Recent Advantages In Neutrosophic Module Theory

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Abstract: This work is dedicated to give the interested reader a good review and background in recent developments in the field of neutrosophic algebraic module theory.

Key words: neutrosophic module, refined neutrosophic module, semi homomorphism, AH-submodule.

We recall the basic definitions and properties of neutrosophic modules.

Definition 1:

Let (N,+,.) be a module over the ring R. Then (N(I),+,.) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I).

Here elements of N(I) have the form $\boldsymbol{x+yI; x, y \in N}$, i.e N(I) can be written as $\boldsymbol{N(I) = N + NI}$.

Definition 2:

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a nonempty set of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

Definition 3:

Let U(I) and W(I) be two strong neutrosophic modules and let $f: U(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

We define the kernel of f by Ker(f) = { x∈ $M(I)$; $f(x)$ = 0 }.

Definition 4:

Let $M(I) = M + MI$ be a strong/weak neutrosophic module, the set

is called an AH-$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are submodules of V submodule of M(I).

If $P = Q$, S is called an AHS-submodule of M(I).

Example 5:

We have M = $Z^2 = Z \times Z$ is a module over R, $P =< (0,1) >, Q =< (1,0) >$, are two submodules of M. The set

is an AH-submodule of M(I).$S = P + QI = \{(0, a) + (b, 0)I; a, b \in Z\}$

The set $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in Z$ is an AHS-submodule of M(I).

Theorem 6:

Let $M(I) = M + MI$ be a neutrosophic weak module over the ring R, and let $S = P + QI$ be an AH-submodule of $M(I)$, then S is a submodule.

Theorem 7:

Let M(I) be a neutrosophic strong module over a neutrosophic ring R(I), let $S = P + PI$ be an AHS-submodule. Then S is a submodule of M(I).

Definition 8:

(a) Let M and W be two modules, $L_M: M \to W$ be a homomorphism. The AHS-homomorphism can be defined as follows:

. $L: M(I) \to W(I); L(a + bI) = L_M(a) + L_M(b)I$

(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S) = L_M(P) + L_M(Q)I.$

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I.$

(d) $AH - Ker(L) = Ker(L_M) + Ker(L_M)\, I = \{x + yI; x, y \in Ker(L_M)\}.$

Theorem 9:

Let W(I) and M(I) be two neutrosophic strong/weak modules, and $L: M(I) \to W(I)$ be an AHS-homomorphism:

(a) $AH - Ker(L)$ is an AHS-submodule of M(I).

(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S)$ is an AH-submodule of W(I).

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S)$ is an AH-submodule of M(I).

Proof:

(a) Since $Ker(L_M)$ is a submodule of M, we find that $AH - Ker(L) = Ker(L_M) + Ker(L_M)I$ is an AHS-submodule of M(I).

(b) We have $L(S) = L_M(P) + L_M(Q)I$; thus L(S) is an AH-submodule of W(I), since $L_M(P), L_M(Q)$are submodules of W.

(c) By regarding that $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I,\ L_W^{-1}(P)$ and $L_W^{-1}(Q)$ are submodules of M, we obtain that $L^{-1}(S)$ is an AH-subModule of M(I).

Theorem 10:

Let W(I) and M(I) be two neutrosophic strong modules over a neutrosophic ring R(I), and $L: M(I) \to W(I)$ be an AHS-homomorphism. Then

, for all $x, y \in M(I), m \in R(I). L(x + y) = L(x) + L(y), L(m.x) = m.L(x)$

Proof:

Suppose $x = a + bI, y = c + dI; a, b, c, d \in M, and\ m = s + tI \in K(I)$, we have

$L(x + y) = L([a + c] + [b + d]I) = L_M(a + c) + L_M(b + d)I = [L_M(a) + L_M(b)I] + [L_M(c) + L_M(d)I] = L(x) + L(y).$

, $L(m.x) = L_M(s.a) + L_M(s.b + t.a + t.b)Im.x = (s.a) + (s.b + t.a + t.b)I$

$= s.L_M(a) + [s.L_M(b) + t.L_M(a) + t.L_M(b)]I = (s + tI).(L_M(a) + L_M(b)I) = m.L(x).$

Theorem 11:

Let (N,+,.) be a module over the commutative ring R, $N(I)$ be the corresponding weak neutrosophic module over R. Then

.$N(I) \cong N \times N$

Proof:

Define $f: N \times N \to N(I); f(x,y) = x + yI; x,y \in N$, it is easy to see that f is well defined bijective map.

Let $(x,y),(z,t) \in M \times M, r \in R$, we have $(x,y) + (z,t) = (x+z, y+t), r.(x,y) = (r.x, r.y)$,

.$f[(x,y) + (z,t)] = (x+z) + (y+t)I = (x+yI) + (z+tI) = f(x,y) + f(z,t)$

. Hence f is a module isomorphism.$f[r.(x,y)] = r.x + r.yI = r.(x+yI) = r.f(x)$

Refined neutrosophic modules

Definition 12: Let $(X\ (I_1,I_2),+,\cdot)$ be any refined neutrosophic algebraic structure where + and $\cdot$ are ordinary addition and multiplication respectively. I_1 *and* I_2 are the split components of the indeterminacy factor I that is $I = \alpha I_1 + \beta I_2$ with $\alpha,\beta \in R$ *or* C . Also, I_1 *and* I_2 are taken to have the properties $I_1^2 = I_1, I_2^2 = I_2$ and $I_1 I_2 = I_2 I_1 = I_1$.

For any two elements,we define

1) $x + y = (a, bI_1, cI_2) + (d, eI_1, fI_2) = (a+d, (b+e)I_1, (c+f)I_2)$

2) $x \cdot y = (a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = \begin{pmatrix} ad, (ae+bd+be+bf+ce)I_1, \\ (af+cd+cf)I_2 \end{pmatrix}$

Definition 13: Let $(M,+,.)$ be any R-module over a refined neutrosophic ring $R(I_1,I_2)$, The triple $(M\ (I_1,I_2),+,\cdot)$ is called a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$, generated by M, I_1 *and* I_2.

Definition14: Let $M(I)$ be a strong neutrosophic R-module, the set $S = P + QI = \{x + yI : x \in P, y \in Q\}$ where P and Q are submodules of M is called an AH-submodule of $M(I)$ and If $P = Q$ then S is called an AHS-submodule of $M(I)$.

Definition 15:

Let $M(I_1,I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)$, P,Q,S be three submodules of M. The set N = $(P, QI_1, SI_2) = \{(a, bI_1, cI_2); a \in P, b \in Q, c \in S\}$ is called a strong AH-submodule of the strong refined neutrosophic module $M(I_1,I_2)$.

If P = Q = S, we call N a strong AHS-submodule.

Theorem 16:

Let $M(I_1,I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)$,

N = (P,PI_1,PI_2) be a strong AHS-submodule. Then N is a submodule by classical meaning.

Definition 17:

Let M,W be two modules over the ring R, $M(I_1,I_2)$ *and* $W(I_1,I_2)$ be the corresponding strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1,I_2)$. Let $f,g,h: M \to W$ be three homomorphisms, then $[f,g,h]: M(I_1,I_2) \to W(I_1,I_2)$; $[f,g,h](a,bI_1,cI_2) = (f(a),g(b)I_1,h(c)I_2)$ is called a strong AH-homomorphism. If $f = g = h$, we get the strong AHS-homomorphism.

Definition 18:

Let $M(I_1,I_2), W(I_1,I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1,I_2)$, $[f,g,h]: M(I_1,I_2) \to W(I_1,I_2)$ be a strong AH-homomorphism, we define

(a) $AH - Ker[f,g,h] = (Ker(f), Ker(g)I_1, Ker(h)I_2) = \{(a,bI_1,cI_2); a \in Ker(f), b \in Ker(g), c \in Ker(h)\}$.

(b) $AH - Im[f,g,h] = (Im(f), Im(g)I_1, Im(h)I_2)$.

Theorem 19:

Let $M(I_1,I_2), W(I_1,I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1,I_2)$, $[f,g,h]: M(I_1,I_2) \to W(I_1,I_2)$ be a strong AH-homomorphism.

(a) If N = (P,QI_1,SI_2) is a strong AH-submodule of $M(I_1,I_2)$, then $[f,g,h](N)$ is a strong AH-submodule of $W(I_1,I_2)$.

(b) $[f,g,h]$ is a classical module homomorphism.

(c) $AH - Ker[f,g,h]$ is a strong AH-submodule of $M(I_1,I_2)$.

(d) $AH - Im[f,g,h]$ is a strong AH-submodule of $W(I_1,I_2)$.

Proof:

(a) Since $f(P), g(Q), h(S)$ are submodules of N, we find that $[f,g,h](N) = (f(P), g(Q)I_1, h(S)I_2)$ is a strong AH-submodule of $W(I_1,I_2)$.

(b) Let $m = (x,yI_1,zI_2), n = (a,bI_1,cI_2)$ be two arbitrary elements in $M(I_1,I_2)$, $r = (t,uI_1,vI_2)$ be any element in $R(I_1,I_2)$,

), $r.m = (tx,[xu+yt+yu+yv+zu]I_1,[xv+zt+zv]I_2)$,$m+n = (x+a,[y+b]I_1,[z+c]I_2$

)=$(f(x),g(y)I_1,h(z)I_2)$ +$[f,g,h](m+n) = (f(x+a),g([y+b])I_1,h([z+c])I_2$
$(f(a),g(b)I_1,h(c)I_2) = [f,g,h](m) + [f,g,h](n)$.

=$[f,g,h](r.m) = (f(tx),g([xu+yt+yu+yv+zu])I_1,h([xv+zt+zv])I_2)$

. Thus $[f,g,h]$ is a classical homomorphism.$(t,uI_1,vI_2).(f(x),g(y)I_1,h(z)I_2) = r.[f,g,h](m)$

(c)Since $Ker(f), Ker(g), Ker(h)$are submodules of M, we get $AH - Ker[f,g,h] = (Ker(f), Ker(g)I_1, Ker(h)I_2)$ as a strong AH-submodule of $M(I_1,I_2)$.

(d) Since $Im(f), Im(g), Im(h)$ are submodules of W, we get $AH - Im[f, g, h] = (Im(f), Im(g)I_1, Im(h)I_2)$ as a strong AH-submodule of $W(I_1, I_2)$.

Example 20:

(a) Let $M = R^2, W = R$ be two modules over the ring R,

are three $f: M \to W; f(x, y) = 2x, g: M \to W; g(x, y) = 3y, h: M \to W; h(x, y) = x + y$ homomorphisms.

(b) $[f, g, h]: M(I_1, I_2) \to W(I_1, I_2); [f, g, h]\big((x, y), (z, t)I_1, (s, m)I_2\big) = (f(x, y), g(z, t)I_1, h(s, m)I_2)$=

is a strong AH-homomorphism, where $x, y, z, t, s, m \in R$.$(2x, 3tI_1, [s + m]I_2)$

(c) $P = \{(0, x); x \in R\}, Q = \{(x, 0); x \in R\}$ are two submodules of M,

is a strong AH-submodule of $M(I_1, I_2)$.$N = (P, PI_1, QI_2) = \{((0, x), (0, y)I_1, (z, 0)I_2; x, y, z \in R\}$

(d) $f(P) = \{0\}, g(P) = \{3y; y \in R\} = R, h(Q) = \{z; z \in R\} = R,$

is a strong AH-$[f, g, h](N) = (f(P), g(P)I_1, h(Q)I_2) = (0, RI_1, RI_2) = \{(0, xI_1, yI_2); x, y \in R\}$ submodule of $W(I_1, I_2)$.

(e) $Ker(f) = \{(0, x); x \in R\}, Ker(g) = \{(x, 0); x \in R\}, Ker(h) = \{(y, -y); y \in R\},$

.$AH - Ker[f, g, h] = (\ker(f), Ker(g)I_1, Ker(h)I_2) = \{(0, x), (y, 0)I_1, (z, -z)I_2); x, y, z \in R\}$

Definition 21:

Let M be a module over a ring R, N be a module over a ring T, $\varphi: M \to N$ be a well defined map, we say that φ is a semi module homomorphism if and only if the following conditions are true:

(a) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$.

(b) There is a ring homomorphism $f: R \to T$ such $\varphi(r.x) = f(r).\varphi(x)$ for all $r \in R, x \in$M.

Remark 22:

(a) The concept of semi homomorphism can be used to study relationships between modules defined over different rings.

(b) It is easy to see that every homomorphism between two modules defined over the same ring is a semi homomorphism.

In the following theorem we show that every neutrosophic module M(I) defined over a neutrosophic ring R(I) is a semi homomorphic image to the corresponding refined neutrosophic module $M(I_1, I_2)$ over the corresponding refined neutrosophic ring $R(I_1, I_2)$.

Theorem 23:

Let M be a module over a ring R, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding stronh refined neutrosophic module over $R(I_1, I_2)$. Then

$\to M(I);\ \varphi(a, bI_1, cI_2) = a + (b + c)I$ is a semi homomorphism.$\varphi: M(I_1, I_2)$

Proof:

Clearly, φ is a well defined map. Let $x = (a, bI_1, cI_2), y = (u, vI_1, wI_2)$ be two arbitrary elements in $M(I_1, I_2)$,

$.\varphi(x + y) = (a + u) + (b + v + c + w)I = [a + (b + c)I] + [u + (v + w)I] = \varphi(x) + \varphi(y)$

The map $f: R(I_1, I_2) \to R(I); f(m, nI_1, tI_2) = m + (n + t)I; m, n, t \in R$ is a ring homomorphism.

Consider $r = (m, nI_1, tI_2) \in R(I_1, I_2)$, we can write

$=\varphi(r.x) = \varphi(m.a, (m.b + n.a + n.b + n.c + t.b)I_1, (t.a + t.c + m.c)I_2)$

$m.a + (m.b + n.a + n.b + n.c + t.b + t.a + t.c + m.c)I = [m + (n + t)I].[a + (b + c)I] =$
. Thus the proof is complete.$f(r).\varphi(x)$

Theorem 24:

Let M be a module over a ring R, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.16. Then

The semi homomorphic image of any strong submodule of $M(I_1, I_2)$ is a strong submodule of M(I).

Proof:

Let N be any strong AH-submodule of $M(I_1, I_2)$, since $(N, +)$ is a subgroup of $(M(I_1, I_2), +)$, we have $(\varphi(N), +)$ is a subgroup of (M(I),+).

Let $y = a + bI$ be an arbitrary element in $\varphi(N)$, $r = u + vI$ be any element in R(I), since φ, f are surjective maps, there are

, i.e $m + n = b$ and $g = (u, zI_1, qI_2) \in R(I_1, I_2); f(g) = r$, $x = (a, mI_1, nI_2) \in M(I_1, I_2);\ \varphi(x) = y$
i.e $z + q = v$,

$=r.y = u.a + (v.a + v.b + u.b)I = u.a + (z.a + q.a + z.m + z.n + q.m + q.n + u.m + u.n)I$

$.f(g).\varphi(x) = \varphi(g.x) \in \varphi(N)$

n-Refined Neutrosophic modules

Definition 25 :

Let (M,+,.) be a module over the ring R, we say that $M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n;\ x_i \in M\}$ is a weak n-refined neutrosophic module over the ring R. Elements of $M_n(I)$ are called n-refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n-refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-refined neutrosophic module over the n-refined neutrosophic ring $M_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

Remark 26:

If we take n=1 we get the classical neutrosophic module.

Addition on $M_n(I)$ is defined as:

$\sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i.$

Multiplication by a scalar $m \in R$ is defined as:

$.m.\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n}(m.a_i)I_i$

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in R_n(I)$ is defined as:

$.\sum_{i=0}^{n} m_i I_i \; .\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n}(m_i.a_i)I_i I_j$

Where $\mathrm{a_i} \in \mathrm{M}, \mathrm{m_i} \in \mathrm{R}, \mathrm{I_i I_j} = \mathrm{I_{min(i,j)}}$.

Example 27:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-refined neutrosophic module over the ring Z_2 is

$.M_n(I) = \{0,1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$

Definition 28:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 29:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 30:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$.

n-Cyclic refined neutrosophic modules

Definition 31 :

Let (M,+,.) be a module over the ring R, we say that $M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n;\ x_i \in M\}$ is a weak n-cyclic refined neutrosophic module over the ring R. Elements of $M_n(I)$ are called n-cyclic refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n- cyclic refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

Remark 32:

Addition on $M_n(I)$ is defined as:

$\sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i$.

Multiplication by a scalar $m \in R$ is defined as:

$.m.\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m.a_i) I_i$

Multiplication by an n-cyclic refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in R_n(I)$ is defined as:

$.\sum_{i=0}^{n} m_i I_i \ .\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m_i.a_i) I_i I_j$

Where $a_i \in M, m_i \in R, I_i I_j = I_{(i+j \bmod n)}$.

Theorem 33 :

Let (M,+,.) be a module over the ring R. Then a weak n-cyclic refined neutrosophic module $M_n(I)$ is a module over the ring R. A strong n-cyclic refined neutrosophic module is a module over the n-cyclic refined neutrosophic ring R_n(I).

Proof:

It is similar to the classical case.

Example 34:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-cyclic refined neutrosophic module over the ring Z_2 is

$.M_n(I) = \{0,1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$

Definition 35:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, a nonempty subset $W_n(I)$ is called a weak n-cyclic refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 36:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-cyclic refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 37:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n-cyclic refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$.

Theorem 38:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a strong n-cyclic refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R_n(I)$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$ over the n-cyclic refined neutrosophic ring $R_n(I)$.

Example 39:

is a module over the ring of real numbers R, $W =< (0,1) >$ is a submodule of M, $R_2^2(I)$ $=M = R^2$ $\{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-cyclic refined neutrosophic module.

is a weak 2-cyclic refined $W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ neutrosophic submodule of the weak 2-cyclic refined neutrosophic module $R_2^2(I)$ over the ring R.

is a strong 2-cyclic refined $W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ neutrosophic submodule of the strong 2-cyclic refined neutrosophic module $R_2^2(I)$ over the n-cyclic refined neutrosophic ring $R_2(I)$.

Definition 40:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if $x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in R, x_i \in M_n(I)$.

Example 41:

Consider the weak 2-cyclic refined neutrosophic module in Example 3.9,

, $x = 2(0,1) + 1.(1,0)I_1 + 3(0,1)I_1$, i.e x is a linear combination of the $x = (0,2) + (1,3)I_1 \in R_2^2(I)$ set $\{(0,1), (1,0)I_1, (0,1)I_1\}$ over the ring R.

Definition 42:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if $x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in R_n(I), x_i \in M_n(I)$.

Example 43:

Consider the strong 2-cyclic refined neutrosophic module $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$,

$x = (0,2) + (2,3)I_1 = (2 + I_1).(0,1) + (1 + I_2).(1,1)I_1 = (0,2) + (0,1)I_1 + (1,1)I_1 + (1,1)I_{(3\ mod\ 2)} = x$

, hence x is a linear combination of the set $\{(0,1), (1,1)I_1\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$.

Definition 44:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-cyclic refined neutrosophic module M_n(I) over the ring R, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0\ implies\ a_i = 0;\ a_i \in R$.

Definition 45:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-cyclic refined neutrosophic module M_n(I) over the n-cyclic refined neutrosophic ring $R_n(I)$, X is a strong linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0\ implies\ a_i = 0;\ a_i \in R_n(I)$.

Definition 46:

Let $M_n(I), W_n(I)$ be two strong n-cyclic refined neutrosophic modules over the n-cyclic refined neutrosophic ring $R_n(I)$, let $f: M_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-cyclic refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$ for all $x, y \in M_n(I), a, b \in R_n(I)$.

A weak n-cyclic refined neutrosophic homomorphism can be defined by the same.

Definition 47:

Let $f: M_n(I) \to U_n(I)$ be a weak/strong n-cyclic refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in M_n(I); f(x) = 0\}$.

(b) $Im(f) = \{y \in U_n(I);\ \exists\ x \in M_n(I)\ and\ y = f(x)\}$.

Theorem 48:

Let $f: M_n(I) \to U_n(I)$ be a weak n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a weak n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a weak n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 49:

Let $f: M_n(I) \to U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 50:

Let $f: M_n(I) \to U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Example 51:

Let $R_2^2(I) = \{x_0 + x_1I_1 + x_2I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1I_1 + y_2I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-cyclic refined neutrosophic modules over the ring of real numbers R. Consider $f: R_2^2(I) \to R_2^3(I)$, where

, f is a weak 2-cyclic refined $f[(a,b)+(m,n)I_1+(k,s)I_2]=(a,0,0)+(m.0.0)I_1+(k,0,0)I_2$ neutrosophic homomorphism over the ring R.

$.Ker(f)=\{(0,b)+(0,n)I_1+(0,s)I_2; b,n,s \in R\}$

$.Im(f)=\{(a,0,0)+(m,0,0)I_1+(k,0,0)I_2; a,m,k \in R\}$

Example 52:

Let $W_2(I)=<(0,0,1)I_1>=\{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I)=<(0,1,0)I_1>=\{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-cyclic refined neutrosophic modules of the strong 2-cyclic refined neutrosophic module $R_2^3(I)$ over 2-refined neutrosophic ring R_2(I). Define $f: W_2(I) \to U_2(I); f[q(0,0,a)I_1]=q(0,a,0)I_1; q \in R_2(I)$.

is a strong 2-cyclic refined neutrosophic homomorphism:f

Let $A=q_1(0,0,a)I_1, B=q_2(0,0,b)I_1 \in W_2(I);\ q_1,q_2 \in R_2(I)$, we have

$A+B=(q_1+q_2)(0,0,a+b)I_1, f(A+B)=(q_1+q_2).(0,a+b,0)I_1=f(A)+f(B)$.

Let $m=c+dI_1+eI_2 \in R_2(I)$ be a 2-cyclic refined neutrosophic scalar, we have

by the definition of f, hence f is a strong 2-cyclic refined neutrosophic $f(m.A)=m.f(A)$ homomorphism.

Conclusion

This paper is a review study for neutrosophic module theory. It provides the interested reader a large background, to help in the development of this field of neutrosophic algebra.

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