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Neutrosophic structured element

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Abstract

This paper presents a new concept in neutrosophic sets (NS) called neutrosophic structured element (NSE). Based on this concept, we define the operational laws, score function, and some aggregation operators of NS. Finally, as an application of this concept, we propose a decision-making method for a multi-attribute decision making (MADM) problem under NSE information. The results indicate that this concept is a useful tool for dealing with neutrosophic decision problems.

KEYWORDS

aggregation operator, multi-attribute decision making, neutrosophic set, operational laws, score function measure, single valued neutrosophic sets

1 | INTRODUCTION

The theory of uncertainty plays a tremendous role in modelling science and engineering issues. However, there may be an essential inquiry regarding how we can easily define or use the concept of uncertainty inside our mathematical modelling. Worldwide researchers characterized numerous ways to deal with describing them and have made recommendations on the use of the uncertainty theory.

Fuzzy logic is an approach to calculating the values based on “degrees of truth” instead of the usual Boolean “true or false” logic. Zadeh (1965) first introduced the term fuzzy sets (FSs) against certain logic, where the membership degree ($\mu(x)$) is indeed a real number on [0, 1]. After this work, many researchers studied this topic; details of some researches can be observed in (Das, Mandal, & Edalatpanah, 2017a, 2017b; Finol, Guo, & Jing, 2001; Hsu, Tsai, & Wu, 2009; Jain & Haynes 1983; Najafi & Edalatpanah, 2013a, 2013b; Najafi, Edalatpanah, & Dutta, 2016; Wang, Lu, & Liu, 2014; Zadeh, 1977). However, fuzzy sets cannot handle some cases where the membership degree is hard to define by a specific value.

To tackle this knowledge shortage, Atanassov (1986), introduced an extension of the FSs that so-called intuitionistic fuzzy sets (IFSs). Although the theory of IFSs can handle incomplete information in various real-world issues, it cannot address all types of uncertainty, such as inconsistent and indeterminate information.

The neutrosophic set (NS) was therefore suggested by Smarandache (1999) as a great overall structure that generalizes the classical set, FSs (Zadeh, 1965), IFSs (Atanassov, 1986), and their interval versions (Atanassov & Gargov, 1989; Turksen, 1986).

Neutrosophic set can deal with indeterminate, uncertain, and indistinguishable information where the indeterminacy is explicitly quantified and also the truth, falsity and indeterminacy memberships are entirely independent (Smarandache, 2003). Moreover, some generalization of neutrosophic sets, including interval neutrosophic set (Broumi & Smarandache, 2013; Gallego Lupiáñez, 2009; Garg, 2018b; Liu & Shi, 2015; Ye, 2014c), bipolar neutrosophic set (Broumi, Smarandache, Talea, & Bakali, 2016; Deli, Yusuf, Smarandache, & Ali, 2016; Uluçay, Deli, & Şahin, 2018), single-valued neutrosophic set (Abdel-Basset & Mohamed, 2018; Biswas, Pramanik, & Giri, 2016; Chakraborty et al., 2018; Edalatpanah, 2018; Liu & Wang, 2014; Şahin & Küçük, 2015; Ye, 2013, 2014a), simplified neutrosophic sets (Edalatpanah & Smarandache, 2019; Peng, Wang, Wang, Zhang, & Chen, 2016; Ye, 2014b, 2015a), multi-valued neutrosophic set (Ji, Zhang, & Wang, 2018; Peng, Wang, Wu, Wang, & Chen, 2015; Peng, Wang, & Yang, 2017), and neutrosophic linguistic set (Garg, 2018a; Ma et al., 2017; Tian, Wang, Wang, & Zhang, 2017; Wang, Yang, & Li, 2018a; Ye, 2015b) have been presented. There are also various neutrosophic decision-making models such as aggregation operator methods, TOPSIS, projection method, α-cut set method, and so forth (see Abdel-Basset, Manogaran, Gamal, & Smarandache, 2019; Basha, Tharwat, Abdalla, & Hassanien, 2019; Dhingra, Kumar, & Joshi, 2019; Guo & Cheng, 2009; Jha et al., 2019a; Jha, Kumar, Priyadarshini, Smarandache, &

Long, 2019b; Kumar, Edalatpanah, Jha, Broumi, & Dey, 2018, 2019; Rivieccio, 2008; Sert & Avci, 2019; Smarandache & Ali, 2018; Smarandache & Pramanik, 2016; Zhang, Zhang, & Cheng, 2010).

However, some methodologies precisely handle original neutrosophic information, which can easily lead to information loss and potentially lead to biased results. In a strict sense, these methodologies have not drawn far from the traditional decision-making field. Furthermore, the calculation process is sometimes disturbed by parameter ergodicity problems. For instance, the α-cut set strategy requires that the parameter be set to [0, 1], but it is actually not realistic. Furthermore, the comparison of neutrosophic numbers depend primarily on the relationship of truth, falsity, and indeterminacy membership functions, but the formulas are complex. Besides, some approaches for comparing two neutrosophic numbers do not satisfy the rational hypothesis of economic man.

It should be noted that these shortcomings also exist in the fuzzy decision-making methods, and there are three main problems during the application of Zadeh's extension principle (Wang, Jin, Deng, & Wang, 2018b): (a) the combination operation process of subjective weight and objective weight is very complicated. It is challenging to get fuzzy combination weights; (b) it is difficult to achieve the analytic expression of calculation results among fuzzy numbers due to the inherent ergodicity problem of extension principle; (c) precise numbers rather than fuzzy numbers were obtained, which was not consistent with the actual situation.

To solve the shortcomings of the extension principle, Guo presented the theory of a fuzzy structured element (Guo, 2002a, 2002b, 2004). The homeomorphic property between the space of fuzzy numbers and the group of bounded functions that have a similar monotone formal on [−1, 1] is the main feature of a fuzzy structured element (Guo, 2009). This theory was applied to characterize fuzzy numbers and operations among them, avoiding the ergodicity of the extension principle. Moreover, the fuzzy inheritance of the calculation process and the analytic expression of calculation results can be realized. In recent years, the FSE applied in various problems (see Cui & Li, 2019; Deng, Zhou, & Wang, 2014; Dong & Zhu, 2009; Hu, Yang, & Guo, 2008; Li & Lei, 2017; Liu & Guo, 2012; Shu & Mo, 2016; Sun & Guo, 2009a, 2009b; Wang, Guo, Bamakan, & Shi, 2015b; Wang, Guo, & Shi, 2015a; Wang, Jin, et al., 2018b; Wang, Wang, & Chen, 2016; Yan & Bao-fu, 2013; Yue & Yan, 2009; Zhao, Yang, & Wan, 2010).

However, FSE cannot be able to define the dilemma, indeterminacy, and falsity details of a real-life problem. In these situations, some information may also be uncertain, indeterminate, and inconsistent. Considering the truth, falsity, and indeterminacy membership functions for each data in the neutrosophic sets help decision-makers to obtain a better interpretation of information. Moreover, by NS, we can obtain a better representation of reality by considering all aspects of the decision-making process. So, in this study, we extend the theory of fuzzy structured element for neutrosophic sets (NSs) and introduce the concept of neutrosophic structured element (NSE). Furthermore, we describe the ordering of neutrosophic numbers using the NSE, which successfully overcame the above-raised challenges. Moreover, we define the operational laws, score function, and some aggregation operators of NSs.

The paper unfolds as follows: some basic knowledge, concepts, and arithmetic operations on fuzzy structured element theory are discussed in section 2. In section 3, we review some concepts of neutrosophic sets and single-valued neutrosophic. In section 4, we introduce the concept of the neutrosophic structured element and define the operational laws, score function, and some aggregation operators of NSs by NSE. In section 5, we propose a decision-making method for a multi-attribute decision making (MADM) problem under NSE information. Concluding remarks and future directions are provided in section 6.

2 | FUZZY STRUCTURED ELEMENT THEORY

Here, a few fundamental concepts of the fuzzy structured element are reviewed (Deng et al., 2014; Guo, 2004; Wang, Wang, & Chen, 2016; Wang, Jin, et al., 2018b).

Definition 2.1 A fuzzy set E in R (where R is the set of real numbers) is said to be a fuzzy structured element, if

1. $\mu_E(0) = 1$, $\mu_E(1 + 0) = \mu_E(-1 - 0) = 0$,
2. $\mu_E(x)$ is a monotone increasing and right continuous function on $[-1,0]$,
3. $\mu_E(x)$ is monotonic decreasing and left continuous on $(0,1]$,
4. $\mu_E(x) = 0, \forall\, x \in (-\infty, -1) \cup (1, +\infty)$,

where $\mu_E(x)$ is called the membership function of E.

Definition 2.2 E is said to be a regular fuzzy structured element, if $\forall\, x \in (-1, 1)$, $\mu_E(x) > 0$, and the membership function of E on $[-1, 0]$ and $(0, 1]$ be strictly monotone increasing and decreasing, respectively.

Definition 2.3 Suppose that $f(x)$ be a continuous and strictly monotone function on [−1, 1], then with the fuzzy structured element E and its membership function, $f(E)$ is a bounded closed fuzzy number in R and the membership function of $f(E)$ is $\mu_E(f^{-1}(x))$.

Lemma 2.1 For any bounded closed fuzzy number $\bar{A}$ and a given regular fuzzy structured element E, there always exists a monotone bounded function $f: [-1, 1] \to [0, 1]$ such that $\bar{A} = f(E)$.

Lemma 2.2 Consider the fuzzy structured element E with the following membership function:

$$\mu_E(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & \text{others.} \end{cases} \tag{1}$$

Then each triangular fuzzy number $\bar{A} = (a,b,c)$ can be generated by E, with the following function:

$$f(x) = \begin{cases} (b-a)x+b, & -1 \le x \le 0, \\ (c-b)x+b, & 0 \le x \le 1, \\ 0, & \text{others.} \end{cases} \tag{2}$$

So, it is easy to see that $\bar{A} = f(E)$.

Lemma 2.3 For the fuzzy numbers $\bar{A}_i = f_i(E), i = 1,2$, the fuzzy arithmetic operations can be defined as

$$\text{(i) } \bar{A}_1 + \bar{A}_2 = f_1(E) + f_2(E),$$

$$\text{(ii) } \bar{A}_1 - \bar{A}_2 = f_1(E) + f_2^{\tau}(E),$$

$$\text{(iii) } k\bar{A}_1 = |k| f_1^{\tau}(E),$$

when $k \ge 0, f_1^{\tau}(E) = f_1(E)$, and $f_2^{\tau}(E) = f_2(E)$; when $k < 0, f_1^{\tau}(E) = -f_1(-E)$, and $f_2^{\tau}(E) = -f_2(-E)$.

Definition 2.4 Suppose that E be a fuzzy structured element on X, and its membership function is $\mu_E(x)$. Then $\forall \alpha \in (0, 1]$, the *α-level* set of E is defined as $E_\alpha = \{x | \mu_E(x) \ge \alpha\} = [e_\alpha^-, e_\alpha^+]$, where $e_\alpha^- \in [-1,0]$ and $e_\alpha^+ \in [0,1]$.

Lemma 2.4 Suppose that $\bar{A} = f(E)$. If f is a monotonic decreasing function, then for $\alpha \in (0, 1]$, the α-level set of $\bar{A}$ is a closed interval on R and it can be denoted as $\bar{A}_\alpha = [f(e_\alpha^+), f(e_\alpha^-)]$. If f is a monotonic increasing function, then $\bar{A}_\alpha = [f(e_\alpha^-), f(e_\alpha^+)]$.

3 | NEUTROSOPHIC SETS

In this section, we recall some definitions and key concepts related to the neutrosophic sets and single-valued neutrosophic numbers that are crucial to the comprehension of this paper.

Definition 3.1 Neutrosophic set (Smarandache, 1999, 2003). A neutrosophic set $\bar{U}$ in $X \subset R$ (where R is the set of real numbers) is a set such that

$$\bar{U} = \{(x, < T_{\bar{u}}(x), I_{\bar{u}}(x), F_{\bar{u}}(x) >) | x \in X\},$$

where $T_{\bar{u}}(x): X \to]0^-, 1^+[$, $I_{\bar{u}}(x): X \to]0^-, 1^+[$, and $F_{\bar{u}}(x): X \to]0^-, 1^+[$ is called the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. Also,

$$0^{-} \le \sup T_{\tilde{u}}(x) + \sup I_{\tilde{u}}(x) + \sup F_{\tilde{u}}(x) \le 3^{+}.$$

Definition 3.2 If the truth, indeterminacy, and falsity membership functions in Definition 3.1 are singleton subintervals/subsets in the real standard [0, 1], then we have a special class of NS that called the single-valued neutrosophic set (SVNS) which satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ (Ye, 2013).

Definition 3.3 For SVSs A and B, $A \subseteq B$ if and only if for every x in X : $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, and $F_A(x) \ge F_B(x)$ (Ye, 2013).

Definition 3.4 Let A and B be two SVNSs. Then the arithmetic relations are given as (Liu & Wang, 2014):

$$\text{(i) } A \oplus B = < T_A(x) + T_B(x) - T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x) >, \tag{3}$$

$$\text{(ii) } A \otimes B = < T_A(x)T_B(x), I_A(x) + I_B(x) - I_A(x).I_B(x), F_A(x) + F_B(x) - F_A(x).F_B(x) >, \tag{4}$$

$$\text{(iii) } \lambda A = < 1 - (1 - T_A(x))^{\lambda}, (I_A(x))^{\lambda}, (F_A(x))^{\lambda} >, \lambda > 0. \tag{5}$$

$$\text{(iv) } A^{\lambda} = < T_A^{\lambda}(x), 1 - (1 - I_A(x))^{\lambda}, 1 - (1 - F_A(x))^{\lambda} >, \quad \lambda > 0. \tag{6}$$

Definition 3.5 Triangular single valued neutrosophic number (TSVNN) is defined as $A^{\aleph} = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle$, whose truth membership function $T_{A^{\aleph}}(x)$, indeterminacy-membership function $I_{A^{\aleph}}(x)$, and falsity-membership function $F_{A^{\aleph}}(x)$ are given as follows (Chakraborty et al., 2018):

$$T_{A^{\aleph}}(x) = \begin{cases} \dfrac{(x - a_1)}{(a_2 - a_1)} & a_1 \le x < a_2, \\ 1 & x = a_2, \\ \dfrac{(a_3 - x)}{(a_3 - a_2)} & a_2 < x \le a_3, \\ 0 & \text{otherwise.} \end{cases}$$

$$I_{A^{\aleph}}(x) = \begin{cases} \dfrac{(b_2 - x)}{(b_2 - b_1)} & b_1 \le x < b_2, \\ 0 & x = b_2, \\ \dfrac{(x - b_2)}{(b_3 - b_2)} & b_2 < x \le b_3, \\ 1 & \text{otherwise.} \end{cases}$$

$$F_{A^{\aleph}}(x) = \begin{cases} \dfrac{(c_2 - x)}{(c_2 - c_1)} & c_1 \le x < c_2, \\ 0 & x = c_2, \\ \dfrac{(x - c_2)}{(c_3 - c_2)} & c_2 < x \le c_3, \\ 1 & \text{otherwise.} \end{cases}$$

where $0 \le T_{A^{\aleph}}(x) + I_{A^{\aleph}}(x) + F_{A^{\aleph}}(x) \le 3, x \in A^{\aleph}$.

Definition 3.6 An interval-valued neutrosophic set (IVNS) A in X can be defined as (Smarandache & Pramanik, 2016)

$$A = \{(x, [\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)]) | x \in X\},$$

where

$$\begin{cases} T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0,1], \\ I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0,1], \\ F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0,1], \end{cases}$$

and also satisfies the condition $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$.

Definition 3.7 The operations between two IVNS of $A = \langle [\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)] \rangle$ and $B = \langle [\inf T_B(x), \sup T_B(x)], [\inf I_B(x), \sup I_B(x)], [\inf F_B(x), \sup F_B(x)] \rangle$ can be defined as follow (Ye, 2014c):

$$\text{(i) } A \oplus B = \langle [\inf T_A(x) + \inf T_b(x) - \inf T_A(x).\inf T_b(x), \sup T_A(x) + \sup T_B(x) - \sup T_A(x).\sup T_B(x)],$$
$$[\inf I_A(x).\inf I_b(x), \sup I_A(x).\sup I_B(x)],$$
$$[\inf F_A(x).\inf F_b(x), \sup F_A(x).\sup F_B(x)] \rangle,$$

$$\text{(ii) } A \otimes B = \langle [\inf T_A(x).\inf T_b(x), \sup T_A(x).\sup T_B(x)],$$
$$[\inf I_A(x) + \inf I_b(x) - \inf I_A(x).\inf I_b(x), \sup I_A(x) + \sup I_B(x) - \sup I_A(x).\sup I_B(x)],$$
$$[\inf F_A(x) + \inf F_b(x) - \inf F_A(x).\inf F_b(x), \sup F_A(x) + \sup F_B(x) - \sup F_A(x).\sup F_B(x)] \rangle,$$

$$\text{(iii) } \lambda A = \langle [1 - (1 - \inf T_A(x))^{\lambda}, 1 - (1 - \sup T_A(x))^{\lambda}],$$
$$\left[(\inf I_A(x))^{\lambda}, (\sup I_A(x))^{\lambda} \right], [(\inf F_A(x))^{\lambda}, (\sup F_A(x))^{\lambda}] \rangle, \lambda > 0,$$

$$\text{(iv) } A^{\lambda} = \langle [(\inf T_A(x))^{\lambda}, (\sup T_A(x))^{\lambda}],$$
$$\left[1 - (1 - \inf I_A(x))^{\lambda}, 1 - (1 - \sup I_A(x))^{\lambda} \right], [1 - (1 - \inf F_A(x))^{\lambda}, 1 - (1 - \sup F_A(x))^{\lambda}] \rangle, \lambda > 0.$$

4 | NEUTROSOPHIC STRUCTURED ELEMENT

Here, we extend the theory of fuzzy structured element for the single-valued neutrosophic set (SVNS) and introduce the concept of neutrosophic structured element (NSE).

Definition 4.1 Consider the TSVNN of $A = \{ (x, T_{A^\aleph}(x), I_{A^\aleph}(x), F_{A^\aleph}(x)) | x \in X \}$, where $T_{A^\aleph}(x) = (a_1, a_2, a_3)$, $I_{A^\aleph}(x) = (b_1, b_2, b_3)$, and $F_{A^\aleph}(x) = (c_1, c_2, c_3)$. Then from Lemma 2.2, for $T_{A^\aleph}(x), I_{A^\aleph}(x)$, and $F_{A^\aleph}(x)$, we can obtain three monotone bounded functions $f, g, h : [-1, 1] \to [0, 1]$, such that $T_{A^\aleph}(x) = f_x(E)$, $I_{A^\aleph}(x) = g_x(E)$, and $F_{A^\aleph}(x) = h_x(E)$.

We call that

$$f_x(E) = \begin{cases} (a_2 - a_1)x + a_2, & -1 \le x \le 0, \\ (a_3 - a_2)x + a_2, & 0 \le x \le 1, \\ 0, & \text{others}, \end{cases} \tag{7}$$

$$g_x(E) = \begin{cases} (b_2 - b_1)x + b_2, & -1 \le x \le 0, \\ (b_3 - b_2)x + b_2, & 0 \le x \le 1, \\ 0, & \text{others}, \end{cases} \tag{8}$$

$$h_x(E) = \begin{cases} (c_2 - c_1)x + c_2, & -1 \le x \le 0, \\ (c_3 - c_2)x + c_2, & 0 \le x \le 1, \\ 0, & \text{others}, \end{cases} \tag{9}$$

are the neutrosophic structured elements (NSEs). Also, $A = \langle f_A(E), g_A(E), h_A(E) \rangle$ is the neutrosophic structured elements number (NSEN), and $A = \{(x, f_x(E), g_x(E), h_x(E)) | \ x \in X\}$ is the neutrosophic structured elements set (NSES).

Definition 4.2 Suppose that $A = \{(x, f_x(E), g_x(E), h_x(E)) | x \in X\}$ be an NSES on X, where the truth, indeterminacy, and falsity-membership functions of $f_x(E)$, $g_x(E)$, and $h_x(E)$ are $\mu_E(x)$, $\gamma_E(x)$, and $\vartheta_E(x)$, respectively. Then $\forall \alpha \in (0, 1]$, we called the $A_\alpha = \{(x, [f_x(E)]_\alpha, [g_x(E)]_\alpha, [h_x(E)]_\alpha) | x \in X\}$ as the *α-level set* of A, where $[f_x(E)]_\alpha = \{x | \mu_E(x) \geq \alpha\}$, $[g_x(E)]_\alpha = \{x | \gamma_E(x) \geq \alpha\}$, and $[h_x(E)]_\alpha = \{x | \vartheta_E(x) \geq \alpha\}$.

We can see that the α-level set of NSES A is an interval-valued neutrosophic set. Also, for the $A = \{(x, f'_x(E), g'_x(E), h'_x(E)) | x \in X\}$ and $B = \{(x, f''_x(E), g''_x(E), h''_x(E)) | x \in X\}$, where $f'_x(E)$, $f''_x(E)$, $g'_x(E)$, $g''_x(E)$, $h'_x(E)$, and $h''_x(E)$ are the same monotone formal functions on [−1, 1] to [0, 1], we have the following theorem.

Theorem 4.1 For every two NSESs $A = \{(x, f'_x(E), g'_x(E), h'_x(E)) | x \in X\}$ and $B = \{(x, f''_x(E), f''_x(E), g''_x(E), h''_x(E)) | x \in X\}$, where $f'_x(E)$, $f''_x(E)$, $g'_x(E)$, $g''_x(E)$, $h'_x(E)$, and $h''_x(E)$ are the same monotone formal functions on *[−1, 1]* to *[0, 1]*, we have:

(i) $A \subseteq B$ if and only $\forall y \in [-1, 1] : f'_x(y) \leq f''_x(y), g'_x(y) \geq g''_x(y)$, and $h'_x(y) \geq h''_x(y)$,

(ii) $A = B$ if and only $A \subset B$ and $A \supset B$,

(iii) $A \cap B = \{(x, (f'_x \wedge f''_x)(E), (g'_x \vee g''_x)(E), (h'_x \vee h''_x)(E)) | x \in X\}$,

(iv) $A \cup B = \{(x, (f'_x \vee f''_x)(E), (g'_x \wedge g''_x)(E), (h'_x \wedge h''_x)(E)) | x \in X\}$,

(v) $A^c = \{(x, h'_x(E), 1 - g'_x(E), f'_x(E)) | x \in X\}$.

Proof From Lemma 2.4 and Definition 3.3, (i) is correct. Based on (i), it is evident that (ii) is correct. So, we will consider (iii).

Let $f'_x(E)$, $f''_x(E)$, $g'_x(E)$, $g''_x(E)$, $h'_x(E)$, and $h''_x(E)$ are the same monotone increasing functions on [−1, 1] to [0, 1]. Then $\forall \alpha \in (0, 1]$, we have

$$\min\left(\inf[f'_x(E)]_\alpha, \inf[f''_x(E)]_\alpha\right) = \min\left(\inf[f'_x(e^-_\alpha), f'_x(e^+_\alpha)], \inf[f''_x(e^-_\alpha), f''_x(e^+_\alpha)]\right) = \min(f'_x(e^-_\alpha), f''_x(e^-_\alpha)) = f'_x(e^-_\alpha) \wedge f''_x(e^-_\alpha).$$

Also,

$$\min\left(\sup[f'_x(E)]_\alpha, \sup[f''_x(E)]_\alpha\right) = \min\left(\sup[f'_x(e^-_\alpha), f'_x(e^+_\alpha)], \sup[f''_x(e^-_\alpha), f''_x(e^+_\alpha)]\right) = \min(f'_x(e^+_\alpha), f''_x(e^+_\alpha)) = f'_x(e^+_\alpha) \wedge f''_x(e^+_\alpha).$$

Therefore, $\forall \alpha \in (0, 1]$, it follows that:

$$\left[\min\left(\inf[f'_x(E)]_\alpha, \inf[f''_x(E)]_\alpha\right), \min\left(\sup[f'_x(E)]_\alpha, \sup[f''_x(E)]_\alpha\right)\right) = [f'_x(e^-_\alpha) \wedge f''_x(e^-_\alpha), f'_x(e^+_\alpha) \wedge f''_x(e^+_\alpha)] = [f'_x(E) \wedge f''_x(E)]_\alpha = (f'_x \wedge f''_x)(E_\alpha).$$

Similarly,

$$\begin{cases} \max\left(\inf[g'_x(E)]_\alpha, \inf[g''_x(E)]_\alpha\right) = g'_x(e^-_\alpha) \vee g''_x(e^-_\alpha), \\ \max\left(\sup[g'_x(E)]_\alpha, \sup[g''_x(E)]_\alpha\right) = g'_x(e^+_\alpha) \vee g''_x(e^+_\alpha). \end{cases}$$

Therefore, $\forall \alpha \in (0, 1]$, it follows that:

$$\left[\max\left(\inf\left[g_x'(E)\right]_\alpha,\inf\left[g_x''(E)\right]_\alpha\right),\max\left(\sup\left[g_x'(E)\right]_\alpha,\sup\left[g_x''(E)\right]_\alpha\right)\right) = \\ \left[g_x'(e_\alpha^-)\vee g_x''(e_\alpha^-),g_x'(e_\alpha^+)\vee g_{xx}''(e_\alpha^+)\right] = \left[g_x'(E)\vee g_{xx}''(E)\right]_\alpha = \left(g_x'\vee g_x''\right)(E_\alpha).$$

Analogously,

$$\left[\max\left(\inf\left[h_x'(E)\right]_\alpha,\inf\left[h_x''(E)\right]_\alpha\right),\max\left(\sup\left[h_x'(E)\right]_\alpha,\sup\left[h_x''(E)\right]_\alpha\right)\right) = \\ \left[h_x'(e_\alpha^-)\vee h_x''(e_\alpha^-),h_x'(e_\alpha^+)\vee h_x''(e_\alpha^+)\right] = \left[h_x'(E)\vee h_x''(E)\right]_\alpha = \left(h_x'\vee h_x''\right)(E_\alpha).$$

So,

$$(A\cap B)_\alpha = \left\{\left(x,\left(f_x'\wedge f_x''\right)(E),\left(g_x'\vee g_x''\right)(E),\left(h_x'\vee h_x''\right)(E)\right)|x\in X\right\}.$$

Therefore, (iii) is correct. Furthermore, (iv) and (v) can prove similarly. □

Next, based on Definition 3.7, we give the corresponding operational laws of NSE.

Theorem 4.2 For every two NSESs A and B we have

(i) $A\oplus B = \left\{\left(x,f_x'(E)+f_x''(E)-f_x'(E)f_x''(E),g_x'(E)g_x''(E),h_x'(E)h_x''(E)\right)|x\in X\right\} = \left\{\left(x,\left(f_x'+f_x''-f_x'f_x''\right)(E),\left(g_x'g_x''\right)(E),\left(h_x'h_x''\right)(E)\right)|x\in X\right\} = \left\{\left(x,\left(f_x'+f_x''+\left(f_x'f_x''\right)^\tau\right)(E),\left(g_x'g_x''\right)(E),\left(h_x'h_x''\right)(E)\right)|x\in X\right\}$,

(ii) $A\otimes B = \left\{\left(x,f_x'(E)f_x''(E),g_x'(E)+g_x''(E)-g_x'(E)g_x''(E),h_x'(E)+h_x''(E)-h_x'(E)h_x''(E)\right)|x\in X\right\} = \left\{\left(x,\left(f_x'f_x''\right)(E),\left(g_x'+g_x''-g_x'g_x''\right)(E),\left(h_x'+h_x''-h_x'h_x''\right)(E)\right)|x\in X\right\} = \left\{\left(x,\left(f_x'f_x''\right)(E),\left(g_x'+g_x''+\left(g_x'g_x''\right)^\tau\right)(E),\left(h_x'+h_x''+\left(h_x'h_x''\right)^\tau\right)(E)\right)|x\in X\right\}$,

(iii) $\lambda A = \left\{\left(x,1-\left[\left(1-f_x'(E)\right)\right]^\lambda,\left[g_x'(E)\right]^\lambda,\left[h_x'(E)\right]^\lambda\right)|x\in X\right\},\lambda>0$,

(iv) $A^\lambda = \left\{\left(x,\left[f_x'(E)\right]^\lambda,\left[1-\left(1-g_x'(E)\right)\right]^\lambda,\left[1-\left(1-h_x'(E)\right)\right]^\lambda\right)|x\in X\right\},\lambda>0$.

Proof Suppose that the expression of NSE of two SVNSs $A = \{(x, T_A(x), I_A(x), F_A(x) \mid x\in X\}$ and $B = \{(x, T_B(x), I_B(x), F_B(x))\mid x\in X\}$ be $A = \left\{\left(x,f_x'(E),g_x'(E),h_x'(E)\right)|x\in X\right\}$ and $B = \left\{\left(x,f_x''(E),g_x''(E),h_x''(E)\right)|x\in X\right\}$, where $f_x'(E)$, $f_x''(E)$, $g_x'(E)$, $g_x''(E)$, $h_x'(E)$, and $h_x''(E)$ are the same monotone increasing functions on [−1, 1] to [0, 1]. Then from Definition 4.2, $\forall\alpha\in(0, 1]$, we get,

$$\begin{cases}\left[f_x'(E)\right]_\alpha = f_x'(E_\alpha) = \left[f_x'(e_\alpha^-),f_x'(e_\alpha^+)\right], & \left[f_x''(E)\right]_\alpha = f_x''(E_\alpha) = \left[f_x''(e_\alpha^-),f_x''(e_\alpha^+)\right],\\ \left[g_x'(E)\right]_\alpha = g_x'(E_\alpha) = \left[g_x'(e_\alpha^-),g_x'(e_\alpha^+)\right], & \left[g_x''(E)\right]_\alpha = g_x''(E_\alpha) = \left[g_x''(e_\alpha^-),g_x''(e_\alpha^+)\right],\\ \left[h_x'(E)\right]_\alpha = h_x'(E_\alpha) = \left[h_x'(e_\alpha^-),h_x'(e_\alpha^+)\right], & \left[h_x''(E)\right]_\alpha = h_x''(E_\alpha) = \left[h_x''(e_\alpha^-),h_x''(e_\alpha^+)\right].\end{cases} \tag{10}$$

Then,

$$\begin{cases}\inf f_x'(E_\alpha) = f_x'(e_\alpha^-),\sup f_x'(E_\alpha) = f_x'(e_\alpha^+), & \inf f_x''(E_\alpha) = f_x''(e_\alpha^-),\sup f_x''(E_\alpha) = f_x''(e_\alpha^+),\\ \inf g_x'(E_\alpha) = g_x'(e_\alpha^-),\sup g_x'(E_\alpha) = g_x'(e_\alpha^+), & \inf g_x''(E_\alpha) = g_x''(e_\alpha^-),\sup g_x''(E_\alpha) = g_x''(e_\alpha^+),\\ \inf h_x'(E_\alpha) = h_x'(e_\alpha^-),\sup h_x'(E_\alpha) = h_x'(e_\alpha^+), & \inf h_x''(E_\alpha) = h_x''(e_\alpha^-),\sup h_x''(E_\alpha) = h_x''(e_\alpha^+).\end{cases} \tag{11}$$

From Equation (11) and Definition 3.7 (i),

$$\begin{cases} \inf[T_A(x)]_\alpha + \inf[T_B(x)]_\alpha - \inf[T_A(x)]_\alpha \inf[T_B(x)]_\alpha = \\ \inf f'_x(E_\alpha) + \inf f''_x(E_\alpha) - \inf f'_x(E_\alpha)\inf f''_x(E_\alpha) = f'_x(e_\alpha^-) + f''_x(e_\alpha^-) - f'_x(e_\alpha^-)f''_x(e_\alpha^-), \end{cases}$$

and

$$\begin{cases} \sup[T_A(x)]_\alpha + \sup[T_B(x)]_\alpha - \sup[T_A(x)]_\alpha \sup[T_B(x)]_\alpha = \\ \sup f'_x(E_\alpha) + \sup f''_x(E_\alpha) - \sup f'_x(E_\alpha)\sup f''_x(E_\alpha) = f'_x(e_\alpha^+) + f''_x(e_\alpha^+) - f'_x(e_\alpha^+)f''_x(e_\alpha^+). \end{cases}$$

So, according to Lemmas 2.3 and 2.4, we have

$$\begin{aligned} &\left[\inf[T_A(x)]_\alpha + \inf[T_B(x)]_\alpha - \inf[T_A(x)]_\alpha \inf[T_B(x)]_\alpha, \sup[T_A(x)]_\alpha + \sup[T_B(x)]_\alpha - \sup[T_A(x)]_\alpha \sup[T_B(x)]_\alpha\right] = \\ &\left[\inf f'_x(E_\alpha) + \inf f''_x(E_\alpha) - \inf f'_x(E_\alpha) + \inf f''_x(E_\alpha), \sup f'_x(E_\alpha) + \sup f''_x(E_\alpha) - \sup f'_x(E_\alpha)\sup f''_x(E_\alpha)\right] = \\ &\left[\, f'_x(e_\alpha^-) + f''_x(e_\alpha^-) - f'_x(e_\alpha^-) + f''_x(e_\alpha^-), f'_x(e_\alpha^+) + f''_x(e_\alpha^+) - f'_x(e_\alpha^+)f''_x(e_\alpha^+)\right] = \\ &\left[\, f'_x(E) + f''_x(E) - f'_x(E)f''_x(E)\right]_\alpha = \left[\, f'_x + f''_x + \left(\, f'_x f''_x\right)^\tau\right](E)_\alpha. \end{aligned}$$

Analogously,

$$\begin{aligned} &\left[\inf[I_A(x)]_\alpha \inf[I_B(x)]_\alpha, \sup[I_A(x)]_\alpha \sup[I_B(x)]_\alpha\right] \\ &\left[\inf g'_x(E_\alpha)\inf g''_x(E_\alpha), \sup g'_x(E_\alpha)\sup g''_x(E_\alpha)\right] = \left[g'_x(e_\alpha^-)g''_x(e_\alpha^-), g'_x(e_\alpha^+)g''_x(e_\alpha^+)\right] = \\ &\left[g'_x(E) + g''_x(E)\right]_\alpha = \left[g'_x g''_x\right](E)_\alpha, \end{aligned}$$

and

$$\begin{aligned} &\left[\inf[F_A(x)]_\alpha \inf[F_B(x)]_\alpha, \sup[F_A(x)]_\alpha \sup[F_B(x)]_\alpha\right] \\ &\left[\inf h'_x(E_\alpha)\inf h''_x(E_\alpha), \sup h'_x(E_\alpha)\sup h''_{xx}(E_\alpha)\right] = \left[h'_x(e_\alpha^-)h''_x(e_\alpha^-), h'_x(e_\alpha^+)h''_{xx}(e_\alpha^+)\right] = \\ &\left[h'_{xx}(E) + h''_x(E)\right]_\alpha = \left[h'_x h''_x\right](E)_\alpha. \end{aligned}$$

Therefore, $\forall \alpha \in (0, 1]$, it follows that:

$$(A \oplus B)_\alpha = \left\{\left(x, \left(\, f'_x + f''_x + \left(\, f'_x f''_x\right)^\tau\right)(E_\alpha), \left(g'_x g''_x\right)(E_\alpha), \left(h'_x h''_x\right)(E_\alpha)\right) | x \in X\right\}.$$

Therefor (i) is correct. The proof of Equations (ii), (iii), and (iv) are similar to (i). □

Theorem 4.3 Let $A = \left\{\left(x, f'_x(E), g'_x(E), h'_x(E)\right) | x \in X\right\}$ and $B = \left\{\left(x, f''_x(E), g''_x(E), h''_x(E)\right) | x \in X\right\}$, be two NSESs, where $f'_x(E)$, $f''_x(E)$, $g'_x(E)$, $g''_x(E)$, $h'_x(E)$, and $h''_x(E)$ are the same monotone increasing functions on *[−1, 1]* to *[0, 1]*. Then*:*

$$(1)\ A \cap B = B \cap A,$$

$$(2)\ A \cup B = B \cup A,$$

$$(3)\ A \oplus B = B \oplus A,$$

$$(4)\ A \otimes B = B \otimes A,$$

$$(5)\ \lambda(A \oplus B) = \lambda A \oplus \lambda B, \lambda > 0,$$

$$(6)\ (A \oplus B)^\lambda = A^\lambda \otimes B^\lambda, \lambda > 0,$$

$$(7)\ \lambda_1 A \oplus \lambda_2 A = (\lambda_1 + \lambda_2)A, \lambda_1, \lambda_2 > 0,$$

$$(8)\ A^{\lambda_1} \otimes A^{\lambda_2} = A^{(\lambda_1 + \lambda_2)}, \lambda_1, \lambda_2 > 0.$$

Proof From Theorem 4.1, it is evident that that (1) and (2) are true.

For (3), from Theorem 4.2:

$$A\oplus B=\left\{\left(x,\left(f'_x+f''_x+\left(f'_xf''_x\right)^{\tau}\right)(E),\left(g'_xg''_x\right)(E),\left(h'_xh''_x\right)(E)\right)|x\in X\right\}=\left\{\left(x,\left(f''_x+f'_x+\left(f''_xf'_x\right)^{\tau}\right)(E),\left(g''_xg'_x\right)(E),\left(h''_xh'_x\right)(E)\right)|x\in X\right\}=B\oplus A. \tag{12}$$

Proof (4) is similar to the proof (3).
For (5), from Theorem 4.2:

$$\lambda A=\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda},\left[g'_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\right)|\,x\in X\right\},\lambda>0,$$

$$\lambda B=\left\{\left(x,1-\left[\left(1-f''_x(E)\right)\right]^{\lambda},\left[g''_x(E)\right]^{\lambda},\left[h''_x(E)\right]^{\lambda}\right)|\,x\in X\right\},\lambda>0.$$

Then, from (12):

$$\lambda(A\oplus B)=\left\{\left(x,\left[1-\left(1-\left[f'_x+f''_x+\left(f'_xf''_x\right)^{\tau}(E)\right]\right)\right]^{\lambda},\left[g'_x(E)\right]^{\lambda}\left[g''_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\left[h''_x(E)\right]^{\lambda}\right)|\,x\in X\right\}.$$

Also,

$$\begin{aligned}&\lambda A\oplus\lambda B=\{(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda}+1-\left[\left(1-f''_x(E)\right)\right]^{\lambda}-\left(1-\left[\left(1-f'_x(E)\right)\right]^{\lambda}\right)\left(1-\left[\left(1-f''_x(E)\right)\right]^{\lambda}\right),\\&\left[g'_x(E)\right]^{\lambda}\left[g''_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\left[h''_x(E)\right]^{\lambda})\,|\,x\in X\}=\\&\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda}\left[\left(1-f''_x(E)\right)\right]^{\lambda},\left[g'_x(E)\right]^{\lambda}\left[g''_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\left[h''_x(E)\right]^{\lambda}\right)|\,x\in X\right\}=\\&\left\{\left(x,1-\left[\left(1-f'_x(E)-f''_x(E)+f'_x(E)f''_x(E)\right)\right]^{\lambda},\left[g'_x(E)\right]^{\lambda}\left[g''_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\left[h''_x(E)\right]^{\lambda}\right)|\,x\in X\right\}=\\&\left\{\left(x,\left[1-\left(1-\left[f'_x+f''_x+\left(f'_xf''_x\right)^{\tau}(E)\right]\right)\right]^{\lambda},\left[g'_x(E)\right]^{\lambda}\left[g''_x(E)\right]^{\lambda},\left[h'_x(E)\right]^{\lambda}\left[h''_x(E)\right]^{\lambda}\right)|\,x\in X\right\}.\end{aligned}$$

Therefore,

$$\lambda(A\oplus B)=\lambda A\oplus\lambda B.$$

Proof (6) is similar to the proof (5).
For (7), $\forall\lambda_1,\lambda_2>0$:

$$\lambda_1A=\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_1},\left[g'_x(E)\right]^{\lambda_1},\left[h'_x(E)\right]^{\lambda_1}\right)|\,x\in X\right\},$$

$$\lambda_2A=\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_2},\left[g'_x(E)\right]^{\lambda_2},\left[h'_x(E)\right]^{\lambda_2}\right)|\,x\in X\right\}.$$

Then,

$$\begin{aligned}&\lambda_1A\oplus\lambda_2A=\{(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_1}+1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_2}-\left(1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_1}\right)\left(1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_2}\right),\\&\left[g'_x(E)\right]^{\lambda_1}\left[g'_x(E)\right]^{\lambda_2},\left[h'_x(E)\right]^{\lambda_1}\left[h'_x(E)\right]^{\lambda_2})\,|\,x\in X\}=\\&\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{\lambda_1}\left[\left(1-f'_x(E)\right)\right]^{\lambda_2},\left[g'_x(E)\right]^{(\lambda_1+\lambda_2)},\left[h'_x(E)\right]^{(\lambda_1+\lambda_2)}\right)|\,x\in X\right\}=\\&\left\{\left(x,1-\left[\left(1-f'_x(E)\right)\right]^{(\lambda_1+\lambda_2)},\left[g'_x(E)\right]^{(\lambda_1+\lambda_2)},\left[h'_x(E)\right]^{(\lambda_1+\lambda_2)}\right)|\,x\in X\right\}=(\lambda_1+\lambda_2)A.\end{aligned}$$

So,

$$\lambda_1A\oplus\lambda_2A=(\lambda_1+\lambda_2)A,\forall\lambda_1,\lambda_2>0.$$

Proof (8) is similar to the proof (7), and the proof is complete. □

For convenience, an NSE number $A = \{(x, f_x(E), g_x(E), h_x(E)) | x \in X\}$ is denoted by $A = \langle f_A(E), g_A(E), h_A(E) \rangle$.

Example 4.1 Consider two TSVNNs as follow:

$$A = \langle (0.5, 0.6, 0.7), (0.1, 0.2, 0.3), (0.3, 0.4, 0.5) \rangle,$$

$$B = \langle (0.4, 0.5, 0.6), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7) \rangle.$$

First, based on Equations (7)–(9), we convert these numbers into the NSE numbers. For $-1 \le x \le 1$, we get,

$$A = \langle (0.1x + 0.6), (0.1x + 0.2), (0.1x + 0.4) \rangle,$$

$$B = \langle (0.1x + 0.5), (0.1x + 0.3), (0.1x + 0.6) \rangle.$$

Figures 1 and 2 show the graphical representation of these two TSVNNs and the related NSENs.

Next, we test the operational laws of these two NSENs. So,

$$A \oplus B = \left\langle -\frac{1}{100}(x^2 - 9x - 80), \frac{1}{100}(x^2 + 5x6), \frac{1}{500}(x + 6)(5x + 2) \right\rangle,$$

$$A \otimes B = \left\langle \frac{1}{100}(x + 5)(x + 6), -\frac{1}{100}(x^2 - 15x - 44), -\frac{1}{500}(5x^2 - 68x - 308) \right\rangle,$$

$$2A = \left\langle -\frac{1}{100}(x + 6)(x - 14), \frac{1}{100}(x + 2)^2, \frac{1}{2500}(5x + 2)^2 \right\rangle,$$

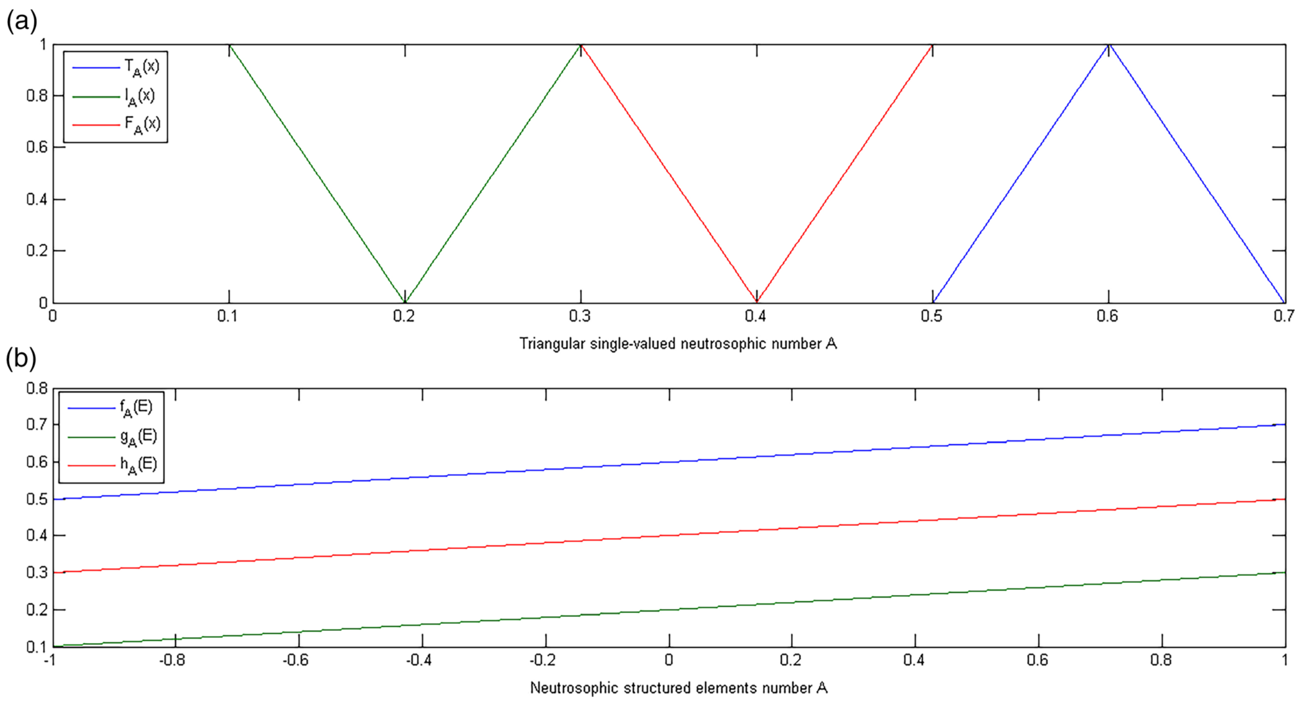

FIGURE 1 TSVNN and the related NSEN $A = < (0.5, 0.6, 0.7), (0.1, 0.2, 0.3), (0.3, 0.4, 0.5) >$

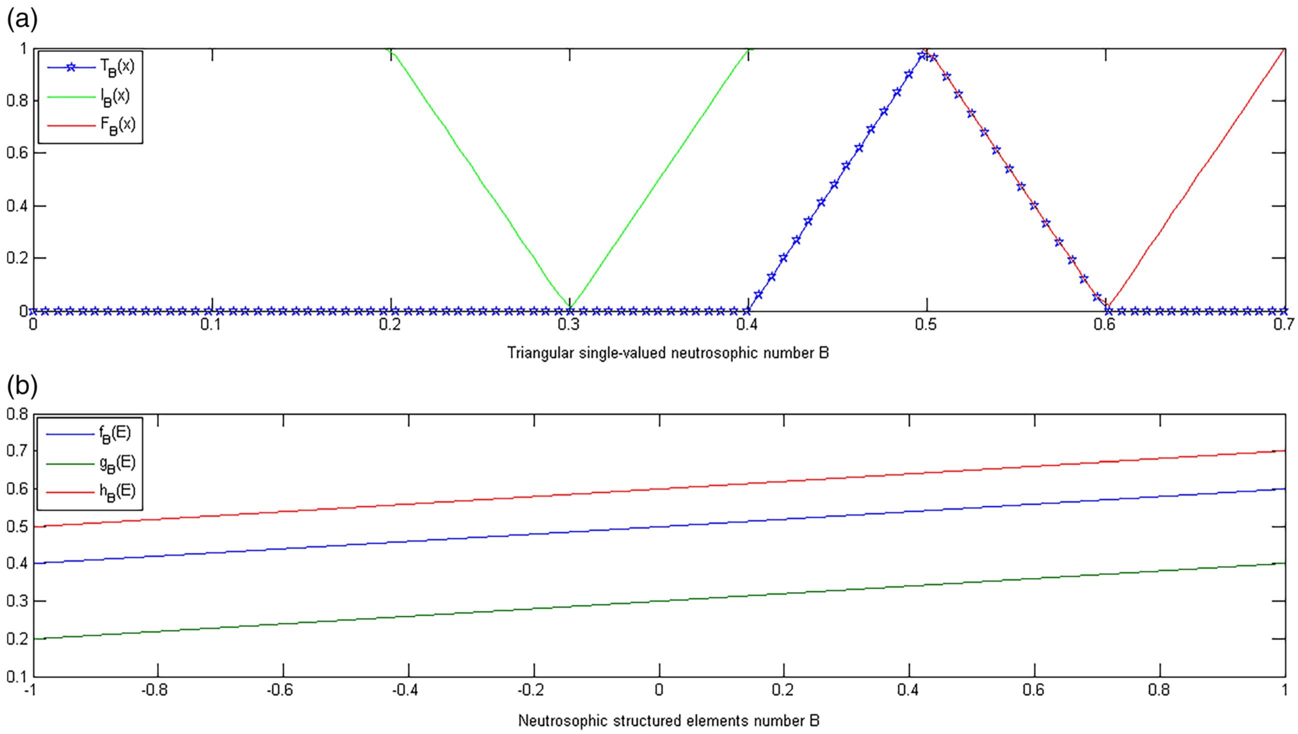

FIGURE 2 TSVNN and the related NSEN $B = \langle (0.4, 0.5, 0.6), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7) \rangle$

$$A^2 = \left\langle \frac{1}{100}(x+6)^2, -\frac{1}{100}(x+2)(x-18), -\frac{1}{2500}(5x+2)(5x-98) \right\rangle.$$

Next, we characterize a strategy to compare two NSE numbers based on the score function and the accuracy function.

Definition 4.3 Let $P = \langle f_A(E), g_A(E), h_A(E) \rangle$ be an NSE number, then we call

$$S(P) = \frac{1}{9}\int_{-1}^{1} E(x)(2 + f_A(x) - g_A(x) - h_A(x))dx, \tag{13}$$

and

$$A(P) = \frac{1}{9}\int_{-1}^{1} E(x)(2 + f_A(x) - g_A(x) + h_A(x))dx, \tag{14}$$

as the score and accuracy function of P, respectively.

Example 4.2 Let $P = \langle (0.1x + 0.6), (0.1x + 0.2), (0.1x + 0.4) \rangle$, be an NSE number. Then,

$$S(P) = \frac{1}{9}\int_{-1}^{1} E(x)(2 + f_A(x) - g_A(x) - h_A(x))dx =$$
$$\frac{1}{9}\left[\left(\int_{-1}^{0}(1-x)\left(-\frac{x}{10} + \frac{59}{25}\right)dx\right) + \left(\int_{0}^{1}(1+x)\left(-\frac{x}{10} + \frac{59}{25}\right)dx\right)\right] = \frac{59}{75},$$

$$A(\mathrm{P})=\frac{1}{9}\int_{-1}^{1}E(x)(2+f_A(x)-g_A(x)+h_A(x))dx=$$
$$\frac{1}{9}\left[\left(\int_{-1}^{0}(1-x)\left(\frac{x}{10}+\frac{61}{25}\right)dx\right)+\left(\int_{0}^{1}(1+x)\left(\frac{x}{10}+\frac{61}{25}\right)dx\right)\right]=\frac{61}{75}.$$

Definition 4.4 Let P and Q be two NSE numbers.

1. If $S(P) < S(Q)$, then P is smaller than Q, denoted by $P \prec Q$.
2. If $S(P) = S(Q)$.

a. If $A(P) < A(Q)$, then P is smaller than Q, denoted by $P \prec Q$.
b. If $A(P) = A(Q)$, then P and Q are the same, denoted by $P = Q$.

Example 4.3 Consider the two NSE numbers of Example 4.1. Since $S(A) = \frac{59}{75}$ and $S(B) = \frac{40}{75}$, then B is smaller than A, and therefore, A f B.

Definition 4.5 Let $A_j = < f_{A_j}(E), g_{A_j}(E), h_{A_j}(E) >$ ($j = 1, 2,..., n$) be an NSE set. The arithmetic average operator is as follows:

$$F_\omega(A_1,\ldots,A_n)=\sum_{j=1}^{n}\omega_j A_j, \tag{15}$$

where $W = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of Aj, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$..

Theorem 4.4 For the NSE weighted arithmetic average operator, the aggregated result is as follows:

$$F_\omega(A_1,\ldots,A_n)=\left\langle 1-\prod_{j=1}^{n}(1-f_{A_j}(E))^{\omega_j},\prod_{j=1}^{n}\left(g_{A_j}(E)\right)^{\omega_j},\prod_{j=1}^{n}(h_{A_j}(E))^{\omega_j}\right\rangle. \tag{16}$$

Proof We proof Equation (16) by using mathematical induction.

1. When $n = 2$, then,

$$\omega_1 A_1=\left\langle 1-[(1-f_{A_1}(E))]^{\omega_1},[g_{A_1}(E)]^{\omega_1},[h_{A_1}(E)]^{\omega_1}\right\rangle,$$

$$\omega_2 A_2=\left\langle 1-[(1-f_{A_2}(E))]^{\omega_2},[g_{A_2}(E)]^{\omega_2},[h_{A_2}(E)]^{\omega_2}\right\rangle.$$

Thus from Theorem 4.2, we obtain

$$\begin{aligned}
&F_\omega(A_1,A_2)=\omega_1A_1\oplus\omega_2A_2\\
&=\left\langle 1-[(1-f_{A_1}(E))]^{\omega_1},[g_{A_1}(E)]^{\omega_1},[h_{A_1}(E)]^{\omega_1}\right\rangle\oplus\left\langle 1-[(1-f_{A_2}(E))]^{\omega_2},[g_{A_2}(E)]^{\omega_2},[h_{A_2}(E)]^{\omega_2}\right\rangle,\\
&=\langle 1-[(1-f_{A_1}(E))]^{\omega_1}+1-[(1-f_{A_2}(E))]^{\omega_2}-(1-[(1-f_{A_1}(E))]^{\omega_1})(1-[(1-f_{A_2}(E))]^{\omega_2}),\\
&[g_{A_1}(E)]^{\omega_1}[g_{A_2}(E)]^{\omega_2},[h_{A_1}(E)]^{\omega_1}[h_{A_2}(E)]^{\omega_2}\rangle\\
&=\langle 1-[(1-f_{A_1}(E))]^{\omega_1}[(1-f_{A_2}(E))]^{\omega_2},[g_{A_1}(E)]^{\omega_1}[g_{A_2}(E)]^{\omega_2},[h_{A_1}(E)]^{\omega_1}[h_{A_2}(E)]^{\omega_2}\rangle\\
&=\left\langle 1-\prod_{j=1}^{2}(1-f_{A_j}(E))^{\omega_j},\prod_{j=1}^{2}\left(g_{A_j}(E)\right)^{\omega_j},\prod_{j=1}^{2}(h_{A_j}(E))^{\omega_j}\right\rangle.
\end{aligned}$$

2. If $n = k$, by applying Equation (16), we get

$$F_{\omega}(A_1,\dots,A_k) = \left\langle 1-\prod_{j=1}^{k}\left(1-f_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k}\left(g_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k}\left(h_{A_j}(E)\right)^{\omega_j} \right\rangle. \tag{17}$$

3. When $n = k + 1$, by applying Equation (17) and Theorem 4.2 we can get

$$\begin{aligned} F_{\omega}(A_1,\dots,A_k,A_{k+1}) &= \langle 1-\prod_{j=1}^{k}\left(1-f_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k}\left(g_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k}\left(h_{A_j}(E)\right)^{\omega_j}\rangle \oplus \langle 1-[(1-f_{A_1}(E))]^{\omega_{k+1}}, [g_{A_1}(E)]^{\omega_{k+1}}, [h_{A_1}(E)]^{\omega_{k+1}}\rangle \\ &= \langle (1-\prod_{j=1}^{k}\left(1-f_{A_j}(E)\right)^{\omega_j}) + (1-[(1-f_{A_1}(E))]^{\omega_{k+1}}) - \left(1-\prod_{j=1}^{k}\left(1-f_{A_j}(E)\right)^{\omega_j}\right)(1-[(1-f_{A_1}(E))]^{\omega_{k+1}}), \prod_{j=1}^{k+1}\left(g_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k+1}\left(h_{A_j}(E)\right)^{\omega_j}\rangle \\ &= \langle (1-\prod_{j=1}^{k+1}\left(1-f_{A_j}(E)\right)^{\omega_j}), \prod_{j=1}^{k+1}\left(g_{A_j}(E)\right)^{\omega_j}, \prod_{j=1}^{k+1}\left(h_{A_j}(E)\right)^{\omega_j}\rangle. \end{aligned}$$

Hence, from the above results, we can conclude that for any n, the Equation (16) is true. □

Definition 4.6 Let $A_j = < f_{A_j}(E), g_{A_j}(E), h_{A_j}(E) >$ ($j = 1, 2,\dots, n$) be an NSE set. The weighted geometric average operator is defined as

$$G_{\omega}(A_1,\dots,A_n) = \prod_{j=1}^{n} A_j^{\omega_j}, \tag{18}$$

where $W = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight vector of Aj, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Theorem 4.5 For the NSE weighted geometric average operator, the aggregated result is as follows:

$$G_{\omega}(A_1,\dots,A_n) = \left\langle \prod_{j=1}^{n}\left(f_{A_j}(E)\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-g_{A_j}(E)\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-h_{A_j}(E)\right)^{\omega_j} \right\rangle. \tag{19}$$

Proof The proof is similar to the proof process of Theorem 4.4. □

5 | APPLICATIONS TO MULTI-ATTRIBUTE DECISION MAKING

Here, we investigate a decision-making method under NSE information using two mentioned aggregation operators and the score function. For it, consider a multi-attribute decision-making problem with "m" different alternatives denoted by $A_i (i = 1, \dots, m)$ and are evaluated under the set of "n" different attributes $C_j (j = 1, \dots, n)$ with weight vector is $W = (\omega_1, \omega_2, \dots, \omega_n)$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\omega_j = 1$..

An expert has evaluated these alternatives and gives their preferences as triangular single-valued neutrosophic numbers (TSVNNs) $\bar{\beta}_{ij} = < \left(a_{ij}^1, a_{ij}^2, a_{ij}^3\right), \left(b_{ij}^1, b_{ij}^2, b_{ij}^3\right), \left(c_{ij}^1, c_{ij}^2, c_{ij}^3\right) >$. First, we convert $\bar{\beta}_{ij}$ into the related NSE numbers $\beta_{ij} = < f_{ij}(E), g_{ij}(E), h_{ij}(E)>$. Then the collection information of all the alternatives are summarized in decision-matrix ψ as

$$\psi = \left(\beta_{ij}\right)_{m\times n} = \left(< f_{ij}(E), g_{ij}(E), h_{ij}(E) > \right)_{m\times n}.$$

Then, by applying Equations (16) or (19) according to each row in the decision matrix $\psi = (\beta_{ij})_{m\times n}$, the aggregating NSE value β_i for $A_i (i = 1, \dots, m)$ is $\beta_i = < f_i(E), g_i(E), h_i(E) > = F_{i\omega}(\beta_{i1}, \dots, \beta_{in})$ or $\beta_i = < f_i(E), g_i(E), h_i(E) > = G_{i\omega}(\beta_{i1}, \dots, \beta_{in})$. To rank alternatives in the decision-making process, we use score function (13) and Definition 4.4. So, the ranking order of all alternatives can be established, and the best one can be easily determined as well.

Therefore, the decision-making method for the proposed method can be obtained as follows:

Algorithm

(1) Convert the TSVNNs $\bar{\beta}_{ij}$ into the related NSE numbers β_{ij}.
(2) Calculate the weighted arithmetic average values by using Equation (16) or the weighted geometric average values by using Equation (19).
(3) Calculate the score degree of all alternatives by using Equation (13).
(4) Give the ranking order of the alternatives from Definition 4.4 and chose the best alternative(s).
(5) End.

Example 5.1 Consider a multi-attribute decision-making problem with four alternatives $A_i(i = 1, \ldots, 4)$ and three attributes $C_j(j = 1, 2, 3)$ that the weight vector of the attributes is given by $W = (0.35, 0.30, 0.35)$ and the related decision-matrix is as follows:

$$\bar{\psi} = \begin{bmatrix} <(0.7,0.8,0.9),(0.1,0.2,0.3),(0.3,0.4,0.5)> & <(0.3,0.4,0.5),(0.2,0.3,0.4),(0.3,0.5,0.7)> & <(0.2,0.4,0.6),(0.2,0.3,0.4),(0.5,0.7,0.9)> \\ <(0.4,0.5,0.6),(0.1,0.3,0.5),(0.2,0.4,0.6)> & <(0.3,0.5,0.7),(0.0,0.2,0.4),(0.6,0.7,0.8)> & <(0.7,0.8,0.9),(0.6,0.7,0.8),(0.5,0.6,0.7)> \\ <(0.6,0.7,0.8),(0.1,0.1,0.1),(0.2,0.3,0.4)> & <(0.5,0.6,0.7),(0.3,0.4,0.5),(0.4,0.6,0.8)> & <(0.1,0.3,0.5),(0.0,0.1,0.2),(0.3,0.5,0.7)> \\ <(0.5,0.6,0.7),(0.0,0.1,0.2),(0.15,0.3,0.45)> & <(0.7,0.8,0.9),(0.0,0.4,0.8),(0.7,0.8,0.9)> & <(0.2,0.3,0.4),(0.1,0.2,0.3),(0.3,0.4,0.5)> \end{bmatrix}.$$

First, we convert the TSVNNs $\bar{\psi}$ into the related NSE numbers ψ:

$$\psi = \begin{bmatrix} <0.1x+0.8,0.1x+0.2,0.1x+0.4)> & <0.1x+0.4,0.1x+0.3,0.2x+0.5> & <0.2x+0.4,0.1x+0.3,0.2x+0.7> \\ <0.1x+0.5,0.2x+0.3,0.2x+0.4> & <0.2x+0.5,0.2x+0.2,0.1x+0.7> & <0.1x+0.8,0.1x+0.7,0.1x+0.6> \\ <0.1x+0.7,0.1,0.1x+0.3> & <0.1x+0.6,0.1x+0.4,0.2x+0.6> & <0.2x+0.3,0.1x+0.1,0.2x+0.5> \\ <0.1x+0.6,0.1x+0.1,0.15x+0.3> & <0.1x+0.8,0.4x+0.4,0.1x+0.8> & <0.1x+0.3,0.1x+0.2,0.2x+0.4> \end{bmatrix}.$$

Now, if we calculate the weighted arithmetic average values by using Equation (16), we get,

$$\beta_1 = \left\langle 1-\left(\frac{3-x}{5}\right)^{0.35}\left(\frac{2-x}{10}\right)^{0.35}\left(\frac{6-x}{10}\right)^{0.3}, \left(\frac{1+x}{5}\right)^{0.35}\left(\frac{3+x}{10}\right)^{0.65}, \left(\frac{5+2x}{10}\right)^{0.3}\left(\frac{4+x}{10}\right)^{0.35}\left(\frac{7+2x}{10}\right)^{0.35} \right\rangle,$$

$$\beta_2 = \left\langle 1-\left(\frac{5-2x}{10}\right)^{0.3}\left(\frac{5-x}{10}\right)^{0.35}\left(\frac{2-x}{10}\right)^{0.35}, \left(\frac{1+x}{5}\right)^{0.3}\left(\frac{3+2x}{10}\right)^{0.35}\left(\frac{7+x}{10}\right)^{0.35}, \left(\frac{2+x}{5}\right)^{0.35}\left(\frac{6+x}{10}\right)^{0.35}\left(\frac{7+x}{10}\right)^{0.3} \right\rangle,$$

$$\beta_3 = \left\langle 1-\left(\frac{4-x}{10}\right)^{0.3}\left(\frac{7-2x}{10}\right)^{0.35}\left(\frac{3-x}{10}\right)^{0.35}, 0.4467\left(\frac{4+x}{10}\right)^{0.3}\left(\frac{1+x}{10}\right)^{0.35}, \left(\frac{5+2x}{10}\right)^{0.35}\left(\frac{3+x}{5}\right)^{0.3}\left(\frac{3+x}{10}\right)^{0.35} \right\rangle,$$

$$\beta_4 = \left\langle 1-\left(\frac{2-x}{10}\right)^{0.3}\left(\frac{4-x}{10}\right)^{0.35}\left(\frac{7-x}{10}\right)^{0.35}, \left(\frac{2+2x}{5}\right)^{0.3}\left(\frac{2+x}{10}\right)^{0.35}\left(\frac{1+x}{10}\right)^{0.35}, \left(\frac{2+x}{5}\right)^{0.35}\left(\frac{8+x}{10}\right)^{0.3}\left(\frac{6+3x}{20}\right)^{0.35} \right\rangle.$$

Then by applying Equation (13), we compute the score degree of all alternative:

$$S(\beta_1) = 0.6074, S(\beta_2) = 0.5840, S(\beta_3) = 0.6585, S(\beta_4) = 0.6582.$$

Hence, the ranking order of the above alternatives is $A_3 > A_4 > A_1 > A_2$.

As a result, we can see that the alternative A_3 is the most excellent choice among all the alternatives. On the other hand, if we calculate the weighted geometric average values by using Equation (19), we have

$$S(\beta_1) = 0.5605, S(\beta_2) = 0.5174, S(\beta_3) = 0.6033, S(\beta_4) = 0.5734.$$

Therefore, the ranking order of four alternatives is $A_3 > A_4 > A_1 > A_2$.

We can also see that the above two kinds of ranking orders and the best alternative are the same.

6 | CONCLUSIONS AND FUTURE WORK

This paper introduced the concept of NSE, which solved the problem of the complex operations of neutrosophic numbers. Then, we proposed operational laws, score function, and some aggregation operators of NSE sets. Finally, as an application of this concept, we proposed a decision-making method for a multi-attribute decision making (MADM) problem under NSE information. The proposed concept has produced promising results from computing efficiency and performance aspects.

The proposed study has some limitations: The indeterminacy, uncertainty, and vagueness in the present study are limited to triangular single-valued neutrosophic numbers, but the other forms of neutrosophic sets such as bipolar neutrosophic set, and interval-valued neutrosophic numbers can also be used to represent variables characterizing neutrosophic essence in real-world problems. Developing the model based on bipolar and interval-valued neutrosophic data is a topic for further studies. Moreover, although the NSE, arithmetic operations, and results presented here demonstrate the effectiveness of this concept, it could also be considered in other decision-making problems. As future researches, we intend to study these problems by NSE information.

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CONFLICT OF INTEREST

The author declares no potential conflict of interest.

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REFERENCES

Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2019). A group decision making framework based on neutrosophic topsis approach for smart medical device selection. *Journal of Medical Systems*, *43*(2), 38.

Abdel-Basset, M., & Mohamed, M. (2018). The role of single valued neutrosophic sets and rough sets in smart city: Imperfect and incomplete information systems. *Measurement*, *124*, 47–55.

Atanassov, K., & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *31*(3), 343–349.

Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96.

Basha, S. H., Tharwat, A., Abdalla, A., & Hassanien, A. E. (2019). Neutrosophic rule-based prediction system for toxicity effects assessment of biotransformed hepatic drugs. *Expert Systems with Applications*, *121*, 142–157.

Biswas, P., Pramanik, S., & Giri, B. C. (2016). TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. *Neural Computing and Applications*, *27*(3), 727–737.

Broumi, S., & Smarandache, F. (2013). Correlation coefficient of interval neutrosophic set. In *Applied mechanics and materials* (Vol. 436, pp. 511–517). Trans Tech.

Broumi, S., Smarandache, F., Talea, M., & Bakali, A. (2016). An introduction to bipolar single valued neutrosophic graph theory. In *Applied mechanics and materials* (Vol. 841, pp. 184–191). Switzerland: Trans Tech.

Chakraborty, A., Mondal, S., Ahmadian, A., Senu, N., Alam, S., & Salahshour, S. (2018). Different forms of triangular neutrosophic numbers, de-neutrosophication techniques, and their applications. *Symmetry*, *10*(8), 327.

Cui, T. J., & Li, S. S. (2019). Study on the construction and application of discrete space fault tree modified by fuzzy structured element. *Cluster Computing*, *22*, 6563–6577.

Das, S. K., Mandal, T., & Edalatpanah, S. A. (2017a). A mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers. *Applied Intelligence*, *46*(3), 509–519.

Das, S. K., Mandal, T., & Edalatpanah, S. A. (2017b). A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming. *RAIRO-Operations Research*, *51*(1), 285–297.

Deli, I., Yusuf, S., Smarandache, F., & Ali, M. (2016). Interval valued bipolar neutrosophic sets and their application in pattern recognition. In *IEEE World Congress on Computational Intelligence*. Vancouver, Canada: IEEE.

Deng, S., Zhou, L., & Wang, X. (2014). Solving the fuzzy bilevel linear programming with multiple followers through structured element method. *Mathematical Problems in Engineering*, *2014*, 418594.

Dhingra, G., Kumar, V., & Joshi, H. D. (2019). A novel computer vision based neutrosophic approach for leaf disease identification and classification. *Measurement*, *135*, 782–794.

Dong, D. X., & Zhu, Y. L. (2009). The study of the fuzzy net present value based on structured element. In *Fuzzy information and engineering* (Vol. 2, pp. 205–211). Berlin-Heidelberg: Springer.

Edalatpanah, S. A. (2018). Neutrosophic perspective on DEA. *Journal of Applied Research on Industrial Engineering*, *5*(4), 339–345.

Edalatpanah, S. A., & Smarandache, F. (2019). Data envelopment analysis for simplified neutrosophic sets. *Neutrosophic Sets and Systems*, *29*, 215–226.

Finol, J., Guo, Y. K., & Jing, X. D. (2001). A rule based fuzzy model for the prediction of petrophysical rock parameters. *Journal of Petroleum Science and Engineering*, *29*(2), 97–113.

Gallego Lupiáñez, F. (2009). Interval neutrosophic sets and topology. *Kybernetes*, *38*(3/4), 621–624.

Garg, H. (2018a). Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision-making. *Journal of Ambient Intelligence and Humanized Computing*, *9*(6), 1975–1997.

Garg, H. (2018b). Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment. *Applied Intelligence*, *48*(8), 2199–2213.

Guo, S. Z. (2002a). Method of structuring element in fuzzy analysis (I). *Journal of Liaoning Technical University*, *21*(5), 670–673.

Guo, S. Z. (2002b). Method of structuring element in fuzzy analysis (II). *Journal of Liaoning Technical University*, *21*(6), 808–810.

Guo, S. Z. (2004). Fuzzy analysis and calculation based on the structured element. *Applied Computational Intelligence*, 128–133.

Guo, S. Z. (2009). Comparison and ordering of fuzzy numbers based on method of structured element. *Systems Engineering-Theory & Practice*, *29*(3), 106–111.

Guo, Y., & Cheng, H. D. (2009). New neutrosophic approach to image segmentation. *Pattern Recognition*, *42*(5), 587–595.

Hsu, T. K., Tsai, Y. F., & Wu, H. H. (2009). The preference analysis for tourist choice of destination: A case study of Taiwan. *Tourism Management*, *30*(2), 288–297.

Hu, J., Yang, Y., & Guo, S. (2008). Fuzzy number intuitionistic fuzzy set and its representation of structured element. In *2008 International Symposium on Intelligent Information Technology Application Workshops* (pp. 349–352). Shanghai, China: IEEE.

Jain, R., & Haynes, S. (1983). Imprecision in computer vision. In *Advances in fuzzy sets, possibility theory, and applications* (pp. 217–236). Boston, MA: Springer.

Jha, S., Kumar, R., Chatterjee, J. M., Khari, M., Yadav, N., & Smarandache, F. (2019a). Neutrosophic soft set decision making for stock trending analysis. *Evolving Systems*, *10*(4), 621–627.

Jha, S., Kumar, R., Priyadarshini, I., Smarandache, F., & Long, H. V. (2019b). Neutrosophic image segmentation with dice coefficients. *Measurement*, *134*, 762–772.

Ji, P., Zhang, H. Y., & Wang, J. Q. (2018). A projection-based TODIM method under multi-valued neutrosophic environments and its application in personnel selection. *Neural Computing and Applications*, *29*(1), 221–234.

Kumar, R., Edalatpanah, S. A., Jha, S., Broumi, S., & Dey, A. (2018). Neutrosophic shortest path problem. *Neutrosophic Sets and Systems*, *23*, 5–15.

Kumar, R., Edalatpanah, S. A., Jha, S., Broumi, S., & Dey, A. (2019). A multi-objective programming approach to solve integer-valued neutrosophic shortest path problems. *Neutrosophic Sets & Systems*, *24*, 134–149.

Li, C., & Lei, T. (2017, July). Fuzzy bi-matrix games based on fuzzy structured element. In *2017 13th international conference on natural computation, fuzzy systems and knowledge discovery (ICNC-FSKD)* (pp. 1107–1111). Guilin, China: IEEE.

Liu, P., & Shi, L. (2015). The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making. *Neural Computing and Applications*, *26*(2), 457–471.

Liu, P., & Wang, Y. (2014). Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. *Neural Computing and Applications*, *25*(7–8), 2001–2010.

Liu, Y. Z., & Guo, S. Z. (2012). Fuzzy multi-objective programming problem with fuzzy structured element solution. In *Fuzzy engineering and operations research* (pp. 97–107). Berlin-Heidelberg: Springer.

Ma, Y. X., Wang, J. Q., Wang, J., & Wu, X. H. (2017). An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options. *Neural Computing and Applications*, *28*(9), 2745–2765.

Najafi, H. S., & Edalatpanah, S. A. (2013a). An improved model for iterative algorithms in fuzzy linear systems. *Computational Mathematics and Modeling*, *24*(3), 443–451.

Najafi, H. S., & Edalatpanah, S. A. (2013b). A note on "a new method for solving fully fuzzy linear programming problems". *Applied Mathematical Modelling*, *37*(14), 7865–7867.

Najafi, H. S., Edalatpanah, S. A., & Dutta, H. (2016). A nonlinear model for fully fuzzy linear programming with fully unrestricted variables and parameters. *Alexandria Engineering Journal*, *55*(3), 2589–2595.

Peng, J. J., Wang, J. Q., Wang, J., Zhang, H. Y., & Chen, X. H. (2016). Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. *International Journal of Systems Science*, *47*(10), 2342–2358.

Peng, J. J., Wang, J. Q., Wu, X. H., Wang, J., & Chen, X. H. (2015). Multi-valued neutrosophic sets and power aggregation operators with their applications in multi-criteria group decision-making problems. *International Journal of Computational Intelligence Systems*, *8*(2), 345–363.

Peng, J. J., Wang, J. Q., & Yang, W. E. (2017). A multi-valued neutrosophic qualitative flexible approach based on are likelihood for multi-criteria decision-making problems. *International Journal of Systems Science*, *48*(2), 425–435.

Rivieccio, U. (2008). Neutrosophic logics: Prospects and problems. *Fuzzy Sets and Systems*, *159*(14), 1860–1868.

Şahin, R., & Küçük, A. (2015). Subsethood measure for single valued neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, *29*(2), 525–530.

Sert, E., & Avci, D. (2019). Brain tumor segmentation using neutrosophic expert maximum fuzzy-sure entropy and other approaches. *Biomedical Signal Processing and Control*, *47*, 276–287.

Shu, T. J., & Mo, Z. W. (2016). Infinitely small quantity and infinitely large quantity of fuzzy valued functions for linear generation of structural elements. In *International workshop on mathematics and decision science* (pp. 82–88). Cham: Springer.

Smarandache, F. (1999). *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic*. Rehoboth, DE: American Research Press.

Smarandache, F. (2003). *A unifying field in logics: Neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics* (3rd ed.). Phoenix, AZ: Xiquan.

Smarandache, F., & Ali, M. (2018). Neutrosophic triplet group. *Neural Computing and Applications*, *29*(7), 595–601.

Smarandache, F., & Pramanik, S. (2016). *New trends in neutrosophic theory and applications*, 1, 15–161.

Sun, X. D., & Guo, S. Z. (2009a). Linear formed general fuzzy linear systems. *Systems Engineering-Theory & Practice*, *29*(9), 92–98.

Sun, X. D., & Guo, S. Z. (2009b). Solving fuzzy linear systems based on the structured element method. In *Fuzzy Information and Engineering* (pp. 270–276). Berlin-Heidelberg: Springer.

Tian, Z. P., Wang, J., Wang, J. Q., & Zhang, H. Y. (2017). Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development. *Group Decision and Negotiation*, *26*(3), 597–627.

Turksen, I. B. (1986). Interval valued fuzzy sets based on normal forms. *Fuzzy Sets and Systems*, *20*(2), 191–210.

Uluçay, V., Deli, I., & Şahin, M. (2018). Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, *29*(3), 739–748.

Wang, H., Guo, S., & Shi, Y. (2015a). Homeomorphism between fuzzy number space and the space of bounded functions with same monotonicity on [−1, 1]. In *International conference on data science* (pp. 70–77). Cham, Switzerland: Springer.

Wang, H. D., Guo, S. C., Bamakan, S. M. H., & Shi, Y. (2015b). Homeomorphism problems of fuzzy real number space and the space of bounded functions with same monotonicity on [−1, 1]. *International Journal of Computers Communications & Control*, *10*(6), 129–143.

Wang, J. Q., Yang, Y., & Li, L. (2018a). Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Computing and Applications*, *30*(5), 1529–1547.

Wang, W. K., Lu, W. M., & Liu, P. Y. (2014). A fuzzy multi-objective two-stage DEA model for evaluating the performance of US bank holding companies. *Expert Systems with Applications*, *41*(9), 4290–4297.

Wang, X., Wang, J., & Chen, X. (2016). Fuzzy multi-criteria decision making method based on fuzzy structured element with incomplete weight information. *Iranian Journal of Fuzzy Systems*, *13*(2), 1–17.

Wang, Y., Jin, Z., Deng, C., & Wang, X. (2018b). Comprehensive decision-making with fuzzy combined weighting and its application on the order of gob management. *Journal of Intelligent & Fuzzy Systems*, *34*(4), 2641–2649.

Yan, Y., & Bao-fu, Z. (2013, November). Fuzzy effective degree analysis in mining machine based on the structured element theory. In *2013 6th international conference on information management, innovation management and industrial engineering (ICIII)* (Vol. 3, pp. 344–349). Xi'an, China: IEEE.

Ye, J. (2013). Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *International Journal of General Systems*, *42*(4), 386–394.

Ye, J. (2014a). Single valued neutrosophic cross-entropy for multicriteria decision making problems. *Applied Mathematical Modelling*, *38*(3), 1170–1175.

Ye, J. (2014b). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, *26*(5), 2459–2466.

Ye, J. (2014c). Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *Journal of Intelligent & Fuzzy Systems*, *26*(1), 165–172.

Ye, J. (2015a). Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. *Artificial Intelligence in Medicine*, *63*(3), 171–179.

Ye, J. (2015b). An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. *Journal of Intelligent & Fuzzy Systems*, *28*(1), 247–255.

Yue, L. Z., & Yan, Y. (2009). The method of fuzzy network shortest path based on the structured element theory. In *Fuzzy information and engineering* (Vol. 2, pp. 213–219). Berlin-Heidelberg: Springer.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.

Zadeh, L. A. (1977). Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering* (pp. 251–299).

Zhang, M., Zhang, L., & Cheng, H. D. (2010). A neutrosophic approach to image segmentation based on watershed method. *Signal Processing*, *90*(5), 1510–1517.

Zhao, H., Yang, J., & Wan, J. (2010). Evaluation model of credibility of e-commerce website using fuzzy multi-attribute group decision making: Based on fuzzy structured element. In *Fuzzy information and engineering 2010* (pp. 417–424). Berlin-Heidelberg: Springer.

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