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On clean neutrosophic rings

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Abstract. A commutative ring is said to be clean if every element of the ring can be written as a sum of a unit and an idempotent. In this paper, we generalize this argument to structure of neutrosophic. We present the structure of clean neutrosophic ring. Some elementary properties of clean neutrosophic ring are also presented.

1. Introduction

The main idea we introduce the concept of clean neutrosophic ring are clean ring and neutrosophic element. By combining the two concepts we can define a new neutrosophic algebraic structure called a clean neutrosophic ring.

Let we have a commutative ring with a multiplicative identity. The ring is said to be clean if every element of this ring is clean. As we know, an element of the ring is called a clean element if it can be expressed as the sum of an idempotent (or idempotent element) and a unit. Clean rings have been widely studied by researchers, among of them by Nicholson in his fundamental article [1]. In this article, he showed that every clean ring is an exchange ring. He also gave necessary and sufficient conditions for a ring with central idempotents is a clean ring, that is, the ring is an exchange ring. Many authors and researchers have studied clean rings their further properties, and their generalizations such as [2, 3, 5] and [4].

The neutrosophic theory, for the first time introduce by Smarandache in 1980. The theory has given way to the construction of the concept of new algebraic structures called the neutrosophic structure. The concept of neutrosophic algebraic structures introduced by Kandasamy and her college, Smarandache in [7]. In this reference, several neutrosophic algebraic structures are introduced and studied. The substance of this reference include neutrosophic groups and n-group, neutrosophic subgroups, neutrosophic loops, neutrosophic groupoids and mixed neutrosophic structures. The study of neutrosophic rings was introduced by Kandasamy and Smarandache in [6]. Then Agboola et.al. in [8, 9] provide answer to the questions in [6] were raised by Kandasamy and Smarandache. In [8] given several aspect related to the neutrosophic algebraic structures and some types of this structure, concerning neutrosophic ring and its elementary properties, neutrosophic polynomial rings, and the unique factorization domain in neutrosophic polynomial rings. In article [9], as the continuation of the work started in [8], present the concept of ideals in neutrosophic rings and studied neutrosophic quotient rings.

Motivated by the result of Anderson and Camillo [2] on commutative clean rings, Nicholson [5] on uniquely clean rings, and Kandasamy and Smarandache [6] on neutrosophic rings. In this article we extend to study clean neutrosophic rings and clean elements, according to some known results of [2] and [6]. Also, we give some elementary properties of a (uniquely) clean neutrosophic ring.

2. Neutrosophic rings and related algebraic aspects

In the second section, we give several main definitions and results about neutrosophic rings. To learn more about neutrosophic ring, the readers can see [6].

Definition 2.1 Let $(R,+,\cdot)$ be a ring. Then the set of all elements $x = a + bI$, with $a, b \in R$ and I is an element such that $I^2 = I$, denoted by $\langle R \cup I \rangle$ is the neutrosophic ring formed by R and I, with respect to the addition and the multipcication on R.

We give the two following examples to explain the existence of this set.

Example 2.2 Let $\mathbb{E} = 2\mathbb{Z}$ be the ring of even integers. The set $\langle \mathbb{E} \cup I \rangle = \{a + bI | a, b \in \mathbb{E}\}$ is called the neutrosophic ring of even integers. Similarly for rings $M(2,\mathbb{Z})$, $\mathbb{Z}[\sqrt{5}]$, and $\mathbb{Q}$, we have $\langle M(2,\mathbb{Z}) \cup I \rangle$, $\langle \mathbb{Z}[\sqrt{5}] \cup I \rangle$, and $\langle \mathbb{Q} \cup I \rangle$ are sets called the neutrosophic ring of 2×2 matrices with integers as entries, numbers $a + b\sqrt{5}$ with a, b are integers, and rationals numbers, respectively.

Example 2.3 The set $\langle \mathbb{Z}_7 \cup I \rangle$ is called the neutrosophic ring of integers modulo 7 generated by ring $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ and an idempotent element I.

In view of the two examples, now we have the following result.

Theorem 2.4 If $R = (R,+,\cdot)$ is any ring , then $(\langle R \cup I \rangle,+,\cdot)$ is a ring and the ring always contains a proper (classical) ring.

Proof: The reader should see [6], pp. 32 – 33. Obviously, $R \subsetneq \langle R \cup I \rangle$.

Now we give two kinds of neurosophic ring, as stated in the definition below.

Definition 2.5 A neutrosophic ring is said to be commutative if its multiplicative operation is commutative. Moreover if there exists a multiplicative identity 1 and satisfies $1 \cdot x = x = x \cdot 1$ for every x in the neutrosophic ring, then we call the multiplicative identity 1 is a unity and the ring is a neutrosophic ring with unity. Thus a unity is simply a multiplicative identity (or an identity for multiplication).

Furthermore we proceed on to define some algebraic aspects related to the neutrosophic ring.

Definition 2.6 Let $\langle R \cup I \rangle$ be a neutrosophic ring and $e \in \langle R \cup I \rangle$. The element e is an idempotent (or idempotent element) for multiplivative operation on $\langle R \cup I \rangle$ if $e^2 = e \cdot e = e$.

Example 2.7 In $\langle \mathbb{Z}_3 \cup I \rangle$, the neutrosophic ring of integer modulo 3, $\overline{0}$ and $\overline{1}$ are idempotent elements.

Definition 2.8 Let $\langle R \cup I \rangle$ be a neutrosophic ring and $e \in \langle R \cup I \rangle$. The element $e = a + bI$ is a neutrosophic idempotent (or neutrosophic idempotent element) for multiplivative operation on $\langle R \cup I \rangle$ if $b \neq 0$ and $e^2 = e$.

Example 2.9 In $\langle \mathbb{Z}_4 \cup I \rangle$, the neutrosophic ring of integer modulo 4, I and $\overline{1} + \overline{3}I$ are neutrosophic idempotents, because $I^2 = I$ and $(\overline{1} + \overline{3}I)^2 = \overline{1} + \overline{3}I$.

Lemma 2.10 Every neutrosophic idempotent element is an idempotent.

Proof: Clearly by Definition 2.6

Definition 2.11 Let $\langle R \cup I\rangle$ be a neutrosophic ring with $1 \neq 0$ and let $u \in \langle R \cup I\rangle$. The element u is a unit of $\langle R \cup I\rangle$ if there is an element $v \in \langle R \cup I\rangle$ and satisfies $u \cdot v = 1 = v \cdot u$.

Example 2.12 The units of $\langle \mathbb{Z} \cup I\rangle$ are 1 and -1.

Definition 2.13 Let $\langle R \cup I\rangle$ be a neutrosophic ring with $1 \neq 0$ and let $u \in \langle R \cup I\rangle$. The element $u = a + bI \in \langle R \cup I\rangle$ with $b \neq 0$ is a neutrosophic unit of $\langle R \cup I\rangle$ if there is an element $v = c + dI \in \langle R \cup I\rangle$ with $d \neq 0$ and satisfies $u \cdot v = 1 = v \cdot u$.

Example 2.14 In $\langle \mathbb{Z}_7 \cup I\rangle$, the neutrosophic ring of integer modulo 7, $\overline{3} + I$ is a neutrosophic unit in $\langle \mathbb{Z}_7 \cup I\rangle$, because there exist $\overline{5} + \overline{4}I \in \langle \mathbb{Z}_7 \cup I\rangle$ and we have $(\overline{3} + I)(\overline{5} + \overline{4}I) = \overline{1} = (\overline{5} + \overline{4}I)(\overline{3} + I)$.

3. Clean neutrosophic rings and their elementary properties

In the third section, unless otherwise stated, all neutrosophic rings are commutative neutrosophic ring with unity.

We will define a clean element.

Definition 3.1 Assume that $\langle R \cup I\rangle$ is a neutrosophic ring and that $x \in \langle R \cup I\rangle$. The element x is said to be clean if $x = e + u$, with e is an idempotent element and u is a unit of $\langle R \cup I\rangle$.

In this section, we use the notation $U(\langle R \cup I\rangle)$ to express the set of all units in $\langle R \cup I\rangle$ and $Id(\langle R \cup I\rangle)$ to the set of all idempotent elements in $\langle R \cup I\rangle$.

Now we go for example of clean elements.

Example 3.2 Review the ring $\langle \mathbb{Z}_2 \cup I\rangle = \{\overline{0}, \overline{1}, I, \overline{1} + I\}$. We have $U(\langle \mathbb{Z}_2 \cup I\rangle) = \{\overline{1}\}$ and $Id(\langle \mathbb{Z}_2 \cup I\rangle) = \{\overline{0}, \overline{1}, I, \overline{1} + I\}$. It is easily seen that each element of $\langle \mathbb{Z}_2 \cup I\rangle$ can be expressed as a sum of an idempotent and a unit in $\langle \mathbb{Z}_2 \cup I\rangle$. Thus all elements of $\langle \mathbb{Z}_2 \cup I\rangle$ are clean elements.

We define the two types of neutrosophic ring, a clean and a uniquely clean, respectively.

Definition 3.3 A neutrosophic ring in which all elements are clean, then the ring is called a clean neutrosophic ring. Furthermore, if each element of the neutrosophic ring is uniquely clean, then the ring is called a uniquely clean neutrosophic ring.

We illustrate the two rings by the following examples.

Example 3.4 According to example 3.2, clearly $\langle \mathbb{Z}_2 \cup I\rangle$ is a clean neutrosophic ring. Since every element of $\langle \mathbb{Z}_2 \cup I\rangle$, its presentation as sum of an idempotent and a unit is unique, so the ring $\langle \mathbb{Z}_2 \cup I\rangle$ to be uniquely clean.

Example 3.5 Review the ring $\langle \mathbb{Z}_3 \cup I\rangle$. We have $Id(\langle \mathbb{Z}_3 \cup I\rangle) = \{\overline{0}, \overline{1}, I, \overline{1} + \overline{2}I\}$ and $U(\langle \mathbb{Z}_3 \cup I\rangle) = \{\overline{1}, \overline{2}, \overline{1} + I, \overline{2} + \overline{2}I\}$. All elements of $\langle \mathbb{Z}_3 \cup I\rangle$ are clean elements. Take $\overline{2} + I$ in $\langle \mathbb{Z}_3 \cup I\rangle$, clearly $\overline{2} + I = I + \overline{2}$, and also we have $\overline{2} + I = \overline{1} + (\overline{1} + I)$. So $\langle \mathbb{Z}_3 \cup I\rangle$ is not the uniquely clean neutrosophic ring. It is only a clean neutrosophic ring.

Now we give some elementary properties of clean neutrosophic rings. The following result will be needed.

Lemma 3.6 Suppose $\langle R \cup I \rangle$ is a neutrosophic ring with 1. If $e \in Id(\langle R \cup I \rangle)$, then $1 - e \in Id(\langle R \cup I \rangle)$, where 1 is unity element in $\langle R \cup I \rangle$.

Proof: Let $e^2 = e \in \langle R \cup I \rangle$ and $f = 1 - e$. We have

$$f^2 = (1-e)^2 = (1-e)(1-e) = (1-e) - (e - e^2) = (1-e) - (e-e) = 1 - e = f$$

Hence $f = 1 - e \in Id(\langle R \cup I \rangle)$.

Definition 3.7 Let $\langle R \cup I \rangle$ be a neutrosophic ring and $e \in \langle R \cup I \rangle$ is an idempotent element. The idempotent e is a central if $e \cdot x = x \cdot e$ for every $x \in \langle R \cup I \rangle$. The set of all central idempotent of $\langle R \cup I \rangle$ is denoted by $C(\langle R \cup I \rangle)$.

Example 3.8 In the neutrosophic ring $\langle \mathbb{Z}_3 \cup I \rangle$, we have $Id(\langle \mathbb{Z}_3 \cup I \rangle) = \{\bar{0}, \bar{1}, I, \bar{1} + \bar{2}I\}$. Since the ring $\langle \mathbb{Z}_3 \cup I \rangle$ is commutative so all idempotent elements of $\langle \mathbb{Z}_3 \cup I \rangle$ are central. Hence $C(\langle \mathbb{Z}_3 \cup I \rangle) = \{\bar{0}, \bar{1}, I, \bar{1} + \bar{2}I\} = Id(\langle \mathbb{Z}_3 \cup I \rangle)$.

Lemma 3.9 Let $\langle R \cup I \rangle$ be a neutrosophic ring with the identity 1. If $e \in C(\langle R \cup I \rangle)$, then $1 - e \in C(\langle R \cup I \rangle)$, where $e \in Id(\langle R \cup I \rangle)$.

Proof: Let $e \in C(\langle R \cup I \rangle)$ and $f = 1 - e$. If $x \in \langle R \cup I \rangle$ then

$$fx = (1-e)x = (1 \cdot x) - (e \cdot x) = (x \cdot 1) - (x \cdot e) = x(1-e) = xf.$$

This prove that $f = 1 - e \in C(\langle R \cup I \rangle)$.

Theorem 3.10 In any neutrosophic ring, a central idempotent is a uniquely clean element.
Proof: Let $e^2 = e$ we have $e = (1-e) + (2e-1)$. Suppose that $e = f + u$, $f^2 = f$, where $u \in U(\langle R \cup I \rangle)$. If $eu = ue$ we obtain $f + u = (f+u)^2 = f + 2fu + u^2$, so $u = 1 - 2f$. Hence $f = 1 - e$, as required.

Back to the uniquely clean neutrosophic ring, we have

Lemma 3.11 Every idempotent element in a uniquely clean neutrosophic ring is central.

Proof: Let e is an idempotent element in $\langle R \cup I \rangle$. If $x \in \langle R \cup I \rangle$ then $e + (ex - exe)$ is an idempotent and $1 + (ex - exe)$ is a unit, so the fact that $[e + (ex - exe)] + 1 = e + [1 + (ex - exe)]$ implies that $e + (ex - exe) = e$ because $\langle R \cup I \rangle$ is uniquely neutrosophic ring. Hence $ex = exe$ and $xe = exe$, so $ex = xe$ as required.

Theorem 3.10 and Lemma 3.11 give immediately the following conjecture.

Conjecture 3.12 Assume that the ring $\langle R \cup I \rangle$ is uniquely clean and that e is an idempotent in $\langle R \cup I \rangle$. Then $e\langle R \cup I \rangle e$ is uniquely clean neutrosophic ring.

To answer this conjecture, note that we never have the ring $e\langle R \cup I \rangle e$ be a (uniquely) clean neutrosophic ring. The following example explains the statement.

Example 3.13 Consider the neutrosophic ring $\langle \mathbb{Z}_2 \cup I \rangle$. Clearly $\langle \mathbb{Z}_2 \cup I \rangle$ uniquely clean and we have $Id(\langle \mathbb{Z}_2 \cup I \rangle) = \{\bar{0}, \bar{1}, I, \bar{1} + I\}$. The ring $(\bar{1} + I)\langle \mathbb{Z}_2 \cup I \rangle(\bar{1} + I) = \{\bar{0}, \bar{1} + I\}$ has not a unity. Hence $(\bar{1} + I)\langle \mathbb{Z}_2 \cup I \rangle(\bar{1} + I)$ cannot be a clean neutrosophic ring.

For the boolean neutrosophic ring, we have the following result. A boolean neutrosophic ring be defined as a neutrosophic ring in which all elements are idempotent; that is $x^2 = x$ for all $x \in \langle R \cup I \rangle$.

Theorem 3.14 A neutrosophic ring $\langle R \cup I \rangle$ is boolean neutrosophic ring $\Leftrightarrow Id(\langle R \cup I \rangle) = \langle R \cup I \rangle$.

Proof : (⇒) Assume that $\langle R \cup I \rangle$ is a boolean neutrosophic ring. Clearly $Id(\langle R \cup I \rangle) \subseteq \langle R \cup I \rangle$. If $x \in \langle R \cup I \rangle$, we have $x^2 = x$ so $x \in Id(\langle R \cup I \rangle)$. Hence $Id(\langle R \cup I \rangle) = \langle R \cup I \rangle$, as required.
(⇐) Let $Id(\langle R \cup I \rangle) = \langle R \cup I \rangle$. Since all elements of $\langle R \cup I \rangle$ are idempotent elements, clearly $\langle R \cup I \rangle$ is boolean.

Finally we have result

Theorem 3.15 Every boolean neutrosophic ring is uniquely clean.

Proof: Suppose $\langle R \cup I \rangle$ is a boolean neutrosophic ring. Since $\langle R \cup I \rangle = Id(\langle R \cup I \rangle)$ and $Id(\langle R \cup I \rangle) = C(\langle R \cup I \rangle)$ we have all elements of $\langle R \cup I \rangle$ are central idempotents. According to Theorem 3.10, all elements of the ring $\langle R \cup I \rangle$ are uniquely clean. Hence $\langle R \cup I \rangle$ is a uniquely clean neutrosophic ring.

4. Conclusion
From this discussion, concluded:

a. A (clean) neutrosophic ring is a (classical) ring.
b. Although $\langle \boldsymbol{R} \cup \boldsymbol{I} \rangle$ is a (uniquely) clean neutrosophic ring, the ring $\boldsymbol{e}\langle \boldsymbol{R} \cup \boldsymbol{I} \rangle \boldsymbol{e}$ may not have a clean ring structure, where $\boldsymbol{e}$ is an idempotent.
c. The neutrosophic ring in which all elements are idempotent is uniquely clean.

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