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SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions

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Abstract: The *n-th* PowerSet of a Set {or $P^n(S)$} better describes our real world, because a system S (which may be a company, institution, association, country, society, set of objects/plants/animals/beings, set of concepts/ideas/propositions, etc.) is formed by sub-systems, which in their turn by sub-sub-systems, and so on. We prove that the SuperHyperFunction is a generalization of classical Function, SuperFunction, and HyperFunction. And the SuperHyperAlgebra, SuperHyperGraph are part of the SuperHyperStructure. Almost all structures in our real world are Neutrosophic SuperHyperStructures since they have indeterminate/incomplete/uncertain/conflicting data.

Keywords: n-th PowerSet; Classical Function; HyperFunction; SuperFunction; SuperHyperFunction; Classical Operation; HyperOperation; SuperHyperOperation; Classical Axiom; HyperAxiom; SuperAxiom; SuperHyperAxiom; Classical Algebra; HyperAlgebra; SuperHyperAlgebra; Neutrosophic SuperHyperAlgebra; SuperHyperGraph; SuperHyperTopology; Classical Structure; HyperStructure; SuperHyperStructure; Neutrosophic SuperHyperStructure.

1. Introduction

In general, a system S (that may be a company, association, institution, society, country, etc.) is formed by sub-systems S_i {or $P(S)$, the PowerSet of S}, and each sub-system S_i is formed by sub-sub-systems S_{ij} {or $P(P(S)) = P^2(S)$} and so on. That's why the n-th PowerSet of a Set S {defined recursively and denoted by $P^n(S) = P(P^{n-1}(S))$} was introduced, to better describes the organization of people, beings, objects etc. in our real world.

The n-th PowerSet, introduced by Smarandache [2] in 2016, was used in defining the SuperHyperOperation, SuperHyperAxiom, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic SuperHyperAxiom in order to build the SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. In general, in any field of knowledge, one in fact encounters *SuperHyperStructures*.

2. The n-th PowerSet of a Set better describes our Real World

(i) ***The n-th PowerSet*** of a set better describes our real world, because a <u>system</u> S (which may be a company, institution, association, country, society, set of objects/plants/animals/beings, set of concepts/ideas/propositions, etc.) is formed by <u>sub-systems</u> S_i, which in their turn are formed by <u>sub-sub-systems</u> S_{i_j}, and so on, up to a needed structural-level n of the system. Afterwards, between the sub-systems, sub-sub-systems and so on there are various inter-relationships (similar to the operations and axioms in general algebraic structures).

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Let's recall the definition of the *n-th* PowerSet of a Set [2], proposed by Smarandache in 2016.

(ii) ***Definition of the n^th^ PowerSet of a Set*** *S* {denoted as $P^n(S)$, where the empty-set ϕ is allowed, and it represents the indeterminacy/uncertainty} is done recursively:

Let *S* be a set.

$P^0(S) \overset{def}{=} S$, be definition.

$P^1(S) = P(S)$ is the PowerSet of *S*, we call it the *1st*-PowerSet of S;

$P^2(S) = P(P(S))$ is the PowerSet of the PowerSet of *S*, or the *2nd*-PowerSet of *S*;

$P^3(S) = P(P^2(S)) = P(P(P(S)))$ is the PowerSet of the PowerSet of the PowerSet of S, or the *3rd*-PowerSet of *S*;

and so on,

$P^n(S) = P(P^{n-1}(S)) = ... = \underbrace{P(P(...P(S)...))}_{n-times}$, where P is repeated *n* times, and the empty-set is allowed.

(iii) ***Example of 2nd-PowerSet of a set S, where the empty-set is allowed.***

Let's consider an easy example to be able to distinguish between several types of functions, algebras, and structures.

Let the set $S = \{1, 2\}$.

Then, the 1st-PowerSet of *S*, with the empty set ϕ included, is: $P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and this $P(S)$ is used for the neutrosophic versions of functions, operations (and operators), axioms, algebras, and structures.

The 2-nd PowerSet of *S*, with the empty set ϕ included, is:

$$P^2(S) = P(P(S)) = P(\{\phi, \{1\}, \{2\}, \{1, 2\}\}) =$$
$$\{\phi; \{1\}; \{2\}; \{1, 2\};$$
$$\{\{\phi, \{1\}\}; \{\phi, \{2\}\}; \{\phi, \{1, 2\}\};$$
$$\{\{1\}, \{2\}\}; \{\{1\}, \{1, 2\}\}; \{\{2\}, \{1, 2\}\};$$
$$\{\phi, \{1\}, \{2\}\}; \{\phi, \{1\}, \{1, 2\}\}; \{\phi, \{2\}, \{1, 2\}\}; \{\{1\}, \{2\}, \{1, 2\}\};$$
$$\{\phi, \{1\}, \{2\}, \{1, 2\}\}\}.$$

(iv) ***Definition of the n^th^ PowerSet of a Set S without the empty-set*** {denoted as $P_*^n(S)$ where the empty-set ϕ is <u>not allowed</u>} is also done recursively:

Let *S* be a non-empty set.

$P_*^0(S) \overset{def}{=} S$, be definition.

$P_*^1(S) = P_*(S)$ is the PowerSet of *S*, without the empty-set, we call it the *1st*-PowerSet of *S*;

$P_*^2(S) = P_*(P_*(S))$ is the PowerSet of the PowerSet of *S*, without the empty-set, or the *2nd* PowerSet of *S*;

$P_*^3(S) = P_*(P_*^2(S)) = P_*(P_*(P_*(S)))$ is the PowerSet of the PowerSet of the PowerSet of *S*, without the empty-set, or the *3rd*-PowerSet of *S*;

and so on,

$$P_*^n(S) = P_*(P_*^{n-1}(S)) = \ldots = \underbrace{P_*(P_*(\ldots P_*(S)\ldots))}_{n-times}$$, where P is repeated *n* times,

and the empty set is not allowed.

(v) ***Example of 2nd-PowerSet of a set S, without the empty-set***

Let's consider an easy example to be able to distinguish between several types of functions, algebras, and structures.

Let $S = \{1, 2\}$ be a set, then the 1st-PowerSet of the set *S*, without the empty set ϕ , is $P_*(S) = \{\{1\}, \{2\}, \{1, 2\}\}$.

The 2nd-PowerSet of the set *S*, without the empty set ϕ , is:

$$P_*^2(S) = P_*(P_*(S)) = P_*(\{\{1\}, \{2\}, \{1, 2\}\}) =$$
$$\{\{1\}; \{2\}; \{1, 2\};$$
$$\{\{1\}, \{2\}\}; \{\{1\}, \{1, 2\}\}; \{\{2\}, \{1, 2\}\};$$
$$\{\{1\}, \{2\}, \{1, 2\}\}\}.$$

What is the distinction between, for example, $A = \{1, 2\}$ and $B = \{\{1\}, \{2\}\}$?

In *A*, the elements *1* and *2* are totally dependent of each other and form together a sub-system called *A*; while in *B*, each of *{1}* and *{2}* are partially independent of each other and as such they are individual sub-sub-systems, and partially dependent of each other and united into a sub-system called *B* (a sub-system of sub-sub-systems).

In the real world, we may consider, for example, *A* as a group of two researchers, denoted by *1* and *2*, that work together (totally dependent on each other) for a common project.

But in *B*, researchers *{1}* and *{2}* work each of them separately for the projects *p1* and respectively *p2* (so they are independent from the point of view of these projects), but the researchers work together for the third common project *p3* (so they are dependent from the point of view of project *p3*).

3. Functions of One Variable

(i) ***Classical Function of One Variable***

The domain and codomain of the function is just *S*.

$f : S \to S$

Example of Classical Function of One Variable

Let's take, as above, *S = {1, 2}.*

$f(1) = 2$ (a single-value) $\in S$;

$f(2) = 1$ (a single-value) $\in S$.

(ii) ***HyperFunction of One Variable***

This is part of the HyperStructures [1], when the domain *S* remains unchanged, while the codomain of the function becomes the PowerSet $P_*(S)$.

$f : S \to P_*(S)$

Example of HyperFunction of One Variable

$f(1) = \{1, 2\}$ (a set-value) $\in P_*(S)$;

$f(2) = 1 \in P_*(S)$.

(iii) ***SuperFunction of One Variable***

This is an extension of the HyperFunction, when the domain *S* remains the same, but the codomain of the function becomes *n*th-PowerSet of the set *S*, i.e. $P_*^n(S)$, $n \geq 2$.

$f : S \to P_*^n(S)$, where integer $n \geq 2$.

(iv) ***Example of SuperFunction of One Variable***

Let's take the easiest case when $n = 2$, the domain of the function remains the same *S*, but one has the 2nd-PowerSet of the set S as codomain of the function:

$f : S \to P_*^2(S)$

$f(1) = \{\{1\}, \{1,2\}\} \in P_*^2(S)$;

$f(2) = \{\{1\}, \{2\}\} \in P_*^2(S)$.

(v) ***SuperHyperFunction of One Variable***

$f : P_*^r S \to P_*^n(S)$, for integers $r, n \geq 0$.

It is part of the SuperHyperStructure [2, 3].

(vi) ***Example of SuperHyperFunction of One Variable***

$f : P_*(S) \to P_*^2(S)$.

$f(\{1\}) = \{\{1\}, \{2\}, \{1,2\}\} \in P_*^2(S)$.

$f(\{2\}) = \{\{1\}, \{2\}\} \in P_*^2(S)$.

$f(\{1,2\}) = \{\{2\}, \{1,2\}\} \in P_*^2(S)$

(vii) ***Theorem 1:***

The SuperHyperFunction of One Variable is the most general form of functions of one variables.

Proof:

For r = 0, and n = 0, one gets the classical Function.

For r = 0, and n = 1, one gets the HyperFunction.

For r = 0, and n ≥ 2, one gets the SuperFunction.

4. Functions of Many Variables

Straightforward generalization of the functions, from one variable to many variables, are provided below.

(i) ***Classical Function of Many Variables***

$f : S^m \to S$, for integer m ≥ 2.

(ii) ***Example of Classical Function of Two Variables***

Let's consider some elementary case, when $m = 2$.

$f : S^2 \to S$

First, $S^2 = \{1,2\} \times \{1.2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

$f(1,1) = 2$ (a single-value) $\in S$

$f(1,2) = 2 \in S$

$f(2,1) = 1 \in S$

$f(2,2) = 1 \in S$

(iii) ***HyperFunction of Many Variables***

This is part of the HyperStructures.

$f : S^m \to P_*(S)$

(iv) ***Example of HyperFunction of Two Variables***

$$f : S^2 \to P_*(S)$$

$f(1,1) = \{1,2\}$ (a set-value) $\in P_*(S)$

$$f(1,2) = \{1\} \in P_*(S)$$
$$f(2,1) = \{1,2\} \in P_*(S)$$
$$f(2,2) = \{2\} \in P_*(S)$$

(v) ***SuperFunction of Many Variables***

$f : S^m \to P_*^n(S)$, where the integers *m,* $n \geq 2$. When the 2-nd PowerSet of the Set *S* is involved.

(vi) ***Example of SuperFunction of Two Variables***

Let's take *m* = *n* = 2 the simplest case.

$$f : S^2 \to P_*^2(S)$$

$$f(1,1) = \{\{1\},\{2\},\{1,2\}\} \in P_*^2(S)$$
$$f(1,2) = \{\{1\},\{2\}\} \in P_*^2(S)$$
$$f(2,1) = \{\{2\},\{1,2\}\} \in P_*^2(S)$$
$$f(2,2) = \{\{1\},\{1,2\}\} \in P_*^2(S)$$

(vii) ***SuperHyperFunction of Many Variable***

$f : (P_*^r S)^m \to P_*^n(S)$, for integers $m \geq 2$ and $r, n \geq 0$.

It is part of the SuperHyperStructure.

(viii) ***Example of SuperHyperFunction of Two Variables***

Let's take *m = 2, r = 1,* and *n = 2.*

$$f : (P_*(S))^2 \to P_*^2(S)$$

Table 1. Values of the above SuperHyperFunction of two variable f(x, y).

x / y	**{1}**	**{2}**	**{1, 2}**
{1}	{{1}, {2}}	{1}	{{1}, {1 2}}
{2}	{{2}, {1, 2}}	{{1}, {1, 2}}	{2}
{1, 2}	{1, 2}		{{1}. {2}, {1, 2}}

For example, *f({1}, {1, 2}) = {{1}, {1 2}}.*

(ix) ***Theorem 2:***

Similarly, the SuperHyperFunction of Many Variables is a generalization of the Classical Function, HyperFunction, and SuperFunction of Many Variables.
Proof is the same as in the previous theorem, by keeping the same *m* (number of variables) value:
For *r = 0,* and *n = 0*, one gets the classical Function of Many Variables.
For *r = 0,* and *n = 1*, one gets the HyperFunction of Many Variables.
For *r = 0,* and *n* ≥ *2*, one gets the SuperFunction of Many Variables.

5. Definition of SuperHyperFunction (SHF$_m$) of $m \geq 2$ Variables

$f_{SH}^{SH} : P_*^{r_1}(S) \times P_*^{r_2}(S) \times \ldots \times P_*^{r_k}(S) \to P_*^{n}(S)$, where integers $r_1, r_2, \ldots, r_k, n \geq 0$ and *SH* stands for *SuperHyper*, the upper *SH* is for the function's domain, and the bottom *SH* is for the function's codomain, meaning that both are some PowerSets of PowerSets etc. of the set S.
For any $x_1 \in P_*^{r_1}(S), x_2 \in P_*^{r_2}(S), \ldots, x_k \in P_*^{r_k}(S)$, one has $f_{SH}^{SH}(x_1, x_2, \ldots, x_k) \in P_*^{n}(S)$.
This is a generalization of all previous functions.

6. Operations / HyperOperations / SuperHyperOperations and Axioms / HyperAxioms / SuperHyperAxioms

Let $m \geq 1$ be an integer.
The **Operations** (and operators) can be treated as *m-ary* functions, while the **Axioms** as logical propositions involving the *m-ary* operations.
Similarly, the **HyperOperations** can be treated as *m-ary* HyperFunctions, while the **HyperAxioms** as logical propositions involving the *m-ary* HyperOperations. And last, the **SuperHyperOperations** can be treated as *m-ary* SuperHyperFunctions, while the **SuperHyperAxioms** (***HSAx***) as logical propositions involving *m-ary* SuperHyperOperations.

7. Structure / HyperStructure / SuperHyperStructure

(i) The classical **Structure** is a structure built on a set *S*, endowed with classical Operations ($\#_C$) $\#_C : S^m \to S$, for integer $m \geq 1$, and classical Axioms (A_C), which are axioms that act on the set S endowed with classical Operations.

(ii) The **HyperStructure** {defined by F. Marty [1] in 1934}, is a structure built on a set *S*, endowed with HyperOperations ($\#_H$), $\#_H : S^m \to P_*(S)$, for integer $m \geq 1$, and HyperAxioms (A_H), which are axioms that act on the set S endowed with HyperOperations. "Hyper" stands for the codomain of the operations, which is $P_*^n(S)$ instead of *S* that is for the classical structure.

(iii) The **SuperStructure** {defined by F. Smarandache [2] in 2016}, is a structure built on $P_*^n(S)$, that is the n^th^-PowerSet of the Set *S*, without the empty-set, endowed with SuperOperations, $\#_S : (P_*^n(S))^m \to P_*^q(S)$, for integers $n \geq 0, q \geq 0$, and SuperAxioms, which are axioms that act on the set $P_*^n(S)$ endowed with SuperOperations. "Super" stands for the codomain of the SuperOperations, which is $P_*^q(S)$, instead of *S* that is for the classical structure or of $P_*^n(S)$ that is for HyperStructure, or for the domain of the SuperOperations, which is $P_*^n(S)$.

(iv) The **SuperHyperStructure** {defined by F. Smarandache [2, 3] in 2016 and 2019}, is a structure built on the n[th]-PowerSet of the Set *S*, $P_*^n(S)$, endowed with SuperHyperOperations and SuperHyperAxioms.

8. The most general form of the SuperHyperAlgebra (SHA) endowed with One Operation and Many Axioms

is:

$(P_*^n(S);\#_{SHO}^m;SHAx_1,SHAx_2,...,SHAx_q)$ where S is a non-empty set, $P_*^n(S)$ is the *n[th]*-PowerSet of the set S, for $n \geq 2$, and $\#_{SHO}^m$ is an ***m-ary*** **SuperHyperOperation (*SHO*)**, acting on $P_*^n(S)$:

$\#_{SHO}^m : \underbrace{P_*^n(S) \times P_*^n(S) \times ... \times P_*^n(S)}_{m-times} \to P_*^n(S)$, where $P_*^n(S)$ is repeated *m times* into the operation domain, and integer $m \geq 1$, and *q* is the number of **SuperHyperAxioms**.

9. The most general form of the SuperHyperAlgebra with Many Operations and Many Axioms

is:

$(P_*^n(S);\#_{SHO1}^{m_1},\#_{SHO2}^{m_2},...,\#_{SHOr}^{m_r};SHAx_1,SHAx_2,...,SHAx_q)$ where the m_i -ary SuperHyperOperations are defined as follows:

$\#_{SHO}^{m_i} : \underbrace{P_*^n(S) \times P_*^n(S) \times ... \times P_*^n(S)}_{m_i-times} \to P_*^n(S)$, with $P_*^n(S)$ being repeated m_i times into the operation domain, $m_i \geq 1$, for $1 \leq i \leq r$, and $r \geq 2$ is the number of m_i - *ary* SuperHyperOperations ($\#_{SHO1}^{m_1},\#_{SHO2}^{m_2},...,\#_{SHOr}^{m_r}$), and $q \geq 1$ is the number of SuperHyperAxioms ($SHAx_1,SHAx_2,...,SHAx_q$).

10. SuperHyperTopology

SuperHyperTopology [5] is a topology built on a SuperHyperAlgebra $(P_*^n(S),\#)$, for integer $n \geq 2$, and it is a collection of SuperHyperSubsets from $P_*^n(S)$ that satisfy the axioms of classical topology.

11. Neutrosophic SuperHyperStructure et al.

All of the above *non-neutrosophic concepts* can easily be extended to the *neutrosophic framework*, therefore:

the Neutrosophic HyperFunction / SuperFunction / SuperHyperFunction of one or many variables, and the Neutrosophic HyperOperation / SuperHyperOperation, and the Neutrosophic HyperAxiom / SuperHyperAxiom, and the Neutrosophic SuperHyperAlgebra / SuperHyperTopology, and, in general, the Neutrosophic Super/Hyper/SuperHyperStructure are built in the same corresponding ways as the above non-neutrosophic concepts, with the only distinction that all $P^k{}_*(S)$, which do not include the empty-set, are replaced by $P^k(S)$, which do include the empty-set (leaving room for indeterminate/incomplete/uncertainty/conflicting data), for all integers $k \geq 1$.

12. Applications

We need to work with the n[th]-PowerSet of a set to better describe the organization of our real world. A system (set) S is composed by sub-systems (the elements of the $P_*(S)$, the PowerSet of S, let's denote them by S_1, S_2, …), and the sub-systems by sub-sub-systems (let's denote them by S_{11}, S_{12}, … respectively S_{21}, S_{22}, …), and so on.

As future possible research work will be to investigate the applicability of many types of SuperHyperStructures and Neutrosophic SuperHyperStructures in the real world.

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Data availability
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest
The authors declare that there is no conflict of interest in the research.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

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