

On Neutrosophic Feebly Frontier in Neutrosophic Topological Spaces

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Abstract

In this paper, we introduce a new class of operator using neutrosophic feebly closed sets namely neutrosophic feebly frontier. Also we study some of their basic properties and their characterizations.

Keywords and Phrases - Neutrosophic feebly closed sets, neutrosophic feebly frontier.

I. INTRODUCTION

Neutrosophy, as a new branch of Philosophy has been introduced by Smarandache [6-9] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set [2]. In 2012, Salama, Alblowi [12-15], introduced the concept of neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of intuitionistic fuzzy topological space and a neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element. In 2014, Salama, Smarandache and Valeri [14] were introduced the concept of neutrosophic closed sets and neutrosophic continuous functions. The authors defined the concept of neutrosophic feebly open sets and neutrosophic feebly closed sets in neutrosophic topological spaces.

In this paper, we have to introduce neutrosophic feebly frontier of a neutrosophic set using neutrosophic feebly closed set. This paper consists of three sections. The introduction given in Section I. The Section II consists of the basic definitions of neutrosophic feebly open sets, neutrosophic feebly closed sets and their properties which are used in the later sections. The Section III deals with the concept of neutrosophic feebly frontier in neutrosophic topological space and their properties.

II. PRELIMINARIES

For basic notations and definitions of Neutrosophic Theory is not given here, the reader can refer [1-16].

Definition 2.1. [11] A neutrosophic subset A of a neutrosophic topological space (X, τ) is neutrosophic feebly open if there is a neutrosophic open set U in X such that $U \leq A \leq NScl(U)$.

Lemma 2.2. [11] A neutrosophic subset A is neutrosophic feebly open if and only if

$$(i) A \leq Nint(Ncl(Nint(A))).$$

$$(ii) A \leq NScl(Nint(A)).$$

Lemma 2.3. [11] Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces such that X is product related to Y . Then the product $A_1 \times A_2$ of a neutrosophic feebly open set A_1 of X and a neutrosophic feebly open set A_2 of Y is a neutrosophic feebly open set of the neutrosophic product space $X \times Y$.

Definition 2.4. [11] A neutrosophic subset A of a neutrosophic topological space (X, τ) is neutrosophic feebly closed if there is a neutrosophic closed set U in X such that $NSint(U) \leq A \leq U$.

Lemma 2.5. [11] A neutrosophic subset A of a neutrosophic topological space (X, τ) is neutrosophic feebly closed if and only if $Ncl(Nint(Ncl(A))) \leq A$.

$$(i) \quad \text{A neutrosophic subset } A \text{ is neutrosophic feebly closed iff } NSint(Ncl(A)) \leq A.$$

- (ii) A neutrosophic subset A is a neutrosophic feebly closed set if and only if A^c is neutrosophic feebly open.

Lemma 2.6. [11] Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces such that X is product related to Y . Then the product $A_1 \times A_2$ of a neutrosophic feebly closed set A_1 of X and a neutrosophic feebly closed set A_2 of Y is a neutrosophic feebly closed set of the neutrosophic product space $X \times Y$.

Definition 2.7. [11] Let (X, τ) be neutrosophic topological space and $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$ be a neutrosophic set in X . Then neutrosophic feebly interior of A is defined by $NFint(A) = \bigvee \{G : G \text{ is a neutrosophic feebly open set in } X \text{ and } G \leq A\}$.

Lemma 2.8. [11] Let (X, τ) be neutrosophic topological space. Then for any neutrosophic feebly subsets A and B of a neutrosophic topological space X , we have

- (i) $NFint(A) \leq A$
- (ii) A is neutrosophic feebly open set in $X \Leftrightarrow NFint(A) = A$
- (iii) $NFint(NFint(A)) = NFint(A)$
- (iv) If $A \leq B, NFint(A) \leq NFint(B)$
- (v) $NFint(A \wedge B) \leq NFint(A) \wedge NFint(B)$.
- (vi) $NFint(A \vee B) \geq NFint(A) \vee NFint(B)$.

Definition 2.9. [11] Let (X, τ) be neutrosophic topological space and $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$ be a neutrosophic set in X . Then the neutrosophic feebly closure is defined by $NFcl(A) = \bigwedge \{K : K \text{ is a neutrosophic feebly closed set in } X \text{ and } A \leq K\}$.

Lemma 2.10. [11] Let (X, τ) be a neutrosophic topological space. Then for any neutrosophic subset A of X ,

- (i) $(NFint(A))^c = NFcl(A^c)$
- (ii) $(NFcl(A))^c = NFint(A^c)$.

Lemma 2.11. [11] Let (X, τ) be a neutrosophic topological space. Then for any neutrosophic subsets A and B of a neutrosophic topological space X ,

- (i) $A \leq NFcl(A)$
- (ii) A is a neutrosophic feebly closed set in $X \Leftrightarrow NFcl(A) = A$
- (iii) $NFcl(NFcl(A)) = NFcl(A)$
- (iv) If $A \leq B$ then $NFcl(A) \leq NFcl(B)$.
- (v) $NFcl(A \vee B) \geq NFcl(A) \vee NFcl(B)$ and
- (vi) $NFcl(A \wedge B) \leq NFcl(A) \wedge NFcl(B)$.
- (vii) $NFcl(A) \times NFcl(B) \geq NFcl(A \times B)$,
- (viii) $NFint(A) \times NFint(B) \leq NFint(A \times B)$.

Lemma 2.12. [11] Let (X, τ) and (Y, σ) be two neutrosophic topological spaces such that X is neutrosophic product related to Y . Then for neutrosophic subsets A of X and B of Y ,

- (i) $NFcl(A \times B) = NFcl(A) \times NFcl(B)$,
- (ii) $NFint(A \times B) = NFint(A) \times NFint(B)$.

III. NEUTROSOPHIC FEEBLY FRONTIER

In this section, we introduce the neutrosophic feebly frontier and their properties in neutrosophic topological spaces.

Definition 3.1. Let A be a neutrosophic subset in the neutrosophic topological space X . Then the neutrosophic feebly frontier of a neutrosophic subset A is defined as neutrosophic subset $NFFr(A) = NFcl(A) \wedge NFcl(A^c)$.

Remark 3.2. For a neutrosophic subset A of X , $NFFr(A)$ is neutrosophic feebly closed.

Theorem 3.3. For a neutrosophic subset A in the neutrosophic topological space X , $NFFr(A) = NFFr(A^c)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(A) = NFcl(A) \wedge NFcl(A^c) = NFcl(A^c) \wedge NFcl(A) = NFcl(A^c) \wedge NFcl((A^c)^c)$. Again by Definition 3.1, this is equal to $NFFr(A^c)$. Hence $NFFr(A) = NFFr(A^c)$.

Theorem 3.4. Let A be a neutrosophic subset in the neutrosophic topological space X . Then $NFFr(A) = NFcl(A) - NFint(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . By Lemma 2.10(ii), $(NFcl(A^c))^c = NFint(A)$ and by Definition

3.1, $NFFr(A) = NFcl(A) \wedge (NFcl(A^c)) = NFcl(A) \wedge (NFint(A^c))^c$ by using $A - B = A \wedge B^c$, $NFFr(A) = NFcl(A) - NFint(A)$. Hence $NFFr(A) = NFcl(A) - NFint(A)$.

Theorem 3.5. A neutrosophic subset A is neutrosophic feebly closed set in X if and only if $NFFr(A) \leq A$.

Proof. Let A be the neutrosophic feebly closed set in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(A) = NFcl(A) \wedge NFcl(A^c) \leq NFcl(A)$. By Lemma 2.11(ii), $NFcl(A) = A$. Hence $NFFr(A) \leq A$, if A is neutrosophic feebly closed in X .

Conversely, Assume that, $NFFr(A) \leq A$. Then $NFcl(A) - NFint(A) \leq A$. Since $NFint(A) \leq A$, then we conclude that $NFcl(A) = A$ and hence A is neutrosophic feebly closed.

Theorem 3.6. If A is neutrosophic feebly open set in X , then $NFFr(A) \leq A^c$.

Proof. Let A be the neutrosophic feebly open set in the neutrosophic topological space X . By Lemma 2.5, A^c is neutrosophic feebly closed set in X . By Theorem 3.5, $NFFr(A^c) \leq A^c$ and by Theorem 3.3, we get $NFFr(A) \leq A^c$.

The converse of the above theorem is not true as shown by the following example.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{0_N, A, B, C, D, 1_N\}$. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are $\tau^c = \{1_N, F, G, H, I, 0_N\}$ where,

$$\begin{aligned} A &= \langle (0.7, 0.2, 0.3), (0.1, 0.8, 0.4), (0.5, 0.6, 0.7) \rangle, \\ B &= \langle (0.9, 0.6, 0.7), (0.5, 0.4, 0.2), (0.8, 0.8, 0.5) \rangle, \\ C &= \langle (0.9, 0.6, 0.3), (0.5, 0.8, 0.2), (0.8, 0.8, 0.5) \rangle, \\ D &= \langle (0.7, 0.2, 0.7), (0.1, 0.4, 0.4), (0.5, 0.6, 0.7) \rangle, \\ E &= \langle (0.9, 0.7, 0.2), (0.5, 0.8, 0.1), (0.8, 0.8, 0.4) \rangle, \\ F &= \langle (0.3, 0.8, 0.7), (0.4, 0.2, 0.1), (0.7, 0.4, 0.5) \rangle, \\ G &= \langle (0.7, 0.4, 0.9), (0.2, 0.6, 0.5), (0.5, 0.2, 0.8) \rangle, \\ H &= \langle (0.3, 0.4, 0.9), (0.2, 0.2, 0.5), (0.5, 0.2, 0.8) \rangle, \end{aligned}$$

$I = \langle (0.7, 0.8, 0.7), (0.4, 0.6, 0.1), (0.7, 0.4, 0.5) \rangle$ and

$J = \langle (0.2, 0.3, 0.9), (0.1, 0.2, 0.5), (0.4, 0.2, 0.8) \rangle$. Here E and J are neutrosophic feebly open and neutrosophic feebly closed set respectively. Some of the neutrosophic feebly open are $0_N, A, B, C, D, E, 1_N$ and neutrosophic feebly-closed set are $1_N, F, G, H, I, J, 0_N$. Therefore $NFFr(C) = H \leq C$. But C is not a neutrosophic feebly closed set. $NFFr(J) \leq J^c = E$. But J is not a neutrosophic feebly open set.

Proposition 3.8. Let $A \leq B$ and B be any neutrosophic feebly closed set in X . Then $NFFr(A) \leq B$.

Proof. By Lemma 2.11(iv), $A \leq B$, $NFcl(A) \leq NFcl(B)$. By Definition 3.1, $NFFr(A) = NFcl(A) \wedge NFcl(A^c) \leq NFcl(B) \wedge NFcl(A^c) \leq NFcl(B)$. Then this is equal to B . Hence $NFFr(A) \leq B$.

Theorem 3.9. Let A be a neutrosophic subset in the neutrosophic topological space X . Then $(NFFr(A))^c = NFint(A) \vee NFint(A^c)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $(NFFr(A))^c = (NFcl(A) \wedge NFcl(A^c))^c = ((NFcl(A))^c \vee (NFcl(A^c))^c)$. By Lemma 2.10(ii), which is equal to $NFint(A^c) \vee NFint(A)$. Hence $(NFFr(A))^c = NFint(A) \vee NFint(A^c)$.

Theorem 3.10. For a neutrosophic subset A in the neutrosophic topological space X , then $NFFr(A) \leq NFr(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Lemma 2.11, $NFcl(A) \leq Ncl(A)$ and $NFcl(A^c) \leq Ncl(A^c)$. Now by Definition 3.1, $NFFr(A) = NFcl(A) \wedge NFcl(A^c) \leq Ncl(A) \wedge Ncl(A^c)$, this is equal to $NFr(A)$. Hence $NFFr(A) \leq NFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.11. From Example 3.7, let $A_1 = \langle (0.4, 0.1, 0.9), (0.1, 0.2, 0.6), (0.1, 0.3, 0.9) \rangle$. Then $A_1^c = \langle (0.9, 0.9, 0.4), (0.6, 0.8, 0.1), (0.9, 0.7, 0.1) \rangle$. Therefore $NFFr(A_1) = H \not\subseteq J = NFr(A_1)$.

Theorem 3.12. For a neutrosophic subset A in the neutrosophic topological space X , $NFcl(NFFr(A)) \leq NFFr(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $NFcl(NFFr(A)) = NFcl(NFcl(A) \wedge NFcl(A^c)) \leq (NFcl(NFcl(A))) \wedge (NFcl(NFcl(A^c)))$. By Lemma 2.11(iii), $NFcl(NFFr(A)) = NFcl(A) \wedge NFcl(A^c)$. By Definition 3.1, this is equal to $NFFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.13. From Example 3.7, $NFFr(A_1) = H \not\subseteq J = NFcl(NFFr(A_1))$.

Theorem 3.14. For a neutrosophic subset A in the neutrosophic topological space X , $NFFr(NFint(A)) \leq NFFr(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(NFint(A)) = NFcl(NFint(A)) \wedge NFcl(NFint(A))^c$. By Lemma 2.10(i), $NFFr(NFint(A)) = NFcl(NFint(A)) \wedge NFcl(NFcl(A^c))$. By Lemma 2.11(iii), $= NFcl(NFint(A)) \wedge NFcl(A^c)$. By Lemma 2.11(i), $\leq NFcl(A) \wedge NFcl(A^c)$. By Definition 3.1,

$$= NFFr(A). \text{Hence } NFFr(NFint(A)) \leq (NFFr(A)).$$

The converse of the above theorem is not true as shown by the following example.

Example 3.15. Let $X = \{a, b, c\}$ and neutrosophic feebly open sets are $0_N, A, B, C, D, E, 1_N$ and neutrosophic feebly closed sets are $1_N, F, G, H, I, J, 0_N$ where

$$\begin{aligned} A &= \langle (0.3, 0.4, 0.2), (0.5, 0.6, 0.7), (0.9, 0.5, 0.2) \rangle, \\ B &= \langle (0.3, 0.5, 0.1), (0.4, 0.3, 0.2), (0.8, 0.4, 0.6) \rangle, \\ C &= \langle (0.3, 0.5, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.2) \rangle, \\ D &= \langle (0.3, 0.4, 0.2), (0.4, 0.3, 0.7), (0.8, 0.4, 0.6) \rangle, \\ E &= \langle (0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2) \rangle, \\ F &= \langle (0.2, 0.6, 0.3), (0.7, 0.4, 0.5), (0.2, 0.5, 0.9) \rangle, \\ G &= \langle (0.1, 0.5, 0.3), (0.2, 0.7, 0.4), (0.6, 0.6, 0.8) \rangle, \\ H &= \langle (0.1, 0.5, 0.3), (0.2, 0.4, 0.5), (0.2, 0.5, 0.9) \rangle, \\ I &= \langle (0.2, 0.6, 0.3), (0.7, 0.7, 0.4), (0.6, 0.6, 0.8) \rangle, \\ J &= \langle (0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9) \rangle \end{aligned}$$

Define $A_1 = \langle (0.2, 0.3, 0.4), (0.4, 0.5, 0.6), (0.3, 0.4, 0.8) \rangle$.

Then $(A_1)^c = \langle (0.4, 0.7, 0.2), (0.6, 0.5, 0.4), (0.8, 0.6, 0.3) \rangle$.

Therefore $NFFr(A_1) = I \notin 0_N = NFFr(NFint(A_1))$.

Proposition 3.16. For a neutrosophic subset A in the neutrosophic topological space X , then $NFFr(NFcl(A)) \leq NFFr(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(NFcl(A)) = NFcl(NFcl(A)) \wedge (NFcl(NFcl(A))^c)$.

By Lemma 2.11(ii),(iii)&(iv), $= NFcl(A) \wedge (NFcl(NFint(A)^c))$. By Lemma 2.10,

$$\begin{aligned} &\leq NFcl(A) \wedge NFcl(A^c). \text{ By Definition 3.1,} \\ &= NFFr(A). \text{ Hence } NFFr(NFcl(A)) \leq NFFr(A). \end{aligned}$$

The converse of the above theorem is not true as shown by the following example.

Example 3.17. From Example 3.15, let $A_2 = \langle (0.2, 0.6, 0.2), (0.3, 0.4, 0.6), (0.3, 0.4, 0.8) \rangle$. Then $(A_2)^c = \langle (0.2, 0.4, 0.2), (0.6, 0.6, 0.3), (0.8, 0.6, 0.3) \rangle$. Therefore $NFFr(A_2) = 1_N \notin 0_N = NFFr(NFcl(A_2))$.

Proposition 3.18. Let A be the neutrosophic subset in the neutrosophic topological space X . Then $NFint(A) \leq A - NFFr(A)$.

Proof. Let A be the neutrosophic subset in the neutrosophic topological space X . Now by Definition 3.1, $A - NFFr(A) = A \wedge (NFFr(A))^c = A \wedge [NFcl(A) \wedge NFcl(A^c)]^c = A \wedge [NFint(A^c) \vee NFint(A)] = [A \wedge NFint(A^c)] \vee [A \wedge NFint(A)] = [A \wedge NFint(A^c)] \vee NFint(A) \geq NFint(A)$. Hence $NFint(A) \leq A - NFFr(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 3.19. From Example 3.15, $A_1 - NFFr(A_1) = \langle (0.2, 0.3, 0.4), (0.4, 0.3, 0.7), (0.3, 0.4, 0.8) \rangle \neq 0_N = NFint(A_1)$.

Remark 3.20. In general topology, the following conditions are hold:

$$\begin{aligned} NFFr(A) \wedge NFint(A) &= 0_N, \\ NFint(A) \vee NFFr(A) &= NFcl(A), \\ NFint(A) \vee NFint(A^c) \vee NFFr(A) &= 1_N. \end{aligned}$$

But the neutrosophic topology, we give counter-examples to show that the condition neutrosophic subset of the above remark may not be hold in general.

Example 3.21. From Example 3.15. Let us take $A_1 = \langle (0.4, 0.6, 0.1), (0.5, 0.8, 0.3), (0.9, 0.6, 0.2) \rangle$. Then $(A_1)^c = \langle (0.1, 0.4, 0.4), (0.3, 0.2, 0.5), (0.2, 0.4, 0.9) \rangle$. It can be shown that, $NFFr(A_1) \wedge (NFint(A_1)) = F \wedge D = \langle (0.2, 0.4, 0.3), (0.4, 0.3, 0.7), (0.2, 0.4, 0.9) \rangle \neq 0_N$.

Now, $NFint(A_1) \vee NFFr(A_1) = D \vee F = \langle (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) \rangle \neq 1_N = NFcl(A_1)$. Further $NFint(A_1) \vee NFint((A_1)^c) \vee (NFFr(A_1)) = D \vee 0_N \vee F = \langle (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) \rangle \neq 1_N$.

Proposition 3.22. Let A and B be neutrosophic subsets in the neutrosophic topological space X . Then $NFFr(A \vee B) \leq NFFr(A) \vee NFFr(B)$.

Proof. Let A and B be neutrosophic subsets in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(A \vee B) = NFcl(A \vee B) \wedge (NFcl(A \vee B))^c = NFcl(A \vee B) \wedge (NFcl(A^c \wedge B^c))$

$$\begin{aligned} &\text{By Lemma 2.12 (i) and (ii), } \leq (NFcl(A) \vee NFcl(B)) \wedge ((NFcl(A^c) \wedge (NFcl(B^c))) \\ &= [(NFcl(A) \vee (NFcl(B))) \wedge (NFcl(A^c))] \wedge [(NFcl(A) \vee (NFcl(B))) \wedge (NFcl(B^c))] \\ &= [(NFcl(A) \wedge NFcl(A^c)) \vee (NFcl(B) \wedge (NFcl(A^c)))] \wedge \end{aligned}$$

$$\begin{aligned}
 & [(NFcl(A) \wedge (NFcl(B^c) \vee ((NFcl(B) \wedge (NFcl(B^c))))] \\
 \text{By Definition 3.1,} &= [NFFr(A) \vee (NFcl(B) \wedge (NFcl(A^c)))] \wedge \\
 & [(NFcl(A) \wedge (NFcl(B^c))) \vee (NFFr(B))] \\
 &= (NFFr(A) \vee (NFFr(B))) \wedge [(NFcl(B) \wedge (NFcl(A^c)) \vee ((NFcl(A) \wedge NFcl(B^c)))] \\
 \leq & NFFr(A) \vee NFFr(B). \text{Hence } NFFr(A \vee B) \leq NFFr(A) \vee NFFr(B).
 \end{aligned}$$

The converse of the above theorem is need not be true as shown by the following example.

Example 3.23. Let $X = \{a\}$ with some neutrosophic feebly open sets are $0_N, A, B, C, D, 1_N$ and neutrosophic feebly closed sets are $1_N, E, F, G, H, 0_N$ where $A = \langle (0.6, 0.8, 0.4) \rangle, B = \langle (0.4, 0.9, 0.7) \rangle, C = \langle (0.6, 0.9, 0.4) \rangle, D = \langle (0.4, 0.8, 0.7) \rangle, E = \langle (0.4, 0.2, 0.6) \rangle,$

$F = \langle (0.7, 0.1, 0.4) \rangle, G = \langle (0.4, 0.1, 0.6) \rangle$ and $H = \langle (0.7, 0.2, 0.4) \rangle$. Now we define $B_1 = \langle (0.7, 0.6, 0.5) \rangle, B_2 = \langle (0.6, 0.8, 0.2) \rangle, B_1 \vee B_2 = B_3 = \langle (0.7, 0.8, 0.2) \rangle$ and $B_1 \wedge B_2 = B_4 = \langle (0.6, 0.6, 0.5) \rangle$. Then $B_1^c = \langle (0.5, 0.4, 0.7) \rangle, B_2^c = \langle (0.2, 0.2, 0.6) \rangle, B_3^c = \langle (0.2, 0.2, 0.7) \rangle$ and $B_4^c = \langle (0.5, 0.4, 0.6) \rangle$. Therefore $NFFr(B_1) \vee NFFr(B_2) = 1_N \vee E = 1_N \not\subseteq E = NFFr(B_3) = NFFr(B_1 \vee B_2)$.

Note 3.24. The following example shows that $NFFr(A \wedge B) \not\subseteq NFFr(A) \wedge NFFr(B)$ and $NFFr(A) \wedge NFFr(B) \not\subseteq NFFr(A \wedge B)$.

Example 3.25. From Example 3.23, we define $A_1 = \langle (0.5, 0.1, 0.9) \rangle, A_2 = \langle (0.3, 0.5, 0.6) \rangle, A_1 \vee A_2 = A_3 = \langle (0.5, 0.5, 0.6) \rangle$ and $A_1 \wedge A_2 = A_4 = \langle (0.3, 0.1, 0.9) \rangle$. Then $(A_1)^c = \langle (0.9, 0.9, 0.5) \rangle, (A_2)^c = \langle (0.6, 0.5, 0.3) \rangle, (A_3)^c = \langle (0.6, 0.5, 0.5) \rangle$ and $(A_4)^c = \langle (0.9, 0.9, 0.3) \rangle$. Therefore $NFFr(A_1) \wedge NFFr(A_2) = F \wedge 1_N = F \not\subseteq G = NFFr(A_4) = NFFr(A_1 \wedge A_2)$.

Theorem 3.26. For any neutrosophic subsets A and B in the neutrosophic topological space $X, NFFr(A \wedge B) \leq (NFFr(A) \wedge (NFcl(B)) \vee (NFFr(B) \wedge NFcl(A)))$.

Proof. Let A and B be neutrosophic subsets in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(A \wedge B) = NFcl(A \wedge B) \wedge (NFcl(A \wedge B))^c$.

By Demorgan Law, $= NFcl(A \wedge B) \wedge (NFcl(A^c) \vee B^c)$

By Lemma 2.12(ii) and (i),

$$\begin{aligned}
 & \leq (NFcl(A) \wedge (NFcl(B))) \wedge (NFcl(A^c) \vee NFcl(B^c)) \\
 &= [NFcl(A) \wedge (NFcl(B))] \wedge (NFcl(A^c)) \vee [(NFcl(A) \wedge (NFcl(B))) \wedge (NFcl(B^c))]
 \end{aligned}$$

By Definition 3.1,

$$= (A) \wedge (NFcl(B)) \vee (NFFr(B) \wedge (NFcl(A)))$$

Hence $NFFr(A \wedge B) \leq (NFFr(A) \wedge (NFcl(B)) \vee (NFFr(B) \wedge (NFcl(A)))$.

The converse of the above theorem is not true as shown by the following example.

Example 3.27. From Example 3.23, $(NFFr(A_1) \wedge (NFcl(A_2)) \vee (NFFr(A_2) \wedge (NFcl(A_1))) = (F \wedge 1_N) \vee (1_N \wedge F) = F \vee F = F \not\subseteq G = NFFr(A_1 \wedge A_2)$.

Corollary 3.28. For any neutrosophic subsets A and B in the neutrosophic topological space $X, NFFr(A \wedge B) \leq NFFr(A) \vee NFFr(B)$.

Proof. Let A and B be neutrosophic subsets in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(A \wedge B) = NFcl(A \wedge B) \wedge (NFcl(A \wedge B))^c$. By Lemma 1.1.10,

$= NFcl(A \wedge B) \wedge (NFcl(A^c) \vee B^c)$. By Proposition 2.3.11 (ii) and (i),

$$\leq (NFcl(A) \wedge NFcl(B)) \wedge (NFcl(A^c) \vee NFcl(B^c))$$

$= (NFcl(A) \wedge NFcl(B)) \wedge (NFcl(A^c) \vee (NFcl(A) \wedge (NFcl(B) \wedge (NFcl(B^c))))$. By Definition 3.1,

$$= (NFFr(A) \wedge (NFcl(B)) \vee ((NFcl(A) \wedge (NFFr(B)) \leq NFFr(A) \vee (NFFr(B))$$

Hence $NFFr(A \wedge B) \leq NFFr(A) \vee NFFr(B)$.

The equality in the above theorem may not hold as seen in the following example.

Example 3.29. From Example 3.23, $NFFr(A_1) \vee NFFr(A_2) = F \vee 1_N = 1_N \not\subseteq G = NFFr(A_4) = NFFr(A_1 \wedge A_2)$.

Theorem 3.30. For any neutrosophic subset A in the neutrosophic topological space X ,

- (1) $NFFr(NFFr(A)) \leq NFFr(A)$,
- (2) $NFFr(NFFr(NFFr(A))) \leq NFFr(NFFr(A))$.

Proof. (1) Let A be the neutrosophic subset in the neutrosophic topological space X . Then by Definition 3.1, $NFFr(NFFr(A)) = NFcl(NFFr(A)) \wedge (NFcl(NFFr(A)))^c$

By Definition 3.1, $= NFcl(NFcl(A) \wedge (NFcl(A^c))) \wedge (NFcl(NFcl(A) \wedge (NFcl(A^c)))^c$

By Lemma 2.10(iii) and by Lemma 2.12 (ii),

$$\leq (NFcl(NFcl(A)) \wedge (NFcl(NFcl(A^c))) \wedge (NFcl(NFint(A^c) \vee NFint(A))$$

By Lemma 2.11(iii), $= (NFcl(A) \wedge (NFcl(A^c))) \wedge ((NFcl(NFint(A) \wedge (A^c)) \vee (NFcl(NFint(A))) \leq NFcl(A) \wedge (NFcl(A^c))$. By Definition 3.1,

$=NFFr(A)$. Therefore $NFFr(NFFr(A)) \leq NFFr(A)$.

Again, $NFFr(NFFr(NFFr(A))) \leq NFFr(NFFr(A))$.

Remark 3.31. From the above theorem, the converse of (1) is need not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 3.32. From Example 3.15, $NFFr(A_2) = 1_N \not\subseteq 0_N = NFFr(NFFr(A_2))$.

Theorem 3.33. Let $X_i = 1, 2, \dots, n$ be a family of neutrosophic product related neutrosophic topological spaces. If each A_i is a neutrosophic subset in X_i , then $NFFr(\prod_{i=1}^n A_i) = [NFFr(A_1) \times (NFcl(A_2)) \times \dots \times (NFcl(A_n))] \vee [NFcl(A_1) \times (NFFr(A_2)) \times (NFcl(A_3)) \times \dots \times (NFcl(A_n))] \vee \dots \vee [NFcl(A_1) \times (NFcl(A_2)) \times \dots \times (NFFr(A_n))]$.

Proof. It suffices to prove this for $i = 2$. Let A_i be the neutrosophic subset in the neutrosophic topological space X_i . Then by Definition 3.1, $NFFr(A_1 \times A_2) = NFcl(A_1 \times A_2) \wedge (NFcl((A_1 \times A_2)^c))$

By Lemma 2.10 (i), $= NFcl(A_1 \times A_2) \wedge (NFint(A_1 \times A_2))^c$

By Theorem 2.14(i) and (ii),

$$\begin{aligned} &= (NFcl(A_1) \times (NFcl(A_2))) \wedge ((NFint(A_1) \times (NFint(A_2)))^c) \\ &= (NFcl(A_1) \times NFcl(A_2)) \wedge [(NFint(A_1) \wedge (NFcl(A_1))) \times ((NFint(A_2) \wedge (NFcl(A_2)))^c)]^c = (NFcl(A_1) \times (NFcl(A_2))) \wedge \\ &[(NFint(A_1) \wedge (NFcl(A_1))^c \times 1_N \vee 1_N \times (NFint(A_2) \wedge (NFcl(A_2))^c))]^c = (NFcl(A_1) \times (NFcl(A_2))) \wedge \\ &[(NFcl(A_1)^c \vee (NFint(A_1)^c)) \vee 1_N \vee 1_N \vee (NFcl(A_2)^c \vee (NFint(A_2)^c))] \\ &= (NFcl(A_1) \times (NFcl(A_2))) \wedge [(NFcl(A_1)^c) \times 1_N \vee (1_N \times NFcl(A_2)^c)] \\ &= [(NFcl(A_1) \times (NFcl(A_2))) \wedge ((NFcl(A_1)^c) \times 1_N)] \\ &\vee [(NFcl(A_1) \times (NFcl(A_2))) \wedge (1_N \times NFcl(A_2)^c)] \\ &= [(NFcl(A_1) \wedge (NFcl(A_1)^c)) \times (1_N \wedge (NFcl(A_2)))] \\ &\vee [(NFcl(A_1) \wedge 1_N) \times (NFcl(A_2) \wedge (NFcl(A_2)^c))] \\ &= (NFFr(A_1) \times (NFcl(A_2))) \vee (NFcl(A_1) \times NFFr(A_2)) \end{aligned}$$

Hence $NFFr(A_1 \times A_2) = (NFFr(A_1) \times (NFcl(A_2))) \vee (NFcl(A_1) \times NFFr(A_2))$.

V. CONCLUSION

So far, we have studied some new operator called neutrosophic feebly frontier with the help of neutrosophic feebly open sets and neutrosophic feebly closed sets in neutrosophic space. We discussed the important properties of them and the relations between them.

REFERENCES

- [1] Albowi, SA, Salama, A, and Mohamed Eisa, 2013, New Concepts of Neutrosophic Sets, International Journal of Mathematics and Computer Applications Research, Vol.3(3), pp. 95-102.
- [2] Athar Kharal, 2014, A Study of Frontier and Semi Frontier in Intuitionistic Fuzzy Topological Spaces, Hindawi Publishing Corporation, The Scientific World Journal, Article ID 674171, 9 pages.
- [3] Azad, KK, 1981, On Fuzzy Semi-Continuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity, J. Math. Anal. Appl, Vol. 82, pp. 14-32.
- [4] ByungSik In, 1998, On Fuzzy FC Compactness, comm. Korean Math. soc. 13(1), pp.137-150.
- [5] Chang, CL, 1968, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24, pp.182-190.
- [6] Florentin Smarandache, 2010, Neutrosophic Set: - A Generalization of Intuitionistic Fuzzy Set, Journal of Defense Resources pp. 107-116.
- [7] Florentin Smarandache, 2005, Neutrosophic set - A Generalization of the Intuitionistic Fuzzy Set, International Journal of Pure and Mathematics, 24(3), pp. 287-297.
- [8] Florentin Smarandache, 2002, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Probability, and statistics, University of New Mexico, Gallup, NM 87301. USA
- [9] Florentin Smarandache, 1999, A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic American Research Press.
- [10] JeyaPuvaneswari, P, Bageerathi, K, 2017, On Neutrosophic Feebly open sets in Neutrosophic topological spaces, International Journal of Mathematics Trends and Technology, Vol. 41(3), pp. 230 – 237.
- [11] JeyaPuvaneswari, P, Bageerathi, K, 2019, On Neutrosophic Feebly Continuous Functions Journal of Emerging Technologies and Innovative Research, Vol. 6 (2).
- [12] Salama, AA, Smarandache, Florentin, and Valeri, K, 2014, Neutrosophic Closed Set and Neutrosophic Continuous Functions, Neutrosophic Sets and Systems, 4, pp. 4-8.
- [13] Salama, AA, and Alblowi, SA, 2012, Generalized Neutrosophic Set and Generalized Neutrosophic Topological Spaces, Journal computer Sci. Engineering, Vol. (2) No. 7.
- [14] Salama, AA, Smarandache, F, Valeri, F, 2014, Neutrosophic Closed Set and Continuous Functions, Neutrosophic Set and Systems, 4, pp. 4-8.
- [15] Salama, AA, Alblowi, AA, 2012, Neutrosophic Set and Neutrosophic Topological Space, ISOR Journal of Mathematics, Vol. (3), Issue (4), pp. 31-35.
- [16] Zadeh, LA, 1965, Fuzzy Sets, Inform and Control, Vol. 8, pp. 338-353.