obviously a *NTS* in the form $\tau = \{ A : \mu_A \in \tau_0 \}$ wherever we identify a fuzzy set in X whose membership function is μ_A with its counterpart.

Remark 1.13 [15] Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces allow more general functions to be members of fuzzy topology.

Example 1.14 [15] Let X = { x } and
$A = \{ \langle x, 0.5, 0.5, 0.4 \rangle : x \in X \}$
$B = \{ \langle x, 0.4, 0.6, 0.8 \rangle : x \in X \}$
$D = \{ \langle x, 0.5, 0.6, 0.4 \rangle : x \in X \}$
$C = \{ \langle x, 0.4, 0.5, 0.8 \rangle : x \in X \}$
Then the family $\tau = \{ 0_N, A, B, C, D, 1_N \}$ of N*S*s in X is neutrosophic topology on X.

Definition 1.15 [15] The complement of A [C (A) for short] of *NOS* is called a neutrosophic closed set [*NCS* for short] in X .

Now, we define neutrosophic closure and neutrosophic interior operations in neutrosophic topological spaces :

Definition 1.16 [15] Let (X, τ) be *NTS* and $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$ be a *NS* in X. Then the neutrosophic closure and neutrosophic interior of A are defined by
$NCl\,(A) = \cap \{ K : K$ is a *NCS* in X and $A \subseteq K \}$
$NInt\,(A) = \cup \{ G : G$ is a *NOS* in X and $G \subseteq A \}$.

It can be also shown that $NCl\,(A)$ is *NCS* and $NInt\,(A)$ is a *NOS* in X.
a) A is *NOS* if and only if $A = NInt\,(A)$.
b) A is *NCS* if and only if $A = NCl\,(A)$.

Proposition 1.17 [15] For any neutrosophic set A in (X, τ) we have
(a) $NCl\,(C\,(A)) = C\,(NInt\,(A))$,
(b) $NInt\,(C\,(A)) = C\,(NCl\,(A))$.

Proposition 1.18 [15] Let (X, τ) be a *NTS* and A, B be two neutrosophic sets in X. Then the following properties are holds :
(a) $NInt\,(A) \subseteq A$,
(b) $A \subseteq NCl\,(A)$,
(c) $A \subseteq B \Rightarrow NInt\,(A) \subseteq NInt\,(B)$,
(d) $A \subseteq B \Rightarrow NCl\,(A) \subseteq NCl\,(B)$,
(e) $NInt\,(NInt\,(A)) = NInt\,(A)$,
(f) $NCl\,(NCl\,(A)) = NCl\,(A)$,
(g) $NInt\,(A \cap B)) = NInt\,(A) \cap NInt\,(B)$,
(h) $NCl\,(A \cup B) = NCl\,(A) \cup NCl\,(B)$,
(i) $NInt\,(0_N) = 0_N$,
(j) $NInt\,(1_N) = 1_N$,
(k) $NCl\,(0_N) = 0_N$,
(l) $NCl\,(1_N) = 1_N$,
(m) $A \subseteq B \Rightarrow C\,(B) \subseteq C\,(A)$,
(n) $NCl\,(A \cap B) \subseteq NCl\,(A) \cap NCl\,(B)$,
(o) $NInt\,(A \cup B) \supseteq NInt\,(A) \cup NInt\,(B)$.

II. PRODUCT RELATED NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we define some basic and important results which are very useful in later sections. In order topology, the product of the closure is equal to the closure of the product and product of the interior is equal to the interior of the product. But this result is not true in neutrosophic topological space. For this reason, we define the product related neutrosophic topological space. Using this definition, we prove the above mentioned result.

Definition 2.1 A subfamily β of *NTS* (X, τ) is called a base for τ if each *NS* of τ is a union of some members of β.

Definition 2.2 Let X, Y be nonempty neutrosophic sets and $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle$ *NSs* of X and Y respectively. Then $A \times B$ is a *NS* of $X \times Y$ is defined by

(P_1) $(A \times B)(x, y) = \langle (x, y), \min(\mu_A(x), \mu_B(y)), \min(\sigma_A(x), \sigma_B(y)), \max(\gamma_A(x), \gamma_B(y)) \rangle$
(P_2) $(A \times B)(x, y) = \langle (x, y), \min(\mu_A(x), \mu_B(y)), \max(\sigma_A(x), \sigma_B(y)), \max(\gamma_A(x), \gamma_B(y)) \rangle$

Notice that
(CP_1) $C((A \times B)(x, y)) = \langle (x, y), \max(\mu_A(x), \mu_B(y)), \max(\sigma_A(x), \sigma_B(y)), \min(\gamma_A(x), \gamma_B(y)) \rangle$
(CP_2) $C((A \times B)(x, y)) = \langle (x, y), \max(\mu_A(x), \mu_B(y)), \min(\sigma_A(x), \sigma_B(y)), \min(\gamma_A(x), \gamma_B(y)) \rangle$

Lemma 2.3 If A is the *NS* of X and B is the *NS* of Y, then
(i) $(A \times 1_N) \cap (1_N \times B) = A \times B$,
(ii) $(A \times 1_N) \cup (1_N \times B) = C\,(C\,(A) \times C\,(B))$,
(iii) $C\,(A \times B) = (C\,(A) \times 1_N) \cup (1_N \times C\,(B))$.

Proof : Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle$.

(i) Since $A \times 1_N = \langle x, \min(\mu_A, 1_N), \min(\sigma_A, 1_N), \max(\gamma_A, 0_N) \rangle = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle = A$ and similarly $1_N \times B = \langle y, \min(1_N, \mu_B), \min(1_N, \sigma_B), \max(0_N, \gamma_B) \rangle = B$, we have $(A \times 1_N) \cap (1_N \times B) = A(x) \cap B(y) = \langle (x, y), \mu_A(x) \wedge \mu_B(y), \sigma_A(x) \wedge \sigma_B(y), \gamma_A(x) \vee \gamma_B(y) \rangle = A \times B$.

(ii) Similarly to (i).

(iii) Obvious by putting A, B instead of C (A), C (B) in (ii).

Definition 2.4 Let X and Y be two nonempty neutrosophic sets and f : X $\to$ Y be a neutrosophic function. (i) If $B = \{ \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle : y \in Y \}$ is a *NS* in Y, then the pre image of B under f is denoted and defined by $f^{-1}(B) = \{ \langle x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\gamma_B)(x) \rangle : x \in X \}$.

(ii) If $A = \{ \langle x, \alpha_A(x), \delta_A(x), \lambda_A(x) \rangle : x \in X \}$ is a *NS* in X, then the image of A under f is denoted and defined by $f(A) = \{ \langle y, f(\alpha_A)(y), f(\delta_A)(y), f\_(\lambda_A)(y) \rangle : y \in Y \}$ where $f\_(\lambda_A) = C(f(C(A)))$.

In (i), (ii), since $\mu_B, \sigma_B, \gamma_B, \alpha_A, \delta_A, \lambda_A$ are neutrosophic sets, we explain that $f^{-1}(\mu_B)(x) = \mu_B(f(x))$,

and $f(\alpha_A)(y) = \begin{cases} \sup \alpha_A(x) & \text{if } x \in f^{-1}(y) \\ 0 & \text{Otherwise} \end{cases}$.

Definition 2.5 Let (X, τ) and (Y, σ) be *NTSs*. The neutrosophic product topological space [*NPTS* for short] of (X, τ) and (Y, σ) is the cartesian product X $\times$ Y of *NSs* X and Y together with the *NT* ξ of X $\times$ Y which is generated by the family $\{P_1^{-1}(A_i), P_2^{-1}(B_j) : A_i \in \tau, B_j \in \sigma$ and P_1, P_2 are projections of X $\times$ Y onto X and Y respectively$\}$ (i.e. the family $\{P_1^{-1}(A_i), P_2^{-1}(B_j) : A_i \in \tau, B_j \in \sigma\}$ is a subbase for *NT* ξ of X $\times$ Y).

Remark 2.6 In the above definition, since $P_1^{-1}(A_i) = A_i \times 1_N$ and $P_2^{-1}(B_j) = 1_N \times B_j$ and $A_i \times 1_N \cap 1_N \times B_j = A_i \times B_j$, the family $\beta = \{ A_i \times B_j : A_i \in \tau, B_j \in \sigma \}$ forms a base for *NPTS* ξ of X $\times$ Y .

Definition 2.7 Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be the two neutrosophic functions. Then the neutrosophic product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$.

Definition 2.8 Let A, A_i ($i \in J$) be *NSs* in X and B, B_j ($j \in K$) be *NSs* in Y and f : X $\to$ Y be the neutrosophic function. Then

(i) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$,

(ii) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$,

(iii) $f^{-1}(1_N) = 1_N$, $f^{-1}(0_N) = 0_N$,

(iv) $f^{-1}(C(B)) = C(f^{-1}(B))$,

(v) $f(\cup A_i) = \cup f(A_i)$.

Definition 2.9 Let f : X $\to$ Y be the neutrosophic function. Then the neutrosophic graph g : X $\to$ X $\times$ Y of f is defined by $g(x) = (x, f(x))$ for all $x \in$ X.

Lemma 2.10 Let $f_i : X_i \to Y_i$ (i = 1, 2) be the neutrosophic functions and A, B be *NSs* of Y_1, Y_2 respectively. Then $(f_1 \times f_2)^{-1} = f_1^{-1}(A) \times f_2^{-1}(B)$.

Proof : Let $A = \langle x_1, \mu_A(x_1), \sigma_A(x_1), \gamma_A(x_1) \rangle$, $B = \langle x_2, \mu_B(x_2), \sigma_B(x_2), \gamma_B(x_2) \rangle$. For each $(x_1, x_2) \in X_1 \times X_2$, we have $(f_1 \times f_2)^{-1}(A, B)(x_1, x_2) = (A \times B)(f_1 \times f_2)(x_1, x_2) = (A \times B)(f_1(x_1), f_2(x_2)) = \langle (f_1(x_1), f_2(x_2)), \min(\mu_A(f_1(x_1)), \mu_B(f_2(x_2))), \min(\sigma_A(f_1(x_1)), \sigma_B(f_2(x_2))), \max(\gamma_A(f_1(x_1)), \gamma_B(f_2(x_2))) \rangle = \langle (x_1, x_2), \min(f_1^{-1}(\mu_A)(x_1), f_2^{-1}(\mu_B)(x_2)), \min(f_1^{-1}(\sigma_A)(x_1), f_2^{-1}(\sigma_B)(x_2)), \max(f_1^{-1}(\gamma_A)(x_1), f_2^{-1}(\gamma_B)(x_2)) \rangle = (f_1^{-1}(A) \times f_2^{-1}(B))(x_1, x_2)$.

Lemma 2.11 Let g : X $\to$ X $\times$ Y be the neutrosophic graph of the neutrosophic function f : X $\to$ Y . If A is the *NS* of X and B is the *NS* of Y, then $g^{-1}(A \times B)(x) = (A \cap f^{-1}(B))(x)$.

Proof : Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle$. For each $x \in$ X, we have $g^{-1}(A \times B)(x) = (A \times B)\, g(x) = (A \times B)(x, f(x)) = \langle (x, f(x)), \min(\mu_A(x), \mu_B(f(x))), \min(\sigma_A(x), \sigma_B(f(x))), \max(\gamma_A(x), \gamma_B(f(x))) \rangle = \langle (x, f(x)), \min(\mu_A(x), f^{-1}(\mu_B)(x)), \min(\sigma_A(x), f^{-1}(\sigma_B)(x)), \max(\gamma_A(x), f^{-1}(\gamma_B)(x)) \rangle = (A \cap f^{-1}(B))(x)$.

Lemma 2.12 Let A, B, C and D be *NSs* in X. Then $A \subseteq B, C \subseteq D \Rightarrow A \times C \subseteq B \times D$.

Proof : Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle$, $C = \langle x, \mu_C(x), \sigma_C(x), \gamma_C(x) \rangle$ and $D = \langle x, \mu_D(x), \sigma_D(x), \gamma_D(x) \rangle$ be *NSs*. Since $A \subseteq B \Rightarrow \mu_A \leq \mu_B$, $\sigma_A \leq \sigma_B$, $\gamma_A \geq \gamma_B$ and also $C \subseteq D \Rightarrow \mu_C \leq \mu_D$, $\sigma_C \leq \sigma_D$, $\gamma_C \geq \gamma_D$, we have $\min(\mu_A, \mu_C) \leq \min(\mu_B, \mu_D)$, $\min(\sigma_A, \sigma_C) \leq \min(\sigma_B, \sigma_D)$ and $\max(\gamma_A, \gamma_C) \geq \max(\gamma_B, \gamma_D)$. Hence the result.

Lemma 2.13 Let (X, τ) and (Y, σ) be any two *NTSs* such that X is neutrosophic product relative to Y. Let A and B be *NCSs* in *NTSs* X and Y respectively. Then $A \times B$ is the *NCS* in the *NPTS* of X $\times$ Y.

Proof : Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle$. From Lemma 2.3,C $(A \times B)(x,y) = (C(A) \times 1_N) \cup (1_N \times C(B))(x, y)$. Since C $(A) \times 1_N$ and $1_N \times$ C (B) are *NOSs* in X and Y respectively. Hence C $(A) \times 1_N \cup 1_N \times$ C (B) is *NOS* of X $\times$ Y. Hence C ($A \times B$) is a *NOS* of X $\times$ Y and consequently $A \times B$ is the *NCS* of X $\times$ Y .

Theorem 2.14 If A and B are *NSs* of *NTSs* X and Y respectively, then

(i) $NCl(A) \times NCl(B) \supseteq NCl(A \times B)$,

(ii) $NInt(A) \times NInt(B) \subseteq NInt(A \times B)$.

Proof : (i) Since $A \subseteq NCl(A)$ and $B \subseteq NCl(B)$, hence $A \times B \subseteq NCl(A) \times NCl(B)$. This implies that $NCl(A \times B) \subseteq NCl(NCl(A) \times NCl(B))$ and from Lemma 2.13, $NCl(A \times B) \subseteq NCl(A) \times NCl(B)$.

(ii) follows from (i) and the fact that $NInt(C(A)) = C(NCl(A))$.

Definition 2.15 Let (X, τ), (Y, σ) be *NTSs* and $A \in \tau$, $B \in \sigma$. We say that (X, τ) is neutrosophic product related to (Y, σ) if for any *NSs* C of X and D of Y , whenever $C(A) \not\supseteq C$ and $C(B) \not\supseteq D \Rightarrow C(A) \times 1_N \cup 1_N \times C(B) \supseteq C \times D$, there exist $A_1 \in \tau$, $B_1 \in \sigma$ such that $C(A_1) \supseteq C$ or $C(B_1) \supseteq D$ and $C(A_1) \times 1_N \cup 1_N \times C(B_1) = C(A) \times 1_N \cup 1_N \times C(B)$.

Lemma 2.16 For *NSs* A_i's and B_j's of *NTSs* X and Y respectively, we have

(i) $\cap \{ A_i, B_j \} = \min(\cap A_i, \cap B_j)$;
$\cup \{ A_i, B_j \} = \max(\cup A_i, \cup B_j)$.

(ii) $\cap \{ A_i, 1_N \} = (\cap A_i) \times 1_N$;
$\cup \{ A_i, 1_N \} = (\cup A_i) \times 1_N$.

(iii) $\cap \{ 1_N \times B_j \} = 1_N \times (\cap B_j)$;
$\cup \{ 1_N \times B_j \} = 1_N \times (\cup B_j)$.

Proof : Obvious.

Theorem 2.17 Let (X, τ) and (Y, σ) be *NTSs* such that X is neutrosophic product related to Y. Then for *NSs* A of X and B of Y , we have

(i) $NCl(A \times B) = NCl(A) \times NCl(B)$,

(ii) $NInt(A \times B) = NInt(A) \times NInt(B)$.

Proof : (i) Since $NCl(A \times B) \subseteq NCl(A) \times NCl(B)$ (By Theorem 2.14) it is sufficient to show that $NCl(A \times B) \supseteq NCl(A) \times NCl(B)$. Let $A_i \in \tau$ and $B_j \in \sigma$. Then $NCl(A \times B) = \langle (x, y), \cap C(\{ A_i \times B_j \}) : C(\{ A_i \times B_j \}) \supseteq A \times B, \cup \{ A_i \times B_j \} : \{ A_i \times B_j \} \subseteq A \times B \rangle = \langle (x, y), \cap (C(A_i) \times 1_N \cup 1_N \times C(B_j)) : C(A_i) \times 1_N \cup 1_N \times C(B_j) \supseteq A \times B, \cup (A_i \times 1_N \cap 1_N \times B_j) : A_i \times 1_N \cap 1_N \times B_j \subseteq A \times B \rangle = \langle (x, y), \cap (C(A_i) \times 1_N \cup 1_N \times C(B_j)) : C(A_i) \supseteq A$ or $C(B_j) \supseteq B, \cup (A_i \times 1_N \cap 1_N \times B_j) : A_i \subseteq A$ and $B_j \subseteq B \rangle = \langle (x, y), \min (\cap \{ C(A_i) \times 1_N \cup 1_N \times C(B_j) : C(A_i) \supseteq A \}, \cap \{ C(A_i) \times 1_N \cup 1_N \times C(B_j) : C(B_j) \supseteq B \}), \max (\cup \{ A_i \times 1_N \cap 1_N \times B_j : A_i \subseteq A \}, \cup \{ A_i \times 1_N \cap 1_N \times B_j : B_j \subseteq B \}) \rangle$. Since $\langle (x, y), \cap \{ C(A_i) \times 1_N \cup 1_N \times C(B_j) : C(A_i) \supseteq A \}, \cap \{ C(A_i) \times 1_N \cup 1_N \times C(B_j) : C(B_j) \supseteq B \} \rangle \supseteq \langle (x, y), \cap \{ C(A_i) \times 1_N : C(A_i) \supseteq A \}, \cap \{ 1_N \times C(B_j) : C(B_j) \supseteq B \} \rangle = \langle (x,y), \cap \{ C(A_i) : C(A_i) \supseteq A \} \times 1_N , 1_N \times \cap \{ C(B_j) : C(B_j) \supseteq B \} \rangle = \langle (x, y), NCl(A) \times 1_N , 1_N \times NCl(B) \rangle$ and $\langle (x, y) , \cup \{ A_i \times 1_N \cap 1_N \times B_j : A_i \subseteq A , \cup \{ A_i \times 1_N \cap 1_N \times B_j : B_j \subseteq B \} \rangle \subseteq \langle (x, y), \cup \{A_i \times 1_N : A_i \subseteq A\}, \cup \{1_N \times B_j : B_j \subseteq B\} \rangle = \langle (x, y), \cup \{ A_i : A_i \subseteq A\} \times 1_N , 1_N \times \cup \{ B_j : B_j \subseteq B \} \rangle = \langle (x, y), NInt(A) \times 1_N , 1_N \times NInt(B) \rangle$, we have $NCl(A \times B) \supseteq \langle (x, y), \min (NCl(A) \times 1_N , 1_N \times NCl(B)), \max (NInt(A) \times 1_N , 1_N \times NInt(B)) \rangle = \langle (x, y), \min (NCl(A), NCl(B)), \max (NInt(A), NInt(B)) \rangle = NCl(A) \times NCl(B)$.

(ii) follows from (i).

III. NEUTROSOPHIC SEMI-OPEN SETS IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, the concepts of the neutrosophic semi-open set is introduced and also discussed their characterizations.

Definition 3.1 Let A be *NS* of a *NTS* X. Then A is said to be neutrosophic semi-open [written *NSO*] set of X if there exists a neutrosophic open set NO such that $NO \subseteq A \subseteq NCl(NO)$.

The following theorem is the characterization of *NSO* set in *NTS*.

Theorem 3.2 A subset A in a *NTS* X is *NSO* set if and only if $A \subseteq NCl(NInt(A))$.

Proof : Sufficiency: Let $A \subseteq NCl(NInt(A))$. Then for $NO = NInt(A)$, we have $NO \subseteq A \subseteq NCl(NO)$. Necessity: Let A be *NSO* set in X. Then $NO \subseteq A \subseteq NCl(NO)$ for some neutrosophic open set NO. But $NO \subseteq NInt(A)$ and thus $NCl(NO) \subseteq NCl(NInt(A))$. Hence $A \subseteq NCl(NO) \subseteq NCl(NInt(A))$.

Theorem 3.3 Let (X, τ) be a *NTS*. Then union of two *NSO* sets is a *NSO* set in the *NTS* X.

Proof : Let A and B are *NSO* sets in X. Then $A \subseteq NCl(NInt(A))$ and $B \subseteq NCl(NInt(B))$. Therefore $A \cup B \subseteq NCl(NInt(A)) \cup NCl(NInt(B)) = NCl(NInt(A) \cup NInt(B)) \subseteq NCl(NInt(A \cup B))$ [By Proposition 1.18 (o)]. Hence $A \cup B$ is *NSO* set in X.

Theorem 3.4 Let (X, τ) be a *NTS*. If $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of *NSO* sets in a *NTS* X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is *NSO* set in X.

Proof : For each $\alpha\in\Delta$, we have a neutrosophic open set NO_α such that $NO_\alpha \subseteq A_\alpha \subseteq NCl(NO_\alpha)$. Then $\bigcup_{\alpha\in\Delta} NO_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} NCl(NO_\alpha) \subseteq NCl(\bigcup_{\alpha\in\Delta} NO_\alpha)$. Hence let $NO = \bigcup_{\alpha\in\Delta} NO_\alpha$.

Remark 3.5 The intersection of any two *NSO* sets need not be a *NSO* set in X as shown by the following example.

Example 3.6 Let X = { *a* , *b* } and
A = ⟨ (0.3, 0.5, 0.4), (0.6, 0.2, 0.5) ⟩
B = ⟨ (0.2, 0.6, 0.7), (0.5, 0.3, 0.1) ⟩
C = ⟨ (0.3, 0.6, 0.4), (0.6, 0.3, 0.1) ⟩
D = ⟨ (0.2, 0.5, 0.7), (0.5, 0.2, 0.5) ⟩ .
Then τ = { 0_N, A, B, C, D, 1_N } is *NTS* on X. Now, we define the two *NSO* sets as follows:
A_1 = ⟨ (0.4, 0.6, 0.4), (0.8, 0.3, 0.4) ⟩ and
A_2 = ⟨ (1, 0.9, 0.2), (0.5, 0.7, 0) ⟩ . Here *NInt* (A_1) = A, *NCl* (*NInt* (A_1)) = 1_N and *NInt* (A_2) = B, *NCl* (*NInt* (A_2)) = 1_N. But $A_1 \cap A_2$ = ⟨(0.4, 0.6, 0.4), (0.5, 0.3, 0.4) ⟩ is not a *NSO* set in X.

Theorem 3.7 Let *A* be *NSO* set in the *NTS* X and suppose $A \subseteq B \subseteq NCl\,(A)$. Then *B* is *NSO* set in X.

Proof : There exists a neutrosophic open set NO such that NO ⊆ *A* ⊆ *NCl* (NO). Then NO ⊆ *B*. But *NCl* (*A*) ⊆ *NCl* (NO) and thus *B* ⊆ *NCl* (NO). Hence NO ⊆ *B* ⊆ *NCl* (NO) and *B* is *NSO* set in X.

Theorem 3.8 Every neutrosophic open set in the *NTS* X is *NSO* set in X.

Proof : Let *A* be neutrosophic open set in *NTS* X. Then *A* = *NInt* (*A*). Also *NInt* (*A*) ⊆ *NCl* (*NInt* (*A*)). This implies that *A* ⊆ *NCl* (*NInt* (*A*)). Hence by Theorem 3.2 , *A* is *NSO* set in X.

Remark 3.9 The converse of the above theorem need not be true as shown by the following example.

Example 3.10 Let X = { *a* , *b* , *c* } with τ = { 0_N, A, B, 1_N }. Some of the *NSO* sets are
A = ⟨(0.4, 0.5, 0.2), (0.3, 0.2, 0.1), (0.9, 0.6, 0.8)⟩
B = ⟨(0.2, 0.4, 0.5), (0.1, 0.1, 0.2), (0.6, 0.5, 0.8)⟩
C = ⟨(0.5, 0.6, 0.1), (0.4, 0.3, 0.1), (0.9, 0.8, 0.5)⟩
D = ⟨(0.3, 0.5, 0.4), (0.1, 0.6, 0.2), (0.7, 0.5, 0.8)⟩
E = ⟨(0.5, 0.6, 0.1), (0.4, 0.6, 0.1), (0.9, 0.8, 0.5)⟩
F = ⟨(0.3, 0.5, 0.4), (0.1, 0.3, 0.2), (0.7, 0.5, 0.8)⟩
G = ⟨(0.4, 0.5, 0.2), (0.3, 0.6, 0.1), (0.9, 0.6, 0.8)⟩
H = ⟨(0.3, 0.5, 0.4), (0.1, 0.2, 0.2), (0.7, 0.5, 0.8)⟩
I = ⟨(0.4, 0.5, 0.2), (0.3, 0.3, 0.1), (0.9, 0.6, 0.8)⟩
J = ⟨(0.3, 0.5, 0.4), (0.1, 0.2, 0.2), (0.7, 0.5, 0.8)⟩.
Here C, D, E, F, G, H, I and J are *NSO* sets but are not neutrosophic open sets.

Proposition 3.11 If X and Y are *NTS* such that X is neutrosophic product related to Y. Then the neutrosophic product *A* × *B* of a neutrosophic semi-open set *A* of X and a neutrosophic semi-open set *B* of Y is a neutrosophic semi-open set of the neutrosophic product topological space X × Y.

Proof : Let $O_1 \subseteq A \subseteq NCl\,(O_1)$ and $O_2 \subseteq B \subseteq NCl\,(O_2)$ where O_1 and O_2 are neutrosophic open sets in X and Y respectively. Then,$O_1 \times O_2 \subseteq A \times B \subseteq NCl\,(O_1) \times NCl\,(O_2)$. By Theorem 2.17 (i) , $NCl\,(O_1) \times NCl\,(O_2) = NCl\,(O_1 \times O_2)$. Therefore $O_1 \times O_2 \subseteq A \times B \subseteq NCl\,(O_1 \times O_2)$. Hence by Theorem 3.1, A × B is neutrosophic semi-open set in X × Y.

IV. NEUTROSOPHIC SEMI-CLOSED SETS IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, the neutrosophic semi-closed set is introduced and studied their properties.

Definition 4.1 Let *A* be *NS* of a *NTS* X. Then *A* is said to be neutrosophic semi-closed [written *NSC*] set of X if there exists a neutrosophic closed set NC such that *NInt* (NC) ⊆ *A* ⊆ NC.

Theorem 4.2 A subset *A* in a *NTS* X is *NCS* set if and only if *NInt* (*NCl* (*A*)) ⊆ *A*.

Proof : Sufficiency: Let *NInt* (*NCl* (*A*)) ⊆ *A*. Then for NC = *NCl* (*A*), we have *NInt* (NC) ⊆ *A* ⊆ NC. Necessity: Let *A* be *NSC* set in X. Then *NInt* (NC) ⊆ *A* ⊆ NC for some neutrosophic closed set NC. But *NCl* (*A*) ⊆ NC and thus *NInt* (*NCl* (*A*)) ⊆ *NInt* (NC)). Hence *NInt* (*NCl* (*A*)) ⊆ *NInt* (NC) ⊆ *A*.

Proposition 4.3 Let (X, τ) be a *NTS* and *A* be a neutrosophic subset of X. Then *A* is *NSC* set if and only if C (*A*) is *NSO* set in X.

Proof : Let *A* be a neutrosophic semi-closed subset of X. Then by Theorem 4.2 , *NInt* (*NCl* (*A*)) ⊆ *A*. Taking complement on both sides, C (*A*) ⊆ C (*NInt* (*NCl* (*A*))) = *NCl* (C (*NCl* (*A*))). By using Proposition 1.17 (b), C (*A*) ⊆ *NCl* (*NInt* (C (*A*))). By Theorem 3.2, C (*A*) is neutrosophic semi-open. Conversely let C (*A*) is neutrosophic semi-open. By Theorem 3.2 , C (A) $\subseteq$ *NCl* (*NInt* (C (A))). Taking complement on both sides, $A \supseteq$ C (*NCl* (*NInt* (C (A))) = *NInt* (C (*NInt* (C (A))). By using Proposition 1.17 (b), $A \supseteq$ *NInt* (*NCl* (A)). By Theorem 4.2, A is neutrosophic semi-closed set.

Theorem 4.4 Let (X, τ) be a *NTS*. Then intersection of two *NSC* sets is a *NSC* set in the *NTS* X.

Proof : Let A and B are *NSC* sets in X. Then *NInt* (*NCl* (A)) $\subseteq A$ and *NInt* (*NCl* (B)) $\subseteq B$. Therefore $A \cap B \supseteq$ *NInt* (*NCl* (A)) $\cap$ *NInt* (*NCl* (B)) = *NInt* (*NCl* (A) $\cap$ *NCl* (B)) $\supseteq$ *NInt* (*NCl* ($A \cap B$)) [By Proposition 1.18 (n)]. Hence $A \cap B$ is *NSC* set in X.

Theorem 4.5 Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of *NSC* sets in a *NTS* X. Then $\bigcap_{\alpha\in\Delta} A_\alpha$ is *NSC* set in X.

Proof : For each $\alpha\in\Delta$, we have a neutrosophic closed set NC_α such that *NInt* (NC_α) $\subseteq A_\alpha \subseteq NC_\alpha$. Then *NInt* $(\bigcap_{\alpha\in\Delta} NC_\alpha) \subseteq \bigcap_{\alpha\in\Delta} NInt\,(NC_\alpha) \subseteq \bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcap_{\alpha\in\Delta} NC_\alpha$Hence let NC = $\bigcap_{\alpha\in\Delta} NC_\alpha$.

Remark 4.6 The union of any two *NSC* sets need not be a *NSC* set in X as shown by the following example.

Example 4.7 Let X = { a } and

A = ⟨ (1, 0.5, 0.7) ⟩

B = ⟨ (0, 0.9, 0.2) ⟩

C = ⟨ (1, 0.9, 0.2) ⟩

D = ⟨ (0, 0.5, 0.7) ⟩ .

Then τ = { 0_N, A, B, C, D, 1_N } is *NTS* on X. Now, we define the two *NSC* sets as follows :

A_1 = ⟨ (0.4, 0.5, 1) ⟩ and

A_2 = ⟨ (0.2, 0, 0.8) ⟩ . Here *NCl* (A_1) = ⟨ (0.7, 0.5, 1) ⟩, *NInt* (*NCl* (A_1)) = 0_N and *NCl* (A_2) = ⟨ (0.2, 0.1, 0) ⟩, *NInt* (*NCl* (A_2)) = 0_N . But $A_1 \cup A_2$ = ⟨ (0.4, 0.5, 0.8) ⟩ is not a *NSC* set in X.

Theorem 4.8 Let A be *NSC* set in the *NTS* X and suppose *NInt* (A) $\subseteq B \subseteq A$. Then B is *NSC* set in X.

Proof : There exists a neutrosophic closed set NC such that *NInt* (NC) $\subseteq A \subseteq$ NC. Then $B \subseteq$ NC. But *NInt* (NC) $\subseteq$ *NInt* (A) and thus *NInt* (NC) $\subseteq B$. Hence *NInt* (NC) $\subseteq B \subseteq$ NC and B is *NSC* set in X.

Theorem 4.9 Every neutrosophic closed set in the *NTS* X is *NSC* set in X.

Proof : Let A be neutrosophic closed set in *NTS* X. Then A = *NCl* (A). Also *NInt* (*NCl* (A)) $\subseteq$ *NCl* (A). This implies that *NInt* (*NCl* (A)) $\subseteq A$. Hence by Theorem 4.2 , A is *NSC* set in X.

Remark 4.10 The converse of the above theorem need not be true as shown by the following example.

Example 4.11 Let X = { a , b , c } with τ = { 0_N, A, B, 1_N } and C (τ) = { 1_N, C, D, 0_N } where

A = ⟨ (0.5, 0.6, 0.3), (0.1, 0.7, 0.9), (1, 0.6, 0.4) ⟩

B = ⟨ (0, 0.4, 0.7), (0.1, 0.6, 0.9), (0.5, 0.5, 0.8) ⟩

C = ⟨ (0.3, 0.4, 0.5), (0.9, 0.3, 0.1), (0.4, 0.4, 1) ⟩

D = ⟨ (0.7, 0.6, 0), (0.9, 0.4, 0.1), (0.8, 0.5, 0.5) ⟩.

E = ⟨ (0.2, 0.4, 0.9), (0, 0.2, 0.9), (0.3, 0.2, 1) ⟩.

Here the *NSC* sets are C, D and E.

Also E is *NSC* set but is not neutrosophic closed set.

Proposition 4.12 If X and Y are neutrosophic spaces such that X is neutrosophic product related to Y. Then the neutrosophic product $A \times B$ of a neutrosophic semi-closed set A of X and a neutrosophic semi-closed set B of Y is a neutrosophic semi-closed set of the neutrosophic product topological space X × Y.

Proof : Let *NInt* (C_1) $\subseteq A \subseteq C_1$ and *NInt* (C_2) $\subseteq B \subseteq C_2$ where C_1 and C_2 are neutrosophic closed sets in X and Y respectively. Then *NInt* (C_1) × *NInt* (C_2) $\subseteq A \times B \subseteq C_1 \times C_2$. By Theorem 2.17 (ii) , *NInt* (C_1) × *NInt* (C_2) = *NInt* ($C_1 \times C_2$). Therefore *NInt* ($C_1 \times C_2$) $\subseteq A \times B \subseteq C_1 \times C_2$. Hence by Theorem 4.1, A × B is neutrosophic semi-closed set in X × Y.

V. NEUTROSOPHIC SEMI-INTERIOR IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we introduce the neutrosophic semi-interior operator and their properties in neutrosophic topological space.

Definition 5.1 Let (X, τ) be a *NTS*. Then for a neutrosophic subset A of X, the neutrosophic semi-interior of A [*NS Int* (A) for short] is the union of all neutrosophic semi-open sets of X contained in A. That is, *NS Int* (A) = ∪ { G : G is a *NSO* set in X and G $\subseteq A$ }.

Proposition 5.2 Let (X, τ) be a *NTS*. Then for any neutrosophic subsets *A* and *B* of a *NTS* X we have
(i) *NS Int* (*A*) $\subseteq$ *A*
(ii) *A* is *NSO* set in X $\Leftrightarrow$ *NS Int* (*A*) = *A*
(iii) *NS Int* (*NS Int* (*A*)) = *NS Int* (*A*)
(iv) If *A* $\subseteq$ *B* then *NS Int* (*A*) $\subseteq$ *NS Int* (*B*)
Proof : (i) follows from Definition 5.1.
Let *A* be *NSO* set in X. Then *A* $\subseteq$ *NS Int* (*A*). By using (i) we get *A* = *NS Int* (*A*). Conversely assume that *A* = *NS Int* (*A*). By using Definition 5.1, *A* is *NSO* set in X. Thus (ii) is proved.
By using (ii), *NS Int* (*NS Int* (*A*)) = *NS Int* (*A*). This proves (iii).
Since *A* $\subseteq$ *B*, by using (i), *NS Int* (*A*) $\subseteq$ *A* $\subseteq$ *B*. That is *NS Int* (*A*) $\subseteq$ *B*. By (iii), *NS Int* (*NS Int* (*A*)) $\subseteq$ *NS Int* (*B*). Thus *NS Int* (*A*) $\subseteq$ *NS Int* (*B*). This proves (iv).

Theorem 5.3 Let (X, τ) be a *NTS*. Then for any neutrosophic subset *A* and *B* of a *NTS*, we have
(i) *NS Int* (*A* $\cap$ *B*) = *NS Int* (*A*) $\cap$ *NS Int* (*B*)
(ii) *NS Int* (*A* $\cup$ *B*) $\supseteq$ *NS Int* (*A*) $\cup$ *NS Int* (*B*).
Proof : Since *A* $\cap$ *B* $\subseteq$ *A* and *A* $\cap$ *B* $\subseteq$ *B*, by using Proposition 5.2 (iv), *NS Int* (*A* $\cap$ *B*) $\subseteq$ *NS Int* (*A*) and *NS Int* (*A* $\cap$ *B*) $\subseteq$ *NS Int* (*B*). This implies that *NS Int* (*A* $\cap$ *B*) $\subseteq$ *NS Int* (*A*) $\cap$ *NS Int* (*B*) -----(1). By using Proposition 5.2 (i), *NS Int* (*A*) $\subseteq$ *A* and *NS Int* (*B*) $\subseteq$ *B*. This implies that *NS Int* (*A*) $\cap$ *NS Int* (*B*) $\subseteq$ *A* $\cap$ *B*. Now applying Proposition 5.2 (iv), *NS Int* ((*NS Int* (*A*) $\cap$ *NS Int* (*B*)) $\subseteq$ *NS Int* (*A* $\cap$ *B*). By (1), *NS Int* (*NS Int* (*A*)) $\cap$ *NS Int* (*NS Int* (*B*)) $\subseteq$ *NS Int* (*A* $\cap$ *B*). By Proposition 5.2 (iii), *NS Int* (*A*) $\cap$ *NS Int* (*B*) $\subseteq$ *NS Int* (*A* $\cap$ *B*) -----(2). From (1) and (2), *NS Int* (*A* $\cap$ *B*) = *NS Int* (*A*) $\cap$ *NS Int* (*B*). This implies (i).
Since *A* $\subseteq$ *A* $\cup$ *B* and *B* $\subseteq$ *A* $\cup$ *B*, by using Proposition 5.2 (iv), *NS Int* (*A*) $\subseteq$ *NS Int* (*A* $\cup$ *B*) and *NS Int* (*B*) $\subseteq$ *NS Int* (*A* $\cup$ *B*). This implies that *NS Int* (*A*) $\cup$ *NS Int* (*B*) $\subseteq$ *NS Int* (*A* $\cup$ *B*). Hence (ii).

The following example shows that the equality need not be hold in Theorem 5.3 (ii).

Example 5.4 Let X = { *a* , *b* , *c* } and τ = { 0_N, A, B, C, D, 1_N } where
A = $\langle$(0.4, 0.7, 0.1), (0.5, 0.6, 0.2), (0.9, 0.7, 0.3)$\rangle$,
B = $\langle$(0.4, 0.6, 0.1), (0.7, 0.7, 0.2), (0.9, 0.5, 0.1)$\rangle$,
C = $\langle$(0.4, 0.7, 0.1), (0.7, 0.7, 0.2), (0.9, 0.7, 0.1)$\rangle$,
D = $\langle$(0.4, 0.6, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.3)$\rangle$.
Then (X, τ) is a *NTS*. Consider the *NSs* are
E = $\langle$(0.7, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.5, 0)$\rangle$ and F = $\langle$(0.4, 0.6, 0.1), (0.5, 0.7, 0.2), (1, 0.7, 0.1)$\rangle$. Then *NS Int* (E) = D and *NS Int* (F) = D. This implies that *NS Int* (E) $\cup$ *NS Int* (F) = D. Now, E $\cup$ F = $\langle$(0.7, 0.6, 0.1), (0.7, 0.7, 0.1), (1, 0.7, 0)$\rangle$, it follows that *NS Int* (E $\cup$ F) = B. Then *NS Int* (E $\cup$ F) $\not\subseteq$ *NS Int* (E) $\cup$ *NS Int* (F).

VI.NEUTROSOPHIC SEMI-CLOSURE IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we introduce the concept of neutrosophic semi-closure operators in a *NTS*.

Definition 6.1 Let (X, τ) be a *NTS*. Then for a neutrosophic subset *A* of X, the neutrosophic semi-closure of *A* [*NS Cl* (*A*) for short] is the intersection of all neutrosophic semi-closed sets of X contained in *A*. That is, *NS Cl* (*A*) = $\cap$ { K : K is a *NSC* set in X and K $\supseteq$ *A* }.

Proposition 6.2 Let (X, τ) be a *NTS*. Then for any neutrosophic subsets *A* of X,
(i) C (*NS Int* (*A*)) = *NS Cl* (C (*A*)),
(ii) C (*NS Cl* (*A*)) = *NS Int* (C (*A*)).
Proof : By using Definition 5.1, *NS Int* (*A*) = $\cup$ { G : G is a *NSO* set in X and G $\subseteq$ *A* }. Taking complement on both sides, C (NS Int (A)) = C ($\cup$ { G : G is a *NSO* set in X and G $\subseteq$ *A* }) = $\cap$ { C (G) : C (G) is a *NSC* set in X and C (*A*) $\subseteq$ C (G) }. Replacing C (G) by K, we get C (*NS Int* (*A*)) = $\cap$ { K : K is a *NSC* set in X and K $\supseteq$ C (*A*) }. By Definition 6.1, C (*NS Int* (*A*)) = *NS Cl* (C (*A*)). This proves (i).
By using (i), C (*NS Int* (C (*A*))) = *NS Cl* (C (C (*A*))) = *NS Cl* (*A*). Taking complement on both sides, we get *NS Int* (C (*A*)) = C (*NS Cl* (*A*)). Hence proved (ii).

Proposition 6.3 Let (X, τ) be a *NTS*. Then for any neutrosophic subsets *A* and *B* of a *NTS* X we have
(i) *A* $\subseteq$ *NS Cl* (*A*)
(ii) *A* is *NSC* set in X $\Leftrightarrow$ *NS Cl* (*A*) = *A*
(iii) *NS Cl* (*NS Cl* (*A*)) = *NS Cl* (*A*)
(iv) If *A* $\subseteq$ *B* then *NS Cl* (*A*) $\subseteq$ *NS Cl* (*B*)
Proof : (i) follows from Definition 6.1.
Let *A* be *NSC* set in X. By using Proposition 4.3, C (*A*) is *NSO* set in X. By Proposition 6.2 (ii), *NS Int* (C (*A*)) = C (*A*) $\Leftrightarrow$ C (*NS Cl* (*A*)) = C (*A*) $\Leftrightarrow$ *NS Cl* (*A*) = *A*. Thus proved (ii).
By using (ii), *NS Cl* (*NS Cl* (*A*)) = *NS Cl* (*A*) . This proves (iii).
Since *A* $\subseteq$ *B*, C (*B*) $\subseteq$ C (*A*). By using Proposition 5.2 (iv), *NS Int* (C (*B*)) $\subseteq$ *NS Int* (C (*A*)). Taking complement on both sides, C (*NS Int* (C (*B*))) $\supseteq$ C (*NS Int* (C (*A*))). By Proposition 6.2 (ii), *NS Cl* (*A*) $\subseteq$ *NS Cl* (*B*). This proves (iv).

Proposition 6.4 Let A be a neutrosophic set in a *NTS* X. Then $NInt\ (A) \subseteq NS\ Int\ (A) \subseteq A \subseteq NS\ Cl\ (A) \subseteq NCl\ (A)$.

Proof : It follows from the definitions of corresponding operators.

Proposition 6.5 Let (X, τ) be a *NTS*. Then for a neutrosophic subset A and B of a *NTS* X, we have
(i) $NS\ Cl\ (A \cup B) = NS\ Cl\ (A) \cup NS\ Cl\ (B)$ and
(ii) $NS\ Cl\ (A \cap B) \subseteq NS\ Cl\ (A) \cap NS\ Cl\ (B)$.

Proof : Since $NS\ Cl\ (A \cup B) = NS\ Cl\ (C\ (C\ (A \cup B)))$, by using Proposition 6.2 (i), $NS\ Cl\ (A \cup B) = C\ (NS\ Int\ (C\ (A \cup B))) = C\ (NS\ Int\ (C\ (A) \cap C\ (B)))$. Again using Proposition 5.3 (i), $NS\ Cl\ (A \cup B) = C\ (NS\ Int\ (C\ (A)) \cap NS\ Int\ (C\ (B))) = C\ (NS\ Int\ (C\ (A))) \cup C\ (NS\ Int\ (C\ (B)))$. By using Proposition 6.2 (i), $NS\ Cl\ (A \cup B) = NS\ Cl\ (C\ (\ C\ (A))) \cup NS\ Cl\ (C\ (C\ (B))) = NS\ Cl\ (A) \cup NS\ Cl\ (B)$. Thus proved (i). Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by using Proposition 6.3 (iv), $NS\ Cl\ (A \cap B) \subseteq NS\ Cl\ (A)$ and $NS\ Cl\ (A \cap B) \subseteq NS\ Cl\ (B)$. This implies that $NS\ Cl\ (A \cap B) \subseteq NS\ Cl\ (A) \cap NS\ Cl\ (B)$. This proves(ii).

The following example shows that the equality need not be hold in Proposition 6.5 (ii).

Example 6.6 Let X = { a , b , c } with τ = { 0_N, A, B, C, D, 1_N } and C (τ) = { 1_N, E, F, G, H, 0_N } where
A = ⟨(0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2)⟩
B = ⟨(0.4, 0.5, 0.2), (0.8, 0.6, 0.3), (0.9, 0.7, 0.3)⟩
C = ⟨(0.4, 0.5, 0.2), (0.6, 0.6, 0.3), (0.9, 0.5, 0.3)⟩
D = ⟨(0.5, 0.6, 0.1), (0.8, 0.7, 0.1), (0.9, 0.7, 0.2)⟩
E = ⟨(0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9)⟩,
F = ⟨(0.2, 0.5, 0.4), (0.3, 0.4, 0.8), (0.3, 0.3, 0.9)⟩,
G = ⟨(0.2, 0.5, 0.4), (0.3, 0.4, 0.6), (0.3, 0.5, 0.9)⟩,
H = ⟨(0.1, 0.4, 0.5), (0.1, 0.3, 0.8), (0.2, 0.3, 0.9)⟩.
Then (X, τ) is a *NTS*. Consider the *NSs* are
I = ⟨(0.1, 0.2, 0.5), (0.2, 0.3, 0.7), (0.3, 0.3, 1)⟩ and J = ⟨(0.2, 0.4, 0.8), (0.1, 0.2, 0.8), (0.2, 0.5, 0.9)⟩. Then $NS\ Cl$ (I) = G and $NS\ Cl$ (J) = G.
This implies that $NS\ Cl\ (I) \cap NS\ Cl\ (J)$ = G. Now, I ∩ J = ⟨(0.1, 0.2, 0.8), (0.1, 0.2, 0.8), (0.2, 0.3, 1)⟩, it follows that $NS\ Cl\ (I \cap J)$ = H. Then $NS\ Cl\ (I) \cap NS\ Cl\ (J) \nsubseteq NS\ Cl\ (I \cap J)$.

Theorem 6.7 If A and B are *NSs* of *NTSs* X and Y respectively, then
(i) $NS\ Cl\ (A) \times NS\ Cl\ (B) \supseteq NS\ Cl\ (A \times B)$,
(ii) $NS\ Int\ (A) \times NS\ Int\ (B) \subseteq NS\ Int\ (A \times B)$.

Proof : (i) Since $A \subseteq NS\ Cl\ (A)$ and $B \subseteq NS\ Cl\ (B)$, hence $A \times B \subseteq NS\ Cl\ (A) \times NS\ Cl\ (B)$. This implies that $NS\ Cl\ (A \times B) \subseteq NS\ Cl\ (NS\ Cl\ (A) \times NS\ Cl\ (B))$ and From Proposition 4.12, $NS\ Cl\ (A \times B) \subseteq NS\ Cl\ (A) \times NS\ Cl\ (B)$.
(ii) follows from (i) and the fact that $NS\ Int\ (C\ (A)) = C\ (NS\ Cl\ (A))$.

Lemma 6.8 For *NSs* A_i's and B_j's of *NTSs* X and Y respectively, we have
(i) $\cap \{ A_i , B_j \} = \min (\cap A_i , \cap B_j)$;
$\cup \{ A_i , B_j \} = \max (\cup A_i , \cup B_j)$.
(ii) $\cap \{ A_i , 1_N \} = (\cap A_i) \times 1_N$;
$\cup \{ A_i , 1_N \} = (\cup A_i) \times 1_N$.
(iii) $\cap \{ 1_N \times B_j \} = 1_N \times (\cap B_j)$;
$\cup \{ 1_N \times B_j \} = 1_N \times (\cup B_j)$.

Proof : Obvious.

Theorem 6.9 Let (X, τ) and (Y, σ) be *NTSs* such that X is neutrosophic product related to Y. Then for *NSs* A of X and B of Y , we have
(i) $NS\ Cl\ (A \times B) = NS\ Cl\ (A) \times NS\ Cl\ (B)$,
(ii) $NS\ Int\ (A \times B) = NS\ Int\ (A) \times NS\ Int\ (B)$.

Proof : (i) Since $NS\ Cl\ (A \times B) \subseteq NS\ Cl\ (A) \times NS\ Cl\ (B)$ (By Theorem 6.7 (i)) it is sufficient to show that $NS\ Cl\ (A \times B) \supseteq NS\ Cl\ (A) \times NS\ Cl\ (B)$. Let $A_i \in \tau$ and $B_j \in \sigma$. Then $NS\ Cl\ (A \times B) = \langle (x, y), \cap\ C\ (\{ A_i \times B_j \}) : C\ (\{ A_i \times B_j \}) \supseteq A \times B, \cup \{ A_i \times B_j \} : \{ A_i \times B_j \} \subseteq A \times B \rangle = \langle (x, y), \cap (C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j)) : C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j) \supseteq A \times B, \cup (A_i \times 1_N \cap 1_N \times B_j) : A_i \times 1_N \cap 1_N \times B_j \subseteq A \times B \rangle = \langle (x, y), \cap (C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j)) : C\ (A_i) \supseteq A$ or $C\ (B_j) \supseteq B, \cup (A_i \times 1_N \cap 1_N \times B_j) : A_i \subseteq A$ and $B_j \subseteq B \rangle = \langle (x, y), \min (\cap \{ C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j) : C\ (A_i) \supseteq A \}, \cap \{ C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j) : C\ (B_j) \supseteq B \})$ $,\max (\cup \{ A_i \times 1_N \cap 1_N \times B_j : A_i \subseteq A \}, \cup \{ A_i \times 1_N \cap 1_N \times B_j : B_j \subseteq B \}) \rangle$. Since $\langle (x, y), \cap \{ C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j) : C\ (A_i) \supseteq A \}, \cap \{ C\ (A_i) \times 1_N \cup 1_N \times C\ (B_j) : C\ (B_j) \supseteq B \} \rangle \supseteq \langle (x, y), \cap \{ C\ (A_i) \times 1_N : C\ (A_i) \supseteq A\}, \cap \{ 1_N \times C\ (B_j) : C\ (B_j) \supseteq B \} \rangle = \langle (x,y), \cap \{ C\ (A_i) : C\ (A_i) \supseteq A\} \times 1_N , 1_N \times \cap \{ C\ (B_j) : C\ (B_j) \supseteq B\} \rangle = \langle (x, y), NS\ Cl\ (A) \times 1_N , 1_N \times NS\ Cl\ (B) \rangle$ and $\langle (x, y) , \cup \{ A_i \times 1_N \cap 1_N \times B_j : A_i \subseteq A , \cup \{ A_i \times 1_N \cap 1_N \times B_j : B_j \subseteq B \} \rangle \subseteq \langle (x, y), \cup \{A_i \times 1_N : A_i \subseteq A\}, \cup \{1_N \times B_j : B_j \subseteq B\} \rangle = \langle (x, y), \cup \{ A_i : A_i \subseteq A\} \times 1_N , 1_N \times \cup \{ B_j : B_j \subseteq B \} \rangle = \langle (x, y), NS\ Int\ (A) \times 1_N , 1_N \times NS\ Int\ (B) \rangle$, we have $NS\ Cl\ (A \times B) \supseteq \langle (x, y), \min (NS\ Cl\ (A) \times 1_N , 1_N \times NS\ Cl\ (B)), \max (NS\ Int\ (A) \times 1_N , 1_N \times NS\ Int\ (B)) \rangle = \langle (x, y), \min (NS\ Cl\ (A), NS\ Cl\ (B)), \max (NS\ Int\ (A), NS\ Int\ (B)) \rangle = NS\ Cl\ (A) \times NS\ Cl\ (B)$.
(ii) follows from (i).

Theorem 6.10 Let (X, τ) be a *NTS*. Then for a neutrosophic subset A and B of X we have,

(i) $NS\ Cl\ (A) \supseteq A \cup NS\ Cl\ (NS\ Int\ (A))$,
(ii) $NS\ Int\ (A) \subseteq A \cap NS\ Int\ (NS\ Cl\ (A))$,
(iii) $NInt\ (NS\ Cl\ (A)) \subseteq NInt\ (NCl\ (A))$,
(iv) $NInt\ (NS\ Cl\ (A)) \supseteq NInt\ (NS\ Cl\ (NS\ Int\ (A)))$.

Proof : By Proposition 6.3 (i), $A \subseteq NS\ Cl\ (A)$ ----- (1). Again using Proposition 5.2 (i), $NS\ Int\ (A) \subseteq A$. Then $NS\ Cl\ (NS\ Int\ (A)) \subseteq NS\ Cl\ (A)$ ----- (2). By (1) & (2) we have, $A \cup NS\ Cl\ (NS\ Int\ (A)) \subseteq NS\ Cl\ (A)$. This proves (i).

By Proposition 5.2 (i), $NS\ Int\ (A) \subseteq A$ ----- (1). Again using proposition 6.3 (i), $A \subseteq NS\ Cl\ (A)$. Then $NS\ Int\ (A) \subseteq NS\ Int\ (NS\ Cl\ (A))$ ----- (2). From (1) & (2), we have $NS\ Int\ (A) \subseteq A \cap NS\ Int\ (NS\ Cl\ (A))$. This proves(ii).

By Proposition 6.4, $NS\ Cl\ (A) \subseteq NCl\ (A)$. We get $NInt\ (NS\ Cl\ (A)) \subseteq NInt\ (NCl\ (A))$. Hence (iii).

By (i), $NS\ Cl\ (A) \supseteq A \cup NS\ Cl\ (NS\ Int\ (A))$. We have $NInt\ (NS\ Cl\ (A)) \supseteq NInt\ (A \cup NS\ Cl\ (NS\ Int\ (A))\ )$. Since $NInt\ (A \cup B) \supseteq NInt\ (A) \cup NInt\ (B)$, $NInt\ (NS\ Cl\ (A) \supseteq NInt\ (A) \cup NInt\ (NS\ Cl\ (NS\ Int\ (A))) \supseteq NInt\ (NS\ Cl\ (NS\ Int\ (A)))$. Hence (iv).

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