Refined neutrosophic quadruple (po-)hypergroups and their fundamental group

M. Al-Tahan[1], B. Davvaz [2,*]

[1]Lebanese International University, Bekaa, 961, Lebanon.

E-mail: madeline.tahan@liu.edu.lb

[2]Yazd University, Yazd, 98, Iran.

E-mail: davvaz@yazd.ac.ir

*Correspondence: B. Davvaz (davvaz@yazd.ac.ir)

Abstract: After introducing the notion of hyperstructures about 80 years ago by F. Marty, a number of researches on its theory, generalization, and it's applications have been done. On the other hand, the theory of Neutrosophy, the study of neutralities, was developed in 1995 by F. Smarandache as an extension of dialectics. This paper aims at finding a connection between refined neutrosophy of sets and hypergroups. In this regard, we define refined neutrosophic quadruple hypergroups, study their properties, and find their fundamental refined neutrosophic quadruple groups. Moreover, some results related to refined neutrosophic quadruple po-hypergroups are obtained.

Keywords: H_v-group, po-hypergroup, refined neutrosophic quadruple number, refined neutrosophic quadruple hypergroup, fundamental group.

1 Introduction

In 1934, Marty [19] introduced the concept of hypergroups by considering the quotient of a group by its subgroup. And this was the birth of an interesting new branch of Mathematics known as "Algebraic hyperstructures" which is considered as a generalization of algebraic structures. In algebraic structure, the composition of two elements is an element whereas in algebraic hyperstructure, the composition of two elements is a non-empty set. Since then, many different kinds of hyperstructures (hyperrings, hypermodules, hypervector spaces, ...) were introduced. And many studies were done on the theory of algebraic hyperstructures as well on their applications to various subjects of Sciences (see [12, 13, 30]). Later, in 1991, Vougioklis [28] generalized hyperstuctures by introducing a larger class known as weak hyperstructures or H_v-structures. For more details about H_v-structures, see [28, 29, 30, 31].

In 1965, Zadeh [32] extended the classical notion of sets by introducing the notion of Fuzzy sets whose elements have degrees of membership. The theory of fuzzy sets is mainly concerned with the measurement of the degree of membership and non-membership of a given abstract situation. Despite its wide range of real life applications, fuzzy set theory can not be applied to models or problems that contain indeterminancy. This is the reason that arose the importance of introducing a new logic known as neutrosophic logic that contains the concept of indeterminancy. It was introduced by F. Smarandache in 1995, studied annd developed by him and by other authors. For more details about neutrosophic theory, we refer to [17, 22, 23, 25]. Recently, many authors are working on the applications of this important concept. For example in [2], Abdel-Baset et al. offered a novel approach for estimating the smart medical devices selection process in a group decision making in a vague decision environment and used neutrosophics in their methodology. Moreover, in [7], R. Alhabib et al. worked on some neutrosophic probability distribution. Other interesting applications of it are found in [1, 3, 4, 15, 20].

In 2015, Smarandache [22] introduced the concept of neutrosophic quadruple numbers and presented some basic operations on the set of neutrosophic quadruple numbers such as, addition, subtraction, multiplication, and scalar multiplication. After that, a connection between neutrosophy and algebraic structures was established where Agboola et al. [5] considered the set of neutrosophic quadruple numbers and used the defined operations on it to discuss neutrosophic quadruple algebraic structures. More results about neutrosophic algebraic structures are found in [11, 26]. A generalization of the latter work was done in 2016 where Akinleye et al. [6] considered the set of neutrosophic quadruple numbers and defined some hyperoperations on it and discussed neutrosophic quadruple hyperstructures. More specifically, the latter papers introduced the notions of neutrosophic groups, neutrosophic rings, neutrosophic hypergroups and neutrosophic hyperrings on a set of real numbers and studied their basic properties.

The authors in [9] discussed neutrosophic quadruple H_v-groups and studied their properties. Then in [10], they found the fundamental group of neutrosophic quadruple H_v-groups and proved that it is a neutrosphic quadruple group. This paper is an extension to the above mentioned results. In Section 2, some definitions related to weak hyperstructures have been presented while section 3 involves the refined neutrosophic quadruple hypergroup and the studying of it's properties. As for section 4, an order on refined neutrosophic quadruple hypergroups is defined and some examples on refined neutrosophic quadruple po-hypergroups are presented. Finally, in section 5, the fundamental refined neutrosophic quadruple group of refined neutrosophic quadruple hypergroups with some important theorems, corollaries and propositions have been submitted.

2 Preliminaries

In this section, some definitions and theorems related to both: hyperstructure theory and neutrosophic theory are presented. (See [12, 13, 30].)

2.1 Basic notions of hypergroups

Definition 2.1. Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*.

In this definition, if A and B are two non-empty subsets of H and $x \in H$, then:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.2. A hypergroupoid $(H, \circ)$ is called a:

1. *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;

2. *quasi-hypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (The latter condition is called the reproduction axiom);

3. *hypergroup* if it is a semihypergroup and a quasi-hypergroup.

Definition 2.3. [13] Let $(H, \star)$ and $(K, \star')$ be two hypergroups. Then $f : H \to K$ is said to be *hypergroup homomorphism* if $f(x \star y) = f(x) \star' f(y)$ for all $x, y \in H$. $(H, \star)$ and $(K, \star')$ are called *isomorphic* H_v*-groups*, and written as $H \cong K$, if there exists a bijective function $f : R \to S$ that is also a homomorphism. The set of all isomorphism of $(H, \star)$ is denoted as $Aut(H)$.

T. Vougiouklis, the pioneer of H_v-structures, generalized the concept of algebraic hyperstructures to weak algebraic hyperstructures [28]. The latter concept is known as "weak" since the equality sign in the definitions of H_v-structures is more likely to be replaced by non-empty intersection. The concepts in H_v-structures are mostly used in representation theory [29].
A hypergroupoid $(H, \circ)$ is called an H_v*-semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$. An element $0 \in H$ is called an *identity* if $x \in (0 \circ x \cap x \circ 0)$ for all $x \in H$ and it is called a *scalar identity* if $x = 0 \circ x = x \circ 0$ for all $x \in H$. If the scalar identity exists then it is unique. A hypergroupoid $(H, \circ)$ is called an H_v*-group* if it is a quasi-hypergroup and an H_v-semigroup. A non empty subset S of an H_v-group $(H, \circ)$ is called H_v*-subgroup of* H if $(S, \circ)$ is an H_v-group.

Definition 2.4. [27] A hypergroup is called *cyclic* if there exist $h \in H$ such that $H = h \cup h^2 \cup \ldots \cup h^i \cup \ldots$ with $i \in \mathbb{N}$. If there exists $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \ldots \cup h^s$ then H is a cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^s = \underbrace{h \star h \star \ldots \star h}_{s \text{ times}}$.

Definition 2.5. [27] A hypergroup is called a *single power cyclic hypergroup* if there exist $h \in H$ and $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \ldots \cup h^s \cup \ldots$ and $h \cup h^2 \cup \ldots \cup h^{m-1} \subset h^m$ for every $m \geq 1$. In this case, h is called a *generator* of H.

2.2 Refined neutrosophic quadruple hypergroups

Let T, I, F, represent the neutrosophic components truth, indeterminacy, and falsehood respectively. Symbolic (or Literal) Neutrosophic theory is referring to the use of these symbols in neutrosophics. In 2013, F. Smarandache [24] introduced the refined neutrosophic components. Where the neutrosophic literal components T, I, F can be split into respectively the following neutrosophic literal subcomponents:

$$T_1, \ldots, T_p; I_{,} \ldots, I_r; F_1, \ldots, F_s,$$

where p, r, s are positive integers with $\max\{p, r, s\} \geq 2$.

Definition 2.6. [25] Let X be a nonempty set and $p, r, s \in \mathbb{N}$ with $(p, r, s) \neq (1, 1, 1)$. A refined neutrosophic quadruple X-number is a number having the following form:

$$a + \sum_{i=1}^{p} b_i T_i + \sum_{j=1}^{r} c_j I_j + \sum_{k=1}^{s} b_k F_k,$$

where $a, b_i, c_j, d_k \in X$ and T, I, F have their usual neutrosophic logic meanings, and T_i, I_j, F_k are refinements of T, I, F respectively.

The set of all refined neutrosophic quadruple X-numbers is denoted by RNQ(X), that is,

$$RNQ(X) = \{a + \sum_{i=1}^{p} b_iT_i + \sum_{j=1}^{r} c_jI_j + \sum_{k=1}^{s} d_kF_k : a, b_i, c_j, d_k \in X\}.$$

For simplicity, we write $a + \sum_{i=1}^{p} b_iT_i + \sum_{j=1}^{r} c_jI_j + \sum_{k=1}^{s} d_kF_k$ as

$$(\ a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k).$$

In what follows, T_i, I_j, F_k are refinements of T, I, F respectively with $1 \leq i \leq p$, $1 \leq j \leq r$ and $1 \leq k \leq s$. Let $(H, +)$ be a hypergroupoid with identity "0" and $0 + 0 = 0$ and define "$\oplus$" on $RNQ(H)$ as follows:

$$\begin{aligned}&(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \oplus (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)\\ &= \{(x, \sum_{i=1}^{p} y_iT_i, \sum_{j=1}^{r} z_jI_j, \sum_{k=1}^{s} w_kF_k) :\ x \in a + a', y_i \in b_i + b'_i, z_j \in c_j + c'_j, w_k \in d_k + d'_k\}.\end{aligned}$$

3 New properties of refined neutrosophic quadruple hypergroups

In this section, refined neutrosophic quadruple hypergroups are defined and their properties are studied.

Proposition 3.1. *Let $(H, +)$ be a hypergroupoid with $0 \in H$ and T_i, I_j, F_k are refinements of T, I, F respectively. Then $(RNQ(H), \oplus)$ is a quasi-hypergroup with identity $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is a quasi-hypergroup with identity 0.*

Proof. Let $(H, +)$ be a quasi-hypergroup. We prove now that $(RNQ(H), \oplus)$ satisfies the reproduction axiom. That is, $\overline{x} \oplus RNQ(H) = RNQ(H) \oplus \overline{x} = RNQ(H)$ for all $\overline{x} = (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \in RNQ(H)$. We prove $\overline{x} \oplus RNQ(H) = RNQ(H)$ and the proof of $RNQ(H) \oplus \overline{x} = RNQ(H)$ is done in a similar manner. Let $\overline{y} = (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k) \in RNQ(H)$, we have $\overline{x} \oplus \overline{y} = (a + a', \sum_{i=1}^{p}(b_i + b'_i)T_i, \sum_{j=1}^{r}(c_j + c'_j)I_j, \sum_{k=1}^{s}(d_k + d'_k)F_k) \subseteq RNQ(H)$ as $(a + a') \cup (b_i + b'_i) \cup (c_j + c'_j) \cup (d_k + d'_k) \subseteq H$. Thus $\overline{x} \oplus RNQ(H) \subseteq RNQ(H)$. Let $\overline{y} = (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k) \in RNQ(H)$. Since $(H, +)$ satisfies the reproduction axiom and $a', b'_i, c'_j, d'_k \in H$, it follows that $a' \in a + H$, $b'_i \in b_i + H$, $c'_j \in c_j + H$ and $d'_k \in d_k + H$. The latter implies that there exist $a^\star, b^\star_i, c^\star_j, d^\star_k \in H$ such that $a' \in a + a^\star$, $b'_i \in b_i + b^\star_i$, $c'_j \in c_j + c^\star_j$ and $d'_k \in d_k + d^\star_k$. It is clear that $\overline{y} \in \overline{x} \oplus \overline{z}$ where $\overline{z} = (a^\star, \sum_{i=1}^{p} b^\star_iT_i, \sum_{j=1}^{r} c^\star_jI_j, \sum_{k=1}^{s} d^\star_kF_k) \in RNQ(H)$. Thus, $(RNQ(H), \oplus)$ satisfies the reproduction axiom.

Conversely, let $(RNQ(H), \oplus)$ be a quasi-hypergroup and $a \in H$. Since $0 \in H$, it follows that $\overline{a} = (a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k) \in RNQ(H)$. Having $(RNQ(H), \oplus)$ a quasi-hypergroup implies that $\overline{a} \oplus RNQ(H) =$

$RNQ(H) \oplus \overline{a} = RNQ(H)$. The latter implies that $a + H = H + a = H$. □

Proposition 3.2. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is a semi-hypergroup with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is a semi-hypergroup with identity element 0.*

Proof. Let $(H, +)$ be a a semi-hypergroup and $\overline{x}, \overline{y}, \overline{z} \in RNQ(H)$ with
$\overline{x} = (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k)$, $\overline{y} = (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$ and
$\overline{z} = (a'', \sum_{i=1}^{p} b''_iT_i, \sum_{j=1}^{r} c''_jI_j, \sum_{k=1}^{s} d''_kF_k)$. Having $a + (a' + a'') = (a + a') + a''$, $b_i + (b'_i + b''_i) = (b_i + b'_i) + b''_i$, $c_j + (c'_j + c''_j) = (c_j + c'_j) + c''_j$ and $d_k + (d'_k + d''_k) = (d_k + d'_k) + d''_k$ implies that $\overline{x} \oplus (\overline{y} \oplus \overline{z}) = (\overline{x} \oplus \overline{y}) \oplus \overline{z}$.
Let $(RNQ(H), \oplus)$ be a semi-hypergroup and $a, b, c \in H$. Then $\overline{a}, \overline{b}, \overline{c} \in RNQ(H)$ with $\overline{a} = (a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$,
$\overline{b} = (b, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ and
$\overline{c} = (c, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$. Having $\overline{a} \oplus (\overline{b} \oplus \overline{c}) = (\overline{a} \oplus \overline{b}) \oplus \overline{c}$ implies that $a + (b + c) = (a + b) + c$. □

Proposition 3.3. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is an H_v-semigroup with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is an H_v-semigroup with identity element* 0.

Proof. The proof is similar to that of Proposition 3.2 but instead of equality we have non-empty intersection. □

Theorem 3.4. *Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus)$ is a hypergroup with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is a hypergroup with identity element 0.*

Proof. The proof is direct from Propositions 3.1 and 3.2. □

Theorem 3.5. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is an H_v-group with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is an H_v-group with an identity element 0.*

Proof. The proof follows from Propositions 3.1 and 3.3. □

Theorem 3.6. *Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus)$ is a commutative hypergroup (H_v-group) with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +)$ is a commutative hypergroup (H_v-group) with an identity 0.*

Proof. The proof is straightforward. □

NOTATION 1. *Let $(H, +)$ be a hypergroup (H_v-group) with identity "0" satisfying $0+0 = 0$. Then $(RNQ(H), \oplus)$ is called a refined neutrosophic quadruple hypergroup (refined neutrosophic quadruple H_v-group).*

Corollary 3.7. *Let $(H,+)$ be a hypergroup (H_v-group) containing an identity element 0 with the property that $0+0=0$. Then there are infinite number of refined neutrosophic quadruple hypergroups (H_v-groups).*

Proof. Let $(H,+)$ be a hypergroup (H_v-group). Theorem 3.4 and Theorem 3.5 implies that $(RNQ(H),\oplus)$ is a neutrosophic quadruple hypergroup (H_v-group) with identity $\overline{0}$ and $\overline{0}\oplus\overline{0}=\overline{0}$. Applying Theorem 3.4 and Theorem 3.5 on $(RNQ(H),\oplus)$, we get $RNQ(RNQ(H))$ is a neutrosophic quadruple hypergroup (H_v-group). Continuing on this pattern, we get $RNQ(RNQ(\ldots(RNQ(H))\ldots)$ is a neutrosophic quadruple hypergroup (H_v-group). □

Proposition 3.8. *Let X be any set with a hyperoperation "$+$". Then $RNQ(X)$ is a cyclic refined neutrosophic quadruple hypergroup if and only if X is a cyclic hypergroup with an identity element "$0\in X$" and $0+0=0$.*

Proof. Let X be a cyclic hypergroup with identity "$0\in X$" and $0+0=0$. Then there exist $a\in X$ such that a is a generator of X. It is clear that $\overline{a}$ is a generator of $RNQ(X)$ where $\overline{a}=(a,\sum_{i=1}^{p}aT_i,\sum_{j=1}^{r}aI_j,\sum_{k=1}^{s}aF_k)\in RNQ(X)$.

Let $RNQ(X)$ be a cyclic quadruple hypergroup. Then there exist $\overline{x}\in RNQ(X)$ such that $\overline{x}=(a,\sum_{i=1}^{p}b_iT_i,\sum_{j=1}^{r}c_jI_j,\sum_{k=1}^{s}d_kF_k)$ is a generator of $RNQ(X)$. It is clear that a is a generator of X. □

Example 3.9. Let T_1,T_2 be refinements of T, I_1,F_1 be refinements of I,F respectively, $H_1=\{0,1\}$ and define $(H_1,+_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	H_1

Since $(H_1,\oplus)$ is a commutative hypergroup with an identity 0, it follows by Theorem 3.6 that $(RNQ(H_1),\oplus)$ is a commutative refined neutrosophic quadruple hypergroup with 32 elements and identity $\overline{0}=(0,0T_1+0T_2,0I_1,0F_1)$. Moreover, having $H_1=1+_1 1$ implies that 1 is a generator of $(H_1,+)$ and $(H_1,+)$ is a single-power cyclic hypergroup of period 2. Theorem 3.8 asserts that $(RNQ(H_1),\oplus)$ is a single power cyclic hypergroup of period 2 and the generator element is $(1,1T_1+1T_2,1I_1,1F_1)$.
It is clear that $(1,0T_1+0T_2,1I_1,1F_1)\oplus(1,0T_1+1T_2,0I_1,1F_1)=\{(1,0T_1+1T_2,1I_1,0F_1),(1,0T_1+1T_2,1I_1,1F_1)\}$.

Definition 3.10. Let $(H,+)$ be a hypergroup (H_v-group). A subset X of $RNQ(H)$ with the property that $\overline{0}=(0,\sum_{i=1}^{p}0T_i,\sum_{j=1}^{r}0I_j,\sum_{k=1}^{s}0F_k)\in X$ is called a *refined neutrosophic subhypergroup (H_v-subgroup) of $RNQ(H)$* if there exists $S\subseteq H$ such that $X=RNQ(S)$ and $(X,\oplus)$ is a refined neutrosophic quadruple hypergroup (H_v-group).

Proposition 3.11. *Let $(H,+)$ be a hypergroup (H_v-group) and $S\subseteq H$. A subset $X=RNQ(S)\subseteq RNQ(H)$ is a refined neutrosophic subhypergroup (H_v-subgroup) of $RNQ(H)$ if the following conditions are satisfied:*

1. $\overline{0}=(0,\sum_{i=1}^{p}0T_i,\sum_{j=1}^{r}0I_j,\sum_{k=1}^{s}0F_k)\in X$;
2. $\overline{x}\oplus X=X\oplus\overline{x}=X$ *for all* $\overline{x}\in X$.

Proof. The proof is straightforward. □

Theorem 3.12. *Let $(H, +)$ be a hypergroup (H_v-group) with identity "0", $S \subseteq H$ and $0 \in S$. Then $(RNQ(S), \oplus)$ is a refined neutrosophic quadruple subhypergroup (H_v-subgroup) of $(RNQ(H), \oplus)$ if and only if $(S, +)$ is a subhypergroup (H_v-subgroup) of $(H, +)$.*

Proof. The proof is straightforward by applying Proposition 3.11. □

Example 3.13. Since $(H_1, +_1)$ in Example 3.9 has only two subhypergroups ($\{0\}$ and H_1), it follows by applying Theorem 3.12 that $(RNQ(H_1), \oplus)$ has only two refined neutrosophic quadruple subhypergroups: $(\{\bar{0}\}, \oplus) = (RNQ(\{0\}), \oplus)$ and $(RNQ(H_1), \oplus)$.

Example 3.14. Let $H_2 = \{0, 1, 2, 3\}$ and define $(H_2, +_2)$ as follows:

$+_2$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	$\{0, 2\}$	1
3	3	0	1	2

It is clear that $(H_2, +_2)$ is a commutative H_v-group that has exactly three non-isomorphic H_v-subgroups containing 0: $\{0\}$, $\{0, 2\}$ and H_2. We can deduce by Theorem 3.12 that $(RNQ(H_2), \oplus)$ is a commutative refined neutrosophic quadruple H_v-group and has three non-isomorphic refined neutrosophic quadruple H_v-subgroups: $RNQ(\{0\}) = \{\bar{0}\}$, $RNQ(\{0, 2\})$ and $RNQ(H_2)$.

Proposition 3.15. *Let $(H, +)$ be a hypergroup and $(S, +)$ be a subhypergroup of $(H, +)$ containing 0. Then $RNQ(S) \oplus RNQ(S) = RNQ(S)$.*

Proof. The proof is straightforward. □

Definition 3.16. Let $(RNQ(H), \oplus_1)$ and $(RNQ(J), \oplus_2)$ be refined neutrosophic quadruple hypergroups with $0_H \in H$ and $0_J \in J$. A function $\phi : RNQ(H) \to RNQ(J)$ is called *refined neutosophic homomorphism* if the following conditions are satisfied:

1. $\phi(0_H, \sum_{i=1}^{p} 0_H T_i, \sum_{j=1}^{r} 0_H I_j, \sum_{k=1}^{s} 0_H F_k) = (0_J, \sum_{i=1}^{p} 0_J T_i, \sum_{j=1}^{r} 0_J I, \sum_{k=1}^{s} 0_J F_k)$;
2. $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$ for all $x, y \in RNQ(H)$.

If ϕ is a refined neutrosophic bijective homomorphism then it is called *refined neutrosophic isomorphism* and we write $RNQ(H) \cong RNQ(J)$.

Example 3.17. Let $(H, +)$ be a hypergroup. Then the function $f : RNQ(H) \to RNQ(H)$ is an isomorphism, where $f(x) = x$ for all $x \in RNQ(H)$.

Example 3.18. Let $(H,+)$ be a hypergroup and $0 \in H$ and $f : RNQ(H) \to RNQ(H)$ be defined as follows:

$$f((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F_k)) = (a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k).$$

Then f is a refined neutrosophic homomorphism.

Proposition 3.19. *Let $(H,+_1)$ and $(J,+_2)$ be hypergroups with $0_H \in H, 0_J \in J$. If there exist a homomorphism $f : H \to J$ with $f(0_H) = 0_J$ then there exist a refined neutrosophic homomorphism from $(RNQ(H), \oplus_1)$ to $(RNQ(J), \oplus_2)$.*

Proof. Let $\phi : RNQ(H) \to RNQ(J)$ be defined as follows:

$$\phi((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F_k)) = (f(a), \sum_{i=1}^{p} f(b_i) T_i, \sum_{j=1}^{r} f(c_j) I_j, \sum_{k=1}^{s} f(d_k) F_k).$$

It is clear that ϕ is a refined neutrosophic homomorphism. □

Corollary 3.20. *Let $(H,+_1)$ and $(J,+_2)$ be isomorphic hypergroups with $0_H \in H, 0_J \in J$. Then $(RNQ(H), \oplus_1)$ and $(RNQ(J), \oplus_2)$ are isomorphic refined neutrosophic quadruple hypergroups.*

Proof. The proof is straightforward by using Proposition 3.19. □

Definition 3.21. Let $(H,+)$ be a commutative hypergroup with an identity element "0" and $S \subseteq R$ be a subhypergroup of H. Then $(H/S, +')$ is a hypergroup with: S as an identity element and $S +' S = S$. Here "$+'$" is defined as follows: For all $x, y \in H$,

$$(x + S) +' (y + S) = (x + y) + S.$$

Proposition 3.22. *Let $(S,+)$ be a subhypergroup of a commutative hypergroup $(H,+)$. Then $(RNQ(H/S), \oplus)$ is a hypergroup.*

Proof. Since $(H,+)$ is commutative, it follows that "$+'$" is well defined. The proof follows from having $(H/S, +')$ a hypergroup with S as an identity, $S +' S = S$ and from Theorem 3.4. □

Proposition 3.23. *Let $(S,+)$ be a subhypergroup of a commutative hypergroup $(H,+)$. Then $(RNQ(H/S), \oplus) \cong (RNQ(H)/RNQ(S), \oplus')$.*

Proof. Let $g : RNQ(H)/RNQ(S) \to RNQ(H/S)$ be defined as follows:

$$\begin{aligned} &g((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F_k) \oplus RNQ(S)) \\ &= (a + S, \sum_{i=1}^{p} (b_i + S) T_i, \sum_{j=1}^{r} (c_j + S) I_j, \sum_{k=1}^{s} (d_k + S) F_k). \end{aligned}$$

Then g is a hypergroup isomorphism. This can be proved easily by applying a similar proof to that of Proposition 3.27 that was done by the authors in [9]. □

Example 3.24. Let $H_3 = \{0, 1, 2, 3, 4\}$ and define "+" on H_3 as follows: $x + y = \{x, y\}$ for all $x, y \in H_3$. It is clear that $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$ and H_3 are the only non-isomorphic subhypergroups of H_3. By applying Proposition 3.23, we get $RNQ(H_3/\{0, 1\}) \cong RNQ(H_3)/RNQ(\{0, 1\})$, $RNQ(H_3/\{0, 1, 2\}) \cong RNQ(H_3)/RNQ(\{0, 1, 2\})$ and $RNQ(H_3/\{0, 1, 2, 3\}) \cong RNQ(H_3)/RNQ(\{0, 1, 2, 3\})$.

4 Ordered refined neutrosophic quadruple hypergroups

In this section, an order on refined neutrosophic quadruple hypergroups is defined and some examples and results on refined neutrosophic quadruple partially ordered hypergroups (po-hypergroups) are presented.

A partial order relation on a set X (Poset) is a binary relation "$\leq$" on X which satisfies conditions reflexivity, antisymmetry and transitivity.

Let $(H, \leq)$ be a partial ordered set and define $(RNQ(H), \unlhd)$ as follows:

$$(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \unlhd (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$$

if and only if $a \leq a'$, $b_i \leq b'_i$, $c_j \leq c'_j$ and $d_k \leq d'_k$. It is clear that $(RNQ(H), \unlhd)$ is a partial ordered set.

Definition 4.1. [16] An algebraic hyperstructure $(H, \circ, \leq)$ is called a *partially ordered hypergroup* or *po-hypergroup*, if $(H, \circ)$ is a hypergroup and $\leq$ is a partial order relation on H such that the monotone condition holds as follows:

$$x \leq y \Rightarrow a \circ x \leq a \circ y \text{ for all } a, x, y \in H.$$

Let A, B be non-empty subsets of $(H, \leq)$. The inequality $A \leq B$ means that for any $a \in A$, there exist $b \in B$ such that $a \leq b$.

Theorem 4.2. *Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus, \unlhd)$ is a refined neutrosophic quadruple po-hypergroup with identity element $\overline{0} = (0, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$ if and only if $(H, +, \leq)$ is a po-hypergroup with identity element 0.*

Proof. Let $(H, +, \leq)$ be a po-hypergroup, $\overline{e} = (e, \sum_{i=1}^{p} f_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} h_kF_k) \in RNQ(H)$ and $(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \unlhd (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$. We need to show that:

$$\overline{e} \oplus (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \unlhd \overline{e} \oplus (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k).$$

Having $a \leq a'$, $b_i \leq b'_i$, $c_j \leq c'_j$, $d_k \leq d'_k$ and $(H, +, \leq)$ a po-hypergroup implies that $e + a \leq e + a'$, $f_i + b_i \leq f_i + b'_i$, $g_j + c_j \leq g_j + c'_j$ and $h_k + d_k \leq h_k + d'_k$. Let $\overline{a^\star} = (a^\star, \sum_{i=1}^{p} b^\star_iT_i, \sum_{j=1}^{r} c^\star_jI_j, \sum_{k=1}^{s} d^\star_kF_k) \in \overline{e} \oplus (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k)$. Then $a^\star \in e + a$, $b^\star_i \in f_i + b_i$, $c^\star_j \in g_j + c_j$ and $d^\star_k \in h_k + d_k$. Having $e + a \leq e + a'$, $f_i + b_i \leq f_i + b'_i$, $g_j + c_j \leq g_j + c'_j$ and $h_k + d_k \leq h_k + d'_k$ implies that there exist $a^{\star\prime} \in e + a'$, $b^{\star\prime}_i \in f_i + b'_i$, $c^{\star\prime}_j \in g_j + c'_j$ and $d^{\star\prime}_k \in h_k + d'_k$ such that $a^\star \leq a^{\star\prime}$, $b^\star_i \leq b^{\star\prime}_i$, $c^\star_j \leq c^{\star\prime}_j$ and $d^\star_k \leq d^{\star\prime}_k$. We get now that $\overline{a^\star} \unlhd \overline{a}^{\star\prime}$ where $\overline{a}^{\star\prime} = (a^{\star\prime}, \sum_{i=1}^{p} b^{\star\prime}_iT_i, \sum_{j=1}^{r} c^{\star\prime}_jI_j, \sum_{k=1}^{s} d^{\star\prime}_kF_k)$ and $\overline{a}^{\star\prime} \in \overline{e} \oplus (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$.

Let $a, b, e \in H$ and $a \leq b$. Having $0 \leq 0$ implies that

$$(a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k) \unlhd (b, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k).$$

Since $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup, it follows that for $\overline{e} = (e, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k)$,

$$\overline{e} \oplus (a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k) \trianglelefteq \overline{e} \oplus (b, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F_k).$$

It is clear that $e + a \leq e + b$. □

Corollary 4.3. *Let $(H, +, \leq)$ be a po-hypergroup containing an identity element 0 with the property that $0 + 0 = 0$. Then there is infinite number of refined neutrosophic quadruple po-hypergroups.*

Proof. The proof is starightforward using Theorem 4.2. □

Example 4.4. Let $H_1 = \{0, 1\}$ and define $(H_1, +_1)$ as in Example 3.9. It is clear that $(H_1, +_1, \leq)$ is a po-hypergroup. Here, the partial order relation "$\leq$" is directed to the set $\{(0, 0), (1, 1)\}$. By using Theorem 4.2, we get $(RNQ(H_1), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.

Example 4.5. Let $(H, \leq)$ be any poset and define $(H, +)$ as the biset hypergroup, i.e. $x + y = \{x, y\}$ for all $x, y \in H$. Then $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.

Theorem 4.6. *[16] Let $(H, \circ)$ be a hypergroup such that there exists an element $0 \in H$ and the following conditions hold:*

1. $0 \circ 0 = 0$;
2. $\{0, x\} \subseteq 0 \circ x$ *for all* $x \in H$;
3. *If* $x \circ 0 = y \circ 0$ *then* $x = y$ *for all* $x, y \in H$.

Then there exist a relation "$\leq$" on H such that $(H, \circ, \leq)$ is a po-hypergroup.

Heidari et al. [16], in their proof of Theorem 4.6, defined the binary relation "$\leq$" on H as follows: $x \leq y \iff x \in y \circ 0,$ for all $x, y \in H$.

Corollary 4.7. *Let $(H, +)$ be a hypergroup satisfying conditions of Theorem 4.6. Then there exist a relation "$\trianglelefteq$" on $RNQ(H)$ such that $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.*

Proof. The proof follows from Theorems 4.2 and 4.6. □

Example 4.8. Let $H = \{0, x, y\}$ and define "$+$" by the following table:

$+$	0	x	y
0	0	$\{0, x\}$	H
c	$\{0, x\}$	H	H
y	H	H	H

Then $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup. Here the partial order relation "$\leq$" is directed to the set $\{(0, 0), (x, x), (y, y), (x, y), (0, x), (0, y)\}$ and $\trianglelefteq$ is defined in the usual way on $RNQ(H)$.

Definition 4.9. Let $(RNQ(H), \oplus_1, \trianglelefteq_1)$ and $(RNQ(J), \oplus_2, \trianglelefteq_2)$ be refined neutrosophic quadruple po-hypergroups. A function $\phi : RNQ(H) \to RNQ(J)$ is called an *ordered refined neutosophic homomorphism* if the following conditions hold:

1. $\phi(0_H, \sum\limits_{i=1}^{p} 0_H T_i, \sum\limits_{j=1}^{r} p0_H I_j, \sum\limits_{k=1}^{s} 0_H F_k) = (0_J, \sum\limits_{i=1}^{p} 0_J T_i, \sum\limits_{j=1}^{r} 0_J I, \sum\limits_{k=1}^{s} 0_J F_k)$;

2. $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$ for all $x, y \in RNQ(H)$;

3. if $x \trianglelefteq_1 y$ then $\phi(x) \trianglelefteq_2 \phi(y)$ for all $x, y \in RNQ(H)$.

If ϕ is an ordered refined neutrosophic homomorphism and is bijective then it is called an *ordered refined neutrosophic isomorphism* and we say $RNQ(H)$ and $RNQ(J)$ are isomorphic refined neutrosophic quadruple po-hypergroups.

Example 4.10. Let $(H, +, \leq)$ be a po-hypergroup. Then $f : RNQ(H) \to RNQ(H)$ is an ordered refined neutrosophic isomorphism, where $f(x) = x$ for all $x \in RNQ(H)$.

Example 4.11. Let $(H, +, \leq)$ be a po-hypergroup, $0 \in H$ and $f : RNQ(H) \to RNQ(H)$ be defined as follows:

$$f((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F)) = (a, \sum_{i=1}^{p} 0T_i, \sum_{j=1}^{r} 0I_j, \sum_{k=1}^{s} 0F).$$

Then f is an ordered refined neutrosophic homomorphism.

Example 4.12. Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be po-hypergroups, $0_H \in H, 0_J \in J$ and $g : H \to J$ be an ordered homomorphism. Then $f : RNQ(H) \to RNQ(J)$ is an ordered refined neutrosophic homomorphism. Here, f is defind as follows:

$$f((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F)) = (g(a), \sum_{i=1}^{p} 0_J T_i, \sum_{j=1}^{r} 0_J I_j, \sum_{k=1}^{s} 0_J F_k).$$

Proposition 4.13. *Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be po-hypergroups with $0_H \in H, 0_J \in J$. If there exist an ordered homomorphism $f : H \to J$ with $f(0_H) = 0_J$ then there exist an ordered refined neutrosophic homomorphism from $(RNQ(H), \oplus_1, \trianglelefteq_1)$ to $(RNQ(J), \oplus_2, \trianglelefteq_2)$.*

Proof. Let $\phi : RNQ(H) \to RNQ(J)$ be defined as follows:

$$\phi((a, \sum_{i=1}^{p} b_i T_i, \sum_{j=1}^{r} c_j I_j, \sum_{k=1}^{s} d_k F_k)) = (f(a), \sum_{i=1}^{p} f(b_i) T_i, \sum_{j=1}^{r} f(c_j) I_j, \sum_{k=1}^{s} f(d_k) F_k).$$

It is clear that ϕ is an ordered refined neutrosophic homomorphism. □

Corollary 4.14. *Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be isomorphic po-hypergroup with $0_H \in H, 0_J \in J$. Then $(RNQ(H), \oplus_1, \trianglelefteq_1)$ and $(RNQ(J), \oplus_2, \trianglelefteq_2)$ are isomorphic refined neutrosophic quadruple po-hypergroups.*

Proof. The proof is straightforward by using Proposition 4.13. □

5 Fundamental group of refined neutrosophic quadruple hypergroups

This section presents the study of fundamental relation on refined neutrosophic quadruple hypergroups and finds their fundamental refined quadruple neutrosophic groups.

Theorem 5.1. *Let $(G,+)$ be a groupoid. Then $(RNQ(G),\oplus)$ is a group with identity element $\overline{0}=(0,\sum_{i=1}^{p}0T_i,\sum_{j=1}^{r}0I_j,\sum_{k=1}^{s}0F_k)$ if and only if $(G,+)$ is a group with identity element 0.*

Proof. It is clear that $\overline{0}=(0,\sum_{i=1}^{p}0T_i,\sum_{j=1}^{r}0I_j,\sum_{k=1}^{s}0F_k)$ is the identity of $(RNQ(G),\oplus)$ if and only if 0 is the identity of G. Let $\overline{x}=(a,\sum_{i=1}^{p}b_iT_i,\sum_{j=1}^{r}c_jI_j,\sum_{k=1}^{s}d_kF_k)\in RNQ(G)$. Then the inverse $-\overline{x}=(a,\sum_{i=1}^{p}(-b_i)T_i,\sum_{j=1}^{r}(-c_j)I_j,\sum_{k=1}^{s}(-d_k)F_k)$ of $\overline{x}$ exists if and only if the inverse $-y$ of y exists in G for all $y\in G$. The proof of $(RNQ(G),\oplus)$ is binary closed if and only if $(G,+)$ is binary closed is similar to that of Proposition 3.1. And the proof of $(RNQ(G),\oplus)$ is associative if and only if $(G,+)$ is associative is similar to that of Proposition 3.2. □

NOTATION 2. *Let $(G,+)$ be a group with identity element "0". Then $(RNQ(G),\oplus)$ is called refined neutrosophic quadruple group.*

Proposition 5.2. *Let G,G' be isomorphic groups. Then $RNQ(G)$ and $RNQ(G')$ are isomorphic neutrosophic quadruple groups.*

Definition 5.3. [14] For all $n>1$, we define the relation β_n on a semihypergroup $(H,\circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1,\ldots,a_n \text{ in } H \text{ such that } \{x,y\}\subseteq\prod_{i=1}^{n}a_i$$

Here, $\prod_{i=1}^{n}a_i=a_1\circ a_2\ldots\circ a_n$. And we set $\beta=\bigcup_{n\geq 1}\beta_n$, where $\beta_1=\{(x,x)\mid x\in H\}$ is the diagonal relation on H.

Koskas [18] introduced this relation as an important tool to connect hypergroups with groups. And due to it's importance in connecting algebraic hyperstructures with algebraic structures, different researchers studied it on various hypergroups and some extended this definition to cover other types of hyperstructures.
Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β. Then $\beta^\star$ is called the *fundamental equivalence relation* on H and it is the smallest strongly regular relation on H. If H is a hypergroup then $\beta=\beta^\star$ and H/β^* is called the *fundamental group*.

Throughout this section, β and $\beta^\star$ are the relation on H and β_N and $\beta_N^\star$ are the relations on $RNQ(H)$.

Theorem 5.4. *Let $(H,+)$ be a hypergroup with identity element element "0" and $0+0=0$ and let a,a', b_i,b_i', c_j,c_j', $d_k,d_k'\in H$. Then $(a,\sum_{i=1}^{p}b_iT_i,\sum_{j=1}^{r}c_jI_j,\sum_{k=1}^{s}d_kF_k)\beta_N(a',\sum_{i=1}^{p}b_i'T_i,\sum_{j=1}^{r}c_j'I_j,\sum_{k=1}^{s}d_k'F_k)$ if and only if $a\beta a'$, $b_i\beta b_i'$, $c_j\beta c_j'$ and $d_k\beta d_k'$.*

Proof. Let $(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k)\beta_N(a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$. Then there exist $(a_t, \sum_{i=1}^{p} b_{it}T_i, \sum_{j=1}^{r} c_{jt}I_j, \sum_{k=1}^{s} d_{kt}F_k)$ with $t = 1, \ldots, n$ such that

$$\{(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k), (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)\}$$

is a subset of

$$(a_1, \sum_{i=1}^{p} b_{i1}T_i, \sum_{j=1}^{r} c_{j1}I_j, \sum_{k=1}^{s} d_kF_k) \oplus \ldots \oplus (a_n, \sum_{i=1}^{p} b_{in}T_i, \sum_{j=1}^{r} c_{jn}I_j, \sum_{k=1}^{s} d_{kn}F_k).$$

The latter implies that $a, a' \in a_1 + \ldots + a_n$, $b_i, b'_i \in b_{i1} + \ldots + b_{in}$, $c_j, c'_j \in c_{j1} + \ldots + c_{jn}$and $d_k, d'_k \in d_{k1} + \ldots + d_{kn}$. Thus, $a\beta a'$, $b_i\beta b'_i$, $c_j\beta c'_j$ and $d_k\beta d'_k$.

Conversely, let $a\beta a'$, $b_i\beta b'_i$, $c_j\beta c'_j$ and $d_k\beta d'_k$. Then there exist $t_1, t_2, t_3, t_4 \in \mathbb{N}$ and $x_1, \ldots, x_{t_1}, y_{i1}, \ldots, y_{it_2}$, $z_{j1}, \ldots, z_{jt_3}, w_{k1}, \ldots, w_{kt_4} \in H$ such that $a, a' \in x_1 + \ldots + x_{t_1}$, $b_i, b'_i \in y_{i1} + \ldots + y_{it_2}$, $c_j, c'_j \in z_{j1} + \ldots + z_{jt_3}$ and $d_k, d'_k \in w_{k1} + \ldots + w_{kt_4}$. By setting $t = \max\{t_1, t_2, t_3, t_4\}$ and $x_m = 0$ for $t_1 < m \leq t$, $y_{im} = 0$ for $t_2 < m \leq t$, $z_{jm} = 0$ for $t_3 < m \leq t$ and $w_{km} = 0$ for $t_4 < m \leq t$ and using the fact that $e \in 0 + e \cap e + 0$ for all $e \in H$, we get $a, a' \in x_1 + \ldots + x_{t_1}$, $b_i, b'_i \in y_{i1} + \ldots + y_{it}$, $c_j, c'_j \in z_{j1} + \ldots + z_{jt}$ and $d_k, d'_k \in w_{k1} + \ldots + w_{kt}$. The latter implies that $\{(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k), (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)\}$ is a subset of $(x_1, \sum_{i=1}^{p} y_{i1}T_i, \sum_{j=1}^{r} z_{j1}I_j, \sum_{k=1}^{s} w_{k1}F_k) \oplus \ldots \oplus (x_t, \sum_{i=1}^{p} y_{it}T_i, \sum_{j=1}^{r} z_{jt}I_j, \sum_{k=1}^{s} w_{kt}F_k)$. Thus,

$$(a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k)\beta_N(a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k).$$

□

Theorem 5.5. *Let $(H, +)$ be a hypergroup with identity "0" and $0 + 0 = 0$. Then $RNQ(H)/\beta_N \cong RNQ(H/\beta)$.*

Proof. Let $\phi : RNQ(H)/\beta_N \to RNQ(H/\beta)$ be defined as

$$\phi(\beta_N((a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k))) = (\beta(a), \sum_{i=1}^{p} \beta(b_i)T_i, \sum_{j=1}^{r} \beta(c_j)I_j, \sum_{k=1}^{s} \beta(d_k)F_k).$$

Theorem 5.4 asserts that ϕ is well-defined and one-to-one. Also, it is clear that ϕ is onto. We need to show that ϕ is a group homomorphism. Let $\overline{a} = (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k)$ and $\overline{a'} = (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k)$. Since $\beta_N(\overline{a}) \boxplus' \beta_N(\overline{a'}) = \beta_N(\overline{x})$ where $\overline{x} = (x, \sum_{i=1}^{p} y_iT_i, \sum_{j=1}^{r} z_jI_j, \sum_{k=1}^{s} w_kF_k) \in (a, \sum_{i=1}^{p} b_iT_i, \sum_{j=1}^{r} c_jI_j, \sum_{k=1}^{s} d_kF_k) \oplus (a', \sum_{i=1}^{p} b'_iT_i, \sum_{j=1}^{r} c'_jI_j, \sum_{k=1}^{s} d'_kF_k) =$

$(a+a', \sum_{i=1}^{p}(b_i+b'_i)T_i, \sum_{j=1}^{r}(c_j+c'_j)I_j, \sum_{k=1}^{s}(d_k+d'_k)F_k)$, it follows that

$$\phi(\beta_N(\overline{a}) \boxplus \beta_N(\overline{a'})) = \phi(\overline{x}) = (\beta(x), \sum_{i=1}^{p}\beta(y_i)T_i, \sum_{j=1}^{r}\beta(z_j)I_j, \sum_{k=1}^{s}\beta(w_k)F_k).$$

Having $\beta(x) = \beta(a) \oplus \beta(a')$, $\beta(y_i) = \beta(b_i) \oplus \beta(b'_i)$, $\beta(z_j) = \beta(c_j) \oplus \beta(c'_j)$ and $\beta(w_k) = \beta(d_k) \boxplus \beta(d'_k)$ imply that $\phi(\beta_N(\overline{a}) \boxplus' \beta_N(\overline{a'})) = \phi(\beta_N(\overline{a})) \oplus' \phi(\beta_N \overline{a'})$. □

Corollary 5.6. *Let $(H, +)$ be a hypergroup with identity element "0" and $0 + 0 = 0$. If G is the fundamental group of H (up to isomorphism) then $RNQ(G)$ is the fundamental group of $RNQ(H)$ (up to isomorphism).*

Proof. The proof follows from Proposition 5.2 and Theorem 5.5. □

Corollary 5.7. *Let $(H, +)$ be a hypergroup with identity element "0" and $0 + 0 = 0$. If H has a trivial fundamental group then $RNQ(H)$ has a trivial fundamental group.*

Proof. The proof is straightforward by applying Corollary 5.6. □

Theorem 5.8. *[8] Every single power cyclic hypergroup has a trivial fundamental group.*

Corollary 5.9. *Let $(H, +)$ be a single power cyclic hypergroup with $0 \in H$ and $0 + 0 = 0$. Then $RNQ(H)$ has a trivial fundamental group.*

Proof. The proof follows from Corollary 5.7 and Theorem 5.8. □

6 Conclusion

This paper contributed to the study of neutrosophic hyperstructures by introducing refined neutrosophic quadruple hypergroups (po-hypergroups) and determining their fundamental refined neutrosophic quadruple groups. Several interesting results related to these new hypergroups were obtained. For future work, it will be interesting to study new properties of other types of refined neutrosophic quadruple hyperstructures.

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Received: April 26, 2019. Accepted: June 04, 2019.