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Positive implicative BMBJ-neutrosophic ideals in BCK-algebras

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Abstract: The concepts of a positive implicative BMBJ-neutrosophic ideal is introduced, and several properties are investigated. Conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal are provided. Relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal are discussed. Characterizations of positive implicative BMBJ-neutrosophic ideal are displayed.

Keywords: MBJ-neutrosophic set; BMBJ-neutrosophic ideal; positive implicative BMBJ-neutrosophic ideal.

1 Introduction

In 1965, L.A. Zadeh [18] introduced the fuzzy set in order to handle uncertainties in many real applications. In 1983, K. Atanassov introdued the notion of intuitionistic fuzzy set as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([13], [14] and [15]). Neutrosophic set is applied to many branchs of sciences. In the aspect of algebraic structures, neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [1], [3], [4], [5], [6], [11], [12], [16] and [17]. In [9], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, and it is applied to BCK/BCI-algebras. Mohseni et al. [9] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. Jun and Roh [7] applied the notion of MBJ-neutrosophic sets to ideals of BCK/BI-algebras, and introduced the concept of MBJ-neutrosophic ideals in BCK/BCI-algebras.

In this article, we introduce the concepts of a positive implicative BMBJ-neutrosophic ideal, and investigate several properties. We provide conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal, and discussed relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal. We consider characterizations of positive implicative BMBJ-neutrosophic ideal.

2 Preliminaries

By a *BCI-algebra*, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0,$

(IV) $x * y = 0,\ y * x = 0\ \Rightarrow\ x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0),$

then X is called a *BCK-algebra.*

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \le y\ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I\ \Rightarrow\ x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* of X (see [8]) if it satisfies (2.5) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I\ \Rightarrow\ x * z \in I)\,. \tag{2.7}$$

Note from [8] that a subset I of a BCK-algebra X is a positive implicative ideal of X if and only if it is an ideal of X which satisfies the condition

$$(\forall x, y \in X)((x * y) * y \in I\ \Rightarrow\ x * y \in I). \tag{2.8}$$

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\mathrm{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\tilde{a}_1 \succeq \tilde{a}_2\ \Leftrightarrow\ a_1^- \ge a_2^-,\ a_1^+ \ge a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [14]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

We refer the reader to the books [2, 8] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X (see [9]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, B\tilde{\ }_A, J_A)$ in X is called a ***BMBJ-neutrosophic ideal*** of X (see [10]) if it satisfies

$$(\forall x \in X)(M_A(x) + B_A^-(x) \leq 1,\, B_A^+(x) + J_A(x) \leq 1), \tag{2.9}$$

$$(\forall x \in X)\begin{pmatrix} M_A(0) \geq M_A(x) \\ B_A^-(0) \leq B_A^-(x) \\ B_A^+(0) \geq B_A^+(x) \\ J_A(0) \leq J_A(x) \end{pmatrix}, \tag{2.10}$$

and

$$(\forall x, y \in X) \begin{pmatrix} M_A(x) \geq \min\{M_A(x*y), M_A(y)\} \\ B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} \\ B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} \\ J_A(x) \leq \max\{J_A(x*y), J_A(y)\} \end{pmatrix}. \tag{2.11}$$

3 Positive implicative BMBJ-neutrosophic ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a *positive implicative BMBJ-neutrosophic ideal* of X if it satisfies (2.9), (2.10) and

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x*z) \geq \min\{M_A((x*y)*z), M_A(y*z)\} \\ B_A^-(x*z) \leq \max\{B_A^-((x*y)*z), B_A^-(y*z)\} \\ B_A^+(x*z) \geq \min\{B_A^+((x*y)*z), B_A^+(y*z)\} \\ J_A(x*z) \leq \max\{J_A((x*y)*z), J_A(y*z)\} \end{pmatrix}. \tag{3.1}$$

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.71	$[0.04, 0.09]$	0.22
1	0.61	$[0.03, 0.08]$	0.55
2	0.51	$[0.02, 0.06]$	0.55
3	0.41	$[0.01, 0.03]$	0.77
4	0.31	$[0.02, 0.05]$	0.99

$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X.

Theorem 3.3. *Every positive implicative BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal.*

Proof. The condition (2.11) is induced by taking $z = 0$ in (3.1) and using (2.1). Hence every positive implica-tive BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4.

Table 4: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.6	$[0.04, 0.09]$	0.3
1	0.5	$[0.03, 0.08]$	0.7
2	0.5	$[0.03, 0.08]$	0.7
3	0.3	$[0.01, 0.03]$	0.5

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. But it is not a positive implicative MBJ-neutrosophic ideal of X since

$$M_A(2 * 1) = 0.5 < 0.6 = \min\{M_A((2 * 1) * 1), M_A(1 * 1)\},$$

Lemma 3.5. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$(\forall x, y \in X) \left(x \leq y \;\Rightarrow \begin{cases} M_A(x) \geq M_A(y), B_A^-(x) \leq B_A^-(y), \\ B_A^+(x) \geq B_A^+(y), J_A(x) \leq J_A(y) \end{cases} \right). \tag{3.2}$$

Proof. Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y),$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} = \max\{B_A^-(0), B_A^-(y)\} = B_A^-(y),$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} = \min\{B_A^+(0), B_A^+(y)\} = B_A^+(y),$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y).$$

This completes the proof. □

We provide conditions for a BMBJ-neutrosophic ideal to be a positive implicative BMBJ-neutrosophic ideal.

Theorem 3.6. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if it is a BMBJ-neutrosophic ideal of X and satisfies the following condition.*

$$(\forall x, y \in X) \begin{pmatrix} M_A(x*y) \geq M_A((x*y)*y) \\ B_A^-(x*y) \leq B_A^-((x*y)*y) \\ B_A^+(x*y) \geq B_A^+((x*y)*y) \\ J_A(x*y) \leq J_A((x*y)*y) \end{pmatrix}. \tag{3.3}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. If z is replaced by y in (3.1), then

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*y), M_A(y*y)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-((x*y)*y), B_A^-(y*y)\} \\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+((x*y)*y), B_A^+(y*y)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*y), J_A(y*y)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of X satisfying the condition (3.3). Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned} &M_A((x*y)*z) \leq M_A(((x*z)*z)*(y*z)), \\ &B_A^-((x*y)*z) \geq B_A^-(((x*z)*z)*(y*z)), \\ &B_A^+((x*y)*z) \leq B_A^+(((x*z)*z)*(y*z)), \\ &J_A((x*y)*z) \geq J_A(((x*z)*z)*(y*z)) \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$. Using (3.3), (2.11) and (3.4), we have

$$\begin{aligned} M_A(x*z) &\geq M_A((x*z)*z) \geq \min\{M_A(((x*z)*z)*(y*z)), M_A(y*z)\} \\ &\geq \min\{M_A((x*y)*z), M_A(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x*z) &\leq B_A^-((x*z)*z) \leq \max\{B_A^-(((x*z)*z)*(y*z)), B_A^-(y*z)\} \\ &\leq \max\{B_A^-((x*y)*z), B_A^-(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x*z) &\geq B_A^+((x*z)*z) \geq \min\{B_A^+(((x*z)*z)*(y*z)), B_A^+(y*z)\} \\ &\geq \min\{B_A^+((x*y)*z), B_A^+(y*z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*z) &\leq J_A((x*z)*z) \leq \max\{J_A(((x*z)*z)*(y*z)), J_A(y*z)\} \\ &\leq \max\{J_A((x*y)*z), J_A(y*z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} &U(M_A; t) := \{x \in X \mid M_A(x) \geq t\}, \\ &L(B_A^-; \alpha^-) := \{x \in X \mid B_A^-(x) \leq \alpha^-\}, \\ &U(B_A^+; \alpha^+) := \{x \in X \mid B_A^+(x) \geq \alpha^+\}, \\ &L(J_A; s) := \{x \in X \mid J_A(x) \leq s\} \end{aligned}$$

where $t, s, \alpha^-, \alpha^+ \in [0, 1]$.

Lemma 3.7 ([10]). *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Theorem 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, B\tilde{}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3. It follows from Lemma 3.7 that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Let

$x, y, a, b, c, d, u, v \in X$ be such that $(x * y) * y \in U(M_A; t)$, $(a * b) * b \in L(B_A^-; \alpha^-)$, $(c * d) * d \in U(B_A^+; \alpha^+)$ and $(u * v) * v \in L(J_A; s)$. Using Theorem 3.6, we have

$$\begin{aligned}
&M_A(x * y) \geq M_A((x * y) * y) \geq t, \text{ that is, } x * y \in U(M_A; t),\\
&B_A^-(a * b) \leq B_A^-((a * b) * b) \leq \alpha^-, \text{ that is, } a * b \in L(B_A^-; \alpha^-),\\
&B_A^+(c * d) \geq B_A^+((c * d) * d) \geq \alpha^+, \text{ that is, } c * d \in U(B_A^+; \alpha^+),\\
&J_A(u * v) \leq J_A((u * v) * v) \leq s, \text{ that is, } u * v \in L(J_A; s).
\end{aligned}$$

Therefore $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.

Conversely, suppose that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Then $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. It follows from Lemma 3.7 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Assume that $M_A(x_0 * y_0) < M_A((x_0 * y_0) * y_0) = t_0$ for some $x_0, y_0 \in X$. Then $(x_0 * y_0) * y_0 \in U(M_A; t_0)$ and $x_0 * y_0 \notin U(M_A; t_0)$, which is a contradiction. Thus $M_A(x * y) \geq M_A((x * y) * y)$ for all $x, y \in X$. Similarly, we have $B_A^+(x * y) \geq B_A^+((x * y) * y)$ for all $x, y \in X$. If there exist $a_0, b_0 \in X$ such that $J_A(a_0 * b_0) > J_A((a_0 * b_0) * b_0) = s_0$, then $(a_0 * b_0) * b_0 \in L(J_A; s_0)$ and $a_0 * b_0 \notin L(J_A; s_0)$. This is impossible, and thus $J_A(a * b) \leq J_A((a * b) * b)$ for all $a, b \in X$. By the similar way, we know that $B_A^-(a * b) \leq B_A^-((a * b) * b)$ for all $a, b \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Theorem 3.9. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is positive implicative if and only if it satisfies the following condition.*

$$(\forall x, y, z \in X)\left(\begin{array}{l}
M_A((x * z) * (y * z)) \geq M_A((x * y) * z),\\
B_A^-((x * z) * (y * z)) \leq B_A^-((x * y) * z),\\
B_A^+((x * z) * (y * z)) \geq B_A^+((x * y) * z),\\
J_A((x * z) * (y * z)) \leq J_A((x * y) * z).
\end{array}\right) \tag{3.5}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3, and satisfies the condition (3.3) by Theorem 3.6. Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned}
&M_A((x * y) * z) \leq M_A(((x * (y * z)) * z) * z),\\
&B_A^-((x * y) * z) \geq B_A^-(((x * (y * z)) * z) * z),\\
&B_A^+((x * y) * z) \leq B_A^+(((x * (y * z)) * z) * z),\\
&J_A((x * y) * z) \geq J_A(((x * (y * z)) * z) * z)
\end{aligned} \tag{3.6}$$

for all $x, y, z \in X$. Using (2.3), (3.3) and (3.6), we have

$$\begin{aligned} M_A((x*z)*(y*z)) &= M_A((x*(y*z))*z) \\ &\geq M_A(((x*(y*z))*z)*z) \\ &\geq M_A((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^-((x*z)*(y*z)) &= B_A^-((x*(y*z))*z) \\ &\leq B_A^-(((x*(y*z))*z)*z) \\ &\leq B_A^-((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^+((x*z)*(y*z)) &= B_A^+((x*(y*z))*z) \\ &\geq B_A^+(((x*(y*z))*z)*z) \\ &\geq B_A^+((x*y)*z), \end{aligned}$$

and

$$\begin{aligned} J_A((x*z)*(y*z)) &= J_A((x*(y*z))*z) \\ &\leq J_A(((x*(y*z))*z)*z) \\ &\leq J_A((x*y)*z). \end{aligned}$$

Hence (3.5) is valid.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X which satisfies the condition (3.5). If we put $z = y$ in (3.5) and use (III) and (2.1), then we obtain the condition (3.3). Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X by Theorem 3.6. □

Theorem 3.10. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X if and only if it satisfies the condition* (2.9), (2.10) *and*

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x*y) \geq \min\{M_A(((x*y)*y)*z), M_A(z)\}, \\ B_A^-(x*y) \leq \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\}, \\ B_A^+(x*y) \geq \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\}, \\ J_A(x*y) \leq \max\{J_A(((x*y)*y)*z), J_A(z)\}. \end{pmatrix} \tag{3.7}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3), and so the conditions (2.9) and (2.10) are valid. Using (2.11), (III), (2.1), (2.3) and (3.5), we have

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*z), M_A(z)\} \\ &= \min\{M_A(((x*z)*y)*(y*y)), M_A(z)\} \\ &\geq \min\{M_A(((x*z)*y)*y), M_A(z)\} \\ &= \min\{M_A(((x*y)*y)*z), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-((x*y)*z), B_A^-(z)\} \\ &= \max\{B_A^-(((x*z)*y)*(y*y)), B_A^-(z)\} \\ &\leq \max\{B_A^-(((x*z)*y)*y), B_A^-(z)\} \\ &= \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+((x*y)*z), B_A^+(z)\} \\ &= \min\{B_A^+(((x*z)*y)*(y*y)), B_A^+(z)\} \\ &\geq \min\{B_A^+(((x*z)*y)*y), B_A^+(z)\} \\ &= \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*z), J_A(z)\} \\ &= \max\{J_A(((x*z)*y)*(y*y)), J_A(z)\} \\ &\leq \max\{J_A(((x*z)*y)*y), J_A(z)\} \\ &= \max\{J_A(((x*y)*y)*z), J_A(z)\} \end{aligned}$$

for all $x, y, z \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies conditions (2.9), (2.10) and (3.7). Then

$$M_A(x) = M_A(x*0) \geq \min\{M_A(((x*0)*0)*z), M_A(z)\} = \min\{M_A(x*z), M_A(z)\},$$

$$B_A^-(x) = B_A^-(x*0) \leq \max\{B_A^-(((x*0)*0)*z), B_A^-(z)\} = \max\{B_A^-(x*z), B_A^-(z)\},$$

$$B_A^+(x) = B_A^+(x*0) \geq \min\{B_A^+(((x*0)*0)*z), B_A^+(z)\} = \min\{B_A^+(x*z), B_A^+(z)\},$$

and

$$J_A(x) = J_A(x*0) \leq \max\{J_A(((x*0)*0)*z), J_A(z)\} = \max\{J_A(x*z), J_A(z)\}$$

for all $x, z \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Taking $z = 0$ in (3.7) and using (2.1) and (2.10) imply that

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A(((x*y)*y)*0), M_A(0)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-(((x*y)*y)*0), B_A^-(0)\} \\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+(((x*y)*y)*0), B_A^+(0)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A(((x*y)*y)*0), J_A(0)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Proposition 3.11. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$x*y \leq z \Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ B_A^-(x) \leq \max\{B_A^-(y), B_A^-(z)\}, \\ B_A^+(x) \geq \min\{B_A^+(y), B_A^+(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases} \tag{3.8}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then

$$M_A(x*y) \geq \min\{M_A((x*y)*z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*z), B_A^-(z)\} = \max\{B_A^-(0), B_A^-(z)\} = B_A^-(z),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*z), B_A^+(z)\} = \min\{B_A^+(0), B_A^+(z)\} = B_A^+(z),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*z), J_A(z)\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \geq \min\{M_A(x*y), M_A(y)\} \geq \min\{M_A(y), M_A(z)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} \leq \max\{B_A^-(y), B_A^-(z)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} \geq \min\{B_A^+(y), B_A^+(z)\},$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} \leq \max\{J_A(y), J_A(z)\}.$$

This completes the proof. □

We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI-algebras.

Theorem 3.12. *Every MBJ-neutrosophic set in X satisfying* (2.9), (2.10) *and* (3.8) *is a BMBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X satisfying (2.9), (2.10) and (3.8). Note that $x * (x * y) \leq y$ for all $x, y \in X$. It follows from (3.8) that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Theorem 3.13. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if (M_A, B_A^-) and (B_A^+, J_A) are intuitionistic fuzzy ideals of X.*

Proof. Straightforward. □

Theorem 3.14. *Given an ideal I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by*

$$M_A(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad B_A^-(x) = \begin{cases} \alpha^- & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$
$$B_A^+(x) = \begin{cases} \alpha^+ & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$

where $t, \alpha^+ \in (0, 1]$ and $s, \alpha^- \in [0, 1)$ with $t + \alpha^- \leq 1$ and $s + \alpha^+ \leq 1$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X such that $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$.

Proof. It is clear that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ satisfies the condition (2.9) and $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$. Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$ and so

$$\begin{aligned} M_A(x) &= t = \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) &= \alpha^- = \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &= \alpha^+ = \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &= s = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If any one of $x * y$ and y is contained in I, say $x * y \in I$, then $M_A(x * y) = t$, $B_A^-(x * y) = \alpha^-$, $J_A(x * y) = s$, $M_A(y) = 0$, $B_A^-(y) = 1$, $B_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x) \geq 0 = \min\{t, 0\} = \min\{M_A(x * y), M_A(y)\} \\ &B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ &B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ &J_A(x) \leq 1 = \max\{s, 1\} = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If $x * y \notin I$ and $y \notin I$, then $M_A(x * y) = 0 = M_A(y)$, $B_A^-(x * y) = 1 = B_A^-(y)$, $B_A^+(x * y) = 0 = B_A^+(y)$ and $J_A(x * y) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x) \geq 0 = \min\{M_A(x * y), M_A(y)\} \\ &B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ &B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ &J_A(x) \leq 1 = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

It is obvious that $M_A(0) \geq M_A(x)$, $B_A^-(0) \leq B_A^-(x)$, $B_A^+(0) \geq B_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Lemma 3.15. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X, then I is an ideal of X.*

Proof. Obviously, $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $M_A(x * y) = t = M_A(y)$, $B_A^-(x * y) = \alpha^- = B_A^-(y)$, $B_A^+(x * y) = \alpha^+ = B_A^+(y)$ and $J_A(x * y) = s = J_A(y)$. Thus

$$\begin{aligned} &M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = t, \\ &B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} = \alpha^-, \\ &B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \alpha^+, \\ &J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = s, \end{aligned}$$

and hence $x \in I$. Therefore I is an ideal of X. □

Theorem 3.16. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then I is a positive implicative ideal of X.*

Proof. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X and satisfies (3.3) by Theorem 3.6. It follows from Lemma 3.15 that I is an ideal of X. Let $x, y \in X$ be such that $(x * y) * y \in I$. Then

$$\begin{aligned} &M_A(x * y) \geq M_A((x * y) * y) = t, B_A^-(x * y) \leq B_A^-((x * y) * y) = \alpha^-, \\ &B_A^+(x * y) \geq B_A^+((x * y) * y) = \alpha^+, J_A(x * y) \leq J_A((x * y) * y) = s, \end{aligned}$$

and so $x * y \in I$. Therefore I is a positive implicative ideal of X. □

Proposition 3.17. *Every positive implicative BMBJ-neutrosophic ideal* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *of* X *satisfies the following condition.*

$$(((x * y) * y) * a) * b = 0 \Rightarrow \begin{cases} M_A(x * y) \geq \min\{M_A(a), M_A(b)\}, \\ B_A^-(x * y) \leq \max\{B_A^-(a), B_A^-(b)\}, \\ B_A^+(x * y) \geq \min\{B_A^+(a), B_A^+(b)\}, \\ J_A(x * y) \leq \max\{J_A(a), J_A(b)\} \end{cases} \tag{3.9}$$

for all $x, y, a, b \in X$.

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3). Let $a, b, x, y \in X$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$M_A(x * y) \geq M_A((x * y) * y) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x * y) \leq \tilde{B}_A((x * y) * y) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x * y) \geq B_A^+((x * y) * y) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and $J_A(x * y) \leq J_A((x * y) * y) \leq \max\{J_A(a), J_A(b)\}$ by Theorem 3.6 and Proposition 3.11. Hence (3.9) is valid. □

Theorem 3.18. *If an MBJ-neutrosophic set* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *in* X *satisfies the conditions* (2.9) *and* (3.9), *then* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *is a positive implicative BMBJ-neutrosophic ideal of* X.

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies the conditions (2.9) and (3.9). It is clear that the condition (2.10) is induced by the condition (3.9). Let $x, a, b \in X$ be such that $x * a \leq b$. Then $(((x * 0) * 0) * a) * b = 0$, and so

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x) = B_A^-(x * 0) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x) = B_A^+(x * 0) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1) and (3.9). Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.12. Since $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ for all $x, y \in X$, we have

$$M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y)$$

by (3.9). It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

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