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Research Article

Study of Two Kinds of Quasi AG-Neutrosophic Extended Triplet Loops

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Abel-Grassmann's groupoid and neutrosophic extended triplet loop are two important algebraic structures that describe two kinds of generalized symmetries. In this paper, we investigate quasi AG-neutrosophic extended triplet loop, which is a fusion structure of the two kinds of algebraic structures mentioned above. We propose new notions of AG-(l,r)-Loop and AG-(r,l)-Loop, deeply study their basic properties and structural characteristics, and prove strictly the following statements: (1) each strong AG-(l,r)-Loop can be represented as the union of its disjoint sub-AG-groups, (2) the concepts of strong AG-(l,r)-Loop, strong AG-(l,l)-Loop, and AG-(l,lr)-Loop are equivalent, and (3) the concepts of strong AG-(r,l)-Loop and strong AG-(r,r)-Loop are equivalent.

1. Introduction

The so-called left almost semigroup (LA-semigroup) was actually the concept of an Abel-Grassmann's groupoid (AG-groupoid), which was put forward by Kazim and Naseeruddin [1] at the first time in 1972. Different classes of AG-groupoids and their concerned characteristics have been studied in [2–5].

Neutrosophic set (NS) was first put forward by Smarandache in [6]. Then, it has been growing promptly over the previous 15 years. Nowadays, NS theory is widely used in a couple of sectors such as professional selection [7], integrated speech and text sentiment analysis [8], finite automata [9], clustering methods [10], and deep learning [11]. Besides, more new theoretical studies on NS in [12–17] have been conducted and a few significant results have been gained.

The concept of Abel-Grassmann's neutrosophic extended triplet loop (AG-NET-Loop), which plays a significant role in neutrosophic triplet algebraic structures, was proposed in [18], that is, an AG-NET-Loop is both an AG-groupoid and a neutrosophic extended triplet loop (NET-Loop). In [19], the concept of neutrosophic triplet elements (NT-elements) and quasi neutrosophic triplet loops were introduced. In [20], two kinds of quasi AG-NET-Loops (AG-(l,l)-Loop and AG-(r,r)-Loop) were proposed and their basic properties were investigated. As a continuation of [20], we propose two other kinds of quasi AG-NET-Loops, which are the AG-(l,r)-Loop and the AG-(r,l)-Loop. We study their properties and analyze their relationship.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on quasi AG-NET-Loop are given. Some properties and structures about the AG-(l,r)-Loop are discussed in Section 3. The relations among four kinds of quasi AG-NET-Loops are analyzed in Section 4. Some properties about the alternative quasi AG-NET-Loops are discussed in Section 5. Lastly, Section 6 presents the summary and the direction of future efforts.

2. Preliminaries

A groupoid $(G, *)$ is called an AG-groupoid if it holds the left invertive law, that is, for all $x, y, z \in G$, $(x * y) * z = (z * y) * x$. In an AG-groupoid $(G, *)$ the medial law holds, for all $x_1, x_2, x_3, x_4 \in G$, $(x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4)$. An AG-groupoid $(G, *)$ is called locally associative if for all

$x \in G$, $(x * x) * x = x * (x * x)$. In an AG-groupoid $(G, *)$, for all $x \in G$, $k \in Z^+$, x^k is defined as follows: $x^1 = x, x^2 = x * x, x^3 = x^2 * x, x^4 = x^3 * x, \ldots, x^k = x^{k-1} * x$.

Definition 1 (see [21]). Let G be a nonempty set together with a binary operation *. Then, G is called a neutrosophic extended triplet set if, for all $x \in G$, there exist a neutral of "x" and an opposite of "x" (denoted by $\text{neut}(x)$ and $\text{anti}(x)$, respectively), such that $\text{neut}(x), \text{anti}(x) \in G$, and $\text{neut}(x) * x = x * \text{neut}(x) = x$, $\text{anti}(x) * x = x * \text{anti}(x) = \text{neut}(x)$. The triplet $(x, \text{neut}(x), \text{anti}(x))$ is called a neutrosophic extended triplet (NET).

Definition 2 (see [18]). An NET set $(G, *)$ is called an NET-Loop, if, for all $x, y \in G$, one has $x * y \in G$.

Definition 3 (see [18]). An AG-groupoid $(G, *)$ is called an AG-NET-Loop if it is an NET-Loop.

An AG-NET-Loop G is called a commutative AG-NET-Loop if for all $x, y \in G, x * y = y * x$.

Theorem 1 (see [18]). *Let $(G, *)$ be an AG-NET-Loop. Then,*

(1) For all $x \in G$, $neut(x)$ is unique

(2) For all $x \in G$, $(neut(x))^2 = neut(x)$

Definition 4 (see [2]). AG-groupoid $(G, *)$ is called regular if, for all $a \in G$, there exists $m \in G$, $a = (a * m) * a$.

Definition 5 (see [20]). Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-*(l,l)*-Loop if, for all $a \in G$, there exist a local *(l,l)*-neutral element of"a" and a local *(l,l)*-opposite element of"a" (denoted by $nll(a)$ and $oll(a)$, respectively), such that $nll(a) \in G, oll(a) \in G$, and $nll(a) * a = a$ and $oll(a) * a = nll(a)$.

Definition 6 (see [20]). Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(r,r)-Loop if, for all $a \in G$, there exist a local (r,r)-neutral element of "a" and a local (r,r)-opposite element of "a" (denoted by $nrr(a)$ and $orr(a)$, respectively), such that $nrr(a) \in G, orr(a) \in G$, and $a * nrr(a) = a$ and $a * orr(a) = nrr(a)$.

Definition 7. Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(l,r)-Loop if, for all $a \in G$, there exist a local (l,r)-neutral element of "a" and a local (l,r)-opposite element of "a" (denoted by $nlr(a)$ and $olr(a)$, respectively), such that $nlr(a) \in G, olr(a) \in G$, and $nlr(a) * a = a$ and $a * olr(a) = nlr(a)$.

Remark 1. For quasi AG-NET-Loop, we will use the notations such as AG-NET-Loop. If $nlr(a)$ and $olr(a)$ are not unique, then the set of all local (l,r)-neutral elements of"a" and the set of all local (l,r)-opposite elements of "a" are denoted by $\{nlr(a)\}$ and $\{olr(a)\}$, respectively.

Definition 8. Let $(G, *)$ be an AG-groupoid. Then, G is called an AG-(r,l)-Loop if, for all $a \in G$, there exist a local (r,l)-neutral element of "a" and a local (r,l)-opposite element of"a" (denoted by $nrl(a)$ and $orl(a)$, respectively), such that $nrl(a) \in G, orl(a) \in G$, and $a * nrl(a) = a$ and $orl(a) * a = nrl(a)$.

Definition 9. Let $(G, *)$ be an AG-(l,r)-Loop. Then, G is called an AG-(l,lr)-Loop if, for all $a \in G$, $olr(a) * a = a * olr(a) = nlr(a)$.

Definition 10 (see [22]). An AG-groupoid G with a left identity is called an AG-group if each $a \in G$ has an inverse element a'.

3. AG-(l,r)-Loop and Strong AG-(l,r)-Loop

Theorem 2. *Let $(G, *)$ be a groupoid. Then, G is an AG-(l,r)-Loop iff it is a regular AG-groupoid.*

Proof. Necessity: if G is an AG-(l,r)-Loop, from Definition 7, for all $a \in G$, there exist $nlr(a), olr(a) \in G, nlr(a) * a = a$, and $a * olr(a) = nlr(a)$. We have $(a * olr(a)) * a = a$. By Definition 4, G is a regular AG-groupoid.

Sufficiency: if G is a regular AG-groupoid, from Definition 4, for all $a \in G$, there exists $m \in G$ and $a = (a * m) * a$. Set $nlr(a) = a * m$, by Definition 7, G is an AG-(l,r)-Loop.

Example 1 illustrates that an AG-groupoid may be neither an AG-(l,l)-Loop nor an AG-(l,r)-Loop nor an AG-*(r,r)*-Loop nor an AG-(r,l)-Loop. ☐

Example 1. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 1. There is no $oll(2), olr(2), orr(2)$, and $orl(2)$ in G. That is, the element "2" in G has no local *(l,l)*-opposite element, no local (l,r)-opposite element, no local *(r,r)*-opposite element, and no local *(r,l)*-opposite element. From Definitions 5–8, G is neither an AG-*(l,l)*-Loop nor an AG-(l,r)-Loop nor an AG-*(r,r)*-Loop nor an AG-(r,l)-Loop.

Example 2 illustrates that an AG-*(l,r)*-Loop may be neither an AG-*(l,l)*-Loop nor an AG-*(r,r)*-Loop nor an AG-*(r,l)*-Loop.

Example 2. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 2. From Definition 7, G is an AG-*(l,r)*-Loop. However, there is no $oll(2), nrr(2)$, and $nrl(2)$ in G. From Definitions 5, 6, and 8, G is neither an AG-*(l,l)*-Loop nor an AG-*(r,r)*-Loop nor an AG-*(r,l)*-Loop.

Definition 11. An AG-*(l,r)*-Loop $(G, *)$ is called a strong AG-*(l,r)*-Loop if, for all $a \in G, nlr(a)^2 = nlr(a)$.

Example 3 illustrates that an AG-*(l,r)*-Loop is not always a strong AG-*(l,r)*-Loop.

Example 3. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 3. From Definition 7, G is an AG-*(l,r)*-Loop. However, $nlr(2) = 3, 3 * 3 = 1$; thus, G is not a strong AG-*(l,r)*-Loop.

Table 1: Table of Example 1.

*	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	2	1	1	1	1
3	1	1	3	1	1	1	1	1
4	1	2	1	4	1	1	1	1
5	1	1	1	1	5	1	1	1
6	1	1	1	1	1	6	8	8
7	1	1	1	1	1	8	7	8
8	1	1	1	1	1	8	8	8

Table 2: Table of Example 2.

*	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	4	4	1	1
3	1	1	3	1	3	3	7
4	1	2	1	1	2	1	1
5	1	2	3	4	5	3	7
6	1	1	3	1	3	6	7
7	1	1	7	1	7	7	7

Table 3: Table of Example 3.

*	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	3	3	1	3	3
3	1	2	1	2	1	2	2
4	1	2	3	4	5	6	7
5	1	1	1	5	5	1	1
6	1	2	3	6	1	6	6
7	1	2	3	7	1	6	7

Example 4 illustrates that a strong AG-*(l,r)*-Loop is not always an AG-NET-Loop.

Example 4. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 4. By Definition 11, G is a strong AG-*(l,r)*-Loop. However, since $1 * 4 \neq 4 * 1$, G is not an AG-NET-Loop.

Theorem 3. *Let $(G, *)$ be a strong AG-(l,r)-Loop. Then,*

(1) For all $a \in G$, $nlr(a)$ is unique

(2) For all $a \in G$, $nlr(nlr(a)) = nlr(a)$

(3) For all $a \in G$ and for any $r \in \{olr(a)\}$, $nlr(a) * r \in \{olr(a)\}$

(4) For all $a, b \in G$, $nlr(a * b) = nlr(a) * nlr(b)$

Proof

(1) If $(G, *)$ is a strong AG-*(l,r)*-Loop, suppose $a \in G$, there exist $nlr_1, nlr_2 \in \{nlr(a)\}$. By Definition 11, $nlr_1 * a = a, nlr_2 * a = a$, $nlr_1 * nlr_1 = nlr_1$, and $nlr_2 * nlr_2 = nlr_2$, and there exist $olr_1, olr_2 \in G$ which satisfy $a * olr_1 = nlr_1$ and $a * olr_2 = nlr_2$. We have

$$
\begin{aligned}
nlr_1 * nlr_2 &= (nlr_1 * nlr_1) * nlr_2 \\
&= (nlr_2 * nlr_1) * nlr_1 \\
&= (nlr_2 * nlr_1) * (a * olr_1) \\
&= (nlr_2 * a) * (nlr_1 * olr_1) \\
&\quad \text{(by the medial law)} \\
&= (nlr_1 * a) * (nlr_1 * olr_1) \\
&= (nlr_1 * nlr_1) * (a * olr_1) \\
&\quad \text{(by the medial law)} \\
&= nlr_1 * nlr_1 = nlr_1, \\
nlr_2 * nlr_1 &= (nlr_2 * nlr_2) * nlr_1 \\
&= (nlr_1 * nlr_2) * nlr_2 \\
&= (nlr_1 * nlr_2) * (a * olr_2) \\
&= (nlr_1 * a) * (nlr_2 * olr_2) \\
&\quad \text{(by the medial law)} \\
&= (nlr_2 * a) * (nlr_2 * olr_2) \\
&= (nlr_2 * nlr_2) * (a * olr_2) \\
&\quad \text{(by the medial law)} \\
&= nlr_2 * nlr_2 = nlr_2, \\
nlr_2 &= nlr_2 * nlr_1 \\
&= (nlr_2 * nlr_2) * nlr_1 \\
&= (nlr_1 * nlr_2) * nlr_2 \\
&= nlr_1 * nlr_2 = nlr_1.
\end{aligned} \tag{1}
$$

We know that $nlr_2 = nlr_1$, and $nlr(a)$ is unique.

(2) If $(G, *)$ is a strong AG-*(l,r)*-Loop, from Definition 11, we have, for all $a \in G$, $nlr(a)^2 = nlr(a)$. Thus, $nlr(nlr(a)) = nlr(a)$.

(3) Suppose $r \in \{olr(a)\}$; then,

$$
\begin{aligned}
a * (nlr(a) * r) &= (nlr(a) * a) * (nlr(a) * r) \\
&= (nlr(a) * nlr(a)) * (a * r) \quad \text{(by the medial law)} \\
&= nlr(a) * nlr(a) \\
&= nlr(a).
\end{aligned} \tag{2}
$$

So, we get $nlr(a) * r \in \{olr(a)\}$.

(4) From Definition 11, we have, for all $a, b \in G$,

Table 4: Table of Example 4.

*	1	2	3	4	5	6	7
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	1
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	1
6	1	1	3	4	1	6	6
7	1	1	3	4	1	6	7

$$\begin{aligned} a * b &= (nlr(a) * a) * (nlr(b) * b) \\ &= (nlr(a) * nlr(b)) * (a * b), \\ nlr(a) * nlr(b) &= (a * olr(a)) * (b * olr(b)) \\ &= (a * b) * (olr(a) * olr(b)). \end{aligned} \tag{3}$$

Therefore, $nlr(a * b) = nlr(a) * nlr(b)$. □

Example 5. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 5. It is a strong AG-*(l,r)*-Loop. We have (corresponding to the results of Theorem 3)

(1) For all $a \in G$, we can verify that $nlr(a)$ is unique.

(2) Being $nlr(nlr(1)) = nlr(1)$, $nlr(nlr(2)) = nlr(2)$, $nlr(nlr(3)) = nlr(3)$, $nlr(nlr(4)) = nlr(4)$, $nlr(nlr(5)) = nlr(5)$, $nlr(nlr(6)) = nlr(6)$, and $nlr(nlr(7)) = nlr(7)$, that is, for all $a \in G$, $nlr(nlr(a)) = nlr(a)$.

(3) For any $a \in G$, let $a = 1$, and we can get $nlr(1) = 1$ and $\{olr(1)\} = \{1, 2, 5, 6, 7\}$. Being $1 * 1 = 1 * 2 = 1 * 5 = 1 * 6 = 1 * 7 = 1 \in \{olr(1)\}$, that is, $nlr(1) * olr(1) \in \{olr(1)\}$, let $a = 3$, and we can get $nlr(3) = 1$, $olr(3) = 3$. Being $1 * 3 = 3 = olr(3)$, that is, $nlr(3) * olr(3) \in \{olr(3)\}$, we can verify other cases; thus, $nlr(a) * r \in \{olr(a)\}$.

(4) For any $a, b \in G$, without loss of generality, let $a = 1$ and $b = 3$; we can get $nlr(1 * 3) = nlr(1) * nlr(3)$. We can verify other cases; thus, $nlr(a * b) = nlr(a) * nlr(b)$.

Table 5: Table of Example 5.

*	1	2	3	4	5	6	7
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	2
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	5
6	1	1	3	4	1	6	1
7	1	2	3	4	5	1	7

Theorem 4. *Let* $(G, *)$ *be a strong AG-(l,r)-Loop. A binary* $\approx$ *on G is introduced as follows:*

$$\text{for all } a, b \in G, a \approx b \Leftrightarrow nlr(a) = nlr(b). \tag{4}$$

Then,

(1) The binary $\approx$ *on G is an equivalence relation, and the equivalent class contained x is denoted by* $[x]_{\approx}$

(2) For all $x \in G$, $[x]_{\approx}$ *is a sub-AG-group*

(3) $G = \cup_{x \in G} [x]_{\approx}$, *that is, each strong AG-(l,r)-Loop can be represented as the union of its disjoint sub-AG-groups*

Proof

(1) From the binary $\approx$ definition, it is easy to verify that $\approx$ has the properties of reflexive, symmetric, and transitive. Thus, it is an equivalence relation.

(2) For all $a \in [x]_{\approx}$, let $nlr(x) = e_x$, and we have $nlr(a) = nlr(x) = e_x$. From Theorem 3 (2), $nlr(e_x) = e_x$, and we have $e_x \in [x]_{\approx}$:

(i) By Definition 11, we have $e_x * a = nlr(a) * a = a$; thus, e_x is a left identity of $[x]_{\approx}$.

(ii) For all $a, b, c \in [x]_{\approx}$, the left invertive law holds directly.

(iii) For all $a, b \in [x]_{\approx}$, $nlr(a) = nlr(b) = e_x$; from Theorem 3 (4), $nlr(a * b) = nlr(a) * nlr(b) = e_x$; thus, $a * b \in [x]_{\approx}$.

(iv) For all $a \in [x]_{\approx}$, let $nlr(a) = e_x$, and suppose $p \in \{olr(a)\}$, $q = nlr(a) * p$; by Theorem 3 (3), we have $q \in \{olr(a)\}$, $a * q = nlr(a) = e_x$, and

$$\begin{aligned} nlr(q) &= nlr(nlr(a) * p) \\ &= nlr(nlr(a)) * nlr(p) \quad (\text{by Theorem 3 (4)}) \\ &= nlr(a) * nlr(p) \quad (\text{by Theorem 3 (2)}) \\ &= nlr(a * p) \quad (\text{by Theorem 3 (4)}) \\ &= nlr(nlr(a)) \\ &= nlr(a) \quad (\text{by Theorem 3 (2)}) \\ &= e_x. \end{aligned} \tag{5}$$

(v) $q * a = (nlr(q) * q) * a = (e_x * q) * a = (a * q) * e_x = e_x$. Thus, $q \in [x]_{\approx}$ and q is an inverse element of a. From Definition 10, $[x]_{\approx}$ is a sub-AG-group of G.

(3) By Theorem 3 (1), for all $a \in [x]_{\approx}$, $nlr(a)$ is unique. Then, $G = \cup_{x \in G} [x]_{\approx}$. □

Example 6. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the definition of operation $*$ on G is shown in Table 6. $[1]_{\approx} = \{1, 2, 3, 4\}$ and $[5]_{\approx} = \{5, 6, 7, 8\}$. $G = [1]_{\approx} \cup [5]_{\approx}$, and $[1]_{\approx}$ and $[5]_{\approx}$ are sub-AG-groups of G.

Let G be an AG-groupoid; then, a is an idempotent in G if $a \in G$, $a^2 = a$. The set of all idempotents in G is denoted by $E(G)$. An AG-groupoid G is called an AG-band if $G = E(G)$.

From now on, we assume that G is a strong AG-*(l,r)*-Loop, which is the same as Theorem 4. Let Y be an AG-band, $Y \subset G$, and for any $\alpha \in Y$, the equivalent class $[\alpha]_{\approx}$, which is defined in Theorem 4, will be denoted by S_α, and the elements of S_α will be denoted by $a_\alpha, b_\alpha, \ldots,$.

Theorem 5. *Let* $(G, *)$ *be a groupoid, Y be an AG-band,* $Y \subset G$. $G = \cup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ *is a strong AG-(l,r)-Loop with a left identity* e_α *for each* $\alpha \in Y$, *and* $S_\alpha \cap S_\beta = \varnothing$, $\alpha, \beta \in Y$ *and* $\alpha \neq \beta$. *If, for all* $a_\alpha \in S_\alpha$, *for all* $b_\beta \in S_\beta$, $a_\alpha * b_\beta = a_\alpha * e_\alpha$, *and* $b_\beta * a_\alpha = a_\alpha$, *then G is a strong AG-(l,r)-Loop.*

Table 6: Table of Example 6.

*	1	2	3	4	5	6	7	8
1	1	2	3	4	1	1	1	1
2	2	1	4	3	2	2	2	2
3	4	3	2	1	4	4	4	4
4	3	4	1	2	3	3	3	3
5	1	2	3	4	5	6	7	8
6	1	2	3	4	6	5	8	7
7	1	2	3	4	8	7	6	5
8	1	2	3	4	7	8	5	6

Proof. Suppose $G = \cup_{\alpha\in Y} S_\alpha$ is the groupoid, Y is an AG-band, for each $\alpha \in Y$, and S_α is a strong AG-*(l,r)*-Loop with a left identity e_α and $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$ in Y.

We first prove that G is an AG-groupoid. Let $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$, and $c_\gamma \in S_\gamma$ be arbitrary elements. Since S_α, S_β, and S_γ are strong AG-*(l,r)*-Loops, we have

$$\begin{aligned}\left(a_\alpha * b_\beta\right) * c_\gamma &= \left(a_\alpha * e_\alpha\right) * c_\gamma \\ &= \left(a_\alpha * e_\alpha\right) * e_\alpha \\ &= \left(e_\alpha * e_\alpha\right) * a_\alpha \quad \text{(by the left invertive law)} \\ &= e_\alpha * a_\alpha = a_\alpha,\end{aligned} \tag{6}$$

where $(c_\gamma * b_\beta) * a_\alpha = b_\beta * a_\alpha = a_\alpha = (a_\alpha * b_\beta) * c_\gamma$. Since S_α is a strong AG-*(l,r)*-Loop, the left invertive law holds directly for elements $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$. Thus, G is an AG-groupoid.

For any $b_\beta \in S_\beta$, we have $nlr(b_\beta) = e_\beta$ and $olr(b_\beta) * b_\beta = b_\beta * olr(b_\beta) = e_\beta$. Let $x \in G - S_\beta$, we denote e_x is the left identity in $[x]_\approx$, $LS_\beta = \left\{x | x * b_\beta = x * e_x, b_\beta * x = x, x \in G - S_\beta\right\}$, and $RS_\beta = \left\{x | x * b_\beta = b_\beta, b_\beta * x = b_\beta * e_\beta, x \in G - S_\beta\right\}$. Being $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$ in Y, we can get $LS_\beta \cap S_\beta \cap RS_\beta = \varnothing$ and $LS_\beta \cup S_\beta \cup RS_\beta = G$.

Depending on S_β, we have three cases to discuss. □

case 1. $LS_\beta = G - S_\beta, RS_\beta = \varnothing, x \in LS_\beta, x * b_\beta = x * e_x$, and $b_\beta * x = x$. Being $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$ in Y, we can get $x * e_x \in [x]_\approx, x * b_\beta \notin S_\beta$. That is, there is no element $x \notin S_\beta$ such that $x * b_\beta = b_\beta$.

case 2. $LS_\beta = \varnothing, RS_\beta = G - S_\beta, x \in RS_\beta$, $x * b_\beta = b_\beta$, and $b_\beta * x = b_\beta * e_\beta$. Being $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$ in Y, we can get $b_\beta * x = b_\beta * e_\beta \in S_\beta$. That is, there is no element $x \notin S_\beta$ such that $x * b_\beta = b_\beta$ and $b_\beta * y = x$, and there exists $y \in G - S_\beta$.

case 3. $LS_\beta \neq \varnothing$ and $RS_\beta \neq \varnothing$, when $x \in LS_\beta, x * b_\beta = x * e_x \notin S_\beta$, and $b_\beta * x = x \notin RS_\beta$; when $x \in RS_\beta, x * b_\beta = b_\beta$, $b_\beta * x = b_\beta * e_\beta \notin RS_\beta$. That is, there is no element $x \notin S_\beta$ such that $x * b_\beta = b_\beta$ and $b_\beta * y = x$, and there exists $y \in G - S_\beta$.

From all the above cases, b_β has a unique $nlr(b_\beta) = e_\beta$ and $\left\{olr(b_\beta)\right\} \subseteq S_\beta$. Consequently, G is a strong AG-*(l,r)*-Loop.

Example 7. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, and the definition of operation $*$ on G is shown in Table 7. An AG-band $Y = \{1, 5, 9\}$ and $S_1 = \{1, 2, 3, 4\}, e_1 = 1$, $S_5 = \{5, 6, 7, 8\}, e_5 = 5$, and $S_9 = \{9, 10, 11, 12, 13\}, e_9 = 9$. For any $a_1 \in S_1, b_5 \in S_5$, and $c_9 \in S_9$, without losing generality, let $a_1 = 3, b_5 = 7$, and $c_9 = 10$, and we have $3 * 7 = 3 * 1$ and $7 * 3 = 3$, $3 * 10 = 3 * 1$ and $10 * 3 = 3$, $7 * 10 = 7 * 5$ and $10 * 7 = 7$, and $(3 * 7) * 10 = (10 * 7) * 3$. The other cases can be verified; thus, G is an AG-groupoid.

Table 7: Table of Example 7.

*	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	3	4	1	1	1	1	1	1	1	1	1
2	2	1	4	3	2	2	2	2	2	2	2	2	2
3	4	3	2	1	4	4	4	4	4	4	4	4	4
4	3	4	1	2	3	3	3	3	3	3	3	3	3
5	1	2	3	4	5	6	7	8	5	5	5	5	5
6	1	2	3	4	6	5	8	7	6	6	6	6	6
7	1	2	3	4	8	7	5	6	8	8	8	8	8
8	1	2	3	4	7	8	6	5	7	7	7	7	7
9	1	2	3	4	5	6	7	8	9	10	11	12	13
10	1	2	3	4	5	6	7	8	10	11	12	13	9
11	1	2	3	4	5	6	7	8	11	12	13	9	10
12	1	2	3	4	5	6	7	8	12	13	9	10	11
13	1	2	3	4	5	6	7	8	13	9	10	11	12

Let $c_9 = 10$, $LS_9 = G - S_9 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $RS_9 = \varnothing$; for all $x \in LS_9$, there is no element x such that $x * 10 = 10$. That is, the element "10" has a unique $nlr(10) = 9$ and $\{olr(10)\} = \{13\} \subseteq S_9$.

Let $a_1 = 3$, $LS_1 = \varnothing, RS_1 = G - S_1 = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$; for all $x \in RS_1$, $3 * x = 3 * e_1 = 3 * 1 = 4 \notin RS_1$; thus, there is no element x such that there exists $y \in RS_1, x * 3 = 3, 3 * y = x$. That is, the element "3" has a unique $nlr(3) = 1$ and $\{olr(3)\} = \{4\} \subseteq S_1$.

Let $b_5 = 7$, $LS_5 = \{1, 2, 3, 4\}$, and $RS_5 = \{9, 10, 11, 12, 13\}$, when $x \in LS_5, x * 7 = x * e_x \notin S_5, 7 * x = x \notin RS_5$; when $x \in RS_5, x * 7 = 7, 7 * x = 7 * e_5 = 7 * 5 = 8 \notin RS_5$. That is, there is no element $x \notin S_5$ such that $x * 7 = 7, 7 * y = x$, and there exists $y \in G - S_5$. The element "7" has a unique $nlr(7) = 5$ and $\{olr(7)\} = \{7\} \subseteq S_5$.

The other cases can be verified; thus, G is a strong AG-*(l,r)*-Loop.

Theorem 6. *Let* $(G, *)$ *be a groupoid,* Y *be an AG-band,* $Y \subset G$. $G = \cup_{\alpha\in Y} S_\alpha$, $(S_\alpha, *)$ *be a strong AG-(l,r)-Loop with a left identity* e_α *for each* $\alpha \in Y$, *and* $S_\alpha \cap S_\beta = \varnothing$, $\alpha, \beta \in Y, \alpha \neq \beta$. *If, for all* $a_\alpha \in S_\alpha$, *for all* $b_\beta \in S_\beta$, $a_\alpha * b_\beta = b_\beta, b_\beta * a_\alpha = b_\beta * e_\beta$, *then* G *is a strong AG-(l,r)-Loop.*

Proof. Theorem 6 is proved similarly to Theorem 5.

The strong AG-(*l*,*r*)-Loop constructed by Theorem 5 is not isomorphic to the strong AG-(*l*,*r*)-Loop constructed by Theorem 6. □

Definition 12 (see [20]). An AG-*(l,l)*-Loop $(G, *)$ is called a strong AG-*(l,l)*-Loop if for all $a \in G, nll(a)^2 = nll(a)$.

Example 8 illustrates that an AG-(*l*,*l*)-Loop is not always a strong AG-(*l*,*l*)-Loop.

Example 8. Let $G = \{1,2,3,4,5,6,7,8\}$, and the definition of operation $*$ on G is shown in Table 8. From Definitions 5 and 7, G is both an AG-*(l,l)*-Loop and an AG-*(l,r)*-Loop. However, $nll(1) = nlr(1) = 3, 3*3 = 4 \neq 3$; thus, it is neither a strong AG-*(l,l)*-Loop nor a strong AG-*(l,r)*-Loop.

Table 8: Table of Example 8.

*	1	2	3	4	5	6	7	8
1	2	4	3	1	7	5	6	8
2	3	1	2	4	6	8	7	5
3	1	3	4	2	8	6	5	7
4	4	2	1	3	5	7	8	6
5	8	6	5	7	6	8	7	5
6	5	7	8	6	7	5	6	8
7	7	5	6	8	5	7	8	6
8	6	8	7	5	8	6	5	7

Theorem 7. *Let $(G, *)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a strong AG-(l,r)-Loop

(2) G is a strong AG-(l,l)-Loop

(3) G is an AG-(l,lr)-Loop

Proof

(1) $\Longrightarrow$ (2). Suppose G is a strong AG-*(l,r)*-Loop; from Definition 11, for all $a \in G$, there exist $nlr(a), olr(a) \in G$, $nlr(a) * a = a$, $a * olr(a) = nlr(a)$, and $nlr(a)^2 = nlr(a)$. Let $d = nlr(a) * olr(a)$, and we have $d * a = (nlr(a) * olr(a)) * a = (a * olr(a)) * nlr(a) = nlr(a)^2 = nlr(a)$. From Definition 12, G is a strong AG-*(l,l)*-Loop.

(2) $\Longrightarrow$ (3). Suppose G is a strong AG-*(l,l)*-Loop; from Definition 12, for all $a \in G$, there exist $nll(a), oll(a) \in G$, $nll(a) * a = a$, $oll(a) * a = nll(a)$, and $nll(a)^2 = nll(a)$. So, $a * oll(a) = (nll(a) * a) * oll(a) = (oll(a) * a) * nll(a) = nll(a)^2 = nll(a)$. By Definition 9, G is an AG-*(l,lr)*-Loop.

(3) $\Longrightarrow$ (1). If G is an AG-*(l,lr)*-Loop, from Definition 9, for all $a \in G$, there exist $nlr(a), olr(a) \in G$, $nlr(a) * a = a$, and $olr(a) * a = a * olr(a) = nlr(a)$. So, $nlr(a) * nlr(a) = (olr(a) * a) * nlr(a) = (nlr(a) * a) * olr(a) = a * olr(a) = nlr(a)$. By Definition 11, G is a strong AG-*(l,r)*-Loop.

Figure 1 shows the relationships among AG-(l,l)-Loop and AG-(l,r)-Loop. Here, A stands for AG-NET-Loop, B stands for strong AG-(l,r)-Loop shown in Example 4 rather than AG-NET-Loop, C stands for AG-(l,r)-Loop and AG-(l,l)-Loop shown in Example 8, which is, however, not strong AG-(l,r)-Loop, D stands for AG-(l,l)-Loop rather than AG-(l,r)-Loop, E stands for AG-(l,r)-Loop shown in Example 2 rather than AG-(l,l)-Loop, and F stands for AG-groupoid shown in Example 1, which is, however, not either AG-(l,l)-Loop or AG-(l,r)-Loop. A + B stands for strong AG-(l,r)-Loop, A + B + C + D stands for AG-(l,l)-Loop, A + B + C + E stands for AG-(l,r)-Loop, and A + B + C + D + E + F stands for AG-groupoid. □

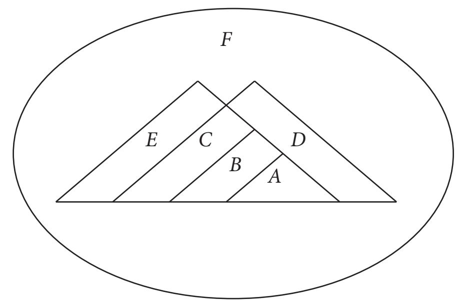

Figure 1: The relationships among AG-*(l,l)*-Loop and AG-*(l,r)*-Loop.

4. AG-*(r,r)*-Loop and AG-*(r,l)*-Loop

Theorem 8. *Let $(G, *)$ be an AG-(r,r)-Loop. Then,*

(1) G is an AG-(r,l)-Loop

(2) G is an AG-(l,l)-Loop

Proof

(1) Suppose G is an AG-*(r,r)*-Loop; from Definition 6, for all $a \in G$, there exist $nrr(a), orr(a) \in G$, $a * nrr(a) = a$, and $a * orr(a) = nrr(a)$. Let $q = orr(a) * nrr(a)$, and we have $q * a = (orr(a) * nrr(a)) * a = (a * nrr(a)) * orr(a) = a * orr(a) = nrr(a)$. By Definition 8, G is an AG-*(r,l)*-Loop.

(2) Suppose G is an AG-*(r,r)*-Loop; from Definition 6, for all $a \in G$, there exist $nrr(a), orr(a) \in G$, $a * nrr(a) = a$, and $a * orr(a) = nrr(a)$. Let $d = nrr(a)^2$ and $q = nrr(a) * orr(a)$, and we have $d * a = (nrr(a) * nrr(a)) * a = (a * nrr(a)) * nrr(a) = a * nrr(a) = a$ and $q * a = (nrr(a) * orr(a)) * a = (a * orr(a)) * nrr(a) = nrr(a) * nrr(a) = d$.

By Definition 5, G is an AG-*(l,l)*-Loop. □

Definition 13. An AG-(r,r)-Loop $(G, *)$ is called a strong AG-(r,r)-Loop if for all $a \in G, nrr(a)^2 = nrr(a)$.

Definition 14. An AG-*(r,l)*-Loop $(G, *)$ is called a strong AG-*(r,l)*-Loop if for all $a \in G, nrl(a)^2 = nrl(a)$.

Example 9 illustrates that an AG-*(r,r)*-Loop is not always a strong AG-*(r,r)*-Loop and an AG-*(r,l)*-Loop is not always a strong AG-(r,l)-Loop.

Example 9. Let $G = \{1,2,3,4,5,6,7,8\}$, and the definition of operation $*$ on G is shown in Table 9. From Definitions 6, 8, 5, and 7, G is both an AG-(r,r)-Loop and an AG-(r,l)-Loop and an AG-(l,l)-Loop and AG-(l,r)-Loop. However, $nrr(1) = 4, nrl(1) = 4, 4 * 4 = 3 \neq 4$; $nll(1) = 3, nlr(1) = 3, 3 * 3 = 4 \neq 3$. Thus, G is neither a strong AG-(r,r)-Loop nor a

Table 9: Table of Example 9.

$*$	1	2	3	4	5	6	7	8
1	2	4	3	1	3	1	2	4
2	3	1	2	4	2	4	3	1
3	1	3	4	2	4	2	1	3
4	4	2	1	3	1	3	4	2
5	1	3	4	2	6	8	7	5
6	4	2	1	3	7	5	6	8
7	2	4	3	1	5	7	8	6
8	3	1	2	4	8	6	5	7

strong AG-(r,l)-Loop nor a strong AG-(l,l)-Loop nor a strong AG-(l,r)-Loop.

Theorem 9. *Let $(G, *)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a strong AG-(r,r)-Loop

(2) G is a strong AG-(r,l)-Loop

(3) G is an AG-NET-Loop

Proof

(1) $\Longrightarrow$(2). Suppose G is a strong AG-*(r,r)*-Loop; from Definition 13, for all $a \in G$, there exist $nrr(a)$, $orr(a) \in G$, $a * nrr(a) = a$, $a * orr(a) = nrr(a)$, and $nrr(a)^2 = nrr(a)$. Let $q = orr(a) * nrr(a)$, and we have $q * a = (orr(a) * nrr(a)) * a = (a * nrr(a)) * orr(a) = a * orr(a) = nrr(a)$. By Definition 14, G is a strong AG-*(r,l)*-Loop.

(2) $\Longrightarrow$(3). Suppose G is a strong AG-*(r,l)*-Loop; from Definition 14, for all $a \in G$, there exist $nrl(a), orl(a) \in G$, $a * nrl(a) = a$, $orl(a) * a = nrl(a)$, and $nrl(a)^2 = nrl(a)$. So, $nrl(a) * a = (nrl(a) * nrl(a)) * a = (a * nrl(a)) * nrl(a) = a * nrl(a) = a$ and $a * orl(a) = (nrl(a) * a) * orl(a) = (orl(a) * a) * nrl(a) = nrl(a)^2 = nrl(a)$. By Definition 3, G is an AG-NET-Loop.

(3) $\Longrightarrow$(1). It is obvious that an AG-NET-Loop is a strong AG-*(r,r)*-Loop.

Figure 2 shows the relationships among AG-(r,l)-Loop and AG-(l,r)-Loop. Here, A stands for AG-NET-Loop, B stands for AG-(r,l)-Loop and strong AG-(l,r)-Loop shown in Example 4, which is, however, not AG-NET-Loop, C stands for AG-(r,l)-Loop and AG-(l,r)-Loop shown in Example 9, which is, however, not strong AG-(l,r)-Loop, D stands for AG-(r,l)-Loop rather than AG-(l,r)-Loop, E stands for strong AG-(l,r)-Loop rather than AG-(r,l)-Loop, F stands for AG-(l,r)-Loop shown in Example 2, which is, however, not either AG-(r,l)-Loop or strong AG-(l,r)-Loop, and G stands for AG-groupoid shown in Example 1, which is, however, not either AG-(l,r)-Loop or AG-(r,l)-Loop. A + B + E stands for strong AG-(l,r)-Loop, A + B + C + D stands for AG-(r,l)-Loop, A + B + C + E + F stands for AG-(l,r)-Loop, and A + B + C + D + E + F + G stands for AG-groupoid.

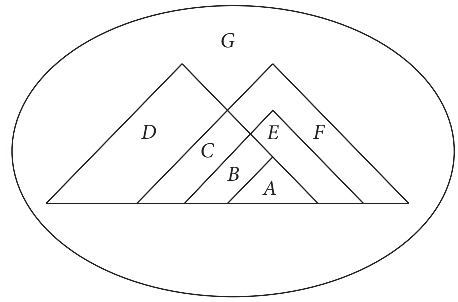

Figure 2: The relationships among AG-*(r,l)*-Loop and AG-*(l,r)*-Loop.

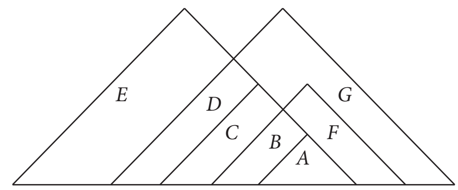

Figure 3: The relationships among AG-(r,l)-Loop and AG-(l,l)-Loop.

Figure 3 shows the relationships among AG-(r,l)-Loop and AG-(l,l)-Loop. Here, A stands for AG-NET-Loop, B stands for AG-(r,r)-Loop and strong AG-(l,l)-Loop shown in Example 4, which is, however, not AG-NET-Loop, C stands for AG-(r,r)-Loop shown in Example 9 rather than strong AG-(l,l)-Loop, D stands for AG-(r,l)-Loop and AG-(l,l)-Loop rather than AG-(r,r)-Loop, E stands for AG-(r,l)-Loop rather than AG-(l,l)-Loop, F stands for strong AG-(l,l)-Loop rather than AG-(r,l)-Loop, and G stands for AG-(l,l)-Loop, which is, however, not either AG-(r,l)-Loop or a strong AG-(l,l)-Loop. A + B + C stands for AG-(r,r)-Loop, A + B + F stands for strong AG-(l,l)-Loop, A + B + C + D + E stands for AG-(r,l)-Loop, and A + B + C + D + F + G stands for AG-(l,l)-Loop. □

5. Alternative Quasi AG-NET-Loop

Definition 15. Let $(G, *)$ be an AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop). Then, G is called a right alternative AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop) if $b * (a * a) = (b * a) * a$, for all $a, b \in G$.

Definition 16. Let $(G, *)$ be an AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop). Then, G is called an alternative AG-NET-Loop (AG-(l,l)-Loop, AG-(l,r)-Loop, AG-(r,r)-Loop, and AG-(r,l)-Loop), if for all $a, b \in G$, $(a * a) * b = a * (a * b)$, $a * (b * b) = (a * b) * b$.

Example 10 illustrates that an AG-NET-Loop is not always an alternative AG-NET-Loop.

Example 10. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 10. By Definition 3, G is

Table 10: Table of Example 10.

$*$	1	2	3	4	5	6	7
1	1	4	2	3	3	1	2
2	3	2	4	1	1	3	4
3	4	1	3	2	2	4	3
4	2	3	1	4	4	2	1
5	2	3	1	4	5	2	1
6	1	4	2	3	3	6	2
7	4	1	3	2	2	4	7

an AG-NET-Loop. However, G is not an alternative AG-NET-Loop because $(3 * 4) * 4 \neq 3 * (4 * 4)$.

Theorem 10. *Let $(G, *)$ be an AG-NET-Loop. Then, the following three statements are equivalent:*

(1) G is a right alternative AG-NET-Loop

(2) G is a commutative AG-NET-Loop

(3) G is an alternative AG-NET-Loop

Proof

(1) ⇒(2). Suppose G is a right alternative AG-NET-Loop; from Definition 15, for all $a, b \in G$,

$$\begin{aligned} a * \operatorname{neut}(b) &= a * (\operatorname{neut}(b) * \operatorname{neut}(b)) && \text{(by Theorem 1 (2))} \\ &= (a * \operatorname{neut}(b)) * \operatorname{neut}(b) && \text{(by the right alternative law)} \\ &= (\operatorname{neut}(b) * \operatorname{neut}(b)) * a && \text{(by the left invertive law)} \\ &= \operatorname{neut}(b) * a, \end{aligned} \tag{7}$$

so

$$\begin{aligned} a * b &= (\operatorname{neut}(a) * a) * (b * \operatorname{neut}(b)) \\ &= (\operatorname{neut}(a) * b) * (a * \operatorname{neut}(b)) \quad \text{(by the medial law)} \\ &= (b * \operatorname{neut}(a)) * (\operatorname{neut}(b) * a) \\ &= (b * \operatorname{neut}(b)) * (\operatorname{neut}(a) * a) \\ &= b * a. \end{aligned} \tag{8}$$

Consequently, G is a commutative AG-NET-Loop.

(2) ⇒(3). If G is a commutative AG-NET-Loop, for all $m, n \in G$, $m * (n * n) = (n * n) * m = (m * n) * n$ and $(m * m) * n = (n * m) * m = m * (n * m) = m * (m * n)$. By Definition 16, G is an alternative AG-NET-Loop.

(3) ⇒(1). It is obvious that an alternative AG-NET-Loop is a right alternative AG-NET-Loop. □

Theorem 11 (see [23]). *Let $(G, *)$ be a locally associative AG-groupoid. If G is finite, then there exists $a \in G, a^2 = a$.*

Theorem 12. *Let $(G, *)$ be a right alternative AG-(r,l)-Loop. If G is finite, then, for all $a \in G$, there exist $s, p \in G, a * s = a, p * a = s$, and $s^2 = s$.*

Proof. If G is a finite right alternative AG-*(r,l)*-Loop. Then, for all $a \in G$, there exist $s, p \in G, a * s = a$, and $p * a = s$, and we have $a * s^2 = a * (s * s) = (a * s) * s = a * s = a$.

When $k \in Z^+$, $k > 2$,

$$\begin{aligned} a * s^k &= (a * s) * \left(s^2 * s^{k-2}\right) \\ &= \left(a * s^2\right) * \left(s * s^{k-2}\right) \quad \text{(by the medial law)} \\ &= a * s^{k-1} \\ &= \cdots\cdots \\ &= a * s^2 = a. \end{aligned} \tag{9}$$

Thus, $s, s^2, s^3, \ldots, s^k, \ldots$ are all right neutral element.

By Theorem 11, we get that there is an idempotent right neutral element in G. □

Theorem 13 (see [23]). *Let $(G, *)$ be a finite alternative AG-(l,l)-Loop. Then, G is a strong AG-(l,l)-Loop.*

Theorem 14. *Let $(G, *)$ be an AG-groupoid. Then, the following three statements are equivalent:*

(1) G is a finite right alternative AG-(r,l)-Loop

(2) G is a finite alternative AG-NET-Loop

(3) G is a finite alternative AG-(l,l)-Loop

Proof

(1) ⇒(2). If G is a finite right alternative AG-*(r,l)*-Loop, applying Theorem 12, we get that G is a strong AG-*(r,l)*-Loop. From Theorem 9, we get that G is a right alternative AG-NET-Loop. Applying Theorem 10, G is a finite alternative AG-NET-Loop.

(2) ⇒(3). It is obvious that a finite alternative AG-NET-Loop is a finite alternative AG-*(l,l)*-Loop.

(3) ⇒(1). If G is a finite alternative AG-*(l,l)*-Loop, applying Theorem 13, we get that G is a strong AG-(l,l)-Loop. From Definition 12, for all $a \in G$, there exist $nll(a), oll(a) \in G$, $nll(a) * a = a$, $oll(a) * a = nll(a)$, and $nll(a)^2 = nll(a)$. We have

Table 11: Table of Example 11.

*	1	2	3	4	5	6	7
1	2	5	4	1	3	1	1
2	5	3	1	2	4	2	2
3	4	1	5	3	2	3	3
4	1	2	3	4	5	4	4
5	3	4	2	5	1	5	5
6	1	2	3	4	5	6	4
7	1	2	3	4	5	4	7

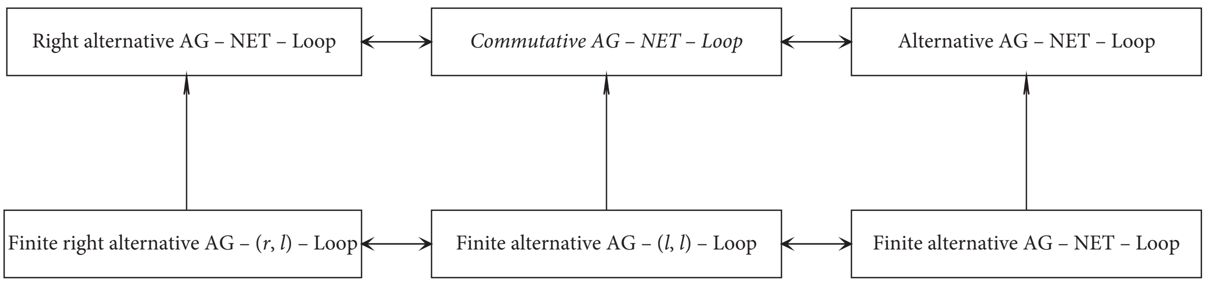

Figure 4: The relationships among alternative AG-NET-Loop and other alternative quasi AG-NET-Loops.

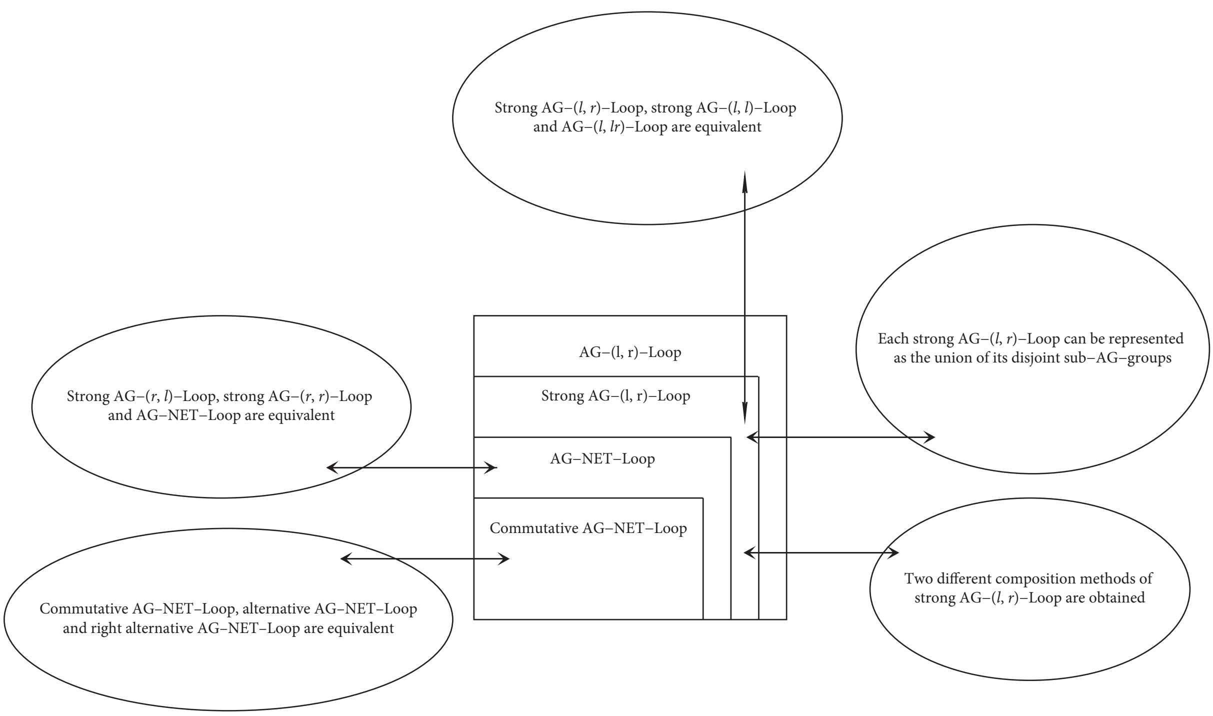

Figure 5: The main results of this paper.

$$
\begin{aligned}
a * nll(a) &= a * (nll(a) * nll(a)) \\
&= (a * nll(a)) * nll(a) \quad \text{(by the right alternative law)} \\
&= (nll(a) * nll(a)) * a \quad \text{(by the left invertive law)} \\
&= nll(a) * a = a.
\end{aligned}
\tag{10}
$$

By Definition 15, G is a finite right alternative AG-*(r,l)*-Loop. □

Example 11. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, and the definition of operation $*$ on G is shown in Table 11. We can easily verify that G satisfies the alternative law. Being each element in G has a neutral element and an opposite element; by Definition 16, G is a finite alternative AG-NET-Loop. Obviously, a finite alternative AG-NET-Loop is both a finite right alternative AG-(r,l)-Loop and a finite alternative AG-(l,l)-Loop. Since for all $a, b \in G$ and $a * b = b * a$, we have G as a commutative AG-NET-Loop.

Figure 4 shows the relationships among alternative AG-NET-Loop and other alternative quasi AG-NET-Loops. In

Figure 4, we prove that the right alternative AG-NET-Loop is equivalent to the commutative AG-NET-Loop, and the commutative AG-NET-Loop is equivalent to the alternative AG-NET-Loop. As the finite right alternative AG-*(r,l)*-Loop is equivalent to the finite alternative AG-*(l,l)*-Loop, the finite alternative AG-*(l,l)*-Loop is equivalent to the finite alternative AG-NET-Loop; therefore, they are equivalent to each other.

6. Conclusion

In this paper, the AG-*(l,r)*-Loop and AG-*(r,l)*-Loop have been introduced, the structure of the quasi AG-NET-Loops have been studied further, and some important results have been obtained. We prove that the strong AG-*(l,r)*-Loop, the strong AG-*(l,l)*-Loop, and the AG-*(l,lr)*-Loop are equivalent (see Theorem 7); the strong AG-(*r,l)*-Loop, the strong AG-*(r,r)*-Loop, and the AG-NET-Loop are equivalent (see Theorem 9); the commutative AG-NET-Loop, the alternative AG-NET-Loop, and the right alternative AG-NET-Loop are equivalent (see Theorem 10). Furthermore, the decomposition theorem of strong AG-*(l,r)*-Loop (see Theorem 4) and two different ways how to make a strong AG-*(l,r)*-Loop are obtained (see Theorem 5 and Theorem 6), thus illuminating the structure of strong AG-*(l,r)*-Loop. Figure 5 shows the main results of this paper. Future efforts will be directed towards discussing the relationship between strong AG-*(l,r)*-Loop and other related AG-groupoid bands, such as root of band, AG-4-band, and AG-3-band (see [24]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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