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Operations on Single-Valued Neutrosophic Graphs

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Abstract

The concept of neutrosophic sets can be utilized as a mathematical tool to deal with imprecise and unspecified information. In this paper, we apply the concept of single-valued neutrosophic sets to graphs. We introduce the notion of single-valued neutrosophic graphs, and present some fundamental operations on single-valued neutrosophic graphs. We explore some interesting properties of single-valued neutrosophic graphs by level graphs. We highlight some flaws in the definitions of Broumi et al. [10] and Shah-Hussain [18]. We also present an application of single-valued neutrosophic graphs in social network.

Keywords: Level graphs, Single-valued Neutrosophic graphs, Characterizations.

1 Introduction

Graph theory has been highly successful in certain academic fields, including natural sciences and engineering. Graph theoretic models can sometimes provide a useful structure upon which analytical techniques can be used. It is often convenient to depict the relationships between pairs of elements of a system by means of a graph or a digraph. The vertices of the graph represent the system elements and its edges or arcs represent the relationships between the elements. This approach is especially useful for transportation, scheduling, sequencing, allocation, assignment, and other problems which can be modelled as networks. Such a graph theoretic model is often useful as an aid in communicating.
Zadeh [23] introduced the concept of fuzzy set. Attanassov [8] introduced the intuitionistic fuzzy sets which is a generalization of fuzzy sets. Fuzzy set theory and intuitionistic fuzzy sets theory are useful models for dealing with uncertainty and incomplete information. But they may not be sufficient in modeling of indeterminate and inconsistent information encountered in real world. In order to cope with this issue, neutrosophic set theory was proposed by Smarandache [17] as a generalization of fuzzy sets and intuitionistic

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fuzzy sets. However, since neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval $]0^-, 1^+[$. There are some difficulties in modeling of some problems in engineering and sciences. To overcome these difficulties, in 2010, concept of single-valued neutrosophic sets and its operations defined by Wang et al. [20] as a generalization of intuitionistic fuzzy sets. Yang *et al* [21] introduced concept of single-valued neutrosophic relation based on single-valued neutrosophic set.
Rosenfeld [15] introduced a new concept known as fuzzy graphs. Later on, Bhattacharya [9] worked on fuzzy graphs. Mordeson and Nair [13] discussed some operations on fuzzy graphs. The complement of a fuzzy graph was defined by Mordeson and Nair [13] and further studied by Sunita and Vijayakumar [16]. Parvathi et al. defined operations on intuitionistic fuzzy graph in [14]. Dudek and Talebi [12] described operations on level graphs of bipolar fuzzy graphs. Akram et al. [1-4] introduced many new concepts, including bipolar fuzzy graphs, interval-valued fuzzy graphs, operations on fuzzy soft graphs and characterization of m-polar fuzzy graphs by level graphs. Yang et al. [21] introduced the concept of single-valued neutrosophic relations. Dhavaseelan et al. [11] defined strong neutrosophic graphs. Broumi et al. [10] proposed single-valued neutrosophic graphs. Akram and Shahzadi [5] introduced the notions of neutrosophic graphs and neutrosophic soft graphs. They also presented application of neutrosophic soft graphs. On the other hand, Akram et al. [7] introduced the notion of single-valued neutrosophic hypergraphs. Ye [22] introduced a multicriteria decision making method using aggregation operators. In this research article, we apply the concept of single-valued neutrosophic sets to graphs. We introduce the notion of single-valued neutrosophic graphs, and present its fundamental operations. We explore some interesting properties of single-valued neutrosophic graphs by level graphs. We highlight some flaws in the definitions of Broumi et al. [10] and Shah-Hussain [18]. We also present an application of single-valued neutrosophic graphs in social network. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [6, 19, 24-25].

2 Operations on Single-Valued Neutrosophic Graphs

Definition 2.1. [17] Let X be a space of points (objects). A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $T_A(x) : X \to \ ]0^-, 1^+[, I_A(x) : X \to \ ]0^-, 1^+[$ and $F_A(x) : X \to \ ]0^-, 1^+[$ and $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.
From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. In real life applications in scientific and engineering problems, it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]0^-, 1^+[$. To apply neutrosophic sets in real-life problems more conveniently, Wang et al. [20] defined single-valued neutrosophic sets which takes the value from the subset of $[0, 1]$.

Definition 2.2. A *single-valued neutrosophic graph* is a pair $G = (A, B)$, where $A : V \to [0, 1]$ is single-valued neutrosophic set in V and $B : V \times V \to [0, 1]$ is single-valued neutrosophic relation on V such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}$$

for all $x, y \in V$. A is called single-valued neutrosophic vertex set of G and B is called single-valued neutrosophic edge set of G, respectively. We note that B is symmetric single-valued neutrosophic relation on A. If B is not symmetric single-valued neutrosophic relation on A, then $G = (A, B)$ is called a *single-valued neutrosophic directed graph*.

Example 2.3. Consider a crisp graph $G^* = (V, E)$ such that $V = \{a, b, c, d, e, f\}$, $E = \{ab, ac, bd, cd, be, cf, ef, bc\}$. Let A and B be the single-valued neutrosophic sets of V and E, respectively, as shown in following Tables. By simple calculations, it is easy to see that $G = (A, B)$ is a single-valued neutrosophic graph as shown in Fig. 2.1.

A	a	b	c	d	e	f
T	0.2	0.3	0.4	0.3	0.5	0.4
I	0.5	0.4	0.5	0.6	0.5	0.6
F	0.7	0.6	0.4	0.8	0.6	0.6

B	ab	ac	bd	cd	be	cf	ef	bc
T	0.2	0.1	0.2	0.3	0.2	0.1	0.4	0.2
I	0.4	0.4	0.2	0.2	0.3	0.4	0.4	0.3
F	0.7	0.5	0.6	0.7	0.5	0.5	0.5	0.6

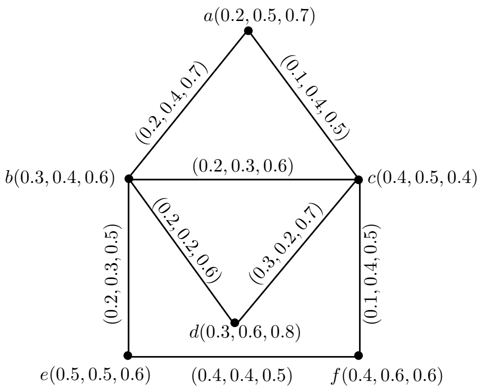

Figure 2.1: Single-valued neutrosophic graph

Definition 2.4. A single-valued neutrosophic graph $G = (A, B)$ is called *complete* if the following conditions are satisfied:

$$T_B(xy) = \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) = \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) = \max\{F_A(x), F_A(y)\}$$

for all $x, y \in V$.

Example 2.5. Consider a single-valued neutrosophic $G = (A, B)$ on the nonempty set $V = \{a, b, c, d, \}$ as shown in Fig. 2.2. By direct calculations, it is easy to see that G is a complete.

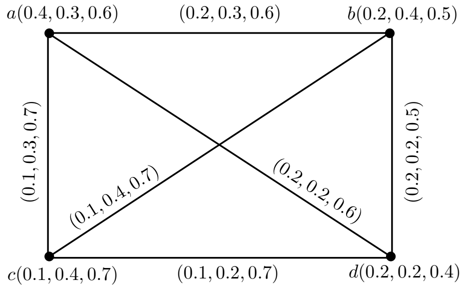

Figure 2.2: Complete single-valued neutrosophic graph

Definition 2.6. Let $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in V\}$ be a single-valued neutrosophic set of the set V. For $\alpha \in [0, 1]$, the α-cut of A is the crisp set A_α defined by

$$A_\alpha = \{x \in V : \text{either } (T_A(x), I_A(x) \geq \alpha) \text{ or } F_A(x) \leq 1 - \alpha\}.$$

Let $B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >\}$ be a neutrosophic set on $E \subseteq V \times V$. For $\alpha \in [0, 1]$, the α-cut is the crisp set B_α defined by

$$B_\alpha = \{xy \in E : \text{either } (T_B(xy), I_B(xy) \geq \alpha) \text{ or } F_B(xy) \leq 1 - \alpha\}.$$

Example 2.7. Consider a single-valued neutrosophic graph $G = (A, B)$ on non-empty set $V = \{a, b, c, d, e\}$ as shown in Figure 2.3.

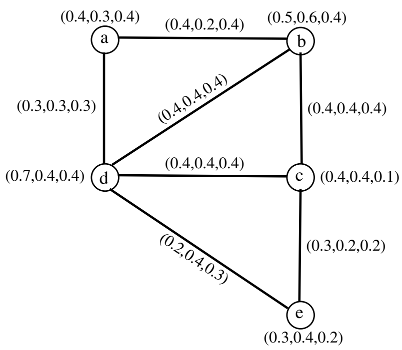

Figure 2.3: Single-valued neutrosophic graph $G = (A, B)$

Take $\alpha = 0.4$. We have $A_{0.4} = \{b, c, d\}$, $B_{0.4} = \{bc, cd, bd\}$. Clearly, the 0.4-level graph $G_{0.4} = (A_{0.4}, B_{0.4})$ is a subgraph of crisp graph $G^* = (V, E)$.

Proposition 2.8. *The level graph $G_\alpha = (A_\alpha, B_\alpha)$ is a crisp graph.*

Theorem 2.9. *$G = (A, B)$ is a single-valued neutrosophic graph if and only if $G_\alpha = (A_\alpha, B_\alpha)$ is a crisp graph for each $\alpha \in [0, 1]$.*

Proof. Let $G = (A, B)$ be a single-valued neutrosophic graph. For each $\alpha \in [0, 1]$, take $xy \in B_\alpha$. Then $\alpha \leq T_B(xy), \alpha \leq I_B(xy)$ or $1 - \alpha \geq F_B(xy)$. Since G is a single-valued neutrosophic graph, it follows that

$$\alpha \leq T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$\alpha \leq I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$1-\alpha \geq F_B(xy) \leq \max\{F_A(x), F_A(y)\}.$$

This shows that $\alpha \leq T_A(x), \alpha \leq T_A(y), \alpha \leq I_A(x), \alpha \leq I_A(y)$ and $1-\alpha \geq F_A(x), 1-\alpha \geq F_A(y)$, that is, $x, y \in A_\alpha$. Therefore, $G_\alpha = (A_\alpha, B_\alpha)$ is a graph for each $\alpha \in [0,1]$.

Conversely, let $G_\alpha = (A_\alpha, B_\alpha)$ be a graph for each $\alpha \in [0,1]$. For every $xy \in V \times V$, let $T_B(xy) = \alpha, I_B(xy) = \alpha$ and $F_B(xy) \leq 1-\alpha$. Then $xy \in B_\alpha$. Since $G_\alpha = (A_\alpha, B_\alpha)$ is a graph, we have $x, y \in A_\alpha$; $T_A(x) \geq \alpha, I_A(x) \geq \alpha$, or $F_A(x) \leq 1-\alpha$ and $T_A(y) \geq \alpha, I_A(y) \geq \alpha$, or $F_A(y) \leq 1-\alpha$, $\min\{T_A(x), T_A(y)\} \geq \alpha, \min\{I_A(x), I_A(y)\} \geq \alpha$, and $\max\{F_A(x), F_A(y)\} \leq 1-\alpha$.

Thus

$$T_B(xy) = \alpha \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) = \alpha \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max\{F_A(x), F_A(y)\} \leq 1-\alpha,$$

that is, $G = (A, B)$ is a single-valued neutrosophic graph. □

Definition 2.10. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The Cartesian product $G_1 \times G_2$ is defined as a pair (A, B) such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,
$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,
$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $T_B((x, x_2)(x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)$,
$I_B((x, x_2)(x, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)$,
$F_B((x, x_2)(x, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1y_1), T_{A_2}(z))$,
$I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1y_1), I_{A_2}(z))$,
$F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1y_1), F_{A_2}(z))$ for all $z \in V_2$ and for all $x_1y_1 \in E_1$.

Proposition 2.11. *The Cartesian product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.12. *Let $G_1 = (A_1, B_1)$and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the Cartesian product of G_1 and G_2 if and only if for each $\alpha \in [0,1]$, the α-level graph G_α is the Cartesian product of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Proof. Let $G = (A, B)$ be the Cartesian product of single-valued neutrosophic graphs G_1 and G_2. For each $\alpha \in [0,1]$, if $(x, y) \in A_\alpha$,

$$\min(T_{A_1}(x), T_{A_2}(y)) = T_A(x, y) \geq \alpha$$

$$\min(I_{A_1}(x), I_{A_2}(y)) = I_A(x, y) \geq \alpha$$

$$\max(F_{A_1}(x), F_{A_2}(y)) = F_A(x, y) \leq 1-\alpha$$

so $x \in (A_1)_\alpha$ and $y \in (A_2)_\alpha$, that is, $(x, y) \in (A_1)_\alpha \times (A_2)_\alpha$. Therefore, $A_\alpha \subseteq (A_1)\alpha \times (A_2)_\alpha$. Let $(x, y) \in (A_1)\alpha \times (A_2)\alpha$, then $x \in (A_1)\alpha$ and $y \in (A_2)\alpha$. It follows that $\min(T_{A_1}(x), T_{A_2}(y)) \geq \alpha, \min(I_{A_1}(x), I_{A_2}(y)) \geq \alpha$, or $\max(F_{A_1}(x), F_{A_2}(y)) \leq 1 - \alpha$. Since (A, B) is the Cartesian product of G_1 and G_2, $T_A(x, y) \geq \alpha, I_A(x, y) \geq \alpha$, or $F_A(x, y) \leq 1 - \alpha$, that is, $(x, y) \in A_\alpha$. Therefore, $(A_1)_\alpha \times (A_2)_\alpha \subseteq A_\alpha$ and so $(A_1)_\alpha \times (A_2)_\alpha = A_\alpha$. We now prove $B_\alpha = E$, where E is the edge set of the Cartesian product $(G_1)_\alpha \times (G_2)_\alpha$ for each $\alpha \in [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_\alpha$. Then, $T_B((x_1, x_2)(y_1, y_2)) \geq \alpha, I_B((x_1, x_2)(y_1, y_2)) \geq \alpha$, or $F_B((x_1, x_2)(y_1, y_2)) \leq 1 - \alpha$. Since (A, B) is Cartesian product of G_1 and G_2, one of the following cases hold:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$.

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.

For the case (i), we have

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{A_1}(x_1), T_{B_2}(x_2y_2)) \geq \alpha,$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{A_1}(x_1), I_{B_2}(x_2y_2)) \geq \alpha,$$

$$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{A_1}(x_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha,$$

so $T_{A_1}(x_1) \geq \alpha, I_{A_1}(x_1) \geq \alpha$ or $F_{A_1}(x_1) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. It follows that $x_1 = y_1 \in (A_1)_\alpha, x_2y_2 \in (B_2)_\alpha$, that is, $(x_1, x_2)(y_1, y_2) \in E$. Similarly, for the case (ii), we conclude that $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_\alpha \subseteq E$. For $(x, x_2)(x, y_2) \in E, T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha$ or $F_{A_1}(x) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. Since (A, B) is the Cartesian product of G_1 and G_2, we have

$$T_B((x, x_2)(x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)) \geq \alpha,$$

$$I_B((x, x_2)(x, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)) \geq \alpha,$$

$$F_B((x, x_2)(x, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)) \leq 1 - \alpha.$$

Therefore, $(x, x_2)(x, y_2) \in B_\alpha$. Similarly for every $(x_1, z)(y_1, z) \in E$, we have $(x_1, z)(y_1, z) \in B_\alpha$. Therefore, $E \subseteq B_\alpha$ and so $E = B_\alpha$.
Conversely, suppose that $G_\alpha = (A_\alpha, B_\alpha)$ is the Cartesian product of $(G_1)_\alpha = ((A_1)_\alpha, (B_1)_\alpha)$ and $(G_2)_\alpha = ((A_2)_\alpha, (B_2)_\alpha)$ for each $\alpha \in [0, 1]$. Let $\min(T_{A_1}(x_1), T_{A_2}(x_2)) = \alpha, \min(I_{A_1}(x_1), I_{A_2}(x_2)) = \alpha$ or $\max(F_{A_1}(x_1), F_{A_2}(x_2)) = 1-\alpha$ for some $(x_1, x_2) \in V_1 \times V_2$. Then $x_1 \in (A_1)_\alpha$ and $x_2 \in (A_2)_\alpha$. By hypothesis, $(x_1, x_2) \in A_\alpha$, hence

$$T_A(x_1, x_2) \geq \alpha = \min(T_{A_1}(x_1), T_{A_2}(x_2)),$$

$$I_A(x_1, x_2) \geq \alpha = \min(I_{A_1}(x_1), I_{A_2}(x_2)),$$

$$F_A(x_1, x_2) \leq 1 - \alpha = \max(F_{A_1}(x_1), F_{A_2}(x_2)).$$

Take $T_A(x_1, x_2) = \beta, I_A(x_1, x_2) = \beta$ or $F_A(x_1, x_2) = 1 - \beta$, then $(x_1, x_2) \in A_\beta$. Since (A_β, B_β) is the Cartesian product of $((A_1)_\beta, (B_1)_\beta)$ and $((A_2)_\beta, (B_2)_\beta)$, then $x_1 \in (A_1)_\beta$ and $x_2 \in (A_2)_\beta$. Hence $T_{A_1}(x_1) \geq \beta, I_{A_1}(x_1) \geq \beta$ or $F_{A_1}(x_1) \leq 1 - \beta$ and $T_{A_2}(x_2) \geq \beta, I_{A_2}(x_2) \geq \beta$ or $F_{A_2}(x_2) \leq 1 - \beta$. It follows that

$$\min(T_{A_1}(x_1), T_{A_2}(x_2)) \geq T_A(x_1, x_2),$$

$$\min(I_{A_1}(x_1), I_{A_2}(x_2)) \geq I_A(x_1, x_2),$$
$$\max(F_{A_1}(x_1), F_{A_2}(x_2)) \geq F_A(x_1, x_2).$$

Therefore,

$$T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2)),$$
$$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2)),$$
$$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2)).$$

for all $(x_1, x_2) \in V_1 \times V_2$. Similarly, for every $x \in V_1$ and every $x_2y_2 \in E_2$, let

$$\min(T_{A_1}(x), T_{B_2}(x_2y_2)) = \alpha,$$
$$\min(I_{A_1}(x), I_{B_2}(x_2y_2)) = \alpha,$$
$$\max(F_{A_1}(x), F_{B_2}(x_2y_2)) = 1 - \alpha,$$

and

$$T_B((x, x_2)(x, y_2)) = \beta,$$
$$I_B((x, x_2)(x, y_2)) = \beta,$$
$$F_B((x, x_2)(x, y_2)) = 1 - \beta.$$

Then we have $T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha$ or $F_{A_1}(x) \leq 1-\alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1-\alpha$, that is, $x \in (A_1)_\alpha, x_2y_2 \in (B_2)_\alpha$ where $\alpha \in [0,1]$ and $(x, x_2)(x, y_2) \in B_\beta$ where $\beta \in [0,1]$. Since (A_α, B_α) (resp. (A_β, B_β)) is the Cartesian product of $((A_1)_\alpha, (B_1)_\alpha)$ and $((A_2)_\alpha, (B_2)_\alpha)$ (resp. $((A_1)_\beta, (B_1)_\beta)$) and $((A_2)_\beta, (B_2)_\beta)$ we have $(x, x_2)(x, y_2) \in B_\alpha, x \in (A_1)_\beta$ and $x_2y_2 \in (B_2)_\beta$, which implies $T_{A_1}(x) \geq \beta, I_{A_1}(x) \geq \beta$ or $F_{A_1}(x) \leq 1-\beta$ and $T_{B_2}(x_2y_2) \geq \beta, I_{B_2}(x_2y_2) \geq \beta$ or $F_{B_2}(x_2y_2) \leq 1-\beta$. It follows that

$$T_B((x, x_2)(x, y_2)) \geq \alpha = \min(T_{A_1}(x), T_{B_2}(x_2y_2),$$
$$I_B((x, x_2)(x, y_2)) \geq \alpha = \min(I_{A_1}(x), I_{B_2}(x_2y_2)),$$
$$F_B((x, x_2)(x, y_2)) \leq 1 - \alpha = \max(F_{A_1}(x), F_{B_2}(x_2y_2)),$$

and

$$\min(T_{A_1}(x), T_{B_2}(x_2y_2)) \geq \beta = T_B((x, x_2)(x, y_2)),$$
$$\min(I_{A_1}(x), I_{B_2}(x_2y_2)) \geq \beta = I_B((x, x_2)(x, y_2)),$$
$$\max(F_{A_1}(x), F_{B_2}(x_2y_2)) \leq 1 - \beta = F_B((x, x_2)(x, y_2)).$$

Therefore,

$$T_B((x, x_2)(x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)),$$
$$I_B((x, x_2)(x, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)),$$
$$F_B((x, x_2)(x, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)).$$

for all $x \in V_1$ and $x_2y_2 \in E_2$. Similarly, we can show that

$$T_B((x_1, z)(x_2, z)) = \min(T_{B_1}(x_1y_1), T_{A_2}(z)),$$
$$I_B((x_1, z)(x_2, z)) = \min(I_{B_1}(x_1y_1), I_{A_2}(z)),$$
$$F_B((x_1, z)(x_2, z)) = \max(F_{B_1}(x_1y_1), F_{A_2}(z)).$$

for all $z \in V_2$ and $x_1y_1 \in E_1$. This ends the proof. □

Definition 2.13. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^*(V_2, E_2)$, respectively. The composition $G_1[G_2]$ is defined as a pair (A, B) such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,
$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,
$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $T_B((x, x_2)(x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2))$,
$I_B((x, x_2)(x, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2))$,
$T_B((x, x_2)(x, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1y_1), T_{A_2}(z))$,
$I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1y_1), I_{A_2}(z))$,
$F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1y_1), F_{A_2}(z))$ for all $z \in V_2$ and for all $x_1y_1 \in E_1$,

(iv) $T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{A_2}(x_2), T_{A_2}(y_2))$,
$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{A_2}(x_2), I_{A_2}(y_2))$,
$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{A_2}(x_2), F_{A_2}(y_2))$ for all $x_2, y_2 \in V_2$, where $x_2 \neq y_2$ and for all $x_1y_1 \in E_1$.

Proposition 2.14. *The composition of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.15. *Let $G_1 = (A_1, B_1)$and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the composition of G_1 and G_2 if and only if for each $\alpha \in [0, 1]$, the α-level graph G_α is the composition of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Proof. Let $G = (A, B)$ be the composition of single-valued neutrosophic graphs G_1 and G_2. By the definition of $G_1[G_2]$ and in the same way as in the proof of Theorem 2.12, we have $A_\alpha = (A_1)_\alpha \times (A_2)_\alpha$. We prove $B_\alpha = E$, where E is the edge set of the composition $(G_1)_\alpha[(G_2)_\alpha]$ for each $\alpha \in [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_\alpha$. Then $T_B((x_1, x_2)(y_1, y_2)) \geq \alpha, I_B((x_1, x_2)(y_1, y_2)) \geq \alpha$ or $F_B((x_1, x_2)(y_1, y_2)) \leq 1 - \alpha$. Since $G = (A, B)$ is the composition $G_1[G_2]$, one of the following cases hold:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$.

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.

(iii) $x_2 \neq y_2$ and $x_1y_1 \in E_1$.

For the cases (i) and (ii), similarly as in the cases (i) and (ii) in the proof of Theorem 2.12, we obtain $(x_1, x_2)(y_1, y_2) \in E$. For the case (iii), we have

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{A_2}(x_2), T_{A_2}(y_2)) \geq \alpha,$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{A_2}(x_2), I_{A_2}(y_2)) \geq \alpha,$$

$$F_B((x_1,x_2)(y_1,y_2)) = \max(F_{B_1}(x_1y_1), F_{A_2}(x_2), F_{A_2}(y_2)) \leq 1-\alpha.$$

Thus, $T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha$ or $T_{B_1}(x_1y_1) \leq 1-\alpha$ and $T_{A_2}(x_2) \geq \alpha, I_{A_2}(x_2) \geq \alpha$ or $F_{A_2}(x_2) \leq 1-\alpha$ and $T_{A_2}(y_2) \geq \alpha, I_{A_2}(y_2) \geq \alpha$ or $F_{A_2}(y_2) \leq 1-\alpha$. It follows that $x_2, y_2 \in (A_2)_\alpha$ and $x_1y_1 \in (B_1)_\alpha$, that is, $(x_1,x_2)(y_1,y_2) \in E$. Therefore, $B_\alpha \subseteq E$. For every $(x,x_2)(x,y_2) \in E$, $T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha$ or $F_{A_1}(x) \leq 1-\alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1-\alpha$. Since $G=(A,B)$ is the composition $G_1[G_2]$, we have

$$T_B((x,x_2)(x,y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)) \geq \alpha,$$

$$I_B((x,x_2)(x,y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)) \geq \alpha,$$

$$F_B((x,x_2)(x,y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)) \leq 1-\alpha.$$

Therefore, $(x,x_2)(x,y_2) \in B_\alpha$. Similarly, for every $(x_1,z)(y_1,z) \in E$, we have $(x_1,z)(y_1,z) \in B_\alpha$. For every $(x_1,x_2)(y_1,y_2) \in E$ where $x_2 \neq y_2, x_1 \neq y_1, T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha$ or $F_{B_1}(x_1y_1) \leq 1-\alpha$ and $T_{A_2}(x_2) \geq \alpha, I_{A_2}(x_2) \geq \alpha$ or $F_{A_2}(x_2) \leq 1-\alpha$ and $T_{A_2}(y_2) \geq \alpha, I_{A_2}(y_2) \geq \alpha$ or $F_{A_2}(y_2) \leq 1-\alpha$. Since $G=(A,B)$ is the composition $G_1[G_2]$, we have

$$T_B((x_1,x_2)(y_1,y_2)) = \min(T_{B_1}(x_1y_1), T_{A_2}(x_2), T_{A_2}(y_2)) \geq \alpha,$$

$$I_B((x_1,x_2)(y_1,y_2)) = \min(I_{B_1}(x_1y_1), I_{A_2}(x_2), I_{A_2}(y_2)) \geq \alpha,$$

$$F_B((x_1,x_2)(y_1,y_2)) = \max(F_{B_1}(x_1y_1), F_{A_2}(x_2), F_{A_2}(y_2)) \leq 1-\alpha.$$

Thus, $(x_1,x_2)(y_1,y_2) \in B_\alpha$. Therefore, $E \subseteq B_\alpha$ and so $E = B_\alpha$. The converse part is obvious, hence we omit its proof. □

Definition 2.16. Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be single-valued neutrosophic graphs of $G_1^*=(V_1,E_1)$ and $G_2^*=(A_2,B_2)$, respectively. The union $G_1 \cup G_2$ is defined as a pair (A,B) such that

(i) $T_A(x) = \begin{cases} T_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ T_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \max(T_{A_1}(x), T_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$

(ii) $I_A(x) = \begin{cases} I_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ I_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \max(I_{A_1}(x), I_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$

(iii) $F_A(x) = \begin{cases} F_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ F_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \min(F_{A_1}(x), F_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$

(iv) $T_B(xy) = \begin{cases} T_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ T_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(T_{B_1}(xy), T_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$

(v) $I_B(xy) = \begin{cases} I_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ I_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(I_{B_1}(xy), I_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$

$$\text{(vi)}\ \ F_B(xy) = \begin{cases} F_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ F_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \min(F_{B_1}(xy), F_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Proposition 2.17. *The union of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.18. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \emptyset$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if each α-level graph G_α is the union of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Proof. Let $G = (A, B)$ be the union of single-valued neutrosophic graphs G_1 and G_2. We have to show that $A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha$ for each $\alpha \in [0, 1]$. Let $x \in A_\alpha$, then $x \in V_1 \setminus V_2$ or $x \in V_2 \setminus V_1$. If $x \in V_1 \setminus V_2$, then $T_{A_1}(x) = T_A(x) \geq \alpha, I_{A_1}(x) = I_A(x) \geq \alpha$ or $F_{A_1}(x) = F_A(x) \leq 1 - \alpha$, which implies $x \in (A_1)_\alpha$. Analogously $x \in V_2 \setminus V_1$ implies $x \in (A_2)_\alpha$. Therefore, $x \in (A_1)_\alpha \cup (A_2)_\alpha$, and so $A_\alpha \subseteq (A_1)_\alpha \cup (A_2)_\alpha$. Now let $x \in (A_1)_\alpha \cup (A_2)_\alpha$. Then $x \in (A_1)_\alpha,\ x \notin (A_2)_\alpha$ or $x \in (A_2)_\alpha,\ x \notin (A_1)_\alpha$. For the first case, we have $T_{A_1}(x) = T_A(x) \geq \alpha, I_{A_1}(x) = I_A(x) \geq \alpha$ or $F_{A_1}(x) = F_A(x) \leq 1 - \alpha$, which implies $x \in A_\alpha$. For the second case, we have $T_{A_2}(x) = T_A(x) \geq \alpha, I_{A_2}(x) = I_A(x) \geq \alpha$ or $F_{A_2}(x) = F_A(x) \leq 1 - \alpha$. Hence $x \in A_\alpha$. Consequently, $(A_1)_\alpha \cup (A_2)_\alpha \subseteq A_\alpha$.
To prove that $B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha$, for each $\alpha \in [0, 1]$, consider $xy \in B_\alpha$. Then $xy \in E_1 \setminus E_2$ or $xy \in E_2 \setminus E_1$. For $xy \in E_1 \setminus E_2$ we have $T_{B_1}(xy) = T_B(xy) \geq \alpha, I_{B_1}(xy) = I_B(xy) \geq \alpha$ or $F_{B_1}(xy) = F_B(xy) \leq 1 - \alpha$. Thus $xy \in (B_1)_\alpha$. Similarly, $xy \in E_2 \setminus E_1$ gives $xy \in (B_2)_\alpha$. Therefore, $B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha$. If $xy \in (B_1)_\alpha \cup (B_2)_\alpha$, then $xy \in (B_1)_\alpha \setminus (B_2)_\alpha$ or $xy \in (B_2)_\alpha \setminus (B_1)_\alpha$. For the first case $T_B(xy) = T_{B_1}(xy) \geq \alpha, I_B(xy) = I_{B_1}(xy) \geq \alpha$ or $F_B(xy) = F_{B_1}(xy) \leq 1 - \alpha$, hence $xy \in B_\alpha$. In the second case we obtain $xy \in B_\alpha$. Therefore, $(B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha$. The converse part is obvious, hence we omit its proof. □

Definition 2.19. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The join $G_1 + G_2$ is defined as a pair (A, B) such that

$$\text{(i)}\ \ T_A(x) = \begin{cases} T_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ T_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \max(T_{A_1}(x), T_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$\text{(ii)}\ \ I_A(x) = \begin{cases} I_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ I_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \max(I_{A_1}(x), I_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$\text{(iii)}\ \ F_A(x) = \begin{cases} F_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ F_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \min(F_{A_1}(x), F_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$\text{(iv)}\ \ T_B(xy) = \begin{cases} T_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ T_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(T_{B_1}(xy), T_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \min(T_{A_1}(x), T_{A_2}(y)) & \text{if } xy \in E'. \end{cases}$$

(v) $I_B(xy) = \begin{cases} I_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ I_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(I_{B_1}(xy), I_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \min(I_{A_1}(x), I_{A_2}(y)) & \text{if } xy \in E'. \end{cases}$

(vi) $F_B(xy) = \begin{cases} F_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ F_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \min(F_{B_1}(xy), F_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \max(F_{A_1}(x), F_{A_2}(y)) & \text{if } xy \in E'. \end{cases}$

Proposition 2.20. *The join of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.21. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (A_2, B_2)$, respectively, and $V_1 \cap V_2 = \emptyset$. Then $G = (A, B)$ is the join of G_1 and G_2 if and only if each α-level graph G_α is the join of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Proof. Let $G = (A, B)$ be the join of single-valued neutrosophic graphs G_1 and G_2. By the definition of union and the proof of Theorem 2.18, $A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha$, for each $\alpha \in [0, 1]$. We show that $B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha$ for each $\alpha \in [0, 1]$, where E'_α is the set of all edges joining the vertices of $(A_1)_\alpha$ and $(A_2)_\alpha$.
From the proof of Theorem 2.18, it follows that $(B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha$. If $xy \in E'_\alpha$, then $T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha$ or $F_{A_1}(x) \leq 1 - \alpha$, and $T_{A_2}(y) \geq \alpha, I_{A_2}(y) \geq \alpha$ or $F_{A_2}(y) \leq 1 - \alpha$. Hence

$$T_B(xy) = \min(T_{A_1}(x), T_{A_2}(y)) \geq \alpha,$$

$$I_B(xy) = \min(I_{A_1}(x), I_{A_2}(y)) \geq \alpha,$$

or

$$F_B(xy) = \max(F_{A_1}(x), F_{A_2}(y)) \leq 1 - \alpha.$$

It follows that $xy \in B_\alpha$. Therefore, $(B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha \subseteq B_\alpha$. For every $xy \in B_\alpha$, if $xy \in E_1 \cup E_2$, then $xy \in (B_1)_\alpha \cup (B_2)_\alpha$, by the proof of Theorem 2.18. Therefore, $B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha$. If $x \in V_1$ and $y \in V_2$, then

$$\min(T_{A_1}(x), T_{A_2}(y)) = T_B(xy) \geq \alpha,$$

$$\min(I_{A_1}(x), I_{A_2}(y)) = I_B(xy) \geq \alpha,$$

or

$$\max(F_{A_1}(x), F_{A_2}(y)) = F_B(xy) \leq 1 - \alpha,$$

so $x \in (A_1)_\alpha$ and $y \in (A_2)_\alpha$. Thus $xy \in E'_\alpha$. Therefore, $B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha$. The converse part is obvious, hence we omit its proof.

□

Definition 2.22. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (A_2, B_2)$, respectively. The cross product $G_1 * G_2$ is defined as a pair (A, B) such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,
$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,
$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2))$,
$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2))$,
$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Example 2.23. Consider G_1 and G_2 are two single-valued neutrosophic graphs as shown in Figure 2.4 such that $A_1 = \{a(0.4,\ 0.6,\ 0.7),\ b(0.9,\ 0.3,\ 0.8)\}$, $A_2 = \{c(0.5,\ 0.7,\ 0.9),\ d(0.2,\ 0.9,\ 0.3),\ e(0.8,\ 0.7,\ 0.6)\}$, $B_1 = \{((ab),\ 0.3,\ 0.2,\ 0.7)\}$, and $B_2 = \{((cd),\ 0.1,\ 0.6,\ 0.8),\ ((de),\ 0.1,\ 0.6,\ 0.5)\}$. Then, we have cross product of G_1 and G_2, defined as $G_1 * G_2 = (A,\ B)$, where $A = A_1 * A_2$ and $B = B_1 * B_2$.

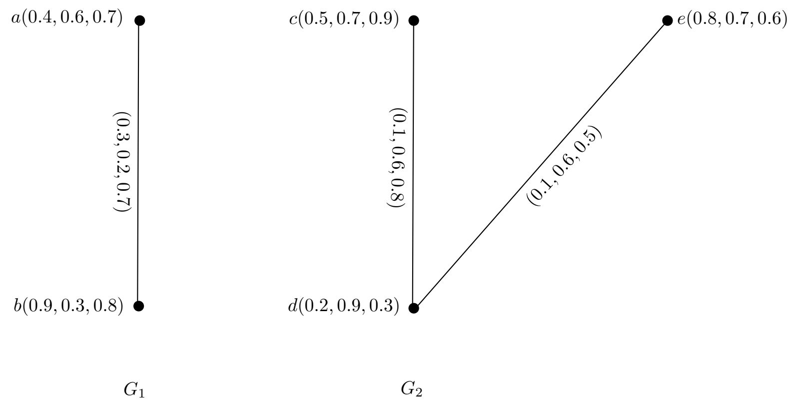

Figure 2.4: (1). G_1 (2). G_2

According to definition 2.19 the degrees of truth, indeterminacy and falsity memberships of vertices and edges are calculated as,

$$T_A(a, c) = \min(T_{A_1}(a), T_{A_2}(c)) = \min(0.4, 0.5) = 0.4,$$
$$I_A(a, c) = \min(I_{A_1}(a), I_{A_2}(c)) = \min(0.6, 0.7) = 0.6,$$
$$F_A(a, c) = \max(F_{A_1}(a), F_{A_2}(c)) = \max(0.7, 0.9) = 0.9,$$

and

$$T_B((a, c)(b, d)) = \min(T_{B_1}(a, b), T_{B_2}(c, d)) = \min(0.3, 0.1) = 0.1,$$
$$I_B((a, c)(b, d)) = \min(I_{B_1}(a, b), I_{B_2}(c, d)) = \min(0.2, 0.6) = 0.2,$$
$$T_B((a, c)(b, d)) = \max(F_{B_1}(a, b), F_{B_2}(c, d)) = \max(0.7, 0.8) = 0.8.$$

All the truth, indeterminacy and falsity membership degrees of vertices and edges of $G_1 * G_2$ are given in Table 1 and Table 2, respectively. Thus, we have the following graph representing the cross product $G_1 * G_2$ of G_1 and G_2.

Table 1: $T_A(x_1,x_2)$, $I_A(x_1,x_2)$, $F_A(x_1,x_2)$ for all $(x_1,x_2) \in V_1 \times V_2$

(x_1,x_2)	$T_A(x_1,x_2)$	$I_A(x_1,x_2)$	$F_A(x_1,x_2)$
(a,c)	0.4	0.6	0.9
(a,d)	0.2	0.6	0.7
(a,e)	0.4	0.6	0.7
(b,c)	0.5	0.3	0.9
(b,d)	0.2	0.3	0.8
(b,e)	0.8	0.3	0.8

Table 2: $T_B((x_1,x_2)(y_1,y_2))$, $I_B((x_1,x_2)(y_1,y_2))$, $F_B((x_1,x_2)(y_1,y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$

$(x_1,x_2)(y_1,y_2)$	$T_B((x_1,x_2)(y_1,y_2))$	$I_B((x_1,x_2)(y_1,y_2))$	$F_B((x_1,x_2)(y_1,y_2))$
$(a,c)(b,d)$	0.1	0.2	0.8
$(a,c)(b,e)$	0	0	0.8
$(a,d)(b,c)$	0.1	0.2	0.8
$(a,d)(b,e)$	0.1	0.2	0.7
$(a,e)(b,c)$	0	0	0.7
$(a,e)(b,d)$	0.1	0.2	0.7

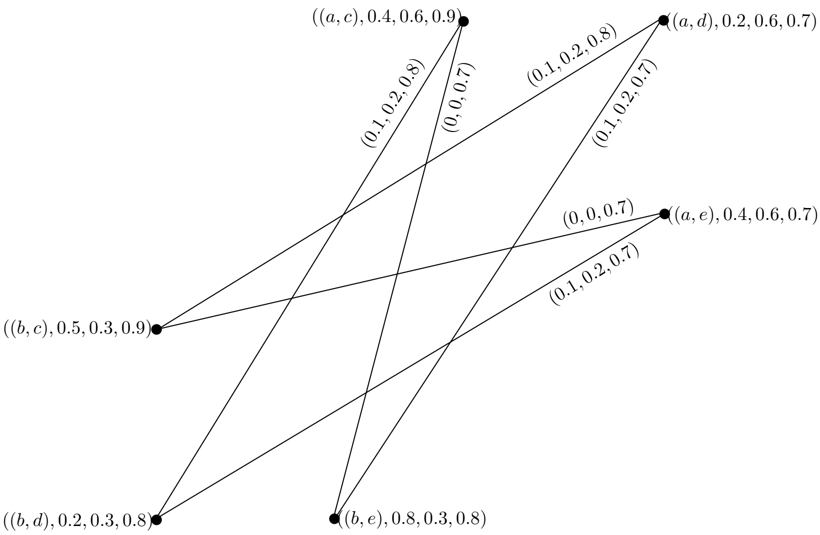

Figure 2.5: $G_1 * G_2$

Proposition 2.24. *The cross product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.25. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (A_2, B_2)$, respectively. Then $G = (A, B)$ is the cross product of G_1 and G_2 if and only if each level* *graph G_α is the cross product of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Proof. Let $G = (A, B)$ be the cross product of G_1 and G_2. By the definition of the Cartesian product and the proof of Theorem 2.12, we have $A_\alpha = (A_1)_\alpha \times (A_2)_\alpha$, for each $\alpha \in [0, 1]$. We show that

$$B_\alpha = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\}$$

for each $\alpha \in [0, 1]$. In fact, if $(x_1, x_2)(y_1, y_2) \in B_\alpha$, then

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) \geq \alpha,$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) \geq \alpha,$$

or

$$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha,$$

so $T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha$ or $F_{B_1}(x_1y_1) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. So, $x_1y_1 \in (B_1)_\alpha$ and $x_2y_2 \in (B_2)_\alpha$. Therefore, $B_\alpha \subseteq \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\}$. Now if $x_1y_1 \in (B_1)_\alpha$ and $x_2y_2 \in (B_2)_\alpha$, then $T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha$ or $F_{B_1}(x_1y_1) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. It follows that

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) \geq \alpha,$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) \geq \alpha,$$

or

$$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha.$$

Since $G = (A, B)$ is the cross product of $G_1 * G_2$. Therefore, $(x_1, x_2)(y_1, y_2) \in B_\alpha$, this implies $\{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\} \subseteq B_\alpha$.
Conversely, let each α-level graph $G_\alpha = (A_\alpha, B_\alpha)$ be the cross product of $(G_1)_\alpha = ((A_1)_\alpha, (B_1)_\alpha)$ and $(G_2)_\alpha = ((A_2)_\alpha, (B_2)_\alpha)$. In view of the fact that the cross product (A_α, B_α) has the same vertex set as the Cartesian product of $((A_1)_\alpha, (B_1)_\alpha)$ and $((A_2)_\alpha, (B_2)_\alpha)$, and by the proof of Theorem 2.12, we have

$$T_A((x_1, x_2)) = \min(T_{A_1}(x_1), T_{A_2}(x_2)),$$

$$I_A((x_1, x_2)) = \min(I_{A_1}(x_1), I_{A_2}(x_2)),$$

$$F_A((x_1, x_2)) = \max(F_{A_1}(x_1), F_{A_2}(x_2)),$$

for all $(x_1, x_2) \in V_1 \times V_2$. Let

$$\min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) = \alpha,$$

$$\min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) = \alpha,$$

or

$$\max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) = 1 - \alpha.$$

and

$$T_B((x_1, x_2)(y_1, y_2)) = \beta,$$

$$I_B((x_1,x_2)(y_1,y_2)) = \beta,$$

or

$$F_B((x_1,x_2)(y_1,y_2)) = 1-\beta,$$

for $x_1y_1 \in E_1,\ x_2y_2 \in E_2$. Then $T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha$ or $F_{B_1}(x_1y_1) \leq 1-\alpha$ and $T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1-\alpha$, hence $x_1y_1 \in (B_1)_\alpha,\ x_2y_2 \in (B_2)_\alpha$, where $\alpha \in [0,1]$ and $(x_1,x_2)(y_1,y_2) \in B_\beta$ where $\beta \in [0,1]$ and consequently $x_1y_1 \in (B_1)_\beta,\ x_2y_2 \in (B_2)_\beta$,
since $B_\beta = \{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in (B_1)_\beta, x_2y_2 \in (B_2)_\beta\}$. It follows that $(x_1,x_2)(y_1,y_2) \in B_\beta,\ T_{B_1}(x_1y_1) \geq \beta,\ I_{B_1}(x_1y_1) \geq \beta$ or $F_{B_1}(x_1y_1) \leq 1-\beta$ and $T_{B_2}(x_2y_2) \geq \beta,\ I_{B_2}(x_2y_2) \geq \beta$ or $F_{B_2}(x_2y_2) \leq 1-\beta$. Therefore,

$$T_B((x_1,x_2)(y_1,y_2)) = \beta \leq \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) = \alpha \leq T_B((x_1,x_2)(y_1,y_2)),$$

$$I_B((x_1,x_2)(y_1,y_2)) = \beta \leq \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) = \alpha \leq I_B((x_1,x_2)(y_1,y_2)),$$

$$F_B((x_1,x_2)(y_1,y_2)) = 1-\beta \geq \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) = 1-\alpha \geq F_B((x_1,x_2)(y_1,y_2)).$$

Hence

$$T_B((x_1,x_2)(y_1,y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)),$$

$$I_B((x_1,x_2)(y_1,y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)),$$

$$F_B((x_1,x_2)(y_1,y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)).$$

This ends the proof. □

Definition 2.26. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs. The lexicographic product $G_1 \bullet G_2$ is the pair (A, B) of single-valued neutrosophic sets defined on the lexicographic product $G_1^* \bullet G_2^*$ such that

(i) $T_A(x_1,x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,
$I_A(x_1,x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,
$F_A(x_1,x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1,x_2) \in V_1 \times V_2$,

(ii) $T_B((x,x_2)(x,y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2))$,
$I_B((x,x_2)(x,y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2))$,
$F_B((x,x_2)(x,y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $T_B((x_1,x_2)(y_1,y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2))$,
$I_B((x_1,x_2)(y_1,y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2))$,
$F_B((x_1,x_2)(y_1,y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Example 2.27. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs with underlying crisp graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively as shown in Figure 2.8. The lexicographic product $G_1 \bullet G_2 = (A, B)$ of G_1 and G_2 is shown in Figure 2.9.

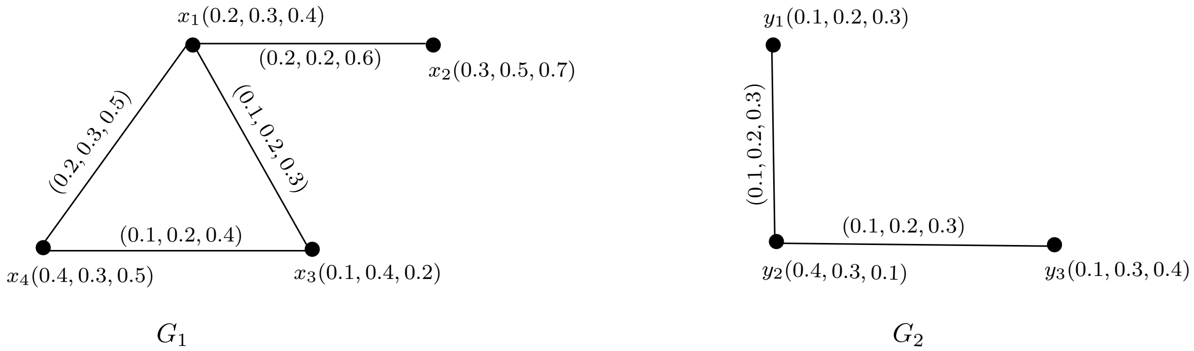

Figure 2.6: Single-valued neutrosophic graphs G_1 and G_2

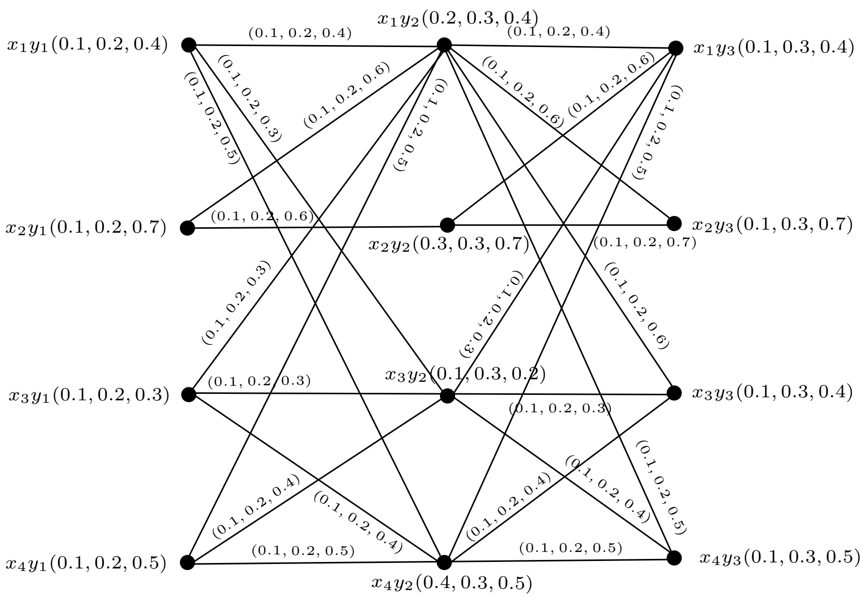

Figure 2.7: Lexicographic product of G_1 and G_2.

Proposition 2.28. *The lexicographic product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

Theorem 2.29. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (A_2, B_2)$, respectively. Then $G = (A, B)$ is the lexicographic product of G_1 and G_2 if and only if $G_\alpha = (G_1)_\alpha \bullet (G_2)_\alpha$ for each $\alpha \in [0, 1]$.*

Proof. Let $G = (A, B) = G_1 \bullet G_2$. By the definition of Cartesian product $G_1 \times G_2$ and the proof of Theorem 2.12, we have $A_\alpha = (A_1)_\alpha \times (A_2)_\alpha$ for each $\alpha \in [0, 1]$. We show that $B_\alpha = E_\alpha \cup E'_\alpha$ for each $\alpha \in [0, 1]$, where $E_\alpha = \{(x, x_2)(y, y_2) \mid x \in V_1, x_2y_2 \in (B_2)_\alpha\}$ is the subset of the edge set of the Cartesian product $(G_1)_\alpha \times (G_2)_\alpha$, and $E'_\alpha = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\}$ is the edge set of the cross product $(G_1)_\alpha * (G_2)_\alpha$. For every $(x_1, x_2)(y_1, y_2) \in B_\alpha,\ x_1 = y_1,\ x_2y_2 \in E_2$ or $x_1y_1 \in E_1,\ x_2y_2 \in E_2$. If

$x_1 = y_1,\ x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E_\alpha$, by the definition of the Cartesian product and the proof of Theorem 2.12. If $x_1y_1 \in E_1,\ x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E'_\alpha$, by the definition of cross product and the proof Theorem 2.25. Therefore, $B_\alpha \subseteq E_\alpha \cup E'_\alpha$. From the definition of the Cartesian product and the proof of Theorem 2.12, we conclude that $E_\alpha \subseteq B_\alpha$, and also from the definition of cross product and the proof Theorem 2.25, we obtain $E'_\alpha \subseteq B_\alpha$. Therefore, $E_\alpha \cup E'_\alpha \subseteq B_\alpha$.

Conversely, let $G_\alpha = (A_\alpha, B_\alpha) = (G_1)_\alpha \bullet (G_2)_\alpha$ for each $\alpha \in [0, 1]$. We know that $(G_1)_\alpha \bullet (G_2)_\alpha$ has the same vertex set as the Cartesian product $(G_1)_\alpha \times (G_2)_\alpha$. Now by the proof of Theorem 2.12, we have

$$T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2)),$$

$$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2)),$$

$$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2)).$$

for all $(x_1, x_2) \in V_1 \times V_2$. Let for $x \in V_1$ and $x_2y_2 \in E_2$ will be $\min(T_{A_1}(x), T_{B_2}(x_2y_2)) = \alpha, \min(I_{A_1}(x), I_{B_2}(x_2y_2)) = \alpha$ or $\max(F_{A_1}(x), F_{B_2}(x_2y_2)) = 1-\alpha$ and $T_B((x, x_2)(x, y_2)) = \beta, I_B((x, x_2)(x, y_2)) = \beta$ or $F_B((x, x_2)(x, y_2)) = 1 - \beta$. Then, in view of the definitions of the Cartesian product and lexicographic product, we have

$$(x, x_2)(x, y_2) \in (B_1)_\alpha \bullet (B_2)_\alpha \Longleftrightarrow (x, x_2)(x, y_2) \in (B_1)_\alpha \times (B_2)_\alpha,$$

$$(x, x_2)(x, y_2) \in (B_1)_\beta \bullet (B_2)_\beta \Longleftrightarrow (x, x_2)(x, y_2) \in (B_1)_\beta \times (B_2)_\beta.$$

From this, by the same way as in the proof of Theorem 2.12, we conclude

$$T_B((x, x_2)(x, x_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)),$$

$$I_B((x, x_2)(x, x_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)),$$

$$F_B((x, x_2)(x, x_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)).$$

Now let $T_B((x_1, x_2)(y_1, y_2)) = \alpha, I_B((x_1, x_2)(y_1, y_2)) = \alpha$ or $F_B((x_1, x_2)(y_1, y_2)) = 1 - \alpha$ and $\min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) = \beta, \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) = \beta$ or $\max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) = 1 - \beta$ for $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then in view of the definitions of cross product and the lexicographic product, we have

$$(x_1, x_2)(y_1, y_2) \in (B_1)_\alpha \bullet (B_2)_\alpha \Longleftrightarrow (x_1, x_2)(y_1, y_2) \in (B_1)_\alpha * (B_2)_\alpha,$$

$$(x_1, x_2)(y_1, y_2) \in (B_1)_\beta \bullet (B_2)_\beta \Longleftrightarrow (x_1, x_2)(y_1, y_2) \in (B_1)_\beta * (B_2)_\beta.$$

By the same way as in the proof of Theorem 2.25, we can conclude

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)),$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)),$$

$$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)).$$

This ends the proof. □

Proposition 2.30. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, such that $V_1 = V_2,\ A_1 = A_2$ and $E_1 \cap E_2 = \emptyset$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if G_α is the union of $(G_1)_\alpha$ and $(G_2)_\alpha$ for each $\alpha \in [0, 1]$.*

Proof. Let $G = (A, B)$ be the union of single-valued neutrosophic graphs G_1 and G_2. Then by the definition of the union and the fact that $V_1 = V_2,\ A_1 = A_2$, we have $A = A_1 = A_2$, hence $A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha$. We now show that $B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha$ for each $\alpha \in [0,1]$. For every $xy \in (B_1)_\alpha$ we have $T_B(xy) = T_{B_1}(xy) \geq \alpha, I_B(xy) = I_{B_1}(xy) \geq \alpha$ or $F_B(xy) = F_{B_1}(xy) \leq 1-\alpha$, hence $xy \in B_\alpha$. Therefore, $(B_1)_\alpha \subseteq B_\alpha$. Similarly we obtain $(B_2)_\alpha \subseteq B_\alpha$. Thus, $(B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha$. For every $xy \in B_\alpha,\ xy \in E_1$ or $xy \in E_2$. If $xy \in E_1$, $T_{B_1}(xy) = T_B(xy) \geq \alpha, I_{B_1}(xy) = I_B(xy) \geq \alpha$ or $F_{B_1}(xy) = F_B(xy) \leq 1-\alpha$ and hence $xy \in (B_1)_\alpha$. If $xy \in E_2$, we have $xy \in (B_2)_\alpha$. Therefore, $B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha$.
Conversely, suppose that the α-level graph $G_\alpha = (A_\alpha, B_\alpha)$ be the union of $(G_1)_\alpha = ((A_1)_\alpha, (B_1)_\alpha)$ and $(G_2)_\alpha = ((A_2)_\alpha, (B_2)_\alpha)$. Let $T_A(x) = \alpha, I_A(x) = \alpha$ or $F_A(x) = 1-\alpha$ and $T_{A_1}(x) = \beta, I_{A_1}(x) = \beta$ or $F_{A_1}(x) = 1-\beta$ for some $x \in V_1 = V_2$. Then $x \in A_\alpha$ where $\alpha \in [0,1]$ and $x \in (A_1)_\beta$ where $\beta \in [0,1]$ so $x \in (A_1)_\alpha$ and $x \in A_\beta$, because $A_\alpha = (A_1)_\alpha$ and $A_\beta = (A_1)_\beta$. It follows that $T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha$ or $F_{A_1}(x) \leq 1-\alpha$ and $T_A(x) \geq \beta, I_A(x) \geq \beta$ or $F_A(x) \leq 1-\beta$. Therefore, $T_{A_1}(x) \geq T_A(x), I_{A_1}(x) \geq I_A(x)$ or $F_{A_1}(x) \leq F_A(x)$ and $T_A(x) \geq T_{A_1}(x), I_A(x) \geq I_{A_1}(x)$ or $F_A(x) \leq F_{A_1}(x)$. Since $A_1 = A_2,\ V_1 = V_2$, then $A = A_1 = A_1 \cup A_2$.
By a similar method, we conclude that

$$\text{(i)} \begin{cases} T_B(xy) = T_{B_1}(xy) & \text{if } xy \in E_1, \\ T_B(xy) = T_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

$$\text{(ii)} \begin{cases} I_B(xy) = I_{B_1}(xy) & \text{if } xy \in E_1, \\ I_B(xy) = I_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

$$\text{(iii)} \begin{cases} F_B(xy) = F_{B_1}(xy) & \text{if } xy \in E_1, \\ F_B(xy) = F_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

This ends the proof. $\square$

Definition 2.31. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The strong product $G_1 \boxtimes G_2$ is defined as a pair (A, B) such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,
$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,
$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $T_B((x, x_2)(x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2))$,
$I_B((x, x_2)(x, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2))$,
$F_B((x, x_2)(x, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1y_1), T_{A_2}(z))$,
$I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1y_1), I_{A_2}(z))$,
$F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1y_1), F_{A_2}(z))$, for all $z \in V_2$ and for all $x_1y_1 \in E_1$,

(iv) $T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2))$,
$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2))$,
$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Example 2.32. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two single-valued neutrosophic graphs of the crisp graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively as shown in Fig. 2.8. The strong product $G_1 \boxtimes G_2 = (A, B)$ of G_1 and G_2 is given in Fig. 2.9.

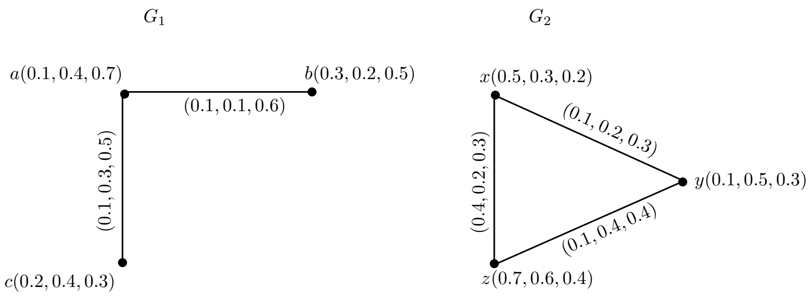

Figure 2.8: SVNGs

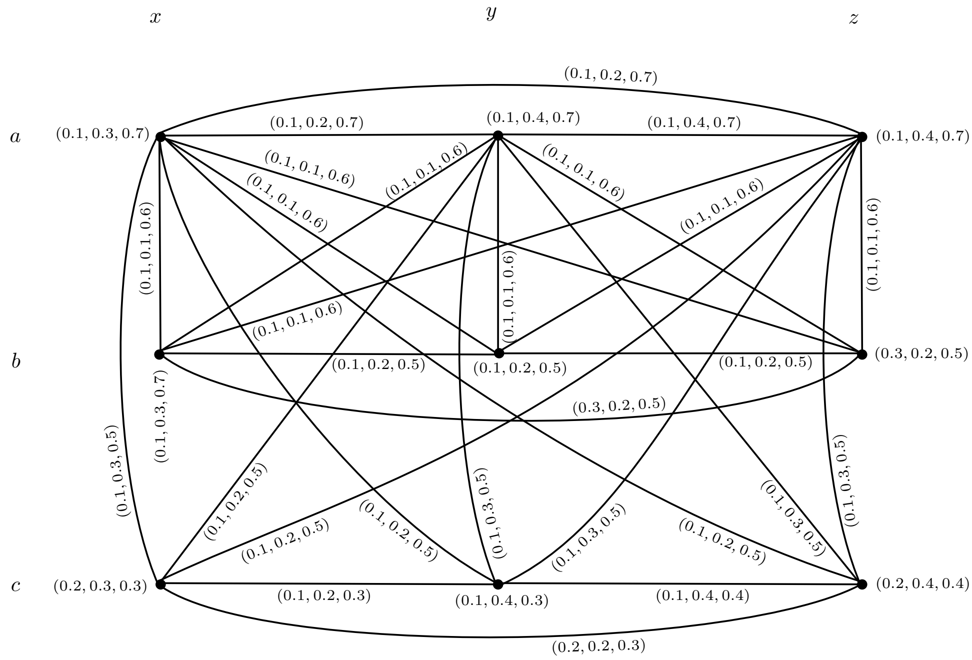

Figure 2.9: Strong product of SVNGs

Proposition 2.33. *The strong product single-valued neutrosophic graphs is a single-valued neutrosophic graph.*

The following theorem is given by without proof.

Theorem 2.34. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then G is the strong product of G_1 and G_2 if and only if G_α, where $\alpha \in [0, 1]$, is the strong product of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Definition 2.35. The complement of a single-valued neutrosophic graph $G = (A, B)$ is a single-valued neutrosophic graph $\overline{G} = (\overline{A}, \overline{B})$, where

1. $\overline{V} = V$
2. $\overline{T}_A(v_i) = T_A(v_i)$,$\overline{I}_A(v_i) = I_A(v_i)$, $\overline{F}_A(v_i) = F_A(v_i)$, for all $v_i \in V$
3.

$$\overline{T}_B(v_i, v_j) = \begin{cases} \min[T_A(v_i), T_A(v_j)] & \text{if } T_B(v_i, v_j) = 0, \\ \min[T_A(v_i), T_A(v_j)] - T_B(v_i, v_j) & \text{if } T_B(v_i, v_j) > 0, \end{cases}$$

$$\overline{I}_B(v_i, v_j) = \begin{cases} \min[I_A(v_i), I_A(v_j)] & \text{if } I_B(v_i, v_j) = 0, \\ \min[I_A(v_i), I_A(v_j)] - I_B(v_i, v_j) & \text{if } I_B(v_i, v_j) > 0, \end{cases}$$

$$\overline{F}_B(v_i, v_j) = \begin{cases} \max[F_A(v_i), F_A(v_j)] & \text{if } F_B(v_i, v_j) = 0, \\ \max[F_A(v_i), F_A(v_j)] - F_B(v_i, v_j) & \text{if } F_B(v_i, v_j) > 0. \end{cases}$$

for all $v_i, v_j \in V$.

Example 2.36. Consider a single-valued neutrosophic graph $G = (A, B)$ on a non-empty set $V = \{v_1, v_2, v_3, v_4\}$. Single-valued neutrosophic graph $G = (A, B)$ and complement of single-valued neutrosophic graph $\overline{G} = (\overline{A}, \overline{B})$ are shown in Figure 2.10.

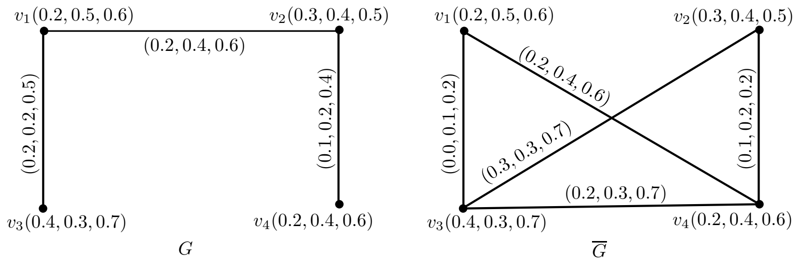

Figure 2.10: Single-valued neutrosophic graph G and its complement $\overline{G}$.

3 Note on definitions of Broumi et al. [10] and Shah-Hussain [18]

Broumi et al. [10] proposed single-valued neutrosophic graphs as follows.

Definition 3.1. [10] A *single-valued neutrosophic graph* is a pair $G = (A, B)$, where A and B are single-valued neutrosophic sets on V and E, respectively, such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \geq \max\{I_A(x), I_A(y)\},$$

$$F_B(xy) \geq \max\{F_A(x), F_A(y)\},$$

$$0 \leq T_B(xy) + I_B(xy) + F_B(xy) \leq 3 \ forallxy \in E.$$

There are some flaws in Definition 3.1. Definition 3.1 violates the definitions of complement and join of single-valued neutrosophic graphs as it can be seen in the following examples.

Example 3.2. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two single-valued neutrosophic graphs. When we apply above definition of join of single-valued neutrosophic graphs G_1 and G_2 then it is easy to note that the indeterminacy-membership values do not satisfy the condition, $I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j))$ as it can be seen in Fig. 3.1. This contradict the definition of single-valued neutrosophic graph.

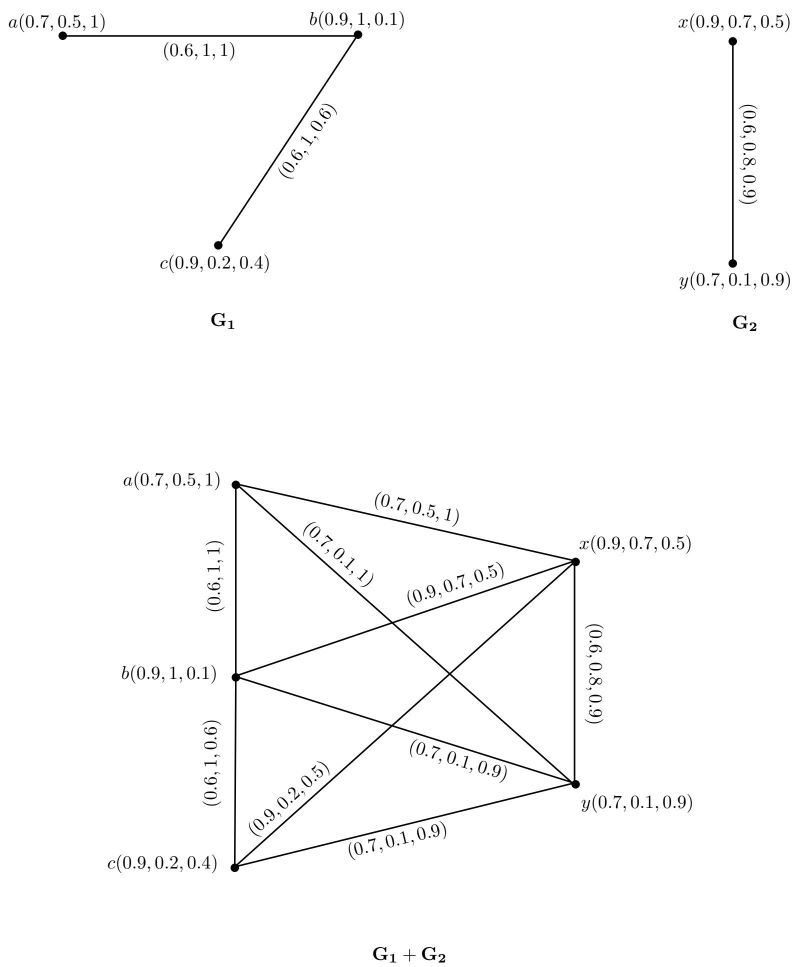

Figure 3.1: Join of a single-valued neutrosophic graph

Example 3.3. Let $G = (A, B)$ be a single-valued neutrosophic graph. When we apply the above definition of complement of a single-valued neutrosophic graph then we see that $\overline{G}$ is not a single-valued neutrosophic graph as it can be seen in Fig. 3.2. Since the indeterminacy-membership and the falsity-membership do not satisfy the conditions, $I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j))$ and $F_B(v_i, v_j) \geq \max(F_A(v_i), F_A(v_j))$, respectively. This contradict the definition of single-valued neutrosophic graph.

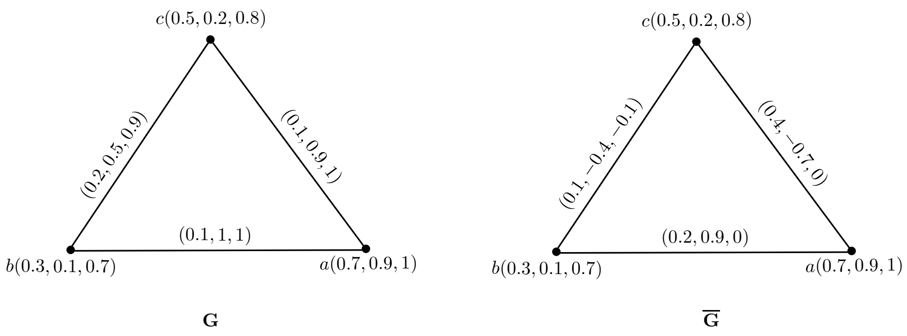

Figure 3.2: Complement of a single-valued neutrosophic graph

Shah and Hussain [18] defined single-valued neutrosophic graphs as follows.

Definition 3.4. [18] A *neutrosophic graph* is a pair $G = (A, B)$, where A and B are neutrosophic sets in V and B, respective;ly, such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \geq \max\{F_A(x), F_A(y)\}$$

for all $xy \in V \times V$.

There are some flaws in Definition 3.4 as it can be seen in the following example.

Example 3.5. Consider a single-valued neutrosophic graph G on a nonempty set $V = \{a, b, c, d\}$ as shown in the Fig. 3.3

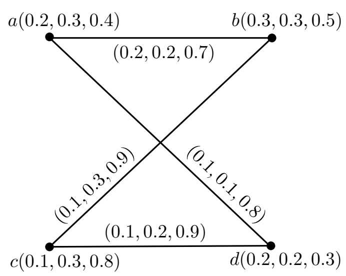

Figure 3.3: A single-valued neutrosophic graph G

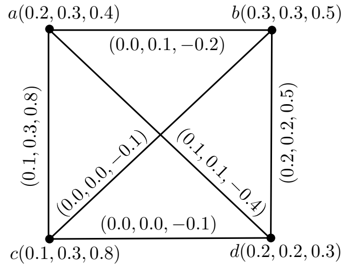

Figure 3.4: Complement of single-valued neutrosophic graph G

It is easy to see that complement of the single-valued neutrosophic graph G shown in Fig. 3.4 is not a single-valued neutrosophic graph. Because in single-valued neutrosophic graph falsity value can never be negative.
Thus, we conclude that our Definition 2.2 on single-valued neutrosophic graphs is more suitable for further study of neutrosophic graphs. Akram and Shahzadi [5] also introduced the concept of neutrosophic soft graphs based on this definition.

4 Application in social network

Graphical models have many applications in our daily life problems. Man is the most adjustable and adapting creature. When human beings interact with each other, more or less they leave an impact(good or bad) on each other. Naturally a human being has influence on others. We can use single-valued neutrosophic diagraph to examine the influence of the people on each other's thinking in a group. We can investigate a person's good influence, bad influence on the thinking of others. We can also examine the percentage of uncertain influence of that person. Single-valued neutrosophic diagraph will also tell us about dominating person and about highly influenced person. We consider a social group on whatsapp.
Consider $I = \{Malik, Haider, Imran, Razi, Ali, Hamza, Aziz\}$ set of seven persons in a social group on whatsapp.
Let $A = \{(Malik, 0.6, 0.4, 0.5), (Haider, 0.5, 0.6, 0.3), (Imran, 0.4, 0.3, 0.2), (Razi, 0.7, 0.6, 0.4),$ $(Ali, 0.4, 0.1, 0.2), (Hamza, 0.6, 0.4, 0.1), (Aziz, 0.7, 0.3, 0.5)\}$ be the single-valued neutrosophic set on the set I, where, truth value of each person represents his good influence on others, falsity value represents his bad influence on others, and indeterminacy value represents uncertainty in his influence.
Let $J = \{(Hamza, Malik), (Hamza, Haider), (Hamza, Razi), (Hamza, Aziz), (Malik, Haider),$ $(Imran, Haider), (Aziz, Malik), (Razi, Imran), (Razi, Ali), (Ali, Aziz)\}$ be the set of edges. Let B be the single-valued neutrosophic set on the set J as shown in Table 3

Table 3: Single-valued neutrosophic set B of edges

Edge	T	I	F
(Hamza, Malik)	0.6	0.4	0.4
(Hamza, Haider)	0.5	0.3	0.3
(Hamza, Razi)	0.3	0.3	0.4
(Hamza, Aziz)	0.3	0.3	0.4
(Malik, Haider)	0.5	0.4	0.5
(Imran, Haider)	0.4	0.3	0.3
(Aziz, Malik)	0.5	0.2	0.5
(Razi, Imran)	0.3	0.3	0.4
(Razi, Ali)	0.4	0.1	0.4
(Ali, Aziz)	0.3	0.1	0.5

The truth, indeterminacy and falsity values of each edge are calculated using:

$$T_B(xy) \leq T_A(x) \wedge T_A(y),\ I_B(xy) \leq I_A(x) \wedge I_A(y),\ F_B(xy) \leq F_A(x) \vee F_A(y).$$

The single-valued neutrosophic diagraph $G = (A, B)$ is shown in Fig. 4.1

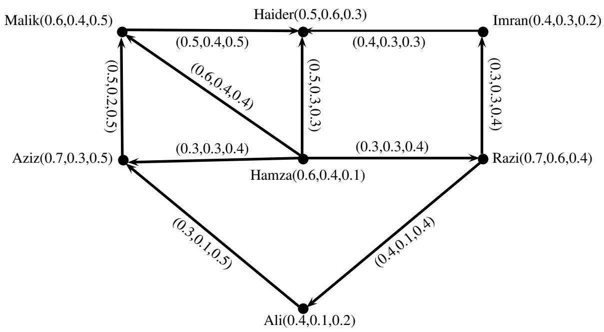

Figure 4.1: Single-valued neutrosophic diagraph

This single-valued neutrosophic diagraph shows that Hamza has influence on Malik, Haider, Razi and Aziz. We can see that Hamza's good influence on Haider is 50%, on Malik is 60%, on Razi is 30%, and on Aziz is 30%. His bad influence on Haider, Malik, Razi and Aziz is 30%, 40%, 40%, and 40%, respectively. Similarly his uncertain influence on Haider, Malik, Razi and Aziz is 30%, 40%, 30%, and 30%, respectively. We can investigate that out-degree of vertex Hamza is highest, that is, four. This shows that Hamza is dominating person in this social group. On the other hand, Haider has highest in-degree, that is, three. It tells us that Haider is highly influenced by others in this social group.
We now explain general procedure of this applications through following algorithm.

Step 1. Input the set of vertices $I = \{I_1, I_2, \ldots, I_n\}$ and a single-valued neutrosophic set A which is defined on set I.

Step 2. Input the set of edges $J = \{J_1, J_2, \ldots, J_n\}$.

Step 3. Compute the truth-membership degree, indeterminacy degree and falsity-membership degree of each edge using: $T_B(xy) \leq T_A(x) \wedge T_A(y)$, $I_B(xy) \leq I_A(x) \wedge I_A(y)$, $F_B(xy) \leq F_A(x) \vee F_A(y)$.

Step 4. Compute the single-valued neutrosophic set B of edges.

Step 5. Obtain a single-valued neutrosophic diagraph $G = (A, B)$.

5 Conclusion

Graph theory is an extremely useful tool in studying and modeling several applications in different areas. A single-valued neutrosophic graph is a generalization of intuitionistic fuzzy graph that is very useful to solve real life problems. In this research article, we have presented certain characterization of single-valued neutrosophic graphs by level graphs. We have aim to extend our work to (1) single-valued neutrosophic soft graphs, (2) single-valued neutrosophic rough fuzzy graphs, (3) single-valued neutrosophic rough fuzzy soft graphs, and (4) single-valued neutrosophic fuzzy soft graphs.

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