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£-Single Valued Extremally Disconnected Ideal Neutrosophic Topological Spaces

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Abstract: This paper aims to mark out new concepts of r-single valued neutrosophic sets, called r-single valued neutrosophic £-closed and £-open sets. The definition of £-single valued neutrosophic irresolute mapping is provided and its characteristic properties are discussed. Moreover, the concepts of £-single valued neutrosophic extremally disconnected and £-single valued neutrosophic normal spaces are established. As a result, a useful implication diagram between the r-single valued neutrosophic ideal open sets is obtained. Finally, some kinds of separation axioms, namely r-single valued neutrosophic ideal-R_i (r-$SVNIR_i$, for short), where $i = \{0, 1, 2, 3\}$, and r-single valued neutrosophic ideal-T_j (r-$SVNIT_j$, for short), where $j = \{1, 2, 2\frac{1}{2}, 3, 4\}$, are introduced. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

Keywords: r-single valued neutrosophic £-closed; £-single valued neutrosophic irresolute mapping; £-single valued neutrosophic extremally disconnected; £-single valued neutrosophic normal; r-$SVNIR_i$; r-$SVNIT_j$

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1. Introduction

In 1999, Smarandache introduced the concept of a neutrosophy [1]. It has been used at various axes of mathematical theories and applications. In recent decades, the theory made an outstanding advancement in the field of topological spaces. Salama et al. and Hur et al. [2–6], for example, among many others, wrote their works in fuzzy neutrosophic topological spaces (FNTS), following Chang [7]'s discoveries in the way of fuzzy topological spaces (FTS).

Šostak, in 1985 [8], marked out a new definition of fuzzy topology as a crisp subfamily of family of fuzzy sets, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Yan, Wang, Nanjing, Liang, and Yan [9,10] developed a parallel theory in the context of intuitionistic I-fuzzy topological spaces.

The idea of "single-valued neutrosophic set" [11] was set out by Wang in 2010. Gayyar [12], in his 2016 paper, foregrounded the concept of a "smooth neutrosophic topological spaces". The ordinary single-valued neutrosophic topology was presented by Kim [13]. Recently, Saber et al. [14,15] familiarized the concepts of single-valued neutrosophic ideal open local function, single-valued neutrosophic topological space, and the connectedness and stratification of single-valued neutrosophic topological spaces.

Neutrosophy, and especially neutrosophic sets, are powerful, general, and formal frameworks that generalize the concept of the ordinary sets, fuzzy sets, and intiuitionistic fuzzy sets from philosophical point of view. This paper sets out to introduce and examine a new class of sets called r-single valued £-closed in the single valued neutrosophic topological spaces in Šostak's sense. More precisely, different attributes, like £-single valued neutrosophic irresolute mapping, £-single valued neutrosophic extremally disconnected, £-single valued neutrosophic normal spaces, and some kinds of separation axioms, were developed. It can be fairly claimed that we have achieved expressive definitions, distinguished theorems, important lemmas, and counterexamples to investigate, in-depth, our consequences and to find out the best results. It is notable to say that different crucial notions in single valued neutrosophic topology were generalized in this article. Different attributes, like extremally disconnected and some kinds of separation axioms, which have a significant impact on the overall topology's notions, were also studied.

It is notable to say that the application aspects to this area of research can be further pointed to. There are many applications of neutrosophic theories in many branches of sciences. Possible applications are to control engineering and to Geographical Information Systems, and so forth, and could be secured, as mentioned by many authors, such as Reference [16–20].

In this study, $\tilde{X}$ is assumed to be a nonempty set, $\xi = [0,1]$ and $\xi_0 = (0,1]$. For $\alpha \in \xi$, $\tilde{\alpha}(\nu) = \alpha$ for all $\nu \in \tilde{X}$. The family of all single-valued neutrosophic sets on $\tilde{X}$ is denoted by $\xi^{\tilde{X}}$.

2. Preliminaries

This section is devoted to provide a complete survey and trace previous studies related to the idea of this research article.

Definition 1 ([21])**.** *Let $\tilde{X}$ be a non-empty set. A neutrosophic set (briefly, NS) in $\tilde{X}$ is an object having the form*

$$\sigma_n = \{\langle \nu, \tilde{\rho}_{\sigma_n}(\nu), \tilde{\varrho}_{\sigma_n}(\nu), \tilde{\eta}_{\sigma_n}(\nu)\rangle : \nu \in \tilde{X}\},$$

where

$$\tilde{\rho} : \tilde{X} \to \rfloor^{-}0, 1^{+}\lfloor,\ \tilde{\varrho} : \tilde{X} \to \rfloor^{-}0, 1^{+}\lfloor,\ \tilde{\eta} : \tilde{X} \to \rfloor^{-}0, 1^{+}\lfloor$$

and

$$^{-}0 \leq \tilde{\rho}_{\sigma_n}(\nu) + \tilde{\varrho}_{\sigma_n}(\nu) + \tilde{\eta}_{\sigma_n}(\nu) \leq 3^{+}$$

represent the degree of membership (namely $\tilde{\rho}_{\sigma_n}(\nu)$), the degree of indeterminacy (namely $\tilde{\varrho}_{\sigma_n}(\nu)$), and the degree of non-membership (namely $\tilde{\eta}_{\sigma_n}(\nu)$), respectively, of any $\nu \in \tilde{X}$ to the set σ_n.

Definition 2 ([11])**.** *Let $\tilde{X}$ be a space of points (objects), with a generic element in $\tilde{X}$ denoted by ν. Then, σ_n is called a single valued neutrosophic set (briefly, SVNS) in $\tilde{X}$, if σ_n has the form $\sigma_n = \langle \tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n}\rangle$, where $\tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n} : \tilde{X} \to [0,1]$. In this case, $\tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n}$ are called truth membership function, indeterminancy membership function, and falsity membership function, respectively.*

Let $\tilde{X}$ be a nonempty set and $\xi = [0,1]$ and $\xi_0 = (0,1]$. A single-valued neutrosophic set σ_n on $\tilde{X}$ is a mapping defined as $\sigma_n = \langle \tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n}\rangle : \tilde{X} \to \xi$ such that $0 \leq \tilde{\rho}_{\sigma_n}(\nu) + \tilde{\varrho}_{\sigma_n}(\nu) + \tilde{\eta}_{\sigma_n}(\nu) \leq 3$.

We denote the single-valued neutrosophic sets $\langle\, 0,1,1\rangle$ and $\langle 1,0,0\rangle$ by $\tilde{0}$ and $\tilde{1}$, respectively.

Definition 3 ([11])**.** *Let $\sigma_n = \langle \tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n}\rangle$ be an SVNS on $\tilde{X}$. The complement of the set σ_n (briefly σ_n^c) is defined as follows:*

$$\tilde{\rho}_{\sigma_n^c}(\nu) = \tilde{\eta}_{\sigma_n}(\nu), \quad \tilde{\varrho}_{\sigma_n^c}(\nu) = [\tilde{\varrho}_{\sigma_n}]^c(\nu), \quad \tilde{\eta}_{\sigma_n^c}(\nu) = \tilde{\rho}_{\sigma_n}(\nu).$$

Definition 4 ([22,23])**.** *Let $\tilde{X}$ be a non-empty set and let $\sigma_n, \gamma_n \in \xi^{\tilde{X}}$ be given by $\sigma_n = \langle \tilde{\rho}_{\sigma_n}, \tilde{\varrho}_{\sigma_n}, \tilde{\eta}_{\sigma_n}\rangle$ and $\gamma_n = \langle \tilde{\rho}_{\gamma_n}, \tilde{\varrho}_{\gamma_n}, \tilde{\eta}_{\gamma_n}\rangle$. Then:*

(1) We say that $\sigma_n \subseteq \gamma_n$ if $\tilde{\rho}_{\sigma_n} \leq \tilde{\rho}_{\gamma_n}$, $\tilde{\varrho}_{\sigma_n} \geq \tilde{\varrho}_{\gamma_n}$, $\tilde{\eta}_{\sigma_n} \geq \tilde{\eta}_{\gamma_n}$.

(2) The intersection of σ_n and γ_n denoted by $\sigma_n \cap \gamma_n$ is an SVNS and is given by

$$\sigma_n \cap \gamma_n = \langle \tilde{\rho}_{\sigma_n} \cap \tilde{\rho}_{\gamma_n},\ \tilde{\varrho}_{\sigma_n} \cup \tilde{\varrho}_{\gamma_n},\ \tilde{\eta}_{\sigma_n} \cup \tilde{\eta}_{\gamma_n}\rangle.$$

(3) The union of σ_n and γ_n denoted by $\sigma_n \cup \gamma_n$ is an SVNS and is given by

$$\sigma_n \cup \gamma_n = \langle \tilde{\rho}_{\sigma_n} \cup \tilde{\rho}_{\gamma_n},\ \tilde{\varrho}_{\sigma_n} \cap \tilde{\varrho}_{\gamma_n},\ \tilde{\eta}_{\sigma_n} \cap \tilde{\eta}_{\gamma_n}\rangle.$$

For any arbitrary family $\{\sigma_n\}_{i\in j} \subseteq \xi^{\tilde{X}}$ of SVNS, the union and intersection are given by

(4) $\bigcap_{i\in j}[\sigma_n]_i = \langle \cap_{i\in j}\tilde{\rho}_{[\sigma_n]_i}, \cup_{i\in j}\tilde{\varrho}_{[\sigma_n]_i}, \cup_{i\in j}\tilde{\eta}_{[\sigma_n]_i}\rangle$,

(5) $\bigcup_{i\in j}[\sigma_n]_i = \langle \cup_{i\in j}\tilde{\rho}_{[\sigma_n]_i}, \cap_{i\in j}\tilde{\varrho}_{[\sigma_n]_i}, \cap_{i\in j}\tilde{\eta}_{[\sigma_n]_i}\rangle$.

Definition 5 ([12]). *A single-valued neutrosophic topological space is an ordered quadruple $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\eta}})$ where $\tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\eta}} : \xi^{\tilde{X}} \to \xi$ are mappings satisfying the following axioms:*

(SVNT1) $\tilde{\tau}^{\tilde{\rho}}(\tilde{0}) = \tilde{\tau}^{\tilde{\rho}}(\tilde{1}) = 1$ *and* $\tilde{\tau}^{\tilde{\varrho}}(\tilde{0}) = \tilde{\tau}^{\tilde{\varrho}}(\tilde{1}) = \tilde{\tau}^{\tilde{\eta}}(\tilde{0}) = \tilde{\tau}^{\tilde{\eta}}(\tilde{1}) = 0$,

(SVNT2) $\tilde{\tau}^{\tilde{\rho}}(\sigma_n \cap \gamma_n) \geq \tilde{\tau}^{\tilde{\rho}}(\sigma_n) \cap \tilde{\tau}^{\tilde{\rho}}(\gamma_n)$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n \cap \gamma_n) \leq \tau^{\tilde{\varrho}}(\sigma_n) \cup \tilde{\tau}^{\tilde{\varrho}}(\gamma_n)$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n \cap \gamma_n) \leq \tilde{\tau}^{\tilde{\eta}}(\sigma_n) \cup \tilde{\tau}^{\tilde{\eta}}(\gamma_n)$, *for all* $\sigma_n, \gamma_n \in \zeta^{\tilde{X}}$,

(SVNT3) $\tilde{\tau}^{\tilde{\rho}}(\cup_{j\in\Gamma}[\sigma_n]_j) \geq \cap_{j\in\Gamma}\tilde{\tau}^{\tilde{\rho}}([\sigma_n]_j)$, $\tilde{\tau}^{\tilde{\varrho}}(\cup_{i\in\Gamma}[\sigma_n]_j) \leq \cup_{j\in\Gamma}\tilde{\tau}^{\tilde{\varrho}}([\sigma_n]_j)$, $\tilde{\tau}^{\tilde{\eta}}(\cup_{j\in\Gamma}[\sigma_n]_j) \leq \cup_{j\in\Gamma}\tilde{\tau}^{\tilde{\eta}}([\sigma_n]_j)$ *for all* $\{[\sigma_n]_j, j \in \Gamma\} \in \zeta^{\tilde{X}}$.

The quadruple $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\eta}})$ is called a single-valued neutrosophic topological space (SVNTS, for short). We will occasionally write $\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}$ for $(\tau^{\tilde{\rho}}, \tau^{\tilde{\varrho}}, \tau^{\tilde{\eta}})$ and it will cause no ambiguity

Definition 6 ([14]). *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\eta}})$ be an SVNTS. Then, for every $\sigma_n \in \zeta^{\tilde{X}}$ and $r \in \zeta_0$, the single valued neutrosophic closure and the single valued neutrosophic interior of σ_n are defined by:*

$$C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, s) = \bigcap\{\gamma_n \in \zeta^{\tilde{X}} : \sigma_n \leq \gamma_n, \quad \tau^{\tilde{\rho}}([\gamma_n]^c) \geq r, \quad \tau^{\tilde{\varrho}}([\gamma_n]^c) \leq 1-r, \quad \tau^{\tilde{\eta}}([\gamma_n]^c) \leq 1-r\},$$

$$\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, s) = \bigcup\{\gamma_n \in \zeta^{\tilde{X}} : \sigma_n \geq \gamma_n, \quad \tau^{\tilde{\rho}}(\gamma_n) \geq r, \quad \tau^{\tilde{\varrho}}(\gamma_n) \leq 1-r, \quad \tau^{\tilde{\eta}}(\gamma_n) \leq 1-r\}.$$

Definition 7 ([24]). *Let $(\tilde{X}, \tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNTS and $r \in \zeta_0, \sigma_n \in \zeta^{\tilde{X}}$. Then,*

(1) *σ_n is r-single valued neutrosophic semiopen (r-SVNSO, for short) iff $\sigma_n \leq C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$,*

(2) *σ_n is r-single valued neutrosophic β-open (r-SVNβO, for short) iff $\sigma_n \leq C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)$.*

The complement of $r-SVNSO$ (resp. r-SVNβO) is said to be an $r-SVNSC$ (resp. r-SVNβC), respectively.

Definition 8 ([14]). *Let $\tilde{X}$ be a nonempty set and $\nu \in \tilde{X}$. If $s \in (0,1]$, $t \in [0,1)$ and $p \in [0,1)$. Then, the single-valued neutrosophic point $x_{s,t,p}$ in $\tilde{X}$ is given by*

$$x_{s,t,p}(\kappa) = \begin{cases} (s,t,p), & \text{if } x = \nu, \\ (0,1,1), & \text{otherwise.} \end{cases}$$

We say $x_{s,t,p} \in \sigma_n$ iff $s < \tilde{\rho}_{\sigma_n}(\nu)$, $t \geq \tilde{\varrho}_{\sigma_n}(\nu)$ and $p \geq \tilde{\eta}_{\sigma_n}(\nu)$. To avoid the ambiguity, we denote the set of all neutrosophic points by $pt(\xi^{\tilde{X}})$.

A single-valued neutrosophic set σ_n is said to be quasi-coincident with another single-valued neutrosophic set γ_n, denoted by $\sigma_n q \gamma_n$, if there exists an element $\nu \in \tilde{X}$ such that

$$\tilde{\rho}_{\sigma_n}(\nu) + \tilde{\rho}_{\gamma_n}(\nu) > 1, \quad \tilde{\varrho}_{\sigma_n}(\nu) + \tilde{\varrho}_{\gamma_n}(\nu) \leq 1, \quad \tilde{\eta}_{\sigma_n}(\nu) + \tilde{\eta}_{\gamma_n}(\nu) \leq 1.$$

Definition 9 ([14]). *A mapping $\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}} = \mathcal{I}^{\tilde{\rho}}, \mathcal{I}^{\tilde{\varrho}}, \mathcal{I}^{\tilde{\eta}} : \xi^{\tilde{X}} \to \xi$ is called single-valued neutrosophic ideal (SVNI) on $\tilde{X}$ if it satisfies the following conditions:*

(I_1) $\mathcal{I}^{\tilde{\rho}}(\tilde{0}) = 1$ *and* $\mathcal{I}^{\tilde{\varrho}}(\tilde{0}) = \mathcal{I}^{\tilde{\eta}}(\tilde{0}) = 0$.

(I_2) *If $\sigma_n \leq \gamma_n$, then $\mathcal{I}^{\tilde{\rho}}(\gamma_n) \leq \mathcal{I}^{\tilde{\rho}}(\sigma_n)$, $\mathcal{I}^{\tilde{\varrho}}(\gamma_n) \geq \mathcal{I}^{\tilde{\varrho}}(\sigma_n)$, and $\mathcal{I}^{\tilde{\eta}}(\gamma_n) \geq \mathcal{I}^{\tilde{\eta}}(\sigma_n)$, for $\gamma_n, \sigma_n \in \xi^{\tilde{X}}$.*

(I_3) $\mathcal{I}^{\tilde{\rho}}(\sigma_n \cup \gamma_n) \geq \mathcal{I}^{\tilde{\rho}}(\sigma_n) \cap \mathcal{I}^{\tilde{\rho}}(\gamma_n)$, $\mathcal{I}^{\tilde{\varrho}}(\sigma_n \cup \gamma_n) \leq \mathcal{I}^{\tilde{\varrho}}(\sigma_n) \cup \mathcal{I}^{\tilde{\varrho}}(\gamma_n)$ *and* $\mathcal{I}^{\tilde{\eta}}(\sigma_n \cup \gamma_n) \leq \mathcal{I}^{\tilde{\eta}}(\sigma_n) \cup \mathcal{I}^{\tilde{\eta}}(\gamma_n)$, *for each* $\sigma_n, \gamma_n \in \xi^{\tilde{X}}$.

The triple $(\tilde{X}, \tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is called a single valued neutrosophic ideal topological space in Šostak's sense (SVNITS, for short).

Definition 10 ([14]). *Let $(\tilde{X}, \tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS for each $\sigma_n \in \xi^{\tilde{X}}$. Then, the single valued neutrosophic ideal open local function $[\sigma_n]^{£}_r(\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ of σ_n is the union of all single-valued neutrosophic points $x_{s,t,k}$ such that, if $\gamma_n \in Q_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,k}, r)$ and $\mathcal{I}^{\tilde{\rho}}(\varsigma_n) \geq r$, $\mathcal{I}^{\tilde{\varrho}}(\varsigma_n) \leq 1-r$, $\mathcal{I}^{\tilde{\eta}}(\varsigma_n) \leq 1-r$, then there is at least one $\nu \in \tilde{X}$ for which $\tilde{\rho}_{\sigma_n}(\nu) + \tilde{\rho}_{\gamma_n}(\nu) - 1 > \tilde{\rho}_{\varsigma_n}(\nu)$, $\tilde{\varrho}_{\sigma_n}(\nu) + \tilde{\varrho}_{\gamma_n}(\nu) - 1 \leq \tilde{\varrho}_{\varsigma_n}(\nu)$, and $\tilde{\eta}_{\sigma_n}(\nu) + \tilde{\eta}_{\gamma_n}(\nu) - 1 \leq \tilde{\eta}_{\varsigma_n}(\nu)$.*

Occasionally, we will write $[\sigma_n]^{£}_r$ for $[\sigma_n]^{£}_r(\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$, and it will cause no ambiguity.

Remark 1 ([14]). *Let $(\tilde{X}, \tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $\sigma_n \in \xi^{\tilde{X}}$. Then,*

$$\mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = \sigma_n \cup [\sigma_n]^{£}_r, \qquad \mathrm{int}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = \sigma_n \cap [(\sigma^c_n)^{£}_r]^c.$$

It is clear that $\mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}$ is a single-valued neutrosophic closure operator and $(\tau^{\tilde{\rho}£}(\mathcal{I}^{\rho}, \tau^{\tilde{\varrho}£}(\mathcal{I}^{\varrho}, \tau^{\tilde{\eta}£}$ $(\mathcal{I}^{\eta})$ is the single-valued neutrosophic topology generated by $\mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}$, i.e.,

$$\tau^{£}(\mathcal{I})(\sigma_n) = \bigcup\{r |\ \mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma^c_n, r) = \sigma^c\}.$$

Theorem 1 ([14]). *Let $\{[\sigma_n]_i\}_{i\in J} \subset \xi^{\tilde{X}}$ be a family of single-valued neutrosophic sets on $\tilde{X}$ and $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an r-SVNITS. Then,*

(1) $(\bigcup([\sigma_n]_i)^{£}_r :\ i \in J) \leq (\bigcup[\sigma_n]_i :\ i \in j)^{£}_r$,

(2) $(\bigcap([\sigma_n]_i) :\ i \in j)^{£}_r \geq (\bigcap([\sigma_n]_i)^{£}_r :\ i \in J)$.

Theorem 2 ([14]). *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $\sigma_n, \gamma_n \in \xi^{\tilde{X}}$, $r \in \xi_0$. Then,*

(1) $\mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \vee \gamma_n, r) \leq \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \vee \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$,

(2) $\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \sigma_n \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$,

(3) $\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r) = [\mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$, *and* $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c = \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r)$,

(4) $\mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n, r) = \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{int}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$.

3. £-Single Valued Neutrosophic Ideal Irresolute Mapping

This section provides the definitions of the r-single-valued neutrosophic £-open set (SVN£O, for short), the r-single-valued neutrosophic £-closed set (SVN£C, for short) and the £-single valued neutrosophic ideal irresolute mapping (£-SVNI-irresolute, for short), in the sense of Šostak. To understand the aim of this section, it is essential to clarify its content and elucidate the context in which the definitions, theorems, and examples are performed. Some results follow.

Definition 11. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an r-SVNITS for every $\sigma_n \in \xi^{\tilde{X}}$ and $r \in \xi_0$. Then, σ_n is called r-SVN£C iff $\mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = \sigma_n$. The complement of the r-SVN£C is called r-SVN£O.*

Proposition 1. *Let $(\tilde{X}, \tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an r-SVNITS and $\sigma_n \in \xi^{\tilde{X}}$. Then,*

(1) σ_n *is r-SVN£C iff* $[\sigma_n]^{£}_r \leq \sigma_n$,

(2) σ_n *is r-SVN£O iff* $([\sigma_n]^{£}_r)^c \geq [\sigma_n]^c$,

(3) *If* $\tau^{\tilde{\rho}}([\sigma_n]^c) \geq r$, $\tau^{\varrho}([\sigma_n]^c) \leq 1-r$, $\tau^{\eta}([\sigma_n]^c) \leq 1-r$, *then* σ_n *is r-SVN£C,*

(4) *If* $\tau^{\tilde{\rho}}(\sigma_n) \geq r$, $\tau^{\varrho}(\sigma_n) \leq 1-r$, $\tau^{\eta}(\sigma_n) \leq 1-r$, *then* σ_n *is r-SVN£O,*

(5) *If* σ_n *is r-SVNSC (resp. r-SVNβC), then* $\mathrm{int}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^{£}_r, r) \leq \sigma_n$ *(resp.* $\mathrm{int}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\mathrm{int}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^{£}_r, r) \leq \sigma_n$*).*

Proof. The proof of (1) and (2) are straightforward from Definition 11.

(3) Let $\tau^{\tilde{\rho}}([\sigma_n]^c) \geq r$, $\tau^{\varrho}([\sigma_n]^c) \leq 1-r$, $\tau^{\eta}([\sigma_n]^c) \leq 1-r$. Then,

$$\sigma_n = \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \geq \mathrm{CI}^{£}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = \sigma_n \cup [\sigma_n]^{£}_r \geq [\sigma_n]^{£}_r.$$

Hence, σ_n is an r-SVN£C.
(4) The proof is direct consequence of (1).
(5) Let σ_n be an r-SVNSC. Then,

$$\begin{aligned}\sigma_n \geq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) &\geq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n \cup [\sigma_n]^{\pounds}_r], r) \\ &\geq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^{\pounds}_r, r).\end{aligned}$$

The another case is similarly proved. □

Example 1. *Suppose that $\tilde{X} = \{a, b\}$. Define $\varepsilon_n, \gamma_n, \varsigma_n \in \xi^{\tilde{X}}$ as follows:*

$$\gamma_n = \langle(0.3, 0.3), (0.3, 0.3), (0.3, 0.3)\rangle; \quad \varepsilon_n = \langle(0.7, 0.7), (0.7, 0.7), (0.7, 0.7)\rangle;$$

$$\varsigma_n = \langle(0.2, 0.2), (0.2, 0.2), (0.2, 0.2)\rangle.$$

Define $\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}} : \xi^{\tilde{X}} \to \xi$ as follows:

$$\tilde{\tau}^{\tilde{\rho}}(\sigma_n) = \begin{cases} 1, & \text{if } \sigma_n = \tilde{0}, \\ 1, & \text{if } \sigma_n = \tilde{1}, \\ \frac{1}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{1}{3}, & \text{if } \sigma_n = \varepsilon_n; \\ 0, & \text{if otherwise}; \end{cases} \qquad \mathcal{I}^{\tilde{\rho}}(\sigma_n) = \begin{cases} 1, & \text{if } \sigma_n = (0,1,1), \\ \frac{1}{3}, & \text{if } \sigma_n = \varsigma_n, \\ \frac{2}{3}, & \text{if } \tilde{0} < \sigma_n < \varsigma_n; \\ 0, & \text{if otherwise}; \end{cases}$$

$$\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = \tilde{0}, \\ 0, & \text{if } \sigma_n = \tilde{1}, \\ \frac{2}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{2}{3}, & \text{if } \sigma_n = \varepsilon_n; \\ 1, & \text{if otherwise}; \end{cases} \qquad \mathcal{I}^{\tilde{\varrho}}(\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = (0,1,1), \\ \frac{2}{3}, & \text{if } \sigma_n = \varsigma_n, \\ \frac{1}{3}, & \text{if } \tilde{0} < \sigma_n < \varsigma_n; \\ 1, & \text{if otherwise}; \end{cases}$$

$$\tilde{\tau}^{\tilde{\eta}}(\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = \tilde{0}, \\ 0, & \text{if } \sigma_n = \tilde{1}, \\ \frac{2}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{2}{3}, & \text{if } \sigma_n = \varepsilon_n; \\ 1, & \text{if otherwise}; \end{cases} \qquad \mathcal{I}^{\tilde{\eta}}(\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = (0,1,1), \\ \frac{2}{3}, & \text{if } \sigma_n = \varsigma_n, \\ \frac{1}{3}, & \text{if } \tilde{0} < \sigma_n < \varsigma_n; \\ 1, & \text{if otherwise}. \end{cases}$$

(1) $G_n = \langle(0.6, 0.6), (0.6, 0.6), (0.6, 0.6)\rangle$ *is* $\frac{1}{3}$*-SVN£C but* $\tilde{\tau}^{\tilde{\rho}}([G_n]^c) \ngeq \frac{1}{3}$, $\tilde{\tau}^{\varrho}([G_n]^c) \nleq \frac{2}{3}$*, and* $\tilde{\tau}^{\eta}([G_n]^c) \nleq \frac{2}{3}$,
(2) $G_n = \langle(0.6, 0.6), (0.6, 0.6), (0.6, 0.6)\rangle \geq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([G_n]^{\pounds}_{\frac{1}{3}}, \frac{1}{3}) = \tilde{0}$ *but G_n is not is* $\frac{1}{3}$*-SVNSC.*

Lemma 1. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS. Then, we have the following.*
(1) Every intersection of r-SVN£C's is r-SVN£C.
(2) Every union of r-SVN£O's is r-SVN£O.

Proof. (1) Let $\{[\sigma_n]_i\}_{i\in j}$ be a family of r-SVN£C's. Then, for every $i \in j$, we obtain $[\sigma_n]_i = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]_i, r)$, and, by Theorem 1(2), we have

$$\begin{aligned}\textstyle\bigcap_{i\in j}[\sigma_n]_i &= \bigcap_{i\in j}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]_i, r), r) = \bigcap_{i\in j}([\sigma_n]_i \cup ([\sigma_n]_i)^{\pounds}_r) \geq \bigcap_{i\in j}[\sigma_n]_i \cup \bigcap_{i\in j}([\sigma_n]_i)^{\pounds}_r \\ &\geq \bigcap_{i\in j}[\sigma_n]_i \cup (\bigcap_{i\in j}[\sigma_n]_i)^{\pounds}_r = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\bigcap_{i\in j}[\sigma_n]_i, r).\end{aligned}$$

Therefore, $\bigcap_{i\in\Gamma}[\sigma_n]_i$ is r-SVN£C.

(2) From Theorem 1(1). □

Lemma 2. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS for each $r \in \zeta_0$. Then,*

(1) *For each r-SVN£O $\sigma_n \in \xi^{\tilde{X}}$, $\sigma_n q \gamma_n$ iff $\sigma_n q \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$,*

(2) *$x_{s,t,k} q \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$ iff $\sigma_n q \gamma_n$ for every r-SVN£O $\sigma_n \in \xi^{\tilde{X}}$ with $x_{s,t,k} \in \sigma_n$.*

Proof. (1) Let σ_n be an r-SVN£O and $\sigma_n \overline{q} \gamma_n$. Then, for any $\nu \in \tilde{X}$, we obtain

$$\tilde{\rho}_{\sigma_n}(\nu) + \tilde{\rho}_{\gamma_n}(\nu) > 1, \quad \tilde{\varrho}_{\sigma_n}(\nu) + \tilde{\varrho}_{\gamma_n}(\nu) \leq 1, \quad \tilde{\eta}_{\sigma_n}(\nu) + \tilde{\eta}_{\gamma_n}(\nu) \leq 1.$$

This implies that $\tilde{\rho}_{\gamma_n} \leq \tilde{\rho}_{[\sigma_n]^c}$, $\tilde{\varrho}_{\gamma_n} \geq \tilde{\varrho}_{[\sigma_n]^c}$ and $\tilde{\eta}_{\gamma_n} \geq \tilde{\eta}_{[\sigma_n]^c}$; hence, $\gamma_n \leq [\sigma_n]^c$. Since σ_n is r-SVN£O, $\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r) = [\sigma_n]^c$, it follows that $\sigma_n \overline{q} \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$.

(2) Let $x_{s,t,k} q \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$. Then, $\sigma_n q \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$ with $x_{s,t,k} \in \sigma_n$. By (1), we have $\gamma_n q \sigma_n$ for each r-SVN£O $\sigma_n \in \xi^{\tilde{X}}$. On the other hand, let $\sigma_n \overline{q} \gamma_n$. Then, $\gamma_n \leq [\sigma_n]^c$. Since σ_n is r-SVN£O,

$$\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r) = [\sigma_n]^c \quad \text{and} \quad \sigma_n \overline{q} \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r).$$

Since $x_{s,t,k} \in \sigma_n$, we obtain $x_{s,t,k} \overline{q} \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$ □

Definition 12. *Suppose that $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \rightarrow (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is a mapping. Then,*

(1) *f is called £-SVNI-irresolute iff $f^{-1}(\sigma_n)$ is r-SVN£O in $\tilde{X}$ for any r-SVN£O σ_n in $\tilde{Y}$,*

(2) *f is called £-SVNI-irresolute open iff $f(\sigma_n)$ is r-SVN£O in $\tilde{Y}$ for any r-SVN£O σ_n in $\tilde{X}$,*

(3) *f is called £-SVNI-irresolute closed iff $f(\sigma_n)$ is r-SVN£C in $\tilde{Y}$ for any r-SVN£C σ_n in $\tilde{X}$.*

Theorem 3. *Let $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \rightarrow (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be a mapping. Then, the following conditions are equivalent:*

(1) *f is £-SVNI-irresolute,*

(2) *$f^{-1}(\sigma_n)$ is r-SVN£C, for each r-SVN£C $\sigma_n \in \tilde{Y}$,*

(3) *$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)$ for each $\sigma_n \in \xi^{\tilde{X}}$, $r \in \zeta_0$,*

(4) *$\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n), r) \leq f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r))$ for each $\gamma_n \in \xi^{\tilde{Y}}$, $r \in \zeta_0$.*

Proof. (1)⇒(2): Let σ_n be an r-SVN£C in $\tilde{Y}$. Then, $[\sigma_n]^c$ is r-SVN£O in $\tilde{Y}$ by (1), we obtain $f^{-1}([\sigma_n]^c)$ is r-SVN£O. But, $f^{-1}([\sigma_n]^c) = [f^{-1}(\sigma_n)]^c$. Then, $f^{-1}(\sigma_n)$ is r-SVN£C in $\tilde{X}$.

(2)⇒(3): For each $\sigma_n \in \xi^{\tilde{X}}$ and $r \in \zeta_0$, since $\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r) = \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)$. From Definition 11, $\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)$ is r-SVN£C in $\tilde{Y}$. By (2), $f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r))$ is r-SVN£C in $\tilde{X}$. Since

$$\sigma_n \leq f^{-1}(f(\sigma_n)) \leq f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)),$$

by Definition 11, we get,

$$\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)), r) = f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r)).$$

Hence,

$$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)) \leq f(f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r))) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n), r).$$

(3)⇒(4): For each $\gamma_n \in \xi^{\tilde{Y}}$ and $r \in \zeta_0$, put $\sigma_n = f^{-1}(\gamma_n)$. By (3),

$$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n), r)) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(f^{-1}(\gamma_n)), r) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r).$$

It implies that $\mathrm{CI}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n),r) \leq f^{-1}(\mathrm{CI}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r))$.

(4)⇒(1): Let γ_n be an r-SVN£O in $\tilde{Y}$. Then, $[\gamma_n]^c$ is an r-SVN£C in $\tilde{Y}$. Hence, $\mathrm{CI}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c,r) = [\gamma_n]^c$, and, by (4), we have,

$$f^{-1}([\gamma_n]^c) = f^{-1}(\mathrm{CI}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c,r)) \geq \mathrm{CI}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}([\gamma_n]^c),r).$$

On the other hand, $f^{-1}([\gamma_n]^c) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}([\gamma_n]^c),r)$. Thus, $f^{-1}([\gamma_n]^c) = \mathrm{CI}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}([\gamma_n]^c),r)$, that is $f^{-1}([\gamma_n]^c)$ is an r-SVN£C set in $\tilde{X}$. Hence, $f^{-1}(\gamma_n)$ is an r-SVN£O set in $\tilde{X}$. □

Theorem 4. *Let $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \to (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be a mapping. Then, the following conditions are equivalent:*

(1) f is £-SVNI-irresolute open,

(2) $f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)) \leq \mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n),r)$ for each $\sigma_n \in \xi^{\tilde{X}}$, $r \in \xi_0$,

(3) $\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}((f^{-1}(\gamma_n),r) \leq f^{-1}(\mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r))$ for each $\gamma_n \in \xi^{\tilde{Y}}$, $r \in \xi_0$,

(4) For any $\gamma_n \in \xi^{\tilde{Y}}$ and any r-SVN£C $\sigma_n \in \xi^{\tilde{X}}$ with $f^{-1}(\gamma_n) \leq \sigma_n$, there exists an r-SVN£C $\varsigma_n \in \xi^{\tilde{Y}}$ with $\gamma_n \leq \varsigma_n$ such that $f^{-1}(\varsigma_n) \leq \sigma_n$.

Proof. (1)⇒(2): For every $\sigma_n \in \xi^{\tilde{X}}$, $r \in \xi_0$ and $\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) \leq \sigma_n$ from Theorem 2(2), we have $f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)) \leq f(\sigma_n)$. By (1), $f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r))$ is r-SVN£O in $\tilde{Y}$. Hence,

$$f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)) = \mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r))) \leq \mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\sigma_n),r).$$

(2)⇒(3): For each $\gamma_n \in \xi^{\tilde{Y}}$ and $r \in \xi_0$, put $\sigma_n = f^{-1}(\gamma_n)$ from (2),

$$f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n),r)) \leq \mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(f^{-1}(\gamma_n)),r) \leq \mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r).$$

It implies that

$$\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n),r) \leq f^{-1}(f(\mathrm{int}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\gamma_n),r))) \leq f^{-1}(\mathrm{int}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r)).$$

(3)⇒(4): Obvious.

(4)⇒(1): Let ε_n be an r-SVN£O in $\tilde{X}$. Put $\gamma_n = [f(\varepsilon_n)]^c$ and $\sigma_n = [\varepsilon_n]^c$ such that σ_n is r-SVN£C in $\tilde{X}$. We obtain

$$f^{-1}(\gamma_n) = f^{-1}([f(\varepsilon_n)]^c) = [f^{-1}(f(\varepsilon_n))]^c \leq [\varepsilon_n]^c = \sigma_n.$$

From (4), there exists r-SVN£O $\varsigma_n \in \xi^{\tilde{Y}}$ with $\gamma_n \leq \varsigma_n$ such that $f^{-1}(\varsigma_n) \leq \sigma_n = [\varepsilon_n]^c$. It implies $\varepsilon_n \leq [f^{-1}(\varsigma)]^c = f^{-1}([\varsigma_n]^c)$. Thus, $f(\varepsilon_n) \leq f(f^{-1}([\varsigma]^c)) \leq [\varsigma_n]^c$. On the other hand, since $\gamma_n \leq \varsigma_n$, we have

$$f(\varepsilon_n) = [\gamma]^c \geq [\varsigma_n]^c.$$

Hence, $f(\varepsilon_n) = [\varsigma_n]^c$, that is, $f(\varepsilon_n)$ is r-SVN£O in $\tilde{Y}$. □

Theorem 5. *Let $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \to (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be a mapping. Then, the following conditions are equivalent:*

(1) f is £-SVNI-irresolute closed.

(2) $f(\mathrm{CI}^{\pounds}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r)) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\gamma_n),r)$ for each $\gamma_n \in \xi^{\tilde{X}}$, $r \in \xi_0$.

Proof. Obvious. □

Theorem 6. *Let $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \to (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be a bijective mapping. Then, the following conditions are equivalent:*

(1) *f is £-SVNI-irresolute closed,*
(2) $\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\sigma_n), r) \leq f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r))$ *for each $\sigma_n \in \xi^{\tilde{Y}},\ r \in \xi_0$.*

Proof. $(1) \Rightarrow (2)$: Suppose that f is an £-SVNI-irresolute closed. From Theorem 5(2), we claim that, for each $\gamma_n \in \xi^{\tilde{X}}$ and $r \in \xi_0$,

$$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\gamma_n), r).$$

Now, for all $\sigma_n \in \xi^{\tilde{Y}},\ r \in \xi_0$, put $\gamma_n = f^{-1}(\sigma_n)$, since f is onto, it implies that $f(f^{-1}(\sigma_n)) = \sigma_n$. Thus,

$$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\sigma_n), r)) \leq \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(f^{-1}(\sigma_n)), r) = \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r).$$

Again, since f is onto, it follows:

$$\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\sigma_n), r) = f^{-1}(f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(\sigma_n), r))) \leq f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)).$$

$(2) \Rightarrow (1)$: Put $\sigma_n = f(\gamma_n)$. By the injection of f, we get

$$\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) = \mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f^{-1}(f(\gamma_n)), r) \leq f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\gamma_n), r)),$$

for the reason that f is onto, which implies that

$$f(\mathrm{CI}^{£}_{\tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)) \leq f(f^{-1}(\mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\gamma_n), r))) = \mathrm{CI}^{£}_{\tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(f(\gamma_n), r).$$

□

4. £-Single Valued Neutrosophic Extremally Disconnected and £-Single Valued Neutrosophic Normal

This section is devoted to introducing £-single valued neutrosophic extremally disconnected (£-SVNE-disconnected, for short) and £-single valued neutrosophic normal (£-SVN-normal, for short), in the sense of Šostak. These definitions and their components, together with a set of criteria for identifying the spaces, are provided to illustrate how the ideas are applied.

Definition 13. *An SVNITS $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is called £-SVNE-disconnected if $\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$ for each $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1 - r$.*

Definition 14. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $r \in \xi_0$. Then, $\sigma_n \in \xi^{\tilde{X}}$ is said to be:*

(1) *r-single valued neutrosophic semi-ideal open set (r-SVNSIO) iff $\sigma_n \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$,*
(2) *r-single valued neutrosophic pre-ideal open set (r-SVNPIO) iff $\sigma_n \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$,*
(3) *r-single valued neutrosophic α-ideal open set (r-SVNαIO) iff $\sigma_n \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}\sigma_n, r), r), r)$,*
(4) *r-single valued neutrosophic β-ideal open set (r-SVNβIO) iff $\sigma_n \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)$,*
(5) *r-single valued neutrosophic β-ideal open (r-SVNSβIO) iff $\sigma_n \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)$,*

(6) *r-single valued neutrosophic regular ideal open set (r-SVNRIO) iff* $\sigma_n = \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$.

The complement of r-SVNSIO (resp. r-SVNPIO, r-SVNαIO, r-SVNβIO, r-SVNSβIO, r-SVNRIO) are called r-SVNSIC (resp. r-SVNPIC, r-SVNαIC, r-SVNβIC, r-SVNSβIC, r-SVNRIC).

Remark 2. *The following diagram can be easily obtained from the above definition:*

$$\begin{array}{ccccccc} & & r-SVN\alpha IO & \Rightarrow & r-SVNSIO & \Rightarrow & r-SVNSO \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ r-SVNRIO & \Rightarrow & r-SVNPIO & \Rightarrow & r-SVN\beta IO & \Rightarrow & r-SVN\beta O \\ & & \Downarrow & & & & \\ & r-SVNSIO & \Rightarrow & r-SVNS\beta IO & \Rightarrow & r-SVN\beta IO. & \end{array}$$

Theorem 7. *Let* $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be an SVNITS and* $r \in \zeta_0$. *Then, the following properties are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is £-SVNE-disconnected,*

(2) $\tilde{\tau}^{\tilde{\rho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)]^c) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)]^c) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)]^c) \leq 1 - r$ *for each* $\tilde{\tau}^{\tilde{\rho}}([\sigma_n]^c) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}([\sigma_n]^c) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}([\sigma_n]^c) \leq 1 - r$,

(3) $\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n.r), r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$, *for each* $\sigma_n \in \zeta^{\tilde{X}}$,

(4) *Every r-SVNSIO set is r-SVNPIO,*

(5) $\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$ *for each r-SVNSβIO* $\sigma_n \in \zeta^{\tilde{X}}$,

(6) *Every r-SVNSβIO set is r-SVNPIO,*

(7) *For each* $\sigma_n \in \zeta^{\tilde{X}}$, σ_n *is r-SVNαIO set iff it is r-SVNSIO.*

Proof. (1) ⇒ (2):The proof is direct consequence of Definition 14.

(2)⇒(3): For each $\sigma_n \in \zeta^{\tilde{X}}$, $\tilde{\tau}^{\tilde{\rho}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\text{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\text{int}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$, and, by (2), we have

$$\tilde{\tau}^{\tilde{\rho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}([\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)]^c, r)]^c) \geq r, \qquad \tilde{\tau}^{\tilde{\varrho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}([\text{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)]^c, r)]^c) \leq 1 - r,$$

$$\tilde{\tau}^{\tilde{\eta}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}([\text{int}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)]^c, r)]^c) \leq 1 - r.$$

Thus,

$$\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}([\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\text{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r), r)) \leq 1 - r, \quad \tilde{\tau}^{\tilde{\eta}}([\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r), r)) \leq 1 - r;$$

hence,

$$\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n.r), r) = \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r).$$

(3)⇒(4): Let σ_n be an r-SVNSIO set. Then, by (4), we have

$$\sigma_n \leq \text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r).$$

Thus, σ_n is an r-SVNPIO set.

(4)⇒(5): Since σ_n is an r-SVNSβIO set, $\sigma_n \leq \text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(CI^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)$. Then, $\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$ is r-SVNSIO, and, by (4), $\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(CI^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$; hence, $\tilde{\tau}^{\tilde{\rho}}\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$.

(5)⇒(6): Let σ_n be an r-SVNβIO set, then, by (5), $\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(Cl^{\star}(\sigma_n, r), r)$. Thus,

$$\sigma_n \leq \text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r).$$

Therefore, σ_n is an r-SVNPIO set.

(6)⇒(7): Let σ_n be an r-SVNSIO. Then, σ_n is r-SVNSβIO, by (6), σ_n is an r-SVNPIO set. Since σ_n is r-SVNSIO and r-SVNPIO, σ_n is r-SVNαIO.

(7) ⇒ (1): Suppose that $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$ is r-SVNSIO, and, by (7), $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$ is r-SVNαIO. Hence,

$$\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(Cl^{\star}(\sigma_n, r), r), r), r) = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r).$$

Hence,

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

Thus, $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected. □

Theorem 8. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS $r \in \zeta_0$ and $\sigma_n \in \zeta^{\tilde{X}}$. Then, the following are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected,

(2) $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$, for every $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$ and every r-SVN£O $\gamma_n \in \zeta^{\tilde{X}}$ with $\sigma_n\overline{q}\gamma_n$,

(3) $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$, for every $\sigma_n \in \zeta^{\tilde{X}}$ and r-SVN£O $\gamma_n \in \zeta^{\tilde{X}}$ with $\sigma_n\overline{q}\gamma_n$.

Proof. (1)⇒(2): Let $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$. Then, by (1),

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

Since $[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$ is an r-SVN£O and $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$, it implies that

$$\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c, r).$$

(2)⇒(1): Let $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$. Since $[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$ is an r-SVN£O, then, by (2),

$$\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c, r).$$

This implies that $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$, so

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

(2)⇒(3): Suppose that $\sigma_n \in \zeta^{\tilde{X}}$ and γ_n is an r-SVN£O with $\sigma_n\overline{q}\gamma_n$. Since

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r), r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r), r)) \leq 1-r.$$

By (2), we have $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$.

(3)⇒(2): Let $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$ and γ_n be an r-SVN£O with $\sigma_n\overline{q}\gamma_n$. Then, by (3), we obtain $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$. Since

$$\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{iny}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r),$$

then, we have $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$. □

Definition 15. *An SVNITS $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is called £-SVN-normal if, for every $[\sigma_n]_1\overline{q}[\sigma_n]_2$ with $\tilde{\tau}^{\tilde{\rho}}([\sigma_n]_1) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}([\sigma_n]_1) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}([\sigma_n]_1) \leq 1-r$ and $[\sigma_n]_2$ is r-SVN£O, there exists*

$[\gamma_n]_j \in \xi^{\tilde{X}}$, for $j = \{1,2\}$ with $\tilde{\tau}^{\tilde{\rho}}([\gamma_n]_1^c) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}([\gamma_n]_1^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}([\gamma_n]_1^c) \leq 1-r$, $[\gamma_n]_2$ is r-SVN£C such that $[\sigma_n]_2 \leq [\gamma_n]_1$, $[\sigma_n]_1 \leq [\gamma_n]_2$ and $[\gamma_n]_1 \overline{q} [\gamma_n]_2$.

Theorem 9. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS; then, the following are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVN-normal.
(2) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVNE-disconnected.

Proof. (1)⇒(2): Let $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$ and $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$ be an r-SVN£O. Then, $\sigma_n \overline{q} [\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$. By the £-SVN-normality of $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$, there exist $[\gamma_n]_i \in \xi^{\tilde{X}}$, for $i = \{1,2\}$ with

$$\tilde{\tau}^{\tilde{\rho}}([\gamma_n]_1^c) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}([\gamma_n]_1^c) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}([\gamma_n]_1^c) \leq 1-r,$$

and $[\gamma_n]_2^c$ r-SVN£C such that $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c \leq [\gamma_n]_1$, $\sigma_n \leq [\gamma_n]_2$ and $[\gamma_n]_1 \overline{q} [\gamma_n]_2$. Since

$$\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]_2, r) = [\gamma_n]_2 \leq [\gamma_n]_1^c \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r),$$

we have $\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = [\gamma_n]_2$. Since $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c \leq [\gamma_n]_1 \leq [\gamma_n]_2^c = [\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$, so $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c = [\gamma_n]_1$. Hence, $\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = [\gamma_n]_1^c$ and

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

Thus, $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVNE-disconnected.

(2)⇒(1): Suppose that $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$ and γ_n is an r-SVN£O with $\sigma_n \overline{q} \gamma_n$. By the £-SVNE-disconnected of $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$, we have

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r,$$

and $[\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$ is r-SVN£O. Since $\sigma_n \overline{q} \gamma_n$, $\sigma_n \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$ and $\gamma_n \leq [\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$. Thus, $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVN-normal. □

Theorem 10. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS, $\sigma_n, \sigma_n \xi^{\tilde{X}}$ and $r \in \xi_0$. Then, the following properties are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected.
(2) If σ_n is r-SVNRIO, then σ_n is r-SVN£C.
(3) If σ_n is r-SVNRIC, then σ_n is r-SVN£O.

Proof. (1)⇒(2): Let σ_n be an r-SVNRIO. Then, $\sigma_n = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$ and $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$. By (1),

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

Hence $\sigma_n = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) = \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$.

(2)⇒(1): Suppose that $\sigma_n = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$, then $\tilde{\tau}^{\tilde{\rho}}(\sigma_n) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\sigma_n) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\sigma_n) \leq 1-r$, by (2), σ_n is r-SVN£C. This implies that

$$\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r), r) = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \leq \mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r).$$

Thus,

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{£}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r,$$

then $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVNE-disconnected.

(2) ⇔ (3): Obvious. □

Remark 3. *The union of two r-SVNRIO sets need not to be an r-SVNRIO.*

Theorem 11. *If* $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is £-SVNE-disconnected and* $\sigma_n, \gamma_n \in \xi^{\tilde{X}}$, $r \in \xi_0$. *Then, the following properties hold:*

(1) *If* σ_n *and* γ_n *are r-SVNRIC, then* $\sigma_n \wedge \gamma_n$ *is r-SVNRIC.*
(2) *If* σ_n *and* γ_n *are r-SVNRIO, then* $\sigma_n \vee \gamma_n$ *is r-SVNRIO.*

Proof. Let σ_n and γ_n be r-SVNRIC. Then, $\tilde{\tau}^{\tilde{\rho}}([\sigma_n]^c) \geq r, \tilde{\tau}^{\tilde{\varrho}}([\sigma_n]^c) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}([\sigma_n]^c) \leq 1-r$ and $\tilde{\tau}^{\tilde{\rho}}([\gamma_n]^c) \geq r, \tilde{\tau}^{\tilde{\varrho}}([\gamma_n]^c) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}([\gamma_n]^c) \leq 1-r$, by Theorem 7, we have

$$\tilde{\tau}^{\tilde{\rho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)]^c) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)]^c) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)]^c) \leq 1-r,$$

and

$$\tilde{\tau}^{\tilde{\rho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)]^c) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\gamma_n, r)]^c) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}([\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\gamma_n, r)]^c) \leq 1-r.$$

This implies that

$$\begin{aligned} \sigma_n \wedge \gamma_n &= C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \wedge C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r) \\ &= \text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) = \text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n, r) \\ &\leq C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n, r), r). \end{aligned}$$

On the other hand,

$$\begin{aligned} C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n, r), r) &= C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r) \\ &\leq C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r) \wedge C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r) \\ &= \sigma_n \wedge \gamma_n. \end{aligned}$$

Thus, $C_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\text{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n, r), r) = \sigma_n \wedge \gamma_n$. Therefore, $\sigma_n \wedge \gamma_n$ is an r-SVNRIC.
(2) The proof is similar to that of (1). □

Theorem 12. *Let* $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be an SVNITS and* $r \in \xi_0$. *Then, the following properties are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is £-SVNE-disconnected,*
(2) $\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r$, *for every r-SVNSIO* $\sigma_n \in \xi^{\tilde{X}}$,
(3) $\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r$, *for every r-SVNPIO* $\sigma_n \in \xi^{\tilde{X}}$,
(4) $\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r$, *for every r-SVNRIO* $\sigma_n \in \xi^{\tilde{X}}$.

Proof. (1) ⇒ (2) and (1) ⇒ (3). Let σ_n be an r-SVNSIO (r-SVNPIO). Then, σ_n is r-SVNSβIO, and, by Theorem 7, we have,

$$\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

(2)⇒(4) and (3)⇒(4). Let σ_n be an r-SVNRIO. Then, σ_n is r-SVNPIO and r-SVNSIO. Thus,

$$\tilde{\tau}^{\tilde{\rho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1-r, \quad \tilde{\tau}^{\tilde{\eta}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1-r.$$

(4)⇒(1). Suppose that

$$\tilde{\tau}^{\tilde{\rho}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^{\tilde{\varrho}}(\text{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^{\tilde{\eta}}(\text{int}_{\tilde{\tau}^{\tilde{\eta}}}(\text{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r), r)) \geq r.$$

Then, by (4), we have

$$\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r),r),r)) \geq r, \qquad \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n,r),r),r)) \geq r,$$
$$\tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r),r),r)) \geq r.$$

Hence,

$$\begin{aligned}\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) &\leq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r),r),r)\\ &= \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r),r),r),r)\\ &= \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r),r) \leq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r).\end{aligned}$$

Thus, $\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)) \geq r, \tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n,r)) \leq 1-r, \tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r)) \leq 1-r$; hence, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an £-SVNE-disconnected. □

Definition 16. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS. Then, σ_n is said to be an r-SVN£SO if $\sigma_n \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r),r)$.*

Definition 17. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS for each $r \in \zeta_0$, $\sigma_n \in \xi^{\tilde{X}}$ and $x_{s,t,p} \in Pt(\xi^{\tilde{X}})$. Then, $x_{s,t,p}$ is called an r-SVN$\delta\mathcal{I}$-cluster point of σ_n if, for every $\gamma_n \in Q_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)$, we have $\sigma_n q \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r),r)$.*

Definition 18. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS for each $r \in \zeta_0$, $\sigma_n \in \xi^{\tilde{X}}$ and $x_{s,t,p} \in Pt(\xi^{\tilde{X}})$. Then, the single-valued neutrosophic $\delta\mathcal{I}$-closure operator is a mapping $\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}} : \xi^{\tilde{X}} \times \zeta_0 \to \xi^{\tilde{X}}$ that is defined as: $\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) = \bigvee\{x_{s,t,p} \in Pt(\xi^{\tilde{X}})$ is r-SVN$\delta\mathcal{I}$-cluster point of $\sigma_n\}$.*

Lemma 3. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS. Then, σ_n is r-SVN£SO iff $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) = \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r),r)$.*

Proof. Obvious. □

Lemma 4. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS for each $\sigma_n \in \xi^{\tilde{x}}$ and $r \in \zeta_0$. Then, $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) \leq \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)$.*

Proof. Obvious. □

Lemma 5. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and σ_n be an r-SVN£SO. Then, $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) = \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)$.*

Proof. We show that $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) \leq \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)$. Suppose that $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r) \not\geq \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)$,; then, there exist $\nu \in \tilde{X}$ and $s,t,p \in \zeta_0$ such that

$$\tilde{\rho}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)}(\nu) < s \leq \tilde{\rho}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)}(\nu), \qquad \tilde{\varrho}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n,r)}(\nu) \geq t > \tilde{\varrho}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n,r)}(\nu), \tag{1}$$
$$\tilde{\eta}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r)}(\nu) \geq p > \tilde{\eta}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r)}(\nu).$$

By the definition of $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}$, there exists $\tilde{\tau}^{\tilde{\rho}}([\gamma_n]^c) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}([\gamma_n]^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}([\gamma_n]^c) \leq 1-r$ with $\sigma_n \leq \gamma_n$ such that

$$\tilde{\rho}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)}(\nu) \leq \tilde{\rho}_{\gamma_n}(\nu) < s < \tilde{\rho}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)}(\nu), \qquad \tilde{\varrho}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n,r)}(\nu) \geq \tilde{\varrho}_{\gamma_n}(\nu) > t > \tilde{\varrho}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}}}(\sigma_n,r)}(\nu),$$
$$\tilde{\eta}_{\mathrm{C}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r)}(\nu) \geq \tilde{\rho}_{\gamma_n}(\nu) > p > \tilde{\eta}_{\mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\eta}}}(\sigma_n,r)}(\nu).$$

Then, $[\gamma_n]^c \in Q_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p}, r)$ and

$$\begin{aligned}[\sigma_n]^c \geq [\gamma_n]^c \quad & \Rightarrow \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r) \geq \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c, r) \\ & \Rightarrow \mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r) \geq \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c, r) \\ & \Rightarrow [\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c \geq [\gamma_n]^c.\end{aligned}$$

Thus, $\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}[\gamma_n]^c$. Hence, $\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c, r), r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$. Since σ_n is an r-SVN£SO, we have $\mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r)\overline{q}\sigma_n$. So, $x_{s,t,p}$ is not an r-SVN$\delta\mathcal{I}$-cluster point of σ_n. It is a contradiction for equation 3. Thus, $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \geq \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$. By Lemma 4, we have $\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) = \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$. □

Theorem 13. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS. Then, the following properties are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected,

(2) If σ_n is r-SVNSβIO and γ_n is r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$,

(3) If σ_n is r-SVNSIO and γ_n is r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$,

(4) $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$, for every r-SVNSIO set $\sigma_n \in \xi^{\tilde{X}}$ and every r-SVN£SO $\gamma_n \in \xi^{\tilde{X}}$ with $\sigma_n\overline{q}\gamma_n$,

(5) If σ_n is an r-SVNPIO and γ_n is an r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$.

Proof. (1)⇒(2): Let σ_n be an r-SVNSβIO and γ_n be an r-SVN£SO, by Theorem 7, $\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r))$
$\geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$. Then,

$$\begin{aligned}\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \quad & \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \wedge \mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r] \\ & \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\gamma_n \wedge \mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r], r] \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\gamma_n \wedge \mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r], r] \\ & \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\gamma_n \wedge \mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r), r] \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}[\gamma_n \wedge \gamma_n, r].\end{aligned}$$

Hence, $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$.

(2)⇒(3): It follows from the fact that every r-SVNSIO set is an r-SVNSβIO.

(3)⇒(4): Clear.

(4)⇒(1): Let σ_n be an r-SVNSIO. Since $[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{int}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\sigma_n]^c, r), r), r)$ we have, $[\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c$ is an r-SVN£SO. Then, by (4), $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c, r)$. Thus, $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \leq [\mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)]^c, r)]^c = \mathrm{int}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r), r)$. Therefore, $\tilde{\tau}^{\tilde{\rho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r$, $\tilde{\tau}^{\tilde{\varrho}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\varrho}}}(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^{\tilde{\eta}}(\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\eta}}}(\sigma_n, r)) \leq 1 - r$. Thus, by Theorem 12, $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected.

(2)⇒(5): It follows from the fact that every r-SVNPIO is an r-SVNSβIO. □

Corollary 1. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS. Then, the following properties are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is £-SVNE-disconnected.

(2) If σ_n is an r-SVNSβIO and γ_n is an r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$.

(3) If σ_n is an r-SVNSIO and γ_n is an r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$.

(4) If σ_n is an r-SVNPIO and γ_n is an r-SVN£SO, then $\mathrm{CI}^{\pounds}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r) \wedge \mathrm{C}_{\delta\mathbf{I}\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \mathrm{C}_{\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n \wedge \gamma_n)$.

Proof. It follows directly from Lemma 3 and 5. □

5. Some Types of Separation Axioms

In this section, some kinds of separation axioms, namely r-single valued neutrosophic ideal-R_i (r-$SVNIR_i$, for short), where $i = \{0, 1, 2, 3\}$, and r-single valued neutrosophic ideal-T_j (r-$SVNIT_j$, for short), where $j = \{1, 2, 2\frac{1}{2}, 3, 4\}$, in the sense of Šostak are defined. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

Definition 19. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $r \in \zeta_0$. Then, $\tilde{X}$ is called:*

(1) *r-$SVNIR_0$ iff $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1}, r)$ implies $y_{s_1,t_1,p_1}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p}, r)$ for any $x_{s,t,p} \neq y_{s_1}$, t_1, p_1.*

(2) *r-$SVNIR_1$ iff $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1}, r)$ implies that there exist r-SVN£O sets $\sigma n, \gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$, $y_{s_1,t_1,p_1} \in \gamma_n$ and $\sigma_n\overline{q}\gamma_n$.*

(3) *r-$SVNIR_2$ iff $x_{s,t.p}\overline{q}\varsigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\varsigma_n, r)$ implies there exist r-SVN£O sets $\sigma n, \gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$, $\varsigma_n \leq \gamma_n$ and $\sigma_n\overline{q}\gamma_n$.*

(4) *r-$SVNIR_3$ iff $[\varsigma_n]_1 = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\varsigma_n]_1, r)\overline{q}[\varsigma_n]_2 = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\varsigma_n]_2, r)$ implies that there exist r-SVN£O sets $\sigma_n, \gamma_n \in \zeta^{\tilde{X}}$ such that $[\varsigma_n]_1 \leq \sigma_n$, $[\varsigma_n]_2 \leq \gamma_n$ and $\sigma_n\overline{q}\gamma_n$.*

(5) *r-$SVNIT_1$ iff $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$ implies that there exists r-SVN£O $\sigma n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$ and $y_{s_1,t_1,p_1}\overline{q}\sigma_n$.*

(6) *r-$SVNIT_2$ iff $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$ implies that there exist r-SVN£O sets $\sigma n, \gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$, $y_{s_1,t_1,p_1} \in \gamma_n$ and $\sigma_n\overline{q}\gamma_n$.*

(7) *r-$SVNIT_{2\frac{1}{2}}$ iff $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$ implies that there exist r-SVN£O sets $\sigma n, \gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$, $y_{s_1,t_1,p_1} \in \gamma_n$ and $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r)$.*

(8) *r-$SVNIT_3$ iff it is r-$SVNITR_2$ and r-$SVNIT_1$.*

(9) *r-$SVNIT_4$ iff it is r-$SVNITR_3$ and r-$SVNIT_1$.*

Theorem 14. *Let $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $r \in \zeta_0$. Then, the following statements are equivalent:*

(1) *$(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_0$.*

(2) *If $x_{s,t,p}\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$, then there exists r-SVN£O $\gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p}\overline{q}\gamma_n$ and $\sigma_n \leq \gamma_n$.*

(3) *If $x_{s,t,p}\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$, then $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p}, r)\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$.*

(4) *If $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1}, r)$, then $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p}, r)\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1}, r)$.*

Proof. (1)⇒(2): Let $x_{s,t,p}\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$. Then,

$$s + \tilde{\rho}_{\sigma_n}(\nu) < 1, \quad t + \tilde{\varrho}_{\sigma_n}(\nu) \geq 1, \quad p + \tilde{\eta}_{\sigma_n}(\nu) \geq 1,$$

for every $y_{s_1,t_1,p_1} \in \sigma_n$, we have $s_1 < \tilde{\rho}_{\sigma_n}(\nu)$, $t_1 \geq \tilde{\varrho}_{\sigma_n}(\nu)$ and $p_1 \geq \tilde{\eta}_{\sigma_n}(\nu)$. Thus, $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1}, r)$. Since $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is an r-$SVNIR_0$, we obtain $y_{s_1,t_1,p_1}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}$ $(x_{s,t,p}, r)$. By Lemma 2(2), there exists an r-SVN£O $\varsigma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p}\overline{q}\varsigma_n$ and $y_{s_1,t_1,p_1} \leq \varsigma_n$. Let

$$\gamma_n = \bigvee_{y_{s_1,t_1,p_1} \in \sigma_n} \{\varsigma_n : x_{s,t,p}\overline{q}\varsigma_n,\ y_{s_1,t_1,p_1} \in \varsigma_n\}.$$

From Lemma 1(1), γ_n is an r-SVN£O. Then, $x_{s,t,p}\overline{q}\gamma_n$, $\sigma_n \leq \gamma_n$.

(2)⇒(3): Let $x_{s,t,p}\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$. Then, there exists an r-SVN£O $\gamma_n \in \zeta^{\tilde{X}}$ such that $x_{s,t,p}\overline{q}\gamma_n$ and $\sigma_n \leq \gamma_n$. Since for every $\nu \in \tilde{X}$,

$$s < 1 - \tilde{\rho}_{\gamma_n}(\nu), \quad t \geq 1 - \tilde{\varrho}_{\gamma_n}(\nu), \quad p \geq 1 - \tilde{\eta}_{\gamma_n}(\nu),$$

we obtain

$$\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p}, r) \leq \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c, r) = [\gamma_n]^c \leq [\sigma_n]^c.$$

Therefore, $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)\overline{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)$.

(3)⇒(4): Let $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$. Then, $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r) = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r),r)$. By (3), $s_1,t_1,p_1(x_{s,t,p},r)\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$.

(4)⇒(1): Clear. □

Theorem 15. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $r\in\zeta_0$. Then, if $\tilde{X}$ is*

(1) [r-$SVNIR_3$ and r-$SVNIR_0$] $\Rightarrow^{(a)}$ r-$SVNIR_2 \Rightarrow^{(b)}$ r-$SVNIR_1 \Rightarrow^{(c)}$ r-$SVNIR_0$.

(2) r-$SVNIT_2 \Rightarrow$ r-$SVNIR_1$.

(3) r-$SVNIT_3 \Rightarrow$ r-$SVNIR_2$.

(4) r-$SVNIT_4 \Rightarrow$ r-$SVNIR_3$.

(5) r-$SVNIT_4 \Rightarrow^{(a)}$ r-$SVNIT_3 \Rightarrow^{(b)}$ r-$SVNIT_{2\frac{1}{2}} \Rightarrow^{(c)}$ r-$SVNIT_2 \Rightarrow^{(d)}$ r-$SVNIT_1$.

Proof. (1_a). Let $x_{s,t.p}\overline{q}\varsigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\varsigma_n,r)$, by Theorem 14(3), $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)\overline{q}\varsigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\varsigma_n,r)$. Since $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_3$ and $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r) = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r),r)$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)\leq\sigma_n$, $\varsigma_n\leq\gamma_n$ and $\sigma_n\overline{q}\gamma_n$. Hence, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_2$.

(1_b). For each $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$, by r-$SVNIR_2$ of $\tilde{X}$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\sigma_n$, $y_{s_1,t_1,p_1},r\in\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r,r)\leq\gamma_n$ and $\sigma_n\overline{q}\gamma_n$. Thus, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_1$.

(1_c). Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be r-$SVNIR_1$. Then, for every $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r,r)$ and $x_{s,t,p}\neq y_{s_1,t_1,p_1}$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\sigma_n$, $y_{s_1,t_1,p_1}\in\gamma_n$ and $\sigma_n\overline{q}\gamma_n$. Hence, $x_{s,t,p}\in\sigma_n\leq[\gamma_n]^c$. Since γ_n is an r-SVN£O set, we obtain $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)\leq\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c,r) = [\gamma_n]^c\leq[y_{s_1,t_1,p_1}]^c$. Thus, $y_{s_1,t_1,p_1}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(x_{s,t,p},r)$ and $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_0$.

(2). Let $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_2,p_1},r)$. Then, $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$. By r-$SVNIT_2$ of $\tilde{X}$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{st,p}\in\sigma_n$, $y_{s_1,t_1,p_1}\in\gamma_n$ and $\sigma_n\overline{q}\gamma_n$. Hence, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_1$.

(3) and (4) The proofs are direct consequence of (2) .

(5_a). The proof is direct consequence of (1).

(5_b). For each $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$, since $\tilde{X}$ is both r-$SVNIR_2$ and r-$SVNIT_1$, then, there exists an r-SVN£O set $\varsigma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\varsigma_n$ and $y_{s_1,t_1,p_1}\overline{q}\varsigma_n$. Then,

$$x_t\in\varsigma_n = \mathrm{int}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\varsigma_n,r)\leq\mathrm{int}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([y_{s_1,t_1,p_1}]^c,r) = [\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)]^c.$$

Hence, $x_{s,t,p}\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$. By r-$SVNIR_2$ of $\tilde{X}$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\sigma_n$, $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)\leq\gamma_n$ and $\sigma_n\overline{q}\gamma_n$. Thus, $\sigma_n\leq[\gamma_n]^c$, so

$$\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)\leq\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]^c,r) = [\gamma_n]^c\leq[\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)]^c.$$

It implies $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$ with $x_{s,t,p}\in\sigma_n$ and $y_{s_1,t_1,p_1}\in\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(y_{s_1,t_1,p_1},r)$. Thus, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIT_{2\frac{1}{2}}$.

(5_c). Let $x_{s,t,p}\overline{q}y_{s_1,t_1,p_1}$. Then, by r-$SVNIT_{2\frac{1}{2}}$ of $\tilde{X}$, there exist r-SVN£O sets $\sigma_n,\gamma_n\in\zeta^{\tilde{X}}$ such that $x_{s,t,p}\in\sigma_n$, $y_{s_1,t_1,p_1}\in\gamma_n$ and $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n,r)\overline{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n,r)$, which implies that $\sigma_n\overline{q}\gamma_n$. Thus, $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIT_2$.

(5_d). Similar to the proof of (5_c). □

Theorem 16. *Let $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ be an SVNITS and $r\in\zeta_0$. Then, the following statements are equivalent:*

(1) $(\tilde{X},\tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}},\mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is r-$SVNIR_2$.

(2) *If* $x_{s,t,p} \in \sigma_n$ *and* σ_n *is r-SVN£O set, then there exists r-SVN£O set* $\gamma_n \in \xi^{\tilde{X}}$ *such that* $x_{s,t,p} \in \gamma_n \leq \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq \sigma_n$.

(3) *If* $x_{s,t,p}\bar{q}\sigma_n = \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\sigma_n, r)$*, then there exists r-SVN£O set* $[\gamma_n]_j \in \xi^{\tilde{X}}$, $j = \{1, 2\}$ *such that* $x_{s,t,p} \in [\gamma_n]_1, \sigma_n \leq [\gamma_n]_2$ *and* $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]_1, r)\bar{q}\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}([\gamma_n]_2, r)$.

Proof. Similar to the proof of Theorem 14. □

Theorem 17. *Let* $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be an SVNITS and* $r \in \zeta_0$*. Then, the following statements are equivalent:*

(1) $(\tilde{X}, \tilde{\tau}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is* $r\text{-}SVNIR_3$.

(2) *If* $[\sigma_n]_1\bar{q}[\sigma_n]_2$ *and* $[\sigma_n]_1, [\sigma_n]_2$ *are r-SVN£C sets, then there exists r-SVN£O set* $\gamma_n \in \xi^{\tilde{X}}$ *such that* $[\sigma_n]_1 \leq \gamma_n$ *and* $\mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq [\sigma_n]_2$.

(3) *For any* $[\sigma_n]_1 \leq [\sigma_n]_2$*, where* $[\sigma_n]_1$ *is an r-SVN£O set, and* $[\sigma_n]_2$ *is an r-SVN£C set, then, there exists an r-SVN£O set* $\gamma_n \in \xi^{\tilde{X}}$ *such that* $[\sigma_n]_1 \leq \gamma_n \leq \mathrm{CI}^{\pounds}_{\tau^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}}(\gamma_n, r) \leq [\sigma_n]_2$.

Proof. Similar to the proof of Theorem 15. □

Theorem 18. *Let* $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \to (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be a £-SVNI-irresolute, bijective, £-SVNI-irresolute open mapping and* $(\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is* $r\text{-}SVNIR_2$*. Then,* $(\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is* $r\text{-}SVNIR_2$.

Proof. Let $y_{s,t,p}\bar{q}\varsigma_n = CI^{\star}(\varsigma_n, r)$. Then, by Definition 11, ς_n is an r-SVN£C set in $\tilde{Y}$. By Theorem 3(2), $f^{-1}(\varsigma_n)$ is an r-SVN£C set in $\tilde{X}$. Put $y_{s,t,p} = f(x_{s,t,p})$. Then, $x_{s,t,p}\bar{q}f^{-1}(\varsigma_n)$. By $r\text{-}SVNIR_2$ of $\tilde{X}$, there exist r-SVN£O sets $\sigma_n, \gamma_n \in \xi^{\tilde{X}}$ such that $x_{s,t,p} \in \sigma_n$, $f^{-1}(\varsigma_n) \leq \gamma_n$ and $\sigma_n\bar{q}\gamma_n$. Since f is bijective and £-SVNI-irresolute open, $y_{s,t,p} \in f(\sigma_n)$, $\varsigma_n \leq f(f^{-1}(\varsigma_n)) \leq f(\gamma_n)$ and $f(\sigma_n)\bar{q}f(\gamma_n)$. Thus, $(\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ is $r\text{-}SVNIR_2$. □

Theorem 19. *Let* $f : (\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}) \to (\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be an £-SVNI-irresolute, bijective, £-SVNI-irresolute open mapping and* $(\tilde{X}, \tilde{\tau}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_1^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *be an* $r\text{-}SVNIR_3$*. Then,* $(\tilde{Y}, \tilde{\tau}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}}, \mathcal{I}_2^{\tilde{\rho}\tilde{\varrho}\tilde{\eta}})$ *is* $r\text{-}SVNIR_3$.

Proof. Similar to the proof of Theorem 18. □

6. Conclusions

In summary, we have introduced the definition of the *r*-single valued neutrosophic £-closed and *r*-single valued neutrosophic £-open sets over single valued neutrosophic ideal topology space in Šostak's sense. Many consequences have been arisen up to show that how far topological structures are preserved by these *r*-single valued neutrosophic £-closed. We also have provided some counterexamples where such properties fail to be preserved. The most important contribution to this area of research is that we have introduced the notion of £-single valued neutrosophic irresolute mapping, £-single valued neutrosophic extremally disconnected spaces, £-single valued neutrosophic normal spaces and that we defined some kinds of separation axioms, namely $r\text{-}SVNIR_i$, where $i = \{0, 1, 2, 3\}$, and $r\text{-}SVNIT_j$, where $j = \{1, 2, 2\frac{1}{2}, 3, 4\}$, in the sense of Šostak. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

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Discussion for Further Works: The theory in this article can be extended in the following natural ways. One may study the properties of neutrosophic metric topological spaces using the concepts defined through this paper.

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