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Simplified Neutrosophic Linguistic Normalized Weighted Bonferroni Mean Operator and Its Application to Multi-Criteria Decision-Making Problems

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Abstract. The main purpose of this paper is to provide a method of multi-criteria decision-making that combines simplified neutrosophic linguistic sets and normalized Bonferroni mean operator to address the situations where the criterion values take the form of simplified neutrosophic linguistic numbers and the criterion weights are known. Firstly, the new operations and comparison method for simplified neutrosophic linguistic numbers are defined and some linguistic scale functions are employed. Subsequently, a Bonferroni mean operator and a normalized weighted Bonferroni mean operator of simplified neutrosophic linguistic numbers are developed, in which some desirable characteristics and special cases with respect to the parameters p and q in Bonferroni mean operator are studied. Then, based on the simplified neutrosophic linguistic normalized weighted Bonferroni mean operator, a multi-criteria decision-making approach is proposed. Finally, an illustrative example is given and a comparison analysis is conducted between the proposed approach and other existing method to demonstrate the effectiveness and feasibility of the developed approach.

1. Introduction

In practice, multi-criteria decision-making (MCDM) methods are widely used to rank alternatives or select the optimal one with respect to several concerned criteria. However, in some cases, it is difficult for decision-makers to explicitly express preference in solving MCDM problems with uncertain or incomplete information. Under these circumstances, fuzzy sets (FSs), proposed by Zadeh [1], where each element has a membership degree represented by a real number in $[0,1]$, are regarded as a significant tool for solving MCDM problems [2,3]. Sometimes, FSs cannot handle the cases where the membership degree is uncertain and hard to be defined by a crisp value. Therefore, interval-valued fuzzy sets (IVFSs) were proposed [4] to capture the uncertainty of membership degree. Generally, if the membership degree is defined, then the non-membership degree can be calculated by default. In order to deal with the uncertainty of non-membership degree, Atanassov [5] introduced the intuitionistic fuzzy sets (IFSs) which is an extension

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of Zadehs FSs, and the corresponding intuitionistic fuzzy logic [6] was proposed. IFSs consider both the membership degree and the non-membership degree simultaneously. So IFSs and intuitionistic fuzzy logic are more flexible in handling information containing uncertainty and incompleteness than traditional FSs. Currently, IFSs have been widely applied in solving MCDM problems [7-9]. Moreover, intuitionistic fuzzy numbers [10], triangular intuitionistic fuzzy numbers [11,12] and intuitionistic trapezoidal fuzzy numbers [13], which are derived form IFSs, are also significant tools to cope with fuzzy and uncertain information. In reality, the degree of membership and non-membership in IFSs may be expressed as interval numbers instead of specific numbers. Hence, interval-valued intuitionistic fuzzy sets (IVIFSs) [14] were proposed, which is an extension of FSs and IVFSs. In recent years, MCDM problems with evaluation information derived from IVIFSs have attracted much attention of researchers [15-19], in which, aggregation operators, prospect score function and possibility degree method are involved.

Although FSs and IFSs have been developed and generalized, they cannot deal with all sorts of fuzziness in real problems. Such as problems that are too complex or ill-defined to be solved by quantitative expressions. The linguistic variable is an effective tool because the use of linguistic information enhances the reliability and flexibility of classical decision models [20,21]. Resent years, linguistic variables have been studied in depth and numerous MCDM methods associated with other theories have been developed. Intuitionistic linguistic sets (ILSs), which combine IFSs and linguistic variables are applied to solve multi-criteria group decision-making problems [22]. ILSs and their extensions [23-25] can describe both a linguistic variable and an intuitionistic fuzzy number, in which the former can provide a qualitative assessment value, whilst the later can define the confidence degree for the given evaluation value. Hesitant fuzzy linguistic sets (HFLSs), which are based on linguistic term sets and hesitant fuzzy sets are used to express decision-makers hesitance that exists in giving the associated membership degrees of one linguistic term [26,27]. Hesitant fuzzy linguistic term sets (HFLTSs), which describe decision-makers preferences by using several linguistic terms are more suitable than traditional fuzzy linguistic sets in expressions [28]. In addition, a method based on the cloud model, which can correctly depict the uncertainty of qualitative concept has been successfully utilized [24,29,30]. A method based on the 2-tuple linguistic information model [31,32], which can effectively avoid the information distortion and has hitherto occurred in linguistic information processing [33]. It is clear that all of those proposals of linguistic variables, promising as they are still need to be refined from a formal point of view. In a word, linguistic variables can only express the uncertain information but not the incomplete or inconsistent one. For example, when a paper is sent to a reviewer, he or she gives the statement that the paper is good. And he or she may say the possibility that the statement is true is 60%, the one that the statement is false is 50% and the degree that he or she is not sure is 20%. This issue cannot be handled effectively with FSs and IFSs. Therefore, some new theories are required.

Smarandache [34,35] coined the neutrosophic logic and neutrosophic sets (NSs). Rivieccio [36] pointed out that a NS is a set where each element of the universe has a truth-membership, indeterminacy-membership and falsity-membership, respectively, and it lies in the non-standard unit interval $]0^-, 1^+[$. In recent years, the NS theory has found practical applications in various fields, such as semantic web services [37], mineral prospectivity prediction [38], image processing [39-41], granular computing [42], medical diagnosis [43] and information fusion [44]. Since it is hard to apply NSs in real scientific and engineering situations, single-valued neutrosophic sets (SVNSs) [45] and interval neutrosophic sets (INSs) [37] were introduced, which are two particular instances of NSs. Subsequently, studies had been conducted in various aspects, which concentrated mainly on defining operations and aggregation operators [46-48], correlation coefficients [49,50], entropy measures [51,52] and similarity measures [53] to cope with opinions of experts or decision-makers in MCDM problems. To overcome the drawback of using linguistic variables associated with FSs and IFSs, the single-valued neutrosophic trapezoid linguistic sets (SVNTLSs), which is based on linguistic term sets, SVNSs and trapezoid fuzzy numbers, were proposed [54]. In addition, the interval neutrosophic linguistic sets (INLSs) , which is based on linguistic term sets and INSs, were introduced [55]. However, the operations proposed in [55] have some limitations that will be discussed in Subsection 3.2. And this paper will define new operations for simplified neutrosophic linguistic sets (SNLSs), which is a reduced form of INLSs. As for the aforementioned example, it can be expressed as $\langle h_5, (0.6, 0.2, 0.5)\rangle$ by means of SNLSs. SNLSs have enabled great progress in describing linguistic information and to some extent may be considered to be an innovative construct.

In general, aggregation operators are important tools for addressing information fusion in MCDM problems. The Bonferroni mean (BM) operator, proposed by Bonferroni [56] has a desirable characteristic that can capture the interrelationship of input arguments. Recently, Xu and Yager [57] extended BM operator to IFSs. Zhou and He developed a intuitionistic fuzzy normalized weighted bonferroni mean (IFNWBM) operator [58], and Liu and Wang introduced a single-valued neutrosophic normalized weighted Bonferroni meam (SVNNWBM) operator [59]. Beliakov and James [60] proposed an extending generalized BM operator to Atanassov orthopairs in MCDM situations. Wei et al. [61] developed an uncertain linguistic Bonferroni mean (ULBM) operator and an uncertain linguistic geometric Bonferroni mean (ULGBM) operator to aggregate the uncertain linguistic information. Obviously, BM operator has been extended to IFSs, SVNSs, Atanassov orthopairs and uncertain linguistic variables. Motivated by INLSs [55] and IFNWBM [58], this paper is to develop a simplified neutrosophic linguistic Bonferroni mean (SNLBM) operator and a simplified neutrosophic linguistic normalized weighted Bonferroni mean (SNLNWBM) operator. In addition, a MCDM approach based on SNLNWBM operator is proposed.

The rest of the paper is organized as follows. In Section 2, linguistic term sets as well as the concepts of NSs and SNSs are briefly reviewed. In Section 3, new operations of simplified neutrosophic linguistic numbers (SNLNs) are provided and a method for comparing SNLNs is proposed based on the linguistic scale function. In Section 4, the traditional BM is extended to the simplified neutrosophic linguistic environment. A SNLBM operator and a SNLNWBM operator are developed and some properties and special cases are discussed. In addition, a MCDM approach based on the SNLNWBM operator is introduced. In Section 5, an illustrative example is given based on the proposed approach and the influence of parameters p and q in SNLNWBM operator on decision-making results is analyzed. In addition, a comparison analysis between the proposed approach and the existing method is conducted. Some summary remarks are given in Section 6.

2. Preliminaries

In this section, some basic concepts and definitions related to SNLSs, including linguistic term sets, linguistic scale functions, NSs and simplified neutrosophic sets (SNSs) are introduced, which will be utilized in the latter analysis.

2.1. The linguistic term set and its extension

Suppose that $H = \{h_0, h_1, h_2, \cdots, h_{2t}\}$ is a finite and totally ordered discrete term set, where t is a nonnegative real number. It is required that h_i and h_j must satisfy the following characteristics [62,63].

(1) The set is orderd: $h_i < h_j$ if and only if $i < j$.

(2) Negation operator: $neg(h_i) = h_{(2t-i)}$.

To preserve all the given information, the discrete linguistic label $H = \{h_0, h_1, h_2, \cdots, h_{2t}\}$ is extended to a continuous label $H = \{h_i | 0 \le i \le L\}$, in which $h_i < h_j$ if and only if $i < j$, and $L(L > 2t)$ is a sufficiently large positive integer. If $h_i \in H$, then h_i is called the original linguistic term; otherwise h_i is called the virtual linguistic term [64].

For any linguistic variables $h_i, h_j \in H$, the operations are defined as below [65].

(1) $\lambda h_i = h_{\lambda \times i}$;

(2) $h_i \oplus h_j = h_{i+j}$;

(3) $h_i \otimes h_j = h_{i \times j}$;

(4) $(h_i)^\lambda = h_{i^\lambda}$.

2.2. Linguisitic scale function

To use data more efficiently and to express the semantics more flexibly, linguistic scale functions assign different semantic values to linguistic terms under different situations [27]. They are preferable in practice because these functions are flexible and can give more deterministic results according to different semantics. For the linguistic term h_i in a linguistic set H, where $H = \{h_i | i = 0, 1, 2, \cdots, 2t\}$, the relationship between the element h_i and its subscript i is strictly monotonically increasing [65]. Those new functions are provided as below.

Definition 1 [27]. If $\theta_i \in [0, 1]$ is a numeric value, then the linguistic scale function f that conducts the mapping from h_i to θ_i $(i = 0, 1, 2, \cdots, 2t)$ is defined as follows:

$$f : h_i \to \theta_i \ (i = 0, 1, 2, \cdots, 2t),$$

where $0 < \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2t}$.

Clearly, the function f is strictly monotonically increasing with respect to subscript i. The symbol θ_i $(i = 0, 1, 2, \cdots, 2t)$ reflects the preference of the decision-makers when they are using the linguistic term $h_i \in H$ $(i = 0, 1, 2, \cdots, 2t)$. Therefore, the function/value in fact denotes the semantics of the linguistic terms.

(1) $f_1(h_i) = \theta_i = \frac{i}{2t}$ $(i = 0, 1, 2, \cdots, 2t)$.

The evaluation scale of the linguistic information given above is divided on average.

(2) $f_2(h_i) = \theta_i = \begin{cases} \frac{\alpha^t - \alpha^{t-i}}{2\alpha^t - 2} \ (i = 0, 1, 2, \cdots, t), \\ \frac{\alpha^t + \alpha^{i-t} - 2}{2\alpha^t - 2} \ (i = t + 1, t + 2, \cdots, 2t). \end{cases}$

With the extension from the middle of the given linguistic term set to both ends, the absolute deviation between adjacent linguistic subscripts also increases.

(3) $f_3(h_i) = \theta_i = \begin{cases} \frac{t^\beta - (t-i)^\beta}{2t^\beta} \ (i = 0, 1, 2, \cdots, t), \\ \frac{t^\gamma + (i-t)^\gamma}{2t^\gamma} \ (i = t + 1, t + 2, \cdots, 2t). \end{cases}$

With the extension from the middle of the given linguistic term set to both ends, the absolute deviation between adjacent linguistic subscripts will decrease.

To preserve all the given information and facilitate the calculation, the above function can be expanded to $f^* : \overline{H} \to R^+$ $(R^+ = \{r | r \geq 0, r \in R\})$, which satisfies $f^*(h_i) = \theta_i$, and is a strictly monotonically increasing and continuous function. Therefore, the mapping from $\overline{H}$ to R^+ is one-to-one because of its monotonicity, and the inverse function of f^* exists and is denoted by f^{*-1}.

Example 1. Assume $t = 3$. Then a linguistic term set H can be given as $H = \{h_0, h_1, h_2, \cdots, h_3\}$ ={very poor, poor, slightly poor, fair, slightly good, good, very good}. And the following results can be obtained.

(1) If $f_1(h_i) = \theta_i = \frac{i}{6}$ $(i = 0, 1, 2, \cdots, 6)$, then $f_1^{-1}(\theta_i) = h_{6\theta_i}$ $(\theta_i \in [0, 1])$.

(2) If $f_2(h_i) = \theta_i = \begin{cases} \frac{\alpha^3 - \alpha^{3-i}}{2\alpha^t - 2} (i = 0, 1, 2, 3) \\ \frac{\alpha^3 + \alpha^{i-3} - 2}{2\alpha^3 - 2} (i = 4, 5, 6) \end{cases}$, then $f_2^{-1}(\theta_i) = \begin{cases} h_{3 - \log_\alpha(\alpha^3 - (2\alpha^3 - 2)\theta_i)} (\theta_i \in [0, 0.5]) \\ h_{3 + \log_\alpha((2\alpha^3 - 2)\theta_i - \alpha^3 + 2)} (\theta_i \in (0.5, 1]) \end{cases}$.

(3) If $f_3(h_i) = \theta_i = \begin{cases} \frac{3^\beta - (3-i)^\beta}{2 \times 3^\beta} (i = 0, 1, 2, 3) \\ \frac{3^\gamma + (i-3)^\gamma}{2 \times 3^\gamma} (i = 4, 5, 6) \end{cases}$, then $f_3^{-1}(\theta_i) = \begin{cases} h_{3 - (3^\beta - 2 \times 3^\beta \theta_i)^{\frac{1}{\beta}}} (\theta_i \in [0, 0.5]) \\ h_{3 + (2 \times 3^\gamma \theta_i - 3^\gamma)^{\frac{1}{\gamma}}} (\theta_i \in (0.5, 1]) \end{cases}$.

2.3. NSs and SNSs

Definition 2 [34]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in X is characterized by a truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. $t_A(x)$, $i_A(x)$ and $f_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is, $t_A(x) : X \to]0^-, 1^+[$, $i_A(x) : X \to]0^-, 1^+[$, and $f_A(x) : X \to]0^-, 1^+[$. There is no restriction on the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$, so $0^- \leq \sup t_A(x) + \sup \imath_A(x) + \sup f_A(x) \leq 3^+$.

Since it is hard to apply NSs to practical problems, Ye [66] reduced NSs of nonstandard interval numbers into a kind of SNSs of standard interval numbers.

Definition 3 [36,66]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in is characterized by $t_A(x)$, $i_A(x)$ and $f_A(x)$, which are single subintervals/subsets in the real standard $[0, 1]$, that is, $t_A(x) : X \to [0, 1]$, $i_A(x) : X \to [0, 1]$, and $f_A(x) : X \to [0, 1]$. And the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$ satisfies the condition $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$. Then a simplification of A is denoted by $A = \{(x, t_A(x), i_A(x), f_A(x)) | x \in X\}$, which is called a SNS. It is a subclass of NSs. If $\|X\| = 1$, a SNS will be degenerated to a simplified neutrosophic number (SNN), denoted by $A = (t_A, i_A, f_A)$.

3. SNLNs and their Operation

SNLNs, as the elements and special cases of SNLSs, are of great significance in information evaluation. In this section, the advantages and applications of SNLNs are firstly introduced, which is on the basis of linguistic tern sets and SNSs. Then, new operations and comparison rules of SNLNs are presented.

3.1. SNLSs

Definition 4. Let X be a space of points (objects) with a generic element in X, denoted by x and $H = \{h_0, h_1, h_2, \cdots, h_{2t}\}$ be a finite and totally ordered discrete term set, where t is a nonnegative real number. A SNLS A in X is characterized as $A = \left\{\left\langle x, h_{\theta(x)}, (t(x), i(x), f(x))\right\rangle | x \in X\right\}$, where $h_{\theta(x)} \in H$, $t(x) \in [0,1]$, $i(x) \in [0,1]$ and $f(x) \in [0,1]$, with the condition $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$ for any $x \in X$. And $t_A(x)$, $i_A(x)$ and $f_A(x)$ represent, respectively, the degree of truth-membership, indeterminacy-membership and falsity-membership of the element x in X to the linguistic term $h_{\theta(x)}$. In addition, if $\|X\| = 1$, a SNLS will be degenerated to a SNLN, denoted by $A = \langle h_\theta, (t, i, f)\rangle$. And A will be degenerated to a linguistic term if $t = 1$, $i = 0$ and $f = 0$.

3.2. Operations of SNLNs

Definition 5 [55]. Let $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ and $a_j = \left\langle h_{\theta_j}, (t_j, i_j, f_j)\right\rangle$ be any two SNLNs and $\lambda \geq 0$. Then the following operations of SNLNs can be defined.

(1) $a_i \oplus a_j = \left\langle h_{\theta_i} \oplus h_{\theta_j}, (t_i + t_j - t_i t_j, i_i i_j, f_i f_j)\right\rangle$;

(2) $a_i \otimes a_j = \left\langle h_{\theta_i} \otimes h_{\theta_j}, (t_i t_j, i_i + i_j - i_i i_j, f_i + f_j - f_i f_j)\right\rangle$;

(3) $\lambda a_i = \left\langle \lambda h_{\theta_i}, \left(1 - (1 - t_i)^\lambda, i_i^\lambda, f_i^\lambda\right)\right\rangle$;

(4) $a_i^\lambda = \left\langle (h_{\theta_i})^\lambda, \left(t_i^\lambda, 1 - (1 - i_i)^\lambda, 1 - (1 - f_i)^\lambda\right)\right\rangle$.

However, the operations presented in Definition 5 have some obvious limitations:

(1) All operations are carried out directly on the basis of the subscripts of linguistic terms, which cannot reveal the critical differences of final results under various semantic situations.

(2) The two parts of SNLNs are processed separately in the additive operation, which ignores the correlation of them. In some situations, it might be irrational. This is shown in the following example.

Example 2. In the example of the performance evaluation of a house under two parameters, in which c_1 stands for the parameter beautiful and c_2 stands for the parameter wooden, and the two parameters are equally important. Suppose that $a_1 = \langle h_5, (1, 0, 0)\rangle$ represents the statement that all the decision-makers think the house is good, and $a_2 = \langle h_5, (0, 1, 1)\rangle$ represents the statement that non of the decision-makers think the house is good. Then the comprehensive performance evaluation can be calculated by using Definition 5, $a_{12} = 0.5a_1 \oplus 0.5a_2 = \langle h_5, (1, 0, 0)\rangle$.

Obviously, the result above are contradictory and unreasonable. Since $(1, 0, 0)$ is the maximum of SNNs and $(0, 1, 1)$ is the minimum of SNNs, $a_1 = \langle h_5, (1, 0, 0)\rangle$ is superior to $a_2 = \langle h_5, (0, 1, 1)\rangle$. It stands to reason that the comprehensive performance evaluation should be between a_1 and a_2. However, according to the result, $a_{12} = a_1$, which is apparently against the logical thinking. In this way, it would be more reasonable if the two parts of SNLNs were taken into account at the same time.

In order to overcome the existing limitations given above, a new definition of operations of SNLNs based on linguistic scale function are defined as below.

Definition 6. Let $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ and $a_j = \langle h_{\theta_j}, (t_j, i_j, f_j)\rangle$ be two SNLNs, f^* be a linguistic scale function and $\lambda \geq 0$. Then the following operations of SNLNs can be defined.

(1) $a_i \oplus a_j = \left\langle f^{*-1}\left(f^*(h_{\theta_i}) + f^*(h_{\theta_j})\right), \left(\frac{f^*(h_{\theta_i})t_i + f^*(h_{\theta_j})t_j}{f^*(h_{\theta_i}) + f^*(h_{\theta_j})}, \frac{f^*(h_{\theta_i})i_i + f^*(h_{\theta_j})i_j}{f^*(h_{\theta_i}) + f^*(h_{\theta_j})}, \frac{f^*(h_{\theta_i})f_i + f^*(h_{\theta_j})f_j}{f^*(h_{\theta_i}) + f^*(h_{\theta_j})}\right)\right\rangle$;

(2) $a_i \otimes a_j = \left\langle f^{*-1}\left(f^*(h_{\theta_i}) f^*(h_{\theta_j})\right), \left(t_i t_j, i_i + i_j - i_i i_j, f_i + f_j - f_i f_j\right)\right\rangle$;

(3) $\lambda a_i = \left\langle f^{*-1}\left(\lambda f^*(h_{\theta_i})\right), (t_i, i_i, f_i)\right\rangle$;

(4) $a_i^\lambda = \left\langle f^{*-1}\left(\left(f^*(h_{\theta_i})\right)^\lambda\right), \left(t_i^\lambda, 1 - (1 - i_i)^\lambda, 1 - (1 - f_i)^\lambda\right)\right\rangle$;

(5) $neg(a_i) = \left\langle f^{*-1}\left(f^*(h_{2t}) - f^*(h_{\theta_i})\right), (f_i, 1 - i_i, t_i)\right\rangle$.

According to Definition 1, it is known that f^* is a mapping from the linguistic term h_i to the numeric value θ_i and f^{*-1} is a mapping from θ_i to h_i. So the first parts of (1-5) are linguistic terms. It is obvious that the second parts of (1-5) are SNNs. In a word, the results obtained by Definition 6 are also SNLNs.

The operations defined above are on the basis of the linguistic scale function, which can present different results when a different linguistic function f^* is employed. Thus, decision-makers can flexibly select f^* depending on their own personal preferences and the actual semantic situations. Furthermore, the new addition operation of SNLNs is more reliable and reasonable. Because the final result can reflect the close combination with each element of the original SNLNs.

In practical applications, $a_i \oplus a_j$, $a_i \otimes a_j$, λa_i and a_i^λ necessarily appear in defining basic operations, but their results have no practical meaning. Only in the aggregation process do $a_i \oplus a_j$ combine with λa_i or $a_i \otimes a_j$ combine with a_i^λ make sense.

Example 3. Assume $H = \{h_0, h_1, h_2, \cdots, h_6\}$, $a_1 = \langle h_2, (0.6, 0.4, 0.2)\rangle$, $a_2 = \langle h_4, (0.5, 0.6, 0.2)\rangle$ and $\lambda = 2$. Then the following results can be calculated.

If $\alpha = 1.4$ and $f_2^*(x) = \begin{cases} \frac{\alpha^t - \alpha^{t-x}}{2\alpha^t - 2} (0 \le x \le t) \\ \frac{\alpha^t - \alpha^{x-t} - 2}{2\alpha^t - 2} (t < x \le 2t) \end{cases}$, then

(1) $a_1 \oplus a_2 = \langle h_6, (0.5385, 0.5229, 0.2)\rangle$;
(2) $a_1 \otimes a_2 = \langle h_{1.0646}, (0.3, 0.76, 0.36)\rangle$;
(3) $2a_1 = \langle h_{4.9754}, (0.6, 0.4, 0.2)\rangle$;
(4) $a_1^2 = \langle h_{0.6215}, (0.36, 0.64, 0.36)\rangle$;
(5) $neg(a_1) = \langle h_4, (0.2, 0.6, 0.6)\rangle$.

As for the issue discussed in Example 2, the comprehensive performance evaluation result can be amended by using Definition 6, $a'_{12} = 0.5a_1 \oplus 0.5a_2 = \langle h_5, (0.5, 0.5, 0.5)\rangle$. It is shown that a'_{12} is inferior to a_1 and superior to a_2, namely, $a_1 > a'_{12} > a_2$, which can normally describe the comprehensive evaluation information and be preferable in practice.

It can be easily proved that all the results given above are also SNLNs. In terms of the corresponding operations of SNLNs, the following theorem can also be easily proved.

Theorem 1. Let $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$, $a_j = \left\langle h_{\theta_j}, (t_j, i_j, f_j)\right\rangle$ and $a_k = \langle h_{\theta_k}, (t_k, i_k, f_k)\rangle$ be three arbitrary SNLNs and f^* be a linguistic scale function. Then the following properties are true.

(1) $a_i \oplus a_j = a_j \oplus a_i$;
(2) $(a_i \oplus a_j) \oplus a_k = a_i \oplus (a_j \oplus a_k)$;
(3) $a_i \otimes a_j = a_j \otimes a_i$;
(4) $(a_i \otimes a_j) \otimes a_k = a_i \otimes (a_j \otimes a_k)$;
(5) $\lambda a_i \oplus \lambda a_j = \lambda(a_i \oplus a_j)$, $\lambda \ge 0$;
(6) $\lambda_1 a_i \oplus \lambda_2 a_i = (\lambda_1 + \lambda_2) a_i$, $\lambda_1, \lambda_2 \ge 0$;
(7) $(a_i \oplus a_j)^\lambda = a_i^\lambda \oplus a_j^\lambda$, $\lambda \ge 0$;
(8) $a_i^{\lambda_1} \otimes a_i^{\lambda_2} = a_i^{\lambda_1 + \lambda_2}$, $\lambda_1, \lambda_2 \ge 0$.

3.3. Comparison method for SNLNs

Based on the score function and accuracy function of ILSs, the score function, accuracy function and certainty function of a SNLN, which are significant indexes for ranking alternatives in decision-making problems are defined.

Definition 7. Let $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ be an SNLN and f^* be a linguistic scale function. Then the score function, accuracy function and certainty function for a_i can be defined, respectively, as below.

(1) $S(a_i) = f^*(h_{\theta_i})(t_i + 1 - i_i + 1 - f_i)$;
(2) $A(a_i) = f^*(h_{\theta_i})(t_i - f_i)$;
(3) $C(a_i) = f^*(h_{\theta_i}) t_i$.

For a SNLN a_i, if the truth-membership t_i with respect to the linguistic term h_{θ_i} is bigger and the determinacy-membership i_i and the falsity-membership f_i corresponding to h_{θ_i} are smaller, then a_i is greater and the reliability of h_{θ_i} is higher. For the accuracy function $A(a_i)$, if the difference between t_i and f_i with respect to h_{θ_i} is bigger, then the statement is more affirmative, i.e., the accuracy of a_i is higher. As for the certainty function $C(a_i)$, the certainty of a_i positively depends on the value of t_i. Obviously, the bigger $S(a_i)$, $A(a_i)$ and $C(a_i)$ are, the greater the corresponding a_i is.

Example 4. Use the data of Example 3. Then the following results can be calculated.

(1) $S(a_1) = 0.7706$, $S(a_2) = 1.0450$;

(2) $A(a_1) = 0.1541$, $A(a_2) = 0.1844$;

(3) $C(a_1) = 0.2312$, $C(a_2) = 0.3074$.

On the basis of Definition 7, the method to compare SNLNs can be defined as below.

Definition 8. Let $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ and $a_j = \left\langle h_{\theta_j}, (t_j, i_j, f_j)\right\rangle$ be two SNLNs and f^* be a linguistic scale function. Then the comparison method can be defined.

(1) If $S(a_i > S(a_j)$, then $a_i > a_j$;

(2) If $S(a_i = S(a_j)$ and $A(a_i > A(a_j)$, then $a_i > a_j$;

(3) If $S(a_i = S(a_j)$, $A(a_i = A(a_j)$ and $C(a_i > C(a_j)$, then $a_i > a_j$;

(4) If $S(a_i = S(a_j)$, $A(a_i = A(a_j)$ and $C(a_i = C(a_j)$, then $a_i = a_j$.

Example 5. Assume $H = \{h_0, h_1, h_2, \cdots, h_6\}$. Then the following results can be calculated.

If $\alpha = 1.4$ and $f_2^*(x) = \begin{cases} \frac{\alpha^t - \alpha^{t-x}}{2\alpha^t - 2} \ (0 \le x \le t) \\ \frac{\alpha^t + \alpha^{x-t} - 2}{2\alpha^t - 2} \ (t < x \le 2t) \end{cases}$, then

(1) For two SNLNs $a_1 = \langle h_4, (0.5, 0.6, 0.2)\rangle$ and $a_2 = \langle h_2, (0.6, 0.4, 0.2)\rangle$, according to Definition 8, $S(a_1) = 1.0450 > S(a_2) = 0.7706$. Therefore, $a_1 > a_2$.

(2) For two SNLNs $a_1 = \langle h_4, (0.8, 0.3, 0.2)\rangle$ and $a_2 = \langle h_4, (0.6, 0.2, 0.1)\rangle$, according to Definition 8, $S(a_1) = S(a_2) = 1.4138$ and $A(a_1) = 0.3688 > A(a_2) = 0.3074$. Therefore, $a_1 > a_2$.

(3) For two SNLNs $a_1 = \langle h_2, (0.8, 0.3, 0.2)\rangle$ and $a_2 = \langle h_2, (0.7, 0.3, 0.1)\rangle$, according to Definition 8, $S(a_1) = S(a_2) = 0.8862$, $A(a_1) = A(a_2) = 0.2312$ and $C(a_1) = 0.3082 > C(a_2) = 0.2312$. Therefore, $a_1 > a_2$.

4. SNLNWBM Operator and its Application in MCDM Problems

In this section, the traditional BM operator is extended to deal with simplified neutrosophic linguisitic information. A SNLBM operator and a SNLNWBM operator are proposed. Further, some desirable characteristics and special cases with respect to the parameters p and q in BM operator are discussed. In addition, a MCDM approach is developed, which is on the basis of the SNLNWBM operator.

4.1. BM and NWBM operator

Definition 9 [56]. Let $p, q \ge 0$ and a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative real numbers. If

$$B^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \left(a_i^p a_j^q \right) \right)^{\frac{1}{p+q}},$$

then $B^{p,q}$ is called the BM operator.

Obviously, the BM operator has the following properties [56].

(1) $B^{p,q}(0, 0, \cdots, 0) = 0$.

(2) $B^{p,q}(a, a, \cdots, a) = a$ if $a_i = a$, for all i.

(3) $B^{p,q}(a'_1, a'_2, \cdots, a'_n) \ge B^{p,q}(a_1, a_2, \cdots, a_n)$, that is, $B^{p,q}$ is monotonic if $a'_i \ge a_i$, for all i.

(4) $\min_i(a_i) \le B^{p,q}(a_1, a_2, \cdots, a_n) \le \max_i(a_i)$.

Especially, if $q = 0$, then the BM operator reduces to the following [56]:

$$B^{p,0}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} a_i^p \right)^{\frac{1}{p}} = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}},$$

which is a generalized mean operator. Particularly, the following cases hold.

(1) If $p = 1$ and $q = 0$, then the BM operator reduces to the well-known average mean operator.

$$B^{1,0}(a_1, a_2, \cdots, a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

(2) If $p \to 0$ and $q = 0$, then the BM operator reduces to the geometric mean operator.

$$\lim_{p\to 0} B^{p,0}(a_1, a_2, \cdots, a_n) = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$

The BM operator discussed above can only consider the interrelationship between a_i and a_j, and ignore their own weights of the aggregation arguments. Nevertheless, in many situations, the importance of each argument should be taken into consideration. So the NWBM operator that can allow the weights of aggregation arguments will be introduced.

Definition 10 [58]. Let $p, q \geq 0$, a_i $(i = 1, 2, \cdots, n)$ be a collection of nonnegative real numbers, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weight vector of a_i $(i = 1, 2, \cdots, n)$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. If

$$NWB_{\omega}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} a_i^p a_j^q\right)\right)^{\frac{1}{p+q}},$$

then $NWB_{\omega}^{p,q}$ is called the NWBM operator.

The NWBM operator can satisfy the properties of reducibility, commutativity, idempotency, monotonicity and boundedness and reflect the interrelationship of input arguments.

4.2. SNLBM and SNLNWBM operator

In this subsection, the traditional BM and NWBM operator are extended to accommodate the situations where the input arguments are SNLNs. Furthermore, a SNLBM operator and a SNLNWBM operator are developed and some desirable properties and special cases are analyzed.

Definition 11. Let $p, q \geq 0$, $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of SNLNs and $SNLB^{p,q} : \Omega^n \to \Omega$. If

$$SNLB^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} (a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}} \quad (1)$$

where Ω is the set of all SNLNs, then $SNLB^{p,q}$ is called the SNLBM operator.

In the following, the SNLNWBM operator will be fully introduced.

Definition 12. Let $p, q \geq 0$, $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of SNLNs and $SNLNWB_{\omega}^{p,q}$: $\Omega^n \to \Omega$. If

$$SNLNWB_{\omega}^{p,q}(a_1, a_2, \cdots, a_n) \left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i} (a_i^p \otimes a_j^p)\right)^{\frac{1}{p+q}}, \quad (2)$$

where Ω is the set of all SNLNs and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight vector of a_i, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then $SNLNWB_{\omega}^{p,q}$ is called the SNLNWBM operator.

According to the operations of SNLNs in Definition 6, the following results can be obtained.

Theorem 2. Let $p, q \geq 0$, $a_i = \langle h_{\theta_i}, (t_i, i_i, f_i)\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of SNLNs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$

be the weight vector of a_i, $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the aggregated result by using Eq. (2) is also a SNLN.

$$SNLNWB_{\omega}^{p,q}(a_1, a_2, \cdots, a_n) = \left\langle f^{*-1}\left(\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\right)^{\frac{1}{p+q}}\right), \left(\left(\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q t_i^p t_j^q}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}}\right),\right.$$
$$1-\left(1-\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_i)^p(1-i_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}},$$
$$\left.1-\left(1-\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_i)^p(1-f_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}}\right\rangle. \tag{3}$$

In the following, Eq. (3) will be proved by using the mathematical induction on n.

Proof. (1) Firstly, the following equation need to be proved.

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q) = \left\langle f^{*-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\right), \left(\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q t_i^p t_j^q}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right),\right.$$
$$\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_i)^p(1-i_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}, \tag{4}$$
$$\left.\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_i)^p(1-f_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right\rangle.$$

(a) When $n = 2$, the following equation can be calculated.

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{2} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q) = \frac{\omega_1\omega_2}{1-\omega_1}(a_1^p \otimes a_2^q) \oplus \frac{\omega_2\omega_1}{1-\omega_2}(a_2^p \otimes a_1^q)$$

$$= \left\langle f^{*-1}\left(\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q\right), \left(t_1^p t_2^q, 1-(1-i_1)^p(1-i_2)^q, 1-(1-f_1)^p(1-f_2)^q\right)\right\rangle$$
$$\oplus \left\langle f^{*-1}\left(\frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q\right), \left(t_2^p t_1^q, 1-(1-i_2)^p(1-i_1)^q, 1-(1-f_2)^p(1-f_1)^q\right)\right\rangle$$
$$= \left\langle f^{*-1}\left(\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q\right),\right.$$
$$\left(\frac{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q t_1^p t_2^q + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q t_2^p t_1^q}{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q},\right.$$
$$\frac{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q\left(1-(1-i_1)^p(1-i_2)^q\right) + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q\left(1-(1-i_2)^p(1-i_1)^q\right)}{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q},$$
$$\left.\left.\frac{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q\left(1-(1-f_1)^p(1-f_2)^q\right) + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q\left(1-(1-f_2)^p(1-f_1)^q\right)}{\frac{\omega_1\omega_2}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_2})\right)^q + \frac{\omega_2\omega_1}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_1})\right)^q}\right)\right\rangle.$$

That is, when $n = 2$, Eq. (4) is right.
(b) Suppose that when $n = k$, Eq. (4) is right. That is,

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q) = \left\langle f^{*-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q\right), \left(\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q t_i^p t_j^q}{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q},\right.\right.$$
$$\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_i)^p(1-i_j)^q\right)}{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q}, \tag{5}$$
$$\left.\left.\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_i)^p(1-f_j)^q\right)}{\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_j})\right)^q}\right)\right\rangle.$$

Then, when $n = k + 1$, the following result can be calculated.

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q) = \bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q)\oplus\bigoplus_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^p\otimes a_{k+1}^q)\oplus\bigoplus_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(a_{k+1}^p\otimes a_j^q). \tag{6}$$

Firstly, the following equation need to be proved.

$$\bigoplus_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^p\otimes a_{k+1}^q) = \left\langle f^{*-1}\left(\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_{k+1}})\right)^q\right), \left(\frac{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_{k+1}})\right)^q t_i^p t_{k+1}^q}{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}\left(f^*(h_{\theta_i})\right)^p\left(f^*(h_{\theta_{k+1}})\right)^q},\right.\right.$$

$$\left.\frac{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{k+1}}))^q\,(1-(1-i_i)^p(1-i_{k+1})^q)}{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{k+1}}))^q},\ \frac{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{k+1}}))^q\,(1-(1-f_i)^p(1-f_{k+1})^q)}{\sum_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{k+1}}))^q}\right)\right\rangle. \tag{7}$$

In the following, Eq. (7) will be proved by using the mathematical induction on k.
(i) When $k=2$, the following result can be calculated.

$$\begin{aligned}
&\bigoplus_{i=1}^{2}\frac{\omega_i\omega_3}{1-\omega_i}(a_i^p\otimes a_3^q)=\frac{\omega_1\omega_3}{1-\omega_1}(a_1^p\otimes a_3^q)\oplus\frac{\omega_2\omega_3}{1-\omega_2}(a_2^p\otimes a_3^q)\\
&=\left\langle f^{*-1}\left(\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q\right),\left(t_1^pt_3^q,1-(1-i_1)^p(1-i_3)^q,1-(1-f_1)^p(1-f_3)^q\right)\right\rangle\\
&\oplus\left\langle f^{*-1}\left(\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q\right),\left(t_2^pt_3^q,1-(1-i_2)^p(1-i_3)^q,1-(1-f_2)^p(1-f_3)^q\right)\right\rangle\\
&=\left\langle f^{*-1}\left(\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q\right),\right.\\
&\left(\frac{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q t_1^pt_3^q+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q t_2^pt_3^q}{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q},\right.\\
&\frac{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q(1-(1-i_1)^p(1-i_3)^q)+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q(1-(1-i_2)^p(1-i_3)^q)}{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q},\\
&\left.\left.\frac{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q(1-(1-f_1)^p(1-f_3)^q)+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q(1-(1-f_2)^p(1-f_3)^q)}{\frac{\omega_1\omega_3}{1-\omega_1}\left(f^*(h_{\theta_1})\right)^p\left(f^*(h_{\theta_3})\right)^q+\frac{\omega_2\omega_3}{1-\omega_2}\left(f^*(h_{\theta_2})\right)^p\left(f^*(h_{\theta_3})\right)^q}\right)\right\rangle.
\end{aligned}$$

That is, when $k=2$, Eq. (7) is right.
(ii) Suppose that when $k=l$, Eq. (7) is right. That is,

$$\begin{aligned}
&\bigoplus_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(a_i^p\otimes a_{l+1}^q)=\left\langle f^{*-1}\left(\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q\right),\left(\frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q t_i^pt_{l+1}^q}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q},\right.\right.\\
&\left.\left.\frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q\,(1-(1-i_i)^p(1-i_{l+1})^q)}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q},\ \frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q\,(1-(1-f_i)^p(1-f_{l+1})^q)}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+1}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+1}}))^q}\right)\right\rangle.
\end{aligned}$$

Then, when $k=l+1$, the following result can be calculated.

$$\bigoplus_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(a_i^p\otimes a_{l+2}^q)=\bigoplus_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(a_i^p\otimes a_{l+2}^q)\oplus\frac{\omega_{l+1}\omega_{l+2}}{1-\omega_{l+1}}(a_{l+1}^p\otimes a_{l+2}^q)$$

$$=\left\langle f^{*-1}\left(\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\right),\left(\frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q t_i^p t_{l+2}^q}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q},\right.\right.$$

$$\frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\,(1-(1-i_i)^p(1-i_{l+2})^q)}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q},$$

$$\left.\left.\frac{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\,(1-(1-f_i)^p(1-f_{l+2})^q)}{\sum_{i=1}^{l}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q}\right)\right\rangle$$

$$\oplus\left\langle f^{*-1}\left(\frac{\omega_{l+1}\omega_{l+2}}{1-\omega_{l+1}}(f^*(h_{\theta_{l+1}}))^p(f^*(h_{\theta_{l+2}}))^q\right),\left(t_{l+1}^p t_{l+2}^q,1-(1-i_{l+1})^p(1-i_{l+2})^q,1-(1-f_{l+1})^p(1-f_{l+2})^q\right)\right\rangle$$

$$=\left\langle f^{*-1}\left(\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\right),\left(\frac{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q t_i^p t_{l+2}^q}{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q},\right.\right.$$

$$\frac{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\,(1-(1-i_i)^p(1-i_{l+2})^q)}{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q},$$

$$\left.\left.\frac{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q\,(1-(1-f_i)^p(1-f_{l+2})^q)}{\sum_{i=1}^{l+1}\frac{\omega_i\omega_{l+2}}{1-\omega_i}(f^*(h_{\theta_i}))^p(f^*(h_{\theta_{l+2}}))^q}\right)\right\rangle.$$

That is, when $k = l + 1$, Eq. (7) is right.
(iii) So, for all , Eq. (7) is right.
Similarly, the following equation can be proved.

$$\bigoplus_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\left(a_{k+1}^p\otimes a_j^q\right)=\left\langle f^{*-1}\left(\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q\right),\left(\frac{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q t_{k+1}^p t_j^q}{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q},\right.\right.$$

$$\frac{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_{k+1})^p(1-i_j)^q\right)}{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q},$$

$$\left.\frac{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_{k+1})^p(1-f_j)^q\right)}{\sum_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(f^*(h_{\theta_{k+1}}))^p\left(f^*(h_{\theta_j})\right)^q}\right)\right\rangle. \tag{8}$$

So, by using Eqs. (5), (7) and (8), Eq. (6) can be transformed as

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^q\right)=\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^q\right)\oplus\bigoplus_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}\left(a_i^p\otimes a_{k+1}^q\right)\oplus\bigoplus_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\left(a_{k+1}^p\otimes a_j^q\right)$$

$$=\left\langle f^{*-1}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\right),\left(\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q t_i^p t_j^q}{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q},\right.\right.$$

$$\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_i)^p(1-i_j)^q\right)}{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q},$$

$$\left.\left.\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_i)^p(1-f_j)^q\right)}{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)\right\rangle.$$

So, when $n=k+1$, Eq. (4) is right. Thus, Eq. (4) is right for all n.
(2) Then, By using Eq. (4), Eq. (3) can be proved right.

$$SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^q\right)\right)^{\frac{1}{p+q}}$$

$$=\left\langle f^{*-1}\left(\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\right)^{\frac{1}{p+q}}\right),\left(\left(\frac{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q t_i^p t_j^q}{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}},\right.\right.$$

$$1-\left(1-\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-i_i)^p(1-i_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}},$$

$$\left.1-\left(1-\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q\left(1-(1-f_i)^p(1-f_j)^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{\theta_i}))^p\left(f^*(h_{\theta_j})\right)^q}\right)^{\frac{1}{p+q}}\right\rangle.$$

The traditional NWBM operator has the properties of reducibility, commutativity, idempotency, monotonicity and boundedness. In the following, that the SNLNWBM operator satisfies those properties will be proved.

Theorem 3 (reducibility). Let $\omega=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$. Then $SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)=SNLB^{p,q}(a_1,a_2,\cdots,a_n)$.

Proof. Since $\omega=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, according to Eq. (2), the following equation can be obtained.

$$\begin{aligned}
SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)&=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^q\right)\right)^{\frac{1}{p+q}}\\
&=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\frac{1}{n^2}}{1-\frac{1}{n}}\left(a_i^p\otimes a_j^q\right)\right)^{\frac{1}{p+q}}\\
&=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{n(n-1)}\left(a_i^p\otimes a_j^q\right)\right)^{\frac{1}{p+q}}\\
&=\left(\frac{1}{n(n-1)}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^p\otimes a_j^q\right)\right)^{\frac{1}{p+q}}\\
&=SNLB^{p,q}(a_1,a_2,\cdots,a_n).
\end{aligned}$$

Theorem 4 (commutativity). Let $(a_1',a_2',\cdots,a_n')$ be any permutation of $(a_1,a_2,\cdots,a_n)$. If $p=q$, then $SNLNWB_{\omega}^{p,q}(a_1',a_2',\cdots,a_n')=SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)$.

Proof. Since $(a_1',a_2',\cdots,a_n')$ is any permutation of $(a_1,a_2,\cdots,a_n)$, according to Eq. (7) in Theorem 1, the following equation can be obtained.

$$\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left((a'_i)^p\otimes(a'_j)^p\right)\right)^{\frac{1}{2p}}=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^p\right)\right)^{\frac{1}{2p}}\Rightarrow\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left((a'_i\otimes a'_j)^p\right)\right)^{\frac{1}{2p}}=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left((a_i\otimes a_j)^p\right)\right)^{\frac{1}{2p}}.$$

Thus, $SNLNWB_{\omega}^{p,q}(a_1',a_2',\cdots,a_n')=SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)$.

Theorem 5 (idempotency). Let $a_i=a$ $(i=1,2,\cdots,n)$. Then $SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)=a$.

Proof. Since $a_i = a$ for all i, according to Eq. (8) in Theorem 1 , the following equation can be obtained.

$$SNLNWB_{\omega}^{p,q}(a_1, a_2, \cdots, a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(a_i^p \otimes a_j^q \right) \right)^{\frac{1}{p+q}} = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} a^{p+q} \right)^{\frac{1}{p+q}} = a \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \right)^{\frac{1}{p+q}} = a.$$

Theorem 6 (monotonicity). Let $a_i = \langle h_{a_i}, (t_{a_i}, i_{a_i}, f_{a_i}) \rangle$ $(i = 1, 2, \cdots, n)$ and $b_i = \langle h_{b_i}, (t_{b_i}, i_{b_i}, f_{b_i}) \rangle$ be two collections of SNLNs. If $h_{a_i} \geq h_{b_i}$, $t_{a_i} \geq t_{b_i}$, $i_{a_i} \leq i_{b_i}$ and $f_{a_i} \leq f_{b_i}$ for all i, then $SNLNWB_{\omega}^{p,q}(a_1, a_2, \cdots, a_n) \geq SNLNWB_{\omega}^{p,q}(b_1, b_2, \cdots, b_n)$.

Proof.

(1) For linguistic term part

Since $h_{a_i} \geq h_{b_i}$ for all i, and both f^* and f^{*-1} are strictly monotonically increasing and continuous functions, then the following inequalities can be obtained.

$$\begin{aligned}
&(f^*(h_{a_i}))^p \geq (f^*(h_{b_i}))^p \; and \; \left(f^*(h_{a_j})\right)^q \geq \left(f^*(h_{b_j})\right)^q \\
&\Rightarrow (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \geq (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q \\
&\Rightarrow \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \geq \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q \\
&\Rightarrow \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \geq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q \\
&\Rightarrow \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \right)^{\frac{1}{p+q}} \geq \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q \right)^{\frac{1}{p+q}} \\
&\Rightarrow f^{*-1}\left(\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \right)^{\frac{1}{p+q}} \right) \geqslant f^{*-1}\left(\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q \right)^{\frac{1}{p+q}} \right).
\end{aligned}$$

(2) For truth-membership part, indeterminacy-membership part and falsity-membership part, they can be proved by using the mathematical induction on n.

$$\left(\frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q t_{a_i}^p t_{a_j}^q}{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q} \right)^{\frac{1}{p+q}} \geq \left(\frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q t_{b_i}^p t_{b_j}^q}{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{b_i}))^p \left(f^*(h_{b_j})\right)^q} \right)^{\frac{1}{p+q}};$$

$$1 - \left(\frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q \left(1 - (1-i_{a_i})^p (1-i_{a_j})^q\right)}{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (f^*(h_{a_i}))^p \left(f^*(h_{a_j})\right)^q} \right)^{\frac{1}{p+q}} \leq$$

$$1-\left(\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{b_i}))^p\left(f^*(h_{b_j})\right)^q\left(1-(1-i_{b_i})^p(1-i_{b_j})^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{b_i}))^p\left(f^*(h_{b_j})\right)^q}\right)^{\frac{1}{p+q}};$$

$$1-\left(\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{a_i}))^p\left(f^*(h_{a_j})\right)^q\left(1-(1-f_{a_i})^p(1-f_{a_j})^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{a_i}))^p\left(f^*(h_{a_j})\right)^q}\right)^{\frac{1}{p+q}}\leq$$

$$1-\left(\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{b_i}))^p\left(f^*(h_{b_j})\right)^q\left(1-(1-f_{b_i})^p(1-f_{b_j})^q\right)}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}(f^*(h_{b_i}))^p\left(f^*(h_{b_j})\right)^q}\right)^{\frac{1}{p+q}}.$$

(3) Comparing $SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)$ with $SNLNWB_{\omega}^{p,q}(b_1,b_2,\cdots,b_n)$

Let $a=\langle h_a,(t_a,i_a,f_a)\rangle = SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)$ and $b=\langle h_b,(t_b,i_b,f_b)\rangle = SNLNWB_{\omega}^{p,q}(b_1,b_2,\cdots,b_n)$. Because $h_a\geq h_b$, $t_a\geq t_b$, $i_a\leq i_b$ and $f_a\leqslant f_b$, then $a\geq b$.

Thus, $SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)\geq SNLNWB_{\omega}^{p,q}(b_1,b_2,\cdots,b_n)$.

Theorem 7 (boundedness). Let $a_i=\langle h_{a_i},(t_{a_i},i_{a_i},f_{a_i})\rangle\,(0,1,2,\cdots,n)$ be a collection of SNLNs, and $a=\left\langle \min_i\{h_{a_i}\},\left(\min_i\{t_{a_i}\},\max_i\{i_{a_i}\},\max_i\{f_{a_i}\}\right)\right\rangle$, $b=\left\langle \max_i\{h_{a_i}\},\left(\max_i\{t_{a_i}\},\min_i\{i_{a_i}\},\min_i\{f_{a_i}\}\right)\right\rangle$.

Then $a\leq SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)\leq b$.

Proof. Since $h_{a_i}\geq \min_i\{h_{a_i}\}$, $t_{a_i}\geq \min_i\{t_{a_i}\}$, $i_{a_i}\leq \max_i\{i_{a_i}\}$ and $f_{a_i}\leq \max_i\{f_{a_i}\}$, then based on Theorem 5 and Theorem 6, the following inequality can be obtained.

$$a=SNLNWB_{\omega}^{p,q}(a,a,\cdots,a)\leq SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n).$$

Similarly, the following inequality can also be obtained.

$$SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)\leq SNLNWB_{\omega}^{p,q}(b,b,\cdots,b)=b.$$

Thus, $a\leq SNLNWB_{\omega}^{p,q}(a_1,a_2,\cdots,a_n)\leq b$.

In the following, some special cases of the SNLNWBM operator will be discussed.

(1) If $q=0$, then Eq. (3) reduces to the simplified neutrosophic linguistic generalized weighted average (SNLGWA) operator as below.

$$SNLNWB_{\omega}^{p,0}(a_1,a_2,\cdots,a_n)=\left(\bigoplus_{i=1}^{n}\omega_i a_i^p\right)^{\frac{1}{p}}=\left\langle f^{*-1}\left(\left(\sum_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p\right)^{\frac{1}{p}}\right),\left(\left(\frac{\sum\limits_{i,j=1}^{n}\omega_i(f^*(h_{\theta_i}))^p t_i^p}{\sum\limits_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p}\right)^{\frac{1}{p}},\right.\right.$$

$$\left.\left.1-\left(1-\frac{\sum\limits_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p\,(1-(1-i_i)^p)}{\sum\limits_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p}\right)^{\frac{1}{p}},\ 1-\left(1-\frac{\sum\limits_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p\,(1-(1-f_i)^p)}{\sum\limits_{i=1}^{n}\omega_i(f^*(h_{\theta_i}))^p}\right)^{\frac{1}{p}}\right)\right\rangle.$$

(2) If $p = 1$ and $q = 0$, then Eq. (3) reduces to the simplified neutrosophic linguistic weighted arithmetic average (SNLWAA) operator.

$$SNLNWB_{\omega}^{1,0}(a_1, a_2, \cdots, a_n) = \bigoplus_{i=1}^{n} \omega_i a_i$$
$$= \left\langle f^{*-1}\left(\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i})\right), \left(\frac{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i}) t_i}{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i})}, \frac{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i}) i_i}{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i})}, \frac{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i}) f_i}{\sum_{i=1}^{n} \omega_i f^*(h_{\theta_i})}\right)\right\rangle.$$

(3) If $p \to 0$ and $q = 0$, then Eq. (3) reduces to the simplified neutrosophic linguistic weighted geometric average (SNLWGA) operator.

$$\lim_{p\to 0} SNLNWB_{\omega}^{p,0}(a_1, a_2, \cdots, a_n) = \bigotimes_{i=1}^{n} a_i^{\omega_i}$$
$$= \left\langle f^{*-1}\left(\prod_{i=1}^{n} (f^*(h_{\theta_i}))^{\omega_i}\right), \left(\prod_{i=1}^{n} t_i^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - i_i)^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - f_i)^{\omega_i}\right)\right\rangle.$$

(4) If $p \to +\infty$ and $q = 0$, then Eq. (3) reduces to the following form.

$$\lim_{p\to +\infty} SNLNWB_{\omega}^{p,0}(a_1, a_2, \cdots, a_n)$$
$$= \left\langle f^{*-1}\left(\max_i \{f^*(h_{\theta_i})\}\right), \left(\max_i\{t_i\}, \min_i \{i_i\}, \min_i\{f_i\}\right)\right\rangle.$$

4.3. MCDM approach based on SNLNWBM operator

In this subsection, the SNLNWBM operator will be applied to solve MCDM problems with simplified neutrosophic linguistic information.

For MCDM problems with simplified neutrosophic linguistic information, let $A = \{a_1, a_2, \cdots, a_m\}$ be a discrete set consisting of m alternatives and let $C = \{c_1, c_2, \cdots, c_n\}$ be a set consisting of n criteria. Assume that the weight of criterion c_j $(j = 1, 2, \cdots, n)$ is ω_j, where $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. And the weight vector of criteria can be expressed as $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$. Let $B = \left[b_{ij}\right]_{m\times n} = \left[\left\langle h_{\theta_{ij}}, \left(t_{ij}, i_{ij}, f_{ij}\right)\right\rangle\right]_{m\times n}$ be the decision matrix, where $\left\langle h_{\theta_{ij}}, \left(t_{ij}, i_{ij}, f_{ij}\right)\right\rangle$ takes the form of the SNLN and represents the assessment information of each alternative a_i $(i = 1, 2, \cdots, m)$ on criterion c_j $(j = 1, 2, \cdots, n)$ with respect to the linguistic term $h_{\theta_{ij}}$. Then the ranking of alternatives is required.

In a word, the main procedures of the above MCDM approach are listed as below.

Step 1. Normalize the decision matrix.

In general, there are two types of criterion called maximizing criteria and minimizing criteria. In order to uniform criterion types, minimizing criteria need to be transformed into maximizing criteria. Suppose the standardized matrix is expressed as $R = \left[r_{ij}\right]_{m\times n}$. The original decision matrix $B = \left[b_{ij}\right]_{m\times n}$ can be converted to $R = \left[r_{ij}\right]_{m\times n}$ by using the negation operator in Definition 6. For convenience, the normalized criterion values of a_i $(i = 1, 2, \cdots, m)$ with respect to c_j $(j = 1, 2, \cdots, n)$ are also expressed as $\left\langle h_{\theta_{ij}}, \left(t_{ij}, i_{ij}, f_{ij}\right)\right\rangle$.

Step 2. Calculate the comprehensive evaluation values for each alternative.

Use Eq. (3) to calculate the comprehensive evaluation values, denoted by r_i $(i = 1, 2, \cdots, m)$ for each alternative a_i.

Step 3. Calculate the score values, accuracy values and certainty values of r_i $(i = 1, 2, \cdots, m)$.

Use equations described in Definition 7 to calculate the score values, accuracy values and certainty values, denoted by $S(r_i)$, $A(r_i)$ and $C(r_i)$ of r_i $(i = 1, 2, \cdots, m)$, respectively.

Step 4. Rank all the alternatives and select the best one(s).

Use the comparison method described in Definition 8 to rank all the alternatives and select the best one(s) according to $S(r_i)$, $A(r_i)$ and $C(r_i)$ $(i = 1, 2, \cdots, m)$.

5. Illustrative Example

In this section, an investment appraisal project is employed to demonstrate the application of the proposed decision-making approach, as well as the validity and effectiveness of the proposed approach.

5.1. Background

The following case is adapted from [29].

ABC Nonferrous Metals Co. Ltd. is a large state-owned company whose main business is producing and selling nonferrous metals. It is also the largest manufacturer of multi-species nonferrous metals in China, with the exception of aluminum. To expand its main business the company is always engaged in overseas investment, and a department which consists of executive managers and several experts in the field has been established specifically to make decisions on global mineral investment.

Recently, the overseas investment department decided to select a pool of alternatives from several foreign countries based on preliminary surveys. After thorough investigation, five countries (alternatives) are taken into consideration, i.e., $\{a_1, a_2, \cdots, a_5\}$. There are many factors that affect the investment environment and four factors are considered based on the experience of the department personnel, including c_1: resources (such as the suitability of the minerals and their exploration); c_2: politics and policy (such as corruption and political risks); c_3: economy (such as development vitality and the stability); and c_4: infrastructure (such as railway and highway facilities). And the weight vector of the factors is $\omega = (0.25, 0.22, 0.35, 0.18)$.

The decision-makers, including experts and executive managers, have gathered to determine the decision information. The linguistic term set $H = \{h_0, h_1, h_2, \cdots, h_6\}$ = {very poor, poor, slightly poor, fair, sightly good, good, very good} is employed here and the evaluation information is given in the form of SNLNs. Consequently, following a heated discussion, they come to a consensus on the final evaluations which are expressed by SNLNs in Table 1.

Table 1: The evaluation values given by the decision-maker

	c_1	c_1	c_1	c_1
a_1	$\langle h_4, (0.6, 0.6, 0.1)\rangle$	$\langle h_5, (0.6, 0.4, 0.3)\rangle$	$\langle h_4, (0.8, 0.5, 0.1)\rangle$	$\langle h_2, (0.8, 0.3, 0.1)\rangle$
a_2	$\langle h_2, (0.7, 0.5, 0.1)\rangle$	$\langle h_4, (0.6, 0.4, 0.2)\rangle$	$\langle h_3, (0.6, 0.2, 0.4)\rangle$	$\langle h_4, (0.7, 0.4, 0.3)\rangle$
a_3	$\langle h_3, (0.5, 0.1, 0.2)\rangle$	$\langle h_4, (0.6, 0.5, 0.3)\rangle$	$\langle h_6, (0.7, 0.6, 0.1)\rangle$	$\langle h_2, (0.5, 0.5, 0.2)\rangle$
a_4	$\langle h_2, (0.4, 0.5, 0.3)\rangle$	$\langle h_3, (0.5, 0.3, 0.4)\rangle$	$\langle h_4, (0.6, 0.8, 0.2)\rangle$	$\langle h_5, (0.9, 0.3, 0.1)\rangle$
a_5	$\langle h_5, (0.6, 0.4, 0.4)\rangle$	$\langle h_5, (0.8, 0.3, 0.1)\rangle$	$\langle h_3, (0.7, 0.5, 0.1)\rangle$	$\langle h_4, (0.6, 0.5, 0.2)\rangle$

5.2. An illustration of the proposed approach

In the following the main procedures of obtaining the optimal ranking of alternatives are presented.

Step 1. Normalize the decision matrix.

Considering all the criteria are maximizing type, the performance values of alternatives a_i $(i = 1, 2, 3, 4, 5)$ do not need to be normalized, i.e., $R = B$.

Step 2. Calculate the comprehensive evaluation values for each alternative.

Use Eq. (3) to calculate the comprehensive evaluation values, denoted by r_i $(i = 1, 2, 3, 4, 5)$ for each alternative a_i (here let $p = q = 1$ and $f_1^*(h_i) = \frac{i}{2t}$).

$r_1 = \langle h_{3.8132}, (0.7161, 0.4783, 0.1584)\rangle$, $r_2 = \langle h_{3.1391}, (0.6408, 0.3592, 0.2710)\rangle$, $r_3 = \langle h_{3.8766}, (0.6011, 0.4606, 0.1886)\rangle$, $r_4 = \langle h_{3.4043}, (0.6159, 0.5370, 0.2358)\rangle$ and $r_5 = \langle h_{4.1215}, (0.6771, 0.4215, 0.2083)\rangle$.

Step 3. Calculate the score values, accuracy values and certainty values of r_i $(i = 1, 2, 3, 4, 5)$.

Use equations described in Definition 7 to calculate the score values, accuracy values and certainty values, denoted by $S(r_i)$, $A(r_i)$ and $C(r_i)$ of r_i $(i = 1, 2, 3, 4, 5)$, respectively.

$$S(r_1) = 1.3215,\ A(r_1) = 0.3545,\ C(r_1) = 0.4551;$$
$$S(r_2) = 1.0519,\ A(r_2) = 0.1935,\ C(r_2) = 0.3353;$$
$$S(r_3) = 1.2611,\ A(r_3) = 0.2665,\ C(r_3) = 0.3884;$$

$$S(r_4) = 1.0457, A(r_4) = 0.2156, C(r_4) = 0.3494;$$
$$S(r_5) = 1.4063, A(r_5) = 0.3220, C(r_5) = 0.4651.$$

Step 4. Rank all the alternatives and select the best one(s).

Use the comparison method described in Definition 8 to rank all the alternatives and select the best one(s) according to $S(r_i)$, $A(r_i)$ and $C(r_i)$ $(i = 1, 2, 3, 4, 5)$.

$a_5 > a_1 > a_3 > a_2 > a_4$ and a_5 is the best one.

5.3. Comparison analysis and discussion

In order to illustrate the influence of the linguistic scale function f^* and parameters p and q on decision-making result of this example, different f^* and values of p and q are taken into consideration. The results are shown in Table 2.

Table 2: Ranking results of the alternatives with different p and q by using f^*

p and q	Ranking results by using f_1^*	Ranking results by using f_2^*	Ranking results by using f_3^*
$p = 1, q = 0$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_3 > a_5 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p \to 0, q = 0$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$
$p = q = 0.5$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$
$p = q = 1$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$	$a_5 > a_1 > a_3 > a_2 > a_4$
$p = 1, q = 2$	$a_5 > a_1 > a_3 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = 2, q = 1$	$a_5 > a_1 > a_3 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 2$	$a_5 > a_1 > a_3 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 3$	$a_5 > a_1 > a_3 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 4$	$a_5 > a_1 > a_3 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 5$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 6$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 7$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 8$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_3 > a_4 > a_1 > a_2$	$a_5 > a_1 > a_3 > a_4 > a_2$
$p = q = 9$	$a_5 > a_3 > a_1 > a_4 > a_2$	$a_5 > a_3 > a_4 > a_1 > a_2$	$a_5 > a_1 > a_4 > a_3 > a_2$
$p = q = 10$	$a_5 > a_3 > a_4 > a_1 > a_2$	$a_5 > a_3 > a_4 > a_1 > a_2$	$a_5 > a_1 > a_4 > a_3 > a_2$

It is shown that the ranking results of alternatives may be different with respect to different linguistic scale function f^* or parameters p and q in the SNLNWBM operator. The best alternative is always a_5 except for one situation where $p = 1$ and $q = 0$ by using f_2^*. And the worst alternative is always a_2 except for the situations where $p \to 0$ and $q = 0$, $p = q = 0.5$ and $p = q = 1$. Furthermore, the ranking may vary as the linguistic scale function f^* changes. In the special cases where at least one of the two parameters p and q take the value of zero, the SNLNWBM operator cannot capture the interrelationship of individual arguments and the ranking results are different. When $p = 1$ and $q = 0$, the SNLNWBM operator reduces to the SNLWAA operator. When $p \to 0$ and $q = 0$, the SNLNWBM operator reduces to the SNLWGA operator. The optimal alternative and the sequence are different by the two methods as it is shown in Table 2.

To further illustrate the advantages of the proposed approach under a simplified neutrosophic linguistic environment, the methods in Ref. [55] are used to solve the same illustrative example given above. In Ref. [55], an INLWAA operator and an INLWGA operator were developed to aggregate interval neutrosophic linguistic information. In order to use the INLWAA operator and INLWGA operator, the evaluation values of this paper need to be transformed into interval neutrosophic linguistic information. That is, $\left\langle h_{\theta_{ij}}, \left(t_{ij}, i_{ij}, f_{ij}\right)\right\rangle$ is replaced as $\left\langle h_{\theta_{ij}}, \left([t_{ij}, t_{ij}], [i_{ij}, i_{ij}], [f_{ij}, f_{ij}]\right)\right\rangle$, where the lower and upper bounds are equal. The ranking result by using INLWAA operator: $a_3 > a_1 > a_5 > a_2 > a_4$ and the ranking result by using INLWGA operator: $a_1 > a_3 > a_5 > a_2 > a_4$.

Obviously, the ranking sequences are inconsistent with the results when $p = 1$ and $q = 0$ and $p \to 0$ and $q = 0$, respectively, in Table 2. This may be caused by different operations and comparison methods. The operations and comparison method for INLNs in Ref. [55] process the linguistic information and interval neutrosophic numbers separately, which may cause information distortion in some situations. And the limitations are discussed in detail in Subsection 3.2. In addition, methods in Ref. [55] consider only one semantic situation, while different linguistic scale functions f^* utilized in this paper are applicable and effective under different semantic environment.

According to the above analysis, the proposed approach for MCDM problems with SNLNs has the following advantages. Firstly, it is flexible to express the evaluation information with SNLNs, which can depict the fuzzy, incomplete and inconsistent information more accurately and retain the completeness of original data. So it can guarantee the accuracy of final result to some extent. Secondly, the operations of SNLNs in this paper are defined on the basis of linguistic scale functions, which can harvest different results when a different linguistic function f^* is involved. Thus, decision-makers can flexibly select the f^* depending on their preferences and actual semantic environment. In addition, the SNLNWBM operator proposed in this paper has the unique characteristic that can capture the interrelationship of the input arguments. In general, the values can be set as $p = q = 1$ or $p = q = 2$, which is not only simple and convenient but also allowing the interrelationship of criteria. Thus, decision-makers can properly select the desirable alternative in accordance with the certain situations and their interests.

6. Conclusion

Linguistic variables can effectively describe qualitative information and SNSs can flexibly express uncertain, imprecise, incomplete and inconsistent information that widely exist in scientific and engineering situations. So it of great significance to study MCDM methods with SNLSs. Consider the limitations in the existing literature, new operations of SNLNs are introduced. Then based on the related research achievements in predecessors, the BM operator is extended to the simplified neutrosophic linguistic environment. Thus, a MCDM approach based on the SNLNWBM aggregation operator is proposed. Finally, an illustrative example is given to demonstrate the application of the proposed approach.

The advantages of this study are that the approach can accommodate situations where decision-making problems involve qualitative variables. In addition, the SNLNWBM operator has a significant advantage that can capture the interrelationship of individual arguments. Whats more, the results may change using different linguistic scale functions and the input parameters p and q may also affect the results. Decision-makers can select the most appropriate linguistic scale function f^* and input parameters according to their interests and actual semantic situations. In a word, this approach has much application potential in deal with MCDM problems in simplified neutrosophic linguistic environment, in which the assessment information of criterion values take the form of SNLNs and criterion weights are known information.

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