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On Neutrosophic Offuninorms

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Abstract: Uninorms comprise an important kind of operator in fuzzy theory. They are obtained from the generalization of the t-norm and t-conorm axiomatic. Uninorms are theoretically remarkable, and furthermore, they have a wide range of applications. For that reason, when fuzzy sets have been generalized to others—e.g., intuitionistic fuzzy sets, interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets, or neutrosophic sets—then uninorm generalizations have emerged in those novel frameworks. Neutrosophic sets contain the notion of indeterminacy—which is caused by unknown, contradictory, and paradoxical information—and thus, it includes, aside from the membership and non-membership functions, an indeterminate-membership function. Also, the relationship among them does not satisfy any restriction. Along this line of generalizations, this paper aims to extend uninorms to the framework of neutrosophic offsets, which are called neutrosophic offuninorms. Offsets are neutrosophic sets such that their domains exceed the scope of the interval [0,1]. In the present paper, the definition, properties, and application areas of this new concept are provided. It is necessary to emphasize that the neutrosophic offuninorms are feasible for application in several fields, as we illustrate in this paper.

Keywords: neutrosophic offset; uninorm; neutrosophic offuninorm; neutrosophic offnorm; neutrosophic offconorm; implicator; prospector; n-person cooperative game

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1. Introduction

Uninorms extend the t-norm and t-conorm axiomatic in fuzzy theory. They retain the axioms of commutativity, associativity, and monotony. Alternatively, they generalize the boundary condition, where the neutral element is any number lying in [0,1]. Thus, t-norm and t-conorm are special cases of uninorms, t-norms have 1 as their neutral element and the neutral element of t-conorms is 0, see [1–3].

Uninorms are theoretically important, and moreover they have also been used as operators in several areas of application; for example, in image processing, to aggregate group decision criteria, among others, see [4–8]. An exhaustive search on uninorm applications made by the authors of this paper yielded more than six hundred scientific articles that have been written in the last five years devoted to this subject.

Rudas et al. in [9] report that uninorms have been applied in diverse applications ranging, e.g., from defining Gross Domestic Product index in economics, to fusing sequences of DNA and RNA or combining information on taxonomies or dendograms in biology, and in the fusion of data provided by sensors of robotics in data mining, and in knowledge-based and intelligent systems. Particularly, they offer many examples in Decision Making, Utility Theory, Fuzzy Inference Systems, Multisensor Data Fusion, network aggregation in sensor networks, image approximation, hardware implementation of parametric operations, in Fuzzy Systems, and as software tools for aggregation problems.

Depaire et al. in [10] proposed a new approach to apply uninorms in Importance Performance Analysis, which is a useful technique to evaluate elements in marketing programs. They proved that their approach was superior when compared with regression and that it matched well with the customer satisfaction theory.

A very recent paper written by Modley et al. in [11] applied uninorms in the market basket analysis. Also, Appel et al. proposed a method based on cross-ratio uninorms as a mechanism to aggregate in the Sentiment Analysis; see [12].

Kamiset al. in [13] implement a geo-uninorm operator in a consensus model. They utilized them to derive a consistently based preference relation from a given reciprocal preference relation. Whereas, Wu et al. in [14] and Ureña et al. in [15] applied uninorms in trust propagation and aggregation methods for group decision making in a social network.

Bordignon and Gomide in [16], introduce a learning approach to train uninorm-based hybrid neural networks using extreme learning concepts. According to them, uninorms bring flexibility and generality to fuzzy neuron models. Wang in [17] and Yang in [18] applied uninorms as a basis to define logics.

Other areas of application can be consulted in González-Hidalgo et al. [19] where uninorms were utilized in edge detection of image processing, in fuzzy morphological associative memories (see [20]), and was also applied in time series prediction.

It is well-known that the minimum is the biggest t-norm and the maximum is the lowest t-conorm, thus they are not compensatory operators; whereas uninorms compensate when the truth values are situated on both sides of the neutral element. The compensation property could be the key factor in the wide range of uninorm applicability, mainly in decision making. Zimmermann experimentally proved in [21], many years before the introduction of uninorms, that often human beings do not make decisions interpreting AND like a t-norm and OR like a t-conorm, but that compensatory operators are more adequate to model human aggregations to signify AND and OR in some situations. The use of means as aggregators to define membership functions can be seen in [22]. However, when the aggregated values are situated on one side with respect to the neutral element, then uninorms operate either like a t-norm or a t-conorm.

Uninorms have been extended to other theories more general than fuzzy logic, due to their applicability. Let us mention intuitionistic fuzzy sets, interval-valued fuzzy sets, and interval-valued intuitionistic fuzzy sets; where the generalizations consist of the inclusion of an independent non-membership function or an interval-valued membership function, or both [23]. They have also been generalized as multi-polar aggregators in [24].

Following this trend, the authors of this paper defined the neutrosophic uninorms, such that the uninorms were extended to the neutrosophy framework [25]. Neutrosophy is the philosophical discipline that studies theories, entities, objects, phenomena, among others, related to neutrality [26]. In particular, neutrosophic sets contain three independent functions, namely, a membership function, a non-membership function, and additionally, an indeterminate-membership function. The last one represents what is unknown, contradictory, and paradoxical. Furthermore, these elements can be intervals.

In addition, the relationship among these three functions has no restriction, contrary to the intuitionist fuzzy sets, which must fulfill the constraint that the sum of the membership truth value with the non-membership truth value of an element to the set does not exceed the unit.

Neutrosophy theory has been used in a wide spectrum of applications such as in image processing, decision making, clustering, among others [27–30]. Therefore, it is not difficult to appreciate the applicability of neutrosophic uninorms.

More recently, other concepts have been defined within the neutrosophy framework, which further generalizes the traditional membership functions, including the axiomatic in probability theory.

They are the undersets, oversets, and offsets, where the basic idea is that negative truth values or truth values greater than 1 are permitted in the calculus [31].

A recurring example in literature is that concerning employment, where the truth value of a worker's effectiveness is measured in working hours. Those workers who have met all of their working hours established for the week will be an effectiveness truth value of 1, those workers who have only partially met their working hours have a truth value between 0 and 1, and other workers who have not attended work throughout the week have the truth value of 0. In addition, those who have performed voluntary overtime after meeting their established hours have a truth value greater than 1, and finally, the workers who have not attended work throughout the week and, moreover, have caused losses to the company, must have a negative truth value.

Other examples take into consideration the relationship between two variables or more, where a negative value represents that they are inversely related, whereas a direct relationship is represented by positive values [31].

The aim of this paper is to extend for the first time the theory of uninorms to the offsets framework—we call them neutrosophic offuninorms—in such a way that they are a generalization of both n-offnorms and n-offconorms equivalently, as fuzzy uninorms generalize both t-norms and t-conorms.

In this paper, definitions and also properties of neutrosophic offuninorms will be given. Additionally, we will emphasize the relationship between these new operators and the aggregation functions used in the well-known medical expert system MYCIN [32], as well as define logical implicators in offset fields and solve voting cooperative games.

In particular, the association of the proposed theory with the aggregation functions used in MYCIN supports the hypothesis that neutrosophic offuninorms are more than an interesting theoretical approach. Historically, within the fuzzy logic framework, some authors have accepted the idea of extending the uninorms domain to [a, b], in order to include the aggregation functions used in MYCIN, [33,34]. This proposal is an important precedent for this investigation because uninorms were there adapted to offsets in the fuzzy theory context. The relationship between uninorms and the PROSPECTOR operator, as well as their application, can be consulted in [35], where they were used in e-arning.

Authors in [33,34] also emphasize that this generalization has important practical advantages because it allows us to naturally apply uninorms in fields like Artificial Neural Networks and Cognitive Maps. These elements certainly suggest that the proposed theory can be applied in fields like Artificial Neural Networks based on neutrosophic sets and in neutrosophic cognitive maps, [36,37].

Let us observe that when uninorms have been extended to other domains they have preserved the property of compensation. Further, we shall prove that offuninorms are not the exception; consequently, the applicability of offuninorms is practically guaranteed. In the discussion section, we insist on this aspect and the advantages that offuninorms have over other generalizations.

This paper is divided as follows. It begins with a preliminary section where concepts such as neutrosophic sets, neutrosophic offsets, neutrosophic uninorms, among other useful aspects, are discussed in order to develop the content of this article. The section on neutrosophic offuninorms is devoted to exposing definitions and properties of these novel operators. Next, the applications section is where the three possible areas of application of this theory are explained. We then finish with the sections of discussion and conclusions.

2. Preliminaries

This section contains the main definitions necessary to develop the theory proposed in this paper. We begin with Definitions 1 and 2, which introduce the neutrosophic sets. These sets are characterized by an independent indeterminacy-membership function that models the unknown, contradictions, inconsistencies in information and so on. Additionally, we have the classic membership and non-membership functions, which are not necessarily dependent on each other.

Definition 1. *Let X be a space of points (objects), with a generic element in X denoted by x. A Neutrosophic Set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, thus, $^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ (see [26]).*

The neutrosophic sets are useful in their nonstandard form only in philosophy, in order to make a distinction between absolute truth (truth in all possible worlds—according to Leibniz) and relative truth (truth in at least one world), but not in technical applications, thus the *Single-Valued Neutrosophic Sets* are defined, see Definition 2.

Definition 2. *Let X be a space of points (objects), with a generic element in X denoted by x. A Single-Valued Neutrosophic Set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$, and $F_A(x)$ are elements of [0,1]. There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, thus, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ (see [38]).*

The domain of the single-valued neutrosophic sets does not surpass the limits of the interval [0,1]. This is a classical condition imposed in previous theories such as probability and fuzzy sets. Despite the past, Smarandache in 2007 proposed the membership >1 and <0 and illustrated this proposal; see [39] (pp. 92–93) and the example given in the introduction of this paper. In the following, the *Single-Valued Neutrosophic Oversets*, *Single-Valued Neutrosophic Undersets*, and *Single-Valued Neutrosophic Offsets* are formally defined.

Definition 3. *Let X be a universe of discourse and the neutrosophic set $A_1 \subset X$. Let T(x), I(x), F(x) be the functions that describe the degree of membership, indeterminate-membership, and non-membership respectively, of a generic element $x \in X$, with respect to the neutrosophic set A_1:*

T, I, F: $X \to [0, \Omega]$, where $\Omega > 1$ is called overlimit, $T(x), I(x), F(x) \in [0, \Omega]$. A Single-Valued Neutrosophic Overset A_1 is defined as $A_1 = \{(x, T(x),\ I(x),\ F(x)), x \in X\}$, such that there exists at least one element in A_1 that has at least one neutrosophic component that is bigger than 1, and no element has neutrosophic components that are smaller than 0 (see [31]).

Definition 4. *Let X be a universe of discourse and the neutrosophic set $A_2 \subset X$. Let T(x), I(x), F(x) be the functions that describe the degree of membership, indeterminate-membership, and non-membership, respectively, of a generic element $x \in X$, with respect to the neutrosophic set A_2:*

T, I, F: $X \to [\Psi, 1]$, where $\Psi < 0$ is called underlimit, $T(x), I(x), F(x) \in [\Psi, 1]$. A Single-Valued Neutrosophic Underset A_2 is defined as $A_2 = \{(x, T(x),\ I(x),\ F(x)), x \in X\}$, such that there exists at least one element in A_2 that has at least one neutrosophic component that is smaller than 0, and no element has neutrosophic components that are bigger than 1 (see [31]).

Definition 5. *Let X be a universe of discourse and the neutrosophic set $A_3 \subset X$. Let T(x), I(x), F(x) be the functions that describe the degree of membership, indeterminate-membership, and non-membership respectively, of a generic element $x \in X$, with respect to the neutrosophic set A_3:*

T, I, F: $X \to [\Psi, \Omega]$, where $\Psi < 0 < 1 < \Omega$, Ψ is called underlimit, while Ω is called overlimit, $T(x), I(x), F(x) \in [\Psi, \Omega]$. A Single-Valued Neutrosophic Offset A_3 is defined as $A_3 = \{(x, T(x),\ I(x),\ F(x)), x \in X\}$, such that there exists at least one element in A_3 that has at least one neutrosophic component that is bigger than 1, and at least another neutrosophic component that is smaller than 0 (see [31]).

Let us note that the oversets, undersets, and offsets cover the three possible cases to characterize. Now, the logical operations over these kinds of sets have to be redefined, in view that the classical ones cannot always be straightforwardly extended to these domains. This is the case of complement given by Smarandache in [31], whereas the union and intersection definitions do not change with respect to those of single-valued neutrosophic sets. This is summarized below:

Let X be a universe of discourse, $A = \{(x, \langle T_A(x), I_A(x), F_A(x)\rangle), x \in X\}$ and $B = \{(x, \langle T_B(x), I_B(x), F_B(x)\rangle), x \in X\}$ be two single-valued neutrosophic oversets/undersets/offsets.

T_A, I_A, F_A, T_B, I_B, F_B: $X \to [\Psi, \Omega]$, where $\Psi \le 0 < 1 \le \Omega$, Ψ is the underlimit, whilst Ω is the overlimit, $T_A(x)$, $I_A(x)$, $F_A(x)$,$T_B(x)$, $I_B(x)$, $F_B(x) \in [\Psi, \Omega]$. Let us remark that the three cases are here comprised, viz., overset when $\Psi = 0$ and $\Omega > 1$, underset when $\Psi < 0$ and $\Omega = 1$, and offset when $\Psi < 0$ and $\Omega > 1$.

Then, the main operators are defined as follows:

$A \cup B = \{(x, \langle \max(T_A(x), T_B(x)),\ \min(I_A(x), I_B(x)),\ \min(F_A(x), F_B(x))\rangle), x \in X\}$ is the union.

$A \cap B = \{(x, \langle \min(T_A(x), T_B(x)),\ \max(I_A(x), I_B(x)),\ \max(F_A(x), F_B(x))\rangle), x \in X\}$ is the intersection,

$C(A) = \{(x, \langle F_A(x),\ \Psi + \Omega - I_A(x), T_A(x)\rangle), x \in X\}$ is the neutrosophic complement of the neutrosophic set.

Let us remark that when $\Psi = 0$ and $\Omega = 1$, the precedent operators convert in the classical ones. With regard to logical operators, e.g., n-norms and n-conorms, their redefinitions in the offsets framework are not so evident. Below, definitions of *offnegation*, *neutrosophic component n-offnorm*, and *neutrosophic component n-offconorm* are provided.

One *offnegation* can be defined as in Equation (1).

$$\overset{\neg}{O}\langle T,\ I,\ F\rangle = \langle\ F,\ \Psi_I + \Omega_I - I,\ T\rangle \tag{1}$$

Definition 6. *Let c be a neutrosophic component (T_O, I_O or F_O). c: $M_O \to [\Psi, \Omega]$, where $\Psi \le 0$ and $\Omega \ge 1$. The neutrosophic component n-offnorm $N_O^n : [\Psi,\ \Omega]^2 \to [\Psi,\ \Omega]$ satisfies the following conditions for any elements x, y, and z $\in M_O$:*

i. $N_O^n(c(x),\ \Psi) = \Psi$, $N_O^n(c(x),\ \Omega) = c(x)$ *(Overbounding Conditions),*
ii. $N_O^n(c(x),\ c(y)) = N_O^n(c(y),\ c(x))$ *(Commutativity),*
iii. *If* $c(x) \le c(y)$ *then* $N_O^n(c(x),\ c(z)) \le N_O^n(c(y),\ c(z))$ *(Monotonicity),*
iv. $N_O^n\left(N_O^n(c(x),\ c(y)), c(z)\right) = N_O^n\left(c(x),\ N_O^n(c(y),\ c(z))\right)$ *(Associativity).*

To simplify the notation, sometimes we use $\langle T_1,\ I_1, F_1\rangle \underset{O}{\wedge} \langle T_2,\ I_2, F_2\rangle = \langle T_1 \underset{O}{\wedge} T_2,\ I_1 \underset{O}{\vee} I_2, F_1 \underset{O}{\vee} F_2\rangle$ instead of $N_O^n(\cdot,\cdot)$.

Let us remark that the definition of the neutrosophic component n-offnorm is valid for every one of the components, thus, we have to apply it three times. Also, Definition 6 contains the definition of n-norm when $\Psi = 0$ and $\Omega = 1$.

Proposition 1. *Let $N_O^n(\cdot,\cdot)$ be a neutrosophic component n-offnorm, then, for any elements x, y $\in M_O$ we have $N_O^n(c(x),\ c(y)) \le min(c(x),\ c(y))$.*

Proof. Because of the monotonicity of the neutrosophic component n-offnorm and one of the overbounding conditions, we have $N_O^n(c(x),\ c(y)) \le N_O^n(c(x),\ \Omega) = c(x)$, hence $N_O^n(c(x),\ c(y)) \le c(x)$ and similarly $N_O^n(c(x),\ c(y)) \le c(y)$ can be proved, therefore, $N_O^n(c(x),\ c(y)) \le \min(c(x),\ c(y))$. □

See that Proposition 1 maintains this property of the n-norms. Likewise to the definition of the neutrosophic component n-offnorm, in Definition 7 it is described the *neutrosophic component n-offconorm.*

Definition 7. *Let c be a neutrosophic component (T_O, I_O or F_O). c: $M_O \to [\Psi, \Omega]$, where $\Psi \leq 0$ and $\Omega \geq 1$. The neutrosophic component n-offconorm $N_O^{co}: [\Psi, \Omega]^2 \to [\Psi, \Omega]$ satisfies the following conditions for any elements x, y, and z $\in M_O$:*

i. $N_O^{co}(c(x), \Omega) = \Omega$, $N_O^{co}(c(x), \Psi) = c(x)$ *(Overbounding Conditions),*
ii. $N_O^{co}(c(x), c(y)) = N_O^{co}(c(y), c(x))$ *(Commutativity),*
iii. *If* $c(x) \leq c(y)$ *then* $N_O^{co}(c(x), c(z)) \leq N_O^{co}(c(y), c(z))$ *(Monotonicity),*
iv. $N_O^{co}\big(N_O^{co}(c(x), c(y)), c(z)\big) = N_O^{co}\big(c(x), N_O^{co}(c(y), c(z))\big)$ *(Associativity).*

To simplify the notation sometimes we use $\langle T_1, I_1, F_1 \rangle \underset{O}{\vee} \langle T_2, I_2, F_2 \rangle = \langle T_1 \underset{O}{\vee} T_2, I_1 \underset{O}{\wedge} I_2, F_1 \underset{O}{\wedge} F_2 \rangle$ instead of $N_O^{co}(\cdot, \cdot)$.

Proposition 2. *Let $N_O^{co}(\cdot, \cdot)$ be a neutrosophic component n-offconorm, then, for any elements x, y$\in M_O$ we have $N_O^{co}(c(x), c(y)) \geq max(c(x), c(y))$.*

Proof. The proof is equivalent to the proof of Proposition 1. □

In this paper, we use the notion of lattice, based on the poset denoted by $\leq_O$, where $\langle T_1, I_1, F_1 \rangle \leq_O \langle T_2, I_2, F_2 \rangle$ if and only if $T_2 \geq T_1$, $I_2 \leq I_1$ and $F_2 \leq F_1$, where the infimum and the supremum of the set are $\langle \Psi, \Omega, \Omega \rangle$ and $\langle \Omega, \Psi, \Psi \rangle$, respectively.

One property that is preserved of n-norms is that the minimum is the biggest neutrosophic component n-offnorm for T_O, as it is demonstrated in Proposition 1. Proposition 2 proved that the maximum is the smallest neutrosophic component n-offconorm for I_O and F_O when we consider $\leq_O$.

Evidently, the minimum is a neutrosophic component n-offnorm and the maximum is a neutrosophic component n-offconorm; see Example 1.

Example 1. *An example of a pair offAND/offOR is, $c(x) \underset{ZO}{\wedge} c(y) = min(c(x), c(y))$ and $c(x) \underset{ZO}{\vee} c(y) = max(c(x), c(y))$, respectively.*

Example 2. *A pair of offAND/offOR is, $c(x) \underset{LO}{\wedge} c(y) = max(\Psi, c(x) + c(y) - \Omega)$ and $c(x) \underset{LO}{\vee} c(y) = min(\Omega, c(x) + c(y))$, respectively.*

Example 2 extends the Łukasiewicz t-norm and t-conorm to the neutrosophic offsets. Let us remark that the simple product t-norm and its dual t-conorm cannot be extended to this new domain.

Finally, we recall the definition of neutrosophic uninorms that appeared in [25], see Definition 8.

Definition 8. *A neutrosophic uninorm $\boldsymbol{U}_N$ is a commutative, increasing, and associative mapping, $U_N: (]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \to]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $U_N\big(x\langle T_x, I_x, F_x\rangle, y\langle T_y, I_y, F_y\rangle\big) = \langle U_N T(x,y), U_N I(x,y), U_N F(x,y)\rangle$, where $\boldsymbol{U}_N T$ means the degree of membership, $\boldsymbol{U}_N I$ the degree of indeterminacy, and $\boldsymbol{U}_N F$ the degree of non-membership of both x and y. Additionally, there exists a neutral element $e \in]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, where $\forall x \in]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, $U_N(e, x) = x$.*

Let us observe that this definition can be restricted to single-valued neutrosophic sets. Neutrosophic uninorms generalize n-norms, n-conorms, uninorms in *L**-fuzzy set theory, and fuzzy uninorms.

3. On Neutrosophic Offuninorms

This section contains the core of the present paper. It is devoted to exposing the definitions and properties of the neutrosophic offuninorms.

Definition 9. *Let c be a neutrosophic component (T_O, I_O or F_O). c: $M_O \to [\Psi, \Omega]$, where $\Psi \le 0$ and $\Omega \ge 1$. The neutrosophic component n-offuninorm $N_O^u : [\Psi, \Omega]^2 \to [\Psi, \Omega]$ satisfies the following conditions for any elements x, y, and z $\in M_O$:*

i. There exists $c(e) \in M_O$, such that $N_O^u(c(x), c(e)) = c(x)$ (Identity),

ii. $N_O^u(c(x), c(y)) = N_O^u(c(y), c(x))$ (Commutativity),

iii. If $c(x) \le c(y)$ then $N_O^u(c(x), c(z)) \le N_O^u(c(y), c(z))$ (Monotonicity),

iv. $N_O^u\left(N_O^u(c(x), c(y)), c(z)\right) = N_O^u\left(c(x), N_O^u(c(y), c(z))\right)$ (Associativity).

The definition of a neutrosophic uninorm is an especial case of neutrosophic offuninorm when $\Psi = 0$ and $\Omega = 1$ (see Definition 8) and, additionally, we are dealing with single-valued neutrosophic sets.

It is easy to prove that the neutral element *e* is unique.

Let *c* be a neutrosophic component (T_O, I_O or F_O). *c*: $M_O \to [\Psi, \Omega]$, where $\Psi \le 0$ and $\Omega \ge 1$. Let us define four useful functions, $\varphi_1 : [\Psi, c(e)] \to [\Psi, \Omega]$, $\varphi_1^{-1} : [\Psi, \Omega] \to [\Psi, c(e)]$, $\varphi_2 : [c(e), \Omega] \to [\Psi, \Omega]$, and $\varphi_2^{-1} : [\Psi, \Omega] \to [c(e), \Omega]$, defined in Equations (2)–(5), respectively.

$$\varphi_1(c(x)) = \left(\frac{\Omega - \Psi}{c(e) - \Psi}\right)(c(x) - \Psi) + \Psi \tag{2}$$

$$\varphi_1^{-1}(c(x)) = \left(\frac{c(e) - \Psi}{\Omega - \Psi}\right)(c(x) - \Psi) + \Psi \tag{3}$$

$$\varphi_2(c(x)) = \left(\frac{\Omega - \Psi}{\Omega - c(e)}\right)(c(x) - c(e)) + \Psi \tag{4}$$

$$\varphi_2^{-1}(c(x)) = \left(\frac{\Omega - c(e)}{\Omega - \Psi}\right)(c(x) - \Psi) + c(e) \tag{5}$$

where, the superscript −1 means it is an inverse mapping. If the condition $c(e) \in (\Psi, \Omega)$ is fulfilled, then the degenerate cases $\Omega = \Psi$, $c(e) = \Psi$ and $c(e) = \Omega$ are excluded. Therefore, $\varphi_1(c(x))$ and $\varphi_2(c(x))$ are well-defined non-constant linear functions. Thus, they are bijective and have inverse mappings defined in Equations (3) and (5), respectively, in the sense that for $c(x) \in [\Psi, \Omega]$, then $\varphi_1\left(\varphi_1^{-1}(c(x))\right) = c(x)$ and $\varphi_2\left(\varphi_2^{-1}(c(x))\right) = c(x)$. Whereas, for $c(x) \in [\Psi, c(e)]$, we have $\varphi_1^{-1}(\varphi_1(c(x))) = c(x)$ and for $c(x) \in [c(e), \Omega]$, $\varphi_2^{-1}(\varphi_2(c(x))) = c(x)$. These properties can be easily verified. Also, it is trivial that they are non-decreasing mappings.

Additionally, let $U_C, U_D : [\Psi, \Omega]^2 \to [\Psi, \Omega]$ be two operators defined by Equations (6) and (7), respectively,

$$U_C(c(x), c(y)) = \begin{cases} \varphi_1^{-1}\left(\varphi_1(c(x)) \overset{\wedge}{\underset{O}{}} \varphi_1(c(y))\right), \text{ if } c(x), c(y) \in [\Psi, c(e)] \\ \varphi_2^{-1}\left(\varphi_2(c(x)) \overset{\vee}{\underset{O}{}} \varphi_2(c(y))\right), \text{ if } c(x), c(y) \in [c(e), \Omega] \\ \min(c(x), c(y)), \text{ otherwise} \end{cases} \tag{6}$$

$$U_D(c(x), c(y)) = \begin{cases} \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{O}{\wedge} \varphi_1(c(y))\right), \text{ if } c(x), c(y) \in [\Psi,\ c(e)] \\ \varphi_2^{-1}\left(\varphi_2(c(x)) \underset{O}{\vee} \varphi_2(c(y))\right), \text{ if } c(x), c(y) \in [c(e),\ \Omega] \\ \max(c(x), c(y)), \text{ otherwise} \end{cases} \quad (7)$$

where, $\underset{O}{\wedge}$ denotes a neutrosophic component n-offnorm and $\underset{O}{\vee}$ denotes a neutrosophic component n-offconorm.

Lemma 1. *Let c be a neutrosophic component (T_O, I_O or F_O). c: $M_O \rightarrow [\Psi, \Omega]$, where $\Psi \leq 0$ and $\Omega \geq 1$. Given $\underset{O}{\wedge}$ a neutrosophic component n-offnorm and $\underset{O}{\vee}$ a neutrosophic component n-offconorm, let us consider $U_C(c(x), c(y))$ and $U_D(c(x), c(y))$ the operators defined in Equations (6) and (7) for $c(e) \in (\Psi,\ \Omega)$. They are commutative, non-decreasing, and c(e) is the neutral element.*

Proof.

i. Commutativity is evidently satisfied due to the commutativity of $\underset{O}{\wedge}$, $\underset{O}{\vee}$, *min*, and *max*.

ii. $\varphi_1(\cdot)$, $\varphi_1^{-1}(\cdot)$, $\varphi_2(\cdot)$, $\varphi_2^{-1}(\cdot)$, $\underset{O}{\wedge}$, $\underset{O}{\vee}$, *min* and *max* are non-decreasing mappings, thus both $U_C(\cdot,\ \cdot)$ and $U_D(\cdot,\ \cdot)$ satisfy monotonicity.

iii. To prove *c(e)* is the neutral element, we have two cases, which are the following:

- If $c(x) \in [\Psi,\ c(e)]$, then, $U_C(c(e), c(x)) = U_D(c(e), c(x)) = \varphi_1^{-1}\left(\varphi_1(c(e)) \underset{O}{\wedge} \varphi_1(c(x))\right) = \varphi_1^{-1}\left(\Omega \underset{O}{\wedge} \varphi_1(c(x))\right) = \varphi_1^{-1}(\varphi_1(c(x))) = c(x)$.
- If $c(x) \in [c(e),\ \Omega]$, then $U_C(c(e), c(x)) = U_D(c(e), c(x)) = \varphi_2^{-1}\left(\varphi_2(c(e)) \underset{O}{\vee} \varphi_2(c(x))\right)\varphi_2^{-1}\left(\Psi \underset{O}{\vee} \varphi_2(c(x))\right) = \varphi_2^{-1}(\varphi_2(c(x))) = c(x)$.

Therefore, identity is satisfied. □

Lemma 2. *Let c be a neutrosophic component (T_O, I_O, or F_O). c: $M_O \rightarrow [\Psi, \Omega]$, where $\Psi \leq 0$ and $\Omega \geq 1$. Given $\underset{O}{\wedge}$ a neutrosophic component n-offnorm and $\underset{O}{\vee}$ a neutrosophic component n-offconorm, let us consider $U_C(c(x), c(y))$ and $U_D(c(x), c(y))$ the operators defined in Equations (6) and (7) for $c(e) \in (\Psi,\ \Omega)$. They are associative.*

Proof. Four cases are possible:

i. Let $c(x), c(y),\ c(z) \in [\Psi,\ c(e)]$, then $U_C(U_C(c(x), c(y)), c(z)) = \varphi_1^{-1}\left(\varphi_1\left(\varphi_1^{-1}\left(\varphi_1(c(x)) \underset{O}{\wedge} \varphi_1(c(y))\right)\right) \underset{O}{\wedge} \varphi_1(c(z))\right) = \varphi_1^{-1}\left(\left[\varphi_1(c(x)) \underset{O}{\wedge} \varphi_1(c(y))\right] \underset{O}{\wedge} \varphi_1(c(z))\right) = \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{O}{\wedge} \left[\varphi_1(c(y)) \underset{O}{\wedge} \varphi_1(c(z))\right]\right) = \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{O}{\wedge} \left[\varphi_1\left(\varphi_1^{-1}\left[\varphi_1(c(y)) \underset{O}{\wedge} \varphi_1(c(z))\right]\right)\right]\right) = \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{O}{\wedge} \varphi_1(U_C(c(y), c(z)))\right) = U_C(c(x), U_C(c(y), c(z)))$.

ii. Let $c(x), c(y),\ c(z) \in [c(e),\ \Omega]$, $U_C(U_C(c(x), c(y)), c(z)) = \varphi_2^{-1}\left(\varphi_2\left(\varphi_2^{-1}\left(\varphi_2(c(x)) \underset{O}{\vee} \varphi_2(c(y))\right)\right) \underset{O}{\vee} \varphi_2(c(z))\right) = \varphi_2^{-1}\left(\left[\varphi_2(c(x)) \underset{O}{\vee} \varphi_2(c(y))\right] \underset{O}{\vee} \varphi_2(c(z))\right) =$

$\varphi_2^{-1}\left(\varphi_2(c(x)) \underset{O}{\vee} \left[\varphi_2(c(y)) \underset{O}{\vee} \varphi_2(c(z))\right]\right) = \varphi_2^{-1}\left(\varphi_2(c(x)) \underset{O}{\vee} \left[\varphi_2\left(\varphi_2^{-1}\left[\varphi_2(c(y)) \underset{O}{\vee} \varphi_2(c(z))\right]\right)\right]\right) =$ $\varphi_2^{-1}\left(\varphi_2(c(x)) \underset{O}{\vee} \varphi_2(U_C(c(y), c(z)))\right) = U_C(c(x), U_C(c(y), c(z)))$. These proofs are also valid for U_D.

iii. Let $c(x), c(y) \in [\Psi,\ c(e)]$ and $c(z) \in [c(e),\ \Omega]$, $U_C(U_C(c(x), c(y)), c(z)) = \min(U_C(c(x), c(y)), c(z)) = U_C(c(x), c(y))$. Also, we have $U_C(c(x), U_C(c(y), c(z))) = U_C(c(x),\ \min(c(y), c(z))) = U_C(c(x), c(y))$, then, it is associative.

iv. Let $c(x), c(y) \in [c(e),\ \Omega]$ and $c(z) \in [\Psi,\ c(e)]$, then $U_C(U_C(c(x), c(y)), c(z)) = \min(U_C(c(x), c(y)), c(z)) = c(z)$. In addition, $U_C(c(x), (U_C(c(y), c(z)))) = U_C(c(x),\ \min(c(y), c(z))) = U_C(c(x), c(z)) = \min(c(x), c(z)) = c(z)$.

Thus, U_C satisfies the associativity.
Similarly, associativity of U_D can be proved.

Let us remark that we applied the properties, $c(x) \underset{O}{\wedge} c(y) \leq \min(c(x),\ c(y))$ and $c(x) \underset{O}{\vee} c(y) \geq \max(c(x),\ c(y))$, as well as $U_C(c(x), c(y)) \leq c(e)$ if $c(x),\ c(y) \in [\Psi,\ c(e)]$ and $U_C(c(x), c(y)) \geq c(e)$ if $c(x),\ c(y) \in [c(e),\ \Omega]$. □

Proposition 3. *Let c be a neutrosophic component (T_O, I_O, or F_O). c: $M_O \to [\Psi,\ \Omega]$, where $\Psi \leq 0$ and $\Omega \geq 1$. Given $\underset{O}{\wedge}$ a neutrosophic component n-offnorm and $\underset{O}{\vee}$ a neutrosophic component n-offconorm, let us consider $U_C(c(x), c(y))$ and $U_D(c(x), c(y))$ the operators defined in Equations 6 and 7 for $c(e) \in (\Psi,\ \Omega)$. Then, $U_C(c(x), c(y))$ and $U_D(c(x), c(y))$ are neutrosophic component n-offuninorms and they satisfy the conditions $U_C(\Psi,\ \Omega) = \Psi$ and $U_D(\Psi,\ \Omega) = \Omega$,* i.e., U_C is a conjunctive neutrosophic component n-offuninorm, and U_D is a disjunctive neutrosophic component n-offuninorm.

Proof. Since Lemma 1, they are commutative, non-decreasing operators, and *c(e)* is the neutral element. Since Lemma 2, they are associative operators. Moreover, it is easy to verify that $U_C(\Psi,\ \Omega) = \Psi$ and $U_D(\Psi,\ \Omega) = \Omega$. □

Example 3. *Two neutrosophic component n-offuninorms can be defined as:*

$$U_{ZC}(c(x), c(y)) = \begin{cases} \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{ZO}{\wedge} \varphi_1(c(y))\right), \text{ if } c(x), c(y) \in [\Psi,\ c(e)] \\ \varphi_2^{-1}\left(\varphi_2(c(x)) \underset{ZO}{\vee} \varphi_2(c(y))\right), \text{ if } c(x), c(y) \in [c(e),\ \Omega] \\ \min(c(x), c(y)), \text{ otherwise} \end{cases}$$

$$U_{ZD}(c(x), c(y)) = \begin{cases} \varphi_1^{-1}\left(\varphi_1(c(x)) \underset{ZO}{\wedge} \varphi_1(c(y))\right), \text{ if } c(x), c(y) \in [\Psi,\ c(e)] \\ \varphi_2^{-1}\left(\varphi_2(c(x)) \underset{ZO}{\vee} \varphi_2(c(y))\right), \text{ if } c(x), c(y) \in [c(e),\ \Omega] \\ \max(c(x), c(y)), \text{ otherwise} \end{cases}$$

where $\underset{ZO}{\wedge}$ and $\underset{ZO}{\vee}$ were defined in the Example 1; $c(e) \in (\Psi,\ \Omega)$.

Then two examples of n-offuninorms are: $U_1(\langle T_1, I_1, F_1\rangle, \langle T_2, I_2, F_2\rangle) = \langle U_{ZC}(T_1, T_2), U_{ZD}(I_1, I_2), U_{ZD}(F_1, F_2)\rangle$ and $U_2(\langle T_1, I_1, F_1\rangle, \langle T_2, I_2, F_2\rangle) = \langle U_{ZD}(T_1, T_2), U_{ZC}(I_1, I_2), U_{ZC}(F_1, F_2)\rangle$.
They satisfy $U_1(\langle\Psi,\ \Omega,\ \Omega\rangle, \langle\Omega, \Psi, \Psi\rangle) = \langle\Psi,\ \Omega, \Omega\rangle$ and $U_2(\langle\Psi,\ \Omega,\ \Omega\rangle, \langle\Omega, \Psi, \Psi\rangle) = \langle\Omega,\ \Psi, \Psi\rangle$.

Example 4. *Two neutrosophic component n-offuninorms can be defined as*

$$\mathrm{U_{LC}}(c(x),c(y)) = \begin{cases} \varphi_1^{-1}\Big(\varphi_1(c(x)) \underset{\mathrm{LO}}{\wedge} \varphi_1(c(y))\Big), \text{ if } c(x), c(y) \in [\Psi,\ c(e)] \\ \varphi_2^{-1}\Big(\varphi_2(c(x)) \underset{\mathrm{LO}}{\vee} \varphi_2(c(y))\Big), \text{ if } c(x), c(y) \in [c(e),\ \Omega] \\ \min(c(x),c(y)), \text{ otherwise} \end{cases}$$

$$\mathrm{U_{LD}}(c(x),c(y)) = \begin{cases} \varphi_1^{-1}\Big(\varphi_1(c(x)) \underset{\mathrm{LO}}{\wedge} \varphi_1(c(y))\Big), \text{ if } c(x), c(y) \in [\Psi,\ c(e)] \\ \varphi_2^{-1}\Big(\varphi_2(c(x)) \underset{\mathrm{LO}}{\vee} \varphi_2(c(y))\Big), \text{ if } c(x), c(y) \in [c(e),\ \Omega] \\ \max(c(x),c(y)), \text{ otherwise} \end{cases}$$

where $\underset{LO}{\wedge}$ *and* $\underset{LO}{\vee}$ *were defined in the Example 2; c(e)∈(Ψ, Ω).*

Now, two examples of n-offuninorms are: $\mathrm{U_3}(\langle \mathrm{T_1}, \mathrm{I_1}, \mathrm{F_1}\rangle, \langle \mathrm{T_2}, \mathrm{I_2}, \mathrm{F_2}\rangle) = \langle \mathrm{U_{LC}}(\mathrm{T_1}, \mathrm{T_2}), \mathrm{U_{LD}}(\mathrm{I_1}, \mathrm{I_2}), \mathrm{U_{LD}}(\mathrm{F_1}, \mathrm{F_2})\rangle$ and $\mathrm{U_4}(\langle \mathrm{T_1}, \mathrm{I_1}, \mathrm{F_1}\rangle, \langle \mathrm{T_2}, \mathrm{I_2}, \mathrm{F_2}\rangle) = \langle \mathrm{U_{LD}}(\mathrm{T_1}, \mathrm{T_2}), \mathrm{U_{LC}}(\mathrm{I_1}, \mathrm{I_2}), \mathrm{U_{LC}}(\mathrm{F_1}, \mathrm{F_2})\rangle$.

They satisfy, $\mathrm{U_3}(\langle \Psi,\ \Omega,\ \Omega\rangle, \langle \Omega, \Psi, \Psi\rangle) = \langle \Psi,\ \Omega, \Omega\rangle$ and $\mathrm{U_4}(\langle \Psi, \Omega, \Omega\rangle, \langle \Omega, \Psi, \Psi\rangle) = \langle \Omega, \Psi, \Psi\rangle$.

Remark 1. *The neutrosophic components n-offuninorms defined by Equations (6) and (7) are idempotent, i.e.,* $N_O^n(c(x), c(x)) = c(x)$, *if and only if they are defined from idempotent neutrosophic component n-offnorms and n-offconorms. Moreover, they are Archimedean, i.e., they satisfy both,* $N_O^u(c(x), c(x)) <_O c(x)$ *when* $\Psi < c(x) < c(e)$ *and* $c(x) <_O N_O^u(c(x), c(x))$ *when* $c(e) < c(x) < \Omega$, *if and only if the neutrosophic component n-offnorm and n-offconorm are Archimedean. Let us observe that* $<_O$ *is the order < defined in the real line when c(x) is* $T_O(x)$ *and it is > when c(x) is* $I_O(x)$ *or* $F_O(x)$.

Proposition 4. *Let c be a neutrosophic component* (T_O, I_O *or* F_O). *c:* $M_O \to [\Psi, \Omega]$, *where* $\Psi < 0$ *and* $\Omega > 1$, *and let a neutrosophic component n-offuninorm* $N_O^u : [\Psi,\ \Omega]^2 \to [\Psi,\ \Omega]$. *Then, for every x, y ∈* M_O, *a neutrosophic component n-offnorm and a neutrosophic component n-offconorm are defined by Equations (8) and (9).*

$$\mathrm{c}(x) \underset{\mathrm{UO}}{\wedge} \mathrm{c}(y) = \varphi_1\Big(\mathrm{N_O^u}\Big(\varphi_1^{-1}(\mathrm{c}\ (x)), \varphi_1^{-1}(\mathrm{c}(y))\Big)\Big) \tag{8}$$

$$\mathrm{c}(x) \underset{\mathrm{UO}}{\vee} \mathrm{c}(y) = \varphi_2\Big(\mathrm{N_O^u}\Big(\varphi_2^{-1}(\mathrm{c}(x)), \varphi_2^{-1}(\mathrm{c}(y))\Big)\Big) \tag{9}$$

Proof. Evidently, both operators are commutative, since $\mathrm{N_O^u}$ is. Also, it is non-decreasing since $\mathrm{N_O^u}$ and the functions in Equations (2)–(5) are. They are associative because of the associativity of $\mathrm{N_O^u}$.

It is easy to verify that the overbounding conditions $\Omega \underset{\mathrm{UO}}{\wedge} c(y) = c(y)$ and $\Psi \underset{\mathrm{UO}}{\vee} c(y) = c(y)$ are also satisfied.

Additionally, we have $\Psi \underset{\mathrm{UO}}{\wedge} c(y) = \varphi_1\Big(\mathrm{N_O^u}\Big(\varphi_1^{-1}(\Psi), \varphi_1^{-1}(c(y))\Big)\Big) = \varphi_1\Big(\mathrm{N_O^u}\Big(\Psi, \varphi_1^{-1}(c(y))\Big)\Big) \le \varphi_1\Big(\mathrm{N_O^u}(\Psi, c(e))\Big) = \varphi_1(\Psi) = \Psi$, then, $\Psi \underset{\mathrm{UO}}{\wedge} c(y) = \Psi$; also, $\Omega \underset{\mathrm{UO}}{\vee} c(y) = \varphi_2\Big(\mathrm{N_O^u}\Big(\varphi_2^{-1}(\Omega), \varphi_2^{-1}(c(y))\Big)\Big) = \varphi_2\Big(\mathrm{N_O^u}\Big(\Omega, \varphi_2^{-1}(c(y))\Big)\Big) \ge \varphi_2\Big(\mathrm{N_O^u}\Big(\Omega, \varphi_2^{-1}(\Psi)\Big)\Big) = \varphi_2\Big(\mathrm{N_O^u}(\Omega, c(e))\Big) = \varphi_2(\Omega) = \Omega$, then, $\Omega \underset{\mathrm{UO}}{\vee} c(y) = \Omega$. □

Proposition 5. *Let (T_O, I_O, or F_O), c_O: M_O→[Ψ, Ω] and (T, I, or F), c_N: MN→[0,1] be a neutrosophic component n-offset and a neutrosophic component, respectively. There exists a bijective mapping such that every neutrosophic component n-offuninormis transformed into a neutrosophic component uninorm and vice versa.*

Proof. Let us define the function $\varphi_3 : [\Psi, \Omega] \to [0, 1]$ and its inverse $\varphi_3^{-1} : [0, 1] \to [\Psi, \Omega]$, expressed in Equations (10) and (11), respectively.

$$\varphi_3(c(x)) = \frac{c(x) - \Psi}{\Omega - \Psi} \tag{10}$$

$$\varphi_3^{-1}(c(x)) = (\Omega - \Psi)c(x) + \Psi \tag{11}$$

Evidently, they are increasing bijective mappings.

If $\hat{\mathrm{U}}_\mathrm{N}(\cdot,\cdot)$ is a neutrosophic uninorm, then we can define the neutrosophic component n-offuninorm $\hat{\mathrm{N}}_\mathrm{O}^\mathrm{u}(\cdot,\cdot)$ as follows:

$$\hat{\mathrm{N}}_\mathrm{O}^\mathrm{u}(c_O(x), c_O(y)) = \varphi_3^{-1}\left(\hat{\mathrm{U}}_\mathrm{N}(\varphi_3(c_O(x)), \varphi_3(c_O(y)))\right)$$

Conversely, if we have $\hat{\mathrm{N}}_\mathrm{O}^\mathrm{u}(\cdot,\cdot)$, we can define $\hat{\mathrm{U}}_\mathrm{N}(\cdot,\cdot)$ as follows:

$$\hat{\mathrm{U}}_\mathrm{N}(c_N(x), c_N(y)) = \varphi_3\left(\hat{\mathrm{N}}_\mathrm{O}^\mathrm{u}\left(\varphi_3^{-1}(c_N(x)), \varphi_3^{-1}(c_N(y))\right)\right)$$

Then, it is easy to prove that $\hat{\mathrm{N}}_\mathrm{O}^\mathrm{u}(c_O(x), c_O(y))$ is a neutrosophic component n-offuninorm and $\hat{\mathrm{U}}_\mathrm{N}(c_N(x), c_N(y))$ is a neutrosophic component uninorm. Moreover, the relationship between the components of their neutral elements $c_O(e_O)$ and $c_N(e_N)$ is $c_N(e_N) = \varphi_3(c_O(e_O))$ and thus $c_O(e_O) = \varphi_3^{-1}(c_N(e_N))$. □

Let us remark that we maintain the definition of inverse mapping that we explained in Equations (3) and (5).

In agreement with Proposition 5, many predefined neutrosophic uninorms can be used to define n-offuninorms. In turn, fuzzy uninorms can be used to define neutrosophic uninorms, thus, it is simply necessary to find examples in the field of fuzzy uninorms; see further Section 4.1. First, let us make reference to some properties of n-offuninorms.

Proposition 6. *Let c be a neutrosophic component (T_O, I_O or F_O). c: M_O→[Ψ, Ω], where Ψ≤ 0 and Ω≥1. Given the neutrosophic component n-offuninorm $N_O^u : [\Psi, \Omega]^2 \to [\Psi, \Omega]$ and the offuninorm $U_O : [\Psi, \Omega]^3 \times [\Psi, \Omega]^3 \to [\Psi, \Omega]^3$ defined from $N_O^u(\cdot, \cdot)$, $U_O(\langle T_O(x), I_O(x), F_O(x)\rangle, \langle T_O(y), I_O(y), F_O(y)\rangle) = \langle N_O^u(T_O(x), T_O(y)), N_O^u(I_O(x), I_O(y)), N_O^u(F_O(x), F_O(y))\rangle$, satisfies the following properties for any $x = \langle T_O(x), I_O(x), F_O(x)\rangle$, denoting $\Psi_O = \langle \Psi, \Omega, \Omega\rangle$ and $\Omega_O = \langle \Omega, \Psi, \Psi\rangle$:*

1. *$U_O(\Psi_O, \Psi_O) = \Psi_O$ and $U_O(\Omega_O, \Omega_O) = \Omega_O$.*
2. *If c(e)≠Ψ, Ω, then, $U_O(\Psi_O, \Omega_O) = U_O(U_O(\Psi_O, \Omega_O), x)$*
3. *If c(e)≠Ψ,Ω, then either $U_O(\Psi_O, \Omega_O) = \Psi_O$ or $U_O(\Psi_O, \Omega_O) = \Omega_O$ or $U_O(\Psi_O, \Omega_O)$ is $\le_O$-incomparable respect to $e = \langle T_O(e), I_O(e), F_O(e)\rangle$.*
4. *If there exists $y = \langle T_O(y), I_O(y), F_O(y)\rangle$, such that either $x \le_O e \le_O y$ or $y \le_O e \le_O x$, then, $min(x, y) \le_O U_O(x, y) \le_O max(x, y)$.*

Proof.

1. Since $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(e)) = \Psi$ and $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Omega, c(e)) = \Omega$ and considering that $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, \cdot)$ and $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Omega, \cdot)$ are non-decreasing, the result is trivial. Then, $\mathrm{U}_\mathrm{O}(\Psi_\mathrm{O}, \Psi_\mathrm{O}) = \Psi_\mathrm{O}$ and $\mathrm{U}_\mathrm{O}(\Omega_\mathrm{O}, \Omega_\mathrm{O}) = \Omega_\mathrm{O}$.
2. First suppose $c(x) \le c(e)$, then $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(x)) \le \mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(e)) = \Psi$, therefore $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(x)) = \Psi$, thus $\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, \Omega) = \mathrm{N}_\mathrm{O}^\mathrm{u}\left(\mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(x)), \Omega\right) = \mathrm{N}_\mathrm{O}^\mathrm{u}\left(\Omega, \mathrm{N}_\mathrm{O}^\mathrm{u}(\Psi, c(x))\right) = \mathrm{N}_\mathrm{O}^\mathrm{u}\left(\mathrm{N}_\mathrm{O}^\mathrm{u}(\Omega, \Psi), c(x)\right)$

$= N_O^u\left(N_O^u(\Psi,\ \Omega), c(x)\right)$. See that we applied the commutativity and associativity of $N_O^u(\cdot,\ \cdot)$. Now, suppose $c(e) \le c(x)$, then $N_O^u(c(x),\Omega) \ge N_O^u(c(x),\Omega) = \Omega$, therefore, $N_O^u(c(x),\ \Omega) = \Omega$, and $N_O^u(\Psi,\ \Omega) = N_O^u\left(\Psi,\ N_O^u(c(x),\Omega)\right) = N_O^u\left(N_O^u(\Omega,\Psi), c(x)\right)$. Suppose x and $e = T_O(e), I_O(e), F_O(e)$ are $\le_O$-incomparable, i.e., $x \nleq_O e$ and $e \nleq_O x$. Then,

$$x \underset{O}{\wedge} e = \min(T_O(x),\ T_O(e)), \max(I_O(x), I_O(e)), \max(F_O(x), F_O(e)) \le_O x \le_O \max(T_O(x),\ T_O(e)), \min(I_O(x), I_O(e)), \min(F_O(x), F_O(e)) = x \underset{O}{\vee} e$$

Then, according to the previous results we have $U_O(\Psi_O,\ \Omega_O) = U_O\left(U_O(\Psi_O,\ \Omega_O), x \underset{O}{\wedge} e\right) = U_O\left(U_O(\Psi_O,\ \Omega_O), x \underset{O}{\vee} e\right)$, thus, for the increasing condition of $U_O(\cdot,\ \cdot)$ it is satisfied $U_O(\Psi_O,\ \Omega_O) = U_O(U_O(\Psi_O,\ \Omega_O), x)$. Then, we proved $U_O(\Psi_O,\ \Omega_O) = U_O(U_O(\Psi_O,\ \Omega_O), x)$.

3. Suppose $U_O(\Psi_O, \Omega_O)$ is $\le_O$-comparable respect to e, then, if $U_O(\Psi_O, \Omega_O) \le_O e$ since the previous proof $U_O(\Psi_O,\ \Omega_O) = U_O(U_O(\Psi_O,\ \Omega_O), \Psi_O) = \Psi_O$. If $e \le_O U_O(\Psi_O, \Omega_O)$ then $U_O(\Psi_O,\ \Omega_O) = U_O(U_O(\Psi_O,\ \Omega_O), \Omega_O) = \Omega_O$.
4. Let us assume without loss of generality that $x \le_O e \le_O y$, then, $x = U_O(x, e) \le_O U_O(x, y) \le_O U_O(e, y) = y$. □

When c_1: $M_O \to [\Psi_1,\ \Omega_1]$ and c_2: $M_O \to [\Psi_2,\ \Omega_2]$ are two neutrosophic components, such that $\Psi_1 \neq \Psi_2$ or $\Omega_1 \neq \Omega_2$, satisfying that at least one of Ψ_1 and Ψ_2 is smaller than 0, or at least one of Ω_1 and Ω_2 is bigger than 1, then, a neutrosophic component n-offuninorm aggregates both of them, according to the interpretation we have to obtain.

For example, if c_1: MO→[−1,1] and c2: MO→[0,1], and the first one means the relationship between two variables like the linear regression coefficient and the second one represents a classical probability, if we need to obtain the aggregation in [−1,1] in the framework of variable relationships, then after transforming $c_2 : M_O \to [0, 1]$ to $\hat{c}_2 : M_O \to [-1,\ 1]$, we aggregate c_1 and $\hat{c}_2$ using $N_O^u :\ [-1,\ 1]^2 \to [-1,\ 1]$, only in the case that it makes sense to rescale c_2, otherwise, because $[0,1] \subset [-1,1]$, we can apply $N_O^u :\ [-1,\ 1]^2 \to [-1,\ 1]$ over c_1 and c_2.

However, if we need to obtain a classical probabilistic interpretation, then we aggregate c_2: $M_O \to [0,1]$ and $\hat{c}_1 : M_O \to [0,\ 1]$, where $\hat{c}_1$ is a transformation obtained from c_1: $M_O \to [-1, 1]$.

Example 5. *Let us revisit Example 3 with* $U_1 : [-0.7,\ 1.2]^3 \times [-0.7,\ 1.2]^3 \to [-0.7,\ 1.2]^3$ *and neutral element* $e = \langle -0.5,\ 0, 0\rangle$, *defined as* $U_1(\langle T_1, I_1, F_1\rangle, \langle T_2, I_2, F_2\rangle) = \langle U_{ZC}(T_1, T_2), U_{ZD}(I_1, I_2), U_{ZD}(F_1, F_2)\rangle$. Then, we have:

$$U_{LC}(T_O(x), T_O(y)) = \begin{cases} \max(T_O(x), T_O(y)), \text{ if } T_O(x), T_O(y) \in [-0.5,\ 1.2] \\ \min(T_O(x), T_O(y)), \text{ otherwise} \end{cases}$$

$$U_{LD}(I_O(x), I_O(y)) = \begin{cases} \min(I_O(x), I_O(y)), \text{ if } I_O(x), I_O(y) \in [-0.7,\ 0] \\ \max(I_O(x), I_O(y)), \text{ otherwise} \end{cases}$$

$$U_{LD}(F_O(x), F_O(y)) = \begin{cases} \min(F_O(x), F_O(y)), \text{ if } F_O(x), F_O(y) \in [-0.7,\ 0] \\ \max(F_O(x), F_O(y)), \text{ otherwise} \end{cases}$$

Let us aggregate the elements of $A = \{(x_1, \langle 1.2, 0.4, -0.1\rangle), (x_2, \langle 0.2, 0.3, -0.7\rangle)\}$ by using $U_1(\cdot,\cdot)$, then, $U_1((x_1, \langle 1.2, 0.4, -0.1\rangle), (x_2, \langle 0.2, 0.3, -0.7\rangle)) = \langle U_{LC}(1.2, 0.2), U_{LD}(0.4, 0.3), U_{LD}(-0.1, -0.7)\rangle = \langle 1.2,\ 0.4, -0.7\rangle$.

4. Applications

In the following, we illustrate the applicability of the present investigation aided by three areas of application.

4.1. N-Offuninorms and MYCIN

Let us start with the parameterized Silvert uninorms, see [40]:

$$u_{N\lambda}(c_N(x), c_N(y)) = \begin{cases} \frac{\lambda c_N(x)c_N(y)}{\lambda c_N(x)c_N(y)+(1-c_N(x))(1-c_N(y))}, \text{ if } (c_N(x),\ c_N(y)) \in [0,\ 1]^2 \backslash \{(0,\ 1), (1,\ 0)\} \\ 0, \text{ otherwise} \end{cases}$$

where $\lambda > 0$ and $c_N(e_\lambda) = \frac{1}{\lambda+1}$. To convert this family to the equivalent one defined into $[-1, 1]$ we have to apply the Equations in Proposition 5. Then, it is obtained $u_{O\lambda}(c_O(x), c_O(y)) = \begin{cases} \frac{(\lambda-1)(1+c_O(x)c_O(y))+(\lambda+1)(c_O(x)+c_O(y))}{(\lambda+1)(1+c_O(x)c_O(y))+(\lambda-1)(c_O(x)+c_O(y))}, \text{ if } (c_O(x),\ c_O(y)) \in [-1,\ 1]^2 \backslash \{(-1,\ 1), (1,\ -1)\} \\ 0, \text{ otherwise} \end{cases}$ where $c_O(e_\lambda) = \frac{1-\lambda}{1+\lambda}$.

Let us note that $\lim_{\lambda \to 0^+} c_O(e_\lambda) = 1$ and $\lim_{\lambda \to +\infty} c_O(e_\lambda) = -1$. Therefore, the closer λ approximates to 0, the closer $u_{O\lambda}(\cdot, \cdot)$ performs like a neutrosophic component n-offnorm; whereas, the greater λ, the closer $u_{O\lambda}(\cdot, \cdot)$ performs like a neutrosophic component n-offconorm.

An additional consequence of these assertions is that inequalities $0<\lambda_1<\lambda_2$ imply $u_{O\lambda_1}(c_O(x), c_O(y)) < u_{O\lambda_2}(c_O(x), c_O(y))$.

Applying Equations (2)–(5) to the conditions of the present example, the following transformations are obtained:

$\hat{\varphi}_{1\lambda}(c_O(x)) = (1+\lambda)c_O(x) + \lambda$, $\hat{\varphi}_{1\lambda}^{-1}(c_O(x)) = \frac{c_O(x)-\lambda}{1+\lambda}$, $\hat{\varphi}_{2\lambda}(c_O(x)) = \frac{(1+\lambda)c_O(x)-1}{\lambda}$ and $\hat{\varphi}_{2\lambda}^{-1}(c_O(x)) = \frac{\lambda c_O(x)+1}{1+\lambda}$.

Then, a neutrosophic component n-offnorm and a neutrosophic component n-offconorm are defined from Equations (8) and (9), as follows:

$c(x) \underset{\lambda O}{\wedge} c(y) = \hat{\varphi}_{1\lambda}\left(u_{O\lambda}\left(\hat{\varphi}_{1\lambda}^{-1}(c(x)), \hat{\varphi}_{1\lambda}^{-1}(c(y))\right)\right)$ and $c(x) \underset{\lambda O}{\vee} c(y) = \hat{\varphi}_{2\lambda}\left(u_{O\lambda}\left(\hat{\varphi}_{2\lambda}^{-1}(c(x)), \hat{\varphi}_{2\lambda}^{-1}(c(y))\right)\right)$, respectively.

Other properties of $u_{O\lambda}(\cdot, \cdot)$ are the following:

1. $u_{O\lambda}(c_O(x), -c_O(x)) = \begin{cases} \frac{\lambda-1}{1+\lambda}, \text{ if } c_O(x) \in (-1,\ 1) \\ -1, \text{ otherwise} \end{cases}$
2. $u_{O\lambda}(\cdot, \cdot)$ is Archimedean. To prove it, given $c_O(x) < c_O(e_\lambda)$, then $u_{O\lambda}(c_O(x), c_O(x)) \leq u_{O\lambda}(c_O(x), c_O(e_\lambda)) = c_O(x)$ and if $c_O(x) > c_O(e_\lambda)$, $u_{O\lambda}(c_O(x), c_O(x)) \geq u_{O\lambda}(c_O(x), c_O(e_\lambda)) = c_O(x)$.

To prove those inequalities are strict, let us suppose the equation $u_{O\lambda}(c_O(x), c_O(x)) = \frac{(\lambda-1)\left(1+c_O^2(x)\right)+2(\lambda+1)c_O(x)}{(\lambda+1)\left(1+c_O^2(x)\right)+2(\lambda-1)c_O(x)} = c_O(x)$ holds, or equivalently $(\lambda-1)\left(1+c_O^2(x)\right) + 2(\lambda+1)c_O(x) = c_O(x)\left[(\lambda+1)\left(1+c_O^2(x)\right) + 2(\lambda-1)c_O(x)\right]$, thus, $(\lambda-1)\left(1-c_O^2(x)\right) + (\lambda+1)c_O(x)\left(1-c_O^2(x)\right) = 0$ and finally, $\left(1-c_O^2(x)\right)(\lambda-1+(\lambda+1)c_O(x)) = 0$, hence the solutions are $c_O(x) = \pm 1$ and $c_O(x) = c_O(e_\lambda)$. Then, we conclude it is Archimedean.

A remarkable case is λ = 1, which converts into Equation (12).

$$\mathrm{u}_{O1}(c_O(x), c_O(y)) = \begin{cases} \frac{c_O(x)+c_O(y)}{1+c_O(x)c_O(y)}, \text{ if } (c_O(x),\ c_O(y)) \in [-1,\ 1]^2 \backslash \{(-1,\ 1), (1,\ -1)\} \\ -1, \text{ otherwise} \end{cases} \tag{12}$$

$u_{O1}(\cdot, \cdot)$ is the function called PROSPECTOR which aggregates hypothesis values or Certainty Factors (CF) related to MYCIN, the well-known medical Expert System; nevertheless, the function used in MYCIN is undefined for the arguments (−1, 1) and (1, −1), see [32–34]. Summarizing, we can say that PROSPECTOR is a neutrosophic component n-offuninorm, such that $c_O(e_1) = 0$, which is an effective and widely used aggregation operator.

$u_{O1}(\cdot,\cdot)$ means the combination of the CFs of two independent experts about the hypothesis H. CF = –1.0 means expert has 100% evidence against H and CF = 1.0 means he or she has 100% evidence to support H. The smaller the CF, the greater the evidence against H; the larger the CF, the greater the evidence supporting H; whereas evidence with degree close to 0 means a borderline degree of evidence. Here, $u_{O1}(c_O(x), -c_O(x)) = 0$, where $u_{O1}(-1, 1) = u_{O1}(1, -1) = -1$ for meaning that the 100% contradiction is assessed as 100% against H. The original $u_{O1}(\cdot,\cdot)$ in [32] accepts they are undefined.

Another function is the *Modified Combining Function* $C(x,y)$, see [34], defined as

$$C(x, y) = \begin{cases} x + y(1 - x), \text{ if } \min(x, y) \geq 0 \\ \frac{x+y}{1-\min(|x|,|y|)}, \text{ if } \min(x, y) < 0 < \max(x, y) \\ x + y(1 + x), \text{ if } \max(x, y) \leq 0 \end{cases}$$

The components n-offnorm and n-offconorm obtained from the PROSPECTOR are the following: $c_O(x) \underset{1O}{\wedge} c_O(y) = \frac{4(c_O(x)+c_O(y)-2)}{4+(c_O(x)-1)(c_O(y)-1)} + 1$ and $c_O(x) \underset{1O}{\vee} c_O(y) = \frac{4(c_O(x)+c_O(y)+2)}{4+(c_O(x)+1)(c_O(y)+1)} - 1$, respectively, see Figures 1 and 2.

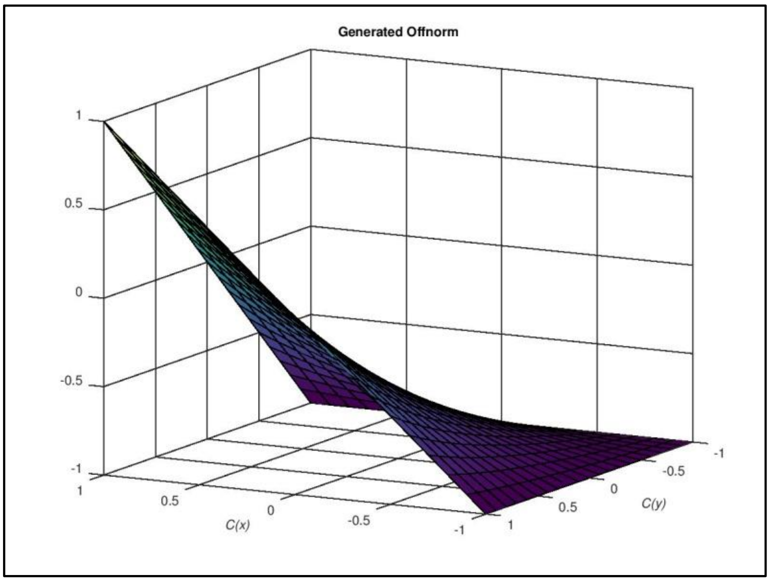

Figure 1. Depiction of the neutrosophic component n-offnorm generated by $u_{O1}(\cdot,\cdot)$.

Hitherto we mostly calculated on neutrosophic components, nevertheless n-offuninorms have to be defined for the three components altogether. For example, given $x, y \in [-1, 1]^3$, $U_{N\lambda}(x, y) = \langle u_{O\lambda_1}(T_O(x), T_O(y)),\ u_{O\lambda_2}(I_O(x), I_O(y)), u_{O\lambda_3}(F_O(x), F_O(y))\rangle$ is an n-offuninorm, which evidently it is not conjunctive, neither is it disjunctive, see that $U_{N\lambda}(\langle -1, 1, 1\rangle, \langle 1, -1, -1\rangle) = \langle -1, -1, -1\rangle$.

Conjunctive and disjunctive neutrosophic component n-offuninorms were illustrated in Example 3; see also Example 5. Example 6 is a hypothetical example to explain the use of this theory in a real-life situation.

Example 6. *Three physicians, denoted by A, B, and C, have to emit a criterion about a patient's disease which suffers from somewhat confusing symptoms. They agree that the Certainty Factor is the better way to express* *their opinions. They use single-valued neutrosophic offsets, instead of a simple CF to increase the accuracy of the criteria.*

After a discussion, they are convinced that it is most likely that the patient has either a thyroid disease or an infectious one. The treatment for each disease is different each other. Therefore, they have two hypotheses; one is H_T which means the patient has thyroid disease and H_I that patient has an infectious disease.

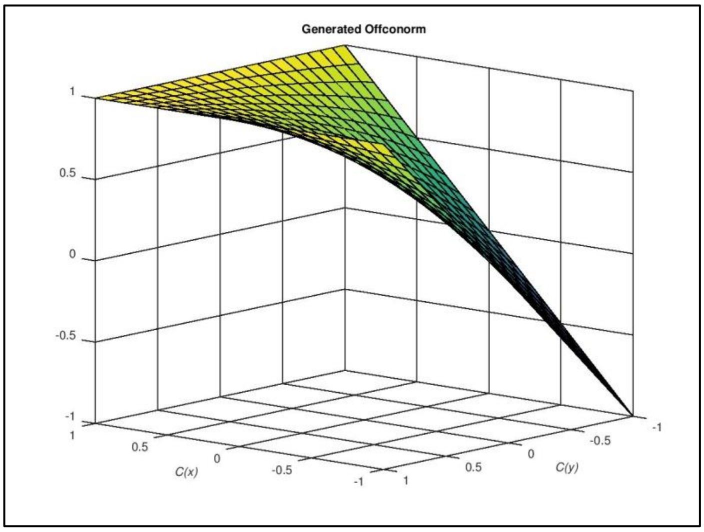

Figure 2. Depiction of the neutrosophic component n-offconorm generated by $u_{O1}(\cdot,\cdot)$.

Physician A thinks that the probability they are dealing with a thyroid disease is A_T = <−0.6, 0.4, 0.6> and that it is an infectious disease is A_I = <0.8, −0.5, −0.8>, thus, A is 60% against H_T and 40% undecided about it; however, A is 80% in favor of H_I and 50% sure about it.

Similarly, we have that B's criteria are, B_T = <−0.1, −0.2,0.1> and B_I = <0.1, 0.8, −0.1>, whereas C's criteria are C_T = <0.7, 0.1, −0.2> and C_I = <−0.6, −0.3, 0.7>.

To decide what is the strongest hypothesis, H_T or H_I, they select the well-known PROSPECTOR function used in MYCIN (see Equation (12)) for each component.

Thus, for H_T we have an aggregated value equal to <0.073684, 0.31064, 0.53043> and for H_I it is <0.46667, 0.23529, −0.32>, therefore, evidently, the infectious disease is the strongest hypothesis, because $\langle 0.073684,\ 0.31064,\ 0.53043\rangle <_O \langle 0.46667,\ 0.23529,\ -0.32\rangle$.

Despite we proved in Proposition 5 that neutrosophic uninorms are mathematically equivalent to offuninorms, it is worthwhile to remark that the reason for using an interval different of [0, 1] is that it could be useful to model real-life problems. The present example is a good one to explain that reason. The advantages arise from the accuracy and compactness of an expert's information. In this example, from an expert's viewpoint, it is easier to express opinions in the scale [−1, 1] with the aforementioned meaning than in the scale [0, 1], which is less clear. Information compactness is given because of only a single offset is semantically equivalent to at least two neutrosophic sets.

Additionally, because of the significance of functions like $u_{O1}(\cdot,\cdot)$ and $C(x,y)$, which were used as aggregation functions in that well-known expert system, some authors have extended the domain of fuzzy uninorms to any interval [a, b], not necessarily restricted to a = 0 and b = 1; see [33,34].

This fact supports the usefulness of the present work, where for the first time the precedent ideas on extending the truth values beyond the scope of [0, 1] naturally associate with the offset concept maintaining the original definitions of the aggregation functions used in MYCIN.

Another powerful reason is the applicability of $u_{O1}(\cdot,\cdot)$ and $C(x,y)$, and hence of the fuzzy uninorms defined in [a, b], as threshold functions of artificial neurons in Artificial Neural Networks, as well as to Fuzzy Cognitive Maps, which are used in fields like decision making, forecasting, and strategic planning [33].

Such applications of uninorms in the fuzzy domain can be explored in the framework of neutrosophy theory, e.g., in Artificial Neural Networks based on neutrosophic sets, in Neutrosophic Cognitive Maps, among others [36,37].

4.2. N-Offuninorms and Implicators

Fuzzy uninorms are used to define implicators (see [41], pp. 151–160). This application was extended to neutrosophic uninorms ([25]). To extend the implication operator in the offuninorm framework, first, we need to consider the notion of offimplication, which has been defined symbolically.

The *Symbolic Neutrosophic Offlogic Operators* or briefly the *Symbolic Neutrosophic Offoperators* extend the Symbolic Neutrosophic Logic Operators, where every one of T, I, F has an under and an over version (see [31], pp. 132–139).

T_O = Over Truth,
T_U = Under Truth;
I_O = Over Indeterminacy,
I_U = Under Indeterminacy;
F_O = Over Falsehood,
F_U = Under Falsehood.

Let $S_N = \{T_O, T, T_U, I_O, I, I_U, F_O, F, F_U\}$ be the set of neutrosophic symbols, an order is defined in S_N as follows: if '<' denotes "more important than", we have the following order, $T_U < I_U < F_U < F < I < T < F_O < I_O < T_O$, where $-\infty < T_U < I_U < F_U < 0$, $0 \le F < I < T \le 1$ and $1 < F_O < I_O < T_O < +\infty$; see Figure 3. Let us note that the proposed order is not the unique one, it depends on the decision maker's objective.

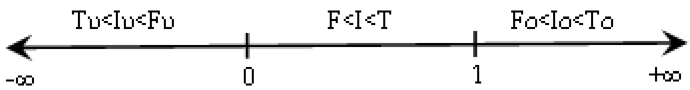

Figure 3. Ordered symbolic neutrosophic components in the neutrosophic offlogic.

Let us observe that I is the center of the elements according to <. For every $\alpha \in S_N$, the *symbolic neutrosophic offcomplement* is denoted by $C_O(\alpha)$ and it is defined as the symmetric element respect to the median centered in I, e.g., $C_{SO}(F_O) = F_U$ and $C_{SO}(F) = T$, hence, given $\alpha \in S_N$ its *symbolic neutrosophic offnegation* is $\underset{SO}{\neg}\alpha = C_{SO}(\alpha)$.

Additionally, for any $\alpha, \beta \in S_N$ the *symbolic neutrosophic offconjunction* is defined as $\alpha \underset{SO}{\wedge} \beta = \min(\alpha, \beta)$, the *symbolic neutrosophic offdisjunction* is defined as $\alpha \underset{SO}{\vee} \beta = \max(\alpha, \beta)$, whereas the *symbolic neutrosophic offimplication* is defined in Equation (13).

$$\alpha \underset{SO}{\rightarrow} \beta = \left(\underset{SO}{\neg} \alpha\right) \underset{SO}{\vee} \beta \tag{13}$$

In this paper, we redefine some of the symbolic neutrosophic offoperators to the continuous quantitative domain. Given $\overline{\alpha} \in [\Psi, \Omega]$, where $\Psi < 0$ or $\Omega > 1$, the *neutrosophic offnegation* is defined by Equation (14).

$$\underset{O}{\neg}\overline{\alpha} = \begin{cases} \min\{\Omega, 1 - \overline{\alpha}\}, \text{ if } \overline{\alpha} \le 0.5 \\ \max\{\Psi, 1 - \overline{\alpha}\}, \text{ if } \overline{\alpha} > 0.5 \end{cases} \tag{14}$$

The *neutrosophic offnegation* satisfies the following properties:

1. It is a non-increasing operator, which extends the classical negation operator in fuzzy logic theory. It is strictly decreasing when $\Omega + \Psi = 1$.
2. It extends the notion of symbolic neutrosophic offnegation because satisfies the following properties:

   2.1 It is centered in 0.5, i.e., $\underset{O}{\neg}\ 0.5 = 0.5$, therefore I = 0.5.

   2.2 If $\overline{\alpha} \in [0,\ 1]$, then $\underset{O}{\neg}\ \overline{\alpha} \in [0,\ 1]$, $\underset{O}{\neg}\ 0 = 1$ and $\underset{O}{\neg}\ 1 = 0$, which is the usual negation operator in fuzzy logic.

   2.3 If $\overline{\alpha} < 0$, then $\underset{O}{\neg}\ \overline{\alpha} \geq 1$. $\underset{O}{\neg}\ \overline{\alpha} = 1$ only when $\Omega = 1$.

   2.4 If $\overline{\alpha} > 1$, then $\underset{O}{\neg}\ \overline{\alpha} \leq 0$. $\underset{O}{\neg}\ \overline{\alpha} = 0$ only when $\Psi = 0$.

   2.5 When $\Omega + \Psi = 1$, we have $\underset{O}{\neg}\ \Psi = \Omega$ and $\underset{O}{\neg}\ \Omega = \Psi$.

3. If $\Omega + \Psi = 1$, then $\underset{O}{\neg}\ \underset{O}{\neg}\ \overline{\alpha} = \overline{\alpha}$, for every $\overline{\alpha} \in [\Psi,\ \Omega]$.

The precedent properties are easy to demonstrate.

Hence, the definition of *offimplication* $\underset{O}{\rightarrow}\ :\ [\Psi,\ \Omega]^3 \times [\Psi,\ \Omega]^3 \rightarrow [\Psi,\ \Omega]^3$ is defined in Equation (15), for every $\overline{\alpha}, \overline{\beta} \in [\Psi,\ \Omega]^3$.

$$\overline{\alpha} \underset{O}{\rightarrow} \overline{\beta} = \left\langle N_O^{co}\left(\underset{O}{\neg}\ T_O(\overline{\alpha}),\ T_O\left(\overline{\beta}\right)\right),\ N_O^{n_1}\left(\underset{O}{\neg}\ I_O(\overline{\alpha}),\ I_O\left(\overline{\beta}\right)\right), N_O^{n_2}\left(\underset{O}{\neg}\ F_O(\overline{\alpha}),\ F_O\left(\overline{\beta}\right)\right)\right\rangle \tag{15}$$

where, $N_O^{n_i}(\cdot,\ \cdot)$ i = 1, 2 are neutrosophic components n-offnorms, $N_O^{co}(\cdot,\ \cdot)$ is a neutrosophic component n-offconorm, and $\underset{O}{}$ is the offnegation defined in Equation (14).

Equation (15) is generalized by using offuninorms, see Equation (16).

$$\overline{\alpha} \underset{UO}{\rightarrow} \overline{\beta} = \left\langle N_O^{u_1}\left(\underset{O}{\neg}\ T_O(\overline{\alpha}),\ T_O\left(\overline{\beta}\right)\right),\ N_O^{u_2}\left(\underset{O}{\neg}\ I_O(\overline{\alpha}),\ I_O\left(\overline{\beta}\right)\right), N_O^{u_3}\left(\underset{O}{\neg}\ F_O(\overline{\alpha}),\ F_O\left(\overline{\beta}\right)\right)\right\rangle \tag{16}$$

where $N_O^{u_i}(\cdot,\ \cdot)$ for i = 1, 2, and 3 are neutrosophic components n-offuninorms.

Example 7. *One illustrative example of Equation (16) is obtained revisiting Section 4.1, by defining the following neutrosophic component n-offnorm:*

$$u_O(c_O(x), c_O(y)) = \begin{cases} \frac{3(c_O(x)+1)(c_O(y)+1)}{(c_O(x)+1)(c_O(y)+1)+(2-c_O(x))(2-c_O(y))} - 1, & \text{if } (c_O(x),\ c_O(y)) \in [-1,\ 2]^2 \backslash \{(-1,\ 2), (2,\ -1)\} \\ -1, & \text{otherwise} \end{cases}$$

This is the transformation of Silvert uninorms to the domain $[-1, 2]^2$ applying the functions in Equations (10) and (11), and the transformation in Proposition 5. Also, let us take $U_{ZD}(c(x), c(y))$ of Example 3. See that [−1, 2] is symmetric respect to 0.5, and the neutral element is 0.5.

Then, we study the offuninorm defined in the following equation: $U_O(\overline{\alpha}, \overline{\beta}) = \langle U_{ZD}(T_O(\overline{\alpha}), T_O(\overline{\beta})),\ u_O(I_O(\overline{\alpha}), I_O(\overline{\beta})), u_O(F_O(\overline{\alpha}), F_O(\overline{\beta}))\rangle$ for $\overline{\alpha} = \langle T_O(\overline{\alpha}),\ I_O(\overline{\alpha}), F_O(\overline{\alpha})\rangle$ and $\overline{\beta} = \langle T_O(\overline{\beta}),\ I_O(\overline{\beta}), F_O(\overline{\beta})\rangle$ in $[-1, 2]^3$.

Thus, we define the offimplication generated by $U_O(\cdot, \cdot)$ according to Equation (16) as follows:

$$\overline{\alpha} \underset{U_O}{\rightarrow} \overline{\beta} = \langle U_{ZD}\left(\underset{O}{\neg} T_O(\overline{\alpha}),\ T_O(\overline{\beta})\right), u_O\left(\underset{O}{\neg} I_O(\overline{\alpha}),\ I_O(\overline{\beta})\right), u_O\left(\underset{O}{\neg} F_O(\overline{\alpha}),\ F_O(\overline{\beta})\right)\rangle.$$

where in this case we have $U_{ZD}\left(T_O(\overline{\alpha}),\ T_O(\overline{\beta})\right) = \begin{cases} \min\left(T_O(\overline{\alpha}),\ T_O(\overline{\beta})\right), \text{ if } T_O(\overline{\alpha}),\ T_O(\overline{\beta}) \in \left[-1,\ \frac{1}{2}\right] \\ \max\left(T_O(\overline{\alpha}),\ T_O(\overline{\beta})\right), \text{ otherwise} \end{cases}$,
see Figure 4, and $u_O(\cdot,\cdot)$ models the neutrosophic n-components I_O and F_O, see Figure 5.

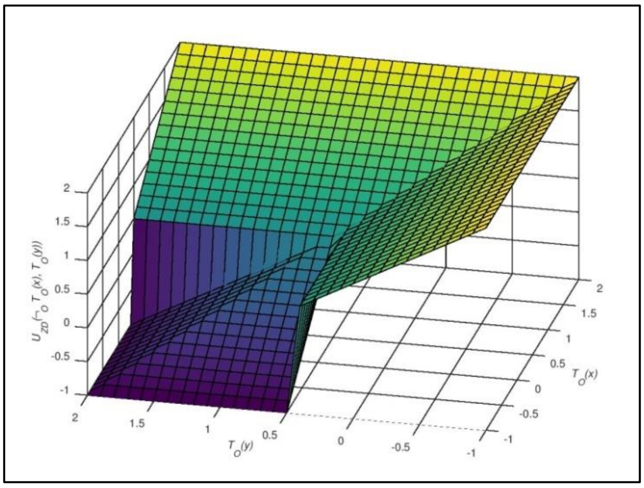

Figure 4. Depiction of the neutrosophic n-offimplication generated by U_{ZD} for T_O.

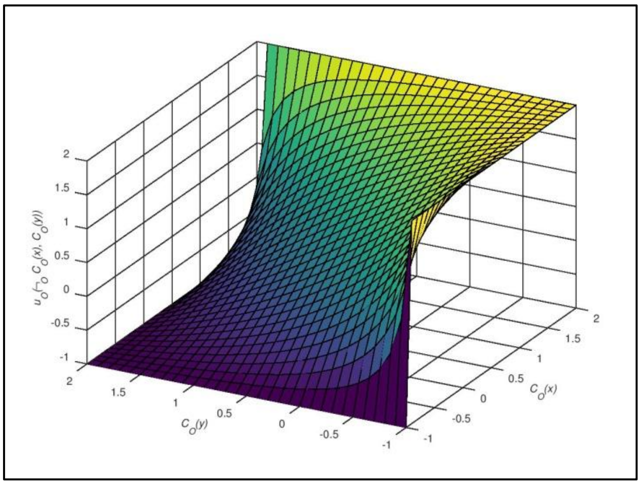

Figure 5. Depiction of the neutrosophic n-offimplication generated by u_O for both, I_O and F_O.

This offimplicator satisfies the overbounding conditions $\langle -1,\ 2,\ 2\rangle \underset{U_O}{\rightarrow} \langle -1,\ 2,\ 2\rangle = \langle -1,\ 2,\ 2\rangle \underset{U_O}{\rightarrow} \langle 2,\ -1,\ -1\rangle = \langle\ 2,\ -1,\ -1\rangle \underset{U_O}{\rightarrow} \langle 2,\ -1,\ -1\rangle = \langle 2,\ -1,\ -1\rangle$, whereas, $\langle 2,\ -1,\ -1\rangle \underset{U_O}{\rightarrow} \langle -1,\ 2,\ 2\rangle = \langle -1,\ 2,\ 2\rangle$.

Also, $\langle 0,\ 1,\ 1\rangle \underset{U_O}{\rightarrow} \langle 0,\ 1,\ 1\rangle = \langle\ 1,\ 0,\ 0\rangle \underset{U_O}{\rightarrow} \langle 1,\ 0,\ 0\rangle = \langle 1,\ 0.5,\ 0.5\rangle$, $\langle 0,\ 1,\ 1\rangle \underset{U_O}{\rightarrow} \langle 1,\ 0,\ 0\rangle = \langle 1,\ -0.4,\ -0.4\rangle$ and $\langle 1,\ 0,\ 0\rangle \underset{U_O}{\rightarrow} \langle 0,\ 1,\ 1\rangle = \langle 0,\ 1.4,\ 1.4\rangle$. Additionally, $\langle 0.5,\ 0.5,\ 0.5\rangle \underset{U_O}{\rightarrow} \langle 0.5,\ 0.5,\ 0.5\rangle = \langle 0.5,\ 0.5,\ 0.5\rangle$ because 0.5 is the neutral element of every neutrosophic component n-offuninorm and $\underset{O}{\neg}\ 0.5 = 0.5$.

It is easy to check that substituting $u_O(\cdot,\cdot)$ by $U_{ZC}(\cdot,\cdot)$ in $\underset{U_O}{\rightarrow}$, we obtain the more classical equations $\langle 0,\ 1,\ 1\rangle \underset{U_O}{\rightarrow} \langle 0,\ 1,\ 1\rangle = \langle 1,\ 0,\ 0\rangle \underset{U_O}{\rightarrow} \langle 1,\ 0,\ 0\rangle = \langle 0,\ 1,\ 1\rangle \underset{U_O}{\rightarrow} \langle 1,\ 0,\ 0\rangle = \langle 1,\ 0,\ 0\rangle$ and $\langle 1,\ 0,\ 0\rangle \underset{U_O}{\rightarrow} \langle 0,\ 1,\ 1\rangle = \langle 0,\ 1,\ 1\rangle$.

4.3. N-Offuninorms and Voting Games

The applicability of uninorms to solve group decision problems is evident. However, the use of them as part of a game theory solution is not so obvious. This subsection is devoted to solving voting games based on n-offuninorms.

A *cooperative game with transferable utility* consists of a pair (N,v), where N = {1, 2, . . . ,n} is a non-empty set of *players*,n ∈ ℕ and v: $2^N \rightarrow \mathbb{R}$, i.e., $v(\cdot)$ is a function of the power set of N such that each *coalition* or $S \subseteq$ N is associated with a real number. v is called *characteristic function* and $v(S)$ represents the conjoint payoff of players in S. Additionally, $v(\emptyset) = 0$ (see [42], p. 2).

A *simple game* models voting situations. It is a cooperative game such that for every coalition S, either $v(S) = 0$ or $v(S) = 1$, and $v(\mathrm{N}) = 1$ (see [42], p. 7).

One solution is the Shapley–Shubik index, which is the Shapley value to simple games (see [42], pp. 6–7). The equation of Shapley value is the following:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [v(S \cup \{i\}) - v(S)] \tag{17}$$

where $|S|$ is the cardinality of coalition S, $|N|$ is the cardinality of the set of players or *grand coalition* and $\phi_i(v)$ is the value assigned to player i in the game.

This is the unique solution which satisfies the following axioms:

- $\sum_{i \in N} \phi_i(v) = v(N)$ (Efficiency),
- If i, j∈N are interchangeable in v, then $\phi_i(v) = \phi_j(v)$ (Symmetry),
- If i is such that for every coalition S the equation $v(S \cup \{i\}) = v(S)$ holds, then $\phi_i(v) = 0$ (Dummy),
- Given v and w two games over N, then $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$ (Additivity).

This value is the sum of the terms $[v(S \cup \{i\}) - v(S)]$, which mean the marginal contribution of player i to the coalitions S, multiplied by $\frac{|S|!(|N|-|S|-1)!}{|N|!}$ which is the probability that $|S| - 1$ players precede player i in the game and $|N| - |S|$ players follow him or her. Thus, the Shapley value of i is the expected marginal contribution of i to the game (see [42], p. 7). The result of the Shapley–Shubik index is interpreted as a measure of each player's power.

In the present paper we basically study voting games with some additional features. We call them *voting n-offgames*. A *voting n-offgame* consists in a pair (N, v), where N = {1, 2, . . . , n} is the set of players; the characteristic function v: $2^N \rightarrow \{1, \ldots, 2^n\} \times \{1, \ldots, 2^n\} \times \{1, \ldots, 2^n\}$ is such that for any coalition S we have $v(S) = (k, l, 2^n\text{-}k+1)$ and $v(\emptyset) = (2^n, 2^n, 1)$.

The n-offgame is interpreted in the following way:

1. Experts forecast that voters will rank coalition S in the k^{th} position of their preference, also they cannot decide if S will be ranked in the l^{th} position. The first place or k = 1 corresponds to the preferred coalition of all and so on. Additionally, the n-offgame must satisfy the following rules:
2. Given any two coalitions S_1 and S_2, $S_1 \neq S_2$, we have the first component that both $v(S_1)$ and $v(S_2)$ are different. Thus, every coalition is associated with a unique number in the order of preference.
3. $v(S) = (k, k, 2^n - k+1)$ means experts have no doubt that coalition S will be voted in the k^{th} position.

Let us observe that it is not a simple game. This game can be interpreted as a multicriteria decision-making problem, where its solution is a measure of every player's power in the game according to the forecasted experts' ranking of the coalitions. Each coalition can represent a bloc of political parties.

Shapley value can be the solution to voting n-offgames, in the form given in Equation (18):

$$\phi_i(v) = -\sum_{S\subseteq N\backslash\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}[v(S\cup\{i\}) - v(S)] \tag{18}$$

Let us note that the minus sign in the expression was taken for convenience because the rank we applied is decreasing respect to the coalition´s significance. Additionally, $v(S\cup\{i\}) - v(S)$ is the difference between two 3-tuple values, thus the operation (k_1, l_1, $2^n - k_1+1$)−(k_2, l_2, $2^n - k_2+1$):= ($k_1 - k_2$, $l_1 - l_2$, $k_2 - k_1$) is defined. Equation (18) means the expected number of places won or lost in voter preference, as predicted by experts.

Apparently, Shapley value cannot be the solution to this problem because $v(\emptyset) \neq 0$ and $v(\cdot)$ is not a game. However, if we take in that $v(S)$ = (k,l,2^n − k+1) in fact represents three games, namely, $v_1(S)$=k, $v_2(S)$= l, and $v_3(S)$ =2^n− k+1, one per component and additionally taking into account they are linear transformations of three games with characteristic functions w_1, $w_{2,}$, and w_3; where $w_1(S)$= $2^n - v_1(S)$, $w_2(S)$= $2^n - v_2(S)$, and $w_3(S) = 1 - v_3(S)$, then, the marginal contributions of the three pairs, $w_1(\cdot)$ and $v_1(\cdot)$, $w_2(\cdot)$ and $v_2(\cdot)$, $w_3(\cdot)$ and $v_3(\cdot)$, are the same except for the sign. Thus, these three pairs have the same Shapley value except for the sign and therefore this property is extended to $v(\cdot)$ and $w(\cdot)$.

Shapley value is a rational solution to the game, nevertheless, it can differ from actual human behavior, as Zhang et al. suggested in [43] to model restrictions in game decisions according to the human behavior based on fuzzy uninorms. Therefore, we propose n-offuninorms to explore other behaviors in human decision making by recursively applying an n-offuninorm to every pair of values $\frac{|S|!(|N|-|S|-1)!}{|N|!}[v(S) - v(S\cup\{i\})]$ in the set of $S \subseteq N\backslash\{i\}$.

Here we explore n-offuninorms defined on [−L, L], L = 2^n−1 and with the PROSPECTOR parameterized function with λ> 0 and neutral element $e = \mathrm{L}\left(\frac{1-\lambda}{1+\lambda}\right)$, see Equation (19).

$$U_{O\lambda}(c(x), c(y)) = \varphi_3^{-1}\left(\frac{\lambda\varphi_3(c(x))\varphi_3(c(y))}{\lambda\varphi_3(c(x))\varphi_3(c(y)) + (1-\varphi_3(c(x)))(1-\varphi_3(c(y)))}\right) \tag{19}$$

where $\varphi_3(\cdot)$ and $\varphi_3^{-1}(\cdot)$ are those defined in Equations (10) and (11), respectively, and now they are $\varphi_3(c(x)) = \frac{c(x)+\mathrm{L}}{2\mathrm{L}}$ and $\varphi_3^{-1}(c(x)) = 2\mathrm{L}c(x) - \mathrm{L}$.

Thus the Algorithm for solving voting n-offgames can be described as follows:

Algorithm 1. Algorithm for solving voting n-offgames

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1. Given (N, v) a voting n-offgame. Fix λ > 0.
2. Fix player i = 1.
3. Let S_j be the set of coalitions not containing i, and j = 1, 2, ... , 2^{n-1}. Let us take $a_{i1} = v(S_1)$ and $a_{i2} = v(S_2)$ and calculate $a_{prev} = U_{O\lambda}\left(\frac{|S_1|!(n-|S_1|-1)!}{n!}[v(S_1) - v(S_1\cup\{i\})], \frac{|S_2|!(n-|S_2|-1)!}{n!}[v(S_2) - v(S_2\cup\{i\})]\right)$, fix j = 3 and go to step 4.
4. If $j<2^{n-1}$, calculate $a_{curr} = U_{O\lambda}\left(a_{prev}, \frac{|S_j|!(n-|S_j|-1)!}{n!}\left[v(S_j) - v(S_j\cup\{i\})\right]\right)$. $a_{prev} = a_{curr}$ and $j = j+1$. Repeat this step. Else, if $j = 2^{n-1}$, $\pi_i(v) = a_{curr}$. Go to Step 5.
5. If i< n, then i = i+ 1 and go to Step 3. Else Finish.

Let us point out that in the precedent algorithm the associativity of n-offuninorms was used. Moreover, the algebraic sum in Shapley value and the n-offuninorms yield to somewhat similar results. Thus, for $U_{o\lambda}(\cdot,\cdot)$ with λ = 1, we have that x, y < 0 imply both $U_{o\lambda}(x,y)<\min(x,y)$ and $x+y<\min(x,y)$, whereas when x, y > 0, we have $U_{o\lambda}(x,y)>\max(x,y)$ and $x+y>\max(x,y)$. For x,y satisfying $x\cdot y<0$, then both $U_{o\lambda}(x,y)$ and $x+y$ are compensatory operators, and finally 0 is the neutral element of them. For λ≠ 1 and hence *e*≠ 0, we obtain other behavioral effects. Let us also recall that $U_{o\lambda}(\cdot,\cdot)$ is a neutrosophic uninorm transformation, which is described as symmetric summation by Silvert in [40].

Example 8. *Let us consider the 3-person voting n-offgame (N, v), where N = {1, 2, 3} and experts predict that coalitions will be ranked according to the positions shown in Table 1.*

Table 1. Position assigned to the coalitions of the 3-person voting n-offgame.

Coalition	Ranking
∅	(8,8,1)
{1}	(3,2,6)
{2}	(4,3,5)
{3}	(7,6,2)
{1,2}	(2,3,7)
{1,3}	(5,6,4)
{2,3}	(6,5,3)
{1,2,3}	(1,1,8)

According to Table 1, the grand coalition N has (1,1,8) as ranking value, i.e., experts think this coalition will undoubtedly be ranked in the first place or k = 1. *v*(∅) = (8,8,1) because it is axiomatically predetermined, which means that to not negotiate at all is the worst option, whereas *v*({2,3}) = (6,5,3) means this coalition shall be ranked in the sixth place and maybe in the fifth one, but never in the third place.

Thus, to calculate each player´s power according to our approach we have to apply the precedent algorithm. We fixed λ=1 in $U_{O\lambda}$ therefore *c*(*e*)= 0, which is defined in [−7, 7].

Table 2 contains the detailed calculus of the Shapley value in Equation (18) and the proposed algorithm to resolve the precedent voting n-offgame.

Table 2. Shapley value and n-offuninorm based solutions to the 3-person voting n-offgame. The final values are written in bold font.

Player*i*	*S*Such That *i*∉*S*	*v*(*S*)–*v*(*S*∪{*i*})	*v*(*S*)–*v*(*S*∪{*i*}) Multiplied by the Probability	Partial Summations of the Shapley Value	Partial Aggregation with U_{o1}
1	∅	(5,6,–5)	(5/3, 2,–5/3)	(5/3,2,–5/3)	(5/3,2,–5/3)
	{2}	(2,0,–2)	(1/3,0,–1/3)	(2,2,–2)	(1.9776,2.0000,–1.9776)
	{3}	(2,0,–2)	(1/3,0,–1/3)	(7/3,2,–7/3)	(2.2802,2.0000,–2.2802)
	{2,3}	(5,4,–5)	(5/3,4/3,–5/3)	**(4,10/3,–4)**	**(3.6628,3.1613,–3.6628)**
2	∅	(4,5,–4)	(4/3,5/3,–4/3)	(4/3,5/3,–4/3)	(4/3,5/3,–4/3)
	{1}	(1,–1,–1)	(1/6,-1/6,–1/6)	(3/2,3/2,–3/2)	(1.4932,1.5086,–1.4932)
	{3}	(1,1,–1)	(1/6,1/6,–1/6)	(5/3,5/3,–5/3)	(1.6515,1.6667,–1.6515)
	{1,3}	(4,5,–4)	(4/3,5/3,–4/3)	**(3,10/3,–3)**	**(2.8565,3.1545,–2.8565)**
3	∅	(1,2,–1)	(1/3,2/3,–1/3)	(1/3,2/3,–1/3)	(1/3,2/3,–1/3)
	{1}	(-2,–4,2)	(-1/3,–2/3,1/3)	(0,0,0)	(0,0,0)
	{2}	(-2,–2,2)	(-1/3,–1/3,1/3)	(-1/3,–1/3,1/3)	(−1/3,–1/3,1/3)
	{1,2}	(1,2,–1)	(1/3,2/3,–1/3)	**(0,1/3,0)**	**(0,0.33485,0)**

According to the results summarized in Table 1, we have that the expected value of places gains by player 1 is 4 with the Shapley value solution and 3.6628 with U_{o1}, whereas the results for player 2 are 3 and 2.8565, respectively, and for player 3 are 0 and 0. Therefore, player 1 is the most powerful of them, followed by player 2 and 3 in this order. Thus, the proposed approach and Shapley value are similar.

Table 3 contains the voting n-offgame solutions comparing U_{o1} with *c*(*e*) = 0, $U_{o99/101}$ with *c*(*e*) = 7/100 and $U_{o101/99}$ with *c*(*e*) = −7/100.

Table 3. Solutions of the 3-person voting n-offgame applying $U_{o\lambda}$ with $\lambda = 1$, 99/101 and 101/99, respectively.

Player	Solution with U_{o1}	Solution with $U_{o99/101}$	Solution with $U_{o101/99}$
1	(3.6628,3.1613,−3.6628)	(3.5079,2.9919,−3.8129)	(3.8129,3.3262,−3.5079)
2	(2.8565,3.1545,−2.8565)	(2.6793,2.9849,−3.0293)	(3.0293,3.3196,−2.6793)
3	(0,0.33485,0)	(−0.20994,0.12509,−0.20994)	(0.20994,0.54402,0.20994)

The solutions in Table 3 prove that the greater λ, the greater the solution values. Thus, when λ is increased, its associated solution models more optimistic behavior with respect to the first component, which is compensated with more pessimistic behavior with respect to the third component.

The advantages of the proposed approach are more evident when it is compared with a classical one restricted to {0, 1}. Here we used a semantic represented with natural numbers and we calculated directly on them. In contrast, for applying classical definitions in {0, 1}, we would need to define eight Boolean functions, one per element. What is more, some operations such as marginal contributions, which is an algebraic difference, cannot be directly applied in the logic sense.

In case we would need to extend the approaches to the continuous gradation, then a continuous ranking can be modeled with the identity line $I_d(x) = x$, but in the classical approach, eight memberships functions would have to be considered, where the simplest ones are triangular (see Figure 6). From Figure 6 we can infer that there exists a transformation between both models; however, the proposed model is the simplest one.

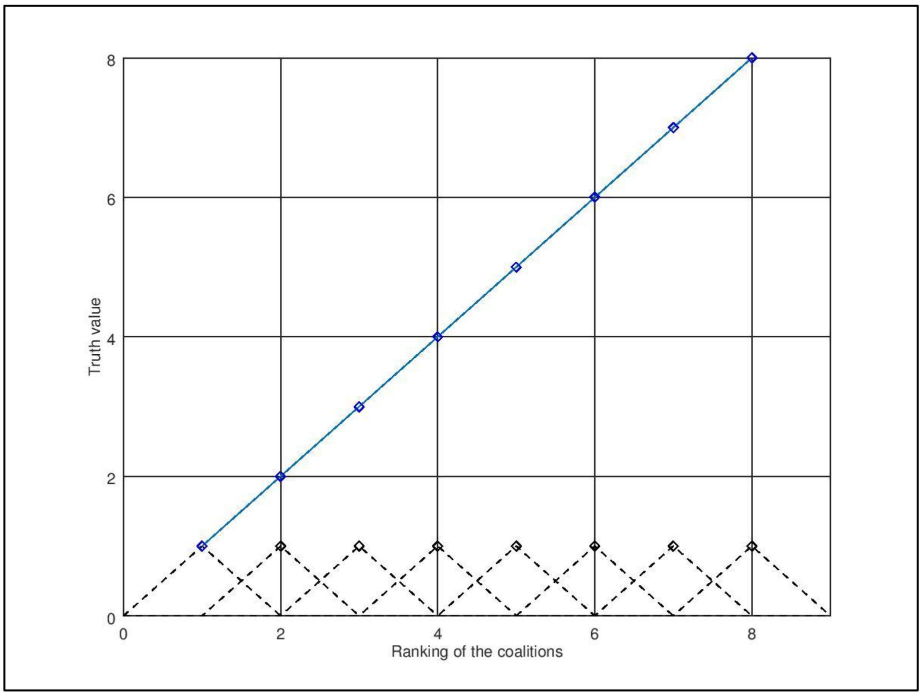

Figure 6. Depiction of two kinds of 3-person game modeling. Classical [0, 1] is represented in dashed lines and triangular membership functions, whereas the solid line represents the solution based on offsets. The points represent the Boolean restrictions.

5. Discussion

Neutrosophic oversets, undersets, and offsets are concepts of a novel and non-conventional theory of uncertainty. Historically, the convention of restricting logic to the interval [0, 1] has dominated fuzzy logic and its generalizations. Possibly this is a legacy of probability and mathematical logic, where, semantically speaking, 0 and 1 have been considered the two extreme opposite sides. Therefore, oversets, undersets, and offsets can be understood as controversial subjects. Nevertheless, Smarandache in [31] illustrates with some examples that such sets, of which their domains surpass the scope of [0, 1], could be useful to represent knowledge in a valid semantic.

This is a recent theory that needs more developing and the scientific community's acknowledgment of its usefulness. One of our aims with this paper is to demonstrate that this theory can be useful. To achieve this end, we introduced the uninorm theory in the neutrosophic offset framework. This union is manifold advantageous, the most evident one being that we have provided a new aggregator operator to these sets. As we mentioned in the introduction, there exists a wide variety of fuzzy uninorm applications, namely, Decision Making [9,14,15], DNA and RNA fusion [9], logic [17], Artificial Neural Networks [16], among others. Uninorm is more flexible than t-norm and t-conorm because it includes the compensatory property in some cases, which is more realistic for modeling human decision making, as was experimentally proved by Zimmermann in [21].

Also, uninorms have enriched other theories when they were generalized to other frameworks. In L^*-fuzzy set theory [23], uninorms also aggregate independent non-membership functions to achieve more precision. Moreover, neutrosophic uninorms aggregate the indeterminate-membership functions [25].

Additionally, some authors have associated uninorms with non-conventional theories. In [33,34] we can find some attempts to extend uninorm domains to an interval [a, b]. The reason is that the PROSPECTOR function related to the MYCIN Expert System is one very important milestone in Artificial Intelligence history. The point is that the PROSPECTOR function is basically a uninorm except it is defined in the interval [−1, 1], thus, we can consider intervals greater than [0, 1]. They have argued that there exist two reasons to maintain the interval [−1, 1]—the first one is the importance of the PROSPECTOR function, the second one is the facility to interchange information among users and decision makers in form of degrees to accept or reject hypotheses.

The second non-conventional approach is the bipolar or Multi-Polar uninorms defined in [24]. The world is (and some people are) is evidently multi-polar; in case of bipolarity they are modeled in [−1, 1]. Especially in [24], we have a multi-polar space consisting of an ordered pair of (k, x), where k∈{1, 2, ... , n} represents a category or class and x∈(0, 1], with the convention **0** = (k, 0) for every category. This is a more complex representation that takes a unique interval [−n, n] where, for x∈[−n, n], the function round(x) represents the category and its fractional part represents the degree of membership to that category. This is a real extension of bipolarity in [−1, 1] to multi-polarity. In [31] (pp. 127, 130) Tripolar offsets and Multi-polar offsets are defined. We illustrated in Example 8 that considering the semantic values belong to {−n,−n+1, ... , 0, 1, ... , n} could be advantageous.

The definition of uninorm-based implicators is not new in literature, they can be seen in [41] (pp. 151–160) for fuzzy uninorms, in [17] it is extended for type 2 fuzzy sets, in [24] for L^*-fuzzy set theory, and in [25] for neutrosophic uninorms. In the present paper, uninorm-based offimplicators are defined, however, we only counted on symbolic offimplication operators (see [31], p. 139). To extend this definition to a continuous framework, we had to extend the symbolic offnegation to a continuous one.

Finally, we preferred to illustrate a voting game solution instead of a group decision method because the relationship of offuninorms with the latter subject is predictable. However, to find any game theory associated with uninorms is uncommon in literature. One remarkable example can be seen in [43], where a behavioral approach has been made to certain kind of games, where uninorms model the humans' restrictions to make the division of gains among the players.

In the present paper, another approach is proposed where an indeterminacy component is taken into account. Also, we proved that modeling with a natural number semantic is simpler than to utilize the classical [0, 1] interval, because of the fact that n membership functions can be substituted by a linear identity function. We basically defined the voting game solution since the Shapley–Shubik index components (see [42], pp. 6–7), where we only changed the algebraic sum by offuninorms. The classical approaches such as the Shapley–Shubik index are interested in a rational and fair solution; nevertheless, many times that does not occur in real negotiations and then behavioral solutions are needed.

6. Conclusions

This paper was devoted to defining for the first time the theory of neutrosophic offuninorms, which is a generalization of both the neutrosophic offnorms and neutrosophic offconorms, where the neutral element lays in the interval [Ψ, Ω]. The properties of these novel operators were proved. Moreover, we defined neutrosophic offuninorms from neutrosophic offnorms and neutrosophic offconorms and vice versa, we also proved their properties. Additionally, we proved the relationship between neutrosophic offuninorms and neutrosophic uninorms.

One of the purposes of this paper is to show the convenience of applying offsets, and to prove that they are not only simple theoretical concepts; furthermore, they are also necessary to define new concepts. This need is demonstrated in this paper by associating offsets with the PROSPECTOR aggregation function, where it is recommendable to extend its domain to the interval [−1, 1]. Some authors in fuzzy logic have suggested the advantages to calculate in the domains [a, b] instead of the classical [0, 1]. Therefore, the use of the idea of the offset in uninorms has some precedence in fuzzy logic.

Additionally, we recommend offsets because they permit more accuracy and compactness. We showed that it is possible to define offimplication operators based on offuninorms. A future direction of this research is to solve problems by using artificial neural networks based on neutrosophic offuninorms, such that neutrosophic offuninorms are utilized as the threshold functions in the neurons or in neutrosophic cognitive maps. For the first time, solutions to cooperative games are defined in the neutrosophic framework—this is an area that it is worthy of development.

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