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Vector Similarity Measures between Refined Simplified Neutrosophic Sets and Their Multiple Attribute Decision-Making Method

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Received: 17 July 2017; Accepted: 8 August 2017; Published: 11 August 2017

Abstract: A refined single-valued/interval neutrosophic set is very suitable for the expression and application of decision-making problems with both attributes and sub-attributes since it is described by its refined truth, indeterminacy, and falsity degrees. However, existing refined single-valued/interval neutrosophic similarity measures and their decision-making methods are scarcely studied in existing literature and cannot deal with this decision-making problem with the weights of both attributes and sub-attributes in a refined interval and/or single-valued neutrosophic setting. To solve the issue, this paper firstly introduces a refined simplified neutrosophic set (RSNS), which contains the refined single-valued neutrosophic set (RSVNS) and refined interval neutrosophic set (RINS), and then proposes vector similarity measures of RSNSs based on the Jaccard, Dice, and cosine measures of simplified neutrosophic sets in vector space, and the weighted Jaccard, Dice, and cosine measures of RSNSs by considering weights of both basic elements and sub-elements in RSNS. Further, a decision-making method with the weights of both attributes and sub-attributes is developed based on the weighted Jaccard, Dice, and cosine measures of RSNSs under RSNS (RINS and/or RSVNS) environments. The ranking order of all the alternatives and the best one can be determined by one of weighted vector similarity measures between each alternative and the ideal solution (ideal alternative). Finally, an actual example on the selecting problem of construction projects illustrates the application and effectiveness of the proposed method.

Keywords: refined simplified neutrosophic set; refined single-valued neutrosophic set; refined interval neutrosophic set; vector similarity measure; decision-making

1. Introduction

Since fuzzy set theory was introduced by Zadeh [1] in 1965, it has been successfully applied to decision-making areas, and fuzzy decision-making has become a research focal point since then. With the increasing complexity of decision-making problems in actual applications, the fuzzy set is not suitable for fuzzy expression, which involves the membership degree and non-membership degree. Hence, an intuitionistic fuzzy set (IFS) [2] and an interval-valued IFS [3] were introduced as the generalization of fuzzy set and applied to decision-making problems. However, the incomplete, indeterminate, and inconsistent problems in real life cannot be explained by means of the IFS and interval-valued IFS. Therefore, Smarandache [4] proposed the concept of a neutrosophic set from a philosophical point of view, which consists of the truth, indeterminacy, and falsity memership functions, denoted by T, I, F, to represent incomplete, indeterminate, and inconsistent information in the real world. Since the truth, indeterminacy, and falsity membership degrees of T, I, F

in the neutrosophic set lie in the real standard/nonstandard interval $]^{-}0, 1^{+}[$, Smarandache [4], Wang et al. [5,6], and Ye [7,8] constrained the three membership degrees in the neutrosophic set to the single-valued membership degrees and the interval membership degrees. These become a single-valued neutrosophic set (SVNS), an interval neutrosophic set (INS), and a simplified neutrosophic set (SNS) (including SVNS and INS), respectively. Obviously, they are subclasses of the neutrosophic set for convenient applications in science and engineering fields, such as decision-making [7–13] and fault diagnosis [14]. However, because there are both arguments and sub-arguments/refined arguments in the truth, indeterminacy, and falsity membership degrees of T, I, F in the neutrosophic set to express complex problems of the real world in detail, one needs to refine truth, indeterminacy, and falsity information. Hence, Smarandache [15] further extended the neutrosophic logic to n-valued refined neutrosophic logic, where he refined/split the truth, indeterminacy, and falsity functions T, I, F into T_1, T_2, ..., T_r, I_1, I_2, ..., I_s, and F_1, F_2, ..., F_t, respectively, and constructed them as a n-valued refined neutrosophic set. Moreover, some researchers extended the neutrosophic set to multi-valued neutrosophic set/neutrosophic multiset/neutrosophic refined sets and applied them to medical diagnoses [16–18] and decision-making [19–21]. In fact, the multi-valued neutrosophic sets/neutrosophic refined sets are neutrosophic multisets in their expressed forms [22,23]. Hence, these multi-valued neutrosophic sets/neutrosophic refined sets, that is, neutrosophic multisets, and their decision-making methods cannot express and deal with decision-making problems with both attributes and sub-attributes. To solve the issue, Ye and Smarandache [22] proposed a refined single-valued neutrosophic set (RSVNS), where the neutrosophic set $\{T, I, F\}$ was refined into the RSVNS $\{(T_1, T_2, ..., \mathrm{T}_\mathrm{r}), (I_1, I_2, ..., I_\mathrm{r}), (F_1, F_2, ..., F_r)\}$, and proposed the similarity measures based on union and intersection operations of RSVNSs to solve decision-making problems with both attributes and sub-attributes. Then, Fan and Ye [23] further presented the cosine measures of RSVNSs and refined interval neutrosophic sets (RINSs) based the distance and cosine function and applied them to the decision-making problems with both attributes and sub-attributes under refined single-value/interval neutrosophic environments. However, these cosine measures cannot handle such a decision-making problem with the weights of both attributes and sub-attributes.

In fact, RINSs and/or RSVNSs are scarcely studied and applied in science and engineering fields. Therefore, it is necessary to develop new similarity measures and their decision-making method in refined interval and/or single-value neutrosophic environments. However, in existing literature [22,23], the similarity measures of RSVNSs and RINSs and their decision-making methods only took into account the basic element (single-valued/interval neutrosophic number in RSVNS/RINS)/attribute weights rather than sub-element/sub-attribute weights (weights of refined elements/refined attributes) in the measures of RSVNSs and RINSs and their decision-making methods. To overcome these drawbacks, this paper firstly introduces a refined simplified neutrosophic set (RSNS), which includes the concepts of RSVNS and RINS, and proposes the vector similarity measures of RSNSs based on the Jaccard, Dice, and cosine measures between SNSs in vector space [8]. Further, a decision-making method is established based on the Jaccard/Dice/cosine measures between RSNSs to solve multiple attribute decision-making problems with both attribute weights and sub-attribute weights under refined simplified (interval and/or single-value) neutrosophic environments. The main advantages of the proposed approach are that it can solve decision-making problems with the weights of both attributes and sub-attributes and extend existing similarity measures and decision-making methods in [22,23], because the existing similarity measures and decision-making methods cannot deal with such a decision-making problem with the weights of both attributes and sub-attributes under RSNS (RINS and/or RSVNS) environments.

The rest of the paper is structured as follows. Section 2 reviews basic concepts of SNSs and vector similarity measures of SNSs. In Section 3, we introduces a RSNS concept, including RSVNS and RINS. Section 4 proposes the Jaccard, Dice, and cosine similarity measures (three vector similarity measures) between RSNSs by considering weights of elements and sub-elements/refined elements in RSNSs. Section 5 develops a multiple attribute decision-making method with both attribute weights and sub-attribute weights based on one of three vector similarity measures under refined simplified (interval and/or single-value) neutrosophic environments. In Section 6, an actual example on the selection problem of construction projects is provided as the multiple attribute decision-making problem with both attribute weights and sub-attribute weights to illustrate the application and effectiveness of the proposed method. Finally, conclusions and future research are contained in Section 7.

2. Basic Concepts of SNSs and Vector Similarity Measures of SNSs

In 1995, Smarandache [4] proposed a concept of neutrosophic sets from a philosophical point of view, which is a part of neutrosophy and extends the concepts of fuzzy sets, interval valued fuzzy sets, IFSs, and interval valued IFSs. A neutrosophic set is characterized independently by the truth, indeterminacy and falsity membership functions, which lie in a real standard interval [0, 1] or a nonstandard interval $]^-0, 1^+[$. For convenient science and engineering applications, we need to constrain them in the real standard interval [0, 1] from a science and engineering point of view. Thus, Ye [7,8] introduced the concept of SNS as a simplified form/subclass of the neutrosophic set.

A SNS A in a universe of discourse X is characterized by its truth, indeterminacy, and falsity membership functions $T_A(x)$, $I_A(x)$, and $F_A(x)$, which is denoted as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton subintervals/subsets in the real standard [0, 1], such that $T_A(x)$: $X \to [0, 1]$, $I_S(x)$: $X \to [0, 1]$, and $F_S(x)$: $X \to [0, 1]$. Then, the SNS A contains SVNS for $T_A(x)$, $I_A(x)$, $F_A(x)$ $\in [0, 1]$ and INS for $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0, 1]$.

For convenient expression, a basic element $\langle x, T_A(x), I_A(x), F_A(x)\rangle$ in A is simply denoted as a simplified neutrosophic number (SNN) $a = <T_a, I_a, F_a>$, where a contains a single-value neutrosophic number (SVNN) for T_a, I_a, $F_a \in [0, 1]$ and an interval neutrosophic number (INN) for T_a, I_a, $F_a \subseteq [0, 1]$.

Assume that two SNSs are $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $a_j = <T_{aj}, I_{aj}, F_{aj}>$ and $b_j = <T_{bj}, I_{bj}, F_{bj}>$ for $j = 1, 2, \ldots, n$ are two collections of SNNs. Based on the Jaccard, Dice, and cosine measures between two vectors, Ye [8] presented the their similarity measures between SNSs (SVNSs and INSs) A and B in vector space, respectively, as follows:

(1) Three vector similarity measures between A and B for SVNSs:

$$M_J(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}}{\left[\left(T_{aj}^2 + I_{aj}^2 + F_{aj}^2\right) + \left(T_{bj}^2 + I_{bj}^2 + F_{bj}^2\right) - \left(T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}\right)\right]} \tag{1}$$

$$M_D(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{aj}\right)}{\left(T_{aj}^2 + I_{aj}^2 + F_{aj}^2\right) + \left(T_{bj}^2 + I_{bj}^2 + F_{bj}^2\right)} \tag{2}$$

$$M_C(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{T_{aj}T_{bj} + I_{aj}I_{bj} + F_{aj}F_{bj}}{\sqrt{T_{aj}^2 + I_{aj}^2 + F_{aj}^2}\sqrt{T_{bj}^2 + I_{bj}^2 + F_{bj}^2}} \tag{3}$$

(2) Three vector similarity measures between A and B for INSs:

$$M_J(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\inf T_{aj}\inf T_{bj} + \sup T_{aj}\sup T_{bj} + \inf I_{aj}\inf I_{bj} + \sup I_{aj}\sup I_{bj} + \inf F_{aj}\inf F_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{l} (\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2 \\ +(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2 \\ -(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj}) \\ -(\sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj}) \end{array}\right)} \tag{4}$$

$$M_D(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj} + \sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{l} (\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2 \\ +(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2 \end{array}\right)} \tag{5}$$

$$M_C(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\inf T_{aj}\inf T_{bj} + \inf I_{aj}\inf I_{bj} + \inf F_{aj}\inf F_{bj} + \sup T_{aj}\sup T_{bj} + \sup I_{aj}\sup I_{bj} + \sup F_{aj}\sup F_{bj}\right)}{\left(\begin{array}{c}\sqrt{(\inf T_{aj})^2 + (\inf I_{aj})^2 + (\inf F_{aj})^2 + (\sup T_{aj})^2 + (\sup I_{aj})^2 + (\sup F_{aj})^2} \\ \sqrt{(\inf T_{bj})^2 + (\inf I_{bj})^2 + (\inf F_{bj})^2 + (\sup T_{bj})^2 + (\sup I_{bj})^2 + (\sup F_{bj})^2}\end{array}\right)} \quad (6)$$

Clearly, Equations (1)–(3) are special cases of Equations (4)–(6) when the upper and lower limits of the interval numbers for T_{aj} = [inf T_{aj}, sup T_{aj}] I_{aj} = [inf I_{aj}, sup I_{aj}], F_{aj} = [inf F_{aj}, sup F_{aj}], T_{bj} = [inf T_{bj}, sup T_{bj}], I_{bj} = [inf I_{bj}, sup I_{bj}], and F_{bj} = [inf F_{bj}, sup F_{bj}] are equal.

Then, the Jaccard, Dice, and cosine measures $M_k(A, B)$ (k = J, D, C) contains the following properties [8]:

(P1) $0 \le M_k(A, B) \le 1$;
(P2) $M_k(A, B) = M_k(B, A)$;
(P3) $M_k(A, B) = 1$ if $A = B$, i.e., $T_{aj}, = T_{bj}$, $I_{aj} = I_{bj}$, and $F_{aj} = F_{bj}$ for $j = 1, 2, \ldots, n$.

3. Refined Simplified Neutrosophic Sets

As the concept of SNS [7,8], a SNS A in a universe of discourse X is denoted as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where the values of its truth, indeterminacy, and falsity membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ for $x \in X$ are single-value and/or interval values in [0, 1]. Then, SNS contain INS and/or SVNS.

If the components $T_A(x)$, $I_A(x)$, $F_A(x)$ in SNS are refined (split) into $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$, respectively, for $x \in X$, $x = \{x_1, x_2, ..., x_r\}$, and a positive integer r, then they can be constructed as RSNS by the refinement of SNS, which is defined below.

Definition 1. *Let X be a universe of discourse, then a RSNS A in X can be defined as*

$$A = \{\langle x, (T_A(x_1), T_A(x_2), \ldots, T_A(x_r)), (I_A(x_1), I_A(x_2), \ldots, I_A(x_r)), (F_A(x_1), F_A(x_2), \ldots, F_A(x_r))\rangle | x \in X, x_j \in x\},$$

where $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$, $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$ for $x \in X$, $x_j \in x = \{x_1, x_2, ..., x_r\}$ ($j = 1, 2, \ldots, r$), and a positive integer r are subintervals/subsets in the real standard interval [0, 1], such that $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$: $X \rightarrow [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$: $X \rightarrow [0, 1]$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$: $X \rightarrow [0, 1]$.

Then, the RSNS A contains the following two concepts:

(1) If $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r) \in [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r) \in [0, 1]$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r) \in [0, 1]$ in A for $x \in X$ and $x_j \in x$ (j =1, 2, ... , r) are considered as single/exact values in [0, 1], then A reduces to RSVNS [22], which satisfies the condition $0 \le T_A(x_j) + I_A(x_j) + F_A(x_j) \le 3$ for $j = 1, 2, \ldots, r$;
(2) If $T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r) \subseteq [0, 1]$, $I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r) \subseteq [0, 1]$, and $F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r) \subseteq [0, 1]$ in A for $x \in X$ and $x_j \in x$ (j =1, 2, ... , r) are considered as interval values in [0, 1], then A reduces to RINS [23], which satisfies the condition $0 \le \sup T_A(x_j) + \sup I_A(x_j) + \sup F_A(x_j) \le 3$ for $j = 1, 2, \ldots, r$.

Particularly when the lower and upper limits of $T_A(x_j)$ = [inf $T_A(x_j)$, sup $T_A(x_j)$], $I_A(x_j)$ = [inf $I_A(x_j)$, sup $I_A(x_j)$] and $F_A(x_j)$ = [inf $F_A(x_j)$, sup $F_A(x_j)$] in A for $x \in X$ and $x_j \in x$ ($j = 1, 2, \ldots, r$) are equal, the RINS A reduces to the RSVNS A. Clearly, RSVNS is a special case of RINS. If some lower and upper limits of $T_A(x_j)$ = [inf $T_A(x_j)$, sup $T_A(x_j)$]/$I_A(x_j)$ = [inf $I_A(x_j)$, sup $I_A(x_j)$]/$F_A(x_j)$ = [inf $F_A(x_j)$, sup $F_A(x_j)$] in RINS are equal, then it can be denoted as a special interval (equal interval of the lower and upper limits) $T_A(x_j) = [T_A(x_j), T_A(x_j)]/I_A(x_j) = [I_A(x_j), I_A(x_j)]/F_A(x_j) = [F_A(x_j), F_A(x_j)]$. Hence, RINS can contain RINS and/or SVNS information (hybrid information of both).

For convenient expression, a basic element <x, ($T_A(x_1)$, $T_A(x_2)$, ..., $T_A(x_r)$), ($I_A(x_1)$, $I_A(x_2)$, ..., $I_A(x_r)$), ($F_A(x_1)$, $F_A(x_2)$, ..., $F_A(x_r)$)> in A is simply denoted as a = <(T_{a1}, T_{a2}, ... , T_{ar}), (I_{a1}, I_{a2}, ... , I_{ar}), (F_{a1}, F_{a2}, ... , F_{ar})>, which is called a refined simplified neutrosophic number (RSNN).

Let two RSNNs be a = <(T_{a1}, T_{a2}, ... , T_{ar}), (I_{a1}, I_{a2}, ... , I_{ar}), (F_{a1}, F_{a2}, ... , F_{ar})> and b = <(T_{b1}, T_{b2}, ... , T_{br}), (I_{b1}, I_{b2}, ... , I_{br}), (F_{b1}, F_{b2}, ... , F_{br})> for T_{aj}, T_{bj}, I_{aj}, I_{bj}, F_{aj}, F_{bj} $\in$ [0, 1] (j = 1, 2, ... , r). Then, there are the following relations between a and b:

(1) Containment: $a \subseteq b$, if and only if $T_{aj} \leq T_{bj}$, $I_{aj} \geq I_{bj}$, $F_{aj} \geq F_{bj}$ for j = 1, 2, ... , r;
(2) Equality: $a = b$, if and only if $a \subseteq b$ *and* $b \subseteq a$, i.e., $T_{aj} = T_{bj}$, $I_{aj} = I_{bj}$, $F_{aj} = F_{bj}$ for j = 1, 2, ... , r;
(3) Union:

$$a \cup b = \langle (T_{a1} \vee T_{b1}, T_{a2} \vee T_{b2}, \ldots, T_{ar} \vee T_{br}), (I_{a1} \wedge I_{b1}, I_{a2} \wedge I_{b2}, \ldots, I_{ar} \wedge I_{br}), (F_{a1} \wedge F_{b1}, F_{a2} \wedge F_{b2}, \ldots, F_{ar} \wedge F_{br}) \rangle;$$

(4) Intersection:

$$a \cap b = \langle (T_{a1} \wedge T_{b1}, T_{a2} \wedge T_{b2}, \ldots, T_{ar} \wedge T_{br}), (I_{a1} \vee I_{b1}, I_{a2} \vee I_{b2}, \ldots, I_{ar} \vee I_{br}), (F_{a1} \vee F_{b1}, F_{a2} \vee F_{b2}, \ldots, F_{ar} \vee F_{br}) \rangle.$$

Let two RSNNs be $a = \langle (T_{a1}, T_{a2}, \ldots, T_{ar}), (I_{a1}, I_{a2}, \ldots, I_{ar}), (F_{a1}, F_{a2}, \ldots, F_{ar}) \rangle$ and $b = \langle (T_{b1}, T_{b2}, \ldots, T_{br}), (I_{b1}, I_{b2}, \ldots, I_{br}), (F_{b1}, F_{b2}, \ldots, F_{br}) \rangle$ for T_{aj}, T_{bj}, I_{aj}, I_{bj}, F_{aj}, F_{bj} $\subseteq$ [0, 1] (j = 1, 2, ... , r). Then, there are the following relations of a and b:

(1) Containment: $a \subseteq b$, if and only if inf $T_{aj} \leq$ inf T_{bj}, sup $T_{aj} \leq$ sup T_{bj}, inf $I_{aj} \geq$ inf I_{bj}, sup $I_{aj} \geq$ sup I_{bj}, inf $F_{aj} \geq$ inf F_{bj}, and sup $F_{aj} \geq$ sup F_{bj} for j = 1, 2, ... , r;
(2) Equality: $a = b$, if and only if $a \subseteq b$ and $b \subseteq a$, i.e., inf T_{aj} = inf T_{bj}, sup T_{aj} = sup T_{bj}, inf I_{aj} = inf I_{bj}, sup I_{aj} = sup I_{bj}, inf F_{aj} = inf F_{bj}, and sup F_{aj} = sup F_{bj} for j = 1, 2, ... , r;
(3) Union:

$$a \cup b = \left\langle \begin{array}{l} ([\inf T_{a1} \vee \inf T_{b1}, \sup T_{a1} \vee \sup T_{b1}], [\inf T_{a2} \vee \inf T_{b2}, \sup T_{a2} \vee \sup T_{b2}], \ldots, [\inf T_{ar} \vee \inf T_{br}, \sup T_{ar} \vee \sup T_{br}]), \\ ([\inf I_{a1} \wedge \inf I_{b1}, \sup I_{a1} \wedge \sup I_{b1}], [\inf I_{a2} \wedge \inf I_{b2}, \sup I_{a2} \wedge \sup I_{b2}], \ldots, [\inf I_{ar} \wedge \inf I_{br}, \sup I_{ar} \wedge \sup I_{br}]), \\ ([\inf F_{a1} \wedge \inf F_{b1}, \sup F_{a1} \wedge \sup F_{b1}], [\inf F_{a2} \wedge \inf F_{b2}, \sup F_{a2} \wedge \sup F_{b2}], \ldots, [\inf F_{ar} \wedge \inf F_{br}, \sup F_{ar} \wedge \sup F_{br}]) \end{array} \right\rangle$$

(4) Intersection:

$$a \cap b = \left\langle \begin{array}{l} ([\inf T_{a1} \wedge \inf T_{b1}, \sup T_{a1} \wedge \sup T_{b1}], [\inf T_{a2} \wedge \inf T_{b2}, \sup T_{a2} \wedge \sup T_{b2}], \ldots, [\inf T_{ar} \wedge \inf T_{br}, \sup T_{ar} \wedge \sup T_{br}]), \\ ([\inf I_{a1} \vee \inf I_{b1}, \sup I_{a1} \vee \sup I_{b1}], [\inf I_{a2} \vee \inf I_{b2}, \sup I_{a2} \vee \sup I_{b2}], \ldots, [\inf I_{ar} \vee \inf I_{br}, \sup I_{ar} \vee \sup I_{br}]), \\ ([\inf F_{a1} \vee \inf F_{b1}, \sup F_{a1} \vee \sup F_{b1}], [\inf F_{a2} \vee \inf F_{b2}, \sup F_{a2} \vee \sup F_{b2}], \ldots, [\inf F_{ar} \vee \inf F_{br}, \sup F_{ar} \vee \sup F_{br}]) \end{array} \right\rangle$$

4. Vector Similarity Measures of RSNSs

Based on the Jaccard, Dice, and cosine measures between SNSs in vector space [8], this section proposes the three vector similarity measures between RSNSs.

Definition 2. *Let two RSNSs be A ={a_1, a_2, ... , a_n} and B ={b_1, b_2, ... , b_n}, where* $a_j = \left\langle (T_{a_j1}, T_{a_j2}, \ldots, T_{a_jr_j}), (I_{a_j1}, I_{a_j2}, \ldots, I_{a_jr_j}), (F_{a_j1}, F_{a_j2}, \ldots, F_{a_jr_j}) \right\rangle$ *and* $b_j = \left\langle (T_{b_j1}, T_{b_j2}, \ldots, T_{b_jr_j}), (I_{b_j1}, I_{b_j2}, \ldots, I_{b_jr_j}), (F_{b_j1}, F_{b_j2}, \ldots, F_{b_jr_j}) \right\rangle$ *for j = 1, 2, ... , n are two collections of RSNNs for* T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} $\in$ [0, 1] *or* T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} $\subseteq$ [0, 1] *(j = 1, 2, ... , n; k = 1, 2, ... ,* r_j*). Then, the Jaccard, Dice, and cosine measures between A and B are defined, respectively, as follows:*

(1) *Three vector similarity measures between A and B for RSVNSs:*

$$R_J(A,B) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\left[\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)-\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)\right]} \quad (7)$$

$$R_D(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{2\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)}{\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)} \tag{8}$$

$$R_C(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\sqrt{T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2}\sqrt{T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2}} \tag{9}$$

(2) *Three vector similarity measures between A and B for RINSs:*

$$R_J(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{\left(\begin{array}{l}\inf T_{a_jk}\inf T_{b_jk}+\sup T_{a_jk}\sup T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}\\+\sup I_{a_jk}\sup I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{array}\right)}{\left(\begin{array}{l}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\\-(\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk})\\-(\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk})\end{array}\right)}, \tag{10}$$

$$R_D(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{2\left(\begin{array}{l}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{array}\right)}{\left(\begin{array}{l}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\end{array}\right)}, \tag{11}$$

$$R_C(A,B)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}\sum_{k=1}^{r_j}\frac{\left(\begin{array}{l}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{array}\right)}{\left(\begin{array}{l}\sqrt{(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2}\\\sqrt{(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2}\end{array}\right)} \tag{12}$$

Clearly, Equations (7)–(9) are special cases of Equations (10)–(12) when the upper and lower limits of the interval numbers for $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \subseteq [0, 1]$ (j = 1, 2, … , n; k = *1, 2,* … *, r_j*) are equal. Especially when k = 1, Equations (7)–(12) are reduced to Equations (1)–(6).

Based on the properties of the Jaccard, Dice, and cosine measures of SNSs [8], it is obvious that the Jaccard, Dice, and cosine measures of RSNSs for $R_s(A, B)$ (s = J, D, C) also contain the following properties (P1)–(P3):

(P1) $0 \le R_s(A, B) \le 1$;
(P2) $R_s(A, B) = R_s(B, A)$;
(P3) $R_s(A, B) = 1$ if $A = B$, i.e., $T_{a_jk} = T_{b_jk}$, $I_{a_jk} = I_{b_jk}$, $F_{a_jk} = F_{b_jk}$ for j = 1, 2, … , n and k = 1, 2, ..., r_j.

When we consider the weights of different elements and sub-elements in RSNS, the weight of elements a_j and b_j (j = 1, 2, … , n) in the RSNSs A and B is given as $w_j \in [0, 1]$ with $\sum_{j=1}^{n} w_j = 1$ and the weight of the refined components (sub-elements) $T_{a_jk}, I_{a_jk}, F_{a_jk}$ and $T_{b_jk}, I_{b_jk}, F_{b_jk}$ (k = 1, 2, … , r_j) in $a_j = \left\langle (T_{a_j1}, T_{a_j2}, \ldots, T_{a_jr_j}), (I_{a_j1}, I_{a_j2}, \ldots, I_{a_jr_j}), (F_{a_j1}, F_{a_j2}, \ldots, F_{a_jr_j}) \right\rangle$ and $b_j = \left\langle (T_{b_j1}, T_{b_j2}, \ldots, T_{b_jr_j}), (I_{b_j1}, I_{b_j2}, \ldots, I_{b_jr_j}), (F_{b_j1}, F_{b_j2}, \ldots, F_{b_jr_j}) \right\rangle$ (j = 1, 2, … , n) is considered as $\omega_k \in [0, 1]$ with $\sum_{k=1}^{r_j} \omega_k = 1$, the weighted Jaccard, Dice, and cosine measures between A and B are presented, respectively, as follows:

(1) Three weighted vector similarity measures between A and B for RSVNSs:

$$R_{WJ}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\left[\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)-\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)\right]} \tag{13}$$

$$R_{WD}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{2\left(T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}\right)}{\left(T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2\right)+\left(T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2\right)} \tag{14}$$

$$R_{WC}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{T_{a_jk}T_{b_jk}+I_{a_jk}I_{b_jk}+F_{a_jk}F_{b_jk}}{\sqrt{T_{a_jk}^2+I_{a_jk}^2+F_{a_jk}^2}\sqrt{T_{b_jk}^2+I_{b_jk}^2+F_{b_jk}^2}} \tag{15}$$

(2) Three weighted vector similarity measures between A and B for RINSs:

$$R_{WJ}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\sup T_{a_jk}\sup T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}\\+\sup I_{a_jk}\sup I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\\-(\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk})\\-(\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk})\end{pmatrix}}, \tag{16}$$

$$R_{WD}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{2\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2\\+(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2\end{pmatrix}}, \tag{17}$$

$$R_{WC}(A,B)=\sum_{j=1}^{n}w_j\sum_{k=1}^{r_j}\omega_k\frac{\begin{pmatrix}\inf T_{a_jk}\inf T_{b_jk}+\inf I_{a_jk}\inf I_{b_jk}+\inf F_{a_jk}\inf F_{b_jk}\\+\sup T_{a_jk}\sup T_{b_jk}+\sup I_{a_jk}\sup I_{b_jk}+\sup F_{a_jk}\sup F_{b_jk}\end{pmatrix}}{\begin{pmatrix}\sqrt{(\inf T_{a_jk})^2+(\inf I_{a_jk})^2+(\inf F_{a_jk})^2+(\sup T_{a_jk})^2+(\sup I_{a_jk})^2+(\sup F_{a_jk})^2}\\\sqrt{(\inf T_{b_jk})^2+(\inf I_{b_jk})^2+(\inf F_{b_jk})^2+(\sup T_{b_jk})^2+(\sup I_{b_jk})^2+(\sup F_{b_jk})^2}\end{pmatrix}}. \tag{18}$$

Clearly, Equations (13)–(15) are special cases of Equations (16)–(18) when the upper and lower limits of the interval numbers for $T_{a_jk}, I_{a_jk}, F_{a_jk}, T_{b_jk}, I_{b_jk}, F_{b_jk} \subseteq [0, 1]$ (j = 1, 2, … , n; k = 1, 2, … , r_j) are equal. Especially when each $w_j = 1/n$ and $\omega_k = 1/r_j$ (j = 1, 2, … , n; k = 1, 2, … , r_j), Equations (13)–(18) are reduced to Equations (7)–(12).

Obviously, the weighted Jaccard, Dice, and cosine measures of RSNSs for $R_{Ws}(A, B)$ (s = J, D, C) also satisfies the following properties (P1)–(P3):

(P1) $0 \leq R_{Ws}(A, B) \leq 1$;
(P2) $R_{Ws}(A, B) = R_{Ws}(B, A)$;
(P3) $R_{Ws}(A, B) = 1$ if $A = B$, i.e., $T_{a_jk} = T_{b_jk}, I_{a_jk} = I_{b_jk}, F_{a_jk} = F_{b_jk}$ for j = 1, 2, … , n and k = 1, 2, ..., r_j.

5. Decision-Making Method Using the Vector Similarity Measures

In a decision-making problem with multiple attributes and sub-attributes, assume that $A = \{A_1, A_2, \ldots, A_m\}$ is a set of m alternatives, which needs to satisfies a set of n attributes $B = \{b_1, b_2, \ldots, b_n\}$, where b_j (j = 1, 2, … , n) may be refined/split into a set of r_j sub-attributes $b_j = \left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ (j = 1, 2, … , n). If the decision-maker provides the suitability evaluation values of attributes $b_j = \left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ (j =1, 2, ..., n) on the alternative A_i (i = 1, 2, … , m) by using RSNS: $A_i = \left\{\left\langle b_j, (T_{A_i}(b_{j1}), T_{A_i}(b_{j2}), \ldots, T_{A_i}(b_{jr_j})), (I_{A_i}(b_{j1}), I_{A_i}(b_{j2}), \ldots, I_{A_i}(b_{jr_j})), (F_{A_i}(b_{j1}), F_{A_i}(b_{j2}), \ldots, F_{A_i}(b_{jr_j}))\right\rangle \middle| b_j \in B, b_{jk} \in b_j\right\}$.

For convenient expression, each basic element in the RSNS A_i is represented by RSNN: $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j})\right\rangle$ for i = 1, 2, … , m and j = 1, 2, … , n.

Hence, we can construct the refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$, as shown in Table 1.

Table 1. Refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$.

	b_1	b_2	...	b_n
	$\{b_{11}, b_{12}, \ldots, b_{1r_1}\}$	$\{b_{21}, b_{22}, \ldots, b_{2r_2}\}$		$\{b_{n1}, b_{n2}, \ldots, b_{nr_n}\}$
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

When the weights of each attribute b_j (j = 1, 2, ... , n) and its sub-attributes are considered as having different importance, the weight vector of the attributes is given by $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and the weight vector for each sub-attribute set $\left\{b_{j1}, b_{j2}, \ldots, b_{jr_j}\right\}$ is given as $\boldsymbol{\omega}_j = \left\{\omega_{j1}, \omega_{j2}, \ldots, \omega_{jr_j}\right\}$ (j = 1, 2, ... , n) with $\omega_{jk} \in [0, 1]$ and $\sum_{k=1}^{r_j} \omega_{jk} = 1$. Thus, the decision steps are described as follows:

Step 1: We determine the ideal solution (ideal RSNN) from the refined simplified neutrosophic decision matrix $M(a_{ij})_{m\times n}$ as follows:

$$a_j^* = \left\langle \begin{array}{c} (\max_i(T_{j1}), \max_i(T_{j2}), \ldots, \max_i(T_{jr_j})), (\min_i(I_{j1}), \min_i(I_{j2}), \\ \ldots, \min_i(I_{jr_j})), (\min_i(F_{j1}), \min_i(F_{j2}), \ldots, \min_i(F_{jr_j})) \end{array} \right\rangle \text{ for RSVNN} \tag{19}$$

$$\text{or } a_j^* = \left\langle \begin{array}{c} ([\max_i(\inf T_{j1}), \max_i(\sup T_{j1})], [\max_i(\inf T_{j2}), \max_i(\sup T_{j2})], \ldots, [\max_i(\inf T_{jr_j}), \max_i(\sup T_{jr_j})]), \\ ([\min_i(\inf I_{j1}), \min_i(\sup I_{j1})], [\min_i(\inf I_{j2}), \min_i(\sup I_{j2})], \ldots, [\min_i(\inf I_{jr_j}), \min_i(\sup I_{jr_j})]), \\ ([\min_i(\inf F_{j1}), \min_i(\sup F_{j1})], [\min_i(\inf F_{j2}), \min_i(\sup F_{j2})], \ldots, [\min_i(\inf F_{jr_j}), \min_i(\sup F_{jr_j})]) \end{array} \right\rangle \text{ for RINN,} \tag{20}$$

which is constructed as the ideal alternative $A^* = \left\{a_1^*, a_2^*, \ldots, a_n^*\right\}$.

Step 2: The similarity measure between each alternative A_i (i = 1, 2, ... , m) and the ideal alternative A^* can be calculated by using one of Equations (13)–(15) or Equations (16)–(18), and obtained as the values of $R_{Ws}(A_i, A^*)$ for i = 1, 2, ... , m and s = J or D or C.

Step 3: According to the values of $R_{Ws}(A_i, A^*)$ for i = 1, 2, ... , m and s = J or D or C, the alternatives are ranked in a descending order. The greater value of $R_{Ws}(A_i, A^*)$ means the best alternative.

Step 4: End.

6. Illustrative Example on the Selection of Construction Projects

In this section, we apply the proposed decision-making method to the selection of construction projects adapted from [23].

Some construction company wants to select one of potential construction projects. The company provides four potential construction projects as their set $A = \{A_1, A_2, A_3, A_4\}$. To select the best one of them, experts or decision-makers need to make a decision of these construction projects corresponding to three attributes and their seven sub-attributes, which are described as follows:

(1) Financial state (b_1) contains two sub-attributes: budget control (b_{11}) and risk/return ratio (b_{12});
(2) Environmental protection (b_2) contains three sub-attributes: public relation (b_{21}), geographical location (b_{22}), and health and safety (b_{23});
(3) Technology (b_3) contains tow sub-attributes: technical know-how (b_{31}), technological capability (b_{32}).

Then, the weight vector of the three attributes is given by $\boldsymbol{W}$ = (0.4, 0.3, 0.3) and the weight vectors of the three sub-attribute sets $\{b_{11}, b_{12}\}$, $\{b_{21}, b_{22}, b_{23}\}$, and $\{b_{31}, b_{32}\}$ are given, respectively, by $\boldsymbol{\omega}_1$ = (0.6, 0.4), $\boldsymbol{\omega}_2$ = (0.25, 0.4, 0.35), and $\boldsymbol{\omega}_3$ = (0.45, 0.55).

In the following, we use the proposed decision-making method for solving the decision-making problem of construction projects under RSVNN and/or RINN environments to show the applications and effectiveness of the proposed decision-making method.

Under RSVNN environment, experts or decision-makers are required to evaluate the four possible alternatives under the above three attributes including seven sub-attributes by suitability judgments, which are described by RSVNN $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j}) \right\rangle$ for $T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j} \in [0, 1]$, $I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j} \in [0, 1]$, and $F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j} \in [0, 1]$ (i = 1, 2, 3, 4; j = 1, 2, 3; r_1 = 2, r_2 = 3, r_3 = 2). Thus, we can construct the following refined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$, which is shown in Table 2.

Table 2. Defined simplified neutrosophic decision matrix $M(a_{ij})_{4\times 3}$ under refined single-valued neutrosophic set (RSVNS) environment.

	b_1	b_2	b_3
	$\{b_{11}, b_{12}\}$	$\{b_{21}, b_{22}, b_{23}\}$	$\{b_{31}, b_{32}\}$
A_1	<(0.6, 0.7), (0.2, 0.1), (0.2, 0.3)>	<(0.9, 0.7, 0.8), (0.1, 0.3, 0.2), (0.2, 0.2, 0.1)>	<(0.6, 0.8), (0.3, 0.2), (0.3, 0.4)>
A_2	<(0.8, 0.7), (0.1, 0.2), (0.3, 0.2)>	<(0.7, 0.8, 0.7), (0.2, 0.4, 0.3), (0.1, 0.2, 0.1)>	<(0.8, 0.8), (0.1, 0.2), (0.1, 0.2)>
A_3	<(0.6, 0.8), (0.1, 0.3), (0.3, 0.4)>	<(0.8, 0.6, 0.7), (0.3, 0.1, 0.1), (0.2, 0.1, 0.2)>	<(0.8, 0.7), (0.4, 0.3), (0.2, 0.1)>
A_4	<(0.7, 0.6), (0.1, 0.2), (0.2, 0.3)>	<(0.7, 0.8, 0.7), (0.2, 0.2, 0.1), (0.1, 0.2, 0.2)>	<(0.7, 0.7), (0.2, 0.3), (0.2, 0.3)>

Under RSVNS environment, the proposed decision-making method is applied to the selection problem of the construction projects. The decision steps are described as follows:

Step 1: By Equation (19), the ideal solution (ideal RSVNS) can be determined as the following ideal alternative: A^* = {<(0.8, 0.8), (0.1, 0.1), (0.2, 0.2)>, <(0.9, 0.8, 0.8), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1)>, <(0.8, 0.8), (0.1, 0.2), (0.1, 0.1)>}.

Step 2: According to one of Equations (13)–(15), the weighted similarity measure values between each alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* can be obtained and all the results are shown in Table 3.

Step 3: In Table 3, since all the measure values are $R_{Ws}(A_2, A^*) > R_{Ws}(A_4, A^*) > R_{Ws}(A_3, A^*) > R_{Ws}(A_1, A^*)$ for s = J, D, C, all the ranking orders of the four alternatives are $A_2 \succ A_4 \succ A_3 \succ A_1$. Hence, the alternative A_2 is the best choice among all the construction projects.

Table 3. All the measure values between A_i (i = 1, 2, 3, 4) and A^* for RSVNSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$W(A_i, A^*)$ [23]	$W(A_1, A^*)$ = 0.9848, $W(A_2, A^*)$ = 0.9938, $W(A_3, A^*)$ = 0.9858, $W(A_4, A^*)$ = 0.9879	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WJ}(A_i, A^*)$	$R_{WJ}(A_1, A^*)$ = 0.9187, $R_{WJ}(A_2, A^*)$ = 0.9610, $R_{WJ}(A_3, A^*)$ = 0.9249, $R_{WJ}(A_4, A^*)$ = 0.9320	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WD}(A_i, A^*)$	$R_{WD}(A_1, A^*)$ = 0.9568, $R_{WD}(A_2, A^*)$ = 0.9797, $R_{WD}(A_3, A^*)$ = 0.9607, $R_{WD}(A_4, A^*)$ = 0.9646	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WC}(A_i, A^{*'})$	$R_{WC}(A_1, A^*)$ = 0.9646, $R_{WC}(A_2, A^*)$ = 0.9832, $R_{WC}(A_3, A^*)$ = 0.9731, $R_{WC}(A_4, A^*)$ = 0.9780	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

Under RINS environment, on the other hand, experts or decision-makers are required to evaluate the four possible alternatives under the above three attributes including seven sub-attributes by suitability judgments, which are described by RINN $a_{ij} = \left\langle (T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j}), (I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j}), (F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j}) \right\rangle$ for $T_{a_{ij}1}, T_{a_{ij}2}, \ldots, T_{a_{ij}r_j} \subseteq [0, 1]$, $I_{a_{ij}1}, I_{a_{ij}2}, \ldots, I_{a_{ij}r_j} \subseteq [0, 1]$, and $F_{a_{ij}1}, F_{a_{ij}2}, \ldots, F_{a_{ij}r_j} \subseteq [0, 1]$ (i = 1, 2, 3, 4; j = 1, 2, 3; r_1 = 2, r_2 = 3,

$r_3 = 2$). Thus, we can construct the following refined simplified neutrosophic decision matrix $M(a_{ij})_{4\times3}$, which is shown in Table 4.

Table 4. Defined simplified neutrosophic decision matrix $M(a_{ij})_{4\times3}$ under refined interval neutrosophic set (RINS) environment.

	b_1	b_2	b_3
	$\{b_{11}, b_{12}\}$	$\{b_{21}, b_{22}, b_{23}\}$	$\{b_{31}, b_{32}\}$
A_1	<([0.6,0.7],[0.7,0.8]),([0.2,0.3],[0.1,0.2]),([0.2,0.3],[0.3,0.4])>	<([0.8,0.9],[0.7,0.8],[0.8,0.9]),([0.1,0.2],[0.3,0.4],[0.2,0.3]),([0.2,0.3],[0.2,0.3],[0.1,0.2])>	<([0.6,0.7],[0.8,0.9]),([0.3,0.4],[0.2,0.3]),([0.3,0.4],[0.4,0.5])>
A_2	<([0.8,0.9],[0.7,0.8]),([0.1,0.2],[0.2,0.3]),([0.3,0.4],[0.2,0.3])>	<([0.7,0.8],[0.8,0.9],[0.7,0.8]),([0.2,0.3],[0.4,0.5],[0.3,0.4]),([0.1,0.2],[0.2,0.3],[0.1,0.2])>	<([0.8,0.9],[0.8,0.9]),([0.1,0.2],[0.2,0.3]),([0.1,0.2],[0.2,0.3])>
A_3	<([0.6,0.7],[0.8,0.9]),([0.1,0.2],[0.3,0.4]),([0.3,0.4],[0.4,0.5])>	<([0.8,0.9],[0.6,0.7],[0.7,0.8]),([0.3,0.4],[0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.1,0.2],[0.2,0.3])>	<([0.8,0.9],[0.7,0.8]),([0.4,0.5],[0.3,0.4]),([0.2,0.3],[0.1,0.2])>
A_4	<([0.7,0.8],[0.6,0.7]),([0.1,0.2],[0.2,0.3]),([0.2,0.3],[0.3,0.4])>	<([0.7,0.8],[0.8,0.9],[0.7,0.8]),([0.2,0.3],[0.2,0.3],[0.1,0.2]),([0.1,0.2],[0.2,0.3],[0.2,0.3])>	<([0.7,0.8],[0.7,0.8]),([0.2,0.3],[0.3,0.4]),([0.2,0.3],[0.3,0.4])>

Under RINS environment, the proposed decision-making method is applied to the selection problem of the construction projects. The decision steps are described as follows:

Step 1: By Equation (20), the ideal solution (ideal RINS) can be determined as the following ideal alternative:

$A^* = \{$<([0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.1, 0.2]), ([0.2, 0.3], [0.2, 0.3])>, <([0.8, 0.9], [0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.1, 0.2], [0.1, 0.2]), ([0.1, 0.2], [0.1, 0.2], [0.1, 0.2])>, <([0.8, 0.9], [0.8, 0.9]), ([0.1, 0.2], [0.2, 0.3]), ([0.1, 0.2], [0.1, 0.2])>$\}$.

Step 2: By using one of Equations (16)–(18), the weighted similarity measure values between each alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* can be calculated, and then all the results are shown in Table 5.

Step 3: In Table 5, since all the measure values are $R_{Ws}(A_2, A^*) > R_{Ws}(A_4, A^*) > R_{Ws}(A_3, A^*) > R_{Ws}(A_1, A^*)$ for s = J, D, C, all the ranking orders of the four alternatives are $A_2 \succ A_4 \succ A_3 \succ A_1$. Hence, the alternative A_2 is the best choice among all the construction projects.

Table 5. All the measure values between A_i (i = 1, 2, 3, 4) and A^* for RINSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$W(A_i, A^*)$ [23]	$W(A_1, A^*) = 0.9848$, $W(A_2, A^*) = 0.9932$, $W(A_3, A^*) = 0.9868$, $W(A_4, A^*) = 0.9886$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WJ}(A_i, A^*)$	$R_{WJ}(A_1, A^*) = 0.9314$, $R_{WJ}(A_2, A^*) = 0.9693$, $R_{WJ}(A_3, A^*) = 0.9369$, $R_{WJ}(A_4, A^*) = 0.9430$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WD}(A_i, A^*)$	$R_{WD}(A_1, A^*) = 0.9639$, $R_{WD}(A_2, A^*) = 0.9841$, $R_{WD}(A_3, A^*) = 0.9672$, $R_{WD}(A_4, A^*) = 0.9705$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
$R_{WC}(A_i, A^*)$	$R_{WC}(A_1, A^*) = 0.9697$, $R_{WC}(A_2, A^*) = 0.9860$, $R_{WC}(A_3, A^*) = 0.9775$, $R_{WC}(A_4, A^*) = 0.9805$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

For convenient comparison with existing related method [23], the decision results based on the cosine function without considering sub-attribute weights in the literature [23] are also indicated in Tables 3 and 5. Obviously, all the ranking orders are identical, which indicate the feasibility and effectiveness of the developed decision-making method based on the proposed measures R_{Ws} for

s = J, D, C. However, the existing related decision-making methods with RSVNSs and RINSs [22,23] cannot deal with such a decision-making problem with both attribute weights and sub-attribute weights in this paper. Although the same computational complexity in decision-making algorithms is shown by comparison of the method of this study with the related methods introduced in [22,23], the developed method in this study extends the methods in [22,23] and is more feasible and more general than the existing related decision-making methods [22,23]. It is obvious that the new developed decision-making method in a RSNS (RINS and/or SVNS) setting is superior to the existing related methods in a RINS or SVNS setting [22,23].

Compared with traditional decision-making approaches without sub-attributes [7–13,19–21], the decision-making approach proposed in this study can deal with decision-making problems with both attributes and sub-attributes; while traditional decision-making approaches [7–13,19–21] cannot deal with such a decision-making problem with both attributes and sub-attributes. Hence, the proposed decision-making approach is superior to traditional ones [7–13,19–21].

However, the study in this paper provides new three vector measures and their decision-making method as the main contributions due to no study of existing literature on the vector similarity measures and decision-making methods with RSNSs (RSVNSs and/or RINSs). Clearly, the main advantages of this study are that it can solve decision-making problems with the weights of both attributes and sub-attributes, which all existing methods cannot deal with, and extend existing similarity measures and decision-making methods.

To analyze the sensitivities of the proposed approach, let us change the RINS of the alternative A_4 into the RSNS $A_{4'}$ = {<([0.7,0.7], [0.6,0.6]), ([0.2,0.2], [0.2,0.2]), ([0.3,0.3], [0.3,0.3])>, <([0.7,0.8], [0.8,0.9], [0.7,0.8]), ([0.2,0.3], [0.2,0.3], [0.1,0.2]), ([0.1,0.2], [0.2,0.3], [0.2,0.3])>, <([0.7,0.8], [0.7,0.8]), ([0.2,0.3], [0.3,0.4]), ([0.2,0.3], [0.3,0.4])> with hybrid information of both RSVNNs and RINNs. Then, by above similar computation steps, we can obtain all the measure values, which are shown in Table 6.

Table 6. All the measure values between A_i (i = 1, 2, 3, 4′) and A^* for RSNSs and ranking orders of the four alternatives.

Measure Method	Measure Value	Ranking Order	The Best Choice
$R_{WJ}(A_i, A_*)$	$R_{WJ}(A_1, A_*) = 0.9314$, $R_{WJ}(A_2, A_*) = 0.9693$, $R_{WJ}(A_3, A_*) = 0.9369$, $R_{WJ}(A_4, A_*) = 0.9356$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
$R_{WD}(A_i, A_*)$	$R_{WD}(A_1, A_*) = 0.9639$, $R_{WD}(A_2, A_*) = 0.9841$, $R_{WD}(A_3, A_*) = 0.9672$, $R_{WD}(A_4, A_*) = 0.9665$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
$R_{WC}(A_i, A_*)$	$R_{WC}(A_1, A_*) = 0.9697$, $R_{WC}(A_2, A_*) = 0.9860$, $R_{WC}(A_3, A_*) = 0.9775$, $R_{WC}(A_4, A_*) = 0.9780$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2

The results of Table 6 demonstrate the ranking orders based on $R_{WJ}(A_i, A^*)$ and $R_{WD}(A_i, A^*)$ are the same, but their decision-making method can change the previous ranking orders and show some difference between two alternatives A_3 and A_4; while the best one is still A_2. Clearly, the decision-making approach based on the Jaccard and Dice measures shows some sensitivity in this case. However, the ranking order based on $R_{WC}(A_i, A^*)$ still keeps the previous ranking order, and then the decision-making approach based on the cosine measure shows some robustness/insensitivity in this case.

In actual decision-making problems, decision-makers can select one of three vector measures of RSNSs to apply it to multiple attribute decision-making problems with weights of attributes and sub-attributes according to their preference and actual requirements.

7. Conclusions

This paper introduced RSNSs, including the concepts of RSVNSs and RINSs, and proposed the vector similarity measures of RSNSs, including the Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) in vector space. It then presented the weighted Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) by considering the weights of basic elements in

RSNSs and the weights of sub-elements (the refined weights) in each RSNN. Further, we established a decision-making method based on the weighted Jaccard/Dice/cosine measures of RSNSs (RSVNSs and RINSs) to deal with multiple attribute decision-making problems with both attribute weights and sub-attribute weights under RSNS (RINS and/or RSVNS) environments. In the decision-making process, through the Jaccard/Dice/cosine measures between each alternative and the ideal alternative, the ranking order of all alternatives and the best one can be determined based on the measure values. Finally, an actual example on the decision-making problem of construction projects with RSNS (RSVNS and/or RINS) information is provided to demonstrate the application and effectiveness of the proposed method. The proposed approach is very suitable for actual applications in decision-making problems with weights of both attributes and sub-attributes under RSNS (RINS and/or RSVNS) environments, and provides a new decision-making method. In the future, we shall further extend the proposed method to group decision-making, clustering analysis, medical diagnosis, fault diagnosis, and so forth.

Acknowledgments: This paper was supported by the National Natural Science Foundation of China (Nos. 71471172, 41427802).

Author Contributions: Jun Ye proposed the vector similarity measures of RSNSs, including the weighted Jaccard, Dice, and cosine measures between RSNSs (RSVNSs and RINSs) in vector space; Jiqian Chen and Jun Ye established a decision-making method based on the weighted Jaccard/Dice/cosine measures of RSNSs (RSVNSs and RINSs) to deal with multiple attribute decision-making problems with both attribute weights and sub-attribute weights under RSNS (RINS and/or RSVNS) environments; Jiqian Chen and Shigui Du gave an actual example on the decision-making problem of construction projects with RSNS (RSVNS and/or RINS) information and its calculation and comparative analysis; we wrote the paper together.

Conflicts of Interest: The authors declare no conflicts of interest.

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