Power Aggregation Operators of Simplified Neutrosophic Sets and Their Use in Multi-attribute Group Decision Making

Chunfang Liu, Yuesheng Luo

***Abstract*—The simplified neutrosophic set (SNS) is a useful generalization of the fuzzy set that is designed for some practical situations in which each element has different truth membership function, indeterminacy membership function and falsity membership function. In this paper, we develop a series of power aggregation operators called simplified neutrosophic number power weighted averaging (SNNPWA) operator, simplified neutrosophic number power weighted geometric (SNNPWG) operator, simplified neutrosophic number power ordered weighted averaging (SNNPOWA) operator and simplified neutrosophic number power ordered weighted geometric (SNNPOWG) operator. We present some useful properties of the operators and discuss the relationships among them. Moreover, an approach to multi-attribute group decision making (MAGDM) within the framework of SNSs is developed by the above aggregation operators. Finally, a practical application of the developed approach to deal with the problem of investment is given, and the result shows that our approach is reasonable and effective in dealing with uncertain decision making problems.**

***Index Terms*—Multi-attribute group decision making (MAGDM), uncertainty, simplified neutrosophic set (SNS), power aggregation operator (PAO)**

I. INTRODUCTION

SINCE Zadeh introduced fuzzy set in 1965, the fuzzy set has been widely utilized in decision making, artificial intelligence, pattern recognition, information fusion, etc [1]−[2]. On the basis of Zadeh's work, several high-order fuzzy sets have been proposed as an extension of fuzzy sets , including interval-valued fuzzy set, type-2 fuzzy set [3], soft set [4], intuitionistic fuzzy set [5], interval-valued intuitionistic fuzzy set [6], hesitant fuzzy set [7] and neutrosophic set (NS) [8]. So far, the proposed high-order fuzzy sets have been successfully utilized in dealing with different uncertain problems, such as decision making [9], pattern recognition [10], artificial intelligence [11], etc.

As a generalization of fuzzy set, neutrosophic sets (NSs) not only provide an effective choice to represent incomplete, uncertain and inconsistent information in application fields but also overcome limitations of the aforementioned three fuzzy sets in representing and handling inconsistent information. In 2014, Ye introduced SNS and discussed its basic operations and proposed the weighted arithmetic aggregation operator and weighted geometric aggregation operator [12]. Then he proposed the similarity measures of SNSs and presented the multiple attribute decision making (MADM) approaches based on these similarity measures [13]−[15]. Later, Peng et al. proposed an approach for ranking SNSs and discussed its application in MADM [16]; Liu et al. proposed the correlated aggregation operators for SNSs and applied them to multi-attribute group decision making [17]. Broumi proposed the single valued neutrosophic trapezoid linguistic aggregation operators [18]. Generally speaking, there is little study on SNSs and its application fields. Due to the merits of SNSs for addressing uncertain and inconsistent information in MADM, it is of great importance to study the relevant theory on MADM based on SNSs. Constructing aggregation operators and ranking alternatives expressed with SNSs belong to two key open problems for MADM within the framework of SNSs. The weighted arithmetic aggregation operator and weighted geometric aggregation operator [9] cannot capture the sophisticated nuances the expert wants to reflect in the aggregated values in the simplified neutrosophic decision making situations. To overcome the defect of existing operators, we develop a series of power aggregation operators (PAO) depending on weighted arithmetic aggregation operator, weighted geometric aggregation operator, and intuitionistic fuzzy power aggregation operators. Power average operator introduced by Yager was able to allow values being aggregated to support and reinforce each other [19]. On the basis of power average operator, Xu developed PAO under intuitionistic fuzzy environment and utilized them to solve the multiple attribute group decision making problems [20]. Except for above PAOs, PAO for multi-valued neutrosophic set, PAO for linguistic term set and other PAOs have been proposed for addressing different uncertain information aggregation problems derived from decision making and pattern recognition. Existing studies indicate that PAOs have been successfully utilized to deal with the aggregations in which a subset of data clustered around

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C. F. Liu is with the College of Science, Northeast Forestry University, Harbin 150040, China, she is also with the College of Automation, Harbin Engineering University, Harbin 150001, China (e-mail: liuchunfang1112@163.com).

Y. S. Luo is with the College of Automation, Harbin Engineering University, Harbin 150001, China (e-mail: luoyuesheng@hrbeu.edu.cn).

a common value can combine in a nonlinear style to act in concert in determining the final aggregated value.

Motivated by the existed PAOs of intuitionistic fuzzy sets, we propose the PAOs of the SNSs and study the properties of the operators, such as idempotency, commutativity and boundedness. Depending on the proposed PAOs of SNSs, we investigate the multi-attribute group decision making problems and present some decision making methods. Furthermore, we utilize a numerical example to verify the effectiveness and feasibility of the proposed decision making methods.

The rest of this paper is organized as follows. In Section 2, we recall the concepts of SNSs and PAOs. In Section 3, we propose simplified neutrosophic number power weighted averaging (SNNPWA) operator, simplified neutrosophic number power weighted geometric (SNNPWG) operator, simplified neutrosophic number power ordered weighted averaging (SNNPOWA) operator and simplified neutrosophic number power ordered weighted geometric (SNNPOWG) operator. Meanwhile, we present some useful properties of the operators and discuss the relationships among them. In Section 4, a series of multi-attribute group decision making methods are proposed based on the PAOs defined in Section 3. Section 5 utilizes an example to validate the proposed decision making methods introduced in Section 4. Finally, a conclusion is given in Section 6.

II. Preliminaries

A. Simplified Neutrosophic Sets (SNSs)

SNS is an extension of neutrosophic set (NS) introduced by Smarandache in 1998. NS allows one to deal with indeterminacy, hesitation and/or uncertainty independent of the membership degree and non-membership degree. Ever since NS was introduced, it has been successfully utilized in dealing with different uncertain decision making problems and pattern recognition problems from applied fields [21]−[26].

Definition 2.1: [8] Assume X be a space of points with a generic element in X denoted by x. A NS A on X is defined by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are defined as

$$T_A(x):X\to]0^-,1^+[$$
$$I_A(x):X\to]0^-,1^+[$$
$$F_A(x):X\to]0^-,1^+[$$

where $0^-\leq \sup T_A(x)+\sup I_A(x)+\sup F_A(x)\leq 3^+$.

Since NS includes both non-standard and standard intervals in its theory and related operations, which restricts its application in applied fields due to complex process for representing uncertain information. For simplification and practical application, Ye proposed the concept of SNS which is a subclass of NS and preserves all the operations on NS. In the following part, we recall SNS and some operations.

Definition 2.2: [6] Assume X be a space of points with a generic element in X denoted by x. A SNS A on X is defined by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are defined as

$$T_A(x):X\to [0,1]$$
$$I_A(x):X\to [0,1]$$
$$F_A(x):X\to [0,1]$$

For convenience, we utilize $A=\{<x,T_A(x),I_A(x),F_A(x)>|x\in X\}$ to denote a SNS A in the following part. In particular, if A has only one element, we call A a simplified neutrosophic number (SNN), and denote it by $A=\{T_A(x),I_A(x),F_A(x)\}$ instead of $A=\{<T_A(x),I_A(x),F_A(x)>\}$ in the following part.

Remark 2.1: In reference [5], Smarandache first introduced the neutrosophic set whose elements involve three functions, truth membership function, indeterminacy membership function and falsity membership function. The range of the function is $]0^-,1^+[$, where $0^-=0-\varepsilon$, $1^+=1+\varepsilon$, $\forall\varepsilon>0$. 0^-, 1^+ are called the nonstandard number, and 0,1 are called the standard number. SNS utilizes $[0,1]$ instead of $]0^-,1^+[$ to denote the range of the three functions $T_A(x)$, $I_A(x)$ and $F_A(x)$. That is to say we can use the negative numbers or the numbers greater than 1 to denote the NSs. The membership functions are not in the interval [0,1]. Ye proposed SNS to restrict the range of the three functions in the unit interval. Compared with NS, SNS is much easier to represent uncertain information.

Definition 2.3: [9] Let A and B be two SNSs on X, $A\subseteq B$ if and only if $T_A(x)\leq T_B(x)$, $I_A(x)\geq I_B(x)$ and $F_A(x)\geq F_B(x)$ for all x in X.

Definition 2.4: [16] Let $A=\{T_A(x),I_A(x),F_A(x)\}$ and $B=\{T_B(x),I_B(x),F_B(x)\}$ be two SNSs on X, $\lambda>0$, the operations on A and B are defined by

$$\begin{aligned}A\oplus B= \;&\{T_A(x)+T_B(x)-T_A(x)T_B(x),I_A(x)I_B(x),\\&F_A(x)F_B(x)\}\end{aligned}$$

$$\begin{aligned}A\otimes B= \;&\{T_A(x)T_B(x),I_A(x)+I_B(x)-I_A(x)I_B(x),\\&F_A(x)+F_B(x)-F_A(x)F_B(x)\};\end{aligned}$$

$$\lambda A=\{1-(1-T_A(x))^\lambda,(I_A(x))^\lambda,(F_A(x))^\lambda\};$$

$$A^\lambda=\{(T_A(x))^\lambda,1-(1-I_A(x))^\lambda,1-(1-F_A(x))^\lambda\}.$$

Definition 2.5: [16] Let $A=\{T_A(x),I_A(x),F_A(x)\}$ be an SNS, then the score function $s(A)$ is defined by

$$s(A)=\frac{T_A(x)+2-I_A(x)-F_A(x)}{3} \tag{1}$$

Depending on the concept of score function on SNS, the method for ranking alternatives expressed by SNSs is as below.

Definition 2.6: [16] Let A and B be two simplified neutrosophic numbers (SNNs), the ranking method is as follows: if $s(A)>s(B)$, then $A>B$.

B. Review of Power Aggregation Operator

Aggregating data using various techniques belongs to key operations in decision making or information fusion process [27]. This situation has become even more challenging with the expansion of the internet and accumulation of its big uncertain data which results in a series of fuzzy decision making problems. It is necessary to present different aggregation operators to suit related decision making situations. Concerning the problem of aggregating data in different environment, the related power aggregation operators have been proposed in the last decades for addressing the decision making or information fusion problem in which argument values support each other in the aggregation process.

Definition 2.7: [19] Let $a_1, a_2, \ldots, a_n$ be a collection of values, the power averaging (PA) operator is the mapping PA: $R^n \to R$, which is defined by

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^n (1 + G(a_i))a_i}{\sum_{i=1}^n (1 + G(a_i))} \tag{2}$$

where

$$G(a_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_i, a_j) \tag{3}$$

$\mathrm{Sup}(a, b)$ denotes the support for a from b. Meanwhile, $\mathrm{Sup}(a, b)$ satisfies the following three properties: (1) $\mathrm{Sup}(a, b) \in [0, 1]$; (2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$; (3) $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)$, if $|a - b| < |x - y|$. It implies that the closer the two sets, the larger the support.

The PA operator is a tool whose weighting vector is determined by the input arguments. To address the geometric mean of the aggregated values, Xu and Yager introduced the power geometric (PG) operator.

Definition 2.8: [25] Let $a_1, a_2, \ldots, a_n$ be a collection of values, the power geometric operator is the mapping PG: $R^n \to R$, which is defined by

$$\mathrm{PG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+G(a_i)}{\sum_{i=1}^n (1+G(a_i))}} \tag{4}$$

where $G(a_i)$ satisfies equation (3).

The POWA operator is proposed by Yager to consider the ordered positions of the elements.

Definition 2.9: [19] Let $a_1, a_2, \ldots, a_n$ be a collection of values, the power order weighted averaging (POWA) operator is the mapping: $R^n \to R$, which is defined by

$$\mathrm{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma_i} \tag{5}$$

where a_{σ_i} $(i = 1, 2, \ldots, n)$ is the ith largest value of a_i $(i = 1, 2, \ldots, n)$, w_i $(i = 1, 2, \ldots, n)$ is defined as

$$w_i = g(\frac{B_i}{TV}) - g(\frac{B_{i-1}}{TV}) \tag{6}$$

where $B_i = \sum_{j=1}^{i} V_{\sigma_j}$, $V_{\sigma_j} = 1 + G(A_{\sigma_j})$ and $TV = \sum_{j=1}^{n} V_{\sigma_j}$. The function $g : [0, 1] \to [0, 1]$ is a basic unit-interval monotonic (BUM) function which satisfies the following three properties: (1) $g(0) = 0$; (2) $g(1) = 1$; (3) if $x \leq y$, then $g(x) \leq g(y)$. If $g(x) = x$, then POWA operator degrades into PA operator.

After the pioneering work of Yager and Xu, a series of power aggregation operators for different fuzzy sets, such as intuitionistic fuzzy set and hesitant fuzzy set, have been successfully proposed in dealing with uncertain decision making problems. SNS, a new high-order fuzzy set, provides a generalization ability to cope with uncertain information. To make use of the capacity of SNS for dealing with decision making or information fusion problems, we present the power aggregation operators on SNS and discuss their application in MAGDM in the next section.

III. Power Aggregation Operators on SNNs

On the basis of PA operator, PG operator and existing power aggregation operators on some different values, we present simplified neutrosophic number power weighted averaging (SNNPWA) operator, simplified neutrosophic number power weighted geometric (SNNPWG) operator, simplified neutrosophic number power ordered weighted averaging (SNNPOWA) operator and simplified neutrosophic number power ordered weighted geometric (SNNPOWG) operator on SNNs in this section.

A. SNNPWA Operator

Definition 3.1: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs, and $w = [w_1, w_2, \ldots, w_n]^T$ be the weight vector of $A = \{A_1, A_2, \ldots, A_n\}$ with $\sum_{i=1}^n w_i = 1$ and $w_i \in [0, 1]$. The simplified neutrosophic number power weighted averaging (SNNPWA) operator is defined as follows:

$$\begin{aligned}&\mathrm{SNNPWA}(A_1, A_2, \ldots, A_n)\\ &= \tfrac{w_1(1+G(A_1))A_1 \oplus \cdots \oplus w_n(1+G(A_n))A_n}{\sum_{i=1}^n w_i(1+G(A_i))}\end{aligned} \tag{7}$$

where

$$G(A_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(A_i, A_j) \tag{8}$$

and $\mathrm{Sup}(A_i, A_j)$ satisfies the aforementioned three properties of $\mathrm{Sup}(a, b)$.

Based on operations on SNSs described in Definition 2.4, we can obtain the following theorem.

Theorem 3.1: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs, then their aggregated value using SNNPWA operator is also an SNN, and

$$\begin{aligned}&\mathrm{SNNPWA}(A_1, A_2, \ldots, A_n)\\ =\ &\{1 - \prod_{i=1}^n (1 - T_{A_i})^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}},\\ &\prod_{i=1}^n I_{A_i}^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}}, \prod_{i=1}^n F_{A_i}^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}}\}\end{aligned} \tag{9}$$

Proof. In the following proof, for convenience, let

$$\xi_i = \frac{w_i(1 + G(A_i))}{\sum_{i=1}^n w_i(1 + G(A_i))} \tag{10}$$

By using the mathematical induction on n, we prove the theorem.

(1) If $n=2$,

$$\begin{aligned} & \mathrm{SNNPWA}(A_1,A_2) \\ = & \ \xi_1 A_1 \oplus \xi_2 A_2 \\ = & \ \{1-(1-T_{A_1})^{\xi_1}, I_{A_1}^{\xi_1}, F_{A_1}^{\xi_1}\} \\ & \oplus\{1-(1-T_{A_2})^{\xi_2}, I_{A_2}^{\xi_2}, F_{A_2}^{\xi_2}\} \\ = & \ \{1-(1-T_{A_1})^{\xi_1}(1-T_{A_2})^{\xi_2}, I_{A_1}^{\xi_1} I_{A_2}^{\xi_2}, F_{A_1}^{\xi_1} F_{A_2}^{\xi_2}\} \end{aligned}$$

(2) If Eq. (9) holds for $n=k$, then

$$\begin{aligned} & \mathrm{SNNPWA}(A_1,A_2,\ldots,A_k) \\ = & \ \{1-\textstyle\prod_{i=1}^{k}(1-T_{A_i})^{\xi_i}, \prod_{i=1}^{k} I_{A_i}^{\xi_i}, \prod_{i=1}^{k} F_{A_i}^{\xi_i}\} \end{aligned}$$

If $n=k+1$, by definition 2.4, we get

$$\begin{aligned} & \mathrm{SNNPWA}(A_1,A_2,\ldots,A_{k+1}) \\ = & \ \tfrac{\sum_{i=1}^{k} w_i(1+G(A_i))\mathrm{SNNPWA}(A_1,A_2,\ldots,A_k)\oplus w_{k+1}(1+G(A_{k+1}))}{\sum_{i=1}^{k+1} w_i(1+G(A_i))} \\ = & \ \{1-\textstyle\prod_{i=1}^{k+1}(1-T_{A_i})^{\xi_i}, \prod_{i=1}^{k+1} I_{A_i}^{\xi_i}, \prod_{i=1}^{k+1} F_{A_i}^{\xi_i}\}. \end{aligned}$$

i.e., Eq. (9) holds for $n=k+1$. Thus, Eq. (9) holds for all n.

It can be easily proved that the SNNPWA operator has the following properties.

Theorem 3.2: (Idempotency) Let $A_i=\{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i=1,2,\ldots,n)$ be a collection of SNNs, and all $A_i\,(i=1,2,\ldots,n)$ are equal, i.e., $A_i=A$ for all $i\in\{1,2,\ldots,n\}$, then

$$\mathrm{SNNPWA}(A_1,A_2,\ldots,A_n)=A.$$

Proof. If $A_i=A$, $i=1,2,\ldots,n$, then

$$\begin{aligned} & \mathrm{SNNPWA}(A_1,A_2,\ldots,A_n) \\ = & \ \mathrm{SNNPWA}(A,A,\ldots,A) \\ = & \ \{1-\textstyle\prod_{i=1}^{n}(1-T_A)^{\frac{w_i(1+G(A))}{\sum_{i=1}^{n} w_i(1+G(A))}}, \\ & \textstyle\prod_{i=1}^{n} I_A^{\frac{w_i(1+G(A))}{\sum_{i=1}^{n} w_i(1+G(A))}}, \prod_{i=1}^{n} F_A^{\frac{w_i(1+G(A))}{\sum_{i=1}^{n} w_i(1+G(A))}}\} \\ = & \ \{T_A, I_A, F_A\}. \end{aligned}$$

Theorem 3.3: (Commutativity) Let $A_i=\{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i=1,2,\ldots,n)$ be a collection of SNNs, and $\acute{A_1},\acute{A_2},\ldots,\acute{A_n}$ be any permutation of $A_1,A_2,\ldots,A_n$, their weight vector $(\acute{w_1},\acute{w_2},\ldots,\acute{w_n})$ be permutation of $(w_1,w_2,\ldots,w_n)$, then

$$\mathrm{SNNPWA}(A_1,A_2,\ldots,A_n)=\mathrm{SNNPWA}(\acute{A_1},\acute{A_2},\ldots,\acute{A_n}).$$

Proof. Let

$$\mathrm{SNNPWA}(A_1,A_2,\ldots,A_n)=\tfrac{\oplus_{i=1}^{n} w_i(1+G(A_i))A_i}{\sum_{i=1}^{n} w_i(1+G(A_i))},$$

$$\mathrm{SNNPWA}(\acute{A_1},\acute{A_2},\ldots,\acute{A_n})=\tfrac{\oplus_{i=1}^{n} \acute{w_i}(1+G(\acute{A_i}))\acute{A_1}}{\sum_{i=1}^{n} \acute{w_i}(1+G(\acute{A_i}))}.$$

Since $\acute{A_1},\acute{A_2},\ldots,\acute{A_n}$ is any permutation of $A_1,A_2,\ldots,A_n$, then we have

$$\sum_{i=1}^{n} w_i(1+G(A_i))=\sum_{i=1}^{n} \acute{w_i}(1+G(\acute{A_i}))$$

and

$$\begin{aligned} & \frac{w_1(1+G(A_1))A_1\oplus\cdots\oplus w_n(1+G(A_n))A_n}{\sum_{i=1}^{n} w_i(1+G(A_i))} \\ = & \ \frac{\acute{w_1}(1+G(\acute{A_1}))\acute{A_1}\oplus\cdots\oplus \acute{w_n}(1+G(\acute{A_n}))\acute{A_n}}{\sum_{i=1}^{n} \acute{w_i}(1+G(\acute{A_i}))}. \end{aligned}$$

Thus, we complete the proof of the theorem.

Theorem 3.4: (Boundedness) Let $A_i=\{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i=1,2,\ldots,n)$ be a collection of SNNs, and let $A_{\min}=\{\min_i T_{A_i}, \max_i I_{A_i}, \max_i F_{A_i}\}$, $A_{\max}=\{\max_i T_{A_i}, \min_i I_{A_i}, \min_i F_{A_i}\}$, then

$$A_{\min}\le \mathrm{SNNPWA}(A_1,A_2,\ldots,A_n)\le A_{\max}.$$

Proof. For convenience, we denote

$$T_{A_s}=\max_i T_{A_i}, T_{A_t}=\min_i T_{A_i},$$

$$I_{A_l}=\max_i I_{A_i}, I_{A_m}=\min_i I_{A_i},$$

$$F_{A_p}=\max_i F_{A_i}, F_{A_q}=\min_i F_{A_i},$$

and let

$$A_{\min}=\{T_{A_t}, I_{A_l}, F_{A_p}\}, A_{\max}=\{T_{A_s}, I_{A_m}, F_{A_q}\}.$$

Since

$$T_{A_t}\le T_{A_i}\le T_{A_s},$$

then

$$1-T_{A_s}\le 1-T_{A_i}\le 1-T_{A_t},$$

$$(1-T_{A_s})^{\xi_i}\le(1-T_{A_i})^{\xi_i}\le(1-T_{A_t})^{\xi_i},$$

and

$$1-\prod_{i=1}^{n}(1-T_{A_t})^{\xi_i}\le 1-\prod_{i=1}^{n}(1-T_{A_i})^{\xi_i}\le 1-\prod_{i=1}^{n}(1-T_{A_s})^{\xi_i},$$

that is

$$T_{A_t}\le 1-\prod_{i=1}^{n}(1-T_{A_i})^{\xi_i}\le T_{A_s}.$$

With the same method, we get

$$I_{A_m}\le\prod_{i=1}^{n} I_{A_i}^{\xi_i}\le I_{A_l},$$

$$F_{A_q}\le\prod_{i=1}^{n} F_{A_i}^{\xi_i}\le F_{A_p}.$$

That means

$$A_{\min}\le \mathrm{SNNPWA}(A_1,A_2,\ldots,A_n)\le A_{\max}.$$

Thus, we complete the proof of the theorem.

B. SNNPWG Operator

Based on the PG operator and SNNPWA operator, here we present a simplified neutrosophic number power weighted geometric (SNNPWG) operator as follows.

Definition 3.2: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs, and $w = [w_1, w_2, \ldots, w_n]^T$ be the weight vector of $A = \{A_1, A_2, \ldots, A_n\}$ with $\sum_{i=1}^n w_i = 1$ and $w_i \in [0, 1]$. The SNNPWG operator is defined as follows:

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2, \ldots, A_n) \\ &= A_1^{\frac{w_1(1+G(A_1))}{\sum_{i=1}^n w_i(1+G(A_i))}} \otimes \cdots \otimes A_n^{\frac{w_n(1+G(A_n))}{\sum_{i=1}^n w_i(1+G(A_i))}} \end{aligned} \tag{11}$$

where

$$G(A_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(A_i, A_j) \tag{12}$$

and $\mathrm{Sup}(A_i, A_j)$ satisfies the aforementioned three properties of $\mathrm{Sup}(a, b)$.

Theorem 3.5: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs, then their aggregated value using SNNPWG operator is also an SNN, and

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2, \ldots, A_n) \\ = &\{\textstyle\prod_{i=1}^n T_{A_i}^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}}, \\ &1 - \textstyle\prod_{i=1}^n (1 - I_{A_i})^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}}, \\ &1 - \textstyle\prod_{i=1}^n (1 - F_{A_i})^{\frac{w_i(1+G(A_i))}{\sum_{i=1}^n w_i(1+G(A_i))}}\} \end{aligned} \tag{13}$$

Proof. By using mathematical induction on n, we prove the theorem.

(1) If $n = 2$,

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2) \\ = &A_1^{\xi_1} \otimes A_2^{\xi_2} \\ = &\{T_{A_1}^{\xi_1}, 1 - (1 - I_{A_1})^{\xi_1}, 1 - (1 - F_{A_1})^{\xi_1}\} \\ &\otimes \{T_{A_2}^{\xi_2}, 1 - (1 - I_{A_2})^{\xi_2}, 1 - (1 - F_{A_2})^{\xi_2}\} \\ = &\{T_{A_1}^{\xi_1} T_{A_2}^{\xi_2}, 1 - (1 - I_{A_1})^{\xi_1}(1 - I_{A_2})^{\xi_2}, \\ &1 - (1 - F_{A_1})^{\xi_1}(1 - F_{A_2})^{\xi_2}\} \end{aligned}$$

(2) If Eq. (13) holds for $n = k$, then

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2, \ldots, A_k) \\ = &\{\textstyle\prod_{i=1}^k T_{A_i}^{\xi_i}, 1 - \prod_{i=1}^k (1 - I_{A_i})^{\xi_i}, \\ &1 - \textstyle\prod_{i=1}^k (1 - F_{A_i})^{\xi_i}\}. \end{aligned}$$

If $n = k + 1$, by definition 2.4, we get

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2, \ldots, A_{k+1}) \\ = &A_1^{\xi_1} \otimes A_2^{\xi_2} \cdots \otimes A_k^{\xi_k} \otimes A_{k+1}^{\xi'_{k+1}} \\ = &(A_1^{\xi_1} \otimes A_2^{\xi_2} \cdots \otimes A_k^{\xi_k}) \otimes A_{k+1}^{\xi'_{k+1}} \\ = &\{\textstyle\prod_{i=1}^k T_{A_i}^{\xi_i}, 1 - \prod_{i=1}^k (1 - I_{A_i})^{\xi_i}, 1 - \prod_{i=1}^k (1 - F_{A_i})^{\xi_i}\} \\ &\otimes \{T_{A_{k+1}}^{\xi'_{k+1}}, 1 - (1 - I_{A_{k+1}})^{\xi'_{k+1}}, 1 - (1 - F_{A_{k+1}})^{\xi'_{k+1}}, \} \\ = &\{\textstyle\prod_{i=1}^{k+1} T_{A_i}^{\xi'_i}, 1 - \prod_{i=1}^{k+1} (1 - I_{A_i})^{\xi'_i}, \\ &1 - \textstyle\prod_{i=1}^{k+1} (1 - F_{A_i})^{\xi'_i}\}. \end{aligned}$$

where

$$\xi_i = \frac{w_i(1 + G(A_i))}{\sum_{i=1}^k w_i(1 + G(A_i))},$$
$$\xi'_i = \frac{w_i(1 + G(A_i))}{\sum_{i=1}^{k+1} w_i(1 + G(A_i))}.$$

i.e., Eq. (13) holds for $n = k+1$. Thus, Eq. (13) holds for all n.

Computation Analysis. Based on the PA and PG operators, we present the SNNPWA and SNNPWG operators. The weight of the attribute values are calculated by the information of each values. That is, according to Eq. (8), we get Eq. (10), which is the weight of each value. Then according Eq. (9), (13), we can aggregate the values.

Since the ordered weighted averaging (OWA) operator provides a family of averaging operators by choosing different weighting vector, it has been successfully utilized in various decision making and information fusion processes [28]. Therefore, we present SNNPOWA operator and SNNPOWG operator on the basis of OWA operator, SNNPWA operator and SNNPWG operator, respectively.

C. SNNPOWA operator

Definition 3.3: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs. The SNNPOWA operator is defined as follows:

$$\mathrm{SNNPOWA}(A_1, A_2, \ldots, A_n) = w_1 A_{\sigma_1} \oplus w_2 A_{\sigma_2} \oplus \cdots \oplus w_n A_{\sigma_n} \tag{14}$$

where A_{σ_i} $(i = 1, 2, \ldots, n)$ is the ith largest value of A_i $(i = 1, 2, \ldots, n)$, w_i $(i = 1, 2, \ldots, n)$ is defined by the following function:

$$w_i = g(\frac{B_i}{TV}) - g(\frac{B_{i-1}}{TV}) \tag{15}$$

where $B_i = \sum_{j=1}^i V_{\sigma_j}$, $V_{\sigma_j} = 1 + G(A_{\sigma_j})$ and $TV = \sum_{j=1}^n V_{\sigma_j}$.

The function $g : [0, 1] \to [0, 1]$ is a BUM function which satisfies the following three properties: (1) $g(0) = 0$; (2) $g(1) = 1$; (3) if $x \le y$, then $g(x) \le g(y)$.

Theorem 3.6: Let $A_i = \{T_{A_i}, I_{A_i}, F_{A_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of SNNs, then their aggregated value using SNNPOWA operator is also an SNN, and

$$\begin{aligned} &\mathrm{SNNPOWA}(A_1, A_2, \ldots, A_n) \\ = &\{1 - \textstyle\prod_{i=1}^n (1 - T_{A_{\sigma_i}})^{w_i}, \prod_{i=1}^n I_{A_{\sigma_i}}^{w_i}, \prod_{i=1}^n F_{A_{\sigma_i}}^{w_i}\} \end{aligned} \tag{16}$$

Proof. By using mathematical induction on n, we prove the theorem.

(1) If $n = 2$, let $A_{\sigma_1} = \max\{A_1, A_2\}$, $A_{\sigma_2} = \min\{A_1, A_2\}$,

$$\begin{aligned} &\mathrm{SNNPOWA}(A_1, A_2) \\ = &w_1 A_{\sigma_1} \oplus w_2 A_{\sigma_2} \\ = &\{1 - (1 - T_{A_{\sigma_1}})^{w_1}, \\ &I_{A_{\sigma_1}}^{w_1}, F_{A_{\sigma_1}}^{w_1}\} \oplus \{1 - (1 - T_{A_{\sigma_2}})^{w_2}, I_{A_{\sigma_2}}^{w_2}, F_{A_{\sigma_2}}^{w_2}\} \\ = &\{1 - (1 - T_{A_{\sigma_1}})^{w_1}(1 - T_{A_{\sigma_2}})^{w_2}, I_{A_{\sigma_1}}^{w_1} I_{A_{\sigma_2}}^{w_2}, F_{A_{\sigma_1}}^{w_1} F_{A_{\sigma_2}}^{w_2}\} \end{aligned}$$

(2) If Eq. (16) holds for $n=k$, then

$$\begin{aligned}&\text{SNNPOWA}(A_{\sigma_1},A_{\sigma_2},\dots,A_{\sigma_k})\\ =\ &\{1-\textstyle\prod_{i=1}^{k}(1-T_{A_{\sigma_i}})^{w_i},\prod_{i=1}^{k}I_{A_{\sigma_i}}^{w_i},\\ &\textstyle\prod_{i=1}^{k}F_{A_{\sigma_i}}^{w_i}\}\end{aligned}$$

If $n=k+1$, by definition 2.4, we get

$$\begin{aligned}&\text{SNNPOWA}(A_{\sigma_1},A_{\sigma_2},\dots,A_{\sigma_{k+1}})\\ =\ &w_1A_{\sigma_1}\oplus w_2A_{\sigma_2}\oplus\dots\oplus w_kA_{\sigma_k}\oplus w_{k+1}A_{\sigma_{k+1}}\\ =\ &(w_1A_{\sigma_1}\oplus w_2A_{\sigma_2}\oplus\dots\oplus w_kA_{\sigma_k})\oplus w_{k+1}A_{\sigma_{k+1}}\\ =\ &\{1-\textstyle\prod_{i=1}^{k}(1-T_{A_{\sigma_i}})^{w_i},\prod_{i=1}^{k}I_{A_{\sigma_i}}^{w_i},\prod_{i=1}^{k}F_{A_{\sigma_i}}^{w_i}\}\\ &\oplus\{1-(1-T_{A_{\sigma_{k+1}}})^{w_{k+1}},I_{A_{\sigma_{k+1}}}^{w_{k+1}},F_{A_{\sigma_{k+1}}}^{w_{k+1}}\}\\ =\ &\{1-\textstyle\prod_{i=1}^{k+1}(1-T_{A_{\sigma_i}})^{w_i},\prod_{i=1}^{k+1}I_{A_{\sigma_i}}^{w_i},\prod_{i=1}^{k+1}F_{A_{\sigma_i}}^{w_i}\}.\end{aligned}$$

i.e., Eq. (16) holds for $n=k+1$. Thus, Eq. (16) holds for all n.

D. SNNPOWG Operator

Definition 3.4: Let $A_i=\{T_{A_i},I_{A_i},F_{A_i}\}$ $(i=1,2,\dots,n)$ be a collection of SNNs. The SNNPOWG operator is defined as follows:

$$\text{SNNPOWG}(A_1,A_2,\dots,A_n)=A_{\sigma_1}^{w_1}\otimes A_{\sigma_2}^{w_2}\otimes\dots\otimes A_{\sigma_n}^{w_n}\tag{17}$$

where A_{σ_i} $(i=1,2,\dots,n)$ is the ith largest value of A_i $(i=1,2,\dots,n)$, w_i $(i=1,2,\dots,n)$ satisfy Eq. (15).

Theorem 3.7: Let $A_i=\{T_{A_i},I_{A_i},F_{A_i}\}$ $(i=1,2,\dots,n)$ be a collection of SNNs, then their aggregated value using SNNPOWG operator is also an SNN, and

$$\begin{aligned}&\text{SNNPOWG}(A_1,A_2,\dots,A_n)\\ =\ &\{\textstyle\prod_{i=1}^{n}T_{A_{\sigma_i}}^{w_i},1-\prod_{i=1}^{n}(1-I_{A_{\sigma_i}})^{w_i},\\ &1-\textstyle\prod_{i=1}^{n}(1-F_{A_{\sigma_i}})^{w_i}\}\end{aligned}\tag{18}$$

Proof. The proof is similar to theorem 3.6, by using mathematical induction on n, we can prove the theorem, however the proof is omitted here.

Computation Analysis. Based on the OWA operators, we present the SNNPOWA and SNNPOWG operators. The weights of the attribute values are calculated. That is, according to score function (1), we rank the alternatives $A_i, i=1,2,\dots,n$, such that

$$A_{\sigma_1}\geq A_{\sigma_2}\geq\dots\geq A_{\sigma_n}$$

then according to Eq. (15), we get the weight of each value, and according to Eq. (16), (18), we can aggregate the values.

IV. Multi-attribute Group Decision Making Method

SNSs provide an effective way to solve uncertain multi-attribute group decision making (MAGDM) problems in which the alternatives on attributes having uncertain information can be classified by truth membership, indeterminacy membership and falsity membership. In this section, we present a MAGDM method based on the operations and aggregation operators on SNSs.

For a MAGDM problem within the framework of SNSs, let $X=\{a_1,a_2,\dots,a_n\}$ be a set of alternatives, $C=\{c_1,c_2,\dots,c_m\}$ be a set of attributes and $w=[w_1,w_2,\dots,w_m]^T$ be the weight vector of C satisfying $\sum_{i=1}^{m}w_i=1$ and $w_i\in[0,1]$ $(i=1,2,\dots,m)$. Suppose that there are s decision makers denoted by $d=\{d_1,d_2,\dots,d_s\}$, the corresponding weight vector is $\lambda=[\lambda_1,\lambda_2,\dots,\lambda_s]^T$. The simplified neutrosophic decision matrix of X on C provided by d_k is $D_k=[\alpha_{ij}^k]_{n\times m}$, where α_{ij}^k is an SNN denoted by $\{T_{\alpha_{ij}^k},I_{\alpha_{ij}^k},F_{\alpha_{ij}^k}\}$.

Generally speaking, the attributes can be classified as benefit attributes and cost attributes in MAGDM. It is necessary to transform the different classes of attributes into one class. Assume that $D_k=[\alpha_{ij}^k]_{n\times m}$ is transformed into $R_k=[r_{ij}^k]_{n\times m}$ as

$$r_{ij}^k=\{T_{\alpha_{ij}^k},I_{\alpha_{ij}^k},F_{\alpha_{ij}^k}\}=\begin{cases}\alpha_{ij}^k & \text{benefit attribute } c_j\\ \bar{\alpha}_{ij}^k & \text{cost attribute } c_j\end{cases}$$

where $\bar{\alpha}_{ij}^k=\{F_{\alpha_{ij}^k},I_{\alpha_{ij}^k},T_{\alpha_{ij}^k}\}$.

Then, we utilize the proposed aggregation operators and some basic operations on SNSs to develop a method in dealing with MAGDM problem within the framework of SNSs, which can be described as follows:

Step 1. Transform the $D_k=[\alpha_{ij}^k]_{n\times m}$ into $R_k=[r_{ij}^k]_{n\times m}$.

Step 2. Calculate the supports $\text{Sup}(r_{ij}^k,r_{ij}^l)$ $(k,l=1,2,\dots,s)$ as follows:

$$\text{Sup}(r_{ij}^k,r_{ij}^l)=1-d(r_{ij}^k,r_{ij}^l)$$

where

$$d(r_{ij}^k,r_{ij}^l)=\sqrt{\frac{1}{3}((T_{r_{ij}^k}-T_{r_{ij}^l})^2+(I_{r_{ij}^k}-I_{r_{ij}^l})^2+(F_{r_{ij}^k}-F_{r_{ij}^l})^2)}$$

Step 3. Calculate the weighted supports $G(r_{ij}^k)=\sum_{l=1,\,l\neq k}^{s}\lambda_l\text{Sup}(r_{ij}^k,r_{ij}^l)$ and the weight ξ_{ij}^k of r_{ij}^k by

$$\xi_{ij}^k=\frac{\lambda_k(1+G(r_{ij}^k))}{\sum_{k=1}^{s}\lambda_k(1+G(r_{ij}^k))}\tag{19}$$

where $\sum_{k=1}^{s}\xi_{ij}^k=1$ with $\xi_{ij}^k\in[0,1]$.

Step 4. Aggregate all the transformed decision making matrices R_k $(k=1,2,\dots,s)$ using aggregation operator on SNSs and obtain a collective matrix denoted by $R=[r_{ij}]_{n\times m}$.

Step 5. Aggregate all the preference values r_{ij} $(j=1,2,\dots,m)$ using aggregation operator and get the overall preference value r_i for the alternative x_i.

Step 6. Calculate the score function $s(r_i)$ $(i=1,2,\dots,n)$ by (1).

Step 7. Rank all the alternatives and select the most desirable one depending on the obtained values of $s(r_i)$ $(i = 1, 2, \ldots, n)$.

V. NUMERICAL EXAMPLE AND ANALYSIS

A. Numerical Example

In this section, an example adapted from [29] is utilized to illustrate the applicability and validity of the proposed MAGDM method. There is an invest problem concerning a financial company with the following four potential alternatives: (1) car company a_1; (2) food company a_2; (3) computer company a_3; (4) arms company a_4. The invest problem must perform a decision making according to the following three attributes: c_1 (risk analysis); c_2 (growth analysis); c_3 (environmental impact analysis). Among the three attributes, both c_1 and c_2 are benefit attributes, and c_3 is a cost attribute. The four possible alternatives are evaluated by three decision makers denoted by d_1, d_2 and d_3 (whose weight vector is $\lambda = [0.5\ 0.3\ 0.2]^T$). Meanwhile, assume that the attribute weight vector is $w = [0.35\ 0.25\ 0.40]^T$. Under the three aforementioned attributes, all the simplified neutrosophic decision making matrices are listed as follows:

$$D_1 = \begin{bmatrix} \{0.4, 0.2, 0.3\} & \{0.4, 0.2, 0.3\} & \{0.2, 0.2, 0.5\} \\ \{0.6, 0.1, 0.2\} & \{0.6, 0.1, 0.2\} & \{0.5, 0.2, 0.2\} \\ \{0.3, 0.2, 0.3\} & \{0.5, 0.2, 0.3\} & \{0.5, 0.3, 0.2\} \\ \{0.7, 0, 0.1\} & \{0.6, 0.1, 0.2\} & \{0.4, 0.3, 0.2\} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} \{0.5, 0.2, 0.2\} & \{0.6, 0.2, 0.3\} & \{0.3, 0.2, 0.4\} \\ \{0.6, 0.1, 0.2\} & \{0.7, 0.2, 0.3\} & \{0.5, 0.2, 0.3\} \\ \{0.4, 0.1, 0.3\} & \{0.5, 0.3, 0.3\} & \{0.6, 0.2, 0.2\} \\ \{0.7, 0.3, 0.1\} & \{0.6, 0.3, 0.2\} & \{0.5, 0.1, 0.2\} \end{bmatrix}$$

$$D_3 = \begin{bmatrix} \{0.5, 0.1, 0.2\} & \{0.5, 0.2, 0.2\} & \{0.3, 0.1, 0.3\} \\ \{0.5, 0.3, 0.2\} & \{0.7, 0.1, 0.3\} & \{0.5, 0.3, 0.3\} \\ \{0.6, 0.2, 0.3\} & \{0.5, 0.1, 0.3\} & \{0.5, 0.1, 0.2\} \\ \{0.5, 0.3, 0.2\} & \{0.7, 0.2, 0.2\} & \{0.7, 0.2, 0.2\} \end{bmatrix}$$

The decision making process is as follows.

Step 1. Calculate the transformed matrices R_1, R_2 and R_3. Since c_1 and c_2 belong to benefit attributes, it is clear that $R_1 = D_1$ and $R_2 = D_2$. We obtain R_3 as

$$R_3 = \begin{bmatrix} \{0.2, 0.1, 0.5\} & \{0.2, 0.2, 0.5\} & \{0.3, 0.1, 0.3\} \\ \{0.2, 0.3, 0.5\} & \{0.3, 0.1, 0.7\} & \{0.3, 0.3, 0.5\} \\ \{0.3, 0.2, 0.6\} & \{0.3, 0.1, 0.5\} & \{0.2, 0.1, 0.5\} \\ \{0.2, 0.3, 0.5\} & \{0.2, 0.2, 0.7\} & \{0.2, 0.2, 0.7\} \end{bmatrix}$$

Step 2. Based on (19), we get the matrix of $\Delta_k = [\xi_{ij}^k]_{4\times 3}$ $(k = 1, 2, 3)$ as below.

$$\Delta_1 = \begin{bmatrix} 0.4647 & 0.4647 & 0.5332 \\ 0.4631 & 0.4640 & 0.4683 \\ 0.4663 & 0.4664 & 0.4647 \\ 0.4654 & 0.4647 & 0.4658 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 0.3139 & 0.3128 & 0.3630 \\ 0.3156 & 0.3137 & 0.3129 \\ 0.3270 & 0.3135 & 0.3154 \\ 0.3156 & 0.3129 & 0.3158 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 0.2213 & 0.2225 & 0.1038 \\ 0.2213 & 0.2223 & 0.2225 \\ 0.2167 & 0.2201 & 0.2199 \\ 0.2190 & 0.2224 & 0.2184 \end{bmatrix}$$

Step 3. Based on SNNPWA operator, we get

$$R = \begin{bmatrix} \{0.4558, 0.1716, 0.2415\} & \{0.4925, 0.2000, 0.2741\} \\ & \{0.4468, 0.2364, 0.3108\} \\ \{0.5798, 0.1275, 0.2000\} & \{0.6572, 0.0418, 0.2486\} \\ & \{0.2558, 0.2154, 0.5000\} \\ \{0.3525, 0.1569, 0.3000\} & \{0.5000, 0.1950, 0.3000\} \\ & \{0.8000, 0.2073, 0.5296\} \\ \{0.6645, 0, 0.1164\} & \{0.6248, 0.1645, 0.2000\} \\ & \{0.2000, 0.1941, 0.4855\} \end{bmatrix}$$

Step 4. Based on SNNWA operator, we get the overall preference value r_i $(i = 1, 2, 3, 4)$ for the alternative x_i $(i = 1, 2, 3, 4)$ as follows.

$$r_1 = \{0.4622, 0.2027, 0.2757\}$$
$$r_2 = \{0.4980, 0.1184, 0.3047\}$$
$$r_3 = \{0.6206, 0.1852, 0.3766\}$$
$$r_4 = \{0.5116, 0, 0.2360\}$$

Step 5. Based on the score function of SNSs, we get $s(r_1) = 0.6613$, $s(r_2) = 0.6916$, $s(r_3) = 0.6863$ and $s(r_4) = 0.7585$.

Step 6. Since $s(r_4) > s(r_2) > s(r_3) > s(r_1)$, the ranking order of all the alternatives is $a_4 \succ a_2 \succ a_3 \succ a_1$ and the most desirable one is a_4.

By applying the method of [14] to the example, the ranking order of all the alternatives is $a_4 \succ a_2 \succ a_1 \succ a_3$ and the most desirable one is a_4. By applying the method of [16] to the example, the ranking result is the same as the one proposed in this paper.

B. Analysis

In this section, we have proposed a method to solve the MAGDM problem expressed with SNSs. From the above example and comparison with the methods in reference [14,16], we see that the main advantages of the proposed aggregation operators provide various choices for addressing different decision making situations. In order to validate the feasibility of our method, an example is conducted to compare with the method of Reference [14,16].

Example. Let

$$A_1 = \{\langle 0.4, 0.2, 0.3\rangle, \langle 0.6, 0.2, 0.1\rangle\},$$
$$A_2 = \{\langle 0.5, 0.2, 0.3\rangle, \langle 0.5, 0.1, 0.3\rangle\},$$
$$A_3 = \{\langle 0.4, 0.1, 0.2\rangle, \langle 0.7, 0.2, 0.3\rangle\}$$

be three alternatives with two attributes, we conduct two methods to choose the best alternative.

Method of Reference [14].
Let

$$A_1 = \{\langle T_{11}, I_{11}, F_{11}\rangle, \langle T_{12}, I_{12}, F_{12}\rangle\},$$

$$A_2 = \{\langle T_{21}, I_{21}, F_{21}\rangle, \langle T_{22}, I_{22}, F_{22}\rangle\},$$
$$A_3 = \{\langle T_{31}, I_{31}, F_{31}\rangle, \langle T_{32}, I_{32}, F_{32}\rangle\}$$

be three alternatives with two attributes, $w = (w_1, w_2)$ be the weight vector of the two attributes. The ideal point of SNSs is defined by

$$\begin{aligned} A^* &= \{T^*, I^*, F^*\} \\ &= \{\langle \max\{T_{11}, T_{21}, T_{31}\}, \min\{I_{11}, I_{21}, I_{31}\}, \\ &\quad \min\{F_{11}, F_{21}, F_{31}\}\rangle, \\ &\quad \langle \max\{T_{12}, T_{22}, T_{32}\}, \min\{I_{12}, I_{22}, I_{32}\}, \\ &\quad \min\{F_{12}, F_{22}, F_{32}\}\rangle\}. \end{aligned}$$

Here suppose $w = (0.5, 0.5)$ is the weight of the attribute values. The ideal point of SNSs is $A^* = \{\langle 0.5, 0.1, 0.2\rangle, \langle 0.7, 0.1, 0.1\rangle\}$.
We use Eq. (9) in Reference [14]:

$$S(A, A^*)=\sum_{i=1}^{n} w_i \frac{T_iT^*+I_iI^*+F_iF^*}{\sqrt{T_i^2+I_i^2+F_i^2}\sqrt{(T^*)^2+(I^*)^2+(F^*)^2}} \quad (20)$$

to calculate the similarity measure between $A_i, (i = 1, 2, 3)$ and A^*. The bigger the similarity measure, the better the alternative.
By Eq. (20), we get

$$S(A_1, A^*)=0.9668, S(A_2, A^*)=0.9503, S(A_3, A^*)=0.9782$$

since

$$S(A_3, A^*) > S(A_1, A^*) > S(A_2, A^*)$$

which means that the similarity measure between A_3 and A^* is the biggest, so A_3 is the best alternative.

Method of Reference [16].
Here, still suppose $w = (0.5, 0.5)$ is the weight vector of the attribute values.
Step 1. Utilise the SNNWA operator to aggregate the attribute value.

$$\begin{aligned} &\mathrm{SNNWA}(A_1, A_2, \ldots, A_n) \\ &= \{1 - \textstyle\prod_{j=1}^{n}(1 - T_j)^{w_j}, \prod_{j=1}^{n}(I_j)^{w_j}, \prod_{j=1}^{n}(F_j)^{w_j}\} \end{aligned} \quad (21)$$

By Eq. (21), we get

$$\tilde{A_1} = \{0.5101, 0.2000, 0.1732\},$$
$$\tilde{A_2} = \{0.5000, 0.1414, 0.3000\},$$
$$\tilde{A_3} = \{0.5757, 0.1414, 0.2449\}$$

Step 2. Calculate the score function of the alternatives.
According to Eq. (1), we get

$$s(\tilde{A_1}) = 0.7123, s(\tilde{A_2}) = 0.6862, s(\tilde{A_3}) = 0.7293$$

Step 3. Rank the alternatives. Since

$$s(\tilde{A_3}) > s(\tilde{A_2}) > s(\tilde{A_1})$$

then A_3 is the best alternative.

The proposed method.
For convenience, we denote

$$r^1_{11} = \langle 0.4, 0.2, 0.3\rangle, r^1_{12} = \langle 0.6, 0.2, 0.1\rangle,$$
$$r^2_{11} = \langle 0.5, 0.2, 0.3\}, r^2_{12} = \langle 0.5, 0.1, 0.3\rangle,$$
$$r^3_{11} = \langle 0.4, 0.1, 0.2\rangle, r^3_{12} = \langle 0.7, 0.2, 0.3\rangle.$$

Step 1. Calculate the supports.

$$\mathrm{Sup}(r^1_{11}, r^2_{11}) = 0.9424, \mathrm{Sup}(r^1_{11}, r^3_{11}) = 0.9184,$$
$$\mathrm{Sup}(r^2_{11}, r^3_{11}) = 0.9000, \mathrm{Sup}(r^1_{12}, r^2_{12}) = 0.8586,$$
$$\mathrm{Sup}(r^1_{12}, r^3_{12}) = 0.8709, \mathrm{Sup}(r^2_{12}, r^3_{12}) = 0.8709.$$

Step 2. Calculate the weight ξ^k_{ij} of each value r^k_{ij}, $i, j = 1, 2,\ k = 1, 2, 3$.

$$\xi^1_{11} = 0.3357, \xi^1_{12} = 0.3324, \xi^2_{11} = 0.3335,$$
$$\xi^2_{12} = 0.3324, \xi^3_{11} = 0.3308, \xi^3_{12} = 0.3352.$$

Step 3. Based on (9), we get

$$\mathring{A}_1 = \mathrm{SNNPWA}(r^1_{11}, r^1_{12}, r^1_{13}) = \{0.3788, 0.3412, 0.3105\},$$
$$\mathring{A}_2 = \mathrm{SNNPWA}(r^2_{11}, r^2_{12}, r^2_{13}) = \{0.3697, 0.2720, 0.4485\},$$
$$\mathring{A}_3 = \mathrm{SNNPWA}(r^3_{11}, r^3_{12}, r^3_{13}) = \{0.4360, 0.2722, 0.3922\}.$$

Step 4. Rank the alternatives. By Eq. (1), we get

$$s(\mathring{A}_1) = 0.5757, s(\mathring{A}_2) = 0.5497, s(\mathring{A}_3) = 0.5905,$$

since

$$s(\mathring{A}_3) > s(\mathring{A}_1) > s(\mathring{A}_2),$$

then A_3 is the best alternative.

We see the above methods have the same result that the best alternative is A_3. Considering the methods of reference [14,16], the weights of the attributes were determined subjectively by the decision makers who have different knowledge structures and experiences. It is sometimes difficult for the decision makers to give the weight of the attribute correctly. The subjective weight only reflects the preference of the decision maker and ignores the objective information included in the decision matrix. In reference [14], in the decision making process, the ideal alternative is defined first, then similarity measures between each alternative and the ideal alternative were calculated, respectively. At last, the final rankings of the alternatives can be determined by the similarity measures. In reference [16], the decision making method based on the SNNWA operator is given, where the weight of the attribute value is given by the decision makers first. They use the SNNWA operator to aggregate the attribute values, and use the score function to rank the alternatives. In fact, with regard to the proposed method, the weight of each attribute is calculated by the attribute information and is allowed the values being aggregated to support and reinforce each other. The weight vector we obtained is more objective which not only benefits from the decision maker' expertise but also the relative importance of attribute information. In the decision making process, we apply the SNNPWA operator to aggregate the data of the attributes on the alternative, then, we use the score function to rank the alternatives.

VI. Conclusion

We have proposed SNNPWA operator, SNNPWG operator, SNNPOWA operator and SNNPOWG operator on the basis of PA operator, PG operator and OWA operator in this paper. Moreover, we have developed a method for addressing the MAGDM problem expressed with SNSs. The prominent characteristic of the proposed aggregation operators is that they take into account not only the supporting degree between attributes but also the ordered weight of aggregation process. The particular emphasis is that these operators provide various choices concerning different MAGDM problems. Moreover, we present the MAGDM models within the framework of SNSs. Finally, a practical numerical example is given to verify the developed MAGDM method and to demonstrate its capacity in dealing with practical and uncertain decision making problems. In the future, we want to further study the power aggregation operators of the interval-valued simplified neutrosophic set and apply them to solve more decision making problems.

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Chunfang Liu graduated from Beihua University, China, in 1999. She received the M.S. degree from Harbin Institute of Technology, China, in 2003. She is studying for her doctorate in Harbin Engineering University. She is currently a lecturer at College of Science, Northeast Forestry University. Her research interests includes system engineering, intelligent control and fuzzy systems. e-mail: liuchunfang1112@163.com

Yuesheng Luo graduated from Harbin Engineering University, China, in 1982. He received the M.S. degree from Jilin University, China, in 1988 and the Ph.D. degree from Harbin Engineering University (HEU), China, in 2003. He is currently a professor at Automobile College, HEU. His research interests includes system engineering, robust control and fuzzy systems.