Introduction:

The Non-Standard Real Unit Interval

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Abstract: In this paper one defines the non-standard real unit interval $]\,^{-}0, 1^{+}[$ as a support for neutrosophy, neutrosophic logic, neutrosophic set, neutrosophic probability and statistics encountered in the next papers.

Keywords and Phrases: Non-standard analysis, hyper-real number, infinitesimal, monad, non-standard real unit interval, operations with sets

0.1. A Small Introduction to Non-Standard Analysis.

In 1960s Abraham Robinson has developed the *non-standard analysis*, a formalization of analysis and a branch of mathematical logic, that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < 1/n$. Let $\varepsilon>0$ be a such infinitesimal number. The *hyper-real number set* is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers. Let's consider the non-standard finite numbers $1^{+} = 1+\varepsilon$, where "1" is its standard part and "ε" its non-standard part, and $^{-}0 = 0-\varepsilon$, where "0" is its standard part and "ε" its non-standard part.

Then, we call $]\,^{-}0, 1^{+}[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. Actually, by "^{-}a" one signifies a monad, i.e. a set of hyper-real numbers in non-standard analysis:

$(^{-}a)= \{a-x: x\in \mathbf{R}^{*}, x \text{ is infinitesimal}\}$,

and similarly "b^{+}" is a monad:

$(b^{+})= \{b+x: x\in \mathbf{R}^{*}, x \text{ is infinitesimal}\}$.

Generally, the left and right borders of a non-standard interval $]\,^{-}a, b^{+}[$ are vague, imprecise, themselves being non-standard (sub)sets (^{-}a) and (b^{+}) as defined above.

Combining the two before mentioned definitions one gets, what we would call, a binad of "$^{-}c^{+}$":

$(^{-}c^{+})= \{c-x: x\in \mathbf{R}^{*}, x \text{ is infinitesimal}\} \cup \{c+x: x\in \mathbf{R}^{*}, x \text{ is infinitesimal}\}$, which is a collection of open punctured neighborhoods (balls) of c.

Of course, $^{-}a < a$ and $b^{+} > b$. No order between $^{-}c^{+}$ and c.

Addition of non-standard finite numbers with themselves or with real numbers:

$^{-}a + b = {}^{-}(a + b)$

$a + b^{+} = (a + b)^{+}$

$^{-}a + b^{+} = {}^{-}(a + b)^{+}$

$^{-}a + {}^{-}b = {}^{-}(a + b)$ (the left monads absorb themselves)

$a^{+} + b^{+} = (a + b)^{+}$ (analogously, the right monads absorb themselves)

Similarly for subtraction, multiplication, division, roots, and powers of non-standard finite numbers with themselves or with real numbers.

By extension let inf $]^-a, b^+[= {}^-a$ and sup$]^-a, b^+[= b^+$.

0.2. Definition of neutrosophic components.

Let T, I, F be standard or non-standard real subsets of $]^-0, 1^+[$,

with sup T = t_sup, inf T = t_inf,
sup I = i_sup, inf I = i_inf,
sup F = f_sup, inf F = f_inf,
and n_sup = t_sup+i_sup+f_sup,
n_inf = t_inf+i_inf+f_inf.

The sets T, I, F are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countably or uncountably) infinite; union or intersection of various subsets; etc.

They may also overlap. The real subsets could represent the relative errors in determining t, i, f (in the case when the subsets T, I, F are reduced to points).

In the next papers, T, I, F, called *neutrosophic components,* will represent the truth value, indeterminacy value, and falsehood value respectively referring to neutrosophy, neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics.

This representation is closer to the human mind reasoning. It characterizes/catches the *imprecision* of knowledge or linguistic inexactitude received by various observers (that's why T, I, F are subsets - not necessarily single-elements), *uncertainty* due to incomplete knowledge or acquisition errors or stochasticity (that's why the subset I exists), and *vagueness* due to lack of clear contours or limits (that's why T, I, F are subsets and I exists; in particular for the appurtenance to the neutrosophic sets).

One has to specify the superior (x_sup) and inferior (x_inf) limits of the subsets because in many problems arises the necessity to compute them.

0.3. Operations with sets.

Let S_1 and S_2 be two (unidimensional) real standard or non-standard subsets, then one defines:

0.3.1. Addition of sets:

$S_1 \oplus S_2 = \{x \mid x=s_1+s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$,
with inf $S_1 \oplus S_2$ = inf S_1 + inf S_2, sup $S_1 \oplus S_2$ = sup S_1 + sup S_2;
and, as some particular cases, we have
$\{a\} \oplus S_2 = \{x \mid x=a+s_2$, where $s_2 \in S_2\}$
with inf $\{a\} \oplus S_2 = a$ + inf S_2, sup $\{a\} \oplus S_2 = a$ + sup S_2;
also $\{1^+\} \oplus S_2 = \{x \mid x=1^+ + s_2$, where $s_2 \in S_2\}$
with inf $\{1^+\} \oplus S_2 = 1^+$ + inf S_2, sup $\{1^+\} \oplus S_2 = 1^+$ + sup S_2.

0.3.2. Subtraction of sets:

$S_1 \ominus S_2 = \{x \mid x=s_1-s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
inf $S_1 \ominus S_2$ = inf S_1 - sup S_2, sup $S_1 \ominus S_2$ = sup S_1 - inf S_2;
and, as some particular cases, we have

$\{a\} \ominus S_2 = \{x \mid x = a - s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{a\} \ominus S_2 = a - \sup S_2$, $\sup \{a\} \ominus S_2 = a - \inf S_2$;
also $\{1^+\} \ominus S_2 = \{x \mid x = 1^+ - s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{1^+\} \ominus S_2 = 1^+ - \sup S_2$, $\sup \{1^+\} \ominus S_2 = 1^+ - \inf S_2$.

0.3.3. Multiplication of sets:
$S_1 \odot S_2 = \{x \mid x = s_1 \cdot s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
$\inf S_1 \odot S_2 = \inf S_1 \cdot \inf S_2$, $\sup S_1 \odot S_2 = \sup S_1 \cdot \sup S_2$;
and, as some particular cases, we have
$\{a\} \odot S_2 = \{x \mid x = a \cdot s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{a\} \odot S_2 = a * \inf S_2$, $\sup \{a\} \odot S_2 = a \cdot \sup S_2$;
also $\{1^+\} \odot S_2 = \{x \mid x = 1^+ \cdot s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{1^+\} \odot S_2 = 1^+ \cdot \inf S_2$, $\sup \{1^+\} \odot S_2 = 1^+ \cdot \sup S_2$.

0.3.4. Division of a set by a number:
Let $k \in \mathbb{R}^*$, then $S_1 \oslash k = \{x \mid x = s_1/k, \text{ where } s_1 \in S_1\}$.

Acknowledgements:

The author would like to thank Drs. Charles T. Le and Ivan Stojmenovic for encouragement and invitation to write this and the following papers.

References:

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