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Neutrosophic Triplet Cosets and Quotient Groups

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Abstract: In this paper, by utilizing the concept of a neutrosophic extended triplet (NET), we define the neutrosophic image, neutrosophic inverse-image, neutrosophic kernel, and the NET subgroup. The notion of the neutrosophic triplet coset and its relation with the classical coset are defined and the properties of the neutrosophic triplet cosets are given. Furthermore, the neutrosophic triplet normal subgroups, and neutrosophic triplet quotient groups are studied.

Keywords: neutosophic extended triplet subgroups; neutrosophic triplet cosets; neutrosophic triplet normal subgroups; neutrosophic triplet quotient groups

1. Introduction

Neutrosophy was first introduced by Smarandache (Smarandache, 1999, 2003) as a branch of philosophy, which studied the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra: (A) is an idea, proposition, theory, event, concept, or entity; anti(A) is the opposite of (A); and (neut-A) means neither (A) nor anti(A), that is, the neutrality in between the two extremes. A notion of neutrosophic set theory was introduced by Smarandache in [1]. By using the idea of the neutrosophic theory, Kandasamy and Smarandache introduced neutrosophic algebraic structures in [2,3]. The neutrosophic triplets were first introduced by Florentin Smarandache and Mumtaz Ali [4–10], in 2014–2016. Florentin Smarandache and Mumtaz Ali introduced neutrosophic triplet groups in [6,11]. A lot of researchers have been dealing with neutrosophic triplet metric space, neutrosophic triplet vector space, neutrosophic triplet inner product, and neutrosophic triplet normed space in [12–22].

A neutrosophic extended triplet, introduced by Smarandache [7,20] in 2016, is defined as the neutral of x (denoted by $e^{neut(x)}$ and called "extended neutral"), which is equal to the classical algebraic unitary element (if any). As a result, the "extended opposite" of x (denoted by $e^{anti(x)}$) is equal to the classical inverse element from a classical group. Thus, the neutrosophic extended triplet (NET) has a form $\left(x, e^{neut(x)}, e^{anti(x)}\right)$ for $x \in N$, where $e^{neut(x)} \in N$ is the extended neutral of x. Here, the neutral element can be equal to or different from the classical algebraic unitary element, if any, such that: $x * e^{neut(x)} = e^{neut(x)} * x = x$, and $e^{anti(x)} \in N$ is the extended opposite of x, where $x * e^{anti(x)} = e^{anti(x)} * x = e^{neut(x)}$. Therefore, we used NET to define these new structures.

In this paper, we deal with neutosophic extended triplet subgroups, neutrosophic triplet cosets, neutrosophic triplet normal subgroups, and neutrosophic triplet quotient groups for the purpose to develop new algebraic structures on NET groups. Additionally, we define the neutrosophic triplet image, neutrosophic triplet kernel, and neutrosophic triplet inverse image. We give preliminaries and results with examples in Section 2, and we introduce neutrosophic extended triplet subgroups in Section 3. Section 4 is dedicated to introduing neutrosophic triplet cosets, with some of their properties, and we show that neutrosophic triplet cosets are different from classical cosets. In Section 5, we introduce neutrosophic triplet normal subgroups and the neutrosophic triplet normal subgroup test. In Section 6, we define the neutrosophic triplet quotient groups and we examine the relationships of these structures with each other. In Section 7, we provide some conclusions.

2. Preliminaries

In this section, the definition of neutrosophic triplets, NET's, and the concepts of NET groups have been outlined.

2.1. Neutrosophic Triplet

Let U be a universe of discourse, and (N, $*$) a set included in it, endowed with a well-defined binary law $*$.

Definition 1 ([1–3]). *A neutrosophic triplet has a form (x, neut(x), anti(x)), for x in N, where neut(x) and anti(x) ∈ N are neutral and opposite to x, which are different from the classical algebraic unitary element, if any, such that:* $x * neut(x) = neut(x) * x = x$ *and* $x * anti(x) = anti(x) * x = neut(x)$*, respectively. In general, x may have more than one neut's and anti's.*

2.2. NET

Definition 2 ([4,7]). *A neutrosophic extended triplet is a neutrosophic triplet, as defined in Definition 1, where the neutral of x (denoted by* $e^{neut(x)}$*and called extended neutral) is equal to the classical algebraic unitary element, if any. As a consequence, the extended opposite of x (denoted by* $e^{anti(x)}$*) is also equal to the classical inverse element from a classical group. Thus, an NET has a form* $\left(x, e^{neut(x)}, e^{anti(x)}\right)$*, for x* $\in$ *N, where* $e^{neut(x)}$ *and* $e^{anti(x)}$ *in N are the extended neutral and opposite of x, respectively, such that:* $x * e^{neut(x)} = e^{neut(x)} * x = x$*, which can be equal to or different from the classical algebraic unitary element, if any, and* $x * e^{anti(x)} = e^{anti(x)} * x = e^{neut(x)}$*. In general, for each* $x \in N$ *there are many* $e^{neut(x)}$*'s and* $e^{anti(x)}$*'s.*

Definition 3 ([1–3]). *The element y in* $(N, *)$ *is the second coordinate of a neutrosophic extended triplet (denoted as neut(y) of a neutrosophic triplet), if there are other elements exist, x and z* $\in$ *N such that:* $x * y = y * x = x$ *and* $x * z = z * x = y$*. The formed neutrosophic triplet is (x, y, z). The element* $z \in (N, *)$*, as the third coordinate, can be defined in the same way.*

Example 1. *Let X = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11), enclosed with the classical multiplication law, (x) modulo 12, which is well defined on X, with the classical unitary element 1. X iss an NET "weak commutative set" see "Table 1".*

Table 1. Neutrosophic triplets of (x) modulo 12.

∗	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	1	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	1	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	3	0	9	6	3	0	9	6	3
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	1

The formed NETs of X are: (0, 0, 0), (0, 0, 1), (0, 0, 2), . . . , (0, 0, 11), (1, 1, 1), (3, 9, 3), (3, 9, 7), (3, 9, 11), (4, 4, 4), (4, 4, 7), (4, 4, 10), (5, 1, 5), (7, 1, 7), (8, 4, 2), (8, 4, 5), (8, 4, 8), (8, 4, 11), (9, 9, 5), (9, 9, 9), (11, 1, 11).

Here, 2, 6, and 10 did not give rise to a neutrosophic triplet, as neut(2) = 1 and 7, however anti(2) did not exist in Z_{12}. In addition, neut(6) = 1, 3, 5, 7, 9, and 11, however anti(6) did not exist in Z_{12}. The neut(10) = 1, however anti(10) did not exist in Z_{12}.

Definition 4 ([4,7])**.** *The set N is called a strong neutrosophic extended triplet set if, for any x in N, $e^{neut(x)} \in N$ and $e^{anti(x)} \in N$ exists.*

Example 2. *The NET's of (x) modulo 12 were as follows:*

(0, 0, 0), (0, 0, 1), (0, 0, 2), . . . , (0, 0, 11), (1, 1, 1), (3, 9, 3), (3, 9, 7), (3, 9, 11), (4, 4, 4), (4, 4, 7), (4, 4, 10), (5, 1, 5), (7, 1, 7), (8, 4, 2), (8, 4, 5), (8, 4, 8), (8, 4, 11), (9, 9, 5), (9, 9, 9), (11, 1, 11).

Definition 5 ([4,7])**.** *The set N is called an NET weak set if, for any $x \in N$, an NET $\left(y,\ e^{neut(y)},\ e^{anti(y)}\right)$ included in N exists, such that:*

$$x = y$$

or

$$x = e^{neut(y)}$$

or

$$x = e^{anti(y)}.$$

Definition 6. *A neutrosophic extended triplet (x, y, z) for $x, y, z \in N$, is called a neutrosophic perfect triplet if both (z, y, x) and (y, y, y) are also neutrosophic triplets.*

Example 3. *The neutrosophic perfect triplets of (x) modulo 12 are described in "Table 1" as follows:*

Here, (0, 0, 0), (1, 1, 1), (3, 9, 3), (4, 4, 4), (5, 1, 5), (7, 1, 7), (8, 4, 8), (9, 9, 9), (11, 1, 11) are neutrosophic perfect triplets of (x) modulo 12.

Definition 7. *An NET (x, y, z) for $x, y, z \in N$, is called a neutrosophic imperfect triplet if at least one of (z, y, x) or (y, y, y) is not a neutrosophic triplet(s).*

Example 4. *The neutrosophic imperfect triplets of (x) modulo 12, from the above table, were as follows:*

$$(0,\ 0,\ 1),\ (0,\ 0,\ 2), \ldots, (0,\ 0,\ 11),\ (3,\ 9,\ 7),\ (3,\ 9,\ 11),\ (4,\ 4,\ 7), (4,\ 4,\ 10), (8,\ 4,\ 2), (8,\ 4,\ 5),\ (8,\ 4,\ 11), (9,\ 9,\ 5).$$

2.3. Neutrosophic Triplet Group (NTG)

Definition 8 ([1–3])**.** *Let (N, ∗) be a neutrosophic strong triplet set. Then, (N, ∗) is called a neutrosophic strong triplet group, if the following classical axioms are satisfied:*

(1) (N, ∗) is well-defined, that is, for any $x, y \in N$, one has $x * y \in N$.
(2) (N, ∗) is associative, that is, for any $x, y, z \in N$, one has $x * (y * z) = (x * y) * z$.

Example 5. *We let Y = (Z_{12}, ×) be a semi-group under product 12. The neutral elements of Z_{12} were 4 and 9. The elements (8, 4, 8), (4, 4, 4), (3, 9, 3), and (9, 9, 9) were NETs.*

NTG, in general, was not a group in the classical sense, because it might not have had a classical unitary element, nor the classical inverse elements. We considered that the neutrosophic neutrals replaced the classical unitary element, and the neutrosophic opposites replaced the classical inverse elements.

Proposition 1 ([3]). *Let (N, ∗) be an NTG with respect to ∗ and a, b, c ∈ N:*

(1) $a * b = a * c \Leftrightarrow neut(a) * b = neut(a) * c$.
(2) $b * a = c * a \Leftrightarrow b * neut(a) = c * neut(a)$.
(3) if $anti(a) * b = anti(a) * c$, then $neut(a) * b = neut(a) * c$.
(4) if $b * anti(a) = c * anti(a)$, then $b * neut(a) = c * neut(a)$.

Theorem 1 ([3]). *Let (N, ∗) be a commutative NET, with respect to ∗ and a, b ∈ N:*

(i) $neut(a) * neut(b) = neut(a * b)$;
(ii) $anti(a) * anti(b) = anti(a * b)$;

Theorem 2 ([3]). *Let (N, ∗) be a commutative NET, with respect to ∗ and a ∈ N:*

(i) $neut(a) * neut(a) = neut(a)$;
(ii) $anti(a) * neut(a) = neut(a) * anti(a) = anti(a)$;

Definition 9 ([3]). *An NET (N, ∗) is called to be cancellable, if it satisfies the following conditions:*

(a) $\forall\, x, y, z \in N, x * y = y * z \Rightarrow y = z$.
(b) $\forall\, x, y, z \in N, y * x = z * x \Rightarrow y = z$.

Definition 10 ([3]). *Let N be an NTG and x ∈ N. N is then called a neutro-cyclic triplet group if $N = \langle a \rangle$. We can say that a is the neutrosophic triplet generator of N.*

Example 6. *We let N = (2, 4, 6) be an NTG with respect to (Z_8, .). Then, N was clearly a neutro-cyclic triplet group as N = $\langle a \rangle$. Therefore, 2 was the neutrosophic triplet generator of N.*

2.4. Neutrosophic Extended Triplet Group (NETG)

Definition 11 ([4,7]). *Let (N, ∗) be an NET strong set. Then, (N, ∗) is called an NETG, if the following classical axioms are satisfied:*

(1) (N, ∗) is well-defined, that is, for any x, y ∈ N, one has $x * y \in N$.
(2) (N, ∗) is associative, that is, for any x, y, z ∈ N, one has

$$x * (y * z) = (x * y) * z.$$

For NETG, the neutrosophic extended neutrals replaced the classical unitary element, and the neutrosophic extended opposites replaced the classical inverse elements. In the case where NETG included a classical group, then NETG enriched the structure of a classical group, since there might have been elements with more extended neutrals and more extended opposites.

Definition 12. *A permutation of a set X is a function σ: x → x that is one to one and onto, that is, a bijective map. Permutation maps, being bijective, have anti neutrals and the maps combine neutrally under composition of maps, which are associative. There is natural neutral permutation σ: x → x, X = (1, 2, 3, . . . , n), which is σ(k) = k. Therefore, all of the permutations of a set X = (1, 2, 3, . . . , n) form an NETG under composition. This group is called the symmetric NETG (e^{S_n}) of degree n.*

Example 7. *We let A = (1, 2, 3). The elements of symmetric group of S_3 were as follows:*

$$\sigma_0 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \mu_2 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \mu_3 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}$$

The operartion of S_3 is defined in Table 2 as follows:

1. *(S_3, ○) is well-defined, that is, for any σ_i, $\mu_i \in S_3$, i = 1,2,3 one has σ_i ○ $\mu_i \in S_3$.*
2. *(S_3, ○) is associative, that is, for any σ_1, μ_1, $\mu_3 \in S_3$, one has the following:*

$$(\sigma_1 \bigcirc \mu_1) \bigcirc \mu_3 = \sigma_1 \bigcirc (\mu_1 \bigcirc \mu_3)$$

$$(\mu_1 \bigcirc \mu_3) = (\sigma_1 \bigcirc \sigma_1) = \sigma_2.$$

Table 2. Neutrosophic triplets of *X*.

○	σ_0	σ_1	σ_2	μ_1	μ_2	μ_3
σ_0	σ_0	σ_1	σ_2	μ_1	μ_2	μ_3
σ_1	σ_1	σ_2	σ_0	μ_2	μ_3	μ_1
σ_2	σ_2	σ_0	σ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_2	μ_3	σ_0	σ_2	σ_1
μ_2	μ_2	μ_1	μ_3	σ_1	σ_0	σ_2
μ_3	μ_3	μ_2	μ_1	σ_2	σ_1	σ_0

The NET's of S_3 (e^{S3}) are as follows:

$$(\sigma_0, \sigma_0, \sigma_0), (\sigma_1, \sigma_0, \sigma_2), (\sigma_2, \sigma_0, \sigma_1), (\mu_1, \sigma_0, \mu_1), (\mu_2, \sigma_0, \mu_2), (\mu_3, \sigma_0, \mu_3).$$

Hence, (S3, ○) is an NET strong group.

Definition 13 ([9–11])**.** *Let (N_1 ∗, N_2 ○) be two NETGs. A mapping f: $N_1 \to N_2$ is called a neutro-homomorphism if:*

(1) *For any x, y ∈ N_1, we have $f(x * y) = f(x)\ f(y)$*
(2) *If (x, neut[x], anti[x]) is an NET from N_1, then,*

$f(neut[x]) = neut(f[x])$ *and* $f(anti[x]) = anti(f[x])$.

Example 8. *We let N_1 be an NETG with respect multiplication modulo 6 in (Z_6, ×), where N_1 = (0, 2, 4), and we let N_2 be another NETG in (Z_{10}, ×), where N_2 = (0, 2, 4, 6, 8). We let f: $N_1 \to N_2$ be a mapping defined as f(0) = 0, f (2) = 4, f (4) = 6. Then, f was clearly a neutro-homomorphism, because condition (1) and (2) were satisfied easily.*

Definition 14. *Let f: $N_1 \to N_2$ be a neutro-homomorphism from an NETG (N_1, ∗) to an NETG (N_2, ∗). The neutrosophic image of f is a subset, as follows:*

$$Im(f) = (f(g){:}g \in N_1, *) \text{ of } N_2.$$

Definition 15. *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from an NETG (N_1, $*$) to an NETG ($N*$, $\bigcirc$) and B $\subseteq N_2$. Then*

$$f^{-1}(B) = (x \in N_1: f(x) \in B)$$

is the neutrosophic inverse image of B under f.

Definition 16. *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from am NETG (N_1, $*$) to an NETG (N_2, $\bigcirc$). The neutrosophic kernel off is a subset*

$$ker(f) = \{x \in N_1 : f(x) = neut(x)\}$$

of N_1, where $neut(x)$ denotes the neutral element of N_2.

Example 9. *We took D_4, the symmetry NETG of the square, which consisted of four rotations and four reflections. We took a set of the four lines through the origin at angles 0, π/4, π/2, and 3π/4, numbered 1, 2, 3, 4, respectively. We let S_4 be the permutation NETG of the set of four lines. Each symmetry s, of the square in particular, gave a permutation φ(s) of the four lines. Then we defined a mapping, as follows:*

$$\Phi: D_4 \rightarrow S_4$$

whose value at the symmetry s $\in D_4$ was the permutation φ(s) of the four lines. Such a process would always define a neutro-homomorphism. We found the kernel and image of φ. The neutral permutation of the square gave the neutral of the four lines. The rotation (1234) of the square gave the permutation (13)(24) of the four lines; the rotation (13)(24) by 180 degrees gave the neutral permutation e^{neut} of the four lines; the rotation (4321) of the square gave the permutation (13)(24) of the four lines again. Thus, the neutrosophic image of the rotation NET subgroup R_4 of D_4 was the NET subgroup (neut, [13][24]) of S_4. The reflections of the square were given by the compositions of the rotations of the square with a reflection, for example, the reflection (13). The reflection (13) of the square (in the vertical axis) gave the permutation (24) of the lines. Thus, the homomorphism φ took the set of reflections $R_4 \bigcirc$ (13) to the following:

$$\varphi(R4) \bigcirc \phi(13) = (neut, [13][24] \bigcirc [24]) = ([24], [13]).$$

The neutrosophic image of φ was the union of the neutrosophic image of the rotations and the reflections, which was Im(φ) = (neut, [13][24], [13], [24]) $\in S_4$. In the work above, we saw that the neutrosophic kernel of φ was as follows:

$$ker(\varphi) = (neut, [13][24]) \text{ of } D_4$$

3. Neutrosophic Extended Triplet Subgroup

In this section, a definition of the neutrosophic extended triplet subgroup and its example have been given.

Definition 17. *Given an NETG (N, $*$), a subset H is called an NET subgroup of N, if it forms an NETG itself under $*$. Explicitly, this means the following:*

(1) The extended neutral element $e^{neut(x)}$ lies $\in$ H.
(2) For any x, y $\in$ H, $x * y \in$ H (H is closed under $*$).
(3) If x $\in$ H, then $e^{anti(x)} \in$ H (H has extended opposites).

We wrote H $\leq$ N whenever H was an NET subgroup of N. $\emptyset \neq$ H $\subseteq$ N, satisfying (2) and (3) of Definition 17, would be an NET subgroup, as we took $x \in$ H and then (2) gave $e^{anti(x)} \in$ H, after which (3) gave $x * e^{anti(x)} = e^{neut(x)} \in$ H.

Example 10. *We let S_4 = (neut, σ_1, σ_2, ... , σ_9, τ_1, τ_2, ... , τ_8, δ_1, δ_2, ... , δ_6) with σ_1 = (1234), σ_2 = (13)(24), σ_3 = (1432), σ_4 = (1243), σ_5 = (14)(23), σ_6 = (1342), σ_7 = (1324), σ_8 = (12)(34),σ_9 = (1432),τ_1 = (234),τ_2 = (243), τ_3 = (134), τ_4 = (143), τ_5 = (124), τ_6 = (142), τ_7 = (123), τ_8 = (132), δ_1 = (12), δ_2 = (13), δ_3 = (14), δ_4 = (23), δ_5 = (24), δ_6 = (34). The trivial neutrosophic extended subgroups of S_4 were the neutral elements, and the non-trivial neutrosophic extended subgroups S_4 of order 2 were as follows: (neut, σ_2), (neut, σ_5), (neut, σ_8), (neut, δ_1), (neut, δ_2), (neut, δ_3), (neut, δ_4), (neut, δ_5), (neut, δ_6), and the neutrosophic extended subgroups, S_4, of order 3 were as follows:*

$$L_{11} = \langle\tau_1\rangle = \langle\tau_2\rangle = (neut, \tau_1, \tau_2)$$

$$L_{12} = \langle\tau_3\rangle = \langle\tau_{14}\rangle = (neut, \tau_3, \tau_4)$$

$$L_{13} = \langle\tau_5\rangle = \langle\tau_6\rangle = (neut, \tau_5, \tau_6)$$

$$L_{14} = \langle\tau_7\rangle = \langle\tau_8\rangle = (neut, \tau_7, \tau_8)$$

it was straightforward to find the neutrosophic extended subgroups of order 4, 6, 8, and 12 of S_4.

4. Neutrosophic Triplet Cosets

In this section, the neutrosophic triplet coset and its properties have been outlined. Furthermore, the difference between the neutrosophic triplet coset and the classical one have been given.

Definition 18. *Let N be an NETG and $H \subseteq N$. $\forall x \in N$, the set $xh/ h \in H$, is denoted by xH, analogously, as follows:*

$$Hx = hx/h \in H$$

and

$$(xH)anti(x) = (xh)anti(x)/h \in H.$$

When $h \leq N$, xH is called the left neutrosophic triplet coset of $H \in N$ containing x, and Hx is called the right neutrosophic triplet coset of $H \in N$ containing x. In this case, the element x is called the neutrosophic triplet coset representative of xH or Hx. $|xH|$ and $|Hx|$ are used to denote the number of elements in xH or Hx, respectively.

Example 11. *When N = S_3 and H = ([1], [12]), the "Table 3" lists the left and right neutrosophic triplet H-cosets of every element of the NETG.*

Table 3. Neutrosophic triplet left and right cosets of S_3.

g	*gH*	*Hg*
(1)	([1], [12])	([1], [12])
(12)	([1], [12])	([1], [12])
(13)	([13], [123])	([13], [132])
(23)	([23], [132])	([23], [123])
(123)	([13], [123])	([23], [123])
(132)	([23], [132])	([23], [123])

First of all, cosets were not usually neutrosophic extended triplet subgroups (some did not even contain the extended neutral). In addition, since (13) $\neq$ H(13), a particular element could have different left and right neutrosophic triplet H-cosets. Since (13)H = H(13), different elements could have the same left neutrosophic triplet H-cosets.

Example 12. *We calculated the neutrosophic triplet cosets of N = (Z_4, +) under addition and let H = (0, 2). The elements (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (1, 1, 1), and (3, 3, 3) were NET's of Z_4 and the classical cosets of N were as follows:*

$$H = H + 0 = H + 2 = (0, 2).$$

and

$$H + 1 = H + 3 = (1, 3).$$

Here, 2 did not give rise to NET, because the neut's of 2 were 1 and 3, however there were no anti's. Therefore, we could not obtain the neutrosophic triplet coset of N. In general, classical cosets were not neutrosophic triplet cosets, because they might not have satisfied the NET conditions.

Similarly to Definition 16, we could define neutrosophic triplet cosets as follows:

Definition 19. *Let N be a neutrosophic triplet group and $H \leq N$. We defined a relation $\equiv \ell(modH)$ on N as follows:*

if x_1, $x_2 \in N$ and anti(x_1)$x_2 \in N$, Then

$$x_1 = l\ x_2(modH)$$

Or, equivalently, if there exists an $h \in H$, such that:

$$anti(x_1) * x_2 = h$$

That is, if $x_2 = x_1 h$ for some $h \in H$.

Proposition 2. *The relation $\equiv \ell(modH)$ is a neutrosophic triplet equivalence relation. The neutrosophic triplet equivalence class containing x is the set $xH = xh/h \in H$.*

Proof.

(1) $\forall x \in N_1$, anti(x) $*$ x = neut(x) $\in$ H. Hence, x = ℓx_1(modH)} and $\equiv \ell$(modH) is reflexive.
(2) İf x = ℓx_2(modH), then anti(x_1) $*$ $x_2 \in$ H. However, since an anti of an element of H is also in H, anti(anti[x_1] $*$ x_2) = anti(x_2) $*$ anti(anti[x_1]) = anti(x_2) $*$ $x_1 \in$ H. Thus, $x_2 = \ell x_1$(modH), hence $\equiv \ell$(modH) is symmetric.
(3) Finally, if $x_1 = \ell x_2$(modH) and $x_2 = \ell x_3$(modH), then anti(x_1) $*$ $x_2 \in$ H and anti(x_2) $*$ $x_3 \in$ H. Since H is closed under taking products, anti(x_1)$x_2$anti(x_2)x_3 = anti(x_1)$x_3 \in$ H. Hence, $x_1 = \ell x_3$(modH) so that $\equiv \ell$(modH) is transitive. Thus, $\equiv \ell$(modH) is a neutrosophic triplet equivalence relation. □

4.1. Properties of Neutrosophic Triplet Cosets

Lemma 1. *Let $H \leq N$ and let x, $y \in N$. Then,*

(1) $x \in xH$.
(2) $xH = H \Leftrightarrow x \in H$.
(3) $xH = yH \Leftrightarrow x \in yH$.
(4) $xH = yH$ *or* $xH \cap yH = \emptyset$.
(5) $xH = yH \Leftrightarrow anti(x)y \in H$.
(6) $xH = Hx \Leftrightarrow H = (xH)anti(x)$.
(7) $xH \subseteq N \Leftrightarrow x \in H$.
(8) $(xy)H = x(yH)$ *and* $H(xy) = (Hx)y$.
(9) $|xH| = |YH|$.

Proof.

(1) $x = x(neut(x)) \in xH$
(2) $\Rightarrow$ Suppose $xH = H$. Then $x = x(neut(x)) \in xH = H$.

$\Leftarrow$ Now assume x in H. Since H is closed, $xH \subseteq H$.

Next, also assume $h \in H$, so $anti(x)h \in H$, since $H \leq N$. Then,

$$h = neut(x)h = x * anti(x)h = x(anti[x])h \in xH,$$

So $H \subseteq xH$. By mutual inclusion, $xH = h$.
(3) $xH = Yh$

$\Rightarrow x = x(neut(x)) \in xH = yH$.

$\Leftarrow x \in yH \Rightarrow x = yh$, where $h \in H \Rightarrow h \in H$, $xH = (yh)H = y(hH) = yH$.
(4) Suppose that $xH \cap yH \neq \emptyset$. Then, $\exists a \in xH \cap yH \Rightarrow \exists h_1 h_2 \in H \ni a = xh_1$

and

$a = yh_2$. Thus, $x = a(anti(h_1)) = yh_2(antih_1)$ and $xH = yh_2(anti(h_1))H$

$= yh_2(anti(h_1)H) = yH$ by (2) of Lemma 1.
(5) $xH = yH \Leftrightarrow H = anti(x)yH \Leftrightarrow$ (2) of Lemma 1, $anti(x)y \in H$.
(6) $xH = Hx \Leftarrow (xH)anti(x) = (Hx)anti(x) = H(x * anti(x) = H \Leftarrow xH(anti(x)) = H$.
(7) (That is, $xH = H$)

Suppose thay xH is a neutrosophic extended triplet subgroup of N. Then

xH contains the identity, so $xH = H$ by (3) of Lemma 1, which holds $\Leftrightarrow x \in H$ by (2) of Lemma 1.

Conversely, if $x \in H$, then $xH = H \leq N$ by (2) of Lemma 1.
(8) $(xy)H = x(yH)$ and $H(xy) = (Hx)y$ follows from the associative

property of group multiplication.
(9) (Find a map α: $xH \longrightarrow xH$ that is one to one and onto)

Consider α: $xH \longrightarrow xH$ defined by $\alpha(xh) = yh$. This is clearly onto yH. Suppose $\alpha(xh_1)$

$= \alpha(xh_2)$. Then $yh_1 = yh_2 \Rightarrow h_1 = h_2$ by left cancellation $\Rightarrow xh_1 = xh_2$, therefore α is one to one. Since α provides a one to one correspondence between xH and yH, $|xH| = |yH|$. □

In classical group theory, cosets were used in the construction of vitali sets (a type of non-measurable set), and in computational group theory cosets were used to decode received data in linear error-correcting codes, to prove Lagrange's theorem. The neutrosophic triplet coset plays a similar role in the theory of neutrosophic extended triplet group, as in the classical group theory. Neutrosophic triplet cosets could be used in areas, such as neutrosophic computational modelling, to prove Lagrange's theorem in the neutrosophic extended triplet, etc.

4.2. The Index and Lagrange's Theorem: $|H|$ divides $|N|$

Theorem 3 *If N is a finite neutrosophic extended triplet group and $H \leq N$, then $|H|/|N|$. Moreover, the number of the distinct left neutrosophic triplet cosets of H in N is $|N|/|H|$.*

Proof. Let $x_1H, x_2H, \ldots, x_rH$ denote the distinct left neutrosophic triplet cosets of H in N. Then, ∀ x ∈ N. xH = x_iH for some i = 1, 2, . . . , r. Considering (1) of Lemma 1, x ∈ xH. Thus, N = $x_1H \cup x_2H \cup, \ldots, \cup x_rH$. Considering (4) of Lemma 1, this union is disjointed:

$$|N| = |x_1H| + |x_2H| + \ldots + |x_rH = r|H|.$$

Therefore: $|x_iH| = |xH|$ for i = 1, 2, . . . , r. □

Example 13. *We let H = ([1], [12]), it had three left neutrosophic triplet cosets in S3, see example 11, [S3:H] = 3 = (H,* [13]*H,* [23]*H) = (H,* [13]*H,* [23]*H).*

5. Neutrosophic Triplet Normal Subgroups

In this section, the neutrosophic triplet normal subgroup and neutrosophic triplet normal subgroup test have been outlined.

Definition 20. *A neutrosophic extended triplet subgroup H of a neutrosophic extended triplet group N is called a neutrosophic triplet normal subgroup of N, if xH = Hx, ∀ x ∈ N and we denote it as H* $\trianglelefteq$ *N.*

Example 14. *The set* $A_n = \sigma \in S_n/\sigma$ *was even a normal subgroup of* S_n*. It was called the alternating neutrosophic extended triplet group on n letters. It was enough to notice that* A_n *= ker(sgn). Since* $|S_n| = n!$*, thus,*

$$|A_n| = n!/2.$$

$$S_n/A_n = n!/n!/2 = 2.$$

Neutrosophic Triplet Normal Subgroup Test

Theorem 4 *A neutrosophic extended triplet subgroup H of N is normal in N if, and only if, anti(x)Hx ⊆ H,*

$$\forall\, x \in N.$$

Proof. *Let H be a neutrosphic extended triplet subgroup of N. Suppose H is neutrosophic extended triplet subgroup of N. Then ∀ x ∈ N, y ∈ H : ∃z ∈ H : xy = zx. Thus (xy)anti(x) = z ∈ H implying (xH)anti(x) ⊆ H.* □

Conversly, suppose ∀ x ∈ N :(xH)anti(x) ⊆ H. Then for n ∈ N, we have (nH)anti(n) ⊆ H, which implies nH ⊆ Hn. Also, for anti(n) ∈ N, we have anti(n)H(anti[anti{n}]) = anti(n)Hn ⊆ H, which implies Hn ⊆ nH. Therefore, nH = Hn, meaning that H $\trianglelefteq$ N.

Example 15. *We let f: N → H be a neutro-homomorphism from a neutrosophic extended triplet group N to a neutrosophic extended triplet group H, Kerf* $\trianglelefteq$ *N.*

(1) If ∀ a, b ∈ kerf, we had to show that a(anti[b]) ∈ kerf. This meant that kerf was a neutrosophic extended triplet subgroup of N. If a ∈ kerf, then

$$f(a) = neut_H$$

and

b ∈ kerf, then

$$f(b) = neut_H$$

Then, we showed that f(a(anti[b]) = $neut_H$*. (f is neutro-homomorphism)*

$f(a(anti(b)) = f(a) \,.\, f(anti(b))$

$= f(a) \,.\, f(anti(b))$

$= neut_H \,.\, anti(neut_H)$

$= neut_H \,.\, neut_H$

$= neut_H$

$\Rightarrow a(anti(b)) \in kerf.$

(2) *We let* $n \in N$ *and* $a \in kerf$*. We had to show that* $n \,.\, a \,.\, (anti(n)) \in kerf$*. (f is neutro-homomorphism)*

$f(n \,.\, a \,.\, (anti(n) = f(n) \,.\, f(a) \,.\, f(anti(n))$

$= f(n) f(a)\, anti(f(n))$

$= h\, neut_H\, (anti(h))$

$= neut_H$

$\Rightarrow n \,.\, a \,.\, (anti(n)) \in kerf$

$\Rightarrow kerf \lhd N.$

Theorem 5 . *A neutrosophic triplet subgroup H of N is a neutrosophic triplet normal subgroup of N if, and only if, each left neutrosophic triplet coset of H in N is a right neutrosophic triplet coset of* $H \in N$.

Proof. *Let H be a neutrosophic triplet normal subgroup of N, then* $xH(anti[x])=H$, $\forall\ x \in N \Rightarrow xH(anti[x])x = Hx$, $\forall x \in N \Rightarrow xH = Hx$, $\forall x \in N$*, since each left neutrosophic triplet coset xH is the right neutrosophic triplet coset Hx.* □

Conversely, let each left neutrosophic triplet coset of H in N be a right neutrosophic triplet coset of H in N. This means that if x is any element of N, then the left neutrosophic triplet coset xH is also a right neutrosophic triplet coset. Now neut(x) ∈ H, therefore $x * neut(x) = x \in xH$. Consequently x must also belong to that right neutrosophic triplet coset, which is equal to left neutrosophic triplet coset xH. However, x is a left neutrosophic triplet coset and needs to contain one common element before they are identical. Therefore, Hx is the unique right neutrosophic triplet coset which is equal to the left neutrosophic triplet coset xH. Therefore, we have xH = xH, ∀x ∈ N ⇒ xH(anti(x)) = Hx(anti(x), ∀x ∈ N ⇒ xH(anti(x)) = H, ∀x ∈ N, since H is a neutrosophic triplet normal subgroup of N.

6. Neutrosophic Triplet Quotient (Factor) Groups

The notion of quotient (factor) groups was one of the central concepts of classical group theory and played an important role in the study of the general structure of groups. Just as in a classical group theory, quotient groups played a similar role in the theory of neutrosophic extended triplet group. In this section, we have introduced the notion of neutrosophic triplet quotient group and its relation to the neutrosophic extended triplet group.

Definition 21. *If N is a neutrosophic extended triplet group and* $H \trianglelefteq N$ *is a neutrosophic triplet normal subgroup, then the neutrosophic triplet quotient group N/H has elements* xH: $x \in N$*, the neutrosophic triplet cosets of H in N, and an operation of* $(xH)(yH) = (xy)H$.

Example 16. *Let's find all of the possible neutrosophic triplet quotient groups for the dihedral group* D_3.

$D_3 = (1, r, r^2, s, sr, sr^2)$*, where* $r^3 = s^2 = rsrs = 1$*. A quotient set* D_3/N *is a neutrosophic triplet group if, and only if,* $N \trianglelefteq D_3$*. Then, all of neutrosophic triplet normal subgroups are* $D_{3\ itself}$*. We always have the trivial ones* $D_3/D_3 = 1 \cong 1$ *and* $D_3/1 \cong D_3$*. The subgroup* $\langle r\rangle = \langle r^2\rangle = (1, r, r^2)$ *is that of index 2 and thus is* *normal. Therefore,* $D_3/\langle r\rangle$ *is also a neutrosophic triplet quotient group. If* $N \trianglelefteq D_3$ *is a different neutrosophic triplet normal subgroup, then* $\langle N\rangle. = 2$*, so either* $N = \langle s\rangle$, $N = \langle sr\rangle$. *or* $N = \langle sr^2\rangle$. *However, none of them are normal, since* $(sr)s(anti(sr)) = sr^2$ *not in* $\langle s\rangle$. *Hence, the only non-triavial neutrosophic triplet quotient group is* $D_{3/}\langle r\rangle$.

Theorem 6 *Let N be a neutrosophic extended triplet group and H be a neutrosophic triplet normal subgroup of N. In the set N/H = xH, x ∈ N is a neutrosophic extended triplet group under the operation of (xH)(yH) = xyH.*

Proof. N/H × N/H → N/H

1. $xH = x'H$ and $yH = y'H$

   $Xh_1 = x'$ and $yh_2 = y'$, $h_1, h_2 \in H$

   $x'y'H = xh_1yh_2H = xh_1yH = x\, h_1Hy = xHy = xyH$.
2. The neutral, for any $x \in H$, is neut(x)H = H. That is, $xH * H = xH * neut(x)H = x * neut(x)H = xH$.
3. An anti of a neutrosophic triplet coset xH is anti(x)H, since $xH*$ anti(x)H = (x ∗ anti(x)H) = neut(x)H = H.
4. Associativity, (xHyH)zH = (xy)HzH = (xy)zH = xH(yz)H = xH(yHzH), ∀x, y, z ∈ N. □

7. Conclusions

The main theme of this paper was to introduce the neutrosophic extended triplets and then to utilize these neutrosophic extended triplets in order to introduce the neutrosophic triplet cosets, neutrosophic triplet normal subgroup, and finally, the neutrosophic triplet quotient group. We also studied some interesting properties of these newly created structures and their application to neutrosophic extended triplet group. We further defined the neutrosophic kernel, neutrosophic-image, and inverse image for neutrosophic extended triplets. As a further generalization, we created a new field of research, called Neutrosophic Triplet Structures (namely, the neutrosophic triplet cosets, neutrosophic triplet normal subgroup, and neutrosophic triplet quotient group).

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